Game Theory

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Contents

Chapter 1

First Chapter

1.1 First Section

DEFINITION 1.1 (Winning Position)**.** Consider a two-player game. We say that a player has a **winning position** if and only if optimal play by that player guarantees a win.

DEFINITION 1.2 (Losing Position)**.** Consider a two-player game. We say that a player has a **losing position** if and only if an optimal move by their opponent guarantees a loss.

PROPOSITION 1.3.

- From a winning position (player to move), there exists a move that leads to a losing position for the other player.
- From a losing position (player to move), every move leads to a winning position for the other player.

1.2 Groups of Games

DEFINITION 1.4 (Equivalent Games)**.** Let G and H be two impartial games. We say that G and H are **equivalent** if and only if for all impartial games J, $G + J$ is a losing position if and only if $H + J$ is a losing position.

- for all impartial games J, $G + J$ is a losing position if and only if $H + J$ is a losing position.
- for all impartial games J, $G + J$ is a winning position if and only if $H + J$ is a winning position.

PROPOSITION 1.5. Game equivalence is an equivalence relation. That is, "$\equiv$" is:

- Reflexive: $\forall G$, we have $G \equiv G$.
- Symmetric: $\forall G, H$, we have $G \equiv H \iff H \equiv G$.
- Transitive: $\forall G, H, K$, we have $((G \equiv H) \wedge (H \equiv K)) \implies G \equiv K$.

PROPOSITION 1.6. $\forall G, H, J$, we have $G \equiv H \implies G + J \equiv H + J$.

PROPOSITION 1.7. $G \equiv H$ implies that G and H are both winning or both losing.

LEMMA 1.8. G is a losing position if and only if $G \equiv *0$.

Proof. **Backward Direction**: Suppose that $G \equiv *0$. Then $\forall J$, $G + J$ is a losing position if and only if $*0 + J$ is a losing position. In particular, take $J := *0$, then $G + *0$ is a losing position if and only if $*0 + *0$ is a losing position. Notice $G + *0 = *0$ and $*0 + *0 = *0$. So G is a losing position if and only if $*0$ is a losing position. We know that $*0$ is indeed a losing position. So G is a losing position.

Forward Direction: Suppose that G is a losing position. I will show that $G \equiv *0$. Let J be an arbitrary impartial game. Notice $*0 + J = J$. So there remains to show that $G + J$ is losing if and only if J is losing.

Suppose that $G+J$ is a losing position. I will show that J is a losing position. Assume for the sake of contradiction that J is not losing. Then J is winning. Let $J \to J'$ be a move such that J' is losing. Since G is losing and J' is losing, we get $G+J'$ is losing. So $G+J$ is winning. However, this contradicts to the assumption that $G+J$ is losing. So J is losing.

Suppose that J is a losing position. I will show that $G+J$ is a losing position. Double strong well-founded induction.

□

G is winning and J is losing, then $G+J$ is winning???

DEFINITION 1.9 (Group of Game)**.** Let $\mathcal{G}$ be a set of games. Let $* : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be a binary operation on $\mathcal{G}$. We say that $(\mathcal{G}, *)$ is a **group** if and only if the following conditions hold:

1. Associativity: $\forall G_1, G_2, G_3 \in \mathcal{G}$, $(G_1 * G_2) * G_3 \equiv G_1 * (G_2 * G_3)$.
2. Identity: $\exists I \in \mathcal{G}$ such that $\forall G \in \mathcal{G}$, $G * I \equiv I * G \equiv G$.
3. Inverse: $\forall G \in \mathcal{G}$, $\exists H \in \mathcal{G}$ such that $G * H \equiv H * G \equiv I$.

LEMMA 1.10. *$G \equiv H$ if and only if $G + H \equiv *0$.*

Proof. **Forward Direction**: Suppose that $G \equiv H$. I will show that $G + H \equiv *0$. Since $G \equiv H$, we get

$$G + H \equiv H + H, \text{ by the}$$

□

LEMMA 1.11. *Let G and H be impartial combinatorial games. Suppose that*

- *For each option G' of G, there exists an option of H which is equivalent to G'.*
- *For each option H' of H, there exists an option of G which is equivalent to H'.*

Then $G \equiv H$.

Proof. Since $G' + H' \equiv *0$, we get $G + H \equiv *0$. □

THEOREM 1.12 (Sum of NIM Heaps)**.** Suppose $n_1, ..., n_k \in \mathbb{Z}_{++}$ are distinct powers of 2. Then we have

$$*(n_1 + ... + n_k) \equiv (*n_1 + ... + *n_k).$$

Proof. **Base Case**: $n = 0$.

Inductive Step: Suppose the theorem holds for all positive integers less than n. Write n as $n = 2^{a_1} + ... + 2^{a_k}$ where $a_1 > ... > a_k$. Define

$$q := n - 2^{a_1} = 2^{a_2} + ... + 2^{a_k}.$$

Note that $q < 2^{a_1} < n$. Apply the induction hypothesis on q, we get

$$*q \equiv *2^{a_1} + ... + *2^{a_k}$$

Now there remains to show that $*n \equiv *2^{a_1} + *q$. Consider the options of $*n$: $\{*(n-1), *(n-2), ..., *0\}$ and the options of $*2^{a_1} + *q$: $\{G + *q, *2^{a_1} + H\}$ where G is some option of $*2^{a_1}$ and H is some option of $*q$.

Consider the set $\{*i + *q : 0 \leq i < 2^{a_1}\}$ of options of $*2^{a_1} + *q$.

Consider the set $\{*2^{a_1} + *i : 0 \leq i < q\}$ of options of $*2^{a_1} + *q$. Write i as $i = 2^{b_1} + 2^{b_2} + ...$. Notice $2^{a_1} + i < 2^{a_1} + q < n$. So by the inductive hypothesis, we get

$$*(2^{a_1} + i) = *(2^{a_1} + 2^{b_1} + 2^{b_2} + ...) = *2^{a_1} + *2^{b_1} + *2^{b_2} + ...$$

So the set of options of $*n$ is equivalent to the set of options for $*2^{a_1} + *q$. So $*n \equiv *2^{a_1} + *2^{a_2} + ...$. □

EXAMPLE 1.13.

$$\begin{aligned}(5, 9, 8) &= *5 + *9 + *8 = *(4 + 1) + *(8 + 1) + *8 \\ &= *4 + *1 + *8 + *1 + *8 = *4.\end{aligned}$$

So the optimal move is to take away the $*4$: $(5, 9, 8) \to (1, 9, 8)$.

DEFINITION 1.14 (Balance, Unbalanced)**.** We say that a NIM position $(a_1, ..., a_q)$ is **balanced** if and only if $a_1 \oplus ... \oplus a_q = 0$. We say that is it **unbalanced** otherwise.

THEOREM 1.15. A NIM position $(a_1, ..., a_q)$ is a losing (winning) position if and only if it is balanced (unbalanced).

DEFINITION 1.16 (Minimum Excludant)**.** Given a subset $S \subsetneq \mathbb{N}$, we define $\text{mex}(S)$ to be the smallest element of $\mathbb{N} \setminus S$.

THEOREM 1.17 (MEX Rule)**.** Let $S \subsetneq \mathbb{N}$. Let G be an impartial game whose options are equivalent to $\{*s : s \in S\}$. Then $G \equiv *(\text{mex}(S))$.

Proof. Let $m := \text{mex}(S)$. By the Generalized Copycat principle, it suffices to show that $G + *m \equiv *0$.

Consider an option of the form $G + *m'$ for some $m' < m$. Since $m = \text{mex}(S)$ and $m' < m$, we have $m' \in S$. Then there exists an option G' of G such that $G' \equiv *m'$. The other player can move to $G' + *m'$. Since $G' \equiv *m'$, the game $G' + *m'$ is a losing position (copycat principle). So $G + *m'$ is winning.

Consider an option of the form $G' + *m$ of $G + *m$. Recall that the options of G are $\{*n : n \in S\}$. Let $k \in S$ be a natural number such that $G' \equiv *k$. Then $G' + *m \equiv *k + *m$. Since $m \notin S$ and $k \in S$, $*k + *m$ is winning. So $G' + *m$ is winning.

Hence all options of $G + *m$ are winning. So $G + *m$ is losing. So $G \equiv *m$. □

COROLLARY 1.18. For every impartial game G, there exists a natural number $n \in \mathbb{N}$ such that $G \equiv *n$.

Proof. We use (well-founded) induction on G.

Base case: If G has no options, then $G \equiv *0$.

Inductive step: Suppose the set of options for G is finite and are $G^1, ..., G^q$. By the induction hypothesis, $\forall i \in \{1, ..., q\}$, we have $G^i \equiv *n_i$ for some $n_i \in \mathbb{N}$. So the set of options of G are equivalent to $\{*n_1, ..., *n_q\}$. Apply the MEX rule with $S := \{n_1, ..., n_q\}$, we have

$$G \equiv *(\text{mex}(S)) = *(\text{mex}(\{n_1, ..., n_q\})).$$

□

Chapter 2

Strategic Games

2.1 Pure Strategies

DEFINITION 2.1 (Extensive Games)**.** Games with game trees are called **extensive games with perfect information**.

DEFINITION 2.2 (Strategy)**.** A **strategy** (for a player) specifies a move for every decision node for that player. i.e., a function that maps each decision node to a move.

DEFINITION 2.3 (Strategy Profile)**.** A **strategy profile** specifies a strategy for every player. We represent a strategy (profile) by concatenating moves.

DEFINITION 2.4 (Strategic Form)**.** The **strategic form** of a game consists of:

- A set $N = \{1, ..., n\}$ of players;
- A set S_i of strategies for $i \in N$;
- A utility function $u_i : S_1 \times ... \times S_n \to \mathbb{R}$, for each $i \in N$.

A strategic form is a $|S_1| \times ... \times |S_n| \times N$ dimensional tensor.

2.2 Nash Equilibrium of Pure Strategies

DEFINITION 2.5 (Nash Equilibrium)**.** Let $N := \{1, ..., n\}$ denote the set of players. Let $\mathcal{S}_i$ denote the set of strategies for player i, for $i \in N$. Let $\mathcal{S} := \mathcal{S}_1 \times ... \times \mathcal{S}_n$. We say that a strategy profile $s^* = (s_1, ..., s_n) \in \mathcal{S}$ is a **Nash equilibrium** if and only if $\forall i \in N$, $\forall s_i' \in \mathcal{S}_i$, we have

$$u_i(s_1, ..., s_i', ..., s_n) \leq u_i(s^*).$$

That is, no one player can improve over their utility in s^* by unilaterally deviating in their strategy.

EXAMPLE 2.6 (Prisoner's Dilemma)**.** The Prisoner's dilemma consists of two players, each with strategies Q and C, with payoffs:

	Q	C
Q	$(2, 2)$	$(0, 3)$
C	$(3, 0)$	$(1, 1)$

- (C, C) is the only Nash equilibrium.
- (C, C) is suboptimal overall.

EXAMPLE 2.7 (Bach-Stravinsky)**.**

	Bach	Stravinsky
Bach	$(2, 1)$	$(0, 0)$
Stravinsky	$(0, 0)$	$(1, 2)$

- (B, B) and (S, S) are both Nash equilibria.

EXAMPLE 2.8 (Matching Pennies).

	Heads	Tails
Heads	$(+1, -1)$	$(-1, +1)$
Tails	$(-1, +1)$	$(+1, -1)$

- Player 1 bets on match; player 2 bets on a mismatch.

- Example of a zero-sum game.
- This game has no Nash equilibrium.
- Later in the course we will see that has a mixed Nash equilibrium.

EXAMPLE 2.9. numbers to be fixed

	L	R
T	$(2,1)$	$(0,0)$
M	$(0,0)$	$(1,2)$
B	$(0,0)$	$(1,2)$

Would player 1 ever choose T?

- No, because M is always better than T.
- In this case, T is strictly dominated by M.

2.3 Domination for Pure Strategies

DEFINITION 2.10 (Strictly Dominate)**.** Let $i \in N$. Let $s_i, s_i' \in \mathcal{S}_i$ be two strategies. Let $\mathcal{S}_{-i} := \bigoplus_{j \neq i} \mathcal{S}_j$. We say that s_i **strictly dominates** s_i' if and only if

$$\forall s_{-i} \in \mathcal{S}_{-i}, \quad u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$

DEFINITION 2.11 (Weakly Dominate)**.** Let $i \in N$. Let $s_i, s_i' \in \mathcal{S}_i$ be two strategies. Let $\mathcal{S}_{-i} := \bigoplus_{j \neq i} \mathcal{S}_j$. We say that s_i **weakly dominates** s_i' if and only if

$$\forall s_{-i} \in \mathcal{S}_{-i}, \quad u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}),$$

and $\exists \bar{s}_{-i}^* \in \mathcal{S}_{-i}$ for which the inequality holds strictly.

DEFINITION 2.12 (Best Response Function)**.** We define the **best response func-**

tion for Player i to be a function $B_i : \bigoplus_{j \neq i} \mathcal{S}_j \to \mathcal{P}(\mathcal{S}_i)$ given by

$$\begin{aligned} B_i(s_{-i}) &:= \{s_i \in \mathcal{S}_i : \forall s_i' \in \mathcal{S}_i, u_i(s_i', s_{-i}) \leq u_i(s_i, s_{-i})\} \\ &= \underset{s_i' \in \mathcal{S}_i}{\operatorname{argmax}} \{u_i(s_i', s_{-i})\}. \end{aligned}$$

In other words, $B_i(s_{-i})$ is the set consisting of all strategies of Player i that yield the maximum payoff against (s_{-i}).

PROPOSITION 2.13. A strategy profile $s^* = (s_1, ..., s_n) \in \mathcal{S}$ is a Nash equilibrium if and only if

$$\forall i \in N, \quad s_i \in B_i(s_{-i}).$$

2.4 Mixed Strategies

DEFINITION 2.14 (Mixed Strategy)**.** Let $\mathcal{S}_i$ denote the set of strategies for player i. We define a **mixed strategy** $x^{(i)}$ over $\mathcal{S}_i$ to be a probability distribution over $\mathcal{S}_i$. That is, $x^{(i)} \in \mathbb{R}^{\mathcal{S}_i}$ is such that $x^{(i)} \geq \mathbb{0}$ and $\mathbb{1}^\top x^{(i)} = 1$.

DEFINITION 2.15 (Mixed Strategy Profile)**.** We define a **mixed strategy profile** to be a vector $\vec{x} = (x^{(1)}, ..., x^{(n)}) \in \mathbb{R}^{\mathcal{S}_1} \times ... \times \mathbb{R}^{\mathcal{S}_n}$ specifying a mixed strategy $x^{(i)} \in \mathbb{R}^{\mathcal{S}_i}$ for each player $i \in N$.

DEFINITION 2.16 (Expected Utility)**.** Let $\vec{x} = (x^{(1)}, ..., x^{(n)})$ denote a mixed strategy profile. We define the **expected utility** of player i in $\vec{x}$, denoted by $u_i(\vec{x})$, to be a number given by

$$u_i(\vec{x}) := \sum_{\vec{s} \in \mathcal{S}} \left[u_i(\vec{s}) \prod_{j \in \{1, \dots, n\}} x_{s_j}^{(j)} \right] = \sum_{s_i \in \mathcal{S}_i} x_{s_i}^{(i)} \sum_{\vec{s} \in \mathcal{S}, \vec{s}_i = s_i} \left[u_i(\vec{s}) \prod_{j \neq i} x_{s_j}^{(j)} \right].$$

We define the **expected utility of strategy** s_i in $\vec{x}$ to be

$$u_i(s_i, \vec{x}) := \sum_{\vec{s} \in \mathcal{S}, \vec{s}_i = s_i} \left[u_i(\vec{s}) \prod_{j \neq i} x_{s_j}^{(j)} \right].$$

2.5 Nash Equilibrium of Mixed Strategies

DEFINITION 2.17 (Mixed Nash Equilibrium)**.** Let $\bar{x} = (\bar{x}^{(1)}, ..., \bar{x}^{(n)})$ be a mixed strategy. We say that $\bar{x}$ is a **mixed Nash equilibrium** if and ony if $\forall i \in \{1, ..., n\}$, for any mixed strategy $x^{(i)}$ over $\mathcal{S}_i$, we have

$$u_i(\bar{x}) \geq u_i(\bar{x}^1, ..., x^i, ..., \bar{x}^n).$$

DEFINITION 2.18 (Best Response)**.** Given a profile $\bar{x}^{-i} = (\bar{x}^1, ..., \bar{x}^{i-1}, \bar{x}^{i+1}, ..., \bar{x}^n)$ of mixed strategies of players in $N \setminus \{i\}$, the best response for $\bar{x}^{-i}$ is the set $B_i(\bar{x}^{-i})$ of all mixed strategies x^i of player i that maximize the expected utility

$$u.$$

PROPOSITION 2.19. Best response functions are continuous.

THEOREM 2.20. A strategy profile is a mixed Nash equilibrium if and only if it lies on both player's best-response graphs.

Optimization problems:

$$\begin{array}{llc} \text{(P)} & \max & \sum_{s \in \mathcal{S}_i} \bar{x}_s^i \cdot u_i(s, \bar{x}^{-i}) \\ & \text{subject to:} & \sum_{s \in \mathcal{S}_i} \bar{x}_s^i = 1, \\ & & \bar{x}^i \geq 0. \end{array}$$

$$\begin{array}{lll} \text{(D)} & \min & y \\ & \text{subject to:} & y \geq u_i(s, \bar{x}^{-i}), \forall s \in \mathcal{S}_i. \end{array}$$

Conversely, we prove that every mixed strategy that chooses from among locally optimal strategies is an optimal strategy...

THEOREM 2.21 (Support Characterization)**.** Given mixed strategies $\bar{x}^{-i}$ of player in $N \setminus \{i\}$, a mixed strategy $\bar{x}^i$ is in $B_i(\bar{x}^{-i})$ if and only if $\bar{x}^i_s > 0$ implies that $s \in \mathcal{S}_i$ is a strategy of maximum expected payoff (against $\bar{x}^{-i}$).

COROLLARY 2.22. The set $B_i(\bar{x}^{-i})$ is a polyhedron.

Proof. Let $S' \subseteq S$ be the subset consisting of pure strategies s that maximize $u_i(s, \bar{x}^{-1})$. Then

$$B_i(\bar{x}^{-i}) = \{x^i : \mathrm{supp}(x^i) \subseteq S' \text{ and } \sum_{s \in S'} x^i_s = 1\}.$$

□

2.6 Domination for Mixed Strategies

DEFINITION 2.23 (Strictly Dominate)**.** A strategy $s_i \in \mathcal{S}_i$ **strictly dominates** strategy $s'_i \in \mathcal{S}_i$ if and only if

$$\forall j \neq i, \forall s_j \in \mathcal{S}_j, \quad u_i(s_1, ..., s_i, ..., s_n) > u_i(s_1, ..., s'_i, ..., s_n).$$

DEFINITION 2.24. Let x^i be a mixed strategy over $\mathcal{S}_i$. Let $s_i \in \mathcal{S}_i$ be a pure strategy. We say that x^i **strictly dominates** s_i if and only if

$$\forall s_{-i} \in \mathcal{S}_{-i}, \quad u_i(x^i, s_{-i}) > u_i(s_i, s_{-i}).$$

THEOREM 2.25. Let $\bar{x} \in \bigoplus_{i=1}^{n} \mathbb{R}_+^{S_i}$ be a mixed Nash equilibrium. Let $s \in \mathcal{S}_i$ be a pure strategy. Suppose that there exists a mixed strategy $x^i \in \mathbb{R}_+^{S_i}$ over $\mathcal{S}_i$ that strictly dominates s, then $\bar{x}^i_s = 0$.

Proof. Assume for the sake of contradiction that $\bar{x}^i_s > 0$.

□

DEFINITION 2.26 (Zero-Sum Game)**.** We say that a game is a **zero-sum game** if and only if

$$\forall s \in \mathcal{S}, \quad \sum_{i=1}^{n} u_i(s) = 0.$$

Player 1's linear program:

$$\begin{array}{lll} (P_1) & \max & \nu_r \\ & \text{subject to:} & x^{(1)\top} A_{\cdot,j} \geq \nu_r, \quad \forall j \in S_2, \\ & & 1^\top x^{(1)} = 1, x^{(1)} \geq 0. \end{array}$$

Player 2's linear program:

$$\begin{array}{lll} (P_2) & \min & \nu_c \\ & \text{subject to:} & A_{i,\cdot} x^{(2)} \leq \nu_c, \quad \forall i \in S_1, \\ & & 1^\top x^{(2)} = 1, x^{(2)} \geq 0. \end{array}$$

They are duals of each other, both feasible and bounded.

These are equivalent to the following programs:

$$\begin{array}{lll} (P_1') & \max & (0^\top_{|S_1|}, 1) \begin{pmatrix} x^{(1)} \\ \nu_r \end{pmatrix} \\ & \text{subject to} & \begin{pmatrix} A^\top & -1_{|S_2|} \\ 1^\top_{|S_1|} & 0 \\ -1^\top_{|S_1|} & 0 \end{pmatrix} \begin{pmatrix} x^{(1)} \\ \nu_r \end{pmatrix} \geq \begin{pmatrix} 0_{|S_2|} \\ 1 \\ -1 \end{pmatrix}, \\ & & x^{(1)} \geq 0_{|S_1|}. \end{array}$$

$$\begin{array}{lll} (P_2') & \min & (0^\top_{|S_2|}, 1) \begin{pmatrix} x^{(2)} \\ \nu_c \end{pmatrix} \\ & \text{subject to:} & \begin{pmatrix} A & 1_{|S_1|} \\ -1^\top_{|S_2|} & 0 \\ 1^\top_{|S_2|} & 0 \end{pmatrix} \begin{pmatrix} x^{(2)} \\ \nu_c \end{pmatrix} \geq \begin{pmatrix} 0_{|S_1|} \\ 1 \\ -1 \end{pmatrix} \\ & & x^{(2)} \leq 0_{|S_2|}. \end{array}$$

THEOREM 2.27. Every finite strategic game has a mixed Nash equilibrium.

Proof. Let $x \in \prod_{i \in N} \Delta_{|\mathcal{S}_i|}$ be a mixed strategy profile. Define for each $i \in \{1, ..., N\}$ and each $s_i \in \mathbb{S}_i$ a function $\Phi^{(i)}_{s_i} : \prod_{i \in N} \Delta_{|\mathcal{S}_i|} \to \mathbb{R}_+$ by

$$\Phi^{(i)}_{s_i}(x) := \max(0, u_i(s_i, x^{-1}) - u_i(x)).$$

Then

- $\Phi^{(i)}_{s_i}(x)$ is positive only if the pure strategy $s_i \in \mathcal{S}_i$ yields higher expected payoff than the mixed strategy $x^{(i)}$;
- By the Support Characterization theorem, $\Phi^{(i)}_{s_i}(x) = 0$ for all $s_i \in \mathbb{S}_i$ if and only if $x^{(i)}$ is a best response to x^{-i};
- $\Phi^{(i)}_{s_i}$ is not necessarily differentiable, but it is continuous.

Define a function $f : \prod_{i \in N} \Delta_{|\mathcal{S}_i|} \to \prod_{i \in N} \Delta_{|\mathcal{S}_i|}$ by $f(x) := \bar{x}$ where $\bar{x}$ is given by:

$$\bar{x}^{(i)}_{s_i} := .$$

Then

-

Let $i \in \{1, ..., n\}$ be arbitrary. Let $s_i \in \mathcal{S}_i$ such that $\hat{x}^{(i)}_{s_i} > 0$ and $u_i(s_i, \hat{x}^{-1}) \leq u_i(\hat{x})$. Then $\Phi^{(i)}_{s_i}(\hat{x}) = 0$ and

$$\hat{x}^{(i)}_{s_i} = (f(\hat{x}))^{(i)}_{s_i} = \frac{\hat{x}^{(i)}_{s_i} + 0}{1 + \sum_{s \in \mathcal{S}_i} \Phi^{(i)}_s(\hat{x})}.$$

So $\forall s \in \mathcal{S}_i$, we have $\Phi^{(i)}_s(\hat{x}) = 0$. So $\forall i \in \{1, ..., n\}$, $\hat{x}^{(i)}$ is a best response to $\hat{x}^{-i}$. So $\hat{x}$ is a Nash equilibrium. □

THEOREM 2.28 (Daskalakis, Goldberg, Papadimitriou (2008))**.** NASH is polynomial parity argument for directed graphs (PPAD)-complete.

REMARK 2.29. NASH, BROUWER, and BORSUK-ULAM are PPAD-complete.

REMARK 2.30. The following problems are NP-complete:

- Find a Nash equilibrium maximizing total utility.
- Find two Nash equilibria (or determine that only one exists).

...

Chapter 3

Lemke-Homson Algorithm

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the strategies for player 1 and player 2, respectively. Let $A, B \in \mathbb{R}^{\mathcal{S}_1 \times \mathcal{S}_2}$ denote the payoff matrices for player 1 and player 2, respectively. Consider the following system

$$
\begin{array}{lll}
(P) & \min & 0 \\
& \text{subject to:} & \mathbb{1}^\top x^{(i)} = 1, \quad \forall i \in \{1,2\}, \\
& & Ax^{(2)} \leq \mathbb{1} v_1, \\
& & B^\top x^{(1)} \leq \mathbb{1} v_2, \\
& & \sum_{i \in \mathcal{S}_i} x_i^{(1)} (v_1 - A_{i\cdot} x^{(2)}) = 0, \\
& & \sum_{j \in \mathcal{S}_j} x_j^{(2)} (v_2 - B_{\cdot j}^\top x^{(1)}) = 0, \\
& & x^{(1)} \in \mathbb{R}^{\mathcal{S}_1}, x^{(2)} \in \mathbb{R}^{\mathcal{S}_2}, v_1, v_2 \in \mathbb{R}.
\end{array}
$$

Note that this is a feasibility problem.

CLAIM 3.1. A non-negative solution to this system is a mixed Nash equilibrium.

Proof. By the Support Characterization theorem, $x^{(1)}$ and $x^{(2)}$ are best responses to each other. □

DEFINITION 3.2 (Lemke-Homson Algorithm)**.** Define $\bar{x}^{(1)} := x^{(1)}/v_2 \in \mathbb{R}^{\mathcal{S}_1}$ and $\bar{x}^{(2)} := x^{(2)}/v_1 \in \mathbb{R}^{\mathcal{S}_2}$. Add slack variables $\gamma^{(1)} \in \mathbb{R}^{\mathcal{S}_1}$ and $\gamma^{(2)} \in \mathbb{R}^{\mathcal{S}_2}$. Then we get the **Lemke-Homson system**:

$$
\begin{array}{lll}
(P) & \min & 0 \\
& \text{subject to:} & A\bar{x}^{(2)} + \gamma^{(1)} = \mathbb{1}, \\
& & B^\top \bar{x}^{(1)} + \gamma^{(2)} = \mathbb{1}, \\
& & \displaystyle\sum_{i \in \mathcal{S}_1} \bar{x}_i^{(1)} \gamma_i^{(1)} = 0, \\
& & \displaystyle\sum_{j \in \mathcal{S}_2} \bar{x}_j^{(2)} \gamma_j^{(2)} = 0, \\
& & \bar{x}^{(1)}, \gamma^{(1)} \in \mathbb{R}_+^{\mathcal{S}_1}, \\
& & \bar{x}^{(2)}, \gamma^{(2)} \in \mathbb{R}_+^{\mathcal{S}_2}.
\end{array}
$$

REMARK 3.3. The first two constraints yield

$$
\begin{bmatrix} 0 & A & I & 0 \\ B^\top & 0 & 0 & I \end{bmatrix}
\begin{pmatrix} \bar{x}^{(1)} \\ \bar{x}^{(2)} \\ \gamma^{(1)} \\ \gamma^{(2)} \end{pmatrix} = \mathbb{1}.
$$

Note that there is a trivial (basic) solution to the above system: $\gamma^{(i)} = \mathbb{1}$ and $\bar{x}^{(i)} = \mathbb{0}$, for $i \in \{1, 2\}$. However, there is no mixed strategy with all entries zero.

REMARK 3.4. Set $v_1 := (\mathbb{1}^\top \bar{x}^{(2)})^{-1}$, $v_2 := (\mathbb{1}^\top \bar{x}^{(1)})^{-1}$, and $x^{(1)} := v_2 \bar{x}^{(1)}$, $x^{(2)} := v_1 \bar{x}^{(2)}$ to get a feasible solution to the original problem.

THEOREM 3.5. For a non-degenerate game, the Lemke-Howson algorithm terminates in a finite number of steps.

Proof Idea. It suffices to show that no basis repeats. □

Chapter 4

Market Models

4.1 Cournot Oligopoly Model

DEFINITION 4.1 (Cournot Oligopoly Model)**.** Let $c \in \mathbb{R}_{++}$ denote the cost of production. Let $\alpha \in \mathbb{R}_{++}$ denote the maximum cost that the buyers are willing to pay. Suppose that $c < \alpha$ and

$$\begin{aligned} C_i(q_i) &:= cq_i, \forall i \in N, \text{ and} \\ P(\vec{q}) &:= \max(\alpha - \sum_{i \in N} q_i, 0). \end{aligned}$$

PROPOSITION 4.2 (Utility Function)**.** The utility for player i, under the Cournot Oligopoly Model, is

$$u_i(\vec{q}) = \begin{cases} q_i(\alpha - c - \sum_{j \in N} q_j), & \text{if } \alpha - \sum_{j \in N} q_j \geq 0 \\ -cq_i, & \text{otherwise.} \end{cases}$$

PROPOSITION 4.3 (Best Response Function)**.** The best response function for player i, under the Cournot Oligopoly Model, is

$$B_i(\vec{q}_{-i}) = \begin{cases} \frac{1}{2}(\alpha - c - \sum_{j \neq i} q_j), & \text{if } \alpha - c - \sum_{j \neq i} q_j \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

PROPOSITION 4.4 (Nash Equilibrium)**.** The Nash equilibrium is $\vec{q}^*$ where $\forall i \in N$,

$$\vec{q}_i^* = \frac{\alpha - c}{n+1}.$$

4.2 Bertrand Oligopoly Model

PROPOSITION 4.5. Let $A := \underset{j \in [n]}{\operatorname{argmin}}\{p_j\}$. Let $m := |A|$. Then the utility function

$$u_i(\vec{p}) = \begin{cases} p_i \frac{D(p_i)}{m} - C_i(\frac{D(p_i)}{m}), & \text{if } i \in A \\ -C_i(0), & \text{otherwise.} \end{cases}$$

4.2.1 Two Player, Linear Cost, Inverse Linear Demand

PROPOSITION 4.6 (Utility Function)**.** Let c denote the cost of production. Let α denote the maximum price that the consumers are willing to pay. Suppose that $n = 2$, $C_i(q_i) = cq_i$, and $D(p) = \max(\alpha - p, 0)$. Then firm i makes a profit of

$$u_i(p_1, p_2) = \begin{cases} (\alpha - p_i)(p_i - c), & \text{if } p_i < p_j \\ \frac{1}{2}(\alpha - p_i)(p_i - c), & \text{if } p_i = p_j \\ 0, & \text{if } p_i > p_j \end{cases}$$

for $i \in \{1, 2\}$ and $j := 3 - i$.

PROPOSITION 4.7 (Best Reponse Function)**.** Let p^* denote the profit-maximizing price in a monopoly. That is, $p^* := \frac{c + \alpha}{2}$ is the value of p that maximizes $(\alpha - p)(p - c)$.

Then the best response function B_i for player i is

$$B_i(p_j) = \begin{cases} \{p_i : p_i > p_j\}, & \text{if } p_j < c \\ \{p_i : p_i \geq c\}, & \text{if } p_j = c \\ \emptyset, & \text{if } c < p_j \leq p^* \\ \{p^*\}, & \text{if } p^* < p_j \end{cases}$$

for $i \in \{1, 2\}$ and $j := 3 - i$.

PROPOSITION 4.8 (Nash Equilibrium)**.** The only point that the graphs of B_1 and B_2 intersect is (c, c).

REMARK 4.9.

- Payoff functions can be discontinuous;
- Best responses can be non-existent;
- Graphs of best response functions can be disconnected.

EXAMPLE 4.10 (Infinite Games with no Nash Equilibrium)**.**

- Non-compact strategy space: $S_1 = S_2 := [0, 1)$, $u_i(s_1, s_2) := s_i$.
- Discontinuous payoff functions: $S_1 = S_2 := [0, 1]$, $u_i(s_1, s_2) := \begin{cases} s_i, & \text{if } s_i < 1 \\ 0, & \text{if } s_i = 1 \end{cases}$.
- Discontinuous pay off functions:

Chapter 5

Routing Games

5.1 Atomic Selfish Routing Game

DEFINITION 5.1 (Atomic Selfish Routing Game). An **atomic selfish routing game** consists of

- A directed graph $G = (V, E)$;
- A set of players $N = \{1, ..., n\}$;
- A source-target pair $(s_i, t_i) \in V \times V$ for each $i \in N$;
- A traffic $r_i \in \mathbb{R}_{++}$ for each $i \in N$;
- A cost function $c_e : \mathbb{R}_{++} \to \mathbb{R}_{++}$ that is continuous and non-decreasing.

REMARK 5.2. Atomic selfish routing game is a special case of finite strategic game. The strategy set $\mathcal{P}_i$ for player i is the set of all $s_i t_i$-paths in G. We assume that $\forall i \in N$, $\mathcal{P}_i \neq \emptyset$. A strategy profile is a vector $\vec{p} = (p_1, ..., p_n)$ of paths. Let $f_e^{\vec{p}}$ denote the total number of units of traffic in $\vec{p}$ on edge e. If $r_i = 1$ for all $i \in N$, then $f_e^{\vec{p}}$ equals the number of occurrences of e in $\vec{p}$. The utility of player i is

$$u_i(\vec{p}) = -\sum_{e \in E(p_i)} c_e(f_e^{\vec{p}}).$$

DEFINITION 5.3 (Flow for Atomic)**.** Let $\mathcal{P} := \bigcup_{i \in N} \mathcal{P}_i$. We define a **fow** to be a function $f : N \times \mathcal{P} \to \mathbb{R}_+$. We say that f is **feasible** if and only if $\forall i \in N$, $\exists p_i \in \mathcal{P}_i$ such that $\forall p \in \mathcal{P}$, we have

$$f(i,p) = \begin{cases} r_i, & \text{if } p = p_i \\ 0, & \text{otherwise.} \end{cases}$$

i.e., each player sets all of its traffic to exactly one path that is available for that player.

DEFINITION 5.4 (Cost for Atomic)**.** We define the **cost of a path** p w.r.t. a flow f, denoted by $c_p(f)$, to be a number given by

$$c_p(f) := \sum_{e \in E(p)} c_e(f_e) \text{ where } f_e := \sum_{q \in \{\mathfrak{q} \in \mathcal{P} : e \in \mathfrak{q}\}} \sum_{i \in N} f(i,q).$$

We define the **cost of a flow** f to be an element of $\mathbb{R}$ given by

$$C(f) := \sum_{e \in E(G)} c_e(f_e) f_e.$$

DEFINITION 5.5 (Equilibrium Flow)**.** We say that a feasible flow f is a **equilibrium flow** if and only if $\forall i \in N$, $\forall p, \tilde{p} \in \mathcal{P}_i$, we have

$$f(i,p) > 0 \implies c_p(f) \leq c_{\tilde{p}}(\tilde{f})$$

where $\tilde{f}$ is the flow identical to f except that $\tilde{f}(i,p) = 0$ and $\tilde{f}(i,\tilde{p}) = r_i$.

5.2 Non-atomic Selfish Routing

DEFINITION 5.6 (Non-atomic Selfish Routing)**.** A **non-atomic selfish routing game** consists of

- A directed graph $G = (V,E)$ (multiple edges are allowed).
- A set of players $N = \{1, ..., n\}$.

- For each player $i \in N$, a source-target pair $(s_i, t_i) \in V \times V$. We assume that $\forall i, j \in N$, $(s_i, t_i) = (s_j, t_j) \implies i = j$.
-

DEFINITION 5.7 (Flow for Non-atomic)**.** Let $\mathcal{P}_i$ denote the set of all $s_i t_i$-paths in G. Let $\mathcal{P} := \bigcup_{i \in N} \mathcal{P}_i$. We define a **flow** to be a function $f : \mathcal{P} \to \mathbb{R}_+$. We say that a flow f is **feasible** if and only if

$$\forall i \in N, \quad \sum_{p \in \mathcal{P}_i} f(p) = r_i.$$

DEFINITION 5.8 (Cost for Non-atomic)**.** Let $f : \mathcal{P} \to \mathbb{R}_+$ be a flow. We define the **cost of a path** p w.r.t. a flow f to be

$$c_p(f) := \sum_{e \in E(p)} c_e(f_e) \text{ where } f_e := \sum_{q \in \mathcal{P} : e \in E(q)} f(q).$$

We define the **cost of a flow** f to be

$$C(f) := \sum_{p \in \mathcal{P}} c_p(f) f(p) = \sum_{e \in E} c_e(f_e) f_e.$$

DEFINITION 5.9 (Equilibrium Flow)**.** We say that a feasible flow is **equilibrium** if and only if

$$\forall i \in N, \forall p, \tilde{p} \in \mathcal{P}_i, \quad f_p > 0 \implies c_p ...$$

THEOREM 5.10. Let $(G, \vec{r}, c)$ be a non-atomic selfish routing instance. Then

1. The instance $(G, \vec{r}, c)$ admits at least one equilibrium flow.
2. If f and $\tilde{f}$ are equilibrium flows for $(G, \vec{r}, c)$, then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge e.

DEFINITION 5.11 (Marginal Cost Functions)**.** Let $e \in E$ and $p \in \mathcal{P}$. We define the **marginal cost functions** to be

$$c_e^*(x) := \frac{d(x \cdot c_e(x))}{dx} = c_e(x) + x \cdot c_e'(x) = \frac{\partial}{\partial f_e} C(f),$$
$$c_p^*(f) := \sum_{e \in E(p)} c_e^*(f_e) = \sum_{e \in E(p)} \frac{\partial}{\partial f_e} C(f).$$

LEMMA 5.12. Let $C : \mathbb{R}^n \to \mathbb{R}$ be a convex differentiable function. Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set. Then a feasible point $x^* \in S$ is optimal for the convex problem

$$\min C(x) \quad s.t. \quad x \in S$$

if and only if

$$\forall x \in S, \quad \nabla C(x^*) \cdot (x - x^*) \geq 0.$$

THEOREM 5.13. Let $(G, \vec{r}, c)$ be a non-atomic selfish routing instance such that for every edge e, the function $x \mapsto x \cdot c_e(x)$ is convex and differentiable. Let c_e^* denote the marginal cost function of the edge e. Then f^* is an optimal flow for $(G, \vec{r}, e)$ if and only if $\forall i \in N$, $\forall p_1, p_2 \in \mathcal{P}_i$ such that $f^*(p_1) > 0$, we have $c_{p_1}^*(f^*) \leq c_{p_2}^*(f^*)$.

Proof. ($\Rightarrow$) Suppose that f^* is optimal. Assume for the sake of contradiction that $\exists i \in N$, $\exists p_1, p_2 \in \mathcal{P}_i$ such that $f^*(p_1) > 0$ and $c_{p_1}^*(f^*) > c_{p_2}^*(f^*)$. Define for each $\varepsilon > 0$ a flow $f : \mathcal{P} \to \mathbb{R}_+$ by

$$f(p) := \begin{cases} f^*(p_1) - \varepsilon, & \text{if } p = p_1 \\ f^*(p_2) + \varepsilon, & \text{if } p = p_2 \\ f^*(p), & \text{otherwise.} \end{cases} \implies (f - f^*)_e = \begin{cases} -\varepsilon, & \text{if } e \in E(p_1) \setminus E(p_2) \\ +\varepsilon, & \text{if } e \in E(p_2) \setminus E(p_1) \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \langle \nabla C(f^*), f - f^* \rangle &= \varepsilon \sum_{e \in E(p_2) \setminus E(p_1)} c_e^*(f_e^*) - \varepsilon \sum_{e \in E(p_1) \setminus E(p_2)} c_e^*(f_e^*) \\ &= \varepsilon \Bigg[\sum_{e \in E(p_2)} c_e^*(f_e^*) - \sum_{e \in E(p_1)} c_e^*(f_e^*) \Bigg] \end{aligned}$$

$$= \varepsilon(c^*_{p_2}(f^*) - c_{p_1}(f^*)) < 0.$$

($\Leftarrow$) Suppose that... I will show that f^* is an optimal flow for $(G, \vec{r}, e)$. Now for any feasible flow $f : \mathcal{P} \to \mathbb{R}_+$ obtained from f^* by shifting ε units...

$$\langle \nabla C(f^*), f - f^* \rangle = \varepsilon(c^*_{p_2}(f^*) - c^*_{p_1}(f^*)) \geq 0.$$

□

Define

$$\Phi(f) := \sum_{e \in E} h_e(f_e) \text{ where } h_e(f_e) := \int_0^{f_e} c_e(x)dx.$$

Then

$$c^*_p(f) = \sum_{e \in E(p)} h'_e(f_e) = \sum_{e \in E(p)} c_e(f_e) = c_p(f).$$

...

THEOREM 5.14. Every non-atomic selfish routing game admits a Nash flow.

Proof. Define

$$h_e(f_e) := \int_0^{f_e} c_e(x)dx.$$

Then h_e is convex and differentiable. Notice that a differentiable function is convex on an interval if and only if its derivative is non-decreasing. So c_e are continuous, non-decreasing, and non-negative. □

Proof. If f and $\tilde{f}$ are equilibrium flows for $(G, \vec{r}, c)$, then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge e. Suppose that f and $\tilde{f}$ are both Nash flows. Then f and $\tilde{f}$ are both minimizers of Φ. So $\Phi(f) = \Phi(\tilde{f})$. Since the feasible set is convex, $\forall \lambda \in [0, 1]$, $\lambda f + (1 - \lambda)\tilde{f}$ is also feasible. Note that $\Phi(f) := \sum_{e \in E} h_e(f_e)$ is a sum of convex function and hence convex. So

$$\Phi(\lambda f + (1 - \lambda)\tilde{f}) \leq \lambda\Phi(f) + (1 - \lambda)\Phi(\tilde{f}) = \Phi(f) = \Phi(\tilde{f}).$$

So $\lambda \mapsto \Phi(\lambda f + (1 - \lambda)\tilde{f})$ is a constant function. For a sum of convex functions to be constant, each summand must be linear. So $h_e(f_e) = \int_0^{f_e} c_e(x)dx$ is linear. So $c_e(x)$ is constant from f to $\tilde{f}$. □

THEOREM 5.15. Suppose $\gamma \geq 1$ satisfies $\forall e \in E$, $\forall x \geq 0$, we have

$$x \cdot c_e(x) \leq \gamma \int_0^x c_e(y)dy.$$

Then the price of anarchy is at most γ.

REMARK 5.16. Note that $\gamma < 1$ is impossible since $\forall e \in E$, $\forall x \geq 0$,

$$\frac{d}{dy}(y \cdot c_e(y)) = c_e(y) + y \cdot c_e'(y) \geq c_e(y)$$
$$\implies x \cdot c_e(x) = \int_0^x \frac{d}{dy}(y \cdot c_e(y))dy \geq \int_0^x c_e(y)dy.$$

Proof. By the previous calculation, $\forall f : \mathcal{P} \to \mathbb{R}$, we have

$$C(f) = \sum_{e \in E} f_e \cdot c_e(f_e) \geq \sum_{e \in E} \int_0^{f_e} c_e(x)dx = \Phi(f).$$

So for all feasible flows f and $\tilde{f}$ where f is a Nash flow, we then have

$$C(f) \leq \gamma\Phi(f) \leq \gamma\Phi(\tilde{f}) \leq \gamma C(\tilde{f}).$$

□

EXAMPLE 5.17. Let $c_e(x)$ be given by $c_e(x) = \sum_{i=0}^{d} a_i x^i$ for some $d \in \mathbb{Z}_{++}$ and $a_0, ..., a_d \in \mathbb{R}_{++}$. Then we have

$$x \cdot c_e(x) = \sum_{i=0}^{d} a_i x^{i+1} \text{ and}$$
$$(d+1)\int_0^x c_e(y)dy = (d+1)\sum_{i=0}^{d} \frac{a_i}{i+1}x^{i+1} = \sum_{i=0}^{d} \frac{d+1}{i+1}a_i x^{i+1} \geq x \cdot c_e(x).$$

Hence we can take $\gamma = d + 1$ in the theorem. So the price of anarchy is at most $d + 1$.

5.3 Potential Function of Atomic Selfish Routing Game

DEFINITION 5.18 (Potential Function)**.** Suppose $r_1 = ... = r_n = r$ for some $r \in \mathbb{R}$. Then there exists a Nash equilibrium. Define $\mathcal{P} := \bigcup_{i=1}^{n} \mathcal{P}_i$. Define for each $e \in E$ a number $f_e^{\vec{p}} \in \mathbb{Z}_+$ by $f_e^{\vec{p}} := |\{i \in N : e \in E(\vec{p}_i)\}|$. We define the **potential function** of an atomic selfish routing game, denoted by Φ, to be a function from $\mathcal{P}_1 \times ... \times \mathcal{P}_n$ to $\mathbb{R}$ given by

$$\Phi(\vec{p}) := \sum_{e \in E} \sum_{i=1}^{f_e^{\vec{p}}} c_e(i).$$

DEFINITION 5.19 (Exact Potential Game)**.** We say that a finite strategic game is an **exact potential game** if and only if there exists a potential function $\Phi : \mathcal{S}_1 \times ... \times \mathcal{S}_n \to \mathbb{R}$ such that $\forall i \in N$, $\forall s_i, s_i' \in \mathcal{S}_i$,

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}).$$

Notice that utilities are negatives of the cost. So if Φ increases, u_i would decrease, and vice versa.

THEOREM 5.20. An atomic selfish routing game is an exact potential game with potential function $\Phi : \mathcal{P}_1 \times ... \times \mathcal{P}_n \to \mathbb{R}$ given by

$$\Phi(\vec{p}) = \sum_{e \in E} \sum_{i=1}^{f_e^{\vec{p}}} c_e(i).$$

Proof. Let $i \in N$, $s_i, s_i' \in \mathcal{S}_i$ be arbitrary. Let $p_i := (s_i, s_{-i})$ and $p_i := (s_i', s_{-i})$. Then

$$\begin{aligned}
\Phi(\vec{p}) - \Phi(\vec{p}') &= \sum_{e \in E} \sum_{i=1}^{f_e^{\vec{p}}} c_e(i) - \sum_{e \in E} \sum_{i=1}^{f_e^{\vec{p}'}} c_e(i) \\
&= \sum_{e \in E(p_i) \setminus E(p_i')} c_e(f_e^{\vec{p}}) - \sum_{e \in E(p_i') \setminus E(p_i)} c_e(f_e^{\vec{p}'}) \\
&= \sum_{e \in E(p_i)} c_e(f_e^{\vec{p}}) - \sum_{e \in E(p_i')} c_e(f_e^{\vec{p}'}) \\
&= (-u_i(\vec{p})) - (-u_i(\vec{p}')) = u_i(\vec{p}') - u_i(\vec{p}).
\end{aligned}$$

$\square$

THEOREM 5.21. Every exact potential game has a Nash equilibrium.

Proof. Notice that the set of strategy profiles $\mathcal{S} = \mathcal{S}_1 \times ... \times \mathcal{S}_n$ is a finite set. Let $\vec{s} \in \mathcal{S}$ be a minimizer of Φ. Assume for the sake of contradiction that $\vec{s}$ is not a Nash equilibrium, then $\exists i \in N$, $\exists s_i' \in \mathcal{S}_i$ such that $u_i(\vec{s}') - u_i(\vec{s}) > 0$. By the preceding theorem we get $\Phi(\vec{s}) - \Phi(\vec{s}') > 0$, which contradicts to the assumption that $\vec{s}$ is a minimizer of Φ. ☐

DEFINITION 5.22 ((λ, μ)-smooth)**.** Let $\lambda \geq 0$ and $\mu < 1$. Let $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$. We say that f is $\boldsymbol{(\lambda, \mu)}$**-smooth** if and only if

$$\forall x, y \in \mathbb{R}_{++}, \quad yf(x) \leq \lambda y f(y) + \mu x f(x).$$

EXAMPLE 5.23. Let $f(x) := ax + b$ for some $a, b \in \mathbb{R}_{++}$. Then f is $(1, 1/4)$-smooth.

Proof. Let $x, y \in \mathbb{R}_{++}$ be arbitrary. Then

$$0 \leq a(\frac{1}{2}x - y)^2 = \frac{1}{4}x^2 - axy + ay^2 \iff axy - ay^2 \leq \frac{1}{4}ax^2$$
$$...$$

☐

THEOREM 5.24 (Variational Inequality Characterization)**.** Let f be a feasible flow. Then f is a Nash flow if and only if

$$\forall \text{ feasible flow } f^*, \quad \sum_{e \in E} c_e(f_e) f_e \leq \sum_{e \in E} c_e(f_e) f_e^*.$$

Proof. Define for any feasible flow f^*

$$H(f^*) = \sum_{e \in E} c_e(f_e) f_e^*$$

Then

$$H(f^*) = \sum_{e \in E} c_e(f_e) f_e^* = \sum_{e \in E} c_e(f_e) \sum_{p \in \mathcal{P} : e \in E(p)} f_p^* = \sum_{e \in E} \sum_{p \in \mathcal{P} : e \in E(p)} c_e(f_e) f_p^*$$

$$= \sum_{p \in \mathcal{P}} \sum_{e \in E : e \in E(p)} c_e(f_e) f_p^* = \sum_{p \in \mathcal{P}} c_p(f) f_p^* = \sum_{i=1}^{N} \sum_{p \in \mathcal{P}_i} c_p(f) f_p^*.$$

($\Rightarrow$) Suppose that f is a Nash flow. Then $\forall p \in \mathcal{P}_i,\ f_p > 0 \implies \forall \hat{p} \in \mathcal{P}_i,\ c_p(f) \leq c_{\hat{p}}(f)$. So the summation in $H(f)$ is a weighted average of minimal possible terms, whereas the summation in $H(f^*)$ is a weighted average of possibly larger terms. So $H(f) \leq H(f^*)$.

($\Leftarrow$) Suppose that f is a minimizer of H. Then the summation in $H(f)$ must only assign positive weights to the smallest possible values of $c_p(f)$. So $\forall p \in \mathcal{P}_i,\ f_p > 0 \implies \forall \hat{p} \in \mathcal{P}_i$, $c_p(f) \leq c_{\hat{p}}(f)$. ☐

THEOREM 5.25. Consider a non-atomic selfish routing game. Suppose that c_e is (λ, μ)-smooth for all $e \in E$ Then

$$C(f) \leq \frac{\lambda}{1-\mu} C(\hat{f})$$

whenever f is a Nash flow ans $\hat{f}$ is an optimal flow.

Proof.

$$\begin{aligned} C(f) &= \sum_{e \in E} c_e(f_e) f_e \leq \sum_{e \in E} c_e(f_e) \hat{f}_e, \text{ by the above theorem} \\ &\leq \lambda \sum_{e \in E} c_e(\hat{f}_e) \hat{f}_e + \mu \sum_{e \in E} c_e(f_e) f_e, \text{ by smoothness of } c_e \\ &= \lambda C(\hat{f}) + \mu C(f). \end{aligned}$$

☐