Home assignment number 1, 2020, in SF2863 Systems Engineering.

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The home assignments are a mandatory part of the course. In total, there are two home assignments and you need to collect 4.0 out of 6.0 points in order to pass this part of the course. Home assignment 1 can give up to 2.0 points.

The home assignments can, in addition, give bonus points on the exam. If a home assignment is handed in before the deadline, then the points awarded on the assignment will also count as bonus points on the exam (hence, home assignment 1 can give up to 2.0 bonus points on the exam). In order to get bonus points on the exam the report should be submitted to canvas before 23:59, Friday, November 13, 2020.

This home assignment should be done in groups of 2 persons, and one report per group is to be submitted to canvas. See the information on canvas on how to form groups and submit your results. State your name, and email address on the front of the report.

In the report your should describe in your own words how the problem was solved. In particular, carefully describe how you defined your states, modelled the problem and implemented the simulations. Do not explain the code, but rather the mathematical ideas. You should not need more than 4 pages for the report. The answers to the questions in the assignment should be given in the main report and numbered as in this text. In particular, it should not be necessary to look at your code to understand how you define and calculate things. Relevant print-outs and plots should be included in the report. You may use any computer program to facilitate the computations, but you have to include your computer code in the submission. We use automated plagiarism-control tools and will also make random checks that you have not copied parts of another groups report or computer code!

Questions should be posed in the discussion group in canvas.

In this home assignment a simple Markov process in continuous time will be examined. The goal is to work with Markov chains, and to see the relation between a continuous time chain and a discrete time chain.

Problem statement

Consider a model of a steam-electric power plant, which uses steam to produce electricity in two independent turbines. We call the turbines 1 and 2. If both of these work the plant produces d_{12} units of power per hour and supplies it to the gird. If turbine 1 breaks the plant can still produce some power, but only d_2 units per hour. Similarly, if turbine 2 breaks the plant can produce d_1 units per hour. The turbines are of different sizes, hence $d_1 < d_2$. If both turbines break, some of the steam can be used in a nearby factory. When the factory uses steam from the power plant it requires less electric power from the grid.

Thus, we count it as power production of d_0 units per hour for the power plant. We model the breaking of the turbines as independent and exponentially distributed with expected time between breaking being $1/\lambda_1$ days for turbine 1, and $1/\lambda_2$ days for turbine 2.
The turbines and the nearby factory are connected to the steam production in the plant via three independent pipes. Hence, repair of broken turbine(s) can take place while still having other production. (We do not investigate the safety of such setup further.)
In the power plant there is a repair crew consisting of the people X, Y, and Z. They can work independently or together on repairing the broken turbines. Since they are individuals their efficiency and skill of cooperation will vary. We model the repair times as independent and exponentially distributed with intensity μ_X, μ_Y, and μ_Z days^{-1} respectively if X, Y, Z works independently. Persons X and Y are more similar in character and hence work with the same intensity, and they work quite well together. If X and Y are assigned the same work the intensity is μ_{XY}. However, person Z is not that efficient when working alone, and although the performance increases when in a team, person Z is not a great team player either. Hence, $\mu_{XZ} = \mu_{YZ}$ are lower than μ_{XY}. Finally it is assumed (perhaps erroneously so) that three people assigned a common task will result in two working and one drinking coffee, and hence such allocation will not be tolerated by the management. Assume that the workers assigned to a broken turbine can be reassigned instantaneously when a repair job is completed or a turbine breaks down.

The system is a Markov process that will depend on the strategy adopted for how to assign the workers to repair the turbine. If only one turbine is broken it is obvious that it is optimal to let X and Y work on it together. However, if both turbines are broken it is not clear how to distribute the workers. We consider a "very long" planning horizon for the electricity production, and your task is to find the optimal strategy for how the three workers X, Y and Z should be assigned. This should be done by determining the electricity production at steady state, for each different repair strategy, and choose the best strategy. The same task should be treated with three different, but related, approaches. One analytic computation, one continuous time simulation, and one simulation based on a discretization.

Parameter values

- $d_{12} = 130$ units of power per hour
- $d_1 = 40$ units of power per hour
- $d_2 = 90$ units of power per hour
- $d_0 = 20$ units of power per hour
- $h = 0.001$
- $\lambda_1 = 0.3$ days^{-1}
- $\lambda_2 = 0.6$ days^{-1}
- $\mu_X = \mu_Y = 0.7$ days^{-1}
- $\mu_Z = 0.5$ days^{-1}
- $\mu_{XY} = 1.8$ days^{-1}
- $\mu_{XZ} = \mu_{YZ} = 1.4$ days^{-1}

Analytic solution

To compare the different strategies analytically you should for each of them:

1. Define the repair strategies considered.
   Hint: You only need to consider 4 different strategies. Motivate why.

2. Define the states of the Markov chain.
   Hint: Which parts of the chain are affected by the choice of strategy?

3. Determine the intensity matrix.
Hint: One for each strategy considered. How do they differ?

4. Motivate why there exists a stationary solution.

5. Determine the stationary distributions.

6. Determine the average production of the plant, based on the stationary distributions.

7. Which strategy is the best?

You may use a computer to solve the systems of equations. Results can be presented as decimal numbers with a reasonable amount of digits.

Continuous time simulation

The problem can also be approached numerically by simulating the system and determining the average power production by ergodic estimates. The simulations should be done for each strategy.

8. Describe the simulation. Specifically, describe in detail how you determine the time to the next jump and where to which state the chain jumps.
Hint: There are two equivalent ways of doing this. You can choose any of them.

9. Run the simulation. One realization of the process can be determined by starting the process in some state and then use the probability description of the process to determine how long it remains in a state and to where it transitions.

The simulation needs to be run for a time interval $[0, T]$, and you need to determine a suitable T. Note that, if you run the simulation twice you would get different results. However, comparing the two different results can give an indication on if the averages have converged or not. Choose T large enough (order of magnitude) so that the differences are less than 1%.

10. Compute the averages to determine estimates of the expected production, for the different strategies. How big are the errors compared to the analytic solution?

Hint: A large number of time steps have to be made to get reasonable ergodic estimates. You may note that it is enough to keep track on how long time the Markov process has spent so far in each state to determine the estimates for the stationary probabilities and the average production.
Hint: If you use Matlab, an exponentially distributed variable can be generated using the command `exprnd`. Note that the input specifies the mean of the variable (inversely proportional to the intensity).

Discretization-approach simulation

We now consider a discretization of the continuous time process. We discretize the time axis in N intervals, each of length h. Hence, we only consider the process at discrete time points $t_k = kh$ for $k = 0, 1, 2, \ldots, N$. (Note that N intervals gives $N + 1$ points, and in relation to the continuous simulation $T = Nh$.) Then the probability of a jump to another state during the time interval $[t_k, t_{k+1}]$ can be determined (approximatively) using the intensity matrix for the continuous time Markov process. (Note that you could use the matrix exponential function to determine the transition probabilities, but we would like you to use the approximative expressions that hold for small h as presented in the lectures.) If the time step h is small enough this approximation is good.

11. Determine the transition matrix of a discrete time Markov chain that will approximate the continuous time process.
Hint: Check if the transition matrix you obtain has row-sums equal to one.

12. Run the simulation. One realization of the process can be determined by starting the process in some state and then use the transition probabilities to determine determine which transitions occur.

The simulation needs to be run for a certain number of steps N, and you need to determine a suitable N. Note that, if you run the simulation twice you would get different results. However, comparing the two different results can give an indication on if the averages have converged or not. Choose N large enough (order of magnitude) so that the differences are less than 1%.

13. Compute the averages to determine estimates of the expected production, for the different strategies. How big are the errors compared to the analytic solution?

Hint: If you use Matlab, a stochastic variable with uniform probability in $[0, 1]$ can be generated using the command `rand`.

Some final questions

14. Assume we are in state with both turbines working. There is a request to give the workers further training, and hence take them off duty for $b = 0.4$ days. To understand the consequences for the power production you want to compute the probability that the plant stays in the state "both working" for at least 0.4 days.
Start by using the continuous time Markov chain to compute this.
Also use the discretized Markov chain to answer the same question (you can assume $k = b/h$ is an integer). What happens when h decreases? As $h \to 0$, do you get the same answer?
Now assume that the workers have been away for $b/3$ days, and that everything is still fine. What is now the probability that none of the turbine break down before they come back? Note that $2b/3$ days remain.

15. Assume that according to some protocol/safety regulations/setup time (which may exist but we have excluded in the rest of the model) the workers cannot abort an initiated attempt to repair a broken turbine, i.e., if they have started repair they continue until they succeed. We assume that X and Y work as a team and they are always sent to the first turbine that breaks. The situation can be modeled as a Markov process. Define states, transitions, and draw the diagram of the process.
Hint: Note that the order that the turbines break will now be important. The Markov model now needs seven states to represent the system.

16. We typically say that a Markov chain is memoryless, but the model above seems to have some memory (see the hint). Explain! Can this be used in general to model processes with memory? What is the drawback?
Hint: For a chain with states (i), consider a new chain with states (k, i), where k and i are states in the previous chain, and k represents the previous state, i.e., (k, i) represents (i), jump $i \to k$, (k) in the original chain.

Good luck!