KTH Royal Institute of Technology

SF2863 Systems Engineering

Home Assignment 1

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Analytic Solution

(1) In the case of one turbine being broken, it is optimal to assign both X and Y to repair it as they have the highest repair intensity. For the case when both turbines are broken, cooperation with Z only has to be considered for either X or Y, as $\mu_{XZ} = \mu_{YZ}$. The remaining possible strategies s_i to be considered are the following

s_1: When the second turbine breaks, X and Y continue and Z starts working on turbine 2.

s_2: When the second turbine breaks, X and Z continue and Y starts working on turbine 2.

s_3: When the second turbine breaks, X continues and X and Z start working on turbine 2.

s_4: When the second turbine breaks, Z takes over the first turbine and Y and X start working on turbine 2.

(2) The possible states X_i of the Markov chain are

X_1: Both turbines work.

X_2: Turbine 1 is broken and turbine 2 works.

X_3: Turbine 2 is broken and turbine 1 works.

X_4: Both turbines are broken.

(3) The intensity matrices Q_i for corresponding strategies s_i are

$$Q_i = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 \\ \mu_{XY} & -(\mu_{XY}+\lambda_2) & 0 & \lambda_2 \\ \mu_{XY} & 0 & -(\mu_{XY}+\lambda_1) & \lambda_1 \\ (\,) & (\,) & (\,) & (\,) \end{bmatrix}$$

where the entries in row 4 differ for the strategies as follows

$$\begin{aligned} s_1 : Q_1(4,:) &= \begin{bmatrix} 0 & \mu_Z & \mu_{XY} & -(\mu_{XY}+\mu_Z) \end{bmatrix} \\ s_2 : Q_2(4,:) &= \begin{bmatrix} 0 & \mu_{YZ} & \mu_X & -(\mu_X+\mu_{YZ}) \end{bmatrix} \\ s_3 : Q_3(4,:) &= \begin{bmatrix} 0 & \mu_{XY} & \mu_Z & -(\mu_{XY}+\mu_Z) \end{bmatrix} \\ s_4 : Q_4(4,:) &= \begin{bmatrix} 0 & \mu_X & \mu_{YZ} & -(\mu_X+\mu_{YZ}) \end{bmatrix} \end{aligned}$$

(4) For a unique stationary solution to exist, the Markov process needs to be both finite and irreducible. In this case, all states are accounted for and they communicate, so there is only one equivalence class and the process is thus both finite and irreducible.

(5) Solving the steady state equations

$$\pi \boldsymbol{Q} = 0 \qquad \sum_i^M \pi_i = 1$$

where $M = 4$ for each strategy allows us to determine their resultant stationary distributions π_i. Matlab was used for this and the result was

$$\begin{aligned} s_1 &: \pi_1 = 0.631,\ \pi_2 = 0.089,\ \pi_3 = 0.226,\ \pi_4 = 0.053 \\ s_2 &: \pi_1 = 0.626,\ \pi_2 = 0.114,\ \pi_3 = 0.199,\ \pi_4 = 0.061 \\ s_3 &: \pi_1 = 0.628,\ \pi_2 = 0.121,\ \pi_3 = 0.193,\ \pi_4 = 0.057 \\ s_4 &: \pi_1 = 0.628,\ \pi_2 = 0.095,\ \pi_3 = 0.218,\ \pi_4 = 0.058 \end{aligned}$$

(6) The average power production for strategies s_i, $d_{av,i}$, can be calculated by taking the scalar product between its π vector and the production vector $prod = [d12\ d2\ d1\ d0]$. The calculated values from the HA1.m file were

$$\boldsymbol{d_{av}} = \begin{bmatrix}100.264 & 100.813 & 101.515 & 100.097\end{bmatrix}.$$

(7) From the obtained values in (6), s_3 is the best repair strategy as it results in the highest average power production.

Continuous time simulation

(8) To simulate the Markov chain for continuous time, the time to transition between states and which state is transitioned to must be determined. The expected dwell time in state i before jumping to state j, $E(T_{ij})$, is exponentially distributed with intensity $E(T_{ij}) = 1/q_{ij}$.

A random variable for T_{ij} that follows the same distribution can then be generated in Matlab using **exprnd**$(1/q_{ij})$ for all accessible states j from i. The lowest generated value will then determine which state should be jumped to, and how long the dwell time is. The simulation was based on the following algorithm.

Algorithm 1 Continuous time simulation

```
while maxT > Time do
    for Column = 1, 2, 3, 4 do
        T = exprnd(State, Column)
    end for
    Time += min(T)
    State = The index where min(T) is
    TimePerState(State) += Time
```

(9) The simulation was run from a starting state of X_1 during a time interval [0,T] where T = 100000. The value for T was selected such that the difference in the computed values compared to the case when T is an order of magnitude lower was less than 1%.

(10) The averages were computed and the estimates of the expected production for each strategy were determined to be the following

$$\boldsymbol{d_{av}} = \begin{bmatrix}100.326 & 100.775 & 101.554 & 99.934\end{bmatrix}.$$

Comparison of this result with the one obtained for the analytic case determined the errors to be

$$|\boldsymbol{d_{av,analytic}} - \boldsymbol{d_{av,continuous}}| = \begin{bmatrix}0.061 & 0.038 & 0.039 & 0.164\end{bmatrix}.$$

Discretization-approach simulation

(11) The transition matrix $\mathbf{P}$ can be determined by using the following equations to the intensity matrix defined in (3).

$$\begin{aligned} p_{ij} &= 1 - hq_{ij}, \quad i = j \\ p_{ij} &= hq_{ij}, \qquad\quad i \neq j \end{aligned}$$

When the equations were applied to Q_1, P_1 was generated:

$$P_1 = \begin{bmatrix} 1-h(\lambda_1+\lambda_2) & h\lambda_1 & h\lambda_2 & 0 \\ h\mu_{XY} & 1-h(\mu_{XY}+\lambda_2) & 0 & h\lambda_2 \\ h\mu_{XY} & 0 & 1-h(\mu_{XY}+\lambda_1) & h\lambda_1 \\ 0 & h\mu_Z & h\mu_{XY} & 1-h(\mu_{XY}+\mu_Z) \end{bmatrix}$$

In order to generate the **P** matrices for all strategies, the same transformation was performed for every **Q** matrix.

(12) To simulate using discretization a random jump indicator was introduced. To make the random jump **rand**() was used to generate a random jump following the algorithm displayed below.

Algorithm 2 Discretization-approach simulation

```
for i = 1, 2, ... n do
    for Column = 1, 2, 3 do
        Pvals = P(State, Column)
    end for
    ran = rand()
    if ran > sum(Pvals) then
        State = 4
    else if ran > Pvals(1) + Pvals(2) then
        State = 3
    else if ran > Pvals(1) then
        State = 2
    else
        State = 1
```

The simulation was run for $n = 10000000$ as the difference for an order of magnitude lower for n was less then 1%.

(13) The average production was calculated using the same method as in (6).

$$\boldsymbol{d_{av}} = \begin{bmatrix} 100.241 & 100.493 & 101.320 & 100.197 \end{bmatrix}.$$

The difference between the analytic solution and the discretization-approach was

$$|\boldsymbol{d_{av,analytic}} - \boldsymbol{d_{av,discrete}}| = \begin{bmatrix} 0.023 & 0.032 & 0.195 & 0.100 \end{bmatrix}.$$

Some final questions

(14) In this problem we are asked to calculate the probability p that the plant stays in state X_1, both working, during a period of 0.4 days. We were asked to simulate this using continuous time Markov chain (p_c) and the discretized Markov chain (p_d) and compare the answers for the discretized case when $h \to 0$.

$$\begin{aligned} p_c &= 0.6988, \; n = 10000 \\ p_d = 0.6995, \; h &= 0.00100, \; n = 10000 \\ p_d = 0.0279, \; h &= 0.00010, \; n = 10000 \\ p_d = 0.0000, \; h &= 0.00001, \; n = 10000 \end{aligned}$$

Here we can see that if we let $h \to 0$ the probability decreases. This is to be expected, because when $h \to 0$ the plant has more chances to jump to another state. And if it takes to many steps it becomes almost guaranteed that it will jump to a different state. For the continuous case and the discretized case using h = 0.001, the probability of the plant working after 0.4 days is almost 0.7.

We were also asked to check the probability for the same period, 0.4, but this time the workers will check the turbines after 0.4/3 days and when it's checked the turbine still works. Since Markov chains don't have any memory, we now want to compute the probability that the plant stays in state both working during a period of 0.4-0.4/3 days.

$$
\begin{aligned}
p_c = 0.7838,&\ \ n = 10000 \\
p_d = 0.7808,\ \ h = 0.00100,&\ \ n = 10000 \\
p_d = 0.0882,\ \ h = 0.00010,&\ \ n = 10000 \\
p_d = 0.0000,\ \ h = 0.00001,&\ \ n = 10000
\end{aligned}
$$

When $h \to 0$ the behaviour is similar to the case where the threshold was 0.4. The explanation for this is the same as before. For the continuous case and the discretized case using h = 0.001, the probability of the plant working after 0.4-0.4/3 days is 0.78.

(15) The new possible states X_i of the Markov chain are

X_1: Both turbines work.

X_2: Turbine 1 is broken and X and Y repair it.

X_3: Turbine 2 is broken and X and Y repair it.

X_4: Both turbines are broken, X and Y repairs turbine 1, Z repairs turbine 2.

X_5: Turbine 1 is healthy, Z repairs turbine 2

X_6: Both turbines are broken, X and Y repairs turbine 2, Z repairs turbine 1.

X_7: Turbine 1 is healthy, X and Y repairs turbine 2.

and the new transition matrix **Q** is

$$
\boldsymbol{Q} = \begin{bmatrix}
-0.9 & 0.3 & 0.6 & 0 & 0 & 0 & 0 \\
1.8 & -2.4 & 0 & 0.6 & 0 & 0 & 0 \\
1.8 & 0 & -2.1 & 0 & 0 & 0.3 & 0 \\
0 & 0.5 & 0 & -2.3 & 1.8 & 0 & 0 \\
0.5 & 0 & 0 & 0.3 & -0.8 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & -2.3 & 1.8 \\
1.8 & 0 & 0 & 0 & 0 & 0.6 & -2.4
\end{bmatrix}
$$

A diagram of this process can be seen in the Appendix.

(16) If states X_i are redefined as $(X_{i,old}, X_{i,new})$, then states X_5 and X_7 become (X_4, X_5) and (X_6, X_7) respectively. So, unlike the others, these states include memory of the past state. Redefining states in this manner could be used to model processes with memory, such that only one state $(X_{i,old}, X_{i,new})$ is possible for each state X_i. A drawback of this, however, is that memory storage only lasts for a single state transition and cannot be accessed at a later one.

1 Appendix