Functional Analysis

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Contents

Chapter 1

Normed Linear Spaces

1.1 Definitions

DEFINITION (Seminorm)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **seminorm** on $\mathfrak{X}$, denoted by ν, to be a map from $\mathfrak{X}$ to $\mathbb{R}$ that satisfies the following conditions.

(1) $\forall x \in \mathfrak{X}, \quad \nu(x) \geq 0.$

(2) $\forall \lambda \in \mathbb{F}, \forall x \in \mathfrak{X}, \quad \nu(\lambda x) = |\lambda|\nu(x).$

(3) $\forall x, y \in \mathfrak{X}, \quad \nu(x+y) \leq \nu(x) + \nu(y).$

The idea behind the seminorm is that we are trying to give our vector space a notion of "length" of vectors.

DEFINITION (Norm)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **norm** on $\mathfrak{X}$, denoted by ν, to be a seminorm on $\mathfrak{X}$ that satisfies the additional condition:

$$\forall x \in \mathfrak{X}, \quad \mu(x) = 0 \iff x = 0.$$

1.2 Properties

PROPOSITION 1.2.1. Let $(V, \|\cdot\|_V)$ be a normed vector space over field $\mathbb{F}$. Then $(V, \|\cdot\|)$ is complete if and only if $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

Proof.

For one direction, assume that $(V, \|\cdot\|)$ is complete.

We are to prove that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

Since $(\overline{B(0,1)}, \|\cdot\|_V)$ is a closed subspace of $(V, \|\cdot\|)$ and $(V, \|\cdot\|)$ is complete, $(\overline{B(0,1)}, \|\cdot\|_V)$ is also complete.

For the reverse direction, assume that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

We are to prove that $(V, \|\cdot\|_V)$ is complete.

Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $(V, \|\cdot\|_V)$.

Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ is bounded in $(V, \|\cdot\|_V)$.

Let λ be a positive upper bound for $\{\|x_i\|_V\}_{i\in\mathbb{N}}$.

Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$ and $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ converges in $(V, \|\cdot\|_V)$.

Since any Cauchy sequence in $(V, \|\cdot\|_V)$ converges in $(V, \|\cdot\|_V)$, $(V, \|\cdot\|_V)$ is complete. ■

PROPOSITION 1.2.2. Proper subspaces of a normed linear space has empty interior.

Proof. Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a proper subspace of $\mathfrak{X}$. Assume for the sake of contradiction that $\mathcal{M}$ has non-empty interior. Then $\exists x_0 \in \mathcal{M}$ and $\exists r > 0$ such that $\mathrm{ball}(x_0, r) \subseteq \mathcal{M}$ where $\mathrm{ball}(x_0, r)$ denotes the open ball centered at point x_0 with radius r. Let x be an arbitrary point in $\mathfrak{X}$. Define a point $y(x)$ as $y(x) := x_0 + \frac{r}{2\|x\|}x$. Then $x = \frac{2\|x\|}{r}(y - x_0)$. It is easy to verify that $\|y - x_0\| = \frac{r}{2} < r$. So $y \in \mathrm{ball}(x_0, r)$. So $y \in \mathcal{M}$. Since $y, x_0 \in \mathcal{M}$ and $\mathcal{M}$ is a subspace, we get $\frac{2\|x\|}{r}(y - x_0) \in \mathcal{M}$. That is, $x \in \mathcal{M}$. So $\forall x \in \mathfrak{X}$, $x \in \mathcal{M}$. So $\mathcal{M} = \mathfrak{X}$. This contradicts to the assumption that $\mathcal{M}$ is a proper subspace of $\mathfrak{X}$. So $\mathcal{M}$ has empty interior. ■

PROPOSITION 1.2.3. Closed proper subspaces of a normed linear space are nowhere dense.

Proof. Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a closed proper subspace of $\mathfrak{X}$. Since $\mathcal{M}$ is closed, $\mathrm{cl}(\mathcal{M}) = \mathcal{M}$. So $\mathrm{cl}(\mathcal{M}) = \mathcal{M}$ is a closed proper subspace of $\mathfrak{X}$. Since $\mathrm{cl}(\mathcal{M})$ is a proper subspace of $\mathfrak{X}$, $\mathrm{int}(\mathrm{cl}(\mathcal{M})) = \emptyset$. So $\mathcal{M}$ is nowhere dense. ■

PROPOSITION 1.2.4. Finite dimensional subspace of a normed linear space is closed.

PROPOSITION 1.2.5. Finite-dimensional normed linear spaces are complete.

1.3 Equivalence of Norms

DEFINITION (Equivalence of Norms)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are **equivalent** if

$$\exists c_1, c_2 > 0, \quad \forall v \in \mathfrak{X}, \quad c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_2.$$

Or equivalently,

$$c_1\|v\|_2 \leq \|v\|_1 \leq c_2\|v\|_2.$$

PROPOSITION 1.3.1. The equivalence of norms is an equivalence relation.

THEOREM 1.1. Let V be a finite dimensional vector space over field $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$. Then any two norms on V are equivalent.

Proof.

Let $\|\cdot\|_p$ be an arbitrary p-norm on V and $\|\cdot\|$ be an arbitrary norm on V.

Let $\mathcal{B}$ be the standard basis for V. Say $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$.

Let v be an arbitrary vector in V.

$$\|v\| = \|\sum_{i=1}^{n} v_i e_i\|$$

$$\begin{aligned}
&\le \sum_{i=1}^{n} |v_i| \|e_i\| \\
&\le \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \\
&= \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \|v\|_p \\
&:= c_1 \|v\|_p.
\end{aligned}$$

■

PROPOSITION 1.3.2. Let X be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if they generate the same metric topology.

Proof. Convergence to 0 is equivalent under either $\|\cdot\|_1$ or $\|\cdot\|_2$. i.e., equivalent norms give rise to the same set of sequences that are convergent. Convergence of sequences defines the topology. ■

PROPOSITION 1.3.3. Let $\mathfrak{X}$ be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathfrak{X}$. Let ι be the identity map from $(\mathfrak{X}, \|\cdot\|_1)$ to $(\mathfrak{X}, \|\cdot\|_2)$. Then if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, ι is continuous, and in fact, a homeomorphism between $(\mathfrak{X}, \|\cdot\|_1)$ and $(\mathfrak{X}, \|\cdot\|_2)$.

1.4 Dual Norms

DEFINITION (Dual Norm)**.** Let $(V, \|\cdot\|)$ be an normed vector space. We define the **dual norm** of $\|\cdot\|$, denoted by $\|\cdot\|_\circ$, to be a function given by

$$\|v\|_\circ := \max_{\|w\|=1} v \cdot w = \max_{\|w\| \neq 0} \frac{|v \cdot w|}{\|w\|}.$$

PROPOSITION 1.4.1. Dual norms of norms are indeed norms.

PROPOSITION 1.4.2. Let $(V, \|\cdot\|)$ be a normed vector space. Let v, w be vectors in the space. Then
$$|v \cdot w| \leq \|v\| \cdot \|w\|_{\circ}.$$

1.5 p-norms

DEFINITION (p-norm)**.** Let V be a finite-dimensional normed vector space over field $\mathcal{F}$. Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis for V where $n = \dim(V)$. Let v be a vector in a normed vector space. For $p \in [1, +\infty)$, we define the **p-norm** of v, denoted by $\|v\|_p$, to be the number given by
$$\|v\|_p = \left(\sum_{i=1}^{n} |(v_{\mathcal{B}})_i|^p \right)^{\frac{1}{p}}.$$

DEFINITION (Infinity Norm - 1)**.** Let $\mathfrak{X} = \mathbb{K}^n$ where $\mathbb{K}$ is a field and $n \in \mathbb{N}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by
$$\|v\|_\infty := \max\{|v_i|\}_{i=1}^n.$$

DEFINITION (Infinity Norm - 2)**.** Let $\mathfrak{X} = \mathbb{K}^{\mathbb{N}}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by
$$\|v\|_\infty := \sup_{i \in \mathbb{N}} |v_i|.$$

DEFINITION (Infinity Norm - 3)**.** Let $\mathfrak{X} = \mathcal{C}([0,1], \mathbb{C})$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by
$$\nu(f) := \sup_{x \in [0,1]} |f(x)|.$$

PROPOSITION 1.5.1. Let $\mathfrak{X} := \mathcal{C}([0,1],\mathbb{C})$. Let x be an arbitrary number in $[0,1]$. Define a function ν_x on $\mathfrak{X}$ by $\nu_x(f) := |f(x)|$. Define a function ν on $\mathfrak{X}$ by $\nu(f) := \sup_{x\in[0,1]} |f(x)|$. Then ν_x is a seminorm on $\mathfrak{X}$ for each x and ν is a norm on $\mathfrak{X}$ and we have $\nu = \sup_{x\in[0,1]} \nu$.

PROPOSITION 1.5.2. p-norms are indeed norms.

PROPOSITION 1.5.3. For any vector v in $\mathbb{R}^n$, we have

$$\lim_{p\to\infty} \|v\|_p = \|v\|_\infty.$$

i.e.,

$$\lim_{p\to\infty} \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}} = \max\{|v_i|\}_{i=1}^{n}.$$

Proof. Let p be an arbitrary number in $[1,+\infty)$. Let k be an arbitrary index in $\{1,..,n\}$. Then

$$|v_k| \leq (\sum_{i=1}^{n} |v_k|^p)^{1/p} = \|v\|_p.$$

So

$$\max\{|v_k|\} = \|v\|_\infty \leq \|v\|_p.$$

So

$$\lim_{p\to\infty} \|v\|_p \geq \|v\|_\infty. \tag{1}$$

On the other hand, note that

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p = \sum_{i=1}^{n} (\frac{|v_i|}{\|v\|_\infty})^p$$

decreases as p increases. So it is bounded above. Say

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p \leq C$$

for some $C \in \mathbb{R}$. Then

$$(\sum_{i=1}^{n} |v_i|^p)^{1/p} = \|v\|_p \leq C^{1/p}\|v\|_\infty.$$

So

$$\lim_{p\to\infty} \|v\|_p \le \lim_{p\to\infty} C^{1/p}\|v\|_\infty = \|v\|_\infty. \tag{2}$$

From (1) and (2) we get

$$\lim_{p\to\infty} \|v\|_p = \|v\|_\infty.$$

■

PROPOSITION 1.5.4. Let p be an arbitrary number in $[1, +\infty)$. Then the dual norm of the p-norm $\|\cdot\|_p$ is the q-norm $\|\cdot\|_q$ where q is such that satisfies

$$\frac{1}{p} + \frac{1}{q} = 1.$$

PROPOSITION 1.5.5. Let p and q be numbers in $[1, +\infty]$. Let v be a vector in $\mathbb{R}^n$. Then if $p \le q$,

$$\|x\|_q \le \|x\|_p \le n^{\frac{1}{p}-\frac{1}{q}} \cdot \|x\|_q.$$

PROPOSITION 1.5.6. Let w and z be vectors in $\mathbb{E}^d$. Then

$$\|w + z\|_2^2 + \|w - z\|_2^2 = 2(\|w\|_2^2 + \|z\|_2^2).$$

Chapter 2

Inner Product Spaces

2.1 Inner Products

2.1.1 Definitions

DEFINITION (Inner Product)**.** Let V be a vector space over field $\mathbb{F}$. We define an ***inner product*** on V, denoted by $\langle \cdot, \cdot \rangle$, to be a scalar-valued function defined on $V \times V$ such that

(1) Positive Definiteness:

$$\forall x \in V, \langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \iff x = 0.$$

(2) Sesqui-Linearity:

$$\begin{aligned} \forall x, y, z, w \in V, &\quad \langle x + y, z + w \rangle = \langle x, z \rangle + \langle y, z \rangle + \langle x, w \rangle + \langle y, w \rangle, \text{ and} \\ \forall a, b \in \mathbb{F}, \forall x, y \in V, &\quad \langle ax, by \rangle = a\bar{b}\langle x, y \rangle. \end{aligned}$$

(3) Conjugate Symmetry:

$$\forall x, y \in V, \quad \langle x, y \rangle = \overline{\langle y, x \rangle}.$$

DEFINITION (Induced Norm)**.** Let $\mathfrak{X}$ be an inner product space over field $\mathbb{K}$. We define the **norm induced by** $\langle \cdot, \cdot \rangle$, denoted by $\| \cdot \|$, to be a function from $\mathfrak{X}$ to $\mathbb{R}_+$ given by

$$\|x\| := \sqrt{\langle x, x \rangle}$$

2.1.2 Examples of Inner Products

DEFINITION (Standard Inner Product)**.** For $V = \mathbb{F}^n$, we define the **standard inner product** by

$$\langle x, y\rangle := \sum_{i=1}^{n} x_i \overline{y_i}.$$

DEFINITION (Frobenius Inner Product)**.** For $V = \mathbb{F}^{n\times n}$, we define the **Frobenius inner product** by

$$\langle M_1, M_2\rangle := \operatorname{tr}(M_2^* M_1).$$

DEFINITION. Let V be the space of continuous scalar-valued functions on $[0, 2\pi]$. We define the inner product on V by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx.$$

2.1.3 Properties

PROPOSITION 2.1.1. Let V be a finite dimensional inner product space. Let $\mathcal{B}$ be a basis for V. Let x and y be vectors in V. Then

$$x = y \iff \forall b \in \mathcal{B}, \quad \langle x, b\rangle = \langle y, b\rangle.$$

2.2 Inner Product Space

DEFINITION (Inner Product Space)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{X}$. We define an **inner product space** to be the pair $(\mathfrak{X}, \langle\cdot,\cdot\rangle)$.

2.3 Inequalities

THEOREM 2.1 (Minkowski)**.**

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}$$

PROPOSITION 2.3.1 (Cauchy-Schwarz Inequality)**.** Let V be an inner product space. Then

$$\forall x, y \in V, \quad |\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

PROPOSITION 2.3.2 (Triangle Inequality)**.** Let V be an inner product space. Then

$$\forall x, y \in V, \quad \|x+y\| \leq \|x\| + \|y\|$$

PROPOSITION 2.3.3 (Parallelogram Law)**.** Let $\mathfrak{X}$ be an inner product space. Then

$$\forall x, y \in \mathfrak{X}, \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Proof.

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&\quad + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\
&= 2\langle x, x\rangle + 2\langle y, y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

That is,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

∎

Chapter 3

Orthogonality

3.1 Orthogonal Sets

DEFINITION (Orthogonality)**.** Let V be an inner product space. We say that points x and y in V are **orthogonal** if $\langle x, y \rangle = 0$.

DEFINITION (Orthogonal Set)**.** Let $\mathfrak{X}$ be an inner product space. Let S be a subset of $\mathfrak{X}$. We say that S is **orthogonal** if

$$\forall x, y \in S : x \neq y, \quad \langle x, y \rangle = 0.$$

i.e., if any two distinct vectors are orthogonal.

DEFINITION (Orthonormal Set)**.** Let $\mathfrak{X}$ be an inner product space. Let S be a set in the space. We say that S is **orthonormal** if S is orthogonal and $\forall x \in S$, $\|x\| = 1$ where $\|\cdot\|$ is the norm induced by the inner product.

PROPOSITION 3.1.1. Orthogonal sets are linearly independent.

3.2 Orthogonal Bases

DEFINITION (Orthogonal Basis)**.** Let V be an inner product space. Let S be a set in the space. We say that S is an **orthogonal basis** for V if it is an ordered basis for V and orthogonal.

DEFINITION (Orthonormal Basis)**.** Let $\mathfrak{X}$ be an inner product space. Let S be a set in the space. We say that S is an **orthonormal basis** for $\mathfrak{X}$ if it is the maximal, with respect to inclusion, in the collection of all orthonormal sets in the space.

PROPOSITION 3.2.1. Let V be an inner product space. Let $S = \{v_1, ..., v_n\}$ be an orthogonal subset of V where each v_i is non-zero. Then

$$\forall y \in \operatorname{span}(S), \quad y = \sum_{i=1}^{n} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

THEOREM 3.1 (Gram-Schmidt Process)**.** Let V be an inner product space. Let $S = \{x_0, ..., x_n\}$ be a linearly independent subset of V. Then the set $T = \{y_0, ..., y_n\}$ given by $y_0 := x_0$ and

$$\forall i \in \{1, ..., n\}, \quad y_i := x_i - \sum_{j=1}^{i-1} \frac{\langle x_i, y_j \rangle}{\|y_j\|} y_j$$

is an orthogonal subset of V consisting of non-zero vectors such that $\operatorname{span}(S) = \operatorname{span}(S')$.

PROPOSITION 3.2.2. Let V be an inner product space and $S = \{v_0, v_1, \ldots, v_n\}$ be an orthogonal subset of V. Then the set $S^{'}$ derived from the Gram-Schmidt process is exactly S.

THEOREM 3.2 (Parseval's Identity)**.** Let V be a finite-dimensional inner product space. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthogonal basis for V. Then

$$\forall x, y \in V, \quad \langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

PROPOSITION 3.2.3. Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. Then $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$ if and only if

$$\forall x \in \mathcal{H}, \quad x = \sum_{e \in \mathcal{E}} \langle x, e \rangle e.$$

3.3 Orthogonal Complements

DEFINITION (Orthogonal Complement)**.** Let $\mathfrak{X}$ be an inner product space. Let S be a non-empty subset of V. We define the **orthogonal complement** of S, denoted by $S^\perp$, to be a set given by

$$S^\perp := \{x \in \mathfrak{X} : \forall s \in S, \langle x, s \rangle = 0\}.$$

i.e., the set of all points in $\mathfrak{X}$ that are orthogonal to all vectors in S.

PROPOSITION 3.3.1. Let V be a finite-dimensional inner product space. Then

(1) $V^\perp = \{O_V\}$

(2) $\{O_V\}^\perp = V$

PROPOSITION 3.3.2. Orthogonal complements are always linear subspaces.

PROPOSITION 3.3.3. Let V be an inner product space and W be a subspace of V with basis β. Then a vector in V is also in $W^\perp$ if and only if it is orthogonal to all vectors in β.

PROPOSITION 3.3.4 (Extension)**.** Let V be an n-dimensional inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal subset of V. Then S can be extended to an orthogonal basis $B = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.

PROPOSITION 3.3.5. Let V be an inner product space. Then

(1) $S \subseteq T$ implies $T^{\perp} \subseteq S^{\perp}$ for any subsets S and T of V.

(2) $S \subseteq (S^{\perp})^{\perp}$ for any subset S of V.

PROPOSITION 3.3.6. Let V be a finite-dimensional inner product space and W be a subspace of V. Then

(1) $W = (W^{\perp})^{\perp}$

(2) $V = W \oplus W^{\perp}$

PROPOSITION 3.3.7. Let V be a finite-dimensional inner product space and W_1 and W_2 be subspaces of V. Then

(1) $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$

(2) $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$

3.4 Orthogonal Projection

DEFINITION (Orthogonal Projection)**.** Let V be a vector space. Let W be a finite-dimensional subspace of V. Let x be a vector in V. We define the **orthogonal projection** of x on W, denoted by (x), to be the vector u in W such that $x = u + v$ where v is another vector in $W^{\perp}$.

3.5 Inequalities in Hilbert Spaces

THEOREM 3.3 (Bessel's Inequality)**.** Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in the space. Then

$$\forall x \in \mathcal{H}, \quad \sum_{e \in \mathcal{E}} |\langle x, e_i \rangle|^2 \leq \|x\|^2.$$

PROPOSITION 3.5.1. Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. Let x be a point in the space. Then the net $\sum_{e \in \mathcal{E}} \langle x, e \rangle e$ converges in $\mathcal{H}$.

Proof. Let $\mathcal{F}$ be the collection of all finite subsets of $\mathcal{E}$, partially ordered by inclusion. Define for each $F \in \mathcal{F}$ a vector y_F as $y_F := \sum_{e \in F} \langle x, e \rangle e$. Let ε be an arbitrary positive number. Since $\mathcal{E}$ is an orthonormal set, the set $\{e \in \mathcal{E} : \langle x, e \rangle \neq 0\}$ is countable. Let $\{e_i\}_{i \in \mathbb{N}}$ denote the set. By the Bessel's inequality, $\exists N \in \mathbb{N}$ such that $\sum_{i=N+1}^{\infty} |\langle x, e_i \rangle|^2 < \varepsilon^2$. Define a set F_0 as $F_0 := \{e_1, ..., e_N\}$. Let F and G be arbitrary elements in $\mathcal{F}$ such that $F_0 \leq F$ and $F_0 \leq G$. Then

$$\begin{aligned}
\|y_F - y_G\|^2 &= \left\| \sum_{e \in F \setminus G} \langle x, e \rangle e - \sum_{e \in G \setminus F} \langle x, e \rangle e \right\|^2 \\
&= \sum_{e \in F \cup G \setminus F \cap G} |\langle x, e \rangle|^2 \\
&= \sum_{e \in F \cup G} |\langle x, e \rangle|^2 - \sum_{e \in F \cap G} |\langle x, e \rangle|^2 \\
&\leq \sum_{i=N+1}^{\infty} |\langle x, e_i \rangle|^2 \\
&< \varepsilon^2.
\end{aligned}$$

So $\{y_F\}_{F \in \mathcal{F}}$ is Cauchy. Since $\mathcal{H}$ is complete and $\{y_F\}_{F \in \mathcal{F}}$ is Cauchy, $\{y_F\}_{F \in \mathcal{F}}$ converges. ∎

Chapter 4

Sequence Spaces

4.1 ℓ^p Space

DEFINITION (ℓ^p Space)**.** We define the ℓ^p space to be the set of all sequences x such that $\|x\|_p$ is finite, equipped with the p-norm $\|\cdot\|_p$.

DEFINITION (Weighted ℓ^p Space)**.** Let $(r_i)_{i\in\mathbb{N}}$ be a sequence of positive integers. We define the **weighted $\boldsymbol{\ell^p}$** space to be the set given by

$$\ell^p := \left\{(x_i)_{i\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sum_{i\in\mathbb{N}} r_i|x_i|^p < +\infty.\right.$$

PROPOSITION 4.1.1. For $p \in [1, +\infty)$, $(\ell^p, \|\cdot\|_p)$ is complete.

Proof.

Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in ℓ^p.

Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in ℓ^p, $\forall\varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $\forall m, n > N$, we have $\|x_m - x_n\|_p < \varepsilon$.

Since $\|x_m - x_n\|_p < \varepsilon$ and $|x_m^{(i)} - x_n^{(i)}| \leq \|x_m - x_n\|_p$ for any $i\in \mathbb{N}$, $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$ for any $i\in \mathbb{N}$.

Since for any $i\in \mathbb{N}$ and any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N$, we have $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$, by definition, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$.

Since $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$ and $\mathbb{F}$ is complete, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ converges.
Let $x_0^{(i)} = x_n^{(i)}$. Let $x_0 = \{x_0^{(i)}\}_{i\in\mathbb{N}}$.

$$\|x_0\|_p = (\sum_{i=1}^{\infty} |x_0^{(i)}|^p)^{\frac{1}{p}}$$

■

4.2 c_0 Space and c_{00} Space

DEFINITION (c_0 Space)**.** We define c_0 to be

$$c_0 := \big\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\big\}.$$

DEFINITION (c_{00} Space)**.** We define c_{00} to be

$$c_{00} := \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}, \forall n > N, x_n = 0\big\}.$$

i.e., the set of all eventually zero sequences of real numbers. i.e., the set of all sequences with finite support.

PROPOSITION 4.2.1. The c_{00} is not complete in $(\ell_1, \|\cdot\|_1)$.

Proof. Define a sequence of vectors $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ by $\mathfrak{x}_i^j := \frac{1}{j^2}$ for $j \in \{1..i\}$ and $\mathfrak{x}_i^j := 0$ for $j > i$. Then $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ converges to something that is not in c_{00}. ■

PROPOSITION 4.2.2. The closure of c_{00} in the space $(\mathbb{R}^\omega, d_1)$ is ℓ_1.

Proof. For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq \ell_1$. Let x be an arbitrary element in $\mathrm{cl}(c_{00})$. Since $x \in \mathrm{cl}(c_{00})$, there exists another element $y \in c_{00}$ such that $d_1(x, y) < 1$. Let $N \in \mathbb{N}$ be such that $\forall n > N$, $y_n = 0$. Then

$$\begin{aligned} & d_1(x, y) < 1 \\ \iff & \sum_{n\in\mathbb{N}} |x_n - y_n| < 1 \end{aligned}$$

$$
\begin{aligned}
&\iff \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n - y_n| < 1 \\
&\iff \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n| < 1 \\
&\implies \sum_{n=1}^{N} \big||x_n| - |y_n|\big| + \sum_{n>N} |x_n| < 1 \\
&\implies \sum_{n=1}^{N} \big(|x_n| - |y_n|\big) + \sum_{n>N} |x_n| < 1 \\
&\implies \sum_{n=1}^{N} |x_n| - \sum_{n=1}^{N} |y_n| + \sum_{n>N} |x_n| < 1 \\
&\iff \sum_{n\in\mathbb{N}} |x_n| - \sum_{n=1}^{N} |y_n| < 1 \\
&\iff \sum_{n\in\mathbb{N}} |x_n| < 1 + \sum_{n=1}^{N} |y_n|.
\end{aligned}
$$

Since $\sum_{n\in\mathbb{N}} |x_n|$ is bounded, $x \in \ell_1$.

For the reverse direction, we are to prove that $\ell_1 \subseteq \mathrm{cl}(c_{00})$. Let x be an arbitrary element in ℓ_1. For $i \in \mathbb{N}$, define $x^i = \{x^i_j\}_{j\in\mathbb{N}}$ as $x^i_j = x_j$ for $j \leq i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}, x^i \in c_{00}$. Then

$$
\begin{aligned}
&\lim_{i\in\mathbb{N}} d_1(x^i, x) \\
=\ &\lim_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |x^i_j - x_j| \\
=\ &\lim_{i\in\mathbb{N}} \sum_{j>i} |x^i_j - x_j| \\
=\ &\lim_{i\in\mathbb{N}} \sum_{j>i} |x_j| \\
=\ &0.
\end{aligned}
$$

That is, $\lim_{i\in\mathbb{N}} d_1(x^i, x) = 0$. So $\lim_{i\in\mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

■

PROPOSITION 4.2.3. The closure of c_{00} in the space $(\mathbb{R}^\omega, d_\infty)$ is c_0.

Proof. For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq c_0$. Let x be an arbitrary element in $\mathrm{cl}(c_{00})$. Let ε be an arbitrary positive real number. Since $x \in \mathrm{cl}(c_{00})$, there exists another element y in c_{00} such that $d_\infty(x,y) < \varepsilon$. That is, $\forall j \in \mathbb{N}, |x_j - y_j| < \varepsilon$. Since $y \in c_{00}$, $\exists N \in \mathbb{N}$ such that $\forall j > N$, $y_j = 0$. So $\forall j > N, |x_j| < \varepsilon$. That is,

$$\forall \varepsilon > 0, \quad , \exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon.$$

By definition of convergence of limits, $\lim_{j \in \mathbb{N}} x_j = 0$. So $x \in c_0$.

For the reverse direction, we are to prove that $c_0 \subseteq \mathrm{cl}(c_{00})$. Let x be an arbitrary element in c_0. For $i \in \mathbb{N}$, define x^i as $x^i_j = x_j$ for $j \leq i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}$, $x^i \in c_{00}$. Let ε be an arbitrary positive real number. Since $x \in c_0$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon/2.$$

Let $i > N$. Then

$$\begin{aligned}
& d_\infty(x^i, x) \\
= & \sup_{j \in \mathbb{N}} |x^i_j - x_j| \\
= & \sup_{j > i} |x^i_j - x_j| \\
= & \sup_{j > i} |x_j| \\
\leq & \ \varepsilon/2 < \varepsilon.
\end{aligned}$$

That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall i > N, \quad d_\infty(x^i, x) < \varepsilon.$$

By definition of convergence of sequences, $\lim_{i \in \mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

■

PROPOSITION 4.2.4. Let $A := \left\{ \{x_n\}_{n \in \mathbb{N}} \in c_{00} : \sum_{n \in \mathbb{N}} x_n = 0 \right\}$. Then A is a subset of ℓ^1 and is closed in (ℓ^1, d_1). i.e. $\mathrm{cl}(A) = A$ in (ℓ^1, d_1).

Proof. Let $x = \{x^i\}_{i \in \mathbb{N}}$ be a sequence in ℓ^1, where each $x^i = \{x^i_j\}_{j \in \mathbb{N}}$ is an element in A, that converges in (ℓ^1, d_1). Say $\lim_{i \to \infty} x^i = x^\infty$.

First I claim that $x^\infty \in c_{00}$.

Now I claim that $\sum_{j \in \mathbb{N}} x^\infty_j = 0$. i.e. $x^\infty \in A$. Since $x^\infty \in c_{00}$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad x^\infty_j = 0.$$

Define $y_i := \sum_{j=1}^N x^i_j$. Define $y_\infty := \sum_{j=1}^N x^\infty_j$. It is easy to see that $\lim_{i \in \mathbb{N}} y_i = y_\infty$. Assume for the sake of contradiction that $y_\infty \neq 0$. i.e. $\{y_i\}_{i \in \mathbb{N}}$ does not converge to 0. Then

$$\exists \varepsilon_0 > 0, \quad \forall M \in \mathbb{N}, \quad \exists i_0 > M, \quad |y_{i_0} - 0| = |y_{i_0}| \geq \varepsilon_0. \tag{1}$$

Since $\lim_{i\to\infty} x^i = x^\infty$,

$$\exists M_0 \in \mathbb{N}, \quad \forall i > M_0, \quad d_1(x^i, x^\infty) < \varepsilon_0. \tag{2}$$

Consider statement (1) for a particular M, M_0, we have

$$\exists i_0 > M_0, \quad |y_{i_0}| \geq \varepsilon_0. \tag{3}$$

That is,

$$\Big|\sum_{j=1}^{N} x_j^{i_0}\Big| \geq \varepsilon_0. \tag{3'}$$

Consider statement (2) for a particular i, i_0, we have

$$d_1(x^{i_0}, x^\infty) < \varepsilon_0. \tag{4}$$

From statement (4) we can derive:

$$\begin{aligned}
& d_1(x^{i_0}, x^\infty) < \varepsilon_0 \\
\iff & \sum_{j\in\mathbb{N}} |x_j^{i_0} - x_j^\infty| < \varepsilon_0 \\
\iff & \sum_{j=1}^{N} |x_j^{i_0} - x_j^\infty| + \sum_{j>N} |x_j^{i_0} - x_j^\infty| < \varepsilon_0 \\
\implies & \sum_{j>N} |x_j^{i_0} - x_j^\infty| < \varepsilon_0 \\
\iff & \sum_{j>N} |x_j^{i_0} - 0| < \varepsilon_0 \\
\iff & \sum_{j>N} |x_j^{i_0}| < \varepsilon_0 \\
\implies & \Big|\sum_{j>N} x_j^{i_0}\Big| < \varepsilon_0 \\
\implies & \Big|\sum_{j>N} x_j^{i_0}\Big| < \varepsilon_0 \\
\iff & \Big|\sum_{j\in\mathbb{N}} x_j^{i_0} - \sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0 \\
\iff & \Big|0 - \sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0 \\
\iff & \Big|\sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0.
\end{aligned}$$

This contradicts to statement (3'). So the original assumption that $y_\infty \neq 0$ is false. i.e. $y_\infty = 0$. It follows that $\sum_{j\in\mathbb{N}} x_j^\infty = 0$. This completes the proof.

■

4.3 Hölder's Inequality

THEOREM 4.1 (Hölder's Inequality)**.** Let $\mathfrak{X} = \mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ be vectors in $\mathfrak{X}$. Then $\forall p, q \in (1, +\infty) : 1/p + 1/q = 1$, $\|xy\|_1 \leq \|x\|_p \|y\|_q$. i.e.,

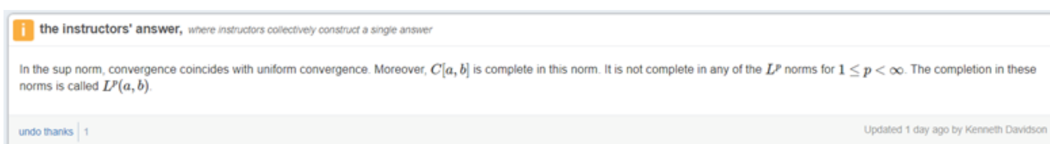
$$\sum_{i=1}^n |x_i y_i| \leq (\sum_{i=1}^n |x_i|^p)^{1/p} (\sum_{i=1}^n |y_i|^q)^{1/q}.$$

Chapter 5

Function Spaces

5.1 The $\mathcal{L}^p$ Norm

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$



the instructors' answer, *where instructors collectively construct a single answer*

In the sup norm, convergence coincides with uniform convergence. Moreover, $C[a,b]$ is complete in this norm. It is not complete in any of the L^p norms for $1 \leq p < \infty$. The completion in these norms is called $L^p(a,b)$.

undo thanks | 1

Updated 1 day ago by Kenneth Davidson

Chapter 6

Quotient Spaces

6.1 Definitions

DEFINITION (Quotient Space)**.** Let $\mathfrak{V}$ be a vector space. Let $\mathfrak{W}$ be a subspace of $\mathfrak{V}$. We define a **quotient space**, denoted by $\mathfrak{V}/\mathfrak{W}$, to be a set $\{v + \mathfrak{W} : v \in \mathfrak{V}\}$ with operations

$$(v_1 + \mathfrak{W}) + (v_2 + \mathfrak{W}) := (v_1 + v_2) + \mathfrak{W} \text{ and}$$

$$\kappa(v + \mathfrak{W}) := (\kappa v) + \mathfrak{W}.$$

DEFINITION (Quotient Map)**.** Let $\mathfrak{X}$ be a vector space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define the **quotient map** on $\mathfrak{X}$ with respect to $\mathfrak{M}$, denoted by $q_{\mathfrak{M}}$, to be a function from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$ given by

$$q_{\mathfrak{M}}(x) := [x] = x + \mathfrak{M}$$

6.2 Quotient Spaces with Seminorms

DEFINITION (Seminorm on Quotient Spaces)**.** Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define a **seminorm** on $\mathfrak{X}/\mathfrak{M}$ to be a function from $\mathfrak{X}/\mathfrak{M}$ to $\mathbb{R}$ given by

$$p(x + \mathfrak{M}) := \inf\{\|x + m\| : m \in \mathfrak{M}\}.$$

PROPOSITION 6.2.1. Seminorms on quotient spaces are indeed seminorms.

PROPOSITION 6.2.2. A seminorm on a quotient space $\mathfrak{X}/\mathfrak{M}$ is a norm if and only if $\mathfrak{M}$ is closed.

PROPOSITION 6.2.3 (Quotient maps are contractive)**.** Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a closed linear subspace of $\mathfrak{X}$. Then

$$\forall x \in \mathfrak{X}, \quad \|x + \mathfrak{M}\|_{\mathfrak{X}/\mathfrak{M}} \leq \|x\|_{\mathfrak{X}}.$$

PROPOSITION 6.2.4. Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a closed linear subspace of $\mathfrak{X}$. Let q denote the canonical quotient map from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$. Then q is a continuous under the norm topology.

Proof. Since q is contractive, q is continuous. ■

6.3 Quotient Spaces with Topologies

DEFINITION (Quotient Toplogy)**.** Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. Let $\mathcal{W}$ be a <u>closed</u> subspace of $\mathcal{V}$. We define the **quotient topology** on the quotient space $\mathcal{V}/\mathcal{W}$ as

$$\{G \subseteq \mathcal{V}/\mathcal{W} : q^{-1}(G) \in \mathcal{T}\}.$$

PROPOSITION 6.3.1. The quotient topology is compatible with the quotient space.

PROPOSITION 6.3.2. The quotient topology is Hausdorff.

PROPOSITION 6.3.3. The quotient map is continuous under the quotient topology.

PROPOSITION 6.3.4. Then

- map. i.e.,
$$\forall \text{ open set } W \subseteq \mathfrak{X}/\mathfrak{M}, \quad q^{-1}(W) \text{ is open in } \mathfrak{X}.$$
- q is an open map. i.e.,
$$\forall \text{ open set } G \subseteq \mathfrak{X}, \quad q(G) \text{ is open in } \mathfrak{X}/\mathfrak{M}.$$

Chapter 7

Banach Space

7.1 Definition

DEFINITION (Banach Space)**.** Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let d be the metric induced by $\|\cdot\|$. We say that $\mathfrak{X}$ is a **Banach space** if $(\mathfrak{X}, d)$ is a complete metric space. i.e., we define a Banach space to be a complete normed linear space.

7.2 Examples of Banach Space

EXAMPLE 7.2.1. $(\mathcal{C}([0,1], \mathbb{F}), \|\cdot\|_\infty)$ is a Banach space.

EXAMPLE 7.2.2 (Disc Algebra)**.** Define $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Define $\mathcal{A}(\mathbb{D}) := \{f \in \mathcal{C}(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is holomorphic }\}$. Define $\|\cdot\|_\infty$ by $\|f\|_\infty := \sup_{z \in \overline{\mathbb{D}}} |f(z)|$. Then $(\mathcal{A}(\mathbb{D}), \|\cdot\|_\infty)$ is a Banach space.

EXAMPLE 7.2.3. Let (X, Ω, μ) be a measure space. Let p be a number in $[1, +\infty)$. Define

$$\mathcal{L}^p(X, \mu) := \text{span}\{f : X \to [0, +\infty] \mid f \text{ is measurable and } \int_X |f|^p < +\infty\}.$$

Define an equivalence relation on $\mathcal{L}^p(X,\mu)$ by $f \equiv g$ if and only if

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

Define a space $L^p(X,\mu) := \mathcal{L}^p(X,\mu)/\equiv$. Then $L^p(X,\mu)$ is a Banach space when equipped with the norm

$$\|[f]\|_p := \left(\int_X |f|^p\right)^{1/p}.$$

EXAMPLE 7.2.4. Let $\mathcal{P}_{\mathbb{C}}[0,1]$ denote the set of all polynomials with complex coefficients. For each $p \in [1,+\infty)$, define a norm

$$\|f\|_p := \left(\int_0^1 |f|^p\right)^{1/p}.$$

For $p = +\infty$, define a norm

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(z)|.$$

7.3 Properties

PROPOSITION 7.3.1. Let $(\mathfrak{X}, \|\cdot\|)$ be a normed vector space over field $\mathbb{F}$. Then $(\mathfrak{X}, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in $\mathfrak{X}$ is summable.

Proof. For one direction, assume that $\mathfrak{X}$ is a Banach space. We are to prove that any absolutely summable series in $\mathfrak{X}$ is summable. Let $\sum_{n\in\mathbb{N}} x_n$ be an absolutely summable series. i.e., $\sum_{n\in\mathbb{N}} \|x_n\| < +\infty$. Define for each $n \in \mathbb{N}$ a vector y_n as $y_n := \sum_{i=1}^n x_i$. Let $\varepsilon > 0$ be arbitrary. Then $\exists N \in \mathbb{N}$ such that $\forall n > N$, $\sum_{i=n}^\infty \|x_i\| < \varepsilon$. Let $n > m > N$ be arbitrary. Then

$$\begin{aligned}\|y_n - y_m\| &= \|\sum_{i=1}^n x_i - \sum_{i=1}^m x_i\| = \|\sum_{i=m+1}^n x_i\| \\ &\leq \sum_{i=m+1}^n \|x_i\| < \sum_{i=m+1}^\infty \|x_i\| \\ &< \varepsilon.\end{aligned}$$

That is, $\|y_n - y_m\| < \varepsilon$. So $(y_n)_{n\in\mathbb{N}}$ is Cauchy. Since $\mathfrak{X}$ is a Banach space and $(y_n)_{n\in\mathbb{N}}$ is Cauchy, it converges. So $\sum_{n\in\mathbb{N}} x_n$ is summable.

For the reverse direction, assume that every absolutely summable series in $\mathfrak{X}$ is summable. We are to prove that $\mathfrak{X}$ is a Banach space. Let $(y_n)_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\mathfrak{X}$. Then $\forall n \in \mathbb{N}$, $\exists N_n \in \mathbb{N}$ such that $\forall k, l \geq N_n$, $\|y_k - y_l\| < \frac{1}{2^n}$. Assume that $N_1 < N_2 < \ldots$. Define $x_1 := y_{N_1}$. Define for each $n \in \mathbb{N}$ a vector x_{n+1} as $x_{n+1} := y_{N_{n+1}} - y_{N_n}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \|x_n\| &= \|x_1\| + \sum_{n=1}^{\infty} \|y_{N_{n+1}} - y_{N_n}\| < \|x_1\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \|x_1\| + 1 < +\infty.\end{aligned}$$

So $\sum_{n\in\mathbb{N}} x_n$ is absolutely summable. By assumption, it is summable. i.e., $(y_n)_{n\in\mathbb{N}}$ converges. Since any Cauchy sequence in $\mathfrak{X}$ converges, $\mathfrak{X}$ is complete and hence a Banach space.

■

PROPOSITION 7.3.2 (Stability of Banach Spaces Under Quotients)**.** Let $\mathfrak{X}$ be a Banach space. Let $\mathcal{M}$ be a closed subspace of $\mathcal{M}$. Then the quotient space $\mathcal{X}/\mathcal{M}$ is again a Banach space.

Proof. **Proof Approach 1**.

Let $(q(x_n))_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\mathfrak{X}/\mathcal{M}$. We are to prove that it converges.

■

Proof. **Proof Approach 2**.

Let q denote the canonical quotient map. Let $\sum_{n\in\mathbb{N}} q(x_n)$ be an arbitrary absolutely summable series in $\mathcal{X}/\mathcal{M}$. Since $\|q(x_n)\|$ is defined to be $\|q(x_n)\| := \inf\{\|x_n + m\| : m \in \mathbb{M}\}$, $\exists m_n \in \mathcal{M}$ such that $\|x_n + m_n\| < \|q(x_n)\| + \frac{1}{2^n}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \|x_n + m_n\| &= \sum_{n=1}^{\infty} \left[\|q(x_n)\| + \frac{1}{2^n}\right] = \sum_{n-1}^{\infty} \|q(x_n)\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} \|q(x_n)\| + 1 < +\infty.\end{aligned}$$

So $\sum_{n\in\mathbb{N}}(x_n + m_n)$ is absolutely summable. Since $\mathfrak{X}$ is a Banach space, $\sum_{n\in\mathbb{N}}(x_n + m_n)$ is summable. Say $\sum_{n\in\mathbb{N}}(x_n + m_n) = x_\bullet$. Then

$$\sum_{n=1}^{\infty} q(x_n) = \sum_{n=1}^{\infty} q(x_n + m_n) = \lim_{N\to\infty} \sum_{n=1}^{N} q(x_n + m_n) = \lim_{N\to\infty} q(\sum_{n=1}^{N}(x_n + m_n))$$

$$= q(\lim_{N\to\infty}\sum_{n=1}^{N}(x_n + m_n)) = q(x_\bullet).$$

So $\sum_{n\in\mathbb{N}} q(x_n)$ is summable. Since any absolutely summable series in $\mathfrak{X}/\mathcal{M}$ is summable, $\mathcal{X}/\mathcal{M}$ is complete. ∎

PROPOSITION 7.3.3. Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a closed subspace of $\mathfrak{X}$. If $\mathcal{M}$ and $\mathfrak{X}/\mathcal{M}$ are both complete, then $\mathfrak{X}$ is a Banach space.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\mathfrak{X}$. We are to prove that it converges. Let q denote the canonical quotient map. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathfrak{X}$, $(q(x_n))_{n\in\mathbb{N}}$ is Cauchy in $\mathfrak{X}/\mathcal{M}$. Since $\mathfrak{X}/\mathcal{M}$ is a Banach space and $(q(x_n))_{n\in\mathbb{N}}$ is Cauchy, $(q(x_n))_{n\in\mathbb{N}}$ converges. Say $\lim_{n\in\mathbb{N}} q(x_n) = q(x_\bullet)$ for some $x_\bullet \in \mathfrak{X}$. By definition of norms in the quotient space, for $n \in \mathbb{N}$, we can choose $m_n \in \mathcal{M}$ such that $\|x_\bullet - x_n - m_n\| \le \|q(x_\bullet) - q(x_n)\| + \frac{1}{n}$. So

$$\lim_{n\in\mathbb{N}} \|x_\bullet - x_n - m_n\| \le \lim_{n\in\mathbb{N}} \|q(x_\bullet) - q(x_n)\| + \lim_{n\in\mathbb{N}} \frac{1}{n} = 0 + 0 = 0.$$

So $(x_n + m_n)_{n\in\mathbb{N}}$ converges to $x_\bullet$. So $(x_n + m_n)_{n\in\mathbb{N}}$ is Cauchy. Since $(x_n)_{n\in\mathbb{N}}$ and $(x_n + m_n)_{n\in\mathbb{N}}$ are both Cauchy, $(m_n)_{n\in\mathbb{N}}$ is Cauchy. Since $\mathcal{M}$ is a Banach space and $(m_n)_{n\in\mathbb{N}}$ is Cauchy, $(m_n)_{n\in\mathbb{N}}$ converges. Say $\lim_{n\in\mathbb{N}} m_N = m_\bullet$. So

$$\begin{aligned}\lim_{n\in\mathbb{N}} x_n &= \lim_{n\in\mathbb{N}}((x_n + m_n) - m_n) = \lim_{n\in\mathbb{N}}(x_n + m_n) - \lim_{n\in\mathbb{N}} m_n \\ &= x_\bullet - m_\bullet.\end{aligned}$$

So $(x_n)_{n\in\mathbb{N}}$ converges. Since any Cauchy sequence in $\mathfrak{X}$ converges, $\mathfrak{X}$ is a Banach space. ∎

PROPOSITION 7.3.4. Any Banach space with a Schauder basis has to be separable.

7.4 Construction of Banach Spaces

DEFINITION. Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be two Banach spaces over field $\mathbb{K}$. Let $p \in [1, +\infty)$. We define

$$\mathfrak{X} \oplus_p \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x, y)\|_p := (\|x\|_{\mathfrak{X}}^p + \|y\|_{\mathfrak{Y}}^p)^{1/p}.$$

For $p = +\infty$, we define

$$\mathfrak{X} \oplus_\infty \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x, y)\|_\infty := \max(\|x\|_{\mathfrak{X}}, \|y\|_{\mathfrak{Y}}).$$

- Note that the norms behave similarly to what the p norm would do.
- We can similarly define the direct sum of finitely many Banach spaces.

PROPOSITION 7.4.1. $\|\cdot, \cdot\|_p$ is a norm on $\mathfrak{X} \oplus_p \mathfrak{Y}$.

PROPOSITION 7.4.2. $\mathfrak{X} \oplus_p \mathfrak{Y}$ is complete with respect to $\|\cdot, \cdot\|_p$.

7.5 Unconditional Convergence in Banach Spaces

DEFINITION (Unconditional Convergence)**.** Let $\mathfrak{X}$ be a Banach space. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a set of vectors in $\mathfrak{X}$. Let $\mathcal{F}$ be the collection of all finite subsets of Λ, partially ordered by inclusion. Define a net $(y_F)_{F \in \mathcal{F}}$ on $\mathcal{F}$ by $y_F := \sum_{\lambda \in F} x_\lambda$. We say that the series $\sum_{\lambda \in \Lambda} x_\lambda$ is **unconditional convergent** if the net $(y_F)_{F \in \mathcal{F}}$ converges.

PROPOSITION 7.5.1 (Equivalent Formulations of Unconditional Convergence)**.** Let $\mathfrak{X}$ be a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathfrak{X}$. Then the following conditions are equivalent.

(1) For any permutation π of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges.

(2) For any subsequence indexing $(k_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} x_{k_n}$ converges.

(3) $\forall\varepsilon > 0$, $\exists\mu \in \mathbb{N}$ such that for all finite subsets F of $\mathbb{N} \setminus \{1..\mu\}$, we have $\|\sum_{n\in F} x_n\| < \varepsilon$.

(4) $\exists y \in \mathfrak{X}$ such that $\forall\varepsilon > 0$, there is a finite subset F_0 of $\mathbb{N}$ such that for all finite F

such that $F_0 \subseteq F \subseteq \mathbb{N}$, we have $\| \sum_{n\in F} x_n - y\| < \varepsilon$.

(5) For any sequence $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \alpha_n x_n$ converges.

(6) For any bounded sequence $(\beta_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \beta_n x_n$ converges.

Proof. **Proof of (1) $\implies$ (5).**

Assume that for any permutation π of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges. We are to prove that for any sequence $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \alpha_n x_n$ converges. Assume for the sake of contradiction that there is some $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \alpha_n x_n$ diverges. i.e., $\exists \varepsilon_0 > 0$ such that $\forall N \in \mathbb{N}$, $\exists k_N > l_N > N$ such that

$$\| \sum_{n=k_N}^{l_N} \alpha_n x_n \| \geq \varepsilon_0. \tag{*}$$

For $N = 1$, find k_1 and l_1. For $N = l_1$, find k_2 and l_2. In general, for $N = l_n$, find k_{n+1} and l_{n+1}. Then we have $k_1 < l_1 < k_2 < l_2 < ....$ For each n, there is an $m_n \in [k_n, l_n]$ and a permutation π_n of $[k_n, l_n]$ such that $\pi_n(i) \in [k_n, m_n]$ if $\alpha_i = 1$ and $\pi_n(i) \in (m_n, l_n]$ if $\alpha_i = -1$. Define a permutation π of $\mathbb{N}$ as $\pi(i) := i$ if $\forall n \in \mathbb{N}$, $i \notin [k_n, l_n]$; and $\pi(i) := \pi_n(i)$ if $i \in [k_n, l_n]$. By assumption, for π, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges. So for ε_0, $\exists N \in \mathbb{N}$ such that $\forall j > i > N$, $\| \sum_{n=i}^{j} x_n \| < \varepsilon_0/2$. So

$$\begin{aligned}
\| \sum_{n=k_N}^{l_N} \alpha_n x_n \| &= \| \sum_{n=k_N}^{m_N} \alpha_n x_n + \sum_{n=m_N+1}^{l_N} \alpha_n x_n \| \\
&= \| \sum_{n=k_N}^{m_N} x_n - \sum_{n=m_N+1}^{l_N} x_n \| \\
&\leq \| \sum_{n=k_N}^{m_N} x_n \| + \| \sum_{n=m_N+1}^{l_N} x_n \| \\
&< \varepsilon_0/2 + \varepsilon_0/2 = \varepsilon_0.
\end{aligned}$$

That is,

$$\| \sum_{n=k_N}^{l_N} \alpha_n x_n \| < \varepsilon_0. \tag{**}$$

Notice (*) and (**) contradict. So the assumption that there is some $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \alpha_n x_n$ diverges does not hold. i.e., for any sequence $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \alpha_n x_n$ converges. ∎

Proof. **Proof of (5) $\implies$ (2).**

Assume that for any sequence $(\alpha_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \alpha_n x_n$ converges. We are to prove that for any subsequence indexing $(k_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} x_{k_n}$ converges. Let $(k_n)_{n\in\mathbb{N}}$ be an arbitrary subsequence indexing. Consider $(\alpha_n)_{n\in\mathbb{N}}$ be given by $\alpha_n := 1$ for all $n \in \mathbb{N}$. Then $\sum_{n\in\mathbb{N}} \alpha_n x_n = \sum_{n\in\mathbb{N}} x_n$ converges. Consider $(\alpha_n)_{n\in\mathbb{N}}$ be given by $\alpha_n := 1$ for $n \in \{k_i\}_{i\in\mathbb{N}}$; and $\alpha_n := -1$ for $n \notin \{k_i\}_{i\in\mathbb{N}}$. Then $\sum_{n\in\mathbb{N}} \alpha_n x_n = \sum_{n\in\{k_i\}_{i\in\mathbb{N}}} x_n - \sum_{n\notin\{k_i\}_{i\in\mathbb{N}}} x_n$ converges Notice

$$\sum_{n\in\mathbb{N}} x_{k_n} = \frac{1}{2}\sum_{n\in\mathbb{N}} x_n + \frac{1}{2}\left(\sum_{n\in\{k_i\}_{i\in\mathbb{N}}} x_n - \sum_{n\notin\{k_i\}_{i\in\mathbb{N}}} x_n\right).$$

So $\sum_{n\in\mathbb{N}} x_{k_n}$ converges. ■

Proof. **Proof of (2) $\implies$ (3).**

Assume that for any subsequence indexing $(k_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} x_{k_n}$ converges. We are to prove that $\forall\varepsilon > 0$, $\exists\mu \in \mathbb{N}$ such that for all finite subsets F of $\mathbb{N} \setminus \{1..\mu\}$, we have $\|\sum_{n\in F} x_n\| < \varepsilon$. Assume for the sake of contradiction that $\exists\varepsilon_0 > 0$ such that $\forall\mu \in \mathbb{N}$, there is some finite subset F of $\mathbb{N} \setminus \{1..\mu\}$ such that $\|\sum_{n\in F} x_n \geq \varepsilon_0$. For $\mu = 1$, find $F_1 \subseteq \mathbb{N} \setminus \{1..\mu\}$ finite. For $\mu = \max\{F_1\}$, find $F_2 \subseteq \mathbb{N} \setminus \{1..\mu\}$ finite. In general, for $\mu = \max\{F_n\}$, find $F_{n+1} \subseteq \mathbb{N} \setminus \{1..\mu\}$ finite. Then we have that the F_n's are disjoint. Define a subsequence indexing $(k_n)_{n\in\mathbb{N}}$ as $(k_n)_{n\in\mathbb{N}} := \bigcup_{n\in\mathbb{N}} F_n$. By assumption, for $(k_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} x_{k_n}$ converges. So for ε_0, $\exists N \in \mathbb{N}$ such that $\forall j > i > N$,

$$\|\sum_{n=i}^{j} x_{k_n}\| < \varepsilon_0. \tag{*}$$

So for N, there is some finite subset F of $\mathbb{N} \setminus \{1..\mu\}$ such that

$$\|\sum_{n\in F} x_n\| \geq \varepsilon_0.$$

Notice $F = \{k_n\}_{n=i_N}^{j_N}$ for some i_N and j_N. So (*) and (**) contradict. So the assumption that $\exists\varepsilon_0 > 0$ such that $\forall\mu \in \mathbb{N}$, there is some finite subset F of $\mathbb{N} \setminus \{1..\mu\}$ such that $\|\sum_{n\in F} x_n \geq \varepsilon_0$ does not hold. i.e., $\forall\varepsilon > 0$, $\exists\mu \in \mathbb{N}$ such that for all finite subsets F of $\mathbb{N} \setminus \{1..\mu\}$, we have $\|\sum_{n\in F} x_n\| < \varepsilon$. ■

Proof. **Proof of (3) $\implies$ (1).**

Assume that $\forall\varepsilon > 0$, $\exists\mu \in \mathbb{N}$ such that for all finite subsets F of $\mathbb{N} \setminus \{1..\mu\}$, we have $\|\sum_{n\in F} x_n\| < \varepsilon$. We are to prove that for any permutation π of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges. Assume for the sake of contradiction that there is some permutation π of $\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ diverges. i.e., $\exists\varepsilon_0 > 0$ such that $\forall N \in \mathbb{N}$, $\exists l_N > k_N > N$ such that $\|\sum_{n=k_N}^{l_N} x_{\pi(n)}\| \geq \varepsilon_0$. Let μ be an arbitrary element of $\mathbb{N}$. Define N as $N := \max\{\pi^{-1}(n)\}_{n=1}^{\mu}$. For N, find $l_N > k_N > N$ such that $\|\sum_{n=k_N}^{l_N} x_{\pi(n)}\| \geq \varepsilon_0$. Define a set F as $F := \{\pi(n)\}_{n=k_N}^{l_N}$. So $F \subseteq \mathbb{N} \setminus \{1..\mu\}$. Then $\|\sum_{n\in F} x_n\| = \|\sum_{n=k_N}^{l_N} x_n\| \geq \varepsilon_0$. So $\exists\varepsilon_0 > 0$ such that $\forall\mu \in \mathbb{N}$, there is some finite subset F of $\mathbb{N} \setminus \{1..\mu\}$ such that

$\|\sum_{n\in F} x_n\| \geq \varepsilon_0$. This contradicts to the assumption. So the assumption that there is some permutation π of $\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ diverges does not hold. So for any permutation π of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges. ∎

Chapter 8

Hilbert Space

8.1 Definition

DEFINITION (Hilbert Space)**.** We define a **Hilbert space**, denoted by $\mathcal{H}$, to be a complete inner product space.

8.2 Examples of Hilbert Space

EXAMPLE 8.2.1. Let (X, μ) be a measure space. Then $L^2(X, \mu)$ is a Hilbert space with inner product given by

$$\langle f, g\rangle := \int_X f(x)\overline{g(x)}d\mu(x).$$

EXAMPLE 8.2.2. $\ell^2(\mathbb{K}) = \left\{(x_i)_{i\in\mathbb{N}} \in \mathbb{K}^\infty : \sum_{i\in\mathbb{N}} |x_i|^2 < +\infty\right\}$ is a Hilbert space with inner product given by

$$\langle (x_i)i \in \mathbb{N}, (y_i)_{i\in\mathbb{N}}\rangle := \sum_{i\in\mathbb{N}} x_n\overline{y_n}.$$

8.3 Properties of Hilbert Space

PROPOSITION 8.3.1. Let $\mathcal{H}$ be a Hilbert space. Let S be a non-empty set in the space. Then $S^{\perp\perp} = \text{clspan}(S)$.

Proof. For one direction, we are to prove that $\text{clspan}(S) \subseteq S^{\perp\perp}$.

For the reverse direction, we are to prove that $S^{\perp\perp} \subseteq \text{clspan}(S)$. Assume for the sake of contradiction that $\exists x \in S^{\perp\perp}$ with $x \neq 0$ such that $x \notin \text{clspan}(S)$. Say $x = m_1 + m_2$ for some $m_1 \in \text{clspan}(S)$ and some $m_2 \in \text{clspan}(S)^{\perp}$. Note that $\text{clspan}(S)^{\perp} = S^{\perp}$. So $m_2 \in S^{\perp}$. Since $x \in S^{\perp\perp}$ and $m_2 \in S^{\perp}$, we should have $\langle x, m_2 \rangle = 0$. However,

$$\begin{aligned}
\langle x, m_2 \rangle &= \langle m_1 + m_2, m_2 \rangle \\
&= \langle m_1, m_2 \rangle + \langle m_2, m_2 \rangle \\
&= 0 + \langle m_2, m_2 \rangle \\
&> 0, \text{ since } m_2 \neq 0.
\end{aligned}$$

This leads to a contradiction. So $S^{\perp\perp} \subseteq \text{clspan}(S)$. ∎

THEOREM 8.1 (The Riesz Representation Theorem)**.** Let $\mathcal{H}$ be a Hilbert space over field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Suppose that $\mathcal{H} \neq \{0\}$. Then for any $\varphi \in \mathcal{H}^*$, $\exists y \in \mathcal{H}$ such that

$$\forall x \in \mathcal{H}, \quad \varphi(x) = \langle x, y \rangle.$$

Proof. Define for each $y \in \mathcal{H}$ a function $\beta_y \in \mathcal{H}^*$ by $\beta_y(x) := \langle x, y \rangle$. We are to prove that $\mathcal{H}^* = \{\beta_y : y \in \mathcal{H}\}$. It is easy to verify that each β_y is linear and bounded. So $\forall y \in \mathcal{H}$, $\beta_y \in \mathcal{H}^*$. i.e., $\{\beta_y : y \in \mathcal{H}\} \subseteq \mathcal{H}^*$. Define a map Θ from $\mathcal{H}$ to $\mathcal{H}^*$ as $\Theta(y) := \beta_y$. It is easy to verify that Θ is linear.

$$\begin{aligned}
\|\Theta(y)\| = \|\beta_y\| &= \sup\{\beta_y(x) : \|x\| = 1\} \\
&= \sup\{\langle x, y \rangle : \|x\| = 1\} \\
&\leq \sup\{\|x\|\|y\| : \|x\| = 1\} \\
&= \|y\|.
\end{aligned}$$

That is, $\|\Theta(y)\| \leq \|y\|$. So $\|\Theta\| \leq 1$. On the other hand, consider an arbitrary point $y_0 \in \mathcal{H}$ with $y_0 \neq 0$:

$$\begin{aligned}
\|\Theta\| &= \sup\left\{\frac{\|\Theta(y)\|}{\|y\|} : y \neq 0\right\} \\
&\geq \left.\frac{\|\Theta(y)\|}{\|y\|}\right|_{y=y_0}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\|\Theta(y_0)\|}{\|y_0\|} \\
&= \frac{\|\beta_{y_0}\|}{\|y_0\|} \\
&= \frac{1}{\|y_0\|} \sup\left\{\frac{\|\beta_{y_0}(y)\|}{\|y\|} : y \neq 0\right\} \\
&\geq \frac{1}{\|y_0\|} \frac{\|\beta_{y_0}(y_0)\|}{\|y_0\|} \\
&\geq \frac{1}{\|y_0\|} \frac{\|y_0\|^2}{\|y_0\|} \\
&= 1.
\end{aligned}
$$

That is, $\|\Theta\| \geq 1$. So $\|\Theta\| = 1$. So Θ is isometric. It immediately follows that Θ is injective. Now it remains to prove that Θ is surjective. Let $\varphi \in \mathcal{H}^*$. If $\varphi = 0$, then $\varphi = \Theta(0)$ and we are done. Otherwise, let $\mathcal{M} := \ker(\varphi)$. Then we have $\operatorname{codim} \mathcal{M} = \dim \mathcal{M}^\perp = 1$. Take $e \in \mathcal{M}^\perp$ such that $\|e\| = 1$. Let P denote the orthogonal projection onto $\mathcal{M}$. Then $1 - P$ is the orthogonal projection onto $\mathcal{M}^\perp$.

$$x = Px + (1-P)x = Px + \langle x, e\rangle e.$$

So for $x \in \mathcal{H}$,

$$\varphi(x) = \varphi(Px) + \langle x, e\rangle \varphi(e) = \langle x, \overline{\varphi(e)} e\rangle = \beta_y(x)$$

where $y := \overline{\varphi(e)} e$. Hence $\varphi = \beta_y$. So Θ is surjective. This completes the proof. ■

PROPOSITION 8.3.2 (Stability of Hilbert Spaces Under Quotients)**.** Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then the quotient space $\mathcal{H}/\mathcal{M}$ is again a Hilbert space.

Chapter 9

Operators

9.1 Bounded Operators

DEFINITION (Bounded Operator)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let T be a linear map from $\mathfrak{X}$ to $\mathfrak{Y}$. We say that T is a **bounded operator** if

$$\exists k \in \mathbb{R}, \quad \forall x \in \mathfrak{X}, \quad \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}.$$

DEFINITION (Operator Norm)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let T be a bounded operator from $\mathfrak{X}$ to $\mathfrak{Y}$. We define the **operator norm** of T, denoted by $\|T\|$, to be the number given by

$$\|T\| := \inf\{k \in \mathbb{R} : \forall x \in \mathfrak{X}, \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}\}.$$

PROPOSITION 9.1.1.

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{X}, \|x\|_{\mathfrak{X}} = 1\}.$$

PROPOSITION 9.1.2. Let X and Y be normed linear spaces. Let T be a linear map from X to Y. Then T is bounded if and only if T is continuous.

9.2 Examples of Bounded Operators

EXAMPLE 9.2.1 (The Multiplication Operator)**.** Let $\mathfrak{X} = \big(\mathcal{C}([0,1],\mathbb{C}), \|\cdot\|_\infty\big)$. Let f be a function in $\mathfrak{X}$. We define the **multiplication operator** on $\mathfrak{X}$, w.r.t. f, denoted by M_f, as

$$M_f(g) = fg.$$

Then M_f is bounded and $\|M_f\| = \|f\|_\infty$.

Proof. Let g be an arbitrary function in $\mathfrak{X}$. Then

$$\begin{aligned}
\|M_f g\|_\infty &= \|fg\|_\infty \\
&= \sup_{x\in[0,1]} |f(x)g(x)| \\
&= \sup_{x\in[0,1]} |f(x)||g(x)| \\
&\leq \sup_{x\in[0,1]} |f(x)| \sup_{x\in[0,1]} |g(x)| \\
&= \|f\|_\infty \|g\|_\infty.
\end{aligned}$$

That is, $\|M_f g\|_\infty \leq \|f\|_\infty \|g\|_\infty$. So $\|f\|_\infty$ is an element of the set $S = \{k \in \mathbb{R} : \forall g \in \mathfrak{X}, \|M_f g\|_{\mathfrak{Y}} \leq k\|g\|_{\mathfrak{X}}\}$. So $\|M_f\| = \inf(S) \leq \|f\|_\infty$. Consider g_0 given by $g_0(x) = 1$. Then g_0 in $\mathfrak{X}$. Then

$$\|M_f g_0\|_\infty = \|fg_0\|_\infty = \|f\|_\infty = \|f\|_\infty \|g_0\|_\infty.$$

Let k be an arbitrary element in S. Assume for the sake of contradiction that $k < \|f\|_\infty$. Then

$$\begin{aligned}
\|f\|_\infty \|g_0\|_\infty &= \|M_f g_0\|_\infty \\
&\leq k\|g_0\|_\infty \\
&< \|f\|_\infty \|g_0\|_\infty.
\end{aligned}$$

This leads to a contradiction. So $\forall k \in S$, $k \geq \|f\|_\infty$. So $\|f\|_\infty$ is a lower bound for the set S. So $\|M_f\| = \inf(S) \geq \|f\|_\infty$. Since $\|M_f\| \leq \|f\|_\infty$ and $\|M_f\| \geq \|f\|_\infty$, we get $\|M_f\| = \|f\|_\infty$. ∎

EXAMPLE 9.2.2 (The Volterra Operator)**.** Let $\mathfrak{X} = \big(\mathcal{C}([0,1],\mathbb{C}), \|\cdot\|_\infty\big)$. Define

$$Vf := x \mapsto \int_0^x f(t)dt.$$

Then the Volterra Operator is bounded and $\|V\| \leq 1$.

Proof. Let f be an arbitrary function in $\mathfrak{X}$ with $\|f\|_\infty = 1$. Then $\forall x \in [0,1]$,

$$\begin{aligned}
|Vf(x)| &= \left|\int_0^x f(t)dt\right| \\
&\leq \int_0^x |f(t)|dt \\
&\leq \int_0^x \sup_{t\in[0,1]} |f(t)|dt \\
&= \int_0^x \|f\|_\infty dt \\
&= \int_0^x 1dt \\
&= x.
\end{aligned}$$

That is, $\forall x \in [0,1]$, $|Vf(x)| \leq 1$. So $\|Vf\|_\infty \leq 1$. Since $\forall f \in \mathfrak{X} : \|f\|_\infty = 1$, $\|Vf\|_\infty \leq 1$, we get $\|V\| \leq 1$. ∎

EXAMPLE 9.2.3 (The Diagonal Operator)**.** Let $\mathfrak{X} = \ell^2(\mathbb{N})$. Let

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & \ddots \end{bmatrix}.$$

Then D is bounded if and only if $(d_i)_{i\in\mathbb{N}}$ is bounded and $\|D\| = \|(d_i)_{i\in\mathbb{N}}\|_\infty$.

Proof. <u>Case 1.</u>

$$\begin{aligned}
\|Dx\|_2^2 &= \sum_{i\in\mathbb{N}} |d_i x_i|^2 \\
&=\leq \sum_{i\in\mathbb{N}} \|(d_j)_{j\in\mathbb{N}}\|_\infty |x_i|^2 \\
&= \|(d_j)_{j\in\mathbb{N}}\|_\infty \sum_{i\in\mathbb{N}} |x_i|^2
\end{aligned}$$

$$= \|(d_j)_{j\in\mathbb{N}}\|_\infty \|x\|_2^2.$$

Case 2.
If $(d_i)_{i\in\mathbb{N}} \notin \ell^\infty$, $\exists (d_{n_i})_{i\in\mathbb{N}} \to \infty$.

$$\begin{aligned}\|De_{n_i}\|_2 &= \|d_{n_i} e_{n_i}\|_2 \\ &= |d_{n_i}| \|e_{n_i}\|_2 \\ &= |d_{n_i}|.\end{aligned}$$

So $\|D\| \geq \|De_{n_i}\|_2 \to \infty$. ■

EXAMPLE 9.2.4 (Weighted Shifts)**.**

- Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(w_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an **unilateral forward weighted shift** W on $\mathcal{H}$ as
$$W(x_n) := (0, w_1x_1, w_2x_2, w_3x_3, ...).$$
i.e.,
$$W = \begin{bmatrix} 0 & & & & \\ w_1 & 0 & & & \\ & w_2 & 0 & & \\ & & w_3 & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.$$
Then W is bounded and $\|W\| = \sup\{|w_n| : n \in \mathbb{N}\}$.

- Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(v_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an **unilateral backward weighted shift** V on $\mathcal{H}$ as
$$V(x_n) := (v_1x_2, v_2x_3, v_3x_4, ...).$$
Then V is bounded and $\|V\| = \sup\{|v_n| : n \in \mathbb{N}\}$.

- Let $\mathcal{H} = \ell^2_{\mathbb{Z}}$. Let $(u_n)_{n\in\mathbb{Z}} \in \ell^\infty_{\mathbb{Z}}$. We define a **bilateral weighted shift** U on $\mathcal{H}$ as
$$U(x_n) := (u_{n-1}x_{n-1})_{n\in\mathbb{Z}}.$$
Then U is bounded and $\|U\| = \sup\{|u_n| : n \in \mathbb{Z}\}$.

EXAMPLE 9.2.5 (The Composition Operators)**.** Let $\mathfrak{X} = \mathcal{C}([0,1], \mathbb{C})$. Let $\varphi \in$

$\mathcal{C}([0,1],[0,1])$. We define the **composition operator** on $\mathfrak{X}$, denoted by C_φ as

$$C_\varphi(f) := f \circ \varphi.$$

Then C_φ is contractive.

Proof.

$$\begin{aligned}\|C_\varphi(f)\| &= \sup_{x\in[0,1]} |(f\circ\varphi)(x)| \\ &\leq \|f\|_\infty.\end{aligned}$$

■

9.3 The Space of Bounded Operators

PROPOSITION 9.3.1. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Then $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ is a vector space and the operator norm is a norm on $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$.

PROPOSITION 9.3.2. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ be the space of bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$. Then if $\mathfrak{Y}$ is complete, $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ is complete.

PROPOSITION 9.3.3. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$. Then $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y},\|\cdot\|_1)$ if and only if $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y},\|\cdot\|_2)$.

9.4 Invertible Bounded Operators

PROPOSITION 9.4.1. Let $(\mathfrak{X},\|\cdot\|_1)$ be a Banach space. Let $S \in \mathcal{B}(\mathfrak{X})$ be a bounded linear map that is invertible. Define a norm $\|\cdot\|_2$ on $\mathfrak{X}$ as

$$\|x\|_2 := \|Sx\|_1.$$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proof. On one hand, since S is bounded, $\exists c_1$ such that $\forall x \in \mathfrak{X}$, $\|Sx\|_1 \le c_1\|x\|_1$. That is, $\|x\|_2 \le c_1\|x\|_1$.

On the other hand, since S is invertible, S^{-1} exists and is also bounded. Since S^{-1} is bounded, $\exists c_2$ such that $\forall x \in \mathfrak{X}$, $\|S^{-1}x\|_1 \le c_2\|x\|_1$. Consider $x = Sx$, we get $\forall x \in \mathfrak{X}$, $\|S^{-1}Sx\|_1 \le c_2\|Sx\|_1$. That is, $\|x\|_1 \le c_2\|x\|_2$.

So $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ■

PROPOSITION 9.4.2. Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space. Let S be a map in $\mathcal{B}(\mathfrak{X})$ that is invertible. Then

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

Proof.

$$\begin{aligned}
\|S^{-1}\| &= \sup\{\|S^{-1}x\| : \|x\| = 1\} \\
&= \sup\left\{\frac{\|S^{-1}x\|}{\|x\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{\|S^{-1}Sx\|}{\|Sx\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{\|x\|}{\|Sx\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{1}{\|Sx\|} : \|x\| = 1\right\} \\
&= (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.
\end{aligned}$$

That is,

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

■

Chapter 10

Dual Space

10.1 Definitions

DEFINITION (Linear Functional)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{K}$. We define a **linear functional** on $\mathfrak{X}$ to be a linear map from $\mathfrak{X}$ to $\mathbb{K}$.

DEFINITION (Algebraic Dual)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{K}$. We define the **algebraic dual** of $\mathfrak{X}$, denoted by $\mathfrak{X}^{\#}$. to be the vector space of all linear functionals on $\mathfrak{X}$.

DEFINITION (Topological Dual)**.** Let $\mathfrak{X}$ be a topological vector space over field $\mathbb{K}$. We define the **topological dual** of $\mathfrak{X}$, denoted by $\mathfrak{X}^*$, to be the vector space of all continuous linear functionals on $\mathfrak{X}$.

PROPOSITION 10.1.1. Let $\mathfrak{X}$ be a normed linear space. Then there exists a contractive map from $\mathfrak{X}$ to its double dual $\mathfrak{X}^{**}$.

10.2 Examples of Dual Spaces

EXAMPLE 10.2.1. $(c_0(\mathbb{N}))^*$ is isometrically isomorphic to $\ell^1(\mathbb{N})$.

EXAMPLE 10.2.2. $(\ell^1(\mathbb{N}))^*$ is isometrically isomorphic to $\ell^\infty(\mathbb{N})$.

10.3 Properties

PROPOSITION 10.3.1. Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let $g, f_1..f_n \in \mathcal{V}^\#$ where $n \in \mathbb{N}$. Then $g \in \operatorname{span}\{f_i\}_{i=1}^n$ if and only if $\bigcap_{i=1}^n \ker(f_i) \subseteq \ker(g)$.

Proof. **Forward Direction**: Assume that $g \in \operatorname{span}\{f_i\}_{i=1}^n$. We are to prove that $\bigcap_{i=1}^n \ker(f_i) \subseteq \ker(g)$. Let x be an arbitrary element of $\bigcap_{i=1}^n \ker(f_i)$. Since $g \in \operatorname{span}\{f_i\}_{i=1}^n$, there exist scalars $\lambda_1..\lambda_n$ such that $g = \sum_{i=1}^n \lambda_i f_i$. Then

$$\begin{aligned} g(x) &= (\sum_{i=1}^n \lambda_i f_i)(x) = \sum_{i=1}^n \lambda_i f_i(x) \\ &= \sum_{i=1}^n \lambda_i \cdot 0, \text{ since } \forall i = 1..n, x \in \ker(f_i) \\ &= 0. \end{aligned}$$

That is, $g(x) = 0$. So $x \in \ker(g)$. So $\bigcap_{i=1}^n \ker(f_i) \subseteq \ker(g)$.

Backward Direction: Assume that $\bigcap_{i=1}^n \ker(f_i) \subseteq \ker(g)$. We are to prove that $g \in \operatorname{span}\{f_i\}_{i=1}^n$. Assume without loss of generality that $\{f_i\}_{i=1}^n$ are linearly independent. Define a set $\mathcal{N}$ by $\mathcal{N} := \bigcap_{i=1}^n \ker(f_i)$. Then $\dim(\mathcal{V}/\mathcal{N}) \leq n$. Define for each $i = 1..n$ a function $F_i : \mathcal{V}/\mathcal{N} \to \mathbb{K}$ by $F_i(x + \mathcal{N}) := f_i(x)$. Then clearly each F_i is linear. Since $\{f_i\}_{i=1}^n$ are linearly independent, $\{F_i\}_{i=1}^n$ are linearly independent. So $\dim(\mathcal{V}/\mathcal{N}) \geq n$. So $\dim(\mathcal{V}/\mathcal{N}) = n$. So $\{F_i\}_{i=1}^n$ is a basis for $(\mathcal{V}/\mathcal{N})^\#$. Define a function $G : \mathcal{V}/\mathcal{N} \to \mathbb{K}$ by $G(x + \mathcal{N}) := g(x)$. Then clearly, G is linear. So $\exists k_1..k_n \in \mathbb{K}$ such that $G = \sum_{i=1}^n k_i F_i$. It follows that $g = \sum_{i=1}^n k_i f_i$. So $g \in \operatorname{span}\{f_i\}_{i=1}^n$.

■

PROPOSITION 10.3.2. Let $\mathcal{V}$ be a topological vector space over field $\mathbb{K}$. Let $\rho \in \mathcal{V}^\#$. Then $\rho \in \mathcal{V}^*$ if and only if $\ker(\rho)$ is a closed set.

Proof. **Forward Direction**: Assume that $\rho \in \mathcal{V}^*$. I will show that $\ker(\rho)$ is closed. Notice $\{0\}$ is closed in $\mathbb{K}$. Since $\rho \in \mathbb{V}^*$, ρ is continuous. So $\rho^{-1}(\{0\})$ is closed. Note that $\rho^{-1}(\{0\}) = \ker(\rho)$. So $\ker(\rho)$ is closed.

Backward Direction: Assume that $\ker(\rho)$ is a closed set. I will show that $\rho \in \mathcal{V}^*$. If $\rho = 0$, then we are done. Otherwise, assume that $\rho \neq 0$. Define a map $\varphi : \mathcal{V}/\ker(\rho) \to \mathbb{K}$ by $\varphi(x + \ker(\rho)) := \rho(x)$. Then clearly φ is linear. Since $\dim(\mathcal{V}/\ker(\rho)) = 1$ and $\dim(\mathbb{K}) = 1$, φ is continuous. Let q denote the canonical quotient map from $\mathcal{V}$ to $\mathcal{V}/\ker(\rho)$. Then q is continuous. Note that $\rho = \varphi \circ q$. So ρ is continuous.

$\blacksquare$

Chapter 11

Balanced Sets and Absorbing Sets

11.1 Definitions

DEFINITION (Balanced Sets)**.** Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We say that S is **balanced** if

$$\forall a \in \mathbb{F} : |a| \leq 1, \quad aS \subseteq S.$$

DEFINITION (Balanced Hull)**.** Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We define the **balanced hull** of S, denoted by $\operatorname{balhull}(S)$, to be the smallest balanced set containing S.

DEFINITION (Balanced Core)**.** Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We define the **balanced core** of S, denoted by $\operatorname{balcore}(S)$, to be the largest balanced set contained in S.

11.2 Properties

PROPOSITION 11.2.1. Let X be a vector space over field $\mathbb{F}$. Let B be a balanced subset of X. Then

$$\forall a, b \in \mathbb{F} : |a| \leq |b|, \quad aB \subseteq bB.$$

PROPOSITION 11.2.2. Balanced sets are path connected.

PROPOSITION 11.2.3 (Act on Other Properties)**.**

- The balanced hull of a compact set is compact.
- The balanced hull of a totally bounded set is totally bounded.
- The balanced hull of a bounded set is bounded.

PROPOSITION 11.2.4 (Act on Other Properties)**.**

- The balanced core of a closed set is closed.

PROPOSITION 11.2.5. Let X be a vector space over field $\mathbb{F}$. Let a be a scalar in field $\mathbb{F}$. Then

$$a \operatorname{balhull}(S) = \operatorname{balhull}(aS).$$

11.3 Stability of Balance

PROPOSITION 11.3.1 (Set Operations)**.**

- The union of balanced sets is also balanced.
- The intersection of balanced sets is also balanced.

PROPOSITION 11.3.2. The convex hull of a balanced set is also a balanced set.

PROPOSITION 11.3.3 (Topological Operations)**.** Let $\mathcal{V}$ be a topological vector space. Let E be a balanced set. Then $\mathrm{cl}(E)$ is also a balanced set.

PROPOSITION 11.3.4 (Linear Mappings)**.**

- The scalar multiple of a balanced set is also balanced.
- The (Minkowski) sum of two balanced sets is also balanced.
- The image of a balanced set under a linear operator is also balanced.
- The inverse image of a balanced set under a linear operator is also balanced.

11.4 Absorbing Sets

DEFINITION (Absorbing Sets)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let S be a subset of X. We say that S is **absorbing** if

$$\forall x \in \mathfrak{X}, \quad \exists r \in \mathbb{R} : r > 0, \quad \forall c \in \mathbb{F} : |c| \geq r, \quad x \in cS.$$

i.e.,

$$\bigcup_{n \in \mathbb{N}} nS = \mathfrak{X}.$$

PROPOSITION 11.4.1. Every absorbing set contains the origin.

PROPOSITION 11.4.2. Let $\mathcal{V}$ be a topological vector space. Let $U \in \mathcal{U}_0$. Then U is absorbing.

Chapter 12

Topological Vector Space

12.1 Definitions

DEFINITION (Compatible)**.** Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let $\mathcal{T}$ be a topology on $\mathcal{V}$. We say that $\mathcal{T}$ is **compatible** with the vector space structure on $\mathcal{V}$ if the addition and scalar multiplication operations on $\mathcal{V}$ are continuous.

DEFINITION (Topological Vector Space)**.** We define a **topological vector space** to be a vector space with a compatible Hausdorff topology.

12.2 Examples

EXAMPLE 12.2.1. Let $\mathfrak{X}$ be a normed linear space. Then $\mathfrak{X}$ is a topological vector space with the topology induced by the norm.

Proof.

$$\begin{aligned}
\|\sigma(x_\alpha, y_\alpha) - \sigma(x, y)\| &= \|(x_\alpha + y_\alpha) - (x + y)\| \\
&= \|(x_\alpha - x) + (y_\alpha - y)\| \\
&\leq \|x_\alpha - x\| + \|y_\alpha - y\| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

So σ is continuous.

$$\begin{aligned}\|\mu(k_\alpha, x_\alpha) - \mu(k, x)\| &= \|k_\alpha x_\alpha - kx\| \\ &= \|k_\alpha x_\alpha - kx_\alpha + kx_\alpha - kx\| \\ &\leq \|k_\alpha x_\alpha - kx_\alpha\| + \|kx_\alpha - kx\| \\ &= |k_\alpha - k|\|x_\alpha + |k|\|x_\alpha - x\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

So μ is continuous. ■

PROPOSITION 12.2.1. Normed linear spaces are Hausdorff.

12.3 Properties

PROPOSITION 12.3.1. Let $(\mathcal{V}, \tau)$ be a topological vector space. Let $U \in \mathcal{U}_0$ be a neighborhood of 0 in $\mathcal{V}$. Then

- $\exists N \in \mathcal{U}_0$ such that $N + N \subseteq U$.
- $\exists M \in \mathcal{U}_0$ and $\exists \varepsilon > 0$ such that $\forall 0 < |k| < \varepsilon$, we have $kM \subseteq U$.
-

PROPOSITION 12.3.2. Let $\mathcal{V}$ be a topological vector space. Every neighborhood of 0 contains a open balanced neighborhood of 0.

Proof. Let U be an arbitrary element of $\mathcal{U}_0^{\mathcal{V}}$. Let μ denote the multiplication operation on $\mathcal{V}$. Then μ is continuous and hence $\mu^{-1}(U)$ is a neighborhood of $(0, 0) \in \mathbb{K} \times \mathcal{V}$. So there exist an $r > 0$ and an element $N \in \mathcal{U}_0^{\mathcal{V}}$ that is open such that $\mathrm{ball}(0, r) \times N \subseteq \mu^{-1}(U)$. Define a set M as $M := \bigcup_{k:0<|k|<r} kN$. Since $\mathrm{ball}(0, r) \times N \subseteq \mu^{-1}(U)$, we have $M \subseteq U$. Since $M = \bigcup_{k:0<|k|<r} kN$ and $N \in \mathcal{T}$, we have $M \in \mathcal{T}$. Since $M \supseteq \frac{r}{2}N$, $\frac{r}{2}N \in \mathcal{T}$, and $0 \in \frac{r}{2}N$, we have $M \in \mathcal{U}_0^{\mathcal{V}}$. Let a be an arbitrary element in $\mathbb{K}$ such that $|a| < 1$. Then

$$aM = a \bigcup_{k:0<|k|<r} kN = \bigcup_{k:0<|k|<r} akN = \bigcup_{k:0<|k|<ar} kN \subseteq \bigcup_{k:0<|k|<r} kN = M.$$

So M is balanced. ■

PROPOSITION 12.3.3. Closure of a linear subspace is a linear subspace.

Proof. Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. Let $\mathcal{W}$ be a linear subspace of $\mathcal{V}$. We are to prove that $\mathrm{cl}(\mathcal{W})$ is a linear subspace.

Let x and y be arbitrary elements of $\mathrm{cl}(\mathcal{W})$. Then there exists a net $(x_\lambda, y_\lambda)_{\lambda \in \Lambda}$ that converges to (x, y). Since the addition operation σ is continuous, we have $\lim_{\lambda \in \Lambda}(x_\lambda + y_\lambda) = x + y$. Since $\mathcal{W}$ is a linear subspace, $x_\lambda + y_\lambda \in \mathcal{W}$. So $x + y \in \mathrm{cl}(\mathcal{W})$.

Let x be an arbitrary element of $\mathrm{cl}(\mathcal{W})$. Let k be an arbitrary element in $\mathbb{K}$. Then there exists a net $(k\lambda, x_\lambda)_{\lambda \in \Lambda}$ that converges to (k, x). Since the scalar multiplication operation μ is continuous, we have $\lim_{\lambda \in \Lambda}(k_\lambda x_\lambda) = kx$. Since $\mathcal{W}$ is a linear subspace, $k_\lambda x_\lambda \in \mathcal{W}$. So $kx \in \mathrm{cl}(\mathcal{W})$. ■

12.4 Operation on Sets in a Topological Vector Space

PROPOSITION 12.4.1 (Stability under Linear Combinations)**.** Let X be a normed vector space over $\mathbb{F}$. Let K be a compact set in the space. Let C be a closed set in the space. Then $\forall \alpha, \beta \in \mathbb{F}$, the set S given by $S := \alpha K + \beta C$ is closed.

Proof. The case where $\beta = 0$ is trivial. I will assume $\beta \neq 0$. Let $\alpha, \beta \in \mathbb{F}$ be arbitrary. Let $\{s_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in S that converges. Say the limit is s_∞. Since $s_i \in S$ for any $i \in \mathbb{N}$ and $S = \alpha K + \beta C$, $s_i = \alpha k_i + \beta c_i$ for some $k_i \in K$ and some $c_i \in C$, for any $i \in \mathbb{N}$. Since $\{k_i\}_{i \in \mathbb{N}}$ is a sequence in K and K is compact, there exists a convergent subsequence $\{k_i\}_{i \in I}$ of $\{k_i\}_{i \in \mathbb{N}}$ in K. Say $\{k_i\}_{i \in I}$ converges to $k_\infty \in K$. Since $\{s_i\}_{i \in \mathbb{N}}$ converges to s_∞, $\{s_i\}_{i \in I}$ also converges to s_∞. Since $s_i = \alpha k_i + \beta c_i$, $c_i = \beta^{-1}(s_i - \alpha k_i)$. Define $c_\infty := \beta^{-1}(s_\infty - \alpha k_\infty)$ Since $\{s_i\}_{i \in I}$ converges to s_∞ and $\{k_i\}_{i \in I}$ converges to k_∞ and $c_i = \beta^{-1}(s_i - \alpha k_i)$, $\{c_i\}_{i \in I}$ converges to c_∞. Since $\{c_i\}_{i \in I}$ is a sequence in C and converges to c_∞ and C is closed, $c_\infty \in C$. Since $s_\infty = \alpha k_\infty + \beta c_\infty$ and $k_\infty \in K$ and $c_\infty \in C$, $s_\infty \in \alpha K + \beta C$. Since for any sequence in S that converges, the limit is also in S, S is closed. ■

Remark. *The sum of two closed sets may not be closed.*

Proof. **Counter-example 1**

Consider $A := \{n : n \in \mathbb{N}\}$ and $B := \{n + \frac{1}{n} : n \in \mathbb{N}\}$.
(`https://math.stackexchange.com/questions/124130/sum-of-two-closed-sets-in-mathbb-r-is-closed`)
Their sum contains the sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ but does not contain 0.

Counter-example 2

Consider $A := \mathbb{R} \times \{0\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x, y > 0, xy \geq 1\}$. Their sum is $\mathbb{R} \times \mathbb{R}_{++}$. ■

PROPOSITION 12.4.2. Let $\mathfrak{X}$ be a normed vector space. Let S be a subset of $\mathfrak{X}$. Let p be a vector in $\mathfrak{X}$. Then we have the followings.

(1) $p + \text{int}(S) = \text{int}(p + S)$,

(2) $p + \text{cl}(S) = \text{cl}(p + S)$.

Proof of (1). For one direction, let x be an arbitrary point in the set $p + \text{int}(S)$. We are to prove that $x \in \text{int}(p + S)$. Since $x \in (p + \text{int}(S))$, $(x - p) \in \text{int}(S)$. Since $(x - p) \in \text{int}(S)$, by definition of interior, there exists a radius r such that

$$B(x - p, r) \subseteq S.$$

It follows that $B(x, r) \subseteq p + S$. Since there exists a radius r such that $B(x, r) \subseteq p + S$, by definition of interior,

$$x \in \text{int}(p + S).$$

For the reverse direction, let x be an arbitrary point in $\text{int}(p + S)$. We are to prove that $x \in p + \text{int}(S)$. Since $x \in \text{int}(p + S)$, by definition of interior, there exists a radius r such that

$$B(x, r) \subseteq (p + S).$$

It follows that $B(x - p, r) \subseteq S$. Since there exists a radius r such that $B(x - p, r) \subseteq S$, by definition of interior,

$$(x - p) \in \text{int}(S).$$

Since $(x - p) \in \text{int}(S)$, we get $x \in (p + \text{int}(S))$. ■

Proof of (2). For one direction, let x be an arbitrary point in the set $p + \text{cl}(S)$. We are to prove that $x \in \text{cl}(p + S)$. Since $x \in (p + \text{cl}(S))$, we get $(x - p) \in \text{cl}(S)$. Since $(x - p) \in \text{cl}(S)$, by definition of closure, for any radius r, we have

$$B(x - p, r) \cap S \neq \emptyset.$$

It follows that $B(x, r) \cap (p + S) \neq \emptyset$. Since for any radius r, $B(x, r) \cap (p + S) \neq \emptyset$, by definition of closure, we get

$$x \in \text{cl}(p + S).$$

For the reverse direction, let x be an arbitrary point in $\mathrm{cl}(p+S)$. We are to prove that $x \in (p + \mathrm{cl}(S))$. Since $x \in \mathrm{cl}(p+S)$, by definition of closure, for any radius r, we have

$$B(x,r) \cap (p+S) \neq \emptyset.$$

It follows that $B(x-p,r) \cap S \neq \emptyset$. Since for any radius r, $B(x-p,r) \cap S \neq \emptyset$, by definition of closure, we get

$$(x-p) \in \mathrm{cl}(S).$$

Since $(x-p) \in \mathrm{cl}(S)$, we get $x \in (p + \mathrm{cl}(S))$. ■

PROPOSITION 12.4.3. Let $(V, \|\cdot\|)$ be a normed vector space. Let S be a subset of V. Let λ be a non-zero real number. Then

(1) $\lambda \,\mathrm{int}(S) = \mathrm{int}(\lambda S)$.

(2) $\lambda \,\mathrm{cl}(S) = \mathrm{cl}(\lambda S)$.

Proof of (1). For one direction, let x be an arbitrary point in $\lambda \,\mathrm{int}(S)$. We are to prove that $x \in \mathrm{int}(\lambda S)$. Since $x \in \lambda \,\mathrm{int}(S)$, we get $x/\lambda \in \mathrm{int}(S)$. Since $x/\lambda \in \mathrm{int}(S)$, by definition of interior, there exists a radius r such that

$$B(x/\lambda, r) \subseteq S.$$

Let y be an arbitrary point in $B(x, \lambda r)$. Since $y \in B(x, \lambda r)$, we get $\|y - x\| \leq \lambda r$. Since $\|y - x\| \leq \lambda r$, we get $\|y/\lambda - x/\lambda\| \leq r$. Since $\|y/\lambda - x/\lambda\| \leq r$, we get $y/\lambda \in B(x/\lambda, r)$. Since $y/\lambda \in B(x/\lambda, r)$ and $B(x/\lambda, r) \subseteq S$, we get $y/\lambda \in S$. Since $y/\lambda \in S$, we get $y \in \lambda S$. Since any point in $B(x, \lambda r)$ is also in λS, we get $B(x, \lambda r) \subseteq \lambda S$. Since there exists a radius r such that $B(x, \lambda r) \subseteq \lambda S$, by definition of interior, we get

$$x \in \mathrm{int}(\lambda S).$$

■

12.5 Finite-Dimensional Topological Vector Spaces

PROPOSITION 12.5.1. Let $\mathcal{V}$ be an n-dimensional topological vector space where

$n \in \mathbb{N}$. Then $\mathcal{V}$ is homeomorphic to $\mathbb{K}^n$ via the map

$$\sum_{i=1}^{n} k_i e_i \mapsto (k_i)_{i=1}^{n}.$$

COROLLARY 12.1. Let $\mathcal{V}$ be a finite-dimensional vector space. Then there is a unique topology $\mathcal{T}$ which makes $\mathcal{V}$ a topological vector space.

Chapter 13

Completeness

13.1 Cauchy Nets

DEFINITION (Cauchy Net)**.** Let $(\mathcal{V}, \tau)$ be a topological vector space. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathcal{V}$. We say that $(x_\lambda)_{\lambda\in\Lambda}$ is a **Cauchy net** if $\forall U \in \mathcal{U}_0$, $\exists\lambda_0 \in \Lambda$ such that $\forall\lambda_1, \lambda_2 \geq \lambda_0$, we have $x_{\lambda_1} - x_{\lambda_2} \in U$.

PROPOSITION 13.1.1. Convergent nets are Cauchy.

Proof. Let $\mathcal{V}$ be a topological vector space. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a convergent net with limit point x. Let U be an arbitrary element in $\mathcal{U}_0$. Let N be an element in $\mathcal{U}_0$ that is balanced and open and that $N - N \subseteq U$. Since $\lim_{\lambda\in\Lambda} x_\lambda = x$, $\exists\lambda_0 \in \Lambda$ such that $\forall\lambda \geq \lambda_0$, $x_\lambda - x \in N$. Let λ_1 and λ_2 be arbitrary elements that are $\geq \lambda_0$. Then

$$x_{\lambda_1} - x_{\lambda_2} = (x_{\lambda_1} - x) - (x_{\lambda_2} - x) \in N - N \subseteq U.$$

That is, $\forall U \in \mathcal{U}_0$, $\exists\lambda_0$ such that $\forall\lambda_1, \lambda_2 \geq \lambda_0$, $x_{\lambda_1} - x_{\lambda_2} \in U$. So $(x_\lambda)_{\lambda\in\Lambda}$ is Cauchy. ∎

13.2 Complete Topological Vector Spaces

DEFINITION (Cauchy Complete)**.** Let $(\mathcal{V}, \tau)$ be a topological vector space. We say that $\mathcal{V}$ is **Cauchy complete** if every Cauchy net in $\mathcal{V}$ converges in $\mathcal{V}$.

PROPOSITION 13.2.1. Let $\mathcal{V}$ be a topological vector space. Let $\mathcal{K}$ be a complete set in $\mathcal{V}$. Then $\mathcal{K}$ is closed in $\mathcal{V}$.

Chapter 14

Seminorms and Locally Convex Spaces

14.1 Preliminaries

DEFINITION (Sublinear Functional)**.** Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let f be a function from $\mathcal{V}$ to $\mathbb{R}$. We say that f is **sublinear** if it satisfies:

- Subadditivity:
$$\forall x, y \in \mathcal{V}, \quad f(x+y) \le f(x) + f(y).$$
- Positive Homogeneity:
$$\forall x \in \mathcal{V}, \forall \lambda \ge 0, \quad f(\lambda x) = \lambda f(x).$$

PROPOSITION 14.1.1. Every seminorm is a sublinear functional.

DEFINITION (Minkowski Functional)**.** Let $\mathcal{V}$ be a topological vector space. Let E be a convex neighborhood of 0. We define the **Minkowski functional** for E, denoted by p_E, to be a function from $\mathcal{V}$ to $\mathbb{R}$ given by

$$p_E(x) := \inf\{r > 0 : x \in rE\}.$$

PROPOSITION 14.1.2. Every Minkowski functional for a convex neighborhood of 0 is a sublinear functional.

PROPOSITION 14.1.3. Every Minkowski functional for a balanced convex neighborhood of 0 is a seminorm.

PROPOSITION 14.1.4. Let $\mathcal{V}$ be a topological vector space. Let E be an open convex neighborhood of 0. Then

$$E = \{x \in \mathcal{V} : p_E(x) < 1\}.$$

Proof. Let F denote the set $\{x \in \mathcal{V} : p_E(x) < 1\}$.

Forward Direction:

Let x be an arbitrary element of E. I will show that $x \in F$. Define a map $f : \mathbb{R} \to \mathcal{V}$ by $f(t) := tx$. Then f is continuous. Since E is open in $\mathcal{V}$ and $f : \mathbb{R} \to \mathcal{V}$ is continuous, we get $f^{-1}(E)$ is open in $\mathbb{R}$. Notice $x = f(1) \in E$. So $1 \in f^{-1}(E)$. Since $f^{-1}(E)$ is open and $1 \in f^{-1}(E)$, $\exists \delta > 0$ such that $1 + \delta \in f^{-1}(E)$. So $f(1 + \delta) \in E$. So $(1 + \delta)x \in E$. So $x \in \frac{1}{1+\delta}E$. So $p_E(x) \leq \frac{1}{1+\delta}$, which further, is < 1. So $x \in F$.

Backward Direction:

Let x be an arbitrary element of F. I will show that $x \in E$. Since $x \in F$, $p_E(x) < 1$. So $\exists r_0 < 1$ such that $x \in r_0E$, which further, is $\subseteq E$. So $x \in E$. ■

DEFINITION (Separating Family of Seminorms)**.** Let $\mathcal{V}$ be a vector space. Let Γ be a family of seminorms on $\mathcal{V}$. We say that Γ is **separating** if $\forall x \in \mathcal{V}$ such that $x \neq 0$, $\exists p \in \Gamma$ such that $p(x) \neq 0$.

DEFINITION (Separating Family of Linear Functionals)**.** Let $\mathcal{V}$ be a vector space. Let $\mathcal{L}$ be a collection of linear functionals on $\mathcal{V}$. Define for each $\varphi \in \mathcal{L}$ a seminorm τ_φ on $\mathcal{V}$ by $\tau_\varphi(x) := |\varphi(x)|$. We say that $\mathcal{L}$ is **separating** if the set Γ given by $\Gamma := \{\tau_\varphi : \varphi \in \mathcal{L}\}$ is a separating family of seminorms.

14.2 Locally Convex Space

DEFINITION (Locally Convex Space)**.** Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. We say that $\mathcal{T}$ is **locally convex** if it admits a base consisting of only convex sets.

PROPOSITION 14.2.1. Let $(\mathcal{V}, \mathcal{T})$ be a locally convex topological vector space. Let $\mathcal{W}$ be a closed subspace of $\mathcal{V}$. Then $\mathcal{V}/\mathcal{W}$ is a locally convex topological vector space in the quotient topology.

Proof. Clearly $\mathcal{V}/\mathcal{W}$ is a topological vector space. It suffices to show that $\mathcal{V}/\mathcal{W}$ admits a neighborhood base at 0 consisting of only convex sets. Let $q := \mathcal{V} \to \mathcal{V}/\mathcal{W}$ denote the canonical quotient map. Then q is linear, continuous and open. Let U be an arbitrary element in $\mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$. Then $q^{-1}(U) \in \mathcal{U}_0^{\mathcal{V}}$. Since $\mathcal{V}$ is locally convex, $\exists N \in \mathcal{U}_0^{\mathcal{V}}$ that is convex and that $N \subseteq q^{-1}(U)$. Define a set M as $M := q(N)$. Since q is open and $N \in \mathcal{U}_0^{\mathcal{V}}$, we have $M \in \mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$. Since q is linear and N is convex, M is convex. Since $N \subseteq q^{-1}(U)$, $M \subseteq U$. So $\forall U \in \mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$, $\exists M \in \mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$ that is convex and that $M \subseteq U$. So $\mathcal{V}/\mathcal{W}$ admits a neighborhood base at 0 consisting of only convex sets. ∎

THEOREM 14.1. Let $\mathcal{V}$ be a vector space. Let Γ be a separating family of seminorms on $\mathcal{V}$. Define a set $\mathcal{B}$ as

$$\mathcal{B} := \{N(x, F, \varepsilon) : x \in \mathcal{V}, \varepsilon > 0, F \subseteq \mathcal{F} \text{ is finite }\}$$

where $N(x, F, \varepsilon)$ is defined as

$$N(x, F, \varepsilon) := \{y \in \mathcal{V} : \forall p \in F, p(x - y) < \varepsilon\}.$$

Then $\mathcal{B}$ is a base for a locally convex topology $\mathcal{T}$ on $\mathcal{V}$. Moreover, each $p \in \Gamma$ is continuous.

THEOREM 14.2. Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. Then there exists a separating family Γ of seminorms on $\mathcal{V}$ that can generate $\mathcal{T}$.

EXAMPLE 14.2.1. The norm topology is exactly the locally convex topology generated by $\Gamma = \{\|\cdot\|\}$.

14.3 Strong Operator Topology

14.4 Weak Operator Topology

Chapter 15

The Hahn-Banach Theorem

15.1 The Extension Results

THEOREM 15.1 (The Hahn-Banach Theorem - 2)**.** Let $\mathcal{V}$ be a vector space. Let $\mathcal{M}$ be a linear manifold of $\mathcal{V}$. Let p be a seminorm on $\mathcal{V}$. Let f be a linear functional on $\mathcal{M}$. Suppose that $\forall m \in \mathcal{M}$, $|f(m)| \leq p(m)$. Then there exists a linear functional g on $\mathcal{V}$ such that $g|_{\mathcal{M}} = f$ and that $\forall x \in \mathcal{V}$, $|g(x)| \leq p(x)$.

COROLLARY 15.1. Let $\mathcal{V}$ be a locally convex space. Let $\mathcal{M}$ be a linear manifold of $\mathcal{V}$. Let $f \in \mathcal{M}^*$. Then $\exists g \in \mathcal{V}^*$ such that $g|_{\mathcal{M}} = f$.

THEOREM 15.2 (The Hahn-Banach Theorem - 3)**.** Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a linear manifold of $\mathfrak{X}$. Let $f \in \mathcal{M}^*$. Then $\exists g \in \mathfrak{X}^*$ such that $g|_M = f$ and that $\|g\| = \|f\|$.

COROLLARY 15.2. Let $\mathcal{V}$ be a locally convex space. Let $\{x_i\}_{i=1}^m$ be a linearly independent set of vectors in $\mathcal{V}$ where $m \in \mathbb{N}$. Let $k_1..k_m$ be arbitrary elements of $\mathbb{K}$. Then $\exists g \in \mathcal{V}^*$ such that $\forall i = 1..m$, $g(x_i) = k_i$.

COROLLARY 15.3. Let $\mathcal{V}$ be a locally convex space. Let $\mathcal{M}$ be a finite-dimensional linear manifold of $\mathcal{V}$. Then $\mathcal{M}$ is topologically complemented.

Proof. Let $\{m_i\}_{i=1}^n$ be a basis for $\mathcal{M}$ where $n = \dim(\mathcal{M})$. Then $\{m_i\}_{i=1}^n$ is a linearly independent set of vectors in $\mathcal{V}$. By Corollary 15.2, for each $i = 1..n$, $\exists \rho_i \in \mathcal{V}^*$ such that $\rho_i(m_j) = \delta_{i,j}$. Define $\mathcal{Y} := \bigcap_{i=1}^m \ker(\rho_i)$. Since the ρ_i's are continuous, the $\ker(\rho_i)$'s are closed. So $\mathcal{Y}$ is closed. Since $\dim(\mathcal{M}) < \infty$, $\mathcal{M}$ is closed.

Now I will show that $\mathcal{V} = \mathcal{M} + \mathcal{Y}$. Let v be an arbitrary element of $\mathcal{V}$. Define for $i = 1..n$ a scalar k_i as $k_i := \rho_i(v)$. Define a point m as $m := \sum_{i=1}^n k_i m_i$. Then $m \in \mathcal{M}$. Define a point y as $y := v - m$. Then $\forall i = 1..n$, we have

$$\begin{aligned}\rho_i(y) = \rho_i(v-m) = \rho_i(v - \sum_{j=1}^n k_j m_j) &= \rho_i(v) - \sum_{j=1}^n k_j \rho_i(m_j)\\ &= k_i - \sum_{j=1}^n k_j \delta_{i,j} = k_i - k_i = 0.\end{aligned}$$

That is, $\rho_i(y) = 0$. So $\forall i = 1..n$, $y \in \ker(\rho_i)$. So $y \in \bigcap_{i=1}^n \ker(\rho_i) = \mathcal{Y}$. So $\forall v \in \mathcal{V}$, $v = m + y$ where $m \in \mathcal{M}$ and $y \in \mathcal{Y}$. So $\mathcal{V} = \mathcal{M} + \mathcal{Y}$.

Now I will show that $\mathcal{M} \cap \mathcal{Y} = \{0\}$. Note that $0 \in \mathcal{M} \cap \mathcal{Y}$. Let z be an arbitrary element of $\mathcal{M} \cap \mathcal{Y}$. Since $z \in \mathcal{M}$, there exist scalars $\{r_j\}_{j=1}^n$ such that $z = \sum_{j=1}^n r_j m_j$. On one hand, since $z = \sum_{j=1}^n r_j m_j$, $\forall i = 1..n$, we have

$$\rho_i(z) = \rho_i(\sum_{j=1}^n r_j m_j) = \sum_{j=1}^n r_j \rho_i(m_j) = \sum_{j=1}^n r_j \delta_{i,j} = r_i.$$

That is, $\rho_i(z) = r_i$. On the other hand, since $z \in \mathcal{Y} = \bigcap_{i=1}^n \ker(\rho_i)$, $\forall i = 1..n$, we have $\rho_i(z) = 0$. So $\forall i = 1..n$, $r_i = 0$. So $z = \sum_{j=1}^n r_j m_j = 0$. So $\mathcal{M} \cap \mathcal{Y} = \{0\}$.

So $\mathcal{M}$ is topologically complemented by $\mathcal{Y}$. ■

COROLLARY 15.4. Let $\mathfrak{X}$ be a normed linear space. Let $x \in \mathfrak{X}$. Then

$$\|x\| = \max\{|x^*(x)| : x^* \in \mathfrak{X}^*, \|x^*\| \le 1\}.$$

i.e., $\exists x^* \in \mathfrak{X}^*$ with $\|x^*\| = 1$ such that $\|x\| = |x^*(x)|$.

COROLLARY 15.5. The canonical embedding $\mathfrak{J} : \mathfrak{X} \to \mathfrak{X}^{**}$ is an isometry.

Proof. Let x be an arbitrary element of $\mathfrak{X}$. We are to prove that $\|x\|_{\mathfrak{X}} = \|\mathfrak{J}x\|_{\mathfrak{X}^{**}}$. Let $\hat{x}$ denote $\mathfrak{J}x$. On one hand, for any $y^* \in \mathfrak{X}^*$, we have

$$|\hat{x}(y^*)| = |y^*(x)| \leq \|y^*\|\|x\|.$$

So $\|\hat{x}\| \leq \|x\|$. On the other hand, by Corollary 15.4, there exists $x^* \in \mathfrak{X}^*$ with $\|x^*\| \leq 1$ such that $|x^*(x)| = \|x\|$. So

$$\|\hat{x}\| \geq |\hat{x}(x^*)| = |x^*(x)| = \|x\|.$$

That is, $\|\hat{x}\| \geq \|x\|$. Since $\forall x \in \mathfrak{X}$, $\|x\| = \|\mathfrak{J}x\|$, we have that $\mathfrak{J}$ is an isometry. ∎

COROLLARY 15.6. Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{Y}$ be a closed subspace of $\mathfrak{X}$. Let $z \in \mathfrak{X} \setminus \mathfrak{Y}$. Then $\exists x^* \in \mathfrak{X}^*$ with $\|x^*\| = 1$ such that $x^*|_{\mathfrak{Y}} = 0$ and $x^*(z) = d(z, \mathfrak{Y})$.

Proof. Since $z \notin \mathfrak{Y}$, $\mathfrak{Y} \neq z + \mathfrak{Y}$. By Corollary 15.4, $\exists \xi^* \in (\mathfrak{X}/\mathfrak{Y})^*$ with $\|\xi^*\| = 1$ such that $|\xi^*(z + \mathfrak{Y})| = \|z + \mathfrak{Y}\| = d(z, \mathfrak{Y})$. Let q be the canonical quotient map from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{Y}$. Define a map from $\mathfrak{X}$ to $\mathbb{K}$ as $x^* := \xi^* \circ q$.

Show that $x^* \in \mathfrak{X}^*$:

Clearly x^* is linear. Recall that $\|\xi^*\| = 1$ and that q is a contraction map and hence $\|q\| \leq 1$. So $\|x^*\| \leq \|\xi^*\|\|q\| \leq 1$. So $x^* \in \mathfrak{X}^*$.

Show that $\|x^*\| = 1$:

Since $\|\xi^*\| = 1$, we can find a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathfrak{X}/\mathfrak{Y}$ such that $\forall n \in \mathbb{N}$, we have $\|t_n\| \leq 1$ and $1 - \frac{1}{n} < |\xi^*(t_n)| \leq 1$. So $\lim_{n\in\mathbb{N}} |\xi^*(t_n)| = 1$. Define for each $n \in \mathbb{N}$ a point $x_n \in \mathfrak{X}$ to be such that $q(x_n) = \frac{n}{n+1}t_n$. Then $\forall n \in \mathbb{N}$, we have

$$\|x_n + \mathfrak{Y}\| = \|q(x_n)\| = \|\frac{n}{n+1}t_n\| = \frac{n}{n+1}\|t_n\| < \|t_n\| \leq 1.$$

That is, $\|x_n + \mathfrak{Y}\| < 1$. So $\forall n \in \mathbb{N}$, $\exists y_n \in \mathfrak{Y}$ such that $\|x_n + y_n\| < 1$. On the other hand, we have

$$\begin{aligned}
\lim_{n\in\mathbb{N}} |x^*(x_n + y_n)| &= \lim_{n\in\mathbb{N}} |\xi^*(q(x_n + y_n))| = \lim_{n\in\mathbb{N}} |\xi^*(x_n + y_n + \mathfrak{Y})| = \lim_{n\in\mathbb{N}} \|x_n + y_n + \mathfrak{Y}\| \\
&= \lim_{n\in\mathbb{N}} \|x_n + \mathfrak{Y}\| = \lim_{n\in\mathbb{N}} |\xi^*(x_n + \mathfrak{Y})| = \lim_{n\in\mathbb{N}} |\xi^*(q(x_n))| \\
&= \lim_{n\in\mathbb{N}} |\xi^*(\frac{n}{n+1}t_n)| = \lim_{n\in\mathbb{N}} |\frac{n}{n+1}\xi^*(t_n)|, \text{ by linearity of } \xi^* \\
&= \lim_{n\in\mathbb{N}} \frac{n}{n+1} \cdot \lim_{n\in\mathbb{N}} |\xi^*(t_n)| = 1 \cdot 1 = 1.
\end{aligned}$$

That is, $\lim_{n\in\mathbb{N}} |x^*(x_n+y_n)| = 1$. Since $\forall n \in \mathbb{N}$, $\|x_n+y_n\| < 1$ and $\lim_{n\in\mathbb{N}} |x^*(x_n+y_n)| = 1$, we get $\|x^*\| \geq 1$. Recall that we have proved $\|\|x^*\| \leq 1$. So $\|\|x^*\| = 1$.

Show that $x^*|_{\mathfrak{Y}} = 0$:

Let y be an arbitrary element of $\mathfrak{Y}$. Then we have

$$x^*(y) = \xi^*(q(y)) = \xi^*(y + \mathfrak{Y}) = d(y, \mathfrak{Y}) = 0.$$

So $x^*|_{\mathfrak{Y}} = 0$.

Show that $x^*(z) = d(z, \mathfrak{Y})$:

Note that

$$x^*(z) = |\xi^*(q(z))| = |\xi^*(z + \mathfrak{Y})| = d(z, \mathfrak{Y}).$$

That is, $x^*(z) = d(z, \mathfrak{Y})$.

■

15.2 Separation Results

(bug)

PROPOSITION 15.2.1. Let $\mathcal{V}$ be a locally convex space over field $\mathbb{K}$. Let G be a non-empty, open, and convex set in $\mathcal{V}$. Suppose that $0 \notin G$. Then there exists a closed hyperplane $\mathcal{M}$ in $\mathcal{V}$ such that $G \cap \mathcal{M} = \emptyset$.

Proof.

Case 1: $\mathbb{K} = \mathbb{R}$.

Since $G \neq \emptyset$, take $x_0 \in G$. Define a set H as $H := x_0 - G$. Then H is non-empty, open, convex, and $0 \in H$. Let p_H denote the Minkowski functional on H. Since H is an open convex neighborhood of 0, $H = \{x \in \mathcal{V} : p_H(x) < 1\}$. Define a set $\mathcal{W}$ by $\mathcal{W} := \mathbb{R}x_0$. Then $\mathcal{W}$ is a linear manifold of $\mathcal{V}$. Define a map $f : \mathcal{W} \to \mathbb{R}$ by $f(kx_0) := kp_H(x_0)$. Then f is a linear functional on $\mathcal{W}$. Note that

$$\begin{aligned} f(kx_0) &= kp_H(x_0) = p_H(kx_0), \text{ for } k \geq 0, \text{ and} \\ f(kx_0) &= kp_H(x_0) < 0 \leq p_H(kx_0), \text{ for } k < 0. \end{aligned}$$

■

not finished

THEOREM 15.3 (The Hahn-Banach Theorem - 4)**.** Let $\mathcal{V}$ be a locally convex space. Let A and B be two non-empty, open, convex, and disjoint sets in $\mathcal{V}$. Then $\exists f \in \mathcal{V}^*$,

$\exists \kappa \in \mathbb{R}$ such that

$$\forall a \in A, b \in B, \quad \Re f(a) > \kappa > \Re f(b).$$

THEOREM 15.4 (The Hahn-Banach Theorem - 5)**.** Let $\mathcal{V}$ be a locally convex space. Let A and B be two non-empty, closed, convex, and disjoint sets in $\mathcal{V}$. Suppose B is compact. Then $\exists f \in \mathcal{V}^*$, $\exists \alpha, \beta \in \mathbb{R}$ such that

$$\forall a \in A, b \in B, \quad \Re f(a) \geq \alpha > \beta \geq \Re f(b).$$

COROLLARY 15.7. Let $\mathcal{V}$ be a locally convex space. Let A be a non-empty set in $\mathcal{V}$. Then the closed convex hull $\mathrm{clconv}(A)$ equals the intersection of all closed half-spaces that contain A.

Proof. Let Ω denote the set of all closed half-spaces that contain A.
Forward Direction:

Note that $\forall \mathcal{S} \in \Omega$, $\mathcal{S}$ is closed and convex. So $\bigcap_{\mathcal{S} \in \Omega} \mathcal{S}$ is closed and convex. Note also that $A \subseteq \bigcap_{\mathcal{S} \in \Omega} \mathcal{S}$. So $\mathrm{clconv}(A) \subseteq \bigcap_{\mathcal{S} \in \Omega} \mathcal{S}$.
Backward Direction:

Let z be an arbitrary element outside $\mathrm{clconv}(A)$. Then $\mathrm{clconv}(A)$ and $\{z\}$ are two non-empty, closed, convex, and disjoint sets and we have that $\{z\}$ is compact. By the Hahn-Banach theorem, version 5, $\exists f \in \mathcal{V}^*$, $\exists \alpha, \beta \in \mathbb{R}$ such that

$$\forall a \in \mathrm{clconv}(A), \quad \Re f(a) \geq \alpha > \beta \geq \Re f(z).$$

Define a set $\mathcal{S}_0$ by $\mathcal{S}_0 := \{x \in \mathcal{V} : \Re f(x) \geq \alpha\}$. Then $\mathcal{S}_0$ is a closed half-space of $\mathcal{V}$ and $z \notin \mathcal{S}_0$. So $z \notin \bigcap_{\mathcal{S} \in \Omega} \mathcal{S}$. So $\bigcap_{\mathcal{S} \in \Omega} \mathcal{S} \subseteq \mathrm{clconv}(A)$.

■

Chapter 16

Weak Topologies

16.1 Definitions

DEFINITION (Dual Pair)**.** Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let $\mathcal{L}$ be a separating family of linear functionals on $\mathcal{V}$. Suppose $\mathcal{L}$ is a linear manifold of $\mathcal{V}^{\#}$. We define a **dual pair** to be the pair $(\mathcal{V}, \mathcal{L})$.

DEFINITION (Weak Topology)**.** Let $\mathcal{V}$ be a locally convex space. Then by the Hahn-Banach theorem, $\mathcal{V}^*$ separates points. So $(\mathcal{V}, \mathcal{V}^*)$ is a dual pair. We define the **weak topology** on $\mathcal{V}$ to be the topology $\sigma(\mathcal{V}, \mathcal{V}^*)$ induced by the family $\mathcal{V}^*$.

DEFINITION (Weak* Topology)**.** Let $\mathcal{V}$ be a locally convex space. Then $\hat{\mathcal{V}}$ is a separating family of linear functionals on $\mathcal{V}^*$ and a linear manifold of $(\mathcal{V}^*)^{\#}$. So $(\mathcal{V}^*, \hat{\mathcal{V}})$ is a dual pair. We define the **week* topology** on $\mathcal{V}^*$ to be the topology $\sigma(\mathcal{V}^*, \hat{\mathcal{V}})$ induced by the family $\hat{\mathcal{V}}$.

16.2 Properties

(bug)

PROPOSITION 16.2.1. Let $\mathfrak{X}$ be a finite-dimensional Banach space. Then the norm, weak, and weak* topologies on $\mathfrak{X}$ all coincide.

PROPOSITION 16.2.2. Let $\mathfrak{X}$ be a Banach space. Let $\mathfrak{X}^*$ denote the dual space of $\mathfrak{X}$. Let τ_* denote the weak topology on $\mathfrak{X}^*$ induced by elements of $\mathfrak{X}$ as

$$\lim_\alpha x_\alpha^* = x^* \iff \forall x \in \mathfrak{X}, \lim_\alpha x_\alpha^*(x) = x^*(x).$$

Then $(\mathfrak{X}^*, \tau_*)$ is a topological vector space.

THEOREM 16.1. Let $\mathcal{V}, \mathcal{L})$ be a dual pair. Then $\mathcal{L} = (\mathcal{V}, \sigma(\mathcal{V}, \mathcal{L}))^*$.

Proof. **Forward Direction**: Let $f \in \mathcal{L}$. I will show that $f \in (\mathcal{V}, \sigma(\mathcal{V}, \mathcal{L}))^*$.

Backward Direction: Let $f \in (\mathcal{V}, \sigma(\mathcal{V}, \mathcal{L}))^*$. I will show that $f \in \mathcal{L}$. ∎

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Chapter 17

Equicontinuity in Metric Spaces

17.1 Definitions

DEFINITION ((Pointwise) Equicontinuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{F}$ be a collection of functions from X to Y. Let x_0 be a point in X. We say that $\mathcal{F}$ is ***(pointwise) equicontinuous*** at point x_0 if for any positive number ε, there exists some number $\delta(x_0, \varepsilon)$ such that for any function f in $\mathcal{F}$ and any point x in X, we have

$$d_Y(f(x), f(x_0)) < \varepsilon$$

whenever $d_X(x, x_0) < \delta(x_0, \varepsilon)$ is satisfied.

DEFINITION (Uniform Equicontinuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{F}$ be a collection of functions from X to Y. We say that $\mathcal{F}$ is ***uniformly equicontinuous*** if for any positive number ε, there exists some number $\delta(\varepsilon)$ such that for any function f in $\mathcal{F}$ and any points x_1 and x_2 in X, we have

$$d_Y(f(x_1), f(x_2)) < \varepsilon$$

whenever $d_X(x_1, x_2) < \delta(\varepsilon)$ is satisfied.

17.2 Sufficient Conditions

PROPOSITION 17.2.1. The closure of an equicontinuous family of functions is equicontinuous.

Proof.

Let (X, d_X) and (Y, d_Y) be metric spaces.

Let $\mathcal{F}$ be an equicontinuous family of functions from X to Y.

We are to prove that $cl(\mathcal{F})$ is equicontinuous.

Let x_0 be an arbitrary point in X.

Let ε be an arbitrary positive number.

Since $\mathcal{F}$ is equicontinuous at point x_0, there exists some $\delta(x_0, \varepsilon)$ such that for any function f in $\mathcal{F}$ and any point x in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon/3$.

Let f be an arbitrary function in $cl(\mathcal{F})$.

Let x be an arbitrary point in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$.

Since $f \in cl(\mathcal{F})$, there exists some function $f_0 \in \mathcal{F}$ such that $d_\infty(f, f_0) < \varepsilon/3$.

Since $d_\infty(f, f_0) < \varepsilon/3$, $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$.

Since $f_0 \in \mathcal{F}$ and $d_X(x, x_0) < \delta(x_0, \varepsilon)$, $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$.

Since $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$ and $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$, $d_Y(f(x), f(x_0)) < \varepsilon$.

Since for any positive number ε, there exists some $\delta(x_0, \varepsilon)$ such that for any function f in $cl(\mathcal{F})$ and any point x in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, by definition of equicontinuous, $cl(\mathcal{F})$ is equicontinuous at point x_0.

Since $cl(\mathcal{F})$ is equicontinuous at point x_0 for any point x_0 in X, $cl(\mathcal{F})$ is equicontinuous.

∎

Chapter 18

Adjoint Operator

18.1 Definitions

DEFINITION (Adjoint Matrix)**.** Let A be an $m \times n$ matrix. We define the **adjoint** of A, denoted by A^*, to be an $n \times m$ matrix given by

$$(A^*)_{ij} := \overline{(A)_{ji}}.$$

DEFINITION (Adjoint Operator)**.** Let V and W be inner product spaces. Let T be a linear map from V to W. We define the **adjoint** of T, denoted by T^*, to be a map from W to V such that

$$\forall x \in V, \forall y \in W, \quad \langle T(x), y \rangle_W = \langle x, T^*(y) \rangle_V.$$

PROPOSITION 18.1.1 (Existence)**.** Let V be a finite-dimensional inner product space and T be a linear operator on V. Then the adjoint of T exists.

PROPOSITION 18.1.2 (Uniqueness)**.** Let V be an inner product space and T be a linear operator on V. Then the adjoint of T is unique, provided that it exists.

18.2 Properties of the Adjoint Operator

PROPOSITION 18.2.1. Let V be an inner product space. Then

(1) $(I_V)^* = I_V$ where I_V is the identity operator on V.

(2) $T^{**} = T$ for any linear operator T on V.

PROPOSITION 18.2.2. Let V be an inner product space and T be a linear operator on V. Then T^* is also linear.

PROPOSITION 18.2.3. Let V be an inner product space. Then

(1) For any linear operators T and U,

$$(T + U)^* = T^* + U^*.$$

(2) For any linear operator T,

$$(cT)^* = \bar{c} \cdot T^*.$$

(3) For any linear operator T and U,

$$(TU)^* = U^* T^*.$$

PROPOSITION 18.2.4. Let V be a finite-dimensional inner product space and T be a linear operator on V. Then if T is invertible, T^* is also invertible.

PROPOSITION 18.2.5. Let V be an inner product space and T be an invertible linear operator on V. Then $(T^{-1})^* = (T^*)^{-1}$.

18.3 Normal Operators

DEFINITION (Normal)**.** Let V be an inner product space and T be a linear operator on V. We say that T is **normal** if $TT^* = T^*T$.

18.4 Self-adjoint

Chapter 19

Convolution

DEFINITION (Convolution)**.** Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. We define the **convolution** of f and g, denoted by $f * g$, to be a function on $\mathbb{R}$ given by

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)dt.$$

Chapter 20

Coercive Functions

20.1 Definitions

DEFINITION (Coercive)**.** Let f be a function from $\mathbb{R}^d$ to $\mathbb{R}^*$. We say that f is **coercive** if $\lim_{\|x\|\to\infty} f(x) = +\infty$.

20.2 Properties

PROPOSITION 20.2.1. Let f be a proper lower semi-continuous function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let K be a compact set in $\mathbb{R}^d$. Assume $K \cap \mathrm{dom}(f) \neq \emptyset$. Then f attains its minimum over K.

Proof.

Define $m := \inf_{x\in K} f(x)$.

Since $m = \inf_{x\in K} f(x)$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in K such that $\lim_{i\to\infty} f(x_i) = m$.

Since K is compact and $\{x_i\}_{i\in\mathbb{N}} \subseteq K$, there exists a convergent subsequence $\{x_i\}_{i\in I}$ in K where I is an infinite subset of $\mathbb{N}$.

Say the limit is x_∞ where $x_\infty \in K$.

Since $\lim_{i\to\infty} f(x_i) = m$, we get $\lim_{i\in I, i\to\infty} f(x_i) = m$.

Since $\lim_{i\in I, i\to\infty} f(x_i) = m$, we get $\liminf_{i\in I, i\to\infty} f(x_i) = m$.

Since f is lower semi-continuous and $\lim_{i\in I, i\to\infty} x_i = x_\infty$, we get $f(x_\infty) \leq \liminf_{i\in I, i\to\infty} x_i$.

That is, $f(x_\infty) \leq m$.

Since $m = \inf_{x \in K} f(x)$, we have $\forall x \in K, f(x) \geq m$.

In particular, $f(x_\infty) \geq m$.

Since $f(x_\infty) \geq m$ and $f(x_\infty) \leq m$, $f(x_\infty) = m$.

Since f is proper, $f(x_\infty) = m \neq -\infty$.

So f attains its minimum at point x_∞.

∎

PROPOSITION 20.2.2. Let f be a proper, lower semi-continuous, and coercive function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let C be a closed subset of $\mathbb{R}^d$. Assume $C \cap \mathrm{dom}(f) \neq \emptyset$. Then f attains its minimum over C.

Proof.

Since $C \cap \mathrm{dom}(f) \neq \emptyset$, take $x \in C \cap \mathrm{dom}(f)$.

Since f is coercive, $\exists R$ such that $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$.

Since $x \in C \cap \mathrm{dom}(f)$ and $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$, the set of minimizers of f over C is the same as the set of minimizers of f over $C \cap \mathrm{ball}[0, R]$.

Since C and $\mathrm{ball}[0, R]$ are both closed, $C \cap \mathrm{ball}[0, R]$ is closed.

Since $\mathrm{ball}[0, R]$ is bounded, $C \cap \mathrm{ball}[0, R]$ is bounded.

Since $C \cap \mathrm{ball}[0, R]$ is closed and bounded, by the Heine-Borel Theorem, $C \cap \mathrm{ball}[0, R]$ is compact.

Since f is proper and lower semi-continuous and $C \cap \mathrm{ball}[0, R]$ is compact, f attains its minimum over $C \cap \mathrm{ball}[0, R]$.

So f attains its minimum over C.

∎

Chapter 21

Unclassified Results

PROPOSITION 21.0.1. Let (X, d) be a compact metric space. Let $L(X)$ be the set of all Lipschitz functions from X to $\mathbb{R}$. Let $C(X)$ be the set of all continuous functions from X to $\mathbb{R}$. Then $L(X)$ is dense in $C(X)$.