

## Math 401 Problem Set 5 (due February 23, 2026)

**Problem 1.** Determine the number of conjugacy classes in (a)  $S_6$ , (b)  $A_6$ .

**Problem 2.** Does  $S_4$  have a normal subgroup of order 3?

**Problem 3.** Let  $G$  be a simple group, and let  $G'$  be any group. Show that any nontrivial homomorphism  $\varphi: G \rightarrow G'$  is injective.

**Problem 4.** Here, we fill in the proof of a proposition (Proposition 2.11.4 in Artin) about product groups which was omitted from the lecture.

Let  $G$  be a group and let  $H$  and  $K$  be subgroups, and let  $f: H \times K \rightarrow G$  be the function

$$f(h, k) = hk.$$

Let  $HK = \{hk : h \in H, k \in K\}$ , which is the image of  $f$ .

- (a) Show that  $f$  is injective if and only if  $H \cap K = \{1\}$ .
- (b) Show that  $f$  is a homomorphism if and only if  $hk = kh$  for all  $h \in H$  and  $k \in K$ .
- (c) Show that if  $H \cap K = \{1\}$ ,  $HK = G$ , and  $H$  and  $K$  are normal, then  $f$  is an isomorphism from the product group  $H \times K$  to  $G$ .

(*Hint.* Two elements  $h$  and  $k$  commute if and only if the *commutator*  $hkh^{-1}k^{-1}$  is equal to 1.)

**Problem 5.** Let  $p$  and  $q$  be distinct primes, and suppose that  $G$  is a group of order  $p^2q$ . Show that  $G$  is not simple.

**Problem 6.** Classify groups of order 2026. (Note that 1013 is prime.)

**Problem 7.** Approximately how long did you spend on this problem set? (Round to the nearest half-hour.)

**Bonus problem (not graded).** Let  $G$  be a group of order 56. Show that  $G$  is not simple.