KTH Engineering Sciences

Computer Exercise 4
Hyperbolic PDEs

In this exercise you shall make some numerical experiments with finite difference methods applied to two different hyperbolic PDE-problems.

Part 1: Model problem

Consider the model problem for a hyperbolic PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad a > 0, \qquad 0 \le x \le D, \qquad t > 0,$$

with initial condition $u(x, 0) \equiv 0$. The boundary condition at $x = 0$ models a a wave with period time τ entering from the left into the domain. The wave can be either a sine wave,

$$u(0, t) = g_{\text{sin}}(t) := \sin(2\pi t/\tau),$$

or a square wave,

$$u(0, t) = g_{\text{sq}}(t) := \begin{cases} 1, & \ell\tau < t \le (\ell + 1/2)\tau, \\ -1, & (\ell + 1/2)\tau < t \le (\ell + 1)\tau, \end{cases} \qquad \ell = 0, 1, 2, \ldots$$

(You can code the last one using MATLAB's `square` command as `square(`$2\pi t/\tau$`))`.)

Your task is to implement the upwind, Lax–Friedrich and Lax–Wendroff methods, and to compare how they perform for the two different boundary conditions g_{sin} and g_{sq}. Use a uniform grid in space and set $\tau = 2.2$, $D = 4.5$ and $a = 1$. For Lax–Wendroff and Lax–Friedrichs you need a *numerical*[1] boundary condition at $x = D$. Use the extrapolation boundary condition described in Edsberg, Section 8.2.6. (If $u_k^n \approx u(x_k, t_n)$ for $k = 1, \ldots, N$, you use $u_N^n = 2u_{N-1}^n - u_{N-2}^n$. OBS! Same time level n for all three terms.)

(a) Run the three methods on the time interval $[0, 6]$. Use 100 grid points in space and choose a stable CFL number $\Delta t/\Delta x$. Present the results in a 2D graph with $u(x, 6)$ as a function of x for *all three methods plotted on top of each other in the same figure together with the exact solution*, for easy comparison. Make one such plot for g_{sin} and one for g_{sq}.

(b) Experiment with different CFL numbers when Δx is kept fixed. Make plots where you show how the solution changes when $\Delta t/\Delta x$ changes, by plotting several solutions in the same figure.[2]

[1] The PDE itself does not have a boundary condition here!

[2] Note: The "magic time step" choice $\Delta t = \Delta x$ is quite special here, as it gives the *exact* solution. This only happens with very simple PDEs like the present one. With the PDE in part 2 the choice does not give the exact solution, for example.

(c) Look at the convergence of the solution when $\Delta x \to 0$ for a fixed CFL number. As in the previous point, show how the solution changes when you decrease Δx, by plotting the different solutions in the same figure.

Draw conclusions and discuss:

- Which method(s) give overly smeared numerical solutions? Which method(s) introduce spurious oscillations? When does this happen?
- How do the smearing and spurious oscillations depend on the CFL number $\Delta t/\Delta x$?
- Which method converges fastest?
- When are the methods stable/unstable? Are your experimental results in agreement with the theoretical stability results?
- Which method is best for the sine and the square wave case, respectively?

Note: In (b) and (c) you should not plot all combinations of methods and boundary conditions. Select a few cases that you think are the most relevant, where the plots best help you support the conclusions that you draw and illustrate the discussion around it. There should be some plot showing an unstable solution, however. Make sure to include information about parameter values, method and boundary condition for all plots.

Part 2: Heat exchanger application

In this part the exchanger in Edsberg, Example 8.1 is studied. A fluid of temperatute $T(x,t)$ is flowing with constant speed v in a pipe. Outside the pipe there is a cooling medium that keeps a constant low temperature T_{cool}. The temperature of the fluid in the pipe is initially cool, i.e. $T = T_{\text{cool}}$ but within a short time period hot fluid with fluctuating temperature enters the pipe. The task is to study how the temperature $T(x,t)$ of the fluid in the pipe depends on x and t.

The following PDE-model is given:

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} + \alpha(T - T_{\text{cool}}) = 0, \qquad 0 \le x \le L, \qquad t > 0, \tag{1}$$

with the initial condition

$$T(x,0) \equiv T_{\text{cool}},$$

and the boundary condition

$$T(0,t) = \begin{cases} T_{\text{cool}} + (T_{\text{hot}} - T_{\text{cool}})\sin(4\pi t), & 0 \le t \le 0.125, \\ T_{\text{hot}}, & 0.125 \le t \le 1, \\ T_{\text{hot}} + T_{\text{cool}}\sin(5\pi(t-1)), & t > 1. \end{cases}$$

The length of the heat exchanger is $L = 5$. The heat exchange parameter $\alpha = 0.5$, the velocity of the fluid is $v = 1$, the cooling temperature $T_{\text{cool}} = 50$ and the hot temperature is $T_{\text{hot}} = 200$.

(a) Use the upwind and the Lax–Wendroff methods to simulate the temperature in the pipe for $0 \le t \le 6$. Choose suitable stepsizes Δx and Δt. Present T as a function of both t and x in two 3D graphs next to each other in two subplots.

Hint: For advection equations with lower order terms and a source,

$$u_t + au_x + bu = c,$$

the upwind method is easy to extend by adding a term $-\Delta t(bu_k^n - c)$. The extension of Lax–Wendroff is more involved. In the Exercise lecture it will be shown to be

$$\begin{aligned} u_k^{n+1} = u_k^n &- \frac{\sigma(1 - b\Delta t)}{2}(u_{k+1}^n - u_{k-1}^n) + \frac{\sigma^2}{2}(u_{k+1}^n - 2u_k^n + u_{k-1}^n) \\ &- \Delta t\left(1 - \frac{b\Delta t}{2}\right)(bu_k^n - c), \end{aligned}$$

where $\sigma = a\Delta t/\Delta x$.

(b) Make 2D plots of $T(x,t)$ for $0 \le x \le L$ at the time points $t \approx 3$ and $t \approx 6$. Plot the solutions from the two methods on top of each other in the same figure/subplot.

Experiment with different Δt and Δx. Compare the two methods. Try to explain your observations using the theoretical results about the methods. Which method is most accurate? (You could for instance compare the results of the methods against a very accurate reference solution.)

(Note: You do not need to plot the solution at exactly $t = 3$ and $t = 6$. Therefore you can choose $\Delta t/\Delta x$ freely.)