Graph Theory

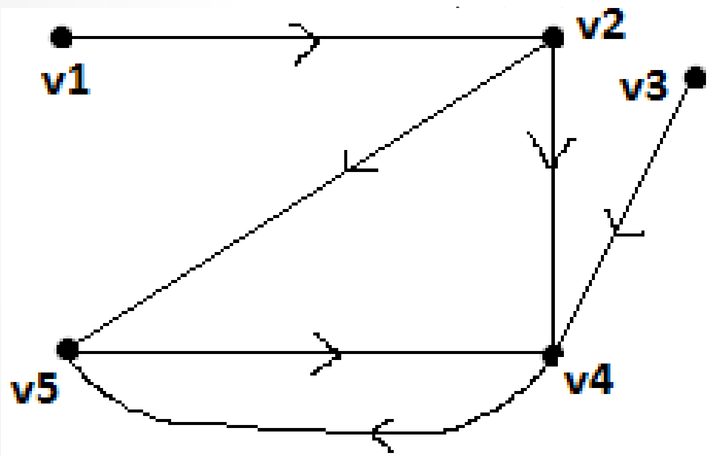
Operations on Graphs

Lecture 3

Adjacency Matrix of a diagraph

It is defined in similar fashion as it defined for undirected graph.

For Example,



$$
\begin{array}{c} \\ v1 \\ v2 \\ v3 \\ v4 \\ v5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} v1 & v2 & v3 & v4 & v5 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

Adjacency Matrix of a diagraph

- Sum of all aij in each row is equal to deg (Vi)
- If sum of all aij = 0, then Vi is an isolated vertex (if 1, then pendant), (if % 2, then even, else odd).
- Parallel edges between vertices will have identical columns
- A disconnected graph (of two components G1, and G2 can be written in block-diagonal form:

$$A(g) = \begin{bmatrix} A_1(g_1) & 0 \\ 0 & A_2(g_2) \end{bmatrix}$$

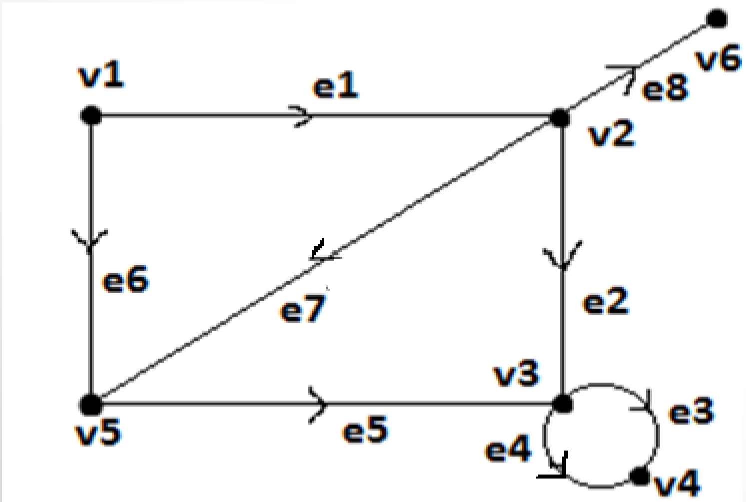
- There cannot be more than two one's in any column of an incidence matrix (2 for normal edges, 1 for loops)

Incident matrix of diagraph

Given a graph G with n, e & no self loops is matrix $x(G)=[X_{ij}]$ or order n*e where n vertices are rows & e edges are columns such that, $X_{ij}=1$, if jth edge e_j is incident out i^{th} vertex v_i

$X_{ij}=-1$, if jth edge e_j is incident into i^{th} vertex v_i

$X_{ij}=0$, if jth edge e_j not incident on i^{th} vertex v_i.



	e1	e2	e3	e4	e5	e6	e7	e8
v1	1	0	0	0	0	1	0	0
v2	-1	1	0	0	0	0	1	1
v3	0	-1	1	1	-1	0	0	0
v4	0	0	-1	-1	0	0	0	0
v5	0	0	0	0	1	-1	-1	0
v6	0	0	0	0	0	0	0	-1

Circuit Matrix

- Circuit can be defined as “A close walk in which no vertex/edge can appear twice”.
- If edge of graph is a part of given circuit then put 1 else 0.

Theorem

If B is a circuit matrix, and A is an incident matrix, then every row of B is orthogonal to every row of A. In other words,

$$A \, . \, B^T = B \, . \, A^T = 0$$

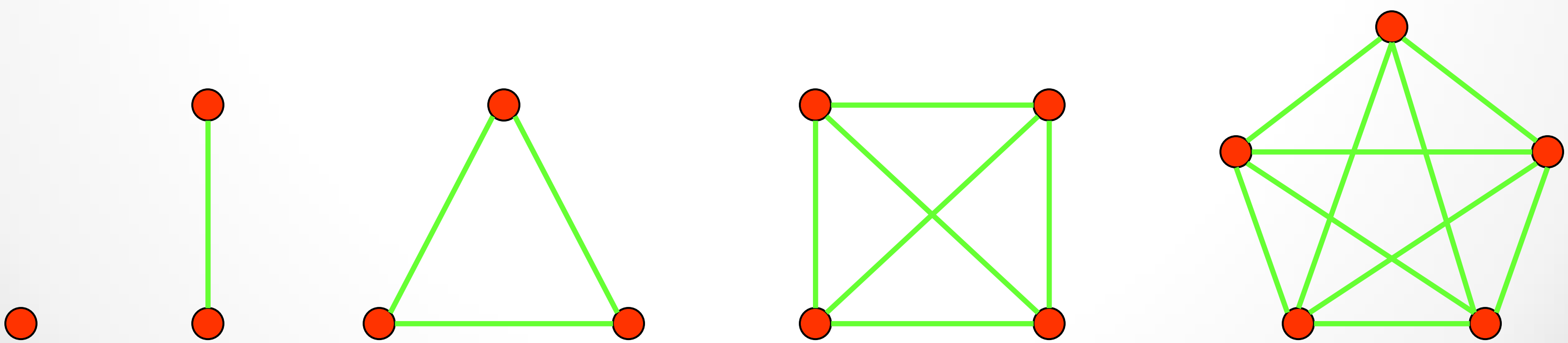
Complete Graph

Definition: Let G be simple graph on n vertices. If the degree of each vertex is (n-1) then the graph is called as **complete graph.**

Complete graph on n vertices, it is denoted by $\mathbf{K_n}$.

Theorem: In complete graph K_n, the number of edges are

n(n-1)/2,For example,

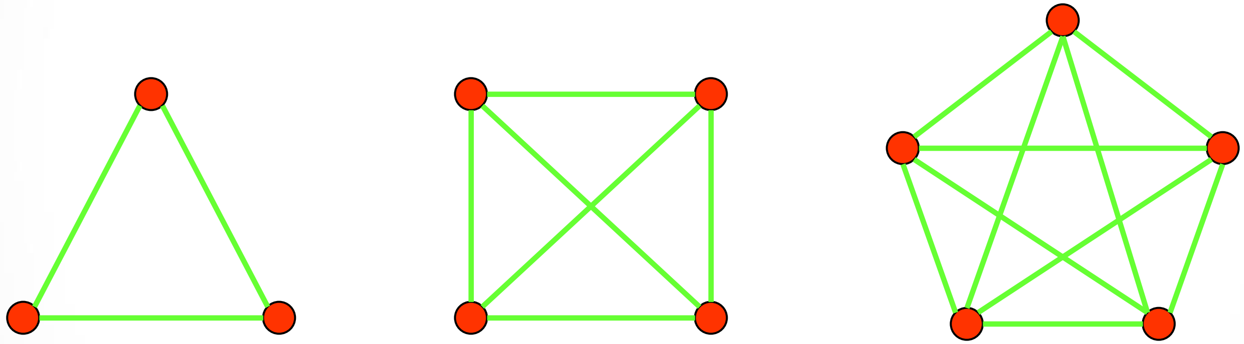


K_1 K_2 K_3 K_4 K_5

Regular Graph

Definition: If the degree of each vertex is same say 'r' in any graph G then the graph is said to be a **regular graph** of degree r.

For example,



K_3 **K_4** **K_5**

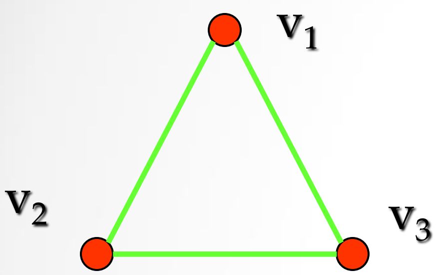
Bipartite Graph

Definition: The graph is called as **bipartite graph** , if its vertex set V can be partitioned into two distinct subset say V1 & V2. such that V1 U V2=V & V1 $\cap$ V2 = $\emptyset$ & also each edge of G joins a vertex of V1 to vertex of V2.

A graph can not have self loop.

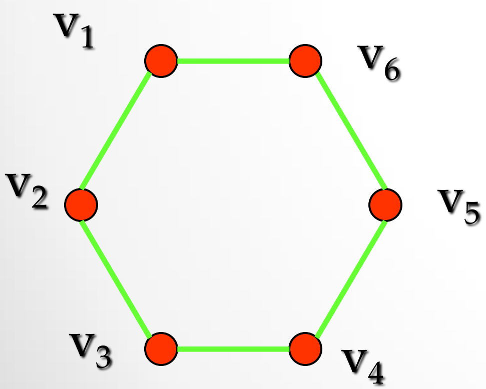
Bipartite Graphs

Example I: Is G1 bipartite?

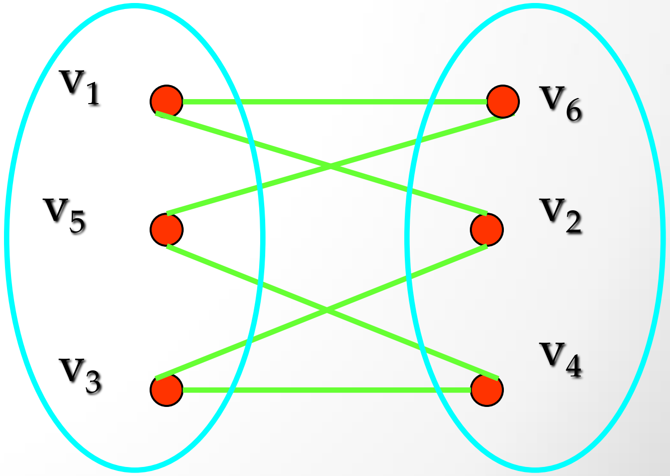


No, because there is no way to partition the vertices into two sets so that there are no edges with both endpoints in the same set.

Example II: Is G2 bipartite?



Yes, because we can display G2 like this:



Handshaking Lemma

Theorem: The graph G with e no. of edges & n no. of vertices, since each edge contributes two degree, the sum of the degrees of all vertices in G is twice no. of edges in G.

i.e. $\sum_{i=1}^{n} d(v_i)=2e$ is called as **Handshaking Lemma**.

Example: How many edges are there in a graph with 10 vertices, each of degree 6? **Solution:** The sum of the degrees of the vertices is 6*10 = 60. According to the Handshaking Theorem, it follows that 2e = 60, so there are 30 edges.

Some more Theorems

Theorem: Theorem 3

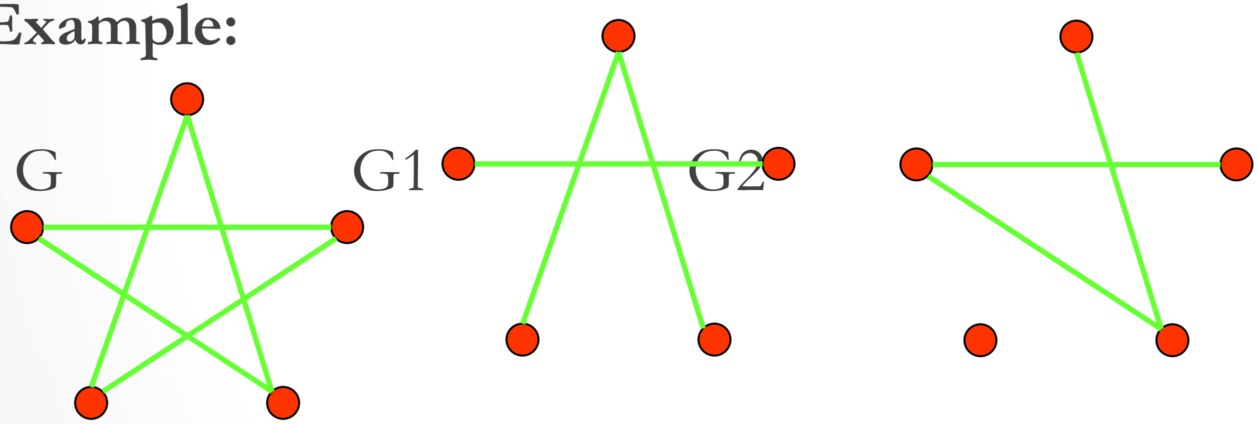
The number of vertices of odd degree in a graph is always even.

Example: How many edges are there in a graph with 10 vertices, each of degree 6? **Solution:** The sum of the degrees of the vertices is 6*10 = 60. According to the Handshaking Theorem, it follows that 2e = 60, so there are 30 edges.

Spanning Graph

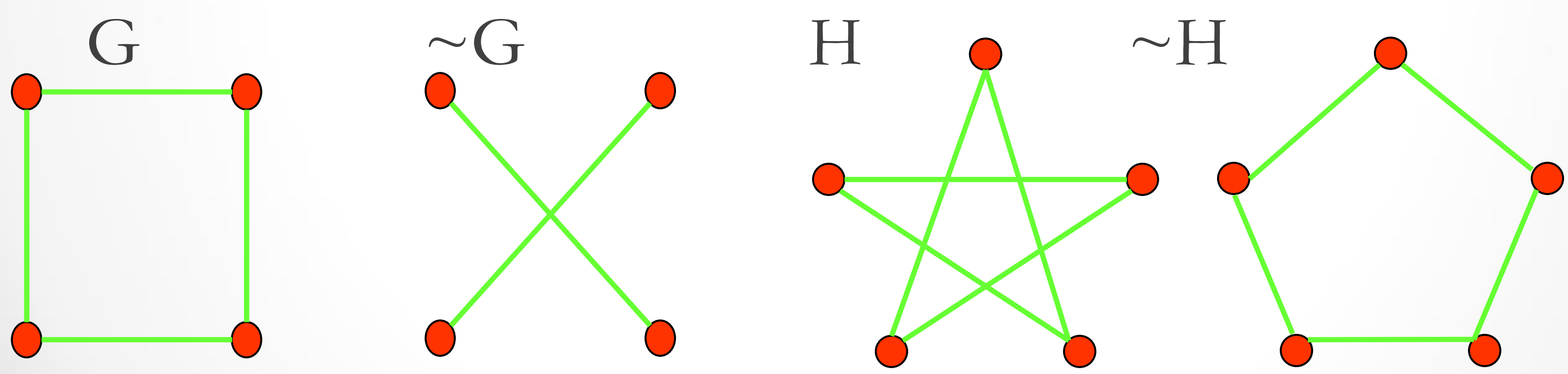
Definition: Let G=(V, E) be any graph. Then G' is said to be the **spanning subgraph** of the graph G if its vertex set V' is equal to vertex set V of G.

For Example:



Complement of a Graph

Definition: Let G is a simple graph. Then **complement of G** denoted by ~G is graph whose vertex set is same as vertex set of G & in which two vertices are adjacent if & only if they are not adjacent in G.**For Example:**



Operations on Graphs

Unary Operations:

- Local changes, e.g., add/delete a vertex, add/delete an edge. Deletion implies removal of vertex, as well as all edges incident to it.
- Edge Contraction: Process of removing an edge eu;v from a graph G while simultaneously merging adjacent vertices u; v into an arbitrary vertex w, such that all adjacent vertices of u are now adjacent to w, and all adjacent vertices of v are now adjacent to w.

Operations on Graphs

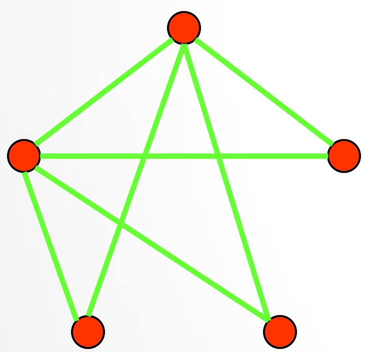
Binary Operations:

- **Union**: of two graphs G1 = (V1; E1) and G2 = (V2; E2) is G3 = G1 [G2, whose set is given as (V3; E3) = (V1 [V2; E1 [E2).
- **Intersection**: of two graphs G1 = (V1; E1) and G2 = (V2; E2) is G3 = G1 \ G2, whose set is given as (V3; E3) = (V1 \ V2; E1 \ E2). (I.e., only includes common vertices and edges of G1 and G2)
- **Ring Sum**: of two graphs G1 = (V1; E1) and G2 = (V2; E2) is G3 = G1 G2, whose vertex set V3 = (V1 [V2), and edge set contains only edges of G1 and G2 that are either in G1 or G2 but not in both.
- **Cartesian product**: of two graphs G1 = (V1; E1) and G2 = (V2; E2) is G3 = G1G2, whose vertex set V3 = V1 V2 is formed by making set V1 adjacent to set V2, and the edge set E3 is formed consequently due to vertex adjacency property.
- **Tensor Product**: of two graphs G1 = (V1; E1), represented as an adjacency matrix [G1]mn and G2 = (V2; E2), represented as an adjacency matrix [G2]pq, is G23 = [G1] [G2], represented as a mp nq block matrix

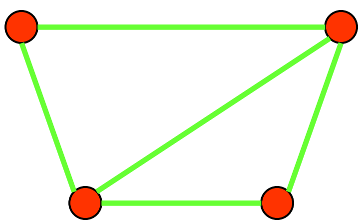
Operations on Graphs

Definition: The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

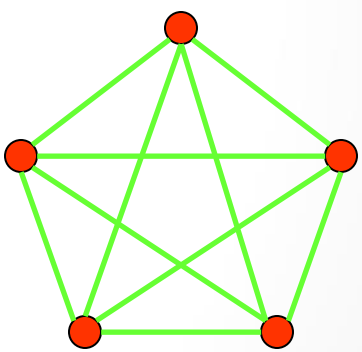
The union of G_1 and G_2 is denoted by $\mathbf{G_1 \cup G_2}$.



$\mathbf{G_1}$



$\mathbf{G_2}$

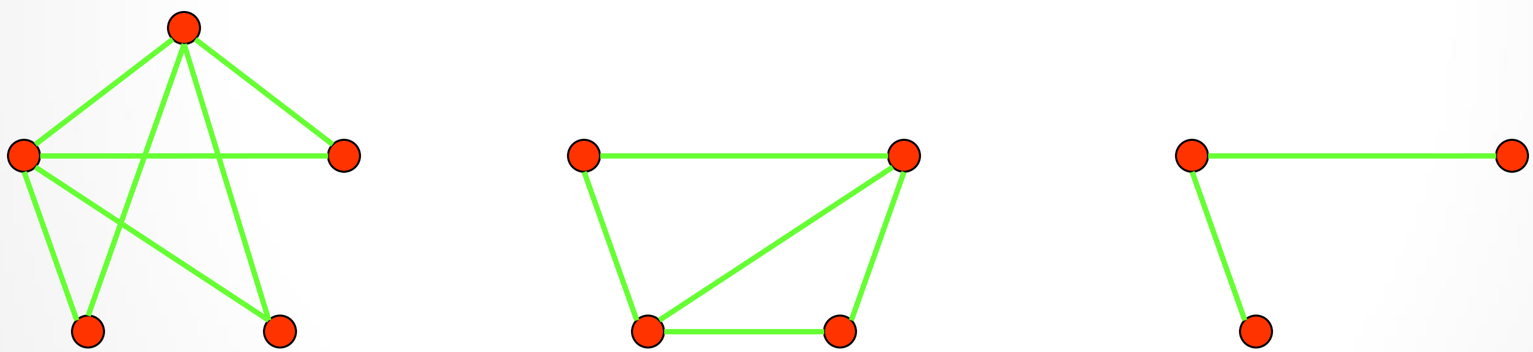


$\mathbf{G_1 \cup G_2}$

Operations on Graphs

Definition: The **Intersection** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$.

The Intersection of G_1 and G_2 is denoted by $\mathbf{G_1 \cap G_2}$.



$\mathbf{G_1}$ $\mathbf{G_2}$ $\mathbf{G_1 \cap G_2}$