

## 1 True / False (5 points each)

Label each statement as true or false. For each true statement, briefly explain why it is true (a single sentence should be enough). For each false statement, provide a counterexample.

- (a) In the dihedral group  $D_n$ , the relation  $r^k s r^k = s$  holds for any integer  $k$ .
- (b) If  $\varphi: G \rightarrow G'$  is a homomorphism of groups such that  $\ker(\varphi) = \{1_G\}$ , then  $\varphi$  is injective.
- (c) Let  $G$  be a group, and let  $a \in G$ . Then the set  $H = \{g \in G : gag^{-1} = a\}$  is a subgroup of  $G$ .
- (d) The map  $*: G \times G \rightarrow G$  given by  $g * a = ag$  defines a group action of  $G$  on  $G$ .
- (e) Let  $G$  be a group, and let  $N$  be a normal subgroup of  $G$ . Then  $G$  contains a subgroup isomorphic to  $G/N$ .
- (f) Let  $G$  be a group. If  $a, b, c \in G$  are elements satisfying  $ab = bc$ , then  $a = c$ .

## 2 Examples (5 points each)

Provide an example for each of the following. (No further explanation needed.)

- (a) A finite nonabelian group.
- (b) A group, and an automorphism of that group which is not the map  $\varphi(g) = g$ .
- (c) A subgroup of  $\mathbb{Z}/63\mathbb{Z}$  which is not equal to  $\{0\}$  or the whole group  $\mathbb{Z}/63\mathbb{Z}$ .
- (d) A subgroup of  $S_3$  which is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ .
- (e) Two groups  $G_1$  and  $G_2$ , and a subgroup of  $G_1 \times G_2$  which is not equal to  $H_1 \times H_2$ , for any subgroups  $H_1$  and  $H_2$  of  $G_1$  and  $G_2$ , respectively.
- (f) A permutation in  $S_6$  with sign  $-1$ .

## 3 Short answer

For each of these problems, provide a short explanation with your answer.

### 3.1 Number of homomorphisms

Find the number of homomorphisms from  $\mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ .

### 3.2 Action of $D_5$ on edges of pentagon

Consider the action of  $D_5$  on the edges of a regular pentagon  $A_1A_2A_3A_4A_5$ .

- (a) What are the orbits of this action?
- (b) What is the stabilizer of the edge  $A_1A_2$ ?

## 4 Proof-based problems

For each of these problems, you should write a complete proof.

### 4.1 Index 2 subgroup is normal

Let  $G$  be a group and suppose  $H$  is a subgroup of  $G$  of index 2. Prove that  $H$  is normal.

### 4.2 Please remember to review the bonus problems

Let  $n > 1$  be a positive integer. Prove that  $n$  does not divide  $3^n - 2^n$ .