1 True / False (5 points each)

Label each statement as true or false. For each true statement, briefly explain why it is true (a single sentence should be enough). For each false statement, provide a counterexample.

(a) In the dihedral group D_n, the relation $r^k s r^k = s$ holds for any integer k.

(b) If $\varphi\colon G \to G'$ is a homomorphism of groups such that $\ker(\varphi) = \{1_G\}$, then φ is injective.

(c) Let G be a group, and let $a \in G$. Then the set $H = \{g \in G : gag^{-1} = a\}$ is a subgroup of G.

(d) The map $*\colon G \times G \to G$ given by $g * a = ag$ defines a group action of G on G.

(e) Let G be a group, and let N be a normal subgroup of G. Then G contains a subgroup isomorphic to G/N.

(f) Let G be a group. If $a, b, c \in G$ are elements satisfying $ab = bc$, then $a = c$.

2 Examples (5 points each)

Provide an example for each of the following. (No further explanation needed.)

(a) A finite nonabelian group.

(b) A group, and an automorphism of that group which is not the map $\varphi(g) = g$.

(c) A subgroup of $\mathbb{Z}/63\mathbb{Z}$ which is not equal to $\{0\}$ or the whole group $\mathbb{Z}/63\mathbb{Z}$.

(d) A subgroup of S_3 which is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

(e) A permutation in S_6 with sign -1.

(f) Two groups G_1 and G_2, and a subgroup of $G_1 \times G_2$ which is not equal to $H_1 \times H_2$, for any subgroups H_1 and H_2 of G_1 and G_2, respectively.

3 Short answer

For each of these problems, provide a short explanation with your answer.

3.1 Number of homomorphisms

Find the number of homomorphisms from $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

3.2 Action of D_5 on edges of pentagon

Consider the action of D_5 on the edges of a regular pentagon $A_1A_2A_3A_4A_5$.

(a) What are the orbits of this action?

(b) What is the stabilizer of the edge A_1A_2?

4 Proof-based problems

For each of these problems, you should write a complete proof.

4.1 Index 2 subgroup is normal

Let G be a group and suppose H is a subgroup of G of index 2. Prove that H is normal.

4.2 Please remember to review the bonus problems

Let $n > 1$ be a positive integer. Prove that n does not divide $3^n - 2^n$.