DGHT Key Population Surveillance Seminar Series

Novel approaches to population size estimation synthesis using Bayesian statistics

August 15th, 2023

Division of Global HIV & TB

UCSF Institute for Global Health Sciences

The Anchored Multiplier calculator

A Bayesian tool to reconcile discrepant population size estimates

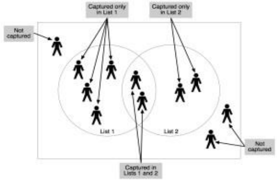
Paul Wesson, PhD
Department of Epidemiology and Biostatistics
University of California, San Francisco

8/17/2023

Population Size Estimation (PSE) is fundamental to public health surveillance:

- Apportionment/Equitable distribution of services and resources
- Determine denominators for calculating measures of disease
- Establish sampling frame for studies
- Quantify a potential public health issue
- Evaluate the reach and coverage of outreach programs and social services
- Set concrete targets for program services

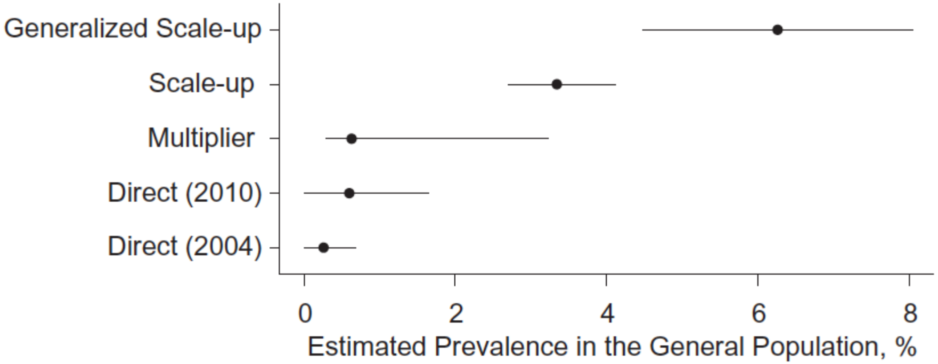
Methods for Population Size Estimation



Opinion	General Population	Venue	Literature	Multiple-Sample	Network
Programme Estimates	Population survey	Mapping and Enumeration	Literature review	Capture-Recapture	Network Scale-up
Wisdom of the Crowd		Reverse Tracking Method		Multiplier Method	Sequential Sampling (SS-PSE)
Modified Delphi					

Wesson P, Reingold A, McFarland W. *Theoretical and Empirical Comparisons of Methods to Estimate the Size of Hard-to-Reach Populations: A Systematic Review*. ***AIDS and Behavior***. 2017 July; 21(7), 2188-2206.

How reliable (consistent) are population size estimates?



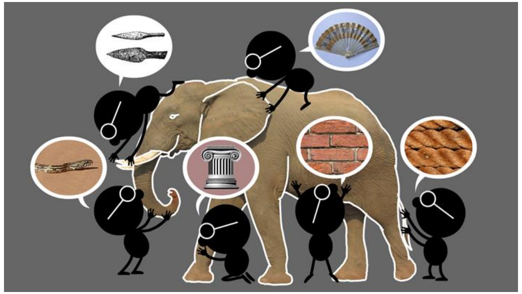
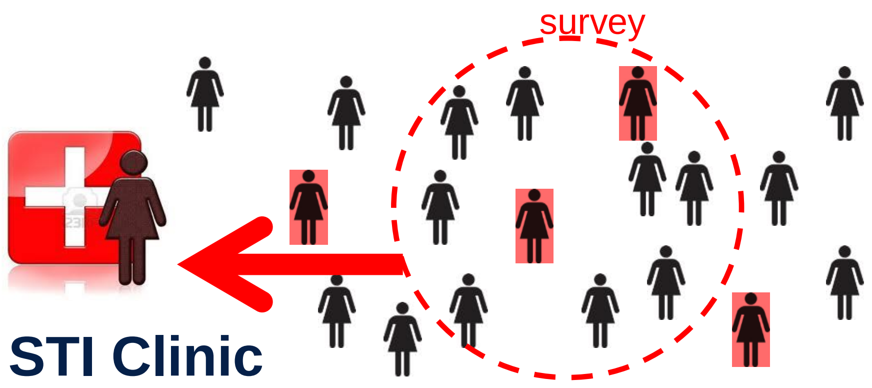


Figure 2. Five estimates of the prevalence of heavy drug use in Curitiba, Brazil, 2004 and 2009–2010. Scale-up and generalized scale-up estimates were substantially higher than those obtained from standard methods (direct estimation and the multiplier method). Estimates of the number of heavy drug users in Curitiba ranged from 4,700 to 114,000. Bars, estimated 95% confidence interval.

Salganik et al. (2011). *American Journal of Epidemiology. 174(10): 1190-1196*

Multiplier Method

- Data
  - 2 data sources
- Estimation
  - N = M/P
- Assumptions
  - Two data sources must be independent (count and survey)
  - Two data sources must be specific to the population (correct person, place, time)
  - At least one source should be representative
  - Limited in- and out-migration
  - No duplicate count (within data source)



$$N = \frac{1}{P} \times M = \frac{M}{P}$$

Member of Hidden P.

Member of Hidden P. who received STI service (in last year) from the clinic

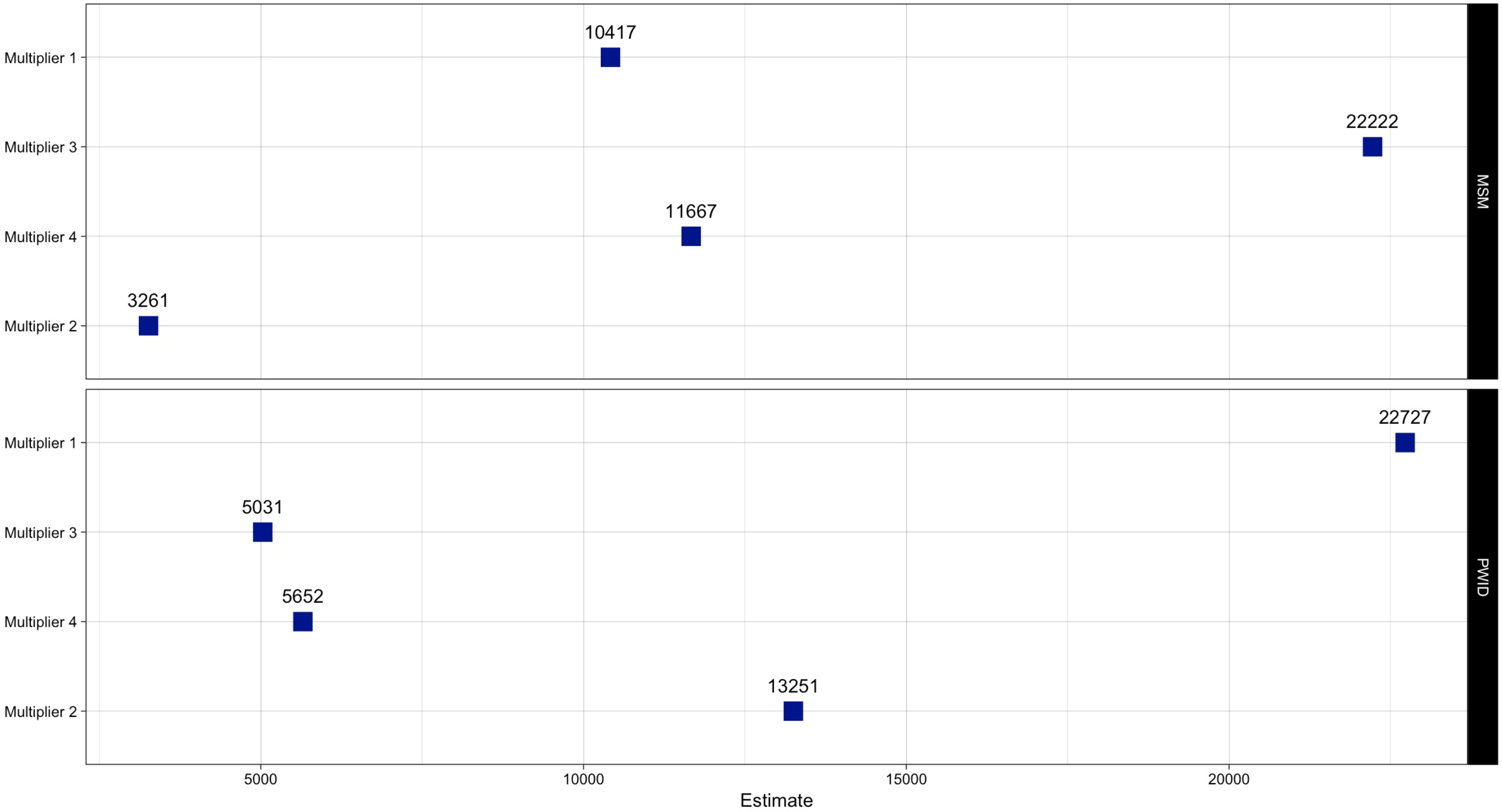
M: total recruited people in your hidden population survey
P: proportion of hidden population who received STI services from the clinic

Johnston LG et al [Sex Transm Dis.] 2013 Apr;40(4):304-10
Slide Credit: Ali Mirzazadeh, PhD

Multiplier Method – Potential Sources of Bias

- Sample from target population may not be representative
- Service providers may report number of visits (not number of unique people)
- Service providers may report total number of people seen (including those who are not members of the key population)
- Study sample participants may misremember receiving specific service
- Dependency (positive or negative) between the benchmark and the multiplier

Nairobi, Kenya 2010 (Okal 2013)



How do you arrive at a single “best” estimate?

- **Take the median**
  - **Pro**: Simple. Presumed to “balance” biases that drive estimates in different directions
  - **Con**: Assumes individual estimates are equally biased (or valid)
- **Delphi approach**
  - **Pro**: Incorporate local knowledge. Final estimate is more “acceptable” to stakeholders
  - **Con**: Vulnerable to subjective biases to increase or decrease PSE

The Anchored Multiplier

- Bayesian framework to synthesize multiple data points (namely, estimates from multiple multipliers) and stakeholder knowledge (or estimates from published literature) as a priori belief.
- Estimates from multiplier methods (**likelihood**) and prior belief (**prior**) converted to probability distributions and combined according to Bayes Theorem.
- Posterior probability distribution reflects updated knowledge of the population size, driven by the strength (i.e. precision) of the individual inputs.

$$p(\theta|x) \propto p(x|\theta)p(\theta)$$

Anchored Multiplier – Variance Adjusted

- Incorporating more data results in increasingly strong priors and extremely narrow 95% credible intervals
- Borrow methodology from meta-analysis literature for calculating random effects variance to estimate additional variance:

$$\tau^2 = \frac{Q\ - df}{C}$$

- Where…

$$Q = \sum_{i=1}^{k} W_i Y_i^2 - \frac{\left(\sum_{i=1}^{k} W_i Y_i\right)^2}{\sum_{i=1}^{k} W_i}; \qquad C = \sum W_i - \frac{\sum W_i^2}{\sum W_i}; \qquad df = k - 1$$

Case study: PWID in San Francisco

- Data Source
  - National HIV Behavioral Surveillance (NHBS) data for PWID in San Francisco
  - 3 rounds of data collection - 2005, 2009, 2012
  - Published estimates from Multiplier Methods (*Chen et al. 2016*)
- Priors
  - Lifetime experience = **2.6% (1.8% - 3.3%)** … *Lansky et al. 2014*
  - Recent drug use = **0.3% (0.19% - 0.41%)** … *Lansky et al. 2014*
  - Vaguely informative = **Uniform (0, 12.88%)**
- Model

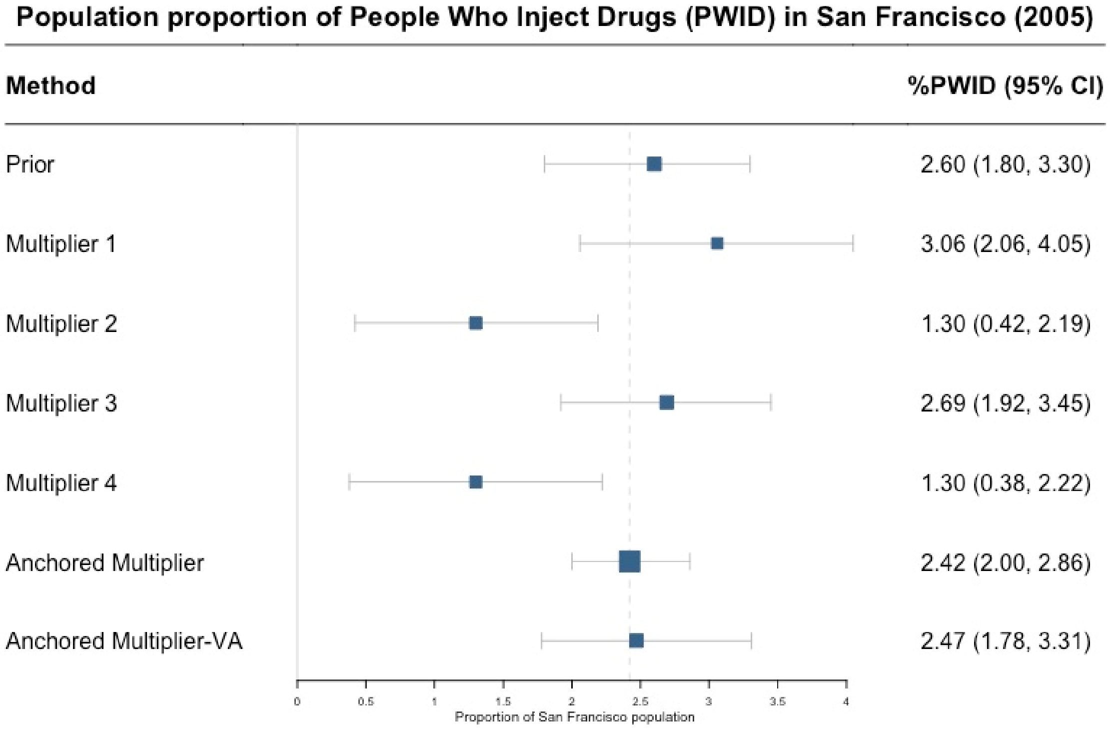
y ~ dbinom(theta, n)

Prior

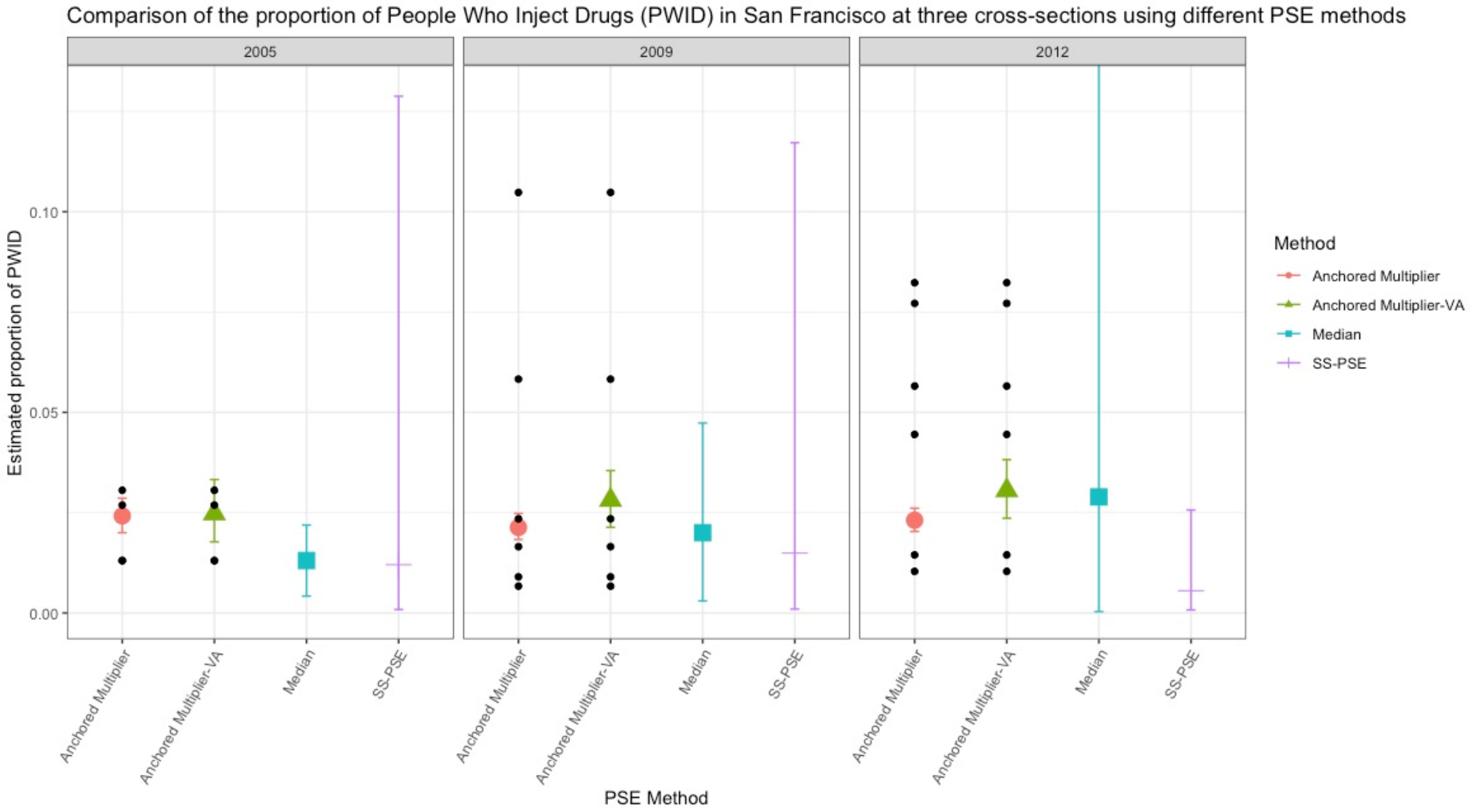
theta ~ dbeta(a, b)

y = population proportion of PWID

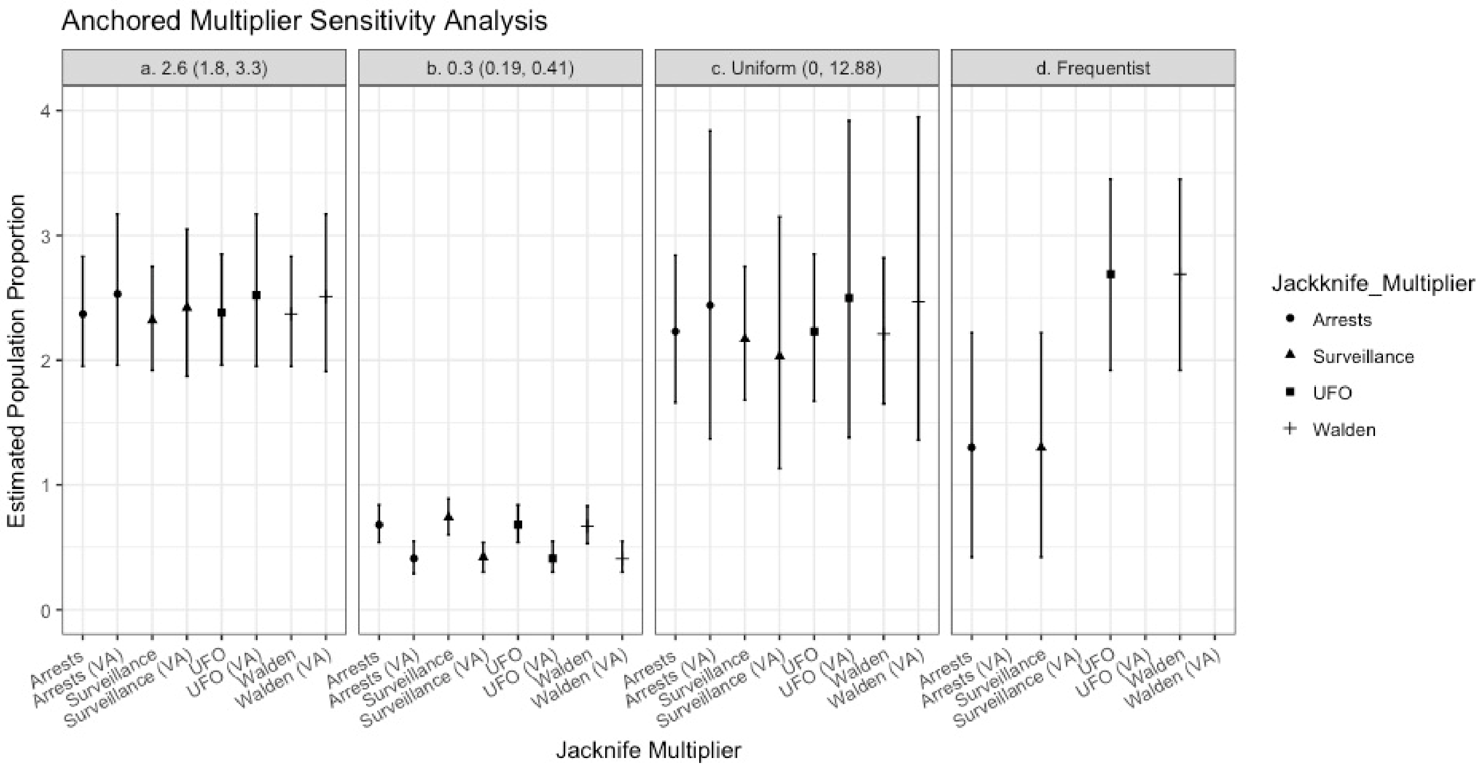
Results: Comparison of estimates (2005)



Results: Comparison of estimates over time



Results: Sensitivity Analysis



Strengths

- Accounts for "strength" of individual estimates, does not treat all estimates as equally precise/valid
- Systematically synthesizes data in way that is transparent and replicable
- Posterior "anchored" by the prior, down-weighted (imprecise) estimates do not pull final estimate towards zero
- Does not require advanced quantitative background

Limitations

- Not a guaranteed fix to biases from the multiplier method (or other PSE methods)

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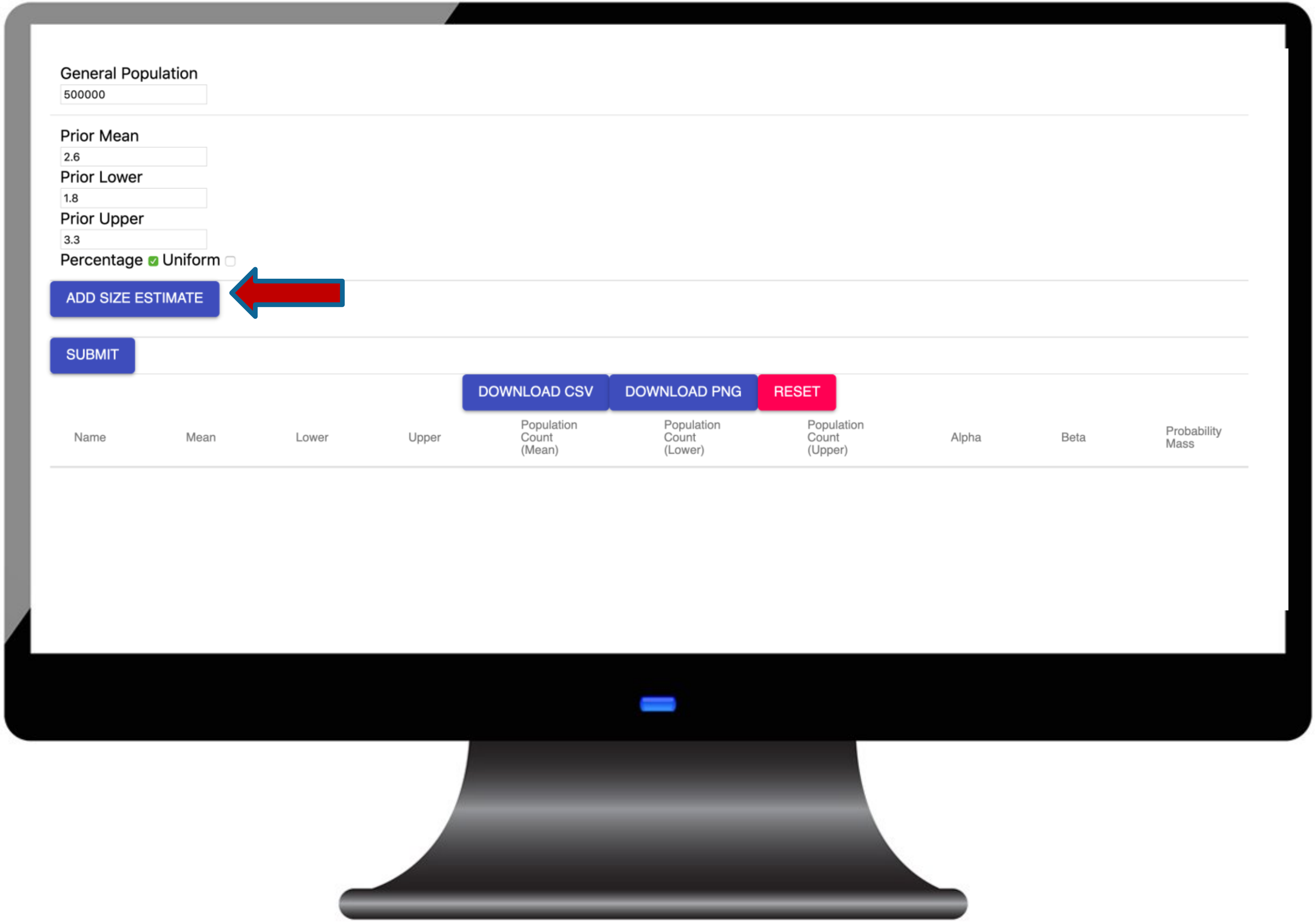
Appropriate Usage

- Population size estimates
  - Using <u>ANY</u> PSE method (not just the multiplier method)
  - Population percentage or absolute counts as input
  - At least two population size estimates required (including the prior)
- Prevalence estimates
- The online calculator is freely available at

https://globalhealthsciences.ucsf.edu/resources/tools


General Population
0
Prior Mean
0
Prior Lower
0
Prior Upper
0
Percentage Uniform
ADD SIZE ESTIMATE
SUBMIT
DOWNLOAD CSV
DOWNLOAD PNG
RESET
Name
Mean
Lower
Upper
Population Count (Mean)
Population Count (Lower)
Population Count (Upper)
Alpha
Beta
Probability Mass


General Population
500000
Prior Mean
2.6
Prior Lower
1.8
Prior Upper
3.3
Percentage
Uniform
ADD SIZE ESTIMATE
SUBMIT
DOWNLOAD CSV
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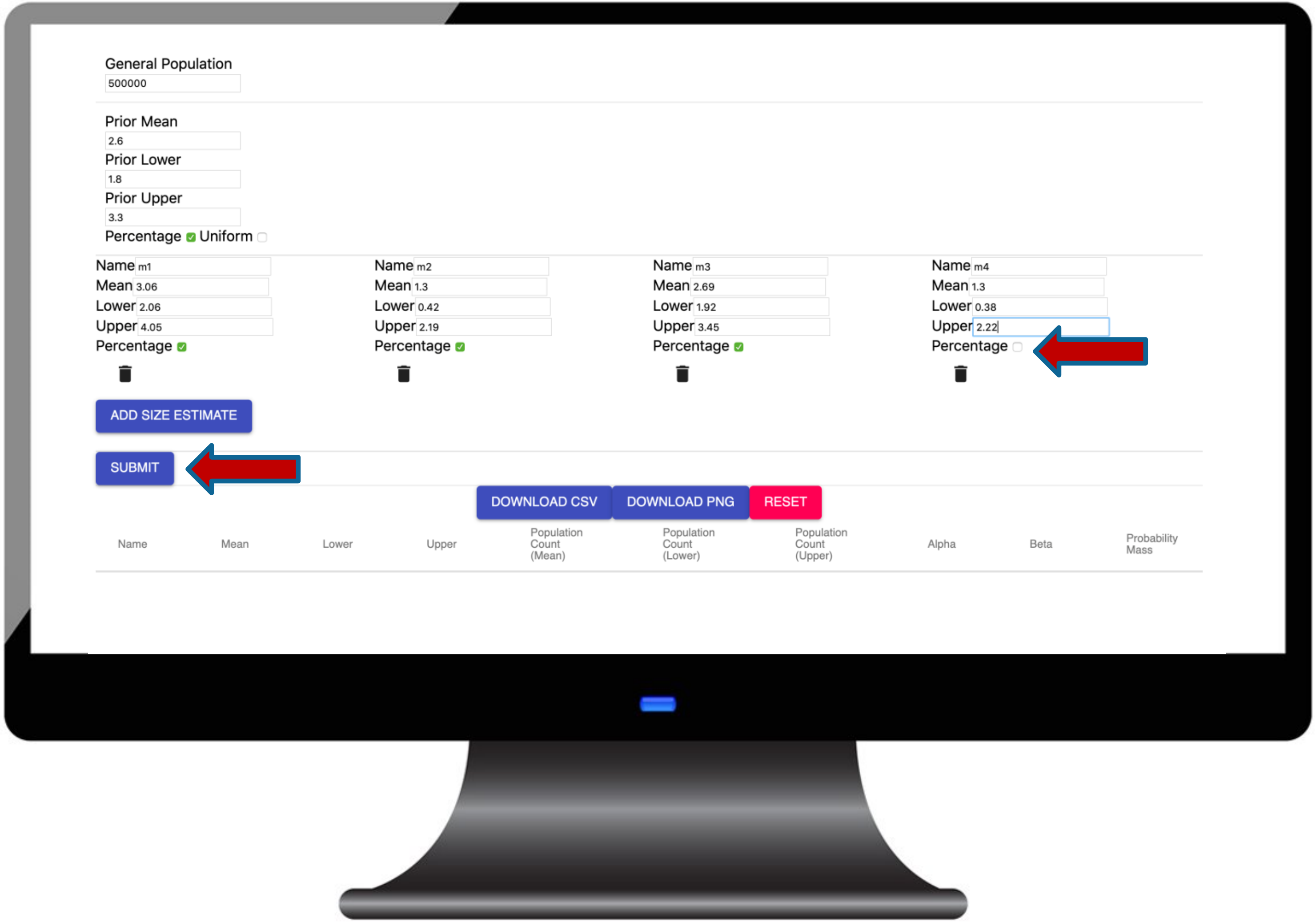
RESET
Name
Mean
Lower
Upper
Population Count (Mean)
Population Count (Lower)
Population Count (Upper)
Alpha
Beta
Probability Mass


General Population
500000
Prior Mean
2.6
Prior Lower
1.8
Prior Upper
3.3
Percentage Uniform
Name m1
Mean 3.06
Lower 2.06
Upper 4.05
Percentage
Name m2
Mean 1.3
Lower 0.42
Upper 2.19
Percentage
Name m3
Mean 2.69
Lower 1.92
Upper 3.45
Percentage
Name m4
Mean 1.3
Lower 0.38
Upper 2.22
Percentage
ADD SIZE ESTIMATE
SUBMIT
DOWNLOAD CSV
DOWNLOAD PNG
RESET
Name
Mean
Lower
Upper
Population Count (Mean)
Population Count (Lower)
Population Count (Upper)
Alpha
Beta
Probability Mass


DOWNLOAD CSV
DOWNLOAD PNG
RESET
Population
Estimates
Method
95% CI
Prior
2.6 (1.8, 3.3)
m1
3.06 (2.06, 4.05)
m2
1.3 (0.42, 2.19)
m3
2.69 (1.92, 3.45)
m4
1.3 (0.38, 2.22)
Anchored Multiplier
2.48 (2.1, 2.89)
Anchored Multiplier-VA
2.54 (1.99, 3.13)

Name	Mean	Lower	Upper	Population Count (Mean)	Population Count (Lower)	Population Count (Upper)	Alpha	Beta	Probability Mass
Anchored Multiplier	2.48	2.1	2.89	12406	10506	14449	145.21	5707.01	0.95
Anchored Multiplier Variance Adjusted	2.54	1.99	3.13	12682	9959	15645	73.07	2807.9	0.95
prior	2.6	1.8	3.3	13000	9000	16500	51.59	1932.81	0.96
m1	3.06	2.06	4.05	15300	10300	20250	35.55	1126.15	0.95
m2	1.3	0.42	2.19	6500	2100	10950	8.08	613.21	0.95
m3	2.69	1.92	3.45	13450	9600	17250	46.81	1693.18	0.95
m4	1.3	0.38	2.22	6500	1900	11100	7.56	573.81	0.96

Questions?

Contact information

- Paul Wesson, PhD
  Paul.Wesson@ucsf.edu

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Publications:

Wesson PD, Mirzazadeh A, McFarland W. **A Bayesian approach to synthesize estimates of the size of hidden populations: the Anchored Multiplier**. Int J Epidemiol. 2018 Oct 1; 47(5):1636-1644.

Wesson PD, McFarland W, Qin CC, Mirzazadeh A. **Software Application Profile: The Anchored Multiplier calculator-a Bayesian tool to synthesize population size estimates**. Int J Epidemiol. 2019 Dec 1; 48(6): 1744-1749.

Rasheed A, Sharifi H, Wesson P, Pashtoon SJ, Tavakoli F, Ghalekhani N, Haghdoost AA, Atarud A, Banehsi MR, Hamdard N, Sadaat SI, McFarland W, Mirzazadeh A. **Mapping and population size estimates of people who inject drugs in Afghanistan in 2019: Synthesis of multiple methods**. PLoS One. 2022 Jan 28; 17(1):e0262405.

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Triangulating Truth

A Flexible Statistical Tool for Reaching Consensus on Population Size, Prevalence and More

Ian Fellows, Carl Corcoran and Anne McIntyre

Key Population Surveillance Team Webinar Series
August 15, 2023

Motivation

- Population size, prevalence, incidence, etc… are critical metrics
- Experts are faced with multiple estimates of varying quality, and estimates are often incongruent with one-another
- Combining estimates is often done in informal, non-transparent ways
  - ‘Pick one’ from multiple estimates at a point in time from one city
  - Group of experts often pick one that makes ‘most sense’
- Synthesizing expert knowledge with empirical estimates can be tricky

Goal

Create a tool for stakeholders to use that:

1. Synthesizes multiple possibly contradictory estimates of a single population quantity.
2. Allows the incorporation of stakeholder knowledge about the population and estimates.
3. Is fully transparent with triangulations that are auditable by any third party.
4. Can be used to triangulate different types of population quantities (e.g. means, proportions, population sizes)
5. Is easy to use and intuitive.

Background

Inspiration

- Inspired by meta-analysis models

	Fixed Effects	**Random Effects**	**Quality Effects**
Weights	$w_j = \frac{v_j^{-1}}{\sum_{j=1}^{k} v_j^{-1}}$	$w'_j = \frac{(v_j + \tau^2)^{-1}}{\sum_{j=1}^{k}(v_j + \tau^2)^{-1}}$	$w''_j = \frac{\left(\frac{Q_j}{v_j} + \hat{\tau}_j\right)}{\sum_{j=1}^{k}\left(\frac{Q_j}{v_j} + \hat{\tau}_j\right)}$
Estimator	$\hat{\theta}_{FE} = \sum_{j=1}^{k} w_j \, \hat{\delta}_j$	$\hat{\theta}_{RE} = \sum_{j=1}^{k} w'_j \, \hat{\delta}_j$	$\hat{\theta}_{QE} = \sum_{j=1}^{k} w''_j \, \hat{\delta}_j$

Notation: $\hat{\delta}_j$ is the estimated effect size of jth study, v_j is the sampling error variance of jth study, τ^2 is the DerSimonian and Laird estimator of the between-study variance, Q_j is the jth study rank (scaled between 0 to 1), and $\hat{\tau}_j = f(Q_j, v_j, N)$ is a bias correction (where N is the number of studies).

Background

Broad Strokes

- Create a Bayesian model similar to the “quality effects” model
  - Scale the study error by a “confidence” score, similar to “quality” score
  - Incorporate between-study variance as a source of variation

Background

Broad Strokes

- **Bayesian hierarchical model**: statistically rigorous framework that allows for *input of user knowledge of population*
  - **Empirical inputs**: *all* estimates of population quantity (and their standard errors)
  - **Expert inputs**: prior knowledge of the population quantity, confidence in estimate quality

Begin Math

Model Structure

Data Model

Study estimate (known)

Study standard error (known)

$$y_i \sim N\left(\theta, \left(\frac{s_i}{c_i}\right)^2 + \tau^2\right)$$

Between-study variance

Population quantity of interest

User-supped confidence (between 0 and 1)

Priors

$\theta \sim N(\mu_0, \sigma_0^2)$ — Stakeholder knowledge about the population quantity

$\tau \sim HC(0.01 \cdot Var(y_i))$ — Prior for the unaccounted for between study variation. The prior is set to be very close to zero most of the time.

Detail: Theta Transformations

- It is often better to triangulate a transformed population quantity (e.g. triangulate **log(Population Size)** rather than **Population Size** directly) because
  - Invalid ranges are prevented (e.g. a proportion of -.05).
  - The normal approximation of the estimates may be more valid.
  - The functional form of the prior may be better.
- This is all handled behind the scenes and is transparent to the user.

Quantity	Transformation	Formula	Untransformed Range	Motivating Model
Mean or Other	None	$y = x$	$[-\infty, \infty]$	Normal
Proportion	Logit	$y = \log(\frac{x}{1-x})$	$[0,1]$	Logistic
Population Size	Log	$y = \log(x)$	$(0, \infty]$	Log-linear

End Math

Web Application

epiapps.com

Outlook | Datim Workbench | EDAV Workbench | Victory

EPIAPPS

THE TRIANGULATOR

The Triangulator is a Shiny user interface designed to help derive consensus estimates of a population quantity (e.g. a population size, a proportion, a mean, etc.) from multiple empirical estimates. Stakeholders may add additional information regarding the methodological quality of the studies and prior knowledge of the metric. Triangulated estimates are statistically defensible, reproducible and openly inspectable.

LAUNCH APPLICATION

Authors:	Ian E. Fellows and Carl Corcoran
Github:	https://github.com/fellstat/triangulator
Manual:	https://fellstat.github.io/triangulator/

Example 1 - PSE

The App

The Triangulator - A Consensus Estimate Calculator

Enter Estimates | Define Prior Beliefs | Synthesis

Estimate Type

Population size

	Name	Estimate	Lower	Upper	Design Confidence
1	Unique Object	679	525	1024	100
2	Event	162	126	249	100
3	Service 1	849	630	1369	100
4	Service 2	2766	1995	4622	100
5	Service 3	668	500	1082	100
6	SS-PSE	674	318	2426	100
7	Source 7				100
8	Source 8				100
9	Source 9				100
10	Source 10				100

More rows

Estimate : An estimate from a study

Lower: 95% confindence interval lower bound

Upper: 95% confindence interval upper bound

Design Confidence : Expert confidence in the design / implementation of the study. This scales the standard error such that a value of 50 will double the standard error.

Example 1 - PSE

The App

The Triangulator - A Consensus Estimate Calculator

Enter Estimates | Define Prior Beliefs | Synthesis

Median

800

75th Percentile

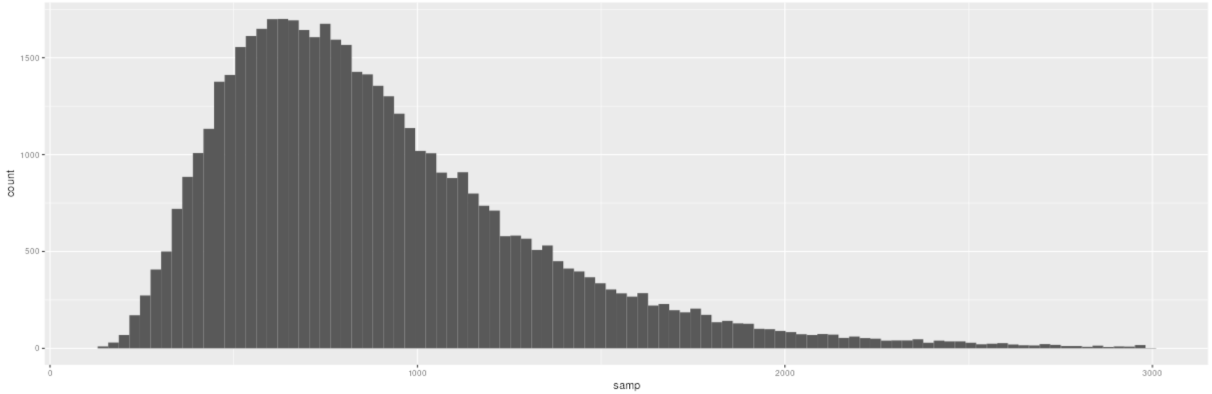
1097

Lower Bound

1

Upper Bound

3000



Summaries

Median	Mean	Standard.Deviation
801.34	890.95	427.37

Quantiles:

5%	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%
371.75	441.26	541.39	626.98	712.96	801.34	901.69	1024.43	1188.70	1459.24	1723.10

Example 1 - PSE

The App

The Triangulator - A Consensus Estimate Calculator

Enter Estimates | Define Prior Beliefs | Synthesis

Tau Multiplier

0.01

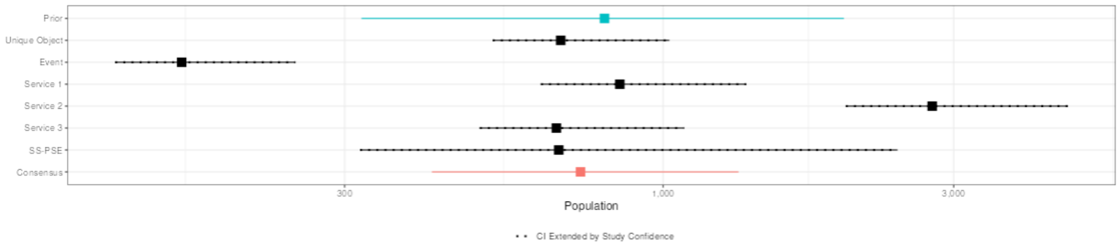
Run

Percent of Estimate Variability Attributable to Unaccounted-for Study Bias: 92%

Consensus Estimate:

Median	Mean	Standard Deviation	95% CI	90% CI
731.82	766.73	233.32	(417, 1330)	(459, 1199)

Posterior:



Posterior Quantiles:

5%	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%
458.93	510.85	582.82	634.97	686.21	731.82	783.24	850.12	926.00	1058.42	1198.68

Example 1 - PSE

The App

Estimate Type

Population size

	Name	Estimate	Lower	Upper	Design Confidence
1	Unique Object	679	525	1024	80
2	Event	162	126	249	5
3	Service 1	849	630	136	60
4	Service 2	2766	1995	462	60
5	Service 3	668	500	1082	60
6	SS-PSE	674	318	2426	70
7	Source 7				100
8	Source 8				100
9	Source 9				100
10	Source 10				100

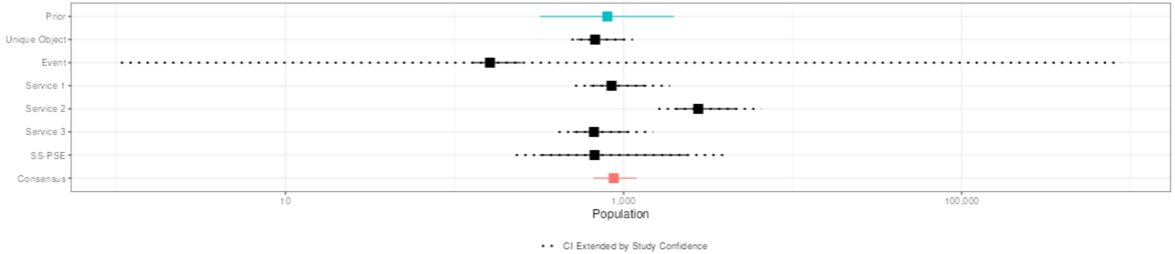
More rows

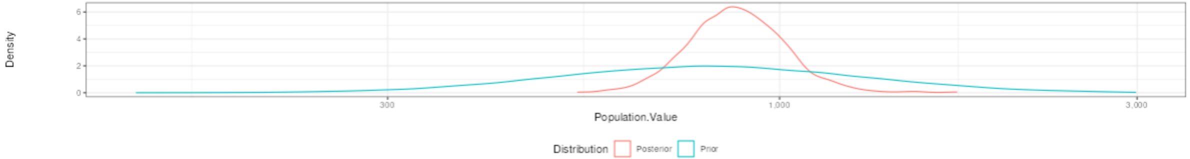
Percent of Estimate Variability Attributable to Unaccounted-for Study Bias: 1%

Consensus Estimate:

Median	Mean	Standard Deviation	95% CI	90% CI
874.23	887.58	138.14	(660, 1194)	(689, 1125)

Posterior:



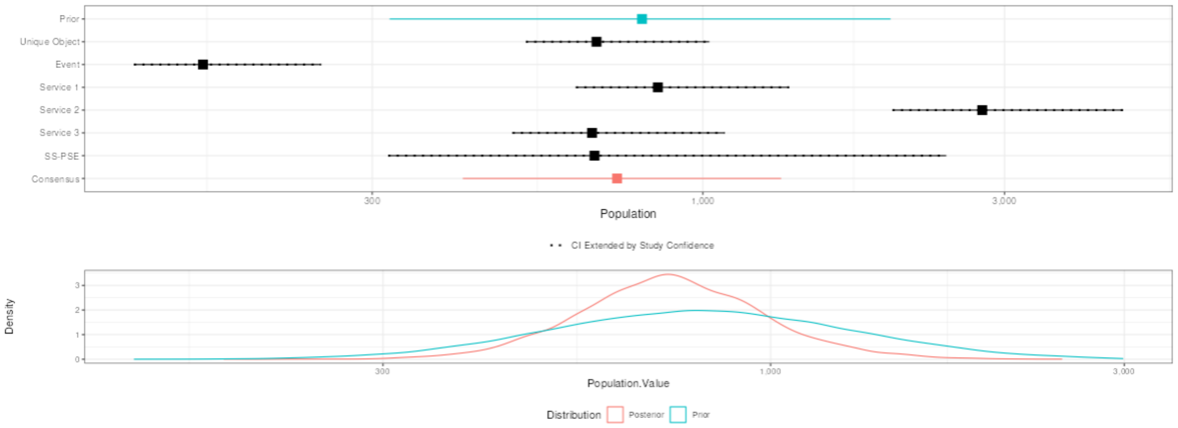


Posterior Quantiles:

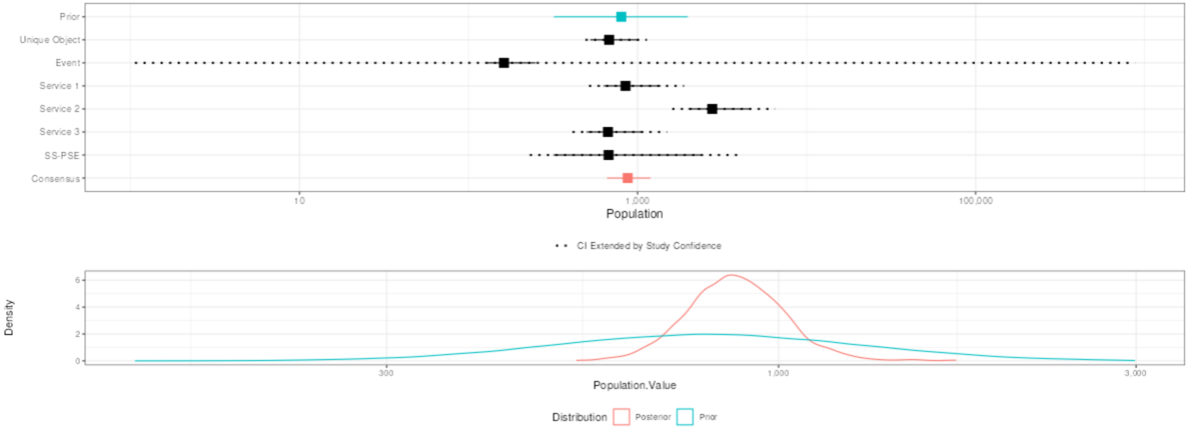
5%	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%
689.01	728.41	776.69	810.40	844.22	874.23	906.29	943.76	988.70	1051.50	1125.35

Example 1 - PSE

Unscaled:
732 (417, 1330)



Confidence Scaling:
847 (660, 1194)



Example 2 – Population Proportion

FSW in Mombasa, Kenya

The Triangulator - A Consensus Estimate Calculator

Enter Estimates | Define Prior Beliefs | Synthesis

Estimate Type

Proportion

	Name	Estimat	Lower	Upper	Desig Confide
1	Object Multiplier	0.0058	0.0054	0.0062	20
2	3S-CRC	0.006	0.0056	0.0066	40
3	Service Multiplier	0.0156	0.0151	0.016	10
4	SS-PSE	0.0107	0.0069	0.0124	25
5	Other	0.034	0.023	0.0628	10

More rows

Estimate : An estimate from a study

Lower: 95% confindence interval lower bound

Upper: 95% confindence interval upper bound

Design Confidence : Expert confidence in the design / implementation of the study. This scales the standard error such that a value of 50 will double the standard error.

Example 2 – Population Proportion

Prior 1: Informative

Enter Estimates | **Define Prior Beliefs** | Synthesis

Median

0.02

75th Percentile

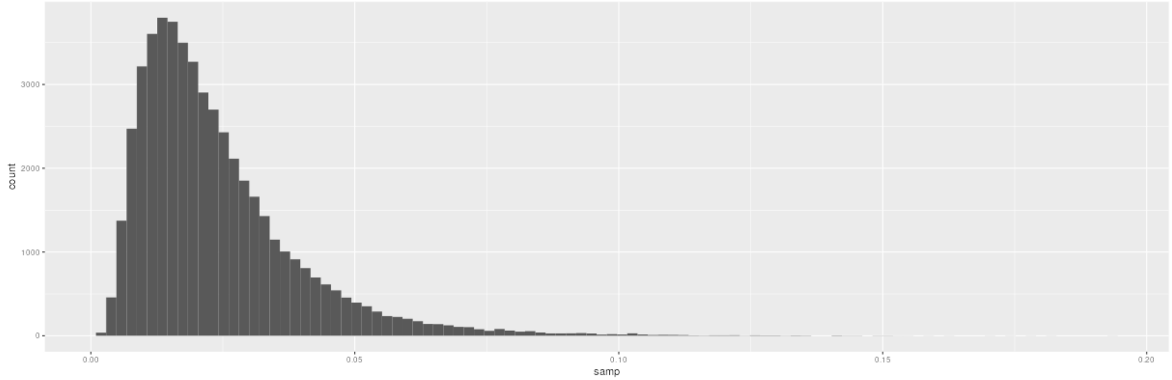
0.03

Lower Bound

0

Upper Bound

1



Summaries

Median	Mean	Standard.Deviation
0.02	0.02	0.02

Quantiles:

5%	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%
0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05

Example 2 – Population Proportion

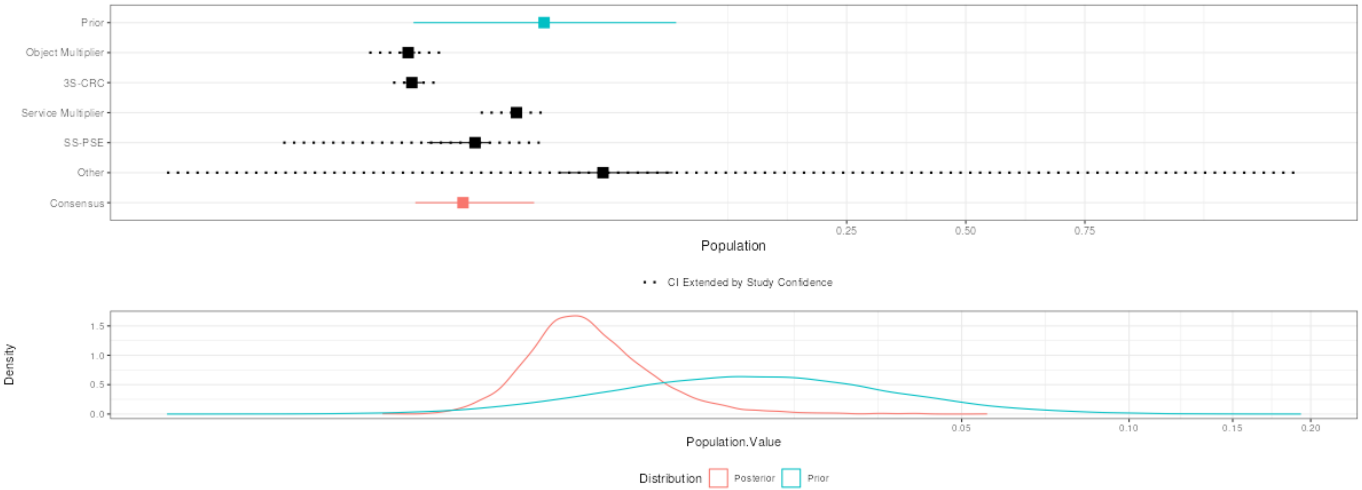
Prior 1: Informative

Percent of Estimate Variability Attributable to Unaccounted-for Study Bias: 26%

Consensus Estimate:

Median	Mean	Standard Deviation	95% CI	90% CI
0.01	0.01	0.00	(0.0062, 0.018)	(0.0067, 0.016)

Posterior:



Example 2 – Population Proportion

Prior 2: Less Informative

The Triangulator - A Consensus Estimate Calculator

Enter Estimates | Define Prior Beliefs | Synthesis

Median

0.5

75th Percentile

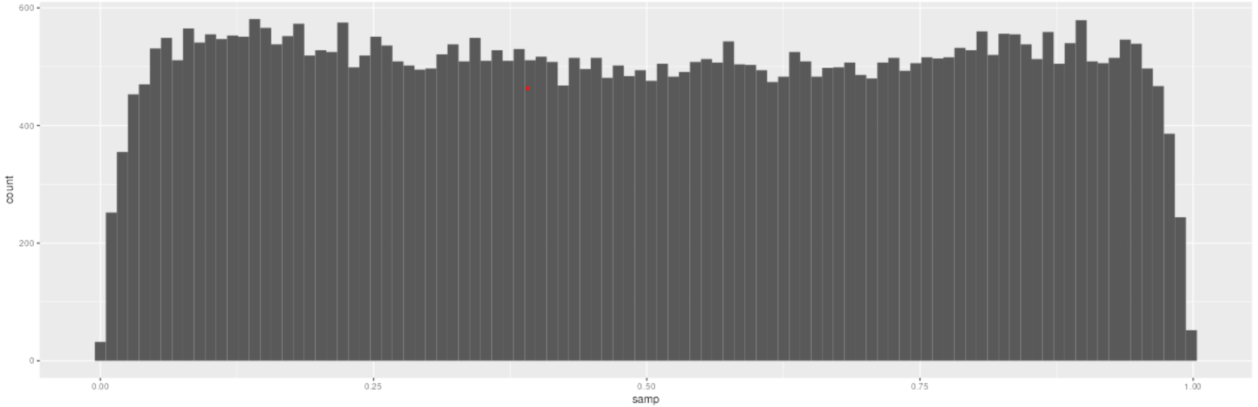
.75

Lower Bound

0

Upper Bound

1



Summaries

Median	Mean	Standard.Deviation
0.49	0.50	0.28

Quantiles:

5%	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%
0.06	0.11	0.20	0.30	0.39	0.49	0.59	0.70	0.80	0.89	0.94

Example 2 – Population Proportion

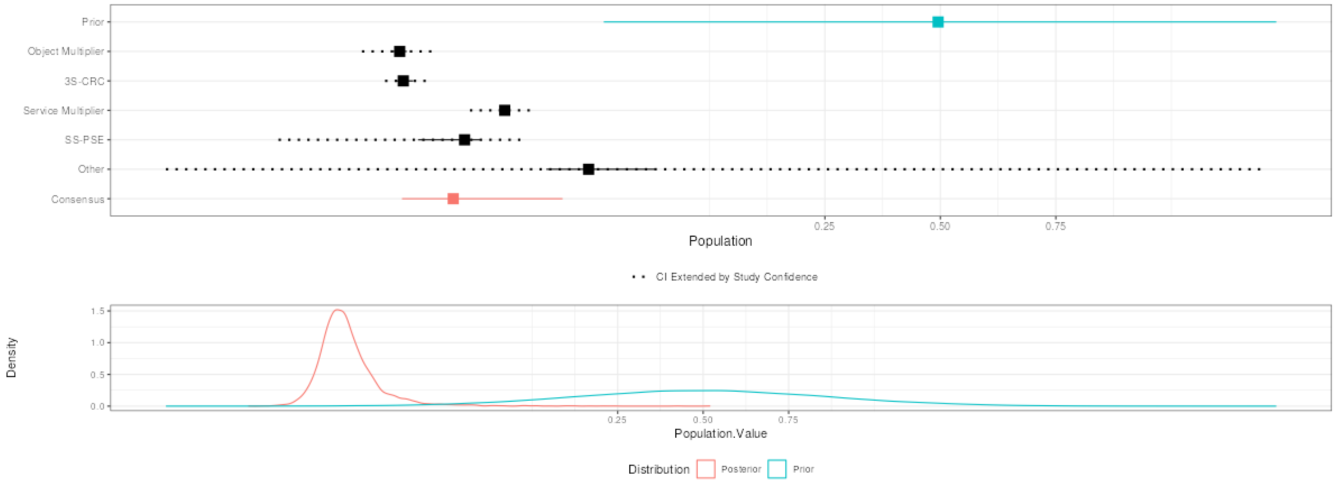
Prior 2: Less Informative

Percent of Estimate Variability Attributable to Unaccounted-for Study Bias: 24%

Consensus Estimate:

Median	Mean	Standard Deviation	95% CI	90% CI
0.01	0.01	0.01	(0.0059, 0.027)	(0.0065, 0.02)

Posterior:



Example 2 – Population Proportion

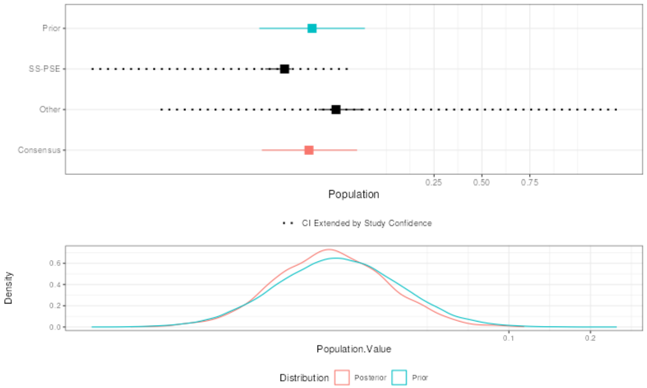
When does the prior have a big impact?

Percent of Estimate Variability Attributable to Unaccounted-for Study Bias: 0%

Consensus Estimate:

Median	Mean	Standard Deviation	95% CI	90% CI
0.02	0.02	0.01	(0.0064, 0.054)	(0.0076, 0.046)

Posterior:



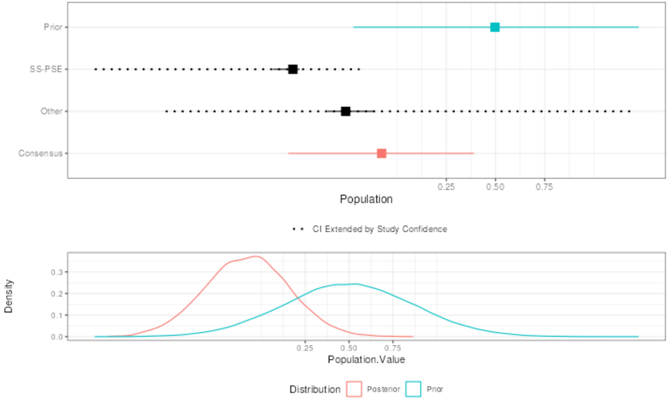
Informative Prior

Percent of Estimate Variability Attributable to Unaccounted-for Study Bias: 0%

Consensus Estimate:

Median	Mean	Standard Deviation	95% CI	90% CI
0.07	0.10	0.10	(0.0096, 0.38)	(0.014, 0.3)

Posterior:



Less Informative Prior

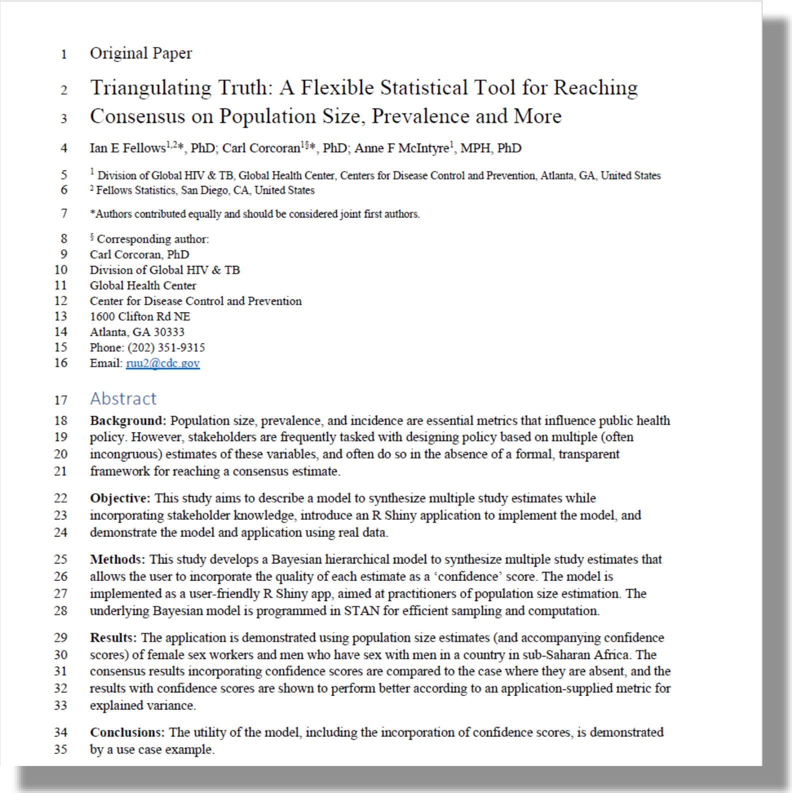
When uncertainty is high (and/or confidence is low)

Benefits and Limitations

- Statistically rigorous framework
- Combine empirical evidence and expert knowledge
- Ease of use
  - https://epiapps.com/shiny/app_direct/shinyproxy_combine_estimates/
  - https://github.com/fellstat/triangulator + manual

- Requires confidence bounds on estimates
  - Triangulator manual on Github has techniques to reconstruct intervals in some cases

Acknowledgements



Original Paper

Triangulating Truth: A Flexible Statistical Tool for Reaching Consensus on Population Size, Prevalence and More

Ian E Fellows[1,2]*, PhD; Carl Corcoran[1§]*, PhD; Anne F McIntyre[1], MPH, PhD

[1] Division of Global HIV & TB, Global Health Center, Centers for Disease Control and Prevention, Atlanta, GA, United States
[2] Fellows Statistics, San Diego, CA, United States

*Authors contributed equally and should be considered joint first authors.

[§] Corresponding author:
Carl Corcoran, PhD
Division of Global HIV & TB
Global Health Center
Center for Disease Control and Prevention
1600 Clifton Rd NE
Atlanta, GA 30333
Phone: (202) 351-9315
Email: ruu2@cdc.gov

Abstract

Background: Population size, prevalence, and incidence are essential metrics that influence public health policy. However, stakeholders are frequently tasked with designing policy based on multiple (often incongruous) estimates of these variables, and often do so in the absence of a formal, transparent framework for reaching a consensus estimate.

Objective: This study aims to describe a model to synthesize multiple study estimates while incorporating stakeholder knowledge, introduce an R Shiny application to implement the model, and demonstrate the model and application using real data.

Methods: This study develops a Bayesian hierarchical model to synthesize multiple study estimates that allows the user to incorporate the quality of each estimate as a 'confidence' score. The model is implemented as a user-friendly R Shiny app, aimed at practitioners of population size estimation. The underlying Bayesian model is programmed in STAN for efficient sampling and computation.

Results: The application is demonstrated using population size estimates (and accompanying confidence scores) of female sex workers and men who have sex with men in a country in sub-Saharan Africa. The consensus results incorporating confidence scores are compared to the case where they are absent, and the results with confidence scores are shown to perform better according to an application-supplied metric for explained variance.

Conclusions: The utility of the model, including the incorporation of confidence scores, is demonstrated by a use case example.

Special thanks to Anne McIntyre, Ray Shiraishi and Oli Stevens

The Triangulator

Explained Variance

- Bayesian Measures of Explained Variance and Pooling in Multilevel (Hierarchical) Models (Gelman and Pardoe, 2006)
- Bayesian hierarchical model equivalent of R^2

$$R^2 = 1 - \frac{E\left(\mathrm{V}_{j=1}^{N}\, y_j - \nu_j\right)}{E\left(\mathrm{V}_{j=1}^{N}\, y_j\right)}$$

- Fraction of estimate variability due to unaccounted-for study bias

Explained Variance

$$R^2 = 1 - \frac{E\left(\mathrm{V}_{j=1}^{N} y_j - \nu_j\right)}{E\left(\mathrm{V}_{j=1}^{N} y_j\right)}$$

$$R^2 = \frac{E\left(\mathrm{V}_{j=1}^{N} \hat{\nu}_j + \frac{1}{N}\Sigma_{j=1}^{N} V_j\right)}{E\left(\mathrm{V}_{j=1}^{N} y_j\right)},$$

$$\hat{\nu}_j = \frac{\tau^2}{\sigma_j^2+\tau^2} y_j + \frac{\sigma_j^2}{\sigma_j^2+\tau^2}\theta \text{ and } V_j = \frac{\sigma_j^2\tau^2}{\sigma_j^2+\tau^2}.$$