Computer Exercise 3
Parabolic equations

A metallic rod of length L [m] is initially of temperature $T = 0$ [C]. At time $t = 0$ a heat pulse of max temperature $T = T_0$ and duration t_P [s] hits the left end (at $y = 0$) of the rod. At the right end (at $y = L$) the rod is isolated. After some time the rod will therefore be warmer in the right end and then cool off again. The following partial differential equation can be set up for the heat diffusion process through the rod:

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2}, \quad t > 0, \quad 0 < y < L.$$

The boundary conditions are

$$T(0,t) = \begin{cases} T_0 \sin\left(\frac{\pi t}{2 t_P}\right), & 0 \le t \le t_P, \\ 0, & t > t_P, \end{cases} \qquad \frac{\partial T}{\partial y}(L,t) = 0,$$

and the initial condition is

$$T(y,0) = 0.$$

In the PDE, ρ is the density [kg/m^3], C_P the heat capacity [J/kg·C] and k is the thermal conductivity [J/m·s·C] of the rod.

Part 1: Rescaling to dimensionless form

With the new variables u, x and τ defined by

$$T = T_0 u, \qquad y = Lx, \qquad t = t_P \tau,$$

the problem can be transformed (scaled) into the following dimensionless form

$$\frac{\partial u}{\partial \tau} = a \frac{\partial^2 u}{\partial x^2}, \qquad \tau > 0, \qquad 0 < x < 1, \tag{1}$$

with boundary conditions

$$u(0,\tau) = \begin{cases} \sin\left(\frac{\pi \tau}{2}\right), & 0 \le \tau \le 1, \\ 0, & \tau > 1, \end{cases} \qquad \frac{\partial u}{\partial x}(1,\tau) = 0,$$

and initial condition

$$u(x,0) = 0.$$

- Show how the rescaling above is done and derive an expression for the only remaining parameter a (in terms of the original parameters) and show that it is dimensionless.
- From now on assume that a has the numerical value $a = 1$. Suppose the rod is made of copper. What length of the rod (L) and the pulse (t_P) would solutions to the rescaled PDE then reflect? (There are many possibilities.)

Part 2: 1D problem with Explicit Euler and MATLAB commands

Discretize the rescaled equation (1) with the method of lines (MoL). Use a constant step size Δx in space and central differences to obtain an ODE-system of the form

$$\frac{d\mathbf{u}}{d\tau} = A\mathbf{u} + \mathbf{b}(\tau), \qquad \mathbf{u}(0) = \mathbf{0}. \tag{2}$$

(a) Solve the system (2) with the Explicit Euler method upto $\tau = 2$ using a constant time step Δt. To make your calculations efficient construct A as a sparse matrix in MATLAB and use vector variables for $\mathbf{u}$ and $\mathbf{b}$ so that $A\mathbf{u} + \mathbf{b}$ can be formed directly (and not from component form). Experiment with different values of the discretization step sizes Δx and Δt and study stability. Visualize the solution with the following plots:

- A 3D-plot of the temperature u as a function of x and τ. (You can for instance use `surf` or `mesh`.) Submit one 3D-plot showing a stable solution and one with an unstable solution. Present the values of Δx, Δt and $\Delta t/\Delta x^2$ in the two cases.
- The temperature distribution in the rod at the fixed time $\tau = 0.5$, i.e. plot $u(x, 0.5)$. (Just for a stable solution.)
- Temperature at the left and right ends of the rod as a function of time, for $0 \le \tau \le 2$, i.e. plot $u(0,\tau)$ and $u(1,\tau)$. Put both plots in the same figure. (Just for a stable solution.)

Comment on the results.

Hint: To make these plots it may help to store the whole approximate solution in a large matrix U in which each column (or row) holds the solution at one time step; the size of U will be (number of time steps)$\times$(number of spatial grid points). Make sure to include initial and boundary conditions in U so that the full solution is shown.

(b) In this part of the exercise you shall solve (2) with MATLAB's built-in ODE solvers and compare the results. The methods are `ode23` (explicit method), `ode23s` (implicit method) and `ode23s` with the option `Jacobian` set to A, the matrix in (2). We denote this method `ode23sJ`. (Use `odeset` to set the option, and `help odeset` to read about its significance.) The three methods shall be used under similar conditions (same problem, same tolerance) and for three step sizes Δx corresponding to $N = 40$, 80 and 160 grid-points on the x-axis. The comparison shall comprise the number of time steps needed to reach $\tau = 2$, and the actual time needed for each computation (e.g. measured with MATLAB's `tic/toc` or `cputime`). Collect your statistics in a table of the type:

N	# time steps			computational time		
	`ode23`	`ode23s`	`ode23sJ`	`ode23`	`ode23s`	`ode23sJ`
40						
80						
160						

Comment on the statistics and explain the observed results for the three methods. Draw conclusions. Which method works best for this parabolic problem, and why? Also, explain why A is the right value to use for the `Jacobian` option in `ode23sJ`.

Part 3: 2D problem with Crank–Nicolson

We now consider a time-dependent version of the problem from Computer Exercise 2. You will be able to reuse a lot of your code from that exercise.

A metal block occupies the region $\Omega = [0 < x < 5,\ 0 < y < 2]$ in the xy-plane. Let $u(x, y, \tau)$ denote the temperature in the point (x, y) at time τ. As before, the block is kept at constant room temperature $u = 20$ at $x = 0$ and at $u = 200$ at $x = 5$. It is insulated at the other two sides. The block has been unheated for a long time when, at $\tau = 0$, the external source modeled by the function f starts to heat it. This means that u satsifies the parabolic equation,

$$\frac{\partial u}{\partial \tau} = \Delta u + f, \quad (x, y) \in \Omega, \quad \tau > 0, \tag{3}$$

$$f(x, y) = 50 + 500 \exp\left(-2(x - 1)^2 - (y - 1.5)^2\right),$$

with boundary conditions

$$\begin{aligned} u(0, y) &= 20, & u(5, y) &= 200, & 0 < y < 2, \\ \frac{\partial u}{\partial y}(x, 0) &= 0, & \frac{\partial u}{\partial y}(x, 2) &= 0, & 0 < x < 5, \end{aligned}$$

and initial data given by

$$u(x, y, 0) = u_0(x, y),$$

where u_0 is the solution to (2a) in Computer Exercise 2. (You may of course use the analytical solution that you derived in (2b).)

- Solve the PDE (3) to $\tau = 10$ using the Crank-Nicolson method. Choose your spatial and time steps in a suitable way. Motivate your choice. It is important here to use sparse format for all matrices, or the code will be very slow.
- Plot the solutions at the fixed times $\tau = 0$ (initial data), $\tau = 0.5$, $\tau = 2$ and $\tau = 10$. I.e. make four plots of $u(x, y, 0) = u_0(x, y)$, $u(x, y, 0.5)$, etc.
- Plot the solution in the point $(x, y) = (3, 1)$ as a function of time for $0 \le \tau \le 10$, i.e. plot $u(3, 1, \tau)$. In the plot, also mark the temperature value you obtained in $(3, 1)$ when you did (2c) in Computer Exercise 2.

Comment on the plots and their relationship with your solution from (2c) in Computer Exercise 2. Explain why Crank–Nicolson is a good method to use for this problem.

Hint: Here you do not need to save the whole solution in a big matrix as in Part 2, just the value at $(3, 1)$.