Graph Theory

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Chapter 1

Graph Basics

DEFINITION (Spanning Subgraph)**.** Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G. We say that H is **spanning** if $W = V$. i.e., if H contains all vertices of G.

Chapter 2

Trees

2.1 Definitions

DEFINITION (Spanning Tree)**.** Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G. We say that H is a **spanning tree** if H is a spanning subgraph of G and is a tree.

2.2 Properties

PROPOSITION 2.2.1. A graph is connected if and only if it has a spanning tree.

Chapter 3

Graph Isomorphism

3.1 Definitions

DEFINITION (Isomorphism)**.** Let G and H be two graphs. We define an **isomorphism** from G to H to be a function f from $V(G)$ to $V(H)$ such that

- f is bijective, and that
- for any pair of vertices $v, w \in V(G)$, $f(v)f(w) \in E(H)$ if and only if $vw \in E(G)$.

i.e., a bijective function that both itself and its inverse preserve adjacency.

DEFINITION (Isomorphic)**.** Let G and H be two graphs. We say that G and H are **isomorphic**, denoted by $G \simeq H$, if there exists an isomorphism from G to H.

PROPOSITION 3.1.1. The relation *simeq* of isomorphism is an equivalence relation. That is, it is reflexive, symmetric, and transitive.

3.2 Properties

PROPOSITION 3.2.1. Let G and H be isomorphic graphs with isomorphism f. Then for any vertex $v \in V(G)$, we have $\deg_G(v) = \deg_H(f(v))$.

Chapter 4

Matchings and Covers

4.1 Matching

DEFINITION (Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of E. We say that M is a **matching** in G if every vertex in the spanning subgraph (V, M) has degree at most one.

DEFINITION (Saturated)**.** Let $(G = (V, E)$ be a graph. Let M be a subset of E. Let v be a vertex of G. We say that v is **M-saturated** if $\deg(v) = 1$ in (V, M).

DEFINITION (Maximal Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximal matching** if it is a matching in G and any other matching is not a superset of it.

DEFINITION (Maximum Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximum matching** if it is a matching in G and any other matching contains edges no more than M.

DEFINITION (Perfect Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **perfect matching** if it matches all vertices of the graph. i.e., any vertex in G is incident to some edge in M.

PROPOSITION 4.1.1. Every maximum matching is maximal.

PROPOSITION 4.1.2. Every perfect matching is maximum.

PROPOSITION 4.1.3. Let $G = (V, E)$ be a graph. Let A and B be two maximal matchings of G. Then both $|A| \leq 2|B|$ and $|B| \leq 2|A|$.

4.2 Cover

DEFINITION (Cover)**.** Let $G = (V, E)$ be a graph. Let C be a subset of V. We say that C is a **cover** of G if any edge has an end in C.

4.3 Relations Between Matchings and Covers

PROPOSITION 4.3.1. Let $G = (V, E)$ be a graph. Let M be a matching of G. Let C be a cover of G. Then $|M| \leq |C|$.

Chapter 5

Bipartite Graphs

5.1 Definitions

DEFINITION (Bipartition)**.** Let $G = (V, E)$ be a graph. Let A and B be two subsets of V. We say the pair (A, B) is a **bipartition** of G if and only if $A \cap B = \emptyset$, $A \cup B = V$, and A and B are both independent.

DEFINITION (Bipartite Graph)**.** Let $G = (V, E)$ be a graph. We say that G is **bipartite** if and only if there exists a bipartition of G.

DEFINITION (Balanced Bipartite Graph)**.** Let $G = (V, E)$ be a bipartite graph with bipartition (A, B). We say that G is **balanced** if and only if $|A| = |B|$.

5.2 Properties of Bipartite Graphs

PROPOSITION 5.2.1. Let $G = (V, E)$ be a bipartite graph with bipartition (A, B). Then

$$\sum_{a \in A} \deg(a) = \sum_{b \in B} \deg(b) = |E|.$$

5.3 Characterizations

PROPOSITION 5.3.1. A graph is bipartite if and only if it has no odd cycles.

PROPOSITION 5.3.2. A graph is bipartite if and only if it is 2-colorable.

Chapter 6

Planar Graphs

6.1 Definitions

DEFINITION (Plane Embedding)**.** Let $G(V, E, B)$ be an undirected multi-graph. A **plane embedding** of G is a pair of sets $(\mathcal{P}, \Gamma)$ such that

6.2 Properties

PROPOSITION 6.2.1. Every subgraph of a planar graph is planar.

PROPOSITION 6.2.2. A multi-graph is planar if and only if its simplification is planar.

PROPOSITION 6.2.3. Let G be a multi-graph and e be an edge in G. Then G is planar if and only if $G \bullet e$ is planar.

THEOREM 6.1. A multi-graph is planar if and only if it does not contain a repeated subdivision of K_5 or $K_{3,3}$ as a subgrph.

6.3 Numerology

DEFINITION (Footprint)**.** Let $G(V, E, B)$ be a planar multi-graph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define the **footprint** of G, denoted by $fp(G)$, to be the union of the points and curves in $\mathbb{R}^2$ representing the vertices and edges in G.

DEFINITION (Face)**.** Let $G(V, E, B)$ be a planar multi-graph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define a **face** of $(\mathcal{P}, \Gamma)$ to be a connected component of the set $\mathbb{R}^2 \setminus fp(G)$.

DEFINITION (Degree)**.** Let $G(V, E, B)$ be a planar multi-graph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define the **degree** of a face to be the sum of the number of edges and the number of bridges in its boundary.

PROPOSITION 6.3.1. An edge e in a planar multi-graph is a bridge if and only if the two faces on either side of the curve γ_e are the same.

Chapter 7

Duality

7.1 Definitions

DEFINITION (Dual Graph)**.** Let $G = (V, E, B)$ be a multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of G. Let $\mathcal{F}$ be the set of faces of G. We define the **dual graph** of this embedding to be the multigraph $G^* = (V^*, E^*, B^*)$ where $V^* = \mathcal{F}$ and $E^* = \{e^* : e \in E\}$.

PROPOSITION 7.1.1. Let $G = (V, E, B)$ be a multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of G. Let $(G^* = (V^*, E^*, B^*)$ be the dual graph of G. Then for any face $f \in \mathcal{F}$, the degree of f as a face of $\mathcal{P}, \Gamma$ equals the degree of f as a vertex of G^*.

PROPOSITION 7.1.2. If G is a connected multigraph embedded in the plane, then G^{**} is isomorphic with G.

Chapter 8

Graph Coloring

8.1 Chromatic Number

DEFINITION ((Proper) Coloring)**.** Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define a **(proper)** X**-coloring** of G to be a function $f : V \to X$ such that if $vw \in E$, then $f(v) \neq f(w)$.

DEFINITION (Chromatic Number)**.** Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define the **chromatic number** of G, denoted by $\chi(G)$, to be the smallest natural number $k \in \mathbb{N}$ for which G has a (proper) k-coloring.

PROPOSITION 8.1.1. The chromatic number exists and $\chi(G) \leq |V|$.

Proof. Take $X = V$. ■

PROPOSITION 8.1.2. G is complete if and only if $\chi(G) = |V(G)|$.

PROPOSITION 8.1.3. The only graph with chromatic number zero is the empty graph.

PROPOSITION 8.1.4. A graph has chromatic number one if and only if it has no edges and at least one vertex.

PROPOSITION 8.1.5. A graph has chromatic number two if and only if it is bipartite and has at least one edge.

PROPOSITION 8.1.6. Let G be a graph. Let $d_{max}(G)$ be the maximum degree of a vertex in G. Then $\chi(G) \leq 1 + d_{max}(G)$.

8.2 5-color Theorem

THEOREM 8.1. Every planar graph is 5-colorable.

Proof. (1890)

True for $|V| \leq 5$.

Inductively, suppose the theorem holds for planar graphs on $n-1$ vertices for $n \geq 5$. Suppose G is a planar graph on n vertices.

Let v be a vertex of degree ≤ 5 in G. This exists by a lemma in our lectures.

Since G is a planar, $G-v$ is planar. By the induction hypothesis, $G-v$ has a 5-coloring.

If some color does not appear on any neighbor of v, we can extend the coloring to a coloring of G.

Otherwise, v has exactly 5 neighbors with different colors.

For each pair i, j of colors, let G_{ij} be the subgraph of $G-v$ induced by the vertices colored i or j.

If the component H of G_{ij} containing x_i does not contain x_j, then we can switch the colors of all vertices in H between i and j to get a coloring of $G-v$ that assigns only 4 colors to neighbors of v, and thus extends to a coloring of G.

So G_{ij} contains a path from x_i to x_j.

Because $G_{2,5}$ and $G_{1,4}$ have disjoint vertex sets, this contradicts the planarity of G.

■

DEFINITION (Near-triangulation)**.** Planar drawing of G where the infinite face is bounded by a cycle, and every other face is bounded by a triangle

THEOREM 8.2. Every planar near-triangulation has a 5-coloring.

Theorem 8.2 $\implies$ Theorem 8.1.

DEFINITION (List Assignment)**.** A **list assignment** L of G is a function that assigns a set $L(v)$ of colors to each $v \in V$.

DEFINITION (L-coloring)**.** An L-coloring of G is a choice of a color in $L(v)$ for each $v \in V$ such that adjacent vertices get different colors.

DEFINITION (5-list-colorable)**.** A graph is **5-list-colorable** if for every list assignment L of G with $|L(v)| \geq 5$, G is L-colorable.

THEOREM 8.3. Every planar near-triangulation is 5-list-colorable.

Theorem 8.3 $\implies$ Theorem 8.2 because coloring is a special case of list coloring.

THEOREM 8.4 (Carsten Thomassen, 1993)**.** If G is a near-triangulation and L is a list assignment such that

(1) $|L(v)| = 5$ for every non-boundary vertex,

(2) $|L(v)| = 3$ for every boundary vertex.

Then G has an L-coloring even if two adjacent boundary vertices have their colors arbitrarily decided in advance.

Proof.

Case 1. There is a "chord" between two boundary vertices.

Let G_1 and G_2 be subgraph of G obtained by "cutting" G along the chord, where G_1 contains the pre-colored vertices.

By applying the inductive hypothesis to G_1, and then applying it to G_2 with the two ends of the chord pre-colored according to the coloring of G_1, we get a coloring of G_1.

Case 2. There is no chord.

Let u and u' be the pre-colored vertices.

Let x, y be the next two vertices occuring in order around the boundary.

■

Theorem 8.4 $\implies$ Theorem 8.3.

Chapter 9

Probability and Edge Density

Q: Let G be a graph on n vertices with no triangles. How many edges can G have?

THEOREM 9.1 (Mantel)**.** If G is triangle-free and has n vertices, then

$$|E| \leq \frac{n^2}{4}.$$

Proof. Let $P_{2,1}$ denote the probability that a pair of distinct vertices chosen uniformly at random, are adjacent.

$$P_{2,1} = |E|/\binom{n}{2}.$$

Let $P_{3,2}$ denote the probability that a randomly chosen triple of vertices contains exactly two edges. Let $P_{3,1}$ denote ... one edge. Let $P_{3,0}$ denote ... no edges. Notice $P_{3,2}+P_{3,1}+P_{3,0} = 1$.

Part 1: Show that $P_{2,1} = \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1}$. Notice that the graph is triangle-free. So $P_{3,3} = 0$. Choosing a pair at random is the same as choosing a triple at random, then choosing a pair at random within that triple.

For a fixed vertex v, let $Q_{v,1}$ denote the probability that a randomly chosen vertex $u \neq v$ is adjacent to v.

$$Q_{v,1} = \frac{deg(v)}{n-1}.$$

Let $Q_{v,2}$ denote the probability that two distinct randomly chosen vertices other than v are both adjacent to v.

$$Q_{v,2} = \binom{deg(v)}{2}/\binom{n-1}{2}.$$

Part 2: Show that $Q_{v,1}^2 \approx Q_{v,2}$. Both give (essentially) the probability that a pair x, y of vertices other than v are both adjacent to v. The LHS allows $x = y$. The RHS does not. But $x = y$ occurs with negligible probability.

Part 3: Show that $P_{2,1} = \frac{1}{n}\sum_v Q_{v,1}$. Both the RHS and LHS are just the probability of an edge between two vertices. The RHS calls the first chosen vertex v.

Part 4: Show that $\frac{1}{3}P_{3,2} = \frac{1}{n}\sum_v Q_{v,2}$. Both sides give the probability that, if we choose 3 vertices at random, and then choose one among those 3 and call it v, that v is adjacent to both the others.

Proof of the theorem.

$$\begin{aligned} P_{2,1} &= \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1} \geq \frac{2}{3}P_{3,2} \\ &= 2\left(\frac{1}{n}\sum_v Q_{v,2}\right) \approx 2\left(\frac{1}{n}\sum_v Q_{v,1}^2\right) \\ &\geq 2\left(\frac{1}{n}\sum_v Q_{v,1}\right)^2 = 2P_{2,1}^2. \end{aligned}$$

So $P_{2,1} \leq \frac{1}{2}$. So $|E| \leq \frac{n^2}{4}$. ■

Q: If G has n vertices, no K_{t+1}-subgraph, how many edges can G have?

THEOREM 9.2 (Turan)**.** If G is a graph on n vertices with no K_{t+1}-subgraph, then

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{t}\right).$$

THEOREM 9.3 (Erdos-Stone)**.** If H is a graph and G is a graph on n vertices without H as a subgraph, then

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{\chi(H) - 1} + \varepsilon(n)\right)$$

where $\varepsilon(n) \to 0$ as $n \to \infty$ and $\chi(H)$ is the chromatic number of H, the fewest number of colors needed to properly color the vertices of H.

Chapter 10

Weird Stuffs

10.1 Geometric Representation of Graphs

DEFINITION (Geometric Representation)**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. We define a **geometric representation** of G to be a map from V to $\mathbb{R}^d$.

DEFINITION (Unit Distance Representation)**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \to \mathbb{R}^d$ be a geometric representation of G. We say that u is a **unit distance representation** of G if and only if $\forall \{i, j\} \in E$, $\|u(i) - u(j)\|_2 = 1$.

DEFINITION (Orthonormal Representation)**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \to \mathbb{R}^d$ be a geometric representation of G. We say that u is an **orthonormal representation** of G if and only if

- $\forall i \in V$, $\|u(i)\|_2 = 1$; and
- $\forall \{i, j\} \in \overline{E}$, $\langle u(i), u(j) \rangle = 0$ where $\overline{E}$ is the edge set of the complement of G.

DEFINITION. We define $t_h(G)$ to be the square radius of the smallest hypersphere that contains a unit distance representation of G.

THEOREM 10.1. Let $G = (V, E)$ be a graph. Then

$$\begin{aligned} t_h(G) \quad = \quad & \min \quad t \\ & \text{subject to:} \quad X_{ii} = t, \forall i \in V \\ & X_{ii} - 2X_{ij} + X_{jj} = 1, \forall \{i, j\} \in E \\ & X \in S_+^V \end{aligned}$$

PROPOSITION 10.1.1. Let $G = (V, E)$ be a graph. Then G is bipartite if and only if $t_h(G) \leq \frac{1}{4}$.

Proof. ■

PROPOSITION 10.1.2. Let $n \in \mathbb{Z}_{++}$. Let K_n denote the n-clique. Then $t_h(K_n) =$.

Proof. ■

10.2 Stable Sets

DEFINITION (Stable Sets)**.** Let $G = (V, E)$ be a graph. Let S be a subset of the vertex set V. We say that S is a **stable set** in G if and only if $\forall \{i, j\} \in E$, at most one of i or j is in S. i.e., S is a set of pairwise non-adjacent vertices.

DEFINITION (Stability Number)**.** Let $G = (V, E)$ be a graph. We define the **stability number** of G, denoted by $\alpha(G)$, to be a number given by

$$\alpha(G) := \max\{|S| : S \text{ is stable in } G\}.$$

DEFINITION (Stable Set Polytope)**.** Let $G = (V, E)$ be a graph. We define the

stable set polytope of G, denoted by $\mathrm{STAB}(G)$, to be a subset of $\mathbb{R}^V$ given by

$$\mathrm{STAB}(G) := \mathrm{conv}\{x : x \text{ is the incidence vector of some stable set in } G\}.$$

DEFINITION (Fractional Stable Set Polytope)**.** Let $G = (V, E)$ be a graph. We define the **fractional stable set polytope** of G, denoted by $\mathrm{FRAC}(G)$, to be a subset of $\mathbb{R}^V$ given by

$$\mathrm{FRAC}(G) := \{x \in [0,1]^V : x_i + x_j \leq 1, \forall \{i,j\} \in E\}.$$

PROPOSITION 10.2.1. Let $G = (V, E)$ be a graph. Then

$$\mathrm{STAB}(G) = \mathrm{conv}(\mathrm{FRAC}(G) \cap \{0,1\}^V).$$

10.3 Theta Bodies

DEFINITION (Theta Body)**.** Let $G = (V, E)$ be a graph. We define the **theta body** of G, denoted by $\mathrm{TH}(G)$, to be a subset of $\mathbb{R}^V_+$ given by

$$\mathrm{TH}(G) := \left\{ x \in \mathbb{R}^V_+ : \sum_{i \in V} (c^\top u(i))^2 x_i \leq 1, \begin{array}{l} \forall c \in \mathbb{R}^V : \|c\|_2 = 1, \\ \forall \text{ orth. repr. } u \text{ of } G \end{array} \right\}.$$

DEFINITION (Lovase Theta Function)**.** Let $G = (V, E)$ be a graph. Let $w \in \mathbb{R}^V_+$. We define the **Lovase Theta function**, denoted by θ, to be a function of G and w given by

$$\theta(G, w) := \max\{w^\top x : x \in \mathrm{TH}(G)\}.$$

DEFINITION (Lovase Theta Number)**.** Let $G = (V, E)$ be a graph. We define the **Lovase Theta number** of G, denoted by $\theta(G)$, to be a number given by

$$\theta(G) := \theta(G, \bar{e}) = \max\{\bar{e}^\top x : x \in \mathrm{TH}(G)\}.$$

THEOREM 10.2. Let $G = (V, E)$ be a graph. Let $w \in \mathbb{R}^V_+$. Then the following quantities are the same:

(1) $\theta(G, w)$;

(2) If $w_i = 0$, define $\frac{w_i}{(c^\top u(i))^2} := 0$,

$$\inf\left\{\max_{i\in V}\left\{\frac{w_i}{(c^\top u(i))^2}\right\} : \begin{array}{l} c \in \mathbb{R}^V, \|c\|_2 = 1, \\ u \text{ is an orth. repr. of } G \end{array}\right\};$$

(3) $\min\{\eta \in \mathbb{R} : S \in \mathbb{S}^V, \operatorname{diag}(S) = 0, S_{ij} = 0, \forall\{i,j\} \in \overline{E}, \eta I - S \succeq W\}$;

(4) $\max\{\operatorname{tr}(WX) : X_{ij} = 0, \forall\{i,j\} \in E, \operatorname{tr}(X) = 1, X \in \mathbb{S}^V_+\}$.

10.4 Product of Graphs

DEFINITION (Strong Product)**.** Let $G = (V, E)$ and $H = (W, F)$ be graphs. We define the **strong product** of G and H, denoted by $G \otimes H$, to be a graph given by $G \otimes H = (V(G \otimes H), E(G \times H))$ where

$$V(G \otimes H) := V \times W \text{ and}$$

$$E(G \otimes H) := \left\{\{(i,u),(j,v)\} : \begin{array}{l} (\{i,j\} \in E \text{ and } \{u,v\} \in F) \text{ or} \\ (\{i,j\} \in E \text{ and } u = v \in W) \text{ or} \\ (i = j \in V \text{ and } \{u,v\} \in F) \end{array}\right\}.$$

PROPOSITION 10.4.1. Let $G = (V, E)$ and $H = (W, F)$ be graphs. Then

$$\theta(G \otimes H) = \theta(G) \times \theta(H).$$

DEFINITION (Shannon Capacity)**.** Let $G = (V, E)$ be a graph. We define the **Shannon capacity** of G, denoted by $\Theta(G)$, to be a number given by

$$\Theta(G) := \limsup_{k\to+\infty}(\alpha(G^{\otimes k}))^{1/k}$$

where $\alpha(G^{\otimes k})$ denotes the stability number of $G^{\otimes k}$.