Math 401 Problem Set 5 (due February 20, 2026)

Problem 1. Determine the number of conjugacy classes in (a) S_6, (b) A_6.

Problem 2. Does S_4 have a normal subgroup of order 3?

Problem 3. Let G be a simple group, and let G' be any group. Show that any nontrivial homomorphism $\varphi\colon G \to G'$ is injective.

Problem 4. Here, we fill in the proof of a proposition (Proposition 2.11.4 in Artin) about product groups which was omitted from the lecture.

Let G be a group and let H and K be subgroups, and let $f\colon H \times K \to G$ be the function

$$f(h, k) = hk.$$

Let $HK = \{hk : h \in H, k \in K\}$, which is the image of f.

(a) Show that f is injective if and only if $H \cap K = \{1\}$.

(b) Show that f is a homomorphism if and only if $hk = kh$ for all $h \in H$ and $k \in K$.

(c) Show that if $H \cap K = \{1\}$, $HK = G$, and H and K are normal, then f is an isomorphism from the product group $H \times K$ to G.

(*Hint.* Two elements h and k commute if and only if the *commutator* $hkh^{-1}k^{-1}$ is equal to 1.)

Problem 5. Let p and q be distinct primes, and suppose that G is a group of order p^2q. Show that G is not simple.

Problem 6. Classify groups of order 2026. (Note that 1013 is prime.)

Problem 7. Approximately how long did you spend on this problem set? (Round to the nearest half-hour.)

Bonus problem (not graded). Let G be a group of order 56. Show that G is not simple.