

## Math 401 Problem Set 2 (due January 23, 2026)

At the top of your submission, write “**sources consulted:**” and list all sources you used while working on this problem set. If no sources were used, write “none.” If one of your sources is an LLM, please also include a copy of your prompt log.

**Problem 1.** Prove that if  $\varphi: G \rightarrow G'$  is an isomorphism, then the order of any element  $a$  of  $G$  is equal to the order of  $\varphi(a)$  in  $G'$ .

**Problem 2.** (a) Show that  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$  is not isomorphic to  $\mathbb{Z}/18\mathbb{Z}$ .

(b) Show that  $(\mathbb{Z}/15\mathbb{Z})^\times$  and  $(\mathbb{Z}/20\mathbb{Z})^\times$  are isomorphic. Show that they are not isomorphic to  $(\mathbb{Z}/24\mathbb{Z})^\times$ .

**Problem 3.** Recall that  $D_5$  is the dihedral group with 10 elements, and  $r$  is a rotation by  $2\pi/5$ , and  $s$  is a reflection.

(a) Is  $\langle r \rangle$  a normal subgroup of  $D_5$ ?

(b) Is  $\langle s \rangle$  a normal subgroup of  $D_5$ ?

**Problem 4.** Show that there are 5 homomorphisms  $\mathbb{Z}/5\mathbb{Z} \rightarrow \mathbb{C}^\times$ , but only 2 homomorphisms  $D_5 \rightarrow \mathbb{C}^\times$ .

**Problem 5.** Give an example of an injective homomorphism  $D_5 \rightarrow S_5$ .

**Problem 6.** An *automorphism* of a group  $G$  is an isomorphism from  $G$  to itself.

(a) Show that for any  $g \in G$ , the map  $\varphi(a) = gag^{-1}$  is an automorphism of  $G$ . (In general, some automorphisms of  $G$  are of this form, and some are not.)

(b) Find the number of automorphisms of  $\mathbb{Z}/10\mathbb{Z}$ .