

1 True / False

Label each statement as true or false. For each true statement, briefly explain why it is true (a single sentence should be enough). For each false statement, provide a counterexample.

- (a) Let a, b, d, r, s be nonzero integers. If $d = ra + sb$, then d divides both a and b .
- (b) If $\varphi: G \rightarrow G'$ is a homomorphism of groups such that $\ker(\varphi) = \{1_G\}$, then φ is injective.
- (c) In the dihedral group D_n , the relation $r^k s r^k = s$ holds for any integer k .
- (d) Let G be a group, and let $a \in G$. Then the set $H = \{g \in G : ga = ag\}$ is a subgroup of G .
- (e) Let G be a group, and let N be a normal subgroup of G . Then G contains a subgroup isomorphic to G/N .
- (f) The map $*$: $G \times G \rightarrow G$ given by $g * a = ag$ defines a group action of G on G .

2 Examples

Provide an example for each of the following. (No further explanation needed.)

- (a) A finite nonabelian group.
- (b) A group, and an automorphism of that group which is not the map $\varphi(g) = g$.
- (c) A subgroup of S_3 which is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.
- (d) A subgroup of $\mathbb{Z}/63\mathbb{Z}$ which is not equal to $\{0\}$ or the whole group $\mathbb{Z}/63\mathbb{Z}$.
- (e) Two groups G_1 and G_2 , and a subgroup of $G_1 \times G_2$ which is not equal to $H_1 \times H_2$, for any subgroups H_1 and H_2 of G_1 and G_2 , respectively.
- (f) A permutation in S_6 with sign -1 .

3 Short answer

For each of these problems, provide a short explanation with your answer.

3.1 Number of homomorphisms

Find the number of homomorphisms from $\mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

3.2 Action of D_5 on edges of pentagon

Consider the action of D_5 on the edges of a regular pentagon $A_1A_2A_3A_4A_5$.

- (a) What are the orbits of this action?
- (b) What is the stabilizer of the edge A_1A_2 ?

4 Proof-based problems

For each of these problems, you should write a complete proof.

4.1 Index 2 subgroup is normal

Let G be a group and suppose H is a subgroup of G of index 2. Prove that H is normal.

4.2 Please remember to review the bonus problems

Let $n > 1$ be a positive integer. Prove that n does not divide $3^n - 2^n$.