Cops and Robbers Meets Chess

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Cops and Robbers Meets Chess

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Outline of Presentation

1. Introduction to Cops & Robbers on Graphs
    - a. Rules of the game
    - b. Terminology and notation
    - c. Some simple examples
    - d. The *Lazy Cops* variant
2. Results about Rooks and Bishops Graphs
    - a. Ordinary Cops: only 2 are needed on an $n \times n$ board
    - b. Lazy Cops: n (Rooks) or $\sim n/2$ (Bishops) are needed on an $n \times n$ board
3. Results about Queens Graphs
    - a. Ordinary Cops: at most 4 are needed on an $n \times n$ board
    - b. Lazy Cops: $\sim n/3$ are needed on an $n \times n$ board
4. Summary and Open Problems

Slides posted on Twitter @professorbrenda (no N!)

The Game of Cops and Robbers

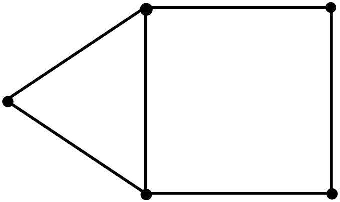
Throughout, the playing board is a specified **graph**, consisting of **vertices** and **edges** between them. A legal **move** is made along an edge.

Definition (Rules of the game)

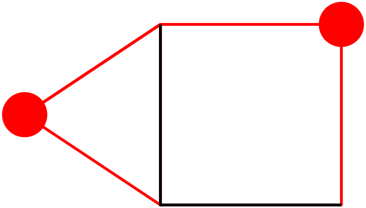
- *Specify a playing board. Specify number of Cops. (One Robber.)*
- *Each Cop chooses where to start.*
- *Robber then gets to choose where to start.*
- **Cops' turn:** *everyone gets to make a move (or stay put).*
- **Robber's turn:** *he gets to make a move (or stay put).*
- *Repeat back and forth.*
- *If a Cop lands on the Robber, their team wins.*
- *If the Robber can* indefinitely *evade the Cops, he wins.*

Important: We want to investigate *optimal strategies*, how idealized players would choose their moves. (While it can be fun to play in person, we really want to explore the *theory of the game.*)

Example: The *Cop Number* of a Graph



- Play against 1 Cop. Who wins?
  - Robber does! Start on a safe vertex of the *4-cycle*. If Cop moves to threaten, respond accordingly; else, stay put.
- Play against 2 Cops. Who wins?



- 2 Cops can **dominate** the graph by guarding every vertex.
  - This is a trivial (but effective!) way for the Cops to win.
  - The **domination number** of this graph G is $\gamma(G) = 2$.

Conclusion: the **Cop number** of this graph G is $c(G) = 2$.

In general, $c(G)$ is *minimum* number of cops required to *guarantee* victory.

Results about Cop Numbers

- The game was introduced independently by Quilliot (1978) [5] and Nowakowski & Winkler (1983) [3].
- Aigner & Fromme (1984) [1] proved several results:
  1. $c(G) = 1 \iff G$ can be dismantled by iteratively removing **pitfalls** (a vertex whose neighborhood is dominated by another vertex).
  2. If G contains no 3- or 4-cycles, then $c(G) \geq \delta(G)$, where $\delta(G)$ is the minimum **degree** (number of incoming edges) amongst all vertices. Furthermore, such graphs can be constructed inductively. Thus, cop numbers can be arbitrarily large.
  3. If G is **planar**, then $c(G) \leq 3$ (!)
- Other ideas:
  1. If H is a **retract** of G, then $c(H) \leq c(G)$.
  2. Computing $c(G)$ is `NP-hard`.
  3. **Meyniel's Conjecture:** For n ($\#$ of vertices) sufficiently large, $\exists d$ such that $c(G) \leq d\sqrt{n}$ (i.e. $c(G) = O(\sqrt{n})$). This is the deepest open problem in the field. See Bonato's book [2].

The *Lazy Cops* Variant

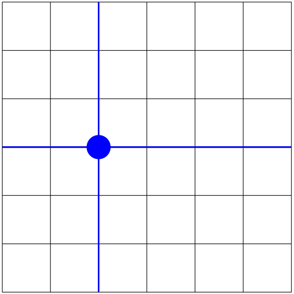
This variant was introduced in 2013 by Offner & Ojakian [4]. The *only* difference in the rules is that, when it is the Cops' turn, **only one of them may move** while the rest stay put (or they all stay put).

Our research group set out to explore the differences between the so-called Ordinary and Lazy versions of the game. Notation: $c(G)$ versus $c_L(G)$.

$$\text{For } \textit{any} \text{ graph:} \quad c(G) \leq c_L(G) \leq \gamma(G)$$

- There are classes of graphs for which $c(G)$ is bounded but $c_L(G)$ grows without bound. We will see some examples today.
- Are there classes of graphs for which both are bounded?
- What are the graphs for which $c(G) = c_L(G)$?
- What are the graphs for which $c_L(G) = \gamma(G)$?
- What properties of graphs have an influence on $c(G)$ and $c_L(G)$?

Chess: The Rooks Graph



R_n is the graph whose ...

- vertices are identified with the squares of an $n \times n$ chessboard
- edges correspond to a legal move by a Rook (any distance horizontally or vertically)

Pictured: R_7 and the **neighborhood** of a vertex.

(Note: We typically put pieces on the *intersections* so that edges are visible. Not like standard Chess!)

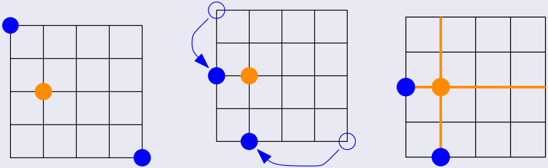
Ordinary Cop Number of Rooks Graphs

Theorem

For any $n \geq 2$, $c(R_n) = 2$.

Proof.

Place the Cops at opposite corners of the board. The Robber starts anywhere. On the Cops' first turn, move one of them to the Robber's *row* and the other to the Robber's *column*. Now, he's trapped!



Note: This only shows $c(R_n) \leq 2$. To complete the proof, we must exhibit a strategy *from the Robber's perspective* to evade one Cop:

Don't start in the Cop's initial row or column. If he moves to your row, go to a different row; likewise for a columnar threat. Otherwise, stay. □

Lazy Cop Number of Rooks Graphs

Theorem

For any n, $c_L(R_n) = n = \gamma(R_n)$.

Proof.

Cop strategy: Place Cops along the entire first column: each one guards his entire row. Thus, $c_L(R_n) \leq n = \gamma(R_n)$ (no smaller domination possible).

Robber strategy: Place $n-1$ Cops anywhere. By Pigeonhole Principle, at least one row is unoccupied by a Cop; then, there is at least one square in that row that is unguarded, as well. Thus, Robber may start.

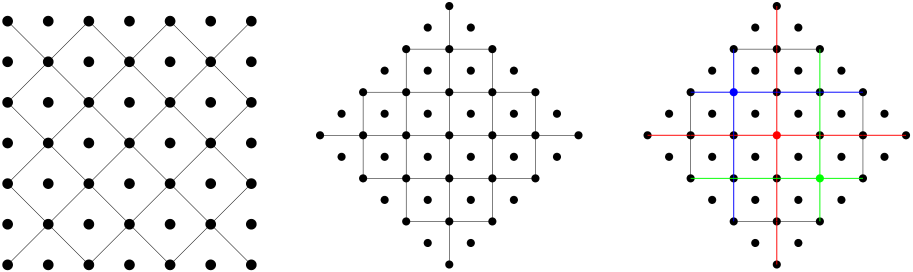
"One Threat" observation: In exhibiting a Robber strategy to evade some number of *Lazy* Cops, we need only consider *one Cop threat at a time:* since only one may move at a time, only one new threat may appear, and if we have two threats, then why didn't the Cops win on their last turn?

Apply "One Threat": if Cop threatens via row (resp. column), look along Robber's column (resp. row); at least one square is unguarded. □

Bishops Graphs: Not Much New!

Pictured below: B_7. Notice this "cuts the board in half" since we're restricted to chessboard squares of the same color (on same diagonals).

Rotate 45°: B_n is an **induced subgraph** of R_n. Ordinary Cops use same strategy as before to win in one turn. Dominating this subgraph is necessary and sufficient for Lazy Cops' victory; need $\lfloor (n+1)/2 \rfloor$ Cops.



Theorem

For any $n \geq 2$, $c(B_n) = 2$ and $c_L(B_n) = \left\lfloor \frac{n+1}{2} \right\rfloor = \gamma(B_n)$.

Summary: Bishops and Rooks Graphs

- For both, the Ordinary Cop number is 2, regardless of the board's size.
- The Lazy Cop number grows with the board's size. For an $n \times n$:
  - $c_L(R_n) = n$
  - $c_L(B_n) = \lfloor (n+1)/2 \rfloor$

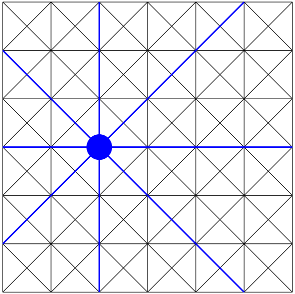
  Both are equal to the *domination number*.
- $|V(R_n)| = n^2$ and $|E(R_n)| = n^3 - n^2$.

  In general, there seems to be a relationship between the Cop number of a graph and its **edge density**.

This is a canonical example of how different the Ordinary and Lazy cop numbers can be, where $c_L = \gamma$ but $c \ll \gamma$.

Open research: What characterizes these situations? We've found some examples, but would like to understand where/why this happens.

Chess: The Queens Graph



Q_n is the graph whose ...

- vertices are identified with the squares of an $n \times n$ chessboard
- edges correspond to a legal move by a Queen (any distance horizontally, vertically, or diagonally)

Pictured: Q_7 and the neighborhood of a vertex.

Note: Q_n is effectively "$B_n \cup R_n$"

More complicated than the Rooks and Bishops combined! With R_n, every vertex has "two neighborhoods", one for each direction of movement, and they're *the same size*. Now, each vertex of Q_n has "four neighborhoods" and they can be *different in size*.

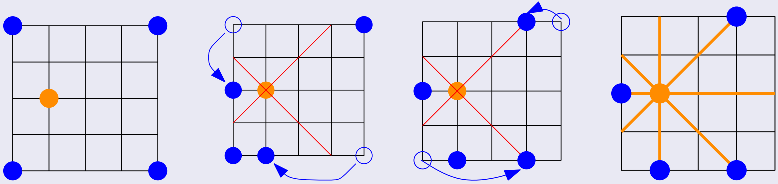
Ordinary Cop Number of Queens Graphs: Upper Bound

Theorem

For any $n \geq 2$, $c(Q_n) \leq 4$.

Proof.

Cop strategy: Place Cops at four corners. Robber starts anywhere. On first turn, capture Robber's row, column, and two diagonals.



□

Proving a *lower bound* by exhibiting a **Robber strategy** is significantly more challenging! This is a phenomenon we noticed throughout our research. Indeed, the literature seems to consist mostly of Cop strategies.

Observations about Queens Graphs: Cops Guarding Lines

Definition

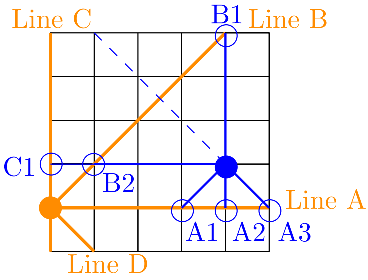
Given a Robber's position, his four **lines** *are the row, column, and two diagonals that contain him. A Cop is* **guarding** *a vertex if he can "see" it from his current vertex. A* **threat** *means a Cop can see the Robber.*

Lemma

A Cop can guard **0, 1, 2, or 3** *vertices on each of the Robber's lines.*

If Cop also threatens, *then he can guard at most 2 vertices on a line.*

In particular, you can't threaten along one line and *guard 3 vertices on another line.*



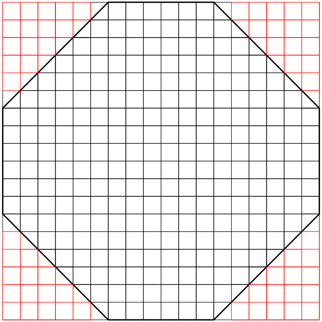
Example: Cop guards 3 on Line A, 2 on Line B, 1 on Line C, and 0 on Line D.

Ordinary Cop Number of Queens Graphs: Lower Bound

Theorem

For any $n \geq 19$, $c(Q_n) > 3$; with previous theorem, this shows $c(Q_n) = 4$.

Proof idea: Use previous observations about guarding lines. If the Robber always has an available line which cannot be fully guarded, then he wins. If the board is large enough, we can find a substructure with this property.



- This octagon is chosen so that the boundary sides have length 7. Using the previous lemma, we can prove that the Robber can stay on the perimeter of this octagon and **evade 3 Cops**.
- The smallest board containing this structure is $n = 19$.
- Could it be smaller? Probably! More careful analysis of cases is required.

Lazy Cop Number of Queens Graphs

Theorem

For any $n \geq 6$, $\lceil n/3 \rceil \leq c_L(Q_n) \leq \lceil 2n/3 \rceil$.

Proof idea: For lower bound, use previous observations about guarding lines, as well as the *one threat at a time* observation about Lazy Cops. Also, appeal to known: $\gamma(Q_n) \geq \frac{n-1}{2} \geq \lceil n/3 \rceil$ (Spencer, from [6]).

For upper bound, appeal to known: $\gamma(Q_n) \leq \lceil 2n/3 \rceil$ (Welch, from [6]).

Proof.

- Play against k Cops (TBD). Provided $k < \frac{n-1}{2}$, Robber may start.
- If Robber not threatened, stay put.
- If Robber threatened, that Cop guards one entire line. There must be *at least one other size-n line*. On that line: Robber occupies 1 vtx, threatening Cop guards ≤ 2 vtxs, any other Cops guard ≤ 3 vtxs.
- So, as long as $k < \frac{1}{3}(n-1-2) = \frac{n-3}{3} < \lceil n/3 \rceil$, Robber can make a move along that line to escape the one threat. □

Summary: Queens Graphs

- Ordinary Cop number is ≤ 4, regardless of board's size.
- The Lazy Cop number grows with the board's size, bounded between $\lceil n/3 \rceil$ and $\lceil 2n/3 \rceil$.
- Known and conjectured values below. A "?" indicates a conjecture. A circled number indicates that it equals γ.

	3	4	5	6	7	8	$\cdots$	14	15	$\cdots$	19
c	1	2	2	2	3	3	$\cdots$	3?	4?	$\cdots$	4
c_L	1	2	2	3	4?	4?	$\cdots$	?	?	$\cdots$	

- *Conjecture:* Q_{14} is the cutoff from $c = 3$ to $c = 4$.
- *Conjecture:* $c(Q_{n-1}) \leq c(Q_n)$. Seems obvious but surprisingly difficult to prove! (Note: *Retract* fact doesn't apply.)
- *Open problems:* c_L for larger n. Characterizing when $c_L = \gamma$.
- Some results about $\gamma(Q_n)$ can help, but all sorts of chessboard problems about Queens are also challenging, in general!

Conclusions

- $c(R_n) = 2$ but $c_L(R_n) = n$
- $c(B_n) = 2$ but $c_L(B_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$
- $c(Q_n) \leq 4$ but $\left\lceil \frac{n}{3} \right\rceil \leq c_L(Q_n) \leq \left\lceil \frac{2n}{3} \right\rceil$
- Kings Graphs: $c = c_L = 1$
- Knights Graphs: ?????
- *Specific open problems:* Tightening bounds on $c_L(Q_n)$. More specific results for $c(Q_n)$ for $n \leq 19$. Knights? Toroidal boards?

  Conjecture: Amongst all graphs with $c_L(G) = n$, R_n is the *smallest*.
- *General open problems:* When is $c_L = \gamma$? When are c and c_L vastly different? Are these answers related to edge density?

These problems are approachable by undergraduates!

References

[1] M. Aigner, M. Fromme, A game of cops and robbers, *Discr.Appl. Math.* **8** (1984) 1–12.

[2] A. Bonato, R. Nowakowski, *The Game of Cops and Robbers on Graphs*, AMS Student Mathematical Library, Providence, RI, 2011.

[3] R. Nowakowski, P. Winkler, Vertex-to-vertex pursuit in a graph, *Discr. Math.* **43** (1983) 235–239.

[4] D. Offner, K. Ojakian, Variations of cops and robbers on the hypercube, *Australas. J. Combin.* **59** (2014) 229–250.

[5] A. Quilliot, Jeux et pointes fixes sur les graphes, *Thèse de 3ème cycle*, Université de Paris VI, 1978, 131-145.

[6] J. J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton Univ. Press, Princeton, NJ, 2012.

THANK YOU

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Questions?

I'm speaking about Cops & Robbers again later this afternoon!

3:00-3:10pm in Room 605, AMS Combinatorics & Graph Theory 1

"Lazy Cops & Robbers on Product Graphs"