

## 1 True / False

Label each statement as true or false. For each true statement, briefly explain why it is true (a single sentence should be enough). For each false statement, provide a counterexample.

- (a) Let  $a, b, d, r, s$  be nonzero integers. If  $d = ra + sb$ , then  $d$  divides both  $a$  and  $b$ .
- (b) If  $\varphi: G \rightarrow G'$  is a homomorphism of groups such that  $\ker(\varphi) = \{1_G\}$ , then  $\varphi$  is injective.
- (c) In the dihedral group  $D_n$ , the relation  $r^k s r^k = s$  holds for any integer  $k$ .
- (d) Let  $G$  be a group, and let  $a \in G$ . Then the set  $H = \{g \in G : ga = ag\}$  is a subgroup of  $G$ .
- (e) Let  $G$  be a group, and let  $N$  be a normal subgroup of  $G$ . Then  $G$  contains a subgroup isomorphic to  $G/N$ .
- (f) The map  $*: G \times G \rightarrow G$  given by  $g * a = ag$  defines a group action of  $G$  on  $G$ .

## 2 Examples

Provide an example for each of the following. (No further explanation needed.)

- (a) A finite nonabelian group.
- (b) A group, and an automorphism of that group which is not the map  $\varphi(g) = g$ .
- (c) A subgroup of  $S_3$  which is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ .
- (d) A subgroup of  $\mathbb{Z}/63\mathbb{Z}$  which is not equal to  $\{0\}$  or the whole group  $\mathbb{Z}/63\mathbb{Z}$ .
- (e) Two groups  $G_1$  and  $G_2$ , and a subgroup of  $G_1 \times G_2$  which is not equal to  $H_1 \times H_2$ , for any subgroups  $H_1$  and  $H_2$  of  $G_1$  and  $G_2$ , respectively.
- (f) A permutation in  $S_6$  with sign  $-1$ .

## 3 Short answer

For each of these problems, provide a short explanation with your answer.

### 3.1 Number of homomorphisms

Find the number of homomorphisms from  $\mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ .

### 3.2 Action of $D_5$ on edges of pentagon

Consider the action of  $D_5$  on the edges of a regular pentagon  $A_1A_2A_3A_4A_5$ .

- (a) What are the orbits of this action?
- (b) What is the stabilizer of the edge  $A_1A_2$ ?

## 4 Proof-based problems

For each of these problems, you should write a complete proof.

### 4.1 Index 2 subgroup is normal

Let  $G$  be a group and suppose  $H$  is a subgroup of  $G$  of index 2. Prove that  $H$  is normal.

### 4.2 Please remember to review the bonus problems

Let  $n > 1$  be a positive integer. Prove that  $n$  does not divide  $3^n - 2^n$ .