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Students/exams Problem

- If a student has studied for an exam, what is the probability that he/she passes the exam?

- $P(\text{pass} \mid \text{study}) = 0.7426$

- Notation:

$$P(A) = P(A,B,C) + P(A,B,\neg C) + P(A,\neg B,C) + P(A,\neg B,\neg C)$$

is abbreviated as

$$P(A) = \sum P(A,B^*,C^*)$$

P(pass | study)

$$\mathrm{P}(\mathrm{pass} \mid \mathrm{study}) = \frac{\mathrm{P}(\mathrm{pass}, \mathrm{study})}{\mathrm{P}(\mathrm{study})}$$

$$= \frac{\sum \mathrm{P}(\mathrm{pass}, \mathrm{study}, \mathrm{prep}^*, \mathrm{fair}^*, \mathrm{smart}^*)}{\mathrm{P}(\mathrm{study})}$$

$$= \frac{\sum \mathrm{P}(\mathrm{pass} \mid \cancel{\mathrm{study}}, \mathrm{prep}^*, \mathrm{fair}^*, \mathrm{smart}^*)\, \mathrm{P}(\mathrm{prep}^* \mid \mathrm{study}, \cancel{\mathrm{fair}^*}, \mathrm{smart}^*)\, \mathrm{P}(\mathrm{study})\, \mathrm{P}(\mathrm{smart}^*)\, \mathrm{P}(\mathrm{fair}^*)}{\mathrm{P}(\mathrm{study})}$$

$$= \frac{\sum \mathrm{P}(\mathrm{pass} \mid \mathrm{prep}^*, \mathrm{fair}^*, \mathrm{smart}^*)\, \mathrm{P}(\mathrm{prep}^* \mid \mathrm{study}, \mathrm{smart}^*)\, \cancel{\mathrm{P}(\mathrm{study})}\, \mathrm{P}(\mathrm{smart}^*)\, \mathrm{P}(\mathrm{fair}^*)}{\cancel{\mathrm{P}(\mathrm{study})}}$$

$$= \sum \mathrm{P}(\mathrm{pass} \mid \mathrm{prep}^*, \mathrm{fair}^*, \mathrm{smart}^*)\, \mathrm{P}(\mathrm{prep}^* \mid \mathrm{study}, \mathrm{smart}^*)\, \mathrm{P}(\mathrm{smart}^*)\, \mathrm{P}(\mathrm{fair}^*)$$

Fig 1

(Cont)

$$\sum \underbrace{\mathrm{P}(\mathrm{pass} \mid \mathrm{prep}^*, \mathrm{fair}^*, \mathrm{smart}^*)}_{A} \underbrace{\mathrm{P}(\mathrm{prep}^* \mid \mathrm{study}, \mathrm{smart}^*)}_{B} \underbrace{\mathrm{P}(\mathrm{smart}^*)}_{C} \underbrace{\mathrm{P}(\mathrm{fair}^*)}_{D}$$

(t) pass	(t) study	smart	prep	fair	A	B	C	D	TOT
		f	f	f	0.1	0.3	0.2	0.1	0.0006
		f	f	t	0.2	0.3	0.2	0.9	0.0108
		f	t	f	0.1	0.7	0.2	0.1	0.0014
		f	t	t	0.7	0.7	0.2	0.9	0.0882
		t	f	f	0.1	0.1	0.8	0.1	0.0008
		t	f	t	0.7	0.1	0.8	0.9	0.0504
		t	t	f	0.1	0.9	0.8	0.1	0.0072
		t	t	t	0.9	0.9	0.8	0.9	0.5832
									0.7426

Fig 2

P(prep | smart)

$$\mathrm{P}(\text{prep} \mid \text{smart}) = \frac{\mathrm{P}(\text{prep, smart})}{\mathrm{P}(\text{smart})}$$

$$= \frac{\mathrm{P}(\text{prep, smart, study}) + \mathrm{P}(\text{prep, smart, } \neg\text{study})}{\mathrm{P}(\text{smart})}$$

$$= \frac{\mathrm{P}(\text{prep} \mid \text{smart, study})\ \mathrm{P}(\text{smart, study}) + \mathrm{P}(\text{prep} \mid \text{smart, } \neg\text{study})\ \mathrm{P}(\text{smart, } \neg\text{study})}{\mathrm{P}(\text{smart})}$$

$$= \frac{\mathrm{P}(\text{prep} \mid \text{smart, study})\ \cancel{\mathrm{P}(\text{smart})}\ \mathrm{P}(\text{study}) + \mathrm{P}(\text{prep} \mid \text{smart, } \neg\text{study})\ \cancel{\mathrm{P}(\text{smart})}\ \mathrm{P}(\neg\text{study})}{\cancel{\mathrm{P}(\text{smart})}}$$

$$= 0.9 \times 0.6 + 0.5 \times 0.4$$

$$= 0.74$$

Fig 3

$\mathbf{P(prep \mid \neg smart)}$

$$
\begin{aligned}
\mathrm{P}(\mathrm{prep} \mid \neg\mathrm{smart}) &= \frac{\mathrm{P}(\mathrm{prep},\, \neg\mathrm{smart})}{\mathrm{P}(\neg\mathrm{smart})} \\
&= \frac{\mathrm{P}(\mathrm{prep},\, \neg\mathrm{smart},\, \mathrm{study}) + \mathrm{P}(\mathrm{prep},\, \neg\mathrm{smart},\, \neg\mathrm{study})}{\mathrm{P}(\neg\mathrm{smart})} \\
&= \frac{\mathrm{P}(\mathrm{prep} \mid \neg\mathrm{smart},\, \mathrm{study})\, \cancel{\mathrm{P}(\neg\mathrm{smart})}\, \mathrm{P}(\mathrm{study}) + \mathrm{P}(\mathrm{prep} \mid \neg\mathrm{smart},\, \neg\mathrm{study})\, \cancel{\mathrm{P}(\neg\mathrm{smart})}\, \mathrm{P}(\neg\mathrm{study})}{\cancel{\mathrm{P}(\neg\mathrm{smart})}} \\
&= 0.7 \times 0.6 + 0.1 \times 0.4 \\
&= 0.46
\end{aligned}
$$

Fig 4

P(pass, prep, fair, smart)

$$
\begin{aligned}
\mathrm{P}(\text{pass, prep, fair, smart}) &= \mathrm{P}(\text{pass} \,|\, \text{prep, fair, smart})\ \mathrm{P}(\text{prep} \,|\, \text{fair, smart})\ \mathrm{P}(\text{fair})\ \mathrm{P}(\text{smart}) \\
&= \mathrm{P}(\text{pass} \,|\, \text{prep, fair, smart})\ \mathrm{P}(\text{prep} \,|\, \text{smart})\ \mathrm{P}(\text{fair})\ \mathrm{P}(\text{smart}) \\
&= 0.9 \times 0.74 \times 0.9 \times 0.8 \\
&\approx 0.479
\end{aligned}
$$

$$
\mathrm{P}(\text{pass} \,|\, \text{prep, fair, smart}) = 0.9
$$

Fig 5

P(pass, ¬prep, ¬fair, ¬smart)

$$
\begin{aligned}
P(\text{pass}, \neg\text{prep}, \neg\text{fair}, \neg\text{smart}) &= P(\text{pass} \mid \neg\text{prep}, \neg\text{fair}, \neg\text{smart})\ P(\neg\text{prep} \mid \neg\text{fair}, \neg\text{smart})\ P(\neg\text{fair})\ P(\neg\text{smart}) \\
&= P(\text{pass} \mid \neg\text{prep}, \neg\text{fair}, \neg\text{smart})\ P(\neg\text{prep} \mid \neg\text{smart})\ P(\neg\text{fair})\ P(\neg\text{smart}) \\
&= 0.1 \times 0.54 \times 0.1 \times 0.2 \\
&\approx 0.001
\end{aligned}
$$

$$P(\text{pass} \mid \neg\text{prep}, \neg\text{fair}, \neg\text{smart}) = 0.1$$

$$P(\neg\text{prep} \mid \neg\text{smart}) = 0.54$$

Fig 12

P(pass, ¬prep, ¬fair, smart)

$$
\begin{aligned}
P(\text{pass}, \neg\text{prep}, \neg\text{fair}, \text{smart}) &= P(\text{pass} \mid \neg\text{prep}, \neg\text{fair}, \text{smart})\ P(\neg\text{prep} \mid \neg\text{fair}, \text{smart})\ P(\neg\text{fair})\ P(\text{smart}) \\
&= P(\text{pass} \mid \neg\text{prep}, \neg\text{fair}, \text{smart})\ P(\neg\text{prep} \mid \text{smart})\ P(\neg\text{fair})\ P(\text{smart}) \\
&= 0.1 \times 0.26 \times 0.1 \times 0.8 \\
&\approx 0.002
\end{aligned}
$$

$$P(\text{pass} \mid \neg\text{prep}, \neg\text{fair}, \text{smart}) = 0.1$$

$$P(\neg\text{prep} \mid \text{smart}) = 0.26$$

Fig 11

P(pass, prep, fair, ¬smart)

$$
\begin{aligned}
\mathrm{P}(\text{pass, prep, fair, } \neg\text{smart}) &= \mathrm{P}(\text{pass} \,|\, \text{prep, fair, } \neg\text{smart})\ \mathrm{P}(\text{prep} \,|\, \text{fair, } \neg\text{smart})\ \mathrm{P}(\text{fair})\ \mathrm{P}(\neg\text{smart}) \\
&= \mathrm{P}(\text{pass} \,|\, \text{prep, fair, } \neg\text{smart})\ \mathrm{P}(\text{prep} \,|\, \neg\text{smart})\ \mathrm{P}(\text{fair})\ \mathrm{P}(\neg\text{smart}) \\
&= 0.7 \times 0.46 \times 0.9 \times 0.2 \\
&\approx 0.058
\end{aligned}
$$

$$
\mathrm{P}(\text{pass} \,|\, \text{prep, fair, } \neg\text{smart}) = 0.7
$$

Fig 6

P(pass, prep, ¬fair, smart)

$$
\begin{aligned}
\mathrm{P}(\text{pass, prep, } \neg\text{fair, smart}) &= \mathrm{P}(\text{pass} \,|\, \text{prep, } \neg\text{fair, smart})\ \mathrm{P}(\text{prep} \,|\, \neg\text{fair, smart})\ \mathrm{P}(\neg\text{fair})\ \mathrm{P}(\text{smart}) \\
&= \mathrm{P}(\text{pass} \,|\, \text{prep, } \neg\text{fair, smart})\ \mathrm{P}(\text{prep} \,|\, \text{smart})\ \mathrm{P}(\neg\text{fair})\ \mathrm{P}(\text{smart}) \\
&= 0.1 \times 0.74 \times 0.1 \times 0.8 \\
&\approx 0.006
\end{aligned}
$$

$$
\mathrm{P}(\text{pass} \,|\, \text{prep, } \neg\text{fair, smart}) = 0.1
$$

$$
\mathrm{P}(\text{prep} \,|\, \text{smart}) = 0.74
$$

Fig 7

P(pass, prep, ¬fair, ¬smart)

$$
\begin{aligned}
\mathrm{P}(\text{pass, prep, } \neg\text{fair, } \neg\text{smart}) &= \mathrm{P}(\text{pass} \,|\, \text{prep, } \neg\text{fair, } \neg\text{smart})\ \mathrm{P}(\text{prep} \,|\, \neg\text{fair, } \neg\text{smart})\ \mathrm{P}(\neg\text{fair})\ \mathrm{P}(\neg\text{smart}) \\
&= \mathrm{P}(\text{pass} \,|\, \text{prep, } \neg\text{fair, } \neg\text{smart})\ \mathrm{P}(\text{prep} \,|\, \neg\text{smart})\ \mathrm{P}(\neg\text{fair})\ \mathrm{P}(\neg\text{smart}) \\
&= 0.1 \times 0.46 \times 0.1 \times 0.2 \\
&\approx 0.001
\end{aligned}
$$

$$
\mathrm{P}(\text{pass} \,|\, \text{prep, } \neg\text{fair, } \neg\text{smart}) = 0.1
$$

$$
\mathrm{P}(\text{prep} \,|\, \neg\text{smart}) = 0.46
$$

Fig 8

P(pass, ¬prep, fair, smart)

$$
\begin{aligned}
\mathrm{P}(\text{pass, } \neg\text{prep, fair, smart}) &= \mathrm{P}(\text{pass} \mid \neg\text{prep, fair, smart})\ \mathrm{P}(\neg\text{prep} \mid \text{fair, smart})\ \mathrm{P}(\text{fair})\ \mathrm{P}(\text{smart}) \\
&= \mathrm{P}(\text{pass} \mid \neg\text{prep, fair, smart})\ \mathrm{P}(\neg\text{prep} \mid \text{smart})\ \mathrm{P}(\text{fair})\ \mathrm{P}(\text{smart}) \\
&= 0.7 \times 0.26 \times 0.9 \times 0.8 \\
&\approx 0.131
\end{aligned}
$$

$$
\mathrm{P}(\text{pass} \mid \neg\text{prep, fair, smart}) = 0.7
$$

$$
\begin{aligned}
\mathrm{P}(\neg\text{prep} \mid \text{smart}) &= 1 - \mathrm{P}(\text{prep} \mid \text{smart}) \\
&= 1 - 0.74 \\
&= 0.26
\end{aligned}
$$

Fig 9

P(pass, ¬prep, fair, ¬smart)

$$
\begin{aligned}
\mathrm{P}(\text{pass}, \neg\text{prep}, \text{fair}, \neg\text{smart}) &= \mathrm{P}(\text{pass} \mid \neg\text{prep}, \text{fair}, \neg\text{smart})\ \mathrm{P}(\neg\text{prep} \mid \text{fair}, \neg\text{smart})\ \mathrm{P}(\text{fair})\ \mathrm{P}(\neg\text{smart}) \\
&= \mathrm{P}(\text{pass} \mid \neg\text{prep}, \text{fair}, \neg\text{smart})\ \mathrm{P}(\neg\text{prep} \mid \neg\text{smart})\ \mathrm{P}(\text{fair})\ \mathrm{P}(\neg\text{smart}) \\
&= 0.2 \times 0.54 \times 0.9 \times 0.2 \\
&\approx 0.019
\end{aligned}
$$

$$
\mathrm{P}(\text{pass} \mid \neg\text{prep}, \text{fair}, \neg\text{smart}) = 0.2
$$

$$
\begin{aligned}
\mathrm{P}(\neg\text{prep} \mid \neg\text{smart}) &= 1 - \mathrm{P}(\text{prep} \mid \neg\text{smart}) \\
&= 1 - 0.46 \\
&= 0.54
\end{aligned}
$$

Fig 10