Math 401 Problem Set 2 (due January 23, 2026)

At the top of your submission, write "**sources consulted:**" and list all sources you used while working on this problem set. If no sources were used, write "none." If one of your sources is an LLM, please also include a copy of your prompt log.

Problem 1. Prove that if $\varphi\colon G \to G'$ is an isomorphism, then the order of any element a of G is equal to the order of $\varphi(a)$ in G'.

Problem 2.

(a) Show that $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/18\mathbb{Z}$.

(b) Show that $(\mathbb{Z}/20\mathbb{Z})^\times$ is not isomorphic to $(\mathbb{Z}/24\mathbb{Z})^\times$.

Problem 3. Recall that D_5 is the dihedral group with 10 elements, and r is a rotation by $2\pi/5$, and s is a reflection.

(a) Is $\langle r \rangle$ a normal subgroup of D_5?

(b) Is $\langle s \rangle$ a normal subgroup of D_5?

Problem 4. Show that there are 5 homomorphisms $\mathbb{Z}/5\mathbb{Z} \to \mathbb{C}^\times$, but only 2 homomorphisms $D_5 \to \mathbb{C}^\times$.

Problem 5. Give an example of an injective homomorphism $D_5 \to S_5$.

Problem 6. (Exerise 2.6.10 and automorphisms) An *automorphism* of a group G is an isomorphism from G to itself.

(a) Find the number of automorphisms of $\mathbb{Z}/10\mathbb{Z}$.

(b) Show that for any $g \in G$, the map $\varphi(a) = gag^{-1}$ is an automorphism of G. (In general, some automorphisms of G are of this form, and some are not.)

(c) Let $G = S_3$. Show that for any two distinct elements $g_1, g_2 \in G$, the maps $\varphi_1(a) = g_1 a g_1^{-1}$ and $\varphi_2(a) = g_2 a g_2^{-1}$ are different automorphisms of G.

(d) Find the number of automorphisms of S_3.