Semidefinite Optimization

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Chapter 1

First Chapter

1.1 Primal Problem and Dual Problem

DEFINITION (Primal Problem)**.** Let $C \in \mathbb{S}^n$. Let $b \in \mathbb{R}^m$. Let $\mathcal{A}$ be a linear transformation from $\mathbb{S}^n$ to $\mathbb{R}^m$. We define the **primal problem** to be the following.

$$\begin{array}{lcl} \text{(P)} & \inf & \langle C, X \rangle \\ & \text{subject to} & \mathcal{A}X = b \\ & & X \succeq 0 \end{array}$$

DEFINITION (Dual Problem)**.** We define the **dual problem** of the above primal problem to be the following.

$$\begin{array}{lcl} \text{(D)} & \sup & b^\top y \\ & \text{subject to} & \mathcal{A}^* y + S = C \\ & & S \succeq 0 \end{array}$$

where $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$.

1.2 Weak Duality

THEOREM 1.1 (The Weak Duality Relation)**.** Let $\bar{X}$ be feasible in (P). Let $(\bar{y}, \bar{S})$

be feasible in (D). Then

$$\langle C, \bar{X} \rangle - b^\top \bar{y} \geq 0.$$

Proof. Since $(\bar{y}, \bar{S})$ is feasible, we have

$$C = \mathcal{A}^* y + S. \tag{1}$$

Since $\bar{X}$ is feasible, we have

$$\mathcal{A}X = b. \tag{2}$$

Then

$$\begin{aligned}
\langle C, \bar{X} \rangle - b^\top \bar{y} &= \langle \mathcal{A}^* y + S, \bar{X} \rangle - b^\top y, \text{ by equation (1)} \\
&= \langle \mathcal{A}^* \bar{y}, \bar{X} \rangle + \langle S, \bar{X} \rangle - b^\top \bar{y}, \text{ by linearity} \\
&= \langle \bar{y}, \mathcal{A}\bar{X} \rangle + \langle \bar{S}, \bar{X} \rangle - b^\top \bar{y}, \text{ by definition of adjoint} \\
&= \langle \bar{y}, b \rangle + \langle \bar{S}, \bar{X} \rangle - b^\top \bar{y}, \text{ by equation (2)} \\
&= \langle \bar{S}, \bar{X} \rangle \\
&\geq 0, \text{ since } X, S \succeq 0.
\end{aligned}$$

That is, $\langle C, \bar{X} \rangle - b^\top \bar{y} \geq 0$. ■

COROLLARY 1.1.

(1) If (P) is unbounded, then (D) is infeasible.

(2) If (D) is unbounded, then (P) is infeasible.

(3) If $\bar{X}$ and $(\bar{y}, \bar{S})$ are feasible in (P) and (D) respectively and $\langle C, \bar{X} \rangle = b^\top \bar{y}$, then $\bar{X}$ is optimal in (P) and $(\bar{y}, \bar{S})$ is optimal in (D).