$f : R^d \to R \tilde{f}(x) = f(x) + \epsilon(x) \epsilon(x) : R^d \to R$

$X \in R^{d\times N} \epsilon(x) E[\epsilon(x)] = 0 V[\epsilon(x)] = \sigma(x)^2 < \infty x \in R^d Cov(\epsilon(x), \epsilon(x')) = 0 x \neq x' \in R^d \epsilon x_0 \epsilon [\|\nabla_{S_{X,x_0}} \tilde{f}(x_0) - \nabla f(x_0)\|_2^2]$

$= \|\nabla_{S_{X,x_0}} f(x_0) - \nabla f(x_0)\|_2^2 + \sum_{i=1}^N \sigma(x_i)^2 \|(SS^T)^{-1}(x_i - x_0)\|_2^2.$

$E[\nabla_S \tilde{f}(x_0)]$

$\|\nabla_S \tilde{f}(x_0) - E[\nabla_S \tilde{f}(x_0)] + E[\nabla_S \tilde{f}(x_0)] - \nabla f(x_0)\|_2^2]$

$= E[\|\nabla_S \tilde{f}(x_0) - E[\nabla_S \tilde{f}(x_0)]\|_2^2]$

$+ \|E[\nabla_S \tilde{f}(x_0)] - \nabla f(x_0)\|_2^2 Duetoexpectationofaconstant.$

$+ E[\Big(\nabla_S \tilde{f}(x_0) - E[\nabla_S \tilde{f}(x_0)]\Big)^T \Big(E[\nabla_S \tilde{f}(x_0)] - \nabla f(x_0)\Big)]$

$E[\nabla_S \tilde{f}(x_0)] = E[\nabla_S f(x_0)] + E[\nabla_S \epsilon(x_0)] \nabla_S f(x_0) E[\nabla_S \tilde{f}(x_0)] = \nabla_S f(x_0)$

$\to E[\|\nabla_S \tilde{f}(x_0) - \nabla_S f(x_0)\|_2^2]$

$+ \|\nabla_S f(x_0) - \nabla f(x_0)\|_2^2$

$+ E[\Big(\nabla_S \tilde{f}(x_0) - \nabla_S f(x_0)\Big)^T \Big(\nabla_S f(x_0) - \nabla f(x_0)\Big)].$

$\|\nabla_S \tilde{f}(x_0) - \nabla_S f(x_0)\|_2^2]$

$= E[\|\nabla_S f(x_0) + \nabla_S \epsilon(x_0) - \nabla_S f(x_0)\|_2^2]$

$= E[\|\nabla_S \epsilon(x_0)\|_2^2] . \|\nabla_S \epsilon(x_0)\|_2^2]$

$= \sum_{i=1}^N E[\|(SS^T)^{-1}(x_i - x_0)(\epsilon(x_i) - \epsilon(x_0))\|_2^2]$

$+ \sum_{j\neq k}^N E[\Big((SS^T)^{-1}(x_j - x_0)(\epsilon(x_j) - \epsilon(x_0))\Big)^T \Big((SS^T)^{-1}(x_k - x_0)(\epsilon(x_k) - \epsilon(x_0))\Big)] . E[\|(SS^T)^{-1}(x_i - x_0)(\epsilon(x_i) - \epsilon(x_0))\|_2^2] =$

$\Big({}^T)^{-1}(x_j - x_0)(\epsilon(x_j) - \epsilon(x_0))\Big)^T \Big((SS^T)^{-1}(x_k - x_0)(\epsilon(x_k) - \epsilon(x_0))\Big)]$

$= \Big(E[\epsilon(x_j)\epsilon(x_k)] - E[\epsilon(x_j)\epsilon(x_0)] - E[\epsilon(x_0)\epsilon(x_k)] + E[\epsilon(x_0)\epsilon(x_0)]\Big)$

$\cdot \Big((SS^T)^{-1}(x_j - x_0)\Big)^T \Big((SS^T)^{-1}(x_k - x_0)\Big)$

$= \sigma(x_0)^2 \Big((SS^T)^{-1}(x_j - x_0)\Big)^T \Big((SS^T)^{-1}(x_k - x_0)\Big).$

$\sum_{i=1}^N (\sigma(x_i)^2 + \sigma(x_0)^2) \|(SS^T)^{-1}(x_i - x_0)\|_2^2$

$+ \sum_{j\neq k}^N \sigma(x_0)^2 \Big((SS^T)^{-1}(x_j - x_0)\Big)^T \Big((SS^T)^{-1}(x_k - x_0)\Big)$

$= \sum_{i=1}^N \sigma(x_i)^2 \|(SS^T)^{-1}(x_i - x_0)\|_2^2$

$+ \sigma(x_0)^2 \sum_{j,k}^N \Big((SS^T)^{-1}(x_j - x_0)\Big)^T \Big((SS^T)^{-1}(x_k - x_0)\Big).$

$\bar{S} = \frac{1}{N} \sum_{i=1}(x_i - x_0) \sum_{i=1}^N \sigma(x_i)^2 \|(SS^T)^{-1}(x_i - x_0)\|_2^2$

$+ \sigma(x_0)^2 \sum_{j,k}^N \Big((SS^T)^{-1}(x_j - x_0)\Big)^T \Big((SS^T)^{-1}(x_k - x_0)\Big).$

$f : \mathcal{K} \to R \mathcal{K} \in R^d x_0 X \mathcal{K} S_{X,x_0} X B \in R C$

$\|\nabla_S f(x_0) - \nabla f(x_0)\|_2$

$\leq \frac{1}{2} B d \frac{1}{\lambda_{min}^2} R(S)^3$

$\|\nabla_S f(x_0) - \nabla f(x_0)\|_2$

$\leq \frac{1}{2} \|(SS^T)^{-1} \sum_{i=1}^N (x_i - x_0)(x_i - x_0)^T \nabla^2 f(x_0)(x_i - x_0)\|_2$

$+ \frac{1}{6} C d^{3/2} \frac{1}{\lambda_{min}^2} R(S)^4$

$\lambda_{min} = \lambda_{min}(\frac{1}{N} SS^T) R(S) = \max_i \|x_i - x_0\|_2$

$f(x) = x^t A x + b^t x + c A \in R^{d\times d} b \in R^d c \in R \|\nabla_{S_{X,x_0}} f(x_0) - \nabla f(x_0)\|_2^2$

$= \|(SS^T)^{-1} \sum_{i=1}^N (f(x_i) - f(x_0))(x_i - x_0) - \nabla f(x_0)\|_2^2$

$= \|(SS^T)^{-1} \sum_{i=1}^N (x_i - x_0)(x_i - x_0)^T \nabla f(x_0) + \frac{1}{2}(x_i - x_0)(x_i - x_0)^T \nabla^2 f(x_0)(x_i - x_0) - \nabla f(x_0)\|_2^2$

$= \|(SS^T)^{-1} \sum_{i=1}^N (x_i - x_0)(x_i - x_0)^T A (x_i - x_0)\|_2^2.$

S

$\|\nabla_{S_{X,x_0}} f(x_0) - \nabla f(x_0)\|_2$

$= \|(SS^T)^{-1} \sum_{i=1}^N (f(x_i) - f(x_0))(x_i - x_0) - \nabla f(x_0)\|_2$

$= \|(SS^T)^{-1} \sum_{i=1}^N (x_i - x_0)(x_i - x_0)^T \nabla f(x_0) + \frac{1}{2}(x_i - x_0)(x_i - x_0)^T \nabla^2 f(x_0)(x_i - x_0)$

$+ \frac{1}{6}(x_i - x_0) D^3 f(\xi_i).(x_i - x_0)^{\otimes 3} - \nabla f(x_0)\|_2$

$= \|(SS^T)^{-1} \sum_{i=1}^N \frac{1}{2}(x_i - x_0)(x_i - x_0)^T \nabla^2 f(x_0)(x_i - x_0) + \frac{1}{6}(x_i - x_0) D^3 f(\xi_i).(x_i - x_0)^{\otimes 3}\|_2$

$\leq \frac{1}{2} \|(SS^T)^{-1} \sum_{i=1}^N (x_i - x_0)(x_i - x_0)^T \nabla^2 f(x_0)(x_i - x_0)\|_2$

$+ \frac{1}{6} \|(SS^T)^{-1} \sum_{i=1}^N (x_i - x_0) D^3 f(\xi_i).(x_i - x_0)^{\otimes 3}\|_2$

$\sum_{i=1}^d |a_i| \leq \sqrt{d} \|\mathbf{a}\|_2 . \sum_{i=1}^d |a_i|$

$= \langle \mathbf{1}, \mathbf{a} \rangle$

$\leq \|\mathbf{1}\|_2 \|\mathbf{a}\|_2$

$= \sqrt{d} \|\mathbf{a}\|_2$

$\|(SS^T)^{-1} \sum_{i=1}^N (x_i - x_0) D^k f(\xi_i).(x_i - x_0)^{\otimes k}\|_2$

$\|(SS^T)^{-1} \sum_{i=1}^N (x_i - x_0) D^k f(\xi_i).(x_i - x_0)^{\otimes k}\|_2$