# Functional Analysis

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# Chapter 1

# Balanced Sets

## 1.1 Definitions

**Definition** (Balanced Sets)**.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $S$ be a subset of $X$. We say that $S$ is **balanced** if*

$$\forall a \in \mathbb{F} : |a| \leq 1, \quad aS \subseteq S.$$

**Definition** (Balanced Hull)**.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $S$ be a subset of $X$. We define the **balanced hull** of $S$, denoted by $\mathrm{balhull}(S)$, to be the smallest balanced set containing $S$.*

**Definition** (Balanced Core)**.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $S$ be a subset of $X$. We define the **balanced core** of $S$, denoted by $\mathrm{balcore}(S)$, to be the largest balanced set contained in $S$.*

## 1.2 Properties

**Proposition 1.2.1.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $B$ be a balanced subset of $X$. Then*
$$\forall a, b \in \mathbb{F} : |a| \leq |b|, \quad aB \subseteq bB.$$

**Proposition 1.2.2.** *Balanced sets are path connected.*

**Proposition 1.2.3** (Act on Other Properties)**.**   *• The balanced hull of a compact set is compact.*

- *The balanced hull of a totally bounded set is totally bounded.*

- *The balanced hull of a bounded set is bounded.*

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**Proposition 1.2.4** (Act on Other Properties).   • *The balanced core of a closed set is closed.*

**Proposition 1.2.5.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $a$ be a scalar in field $\mathbb{F}$. Then*

$$a\,\mathrm{balhull}(S) = \mathrm{balhull}(aS).$$

## 1.3   Stability of Balance

**Proposition 1.3.1** (Set Operations).   • *The union of balanced sets is also balanced.*

   • *The intersection of balanced sets is also balanced.*

**Proposition 1.3.2** (Linear Mappings).   • *The scalar multiple of a balanced set is also balanced.*

   • *The (Minkowski) sum of two balanced sets is also balanced.*

   • *The image of a balanced set under a linear operator is also balanced.*

   • *The inverse image of a balanced set under a linear operator is also balanced.*

**Proposition 1.3.3** (Topological Operations). *The closure of a balanced set is also balanced.*

**Proposition 1.3.4.** *The convex hull of a balanced set is also balanced (and also convex).*

## 1.4   Absorbing Sets

**Definition** (Absorbing Sets). *Let $X$ be a vector space over field $\mathbb{F}$. Let $S$ be a subset of $X$. We say that $S$ is **absorbing** if*

$$\forall x \in X, \quad \exists r \in \mathbb{R} : r > 0, \quad \forall c \in \mathbb{F} : |c| \geq r, \quad x \in cA.$$

**Proposition 1.4.1.** *Every absorbing set contains the origin.*

# Chapter 2

# Inner Product Spaces

## 2.1 Inner Products

### 2.1.1 Definitions

**Definition** (Inner Product). *Let $V$ be a vector space over field $\mathbb{F}$. We define an* **inner product** *on $V$, denoted by $\langle \cdot, \cdot \rangle$, to be a scalar-valued function defined on $V \times V$ such that*

*(1) Positive Definiteness*

$$\forall x, y \in V, \quad \langle x, x \rangle \geq 0, \text{ and}$$
$$\forall x \in V, \quad \langle x, x \rangle = 0 \iff x = O_V.$$

*(2) Sesqui-Linearity*

$$\forall x, y, z, w \in V, \quad \langle x + y, z + w \rangle = \langle x, z \rangle + \langle y, z \rangle + \langle x, w \rangle + \langle y, w \rangle, \text{ and}$$
$$\forall a, b \in \mathbb{F}, \forall x, y \in V, \quad \langle ax, by \rangle = a\bar{b}\langle x, y \rangle.$$

*(3) Conjugate Symmetry*

$$\forall x, y \in V, \quad \langle x, y \rangle = \overline{\langle y, x \rangle}.$$

**Definition** (Norm). *Let $V$ be an inner product space over field $\mathbb{F}$. We define the* **norm**, *denoted by $\| \cdot \|$, to be a function from $V$ to $\mathbb{R}_+$ given by*

$$\|x\| := \sqrt{\langle x, x \rangle}$$

**Definition** (Orthogonal Vectors). *Let $V$ be an inner product space. Let $x$ and $y$ be vectors in $V$. We say that $x$ and $y$ are* **orthogonal** *if $\langle x, y \rangle = 0$.*

**Definition** (Orthogonal Sets). *Let $S$ be a subset of $V$. We say that $S$ is* **orthogonal** *if*

$$\forall x, y \in S, \quad \langle x, y \rangle = 0.$$

### 2.1.2   Examples

**Definition** (Standard Inner Product)**.** *For $V = \mathbb{F}^n$, we define the **standard inner product** by*

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i \overline{y_i}.$$

**Definition** (Frobenius Inner Product)**.** *For $V = \mathbb{F}^{n \times n}$, we define the **Frobenius inner product** by*

$$\langle M_1, M_2 \rangle := \operatorname{tr}(M_2^* M_1).$$

**Definition.** *Let $V$ be the space of continuous scalar-valued functions on $[0, 2\pi]$. We define the inner product on $V$ by*

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} dx.$$

### 2.1.3   Properties

**Proposition 2.1.1.** *Let $V$ be a finite dimensional inner product space. Let $\mathcal{B}$ be a basis for $V$. Let $x$ and $y$ be vectors in $V$. Then*

$$x = y \iff \forall b \in \mathcal{B}, \quad \langle x, b \rangle = \langle y, b \rangle.$$

## 2.2   Inequalities

**Theorem 1** (Minkowski)**.**

$$\left( \sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}$$

**Proposition 2.2.1** (Cauchy-Schwarz Inequality)**.** *Let $V$ be an inner product space. Then*

$$\forall x, y \in V, \quad |\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

**Proposition 2.2.2** (Triangle Inequality)**.** *Let $V$ be an inner product space. Then*

$$\forall x, y \in V, \quad \|x + y\| \leq \|x\| + \|y\|$$

**Proposition 2.2.3** (Parallelogram Law)**.** *Let $V$ be an inner product space. Then*

$$\forall x, y \in V, \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

## 2.3 Orthogonality

### 2.3.1 Orthogonal Sets

**Definition** (Orthogonality). *Let $V$ be an inner product space. We say that points $x$ and $y$ in $V$ are **orthogonal** if $\langle x, y \rangle = 0$.*

**Definition** (Orthogonal Sets). *Let $V$ be an inner product space and $S$ be a subset of $V$. We say that $S$ is **orthogonal** if any two vectors in $S$ are orthogonal.*

**Proposition 2.3.1.** *Orthogonal sets are linearly independent.*

### 2.3.2 Orthogonal Bases

**Definition** (Orthogonal Basis). *Let $V$ be an inner product space and $S$ be a subset of $V$. We say that $S$ is an **orthogonal basis** for $V$ if it is an ordered basis for $V$ and orthogonal.*

**Proposition 2.3.2.** *Let $V$ be an inner product space. Let $S = \{v_1, ..., v_n\}$ be an orthogonal subset of $V$ where each $v_i$ is non-zero. Then*

$$\forall y \in \mathrm{span}(S), \quad y = \sum_{i=1}^{n} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

**Theorem 2** (Gram-Schmidt Process). *Let $V$ be an inner product space. Let $S = \{v_0, ..., v_n\}$ be a linearly independent subset of $V$. Then the set $S' = \{v'_0, ..., v'_n\}$ given by $v'_0 := v_0$ and*

$$\forall i \in \{1, ..., n\}, \quad v'_i := v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, v'_j \rangle}{\|v'_j\|} v'_j$$

*is an orthogonal subset of $V$ consisting of non-zero vectors. Furthermore, we have $\mathrm{span}(S') = \mathrm{span}(S)$.*

**Proposition 2.3.3.** *Let $V$ be an inner product space and $S = \{v_0, v_1, \ldots, v_n\}$ be an orthogonal subset of $V$. Then the set $S'$ derived from the Gram-Schmidt process is exactly $S$.*

**Theorem 3** (Parseval's Identity). *Let $V$ be a finite-dimensional inner product space. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthogonal basis for $V$. Then*

$$\forall x, y \in V, \quad \langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

**Theorem 4** (Bessel's Inequality). *Let $V$ be a finite-dimensional inner product space. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthogonal subset for $V$. Then*

$$\forall x \in V, \quad \|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$

### 2.3.3    Orthogonal Complements

**Definition** (Orthogonal Complement)**.** *Let $V$ be an inner product space and $S$ be a non-empty subset of $V$. We define the **orthogonal complement** of $S$, denoted by $S^\perp$, to be the set of all points in $V$ that are orthogonal to all vectors in $S$.*

**Proposition 2.3.4.** *Let $V$ be a finite-dimensional inner product space. Then*

*(1) $V^\perp = \{O_V\}$*

*(2) $\{O_V\}^\perp = V$*

**Proposition 2.3.5.** *Orthogonal complements are always linear subspaces.*

**Proposition 2.3.6.** *Let $V$ be an inner product space and $W$ be a subspace of $V$ with basis $\beta$. Then a vector in $V$ is also in $W^\perp$ if and only if it is orthogonal to all vectors in $\beta$.*

**Proposition 2.3.7** (Extension)**.** *Let $V$ be an $n$-dimensional inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal subset of $V$. Then $S$ can be extended to an orthogonal basis $B = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for $V$.*

### 2.3.4    Properties of the Orthogonal Complement Operator

**Proposition 2.3.8.** *Let $V$ be an inner product space. Then*

*(1) $S \subseteq T$ implies $T^\perp \subseteq S^\perp$ for any subsets $S$ and $T$ of $V$.*

*(2) $S \subseteq (S^\perp)^\perp$ for any subset $S$ of $V$.*

**Proposition 2.3.9.** *Let $V$ be a finite-dimensional inner product space and $W$ be a subspace of $V$. Then*

*(1) $W = (W^\perp)^\perp$*

*(2) $V = W \oplus W^\perp$*

**Proposition 2.3.10.** *Let $V$ be a finite-dimensional inner product space and $W_1$ and $W_2$ be subspaces of $V$. Then*

*(1) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$*

*(2) $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$*

### 2.3.5    Orthogonal Projection

**Definition** (Orthogonal Projection)**.** *Let $V$ be a vector space. Let $W$ be a finite-dimensional subspace of $V$. Let $x$ be a vector in $V$. We define the **orthogonal projection** of $x$ on $W$, denoted by $(x)$, to be the vector $u$ in $W$ such that $x = u + v$ where $v$ is another vector in $W^\perp$.*

# Chapter 3

# Normed Linear Spaces

## 3.1 Definitions

**Definition** (Seminorm). *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **seminorm** on $\mathfrak{X}$, denoted by $\nu$, to be a map from $\mathfrak{X}$ to $\mathbb{R}$ that satisfies the following conditions.*

*(1) $\forall x \in \mathfrak{X}, \quad \nu(x) \geq 0$.*

*(2) $\forall \lambda \in \mathbb{F}, \forall x \in \mathfrak{X}, \quad \nu(\lambda x) = \lambda \nu(x)$.*

*(3) Triangle Inequality.*
$$\forall x, y \in \mathfrak{X}, \quad \nu(x + y) \leq \nu(x) + \nu(y).$$

*The idea behind the seminorm is that we are trying to give our vector space a notion of "length" of vectors.*

**Definition** (Norm). *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **norm** on $\mathfrak{X}$, denoted by $\nu$, to be a seminorm on $\mathfrak{X}$ that satisfies the additional condition:*

$$\forall x \in \mathfrak{X}, \quad \mu(x) = 0 \iff x = 0.$$

## 3.2 Properties

**Proposition 3.2.1.** *Let $(V, \| \cdot \|_V)$ be a normed vector space over field $\mathbb{F}$. Then $(V, \| \cdot \|)$ is complete if and only if $(\overline{B(0,1)}, \| \cdot \|_V)$ is complete.*

*Proof.*
For one direction, assume that $(V, \| \cdot \|)$ is complete.
We are to prove that $(\overline{B(0,1)}, \| \cdot \|_V)$ is complete.

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Since $(\overline{B(0,1)}, \|\cdot\|_V)$ is a closed subspace of $(V, \|\cdot\|)$ and $(V, \|\cdot\|)$ is complete, $(\overline{B(0,1)}, \|\cdot\|_V)$ is also complete.

For the reverse direction, assume that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

We are to prove that $(V, \|\cdot\|_V)$ is complete.

Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $(V, \|\cdot\|_V)$.

Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ is bounded in $(V, \|\cdot\|_V)$.

Let $\lambda$ be a positive upper bound for $\{\|x_i\|_V\}_{i\in\mathbb{N}}$.

Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$ and $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ converges in $(V, \|\cdot\|_V)$.

Since any Cauchy sequence in $(V, \|\cdot\|_V)$ converges in $(V, \|\cdot\|_V)$, $(V, \|\cdot\|_V)$ is complete.

$\blacksquare$

## 3.3　Equivalence of Norms

**Definition** (Equivalence of Norms)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $V$. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are* ***equivalent*** *if*

$$\exists c_1, c_2 > 0, \quad \forall v \in \mathfrak{X}, \quad c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_2.$$

*Or equivalently,*

$$c_1\|v\|_2 \leq \|v\|_1 \leq c_2\|v\|_2.$$

**Proposition 3.3.1.** *The equivalence of norms is an equivalence relation.*

**Theorem 5.** *Let $V$ be a finite dimensional vector space over field $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$. Then any two norms on $V$ are equivalent.*

*Proof.*

Let $\|\cdot\|_p$ be an arbitrary $p$-norm on $V$ and $\|\cdot\|$ be an arbitrary norm on $V$.

Let $\mathcal{B}$ be the standard basis for $V$. Say $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$.

Let $v$ be an arbitrary vector in $V$.

$$\|v\| = \|\sum_{i=1}^{n} v_i e_i\|$$

$$\leq \sum_{i=1}^{n} |v_i| \|e_i\|$$

$$\leq \left( \sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}}$$

$$= \left( \sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \|v\|_p$$

$$:= c_1 \|v\|_p.$$

∎

**Proposition 3.3.2.** *Let $X$ be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $X$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if they generate the same metric topology.*

*Proof.* Convergence to 0 is equivalent under either $\|\cdot\|_1$ or $\|\cdot\|_2$. i.e., equivalent norms give rise to the same set of sequences that are convergent. Convergence of sequences defines the topology. ∎

**Proposition 3.3.3.** *Let $\mathfrak{X}$ be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathfrak{X}$. Let $i$ be the identity map from $(\mathfrak{X}, \|\cdot\|_1)$ to $(\mathfrak{X}, \|\cdot\|_2)$. Then if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, $i$ is continuous.*

## 3.4 Dual Norms

**Definition** (Dual Norm). *Let $(V, \|\cdot\|)$ be an normed vector space. We define the **dual norm** of $\|\cdot\|$, denoted by $\|\cdot\|_\circ$, to be a function given by*

$$\|v\|_\circ := \max_{\|w\|=1} v \cdot w = \max_{\|w\| \neq 0} \frac{|v \cdot w}{\|w\|}.$$

**Proposition 3.4.1.** *The dual norms of norms are indeed norms.*

**Proposition 3.4.2.** *Let $(V, \|\cdot\|)$ be a normed vector space. Let $v, w$ be vectors in the space. Then*

$$|v \cdot w| \leq \|v\| \cdot \|w\|_\circ.$$

## 3.5   $p$-norms

**Definition** ($p$-norm)**.** *Let $V$ be a finite-dimensional normed vector space over field $\mathcal{F}$. Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis for $V$ where $n = \dim(V)$. Let $v$ be a vector in a normed vector space. For $p \in [1, +\infty)$, we define the $p$-**norm** of $v$, denoted by $\|v\|_p$, to be the number given by*

$$\|v\|_p = \left( \sum_{i=1}^{n} |(v_{\mathcal{B}})_i|^p \right)^{\frac{1}{p}}.$$

**Definition** (Infinity Norm - 1)**.** *Let $\mathfrak{X} = \mathbb{K}^n$ where $\mathbb{K}$ is a field and $n \in \mathbb{N}$. We define the* ***infinity norm*** *on $\mathfrak{X}$, denoted by $\| \cdot \|_\infty$, to be a function given by*

$$\|v\|_\infty := \max\{|v_i|\}_{i=1}^{n}.$$

**Definition** (Infinity Norm - 2)**.** *Let $\mathfrak{X} = \mathbb{K}^{\mathbb{N}}$. We define the* ***infinity norm*** *on $\mathfrak{X}$, denoted by $\| \cdot \|_\infty$, to be a function given by*

$$\|v\|_\infty := \sup_{i \in \mathbb{N}} |v_i|.$$

**Definition** (Infinity Norm - 3)**.** *Let $\mathfrak{X} = \mathcal{C}([0,1], \mathbb{C})$. We define the* ***infinity norm*** *on $\mathfrak{X}$, denoted by $\| \cdot \|_\infty$, to be a function given by*

$$\nu(f) := \sup_{x \in [0,1]} |f(x)|.$$

**Proposition 3.5.1.** *Let $\mathfrak{X} := \mathcal{C}([0,1], \mathbb{C})$. Let $x$ be an arbitrary number in $[0,1]$. Define a function $\nu_x$ on $\mathfrak{X}$ by $\nu_x(f) := |f(x)|$. Define a function $\nu$ on $\mathfrak{X}$ by $\nu(f) := \sup_{x \in [0,1]} |f(x)|$. Then $\nu_x$ is a seminorm on $\mathfrak{X}$ for each $x$ and $\nu$ is a norm on $\mathfrak{X}$ and we have $\nu = \sup_{x \in [0,1]} \nu$.*

**Proposition 3.5.2.** *$p$-norms are indeed norms.*

**Proposition 3.5.3.** *For any vector $v$ in $\mathbb{R}^n$, we have*

$$\lim_{p \to \infty} \|v\|_p = \|v\|_\infty.$$

*i.e.,*

$$\lim_{p \to \infty} \left( \sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}} = \max\{|v_i|\}_{i=1}^{n}.$$

*Proof.* Let $p$ be an arbitrary number in $[1, +\infty)$. Let $k$ be an arbitrary index in $\{1, .., n\}$. Then

$$|v_k| \leq (\sum_{i=1}^{n} |v_k|^p)^{1/p} = \|v\|_p.$$

So

$$\max\{|v_k|\} = \|v\|_\infty \leq \|v\|_p.$$

So

$$\lim_{p \to \infty} \|v\|_p \geq \|v\|_\infty. \tag{1}$$

On the other hand, note that

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p = \sum_{i=1}^{n} (\frac{|v_i|}{\|v\|_\infty})^p$$

decreases as $p$ increases. So it is bounded above. Say

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p \leq C$$

for some $C \in \mathbb{R}$. Then

$$(\sum_{i=1}^{n} |v_i|^p)^{1/p} = \|v\|_p \leq C^{1/p}\|v\|_\infty.$$

So

$$\lim_{p \to \infty} \|v\|_p \leq \lim_{p \to \infty} C^{1/p}\|v\|_\infty = \|v\|_\infty. \tag{2}$$

From (1) and (2) we get

$$\lim_{p \to \infty} \|v\|_p = \|v\|_\infty.$$

∎

**Proposition 3.5.4.** *Let $p$ be an arbitrary number in $[1, +\infty)$. Then the dual norm of the $p$-norm $\| \cdot \|_p$ is the $q$-norm $\| \cdot \|_q$ where $q$ is such that satisfies*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

**Proposition 3.5.5.** *Let $p$ and $q$ be numbers in $[1, +\infty]$. Let $v$ be a vector in $\mathbb{R}^n$. Then if $p \leq q$,*

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \cdot \|x\|_q.$$

**Proposition 3.5.6.** *Let $w$ and $z$ be vectors in $\mathbb{E}^d$. Then*

$$\|w + z\|_2^2 + \|w - z\|_2^2 = 2(\|w\|_2^2 + \|z\|_2^2).$$

## 3.6  Banach Spaces

**Definition** (Banach Space)**.** *Let $(\mathfrak{X}, \| \cdot \|)$ be a normed linear space. Let $d$ be the metric induced by $\| \cdot \|$. We say that $\mathfrak{X}$ is a **Banach space** if $(\mathfrak{X}, d)$ is a complete metric space. i.e., we define a Banach space to be a complete normed linear space.*

**Proposition 3.6.1.** *Let $(\mathfrak{X}, \| \cdot \|)$ be a normed vector space over field $\mathbb{F}$. Then $(\mathfrak{X}, \| \cdot \|)$ is a Banach space if and only if every absolutely summable series in $X$ is summable.*

**Example 3.6.1.** $(\mathcal{C}([0,1], \mathbb{F}), \|\cdot\|_\infty)$ *is a Banach space.*

**Example 3.6.2** (Disc Algebra)**.** *Define* $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. *Define* $\mathcal{A}(\mathbb{D}) := \{f \in \mathcal{C}(\overline{\mathbb{D}}) : f|_\mathbb{D} \text{ is holomorphic }\}$. *Define* $\|\cdot\|_\infty$ *by* $\|f\|_\infty := \sup_{z \in \overline{\mathbb{D}}} |f(z)|$. *Then* $(\mathcal{A}(\mathbb{D}), \|\cdot\|_\infty)$ *is a Banach space.*

**Example 3.6.3.** *Let* $(X, \Omega, \mu)$ *be a measure space. Let* $p$ *be a number in* $[1, +\infty)$. *Define*

$$\mathcal{L}^p(X, \mu) := \text{span}\{f : X \to [0, +\infty] \mid f \text{ is measurable and } \int_X |f|^p < +\infty\}.$$

*Define an equivalence relation on* $\mathcal{L}^p(X, \mu)$ *by* $f \equiv g$ *if and only if*

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

*Define a space* $L^p(X, \mu) := \mathcal{L}^p(X, \mu)/ \equiv$. *Then* $L^p(X, \mu)$ *is a Banach space when equipped with the norm*

$$\|[f]\|_p := \left(\int_X |f|^p\right)^{1/p}.$$

**Example 3.6.4.** *Let* $\mathcal{P}_\mathbb{C}[0,1]$ *denote the set of all polynomials with complex coefficients. For each* $p \in [1, +\infty)$, *define a norm*

$$\|f\|_p := \left(\int_0^1 |f|^p\right)^{1/p}.$$

*For* $p = +\infty$, *define a norm*

$$\|f\|_\infty := \sup_{x \in [0,1]} |f(z)|.$$

## 3.7 Construction of Banach Spaces

**Definition.** *Let* $(\mathfrak{X}, \|\cdot\|_\mathfrak{X})$ *and* $(\mathfrak{Y}, \|\cdot\|_\mathfrak{Y})$ *be two Banach spaces over field* $\mathbb{K}$. *Let* $p \in [1, +\infty)$. *We define*

$$\mathfrak{X} \oplus_p \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

*and*

$$\|(x, y)\|_p := (\|x\|_\mathfrak{X}^p + \|y\|_\mathfrak{Y}^p)^{1/p}.$$

*For* $p = +\infty$, *we define*

$$\mathfrak{X} \oplus_\infty \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

*and*

$$\|(x, y)\|_\infty := \max(\|x\|_\mathfrak{X}, \|y\|_\mathfrak{Y}).$$

- *Note that the norms behave similarly to what the p norm would do.*

- *We can similarly define the direct sum of finitely many Banach spaces.*

**Proposition 3.7.1.** $\|\cdot,\cdot\|_p$ *is a norm on* $\mathfrak{X} \oplus_p \mathfrak{Y}$.

**Proposition 3.7.2.** $\mathfrak{X} \oplus_p \mathfrak{Y}$ *is complete with respect to* $\|\cdot,\cdot\|_p$.

# Chapter 4

# Quotient Spaces

## 4.1 Definitions

**Definition.** *Let $\mathfrak{V}$ be a vector space. Let $\mathfrak{W}$ be a subspace of $\mathfrak{V}$. We define a **quotient space**, denoted by $\mathfrak{V}/\mathfrak{W}$, to be a set $\{v + \mathfrak{W} : v \in \mathfrak{V}\}$ with operations*

$$(v_1 + \mathfrak{W}) + (v_2 + \mathfrak{W}) := (v_1 + v_2) + \mathfrak{W} \ and$$

$$\kappa(v + \mathfrak{W}) := (\kappa v) + \mathfrak{W}.$$

**Definition** (Quotient Map)**.** *Let $(\mathfrak{X}, \| \cdot \|)$ be a normed linear space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define the **quotient map** on $\mathfrak{X}$ with respect to $\mathfrak{M}$, denoted by $q_{\mathfrak{M}}$, to be a function from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$ given by*

$$q_{\mathfrak{M}}(x) := x + \mathfrak{M}$$

**Proposition 4.1.1.** *Quotient maps are contractive. i.e.,*

$$\forall x \in \mathfrak{X}, \quad \|x + \mathfrak{M}\|_{\mathfrak{X}/\mathfrak{M}} \leq \|x\|_{\mathfrak{X}}.$$

**Proposition 4.1.2.** *Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a closed subspace of $\mathfrak{X}$. Let $q$ be the canonical quotient map from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$. Then*

- *$q$ is a continuous map. i.e.,*

$$\forall \text{ open set } W \subseteq \mathfrak{X}/\mathfrak{M}, \quad q^{-1}(W) \text{ is open in } \mathfrak{X}.$$

- *$q$ is an open map. i.e.,*

$$\forall \text{ open set } G \subseteq \mathfrak{X}, \quad q(G) \text{ is open in } \mathfrak{X}/\mathfrak{M}.$$

*Proof.* Since $q$ is contractive, $q$ is continuous and hence (1).                        ∎

**Definition** (Seminorm on Quotient Spaces). *Let* $(\mathfrak{X}, \|\cdot\|)$ *be a normed linear space. Let* $\mathfrak{M}$ *be a linear manifold in* $\mathfrak{X}$. *We define a* ***seminorm*** *on* $\mathfrak{X}/\mathfrak{M}$ *to be a function from* $\mathfrak{X}/\mathfrak{M}$ *to* $\mathbb{R}$ *given by*

$$p(x + \mathfrak{M}) := \inf\{\|x + m\| : m \in \mathfrak{M}\}.$$

**Proposition 4.1.3.** *Seminorms on quotient spaces are indeed seminorms.*

**Proposition 4.1.4.** *A seminorm on a quotient space* $\mathfrak{X}/\mathfrak{M}$ *is a norm if and only if* $\mathfrak{M}$ *is closed.*

# Chapter 5

# Topological Vector Spaces

## 5.1 Definitions

## 5.2 Topological Vector Spaces

**Definition** (Vector Topology)**.** *Let $X$ be a vector space over a topological field $\mathbb{K}$. We define a **vector topology** on $X$ to be a topology on $X$ such that vector addition and scalar multiplication are continuous.*

**Proposition 5.2.1** (Stability under Linear Combinations)**.** *Let $X$ be a normed vector space over $\mathbb{F}$. Let $K$ be a compact set in the space. Let $C$ be a closed set in the space. Then $\forall \alpha, \beta \in \mathbb{F}, S := \alpha K + \beta C$ is closed.*

*Proof.*
The case where $\beta = 0$ is trivial. I will assume $\beta \neq 0$.
Let $\alpha, \beta \in \mathbb{F}$ be arbitrary.
Let $\{s_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in $S$ that converges.
Say the limit is $s_\infty$.
Since $s_i \in S$ for any $i \in \mathbb{N}$ and $S = \alpha K + \beta C$, $s_i = \alpha k_i + \beta c_i$ for some $k_i \in K$ and some $c_i \in C$, for any $i \in \mathbb{N}$.
Since $\{k_i\}_{i \in \mathbb{N}}$ is a sequence in $K$ and $K$ is compact, there exists a convergent subsequence $\{k_i\}_{i \in I}$ of $\{k_i\}_{i \in \mathbb{N}}$ in $K$.
Say $\{k_i\}_{i \in I}$ converges to $k_\infty \in K$.
Since $\{s_i\}_{i \in \mathbb{N}}$ converges to $s_\infty$, $\{s_i\}_{i \in I}$ also converges to $s_\infty$.
Since $s_i = \alpha k_i + \beta c_i$, $c_i = \beta^{-1}(s_i - \alpha k_i)$.
Define $c_\infty := \beta^{-1}(s_\infty - \alpha k_\infty)$
Since $\{s_i\}_{i \in I}$ converges to $s_\infty$ and $\{k_i\}_{i \in I}$ converges to $k_\infty$ and $c_i = \beta^{-1}(s_i - \alpha k_i)$, $\{c_i\}_{i \in I}$ converges to $c_\infty$.

Since $\{c_i\}_{i\in I}$ is a sequence in $C$ and converges to $c_\infty$ and $C$ is closed, $c_\infty \in C$.

Since $s_\infty = \alpha k_\infty + \beta c_\infty$ and $k_\infty \in K$ and $c_\infty \in C$, $s_\infty \in \alpha K + \beta C$.

Since for any sequence in $S$ that converges, the limit is also in $S$, $S$ is closed.

■

**Remark.** *The sum of two closed sets may not be closed.*

*Proof.*

**Counter-example 1**

Consider $A := \{n : n \in \mathbb{N}\}$ and $B := \{n + \frac{1}{n} : n \in \mathbb{N}\}$.

(`https://math.stackexchange.com/questions/124130/sum-of-two-closed-sets-in-mathbb-r-is-closed`)

Their sum contains the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ but does not contain 0.

**Counter-example 2**

Consider $A := \mathbb{R} \times \{0\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x, y > 0, xy \geq 1\}$.

Their sum is $\mathbb{R} \times \mathbb{R}_{++}$.

■

## 5.3   Neighborhoods

# Chapter 6

# Sequence Spaces

## 6.1 $\ell_p$ Space

**Definition** ($\ell_p^{(n)}$ Space). *We define the $\ell_p^{(n)}$ space to be the set of all sequences $\{x_i\}_{i=1}^{i=n}$ such that*

**Definition** ($\ell_p$ Space). *We define the $\ell_p$ space to be the set of all sequences $x$ such that $\|x\|_p$ is finite, equipped with the p-norm $\|\cdot\|_p$.*

**Proposition 6.1.1.** *For $p \in [1, +\infty)$, $(\ell_p, \|\cdot\|_p)$ is complete.*

*Proof.*

Let $\{x_n\}_{n \in \mathbb{N}}$ be an arbitrary Cauchy sequence in $\ell_p$.

Since $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy in $\ell_p$, $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $\forall m, n > N$, we have $\|x_m - x_n\|_p < \varepsilon$.

Since $\|x_m - x_n\|_p < \varepsilon$ and $|x_m^{(i)} - x_n^{(i)}| \leq \|x_m - x_n\|_p$ for any $i \in \mathbb{N}$, $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$ for any $i \in \mathbb{N}$.

Since for any $i \in \mathbb{N}$ and any positive number $\varepsilon$, there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N$, we have $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$, by definition, $\{x_n^{(i)}\}_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{F}$.

Since $\{x_n^{(i)}\}_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{F}$ and $\mathbb{F}$ is complete, $\{x_n^{(i)}\}_{n \in \mathbb{N}}$ converges.

Let $x_0^{(i)} = x_n^{(i)}$. Let $x_0 = \{x_0^{(i)}\}_{i \in \mathbb{N}}$.

$$\|x_0\|_p = \left(\sum_{i=1}^{\infty} |x_0^{(i)}|^p\right)^{\frac{1}{p}}$$

∎

## 6.2   $c_0$ Space and $c_{00}$ Space

**Definition** ($c_0$ Space). *We define $c_0$ to be*

$$c_0 := \big\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\big\}.$$

**Definition** ($c_{00}$ Space). *We define $c_{00}$ to be*

$$c_{00} := \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}, \forall n > N, x_n = 0\big\}.$$

*i.e., the set of all eventually zero sequences of real numbers. i.e., the set of all sequences with finite support.*

**Proposition 6.2.1.** *The $c_{00}$ is not complete in $(\ell_1, \|\cdot\|_1)$.*

*Proof.* Define a sequence of vectors $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ by $\mathfrak{x}_i^j := \frac{1}{j^2}$ for $j \in \{1..i\}$ and $\mathfrak{x}_i^j := 0$ for $j > i$. Then $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ converges to something that is not in $c_{00}$. ∎

**Proposition 6.2.2.** *The closure of $c_{00}$ in the space $(\mathbb{R}^\omega, d_1)$ is $\ell_1$.*

*Proof.* For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq \ell_1$. Let $x$ be an arbitrary element in $\mathrm{cl}(c_{00})$. Since $x \in \mathrm{cl}(c_{00})$, there exists another element $y \in c_{00}$ such that $d_1(x, y) < 1$. Let $N \in \mathbb{N}$ be such that $\forall n > N, y_n = 0$. Then

$$
\begin{aligned}
& d_1(x, y) < 1 \\
\iff\; & \sum_{n\in\mathbb{N}} |x_n - y_n| < 1 \\
\iff\; & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n - y_n| < 1 \\
\iff\; & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n| < 1 \\
\implies\; & \sum_{n=1}^{N} \big||x_n| - |y_n|\big| + \sum_{n>N} |x_n| < 1 \\
\implies\; & \sum_{n=1}^{N} \big(|x_n| - |y_n|\big) + \sum_{n>N} |x_n| < 1 \\
\implies\; & \sum_{n=1}^{N} |x_n| - \sum_{n=1}^{N} |y_n| + \sum_{n>N} |x_n| < 1 \\
\iff\; & \sum_{n\in\mathbb{N}} |x_n| - \sum_{n=1}^{N} |y_n| < 1 \\
\iff\; & \sum_{n\in\mathbb{N}} |x_n| < 1 + \sum_{n=1}^{N} |y_n|.
\end{aligned}
$$

Since $\sum_{n\in\mathbb{N}}|x_n|$ is bounded, $x\in\ell_1$.

For the reverse direction, we are to prove that $\ell_1\subseteq\mathrm{cl}(c_{00})$. Let $x$ be an arbitrary element in $\ell_1$. For $i\in\mathbb{N}$, define $x^i=\{x^i_j\}_{j\in\mathbb{N}}$ as $x^i_j=x_j$ for $j\le i$ and $x^i_j=0$ for $j>i$. Then $\forall i\in\mathbb{N}, x^i\in c_{00}$. Then

$$\lim_{i\in\mathbb{N}}d_1(x^i,x)$$
$$=\lim_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}|x^i_j-x_j|$$
$$=\lim_{i\in\mathbb{N}}\sum_{j>i}|x^i_j-x_j|$$
$$=\lim_{i\in\mathbb{N}}\sum_{j>i}|x_j|$$
$$=0.$$

That is, $\lim_{i\in\mathbb{N}}d_1(x^i,x)=0$. So $\lim_{i\in\mathbb{N}}x^i=x$. So $x\in\mathrm{cl}(c_{00})$.

∎

**Proposition 6.2.3.** *The closure of $c_{00}$ in the space $(\mathbb{R}^\omega,d_\infty)$ is $c_0$.*

*Proof.* For one direction, we are to prove that $\mathrm{cl}(c_{00})\subseteq c_0$. Let $x$ be an arbitrary element in $\mathrm{cl}(c_{00})$. Let $\varepsilon$ be an arbitrary positive real number. Since $x\in\mathrm{cl}(c_{00})$, there exists another element $y$ in $c_{00}$ such that $d_\infty(x,y)<\varepsilon$. That is, $\forall j\in\mathbb{N}, |x_j-y_j|<\varepsilon$. Since $y\in c_{00}$, $\exists N\in\mathbb{N}$ such that $\forall j>N, y_j=0$. So $\forall j>N, |x_j|<\varepsilon$. That is,

$$\forall\varepsilon>0,\quad,\exists N\in\mathbb{N},\quad\forall j>N,\quad|x_j|<\varepsilon.$$

By definition of convergence of limits, $\lim_{j\in\mathbb{N}}x_j=0$. So $x\in c_0$.

For the reverse direction, we are to prove that $c_0\subseteq\mathrm{cl}(c_{00})$. Let $x$ be an arbitrary element in $c_0$. For $i\in\mathbb{N}$, define $x^i$ as $x^i_j=x_j$ for $j\le i$ and $x^i_j=0$ for $j>i$. Then $\forall i\in\mathbb{N}, x^i\in c_{00}$. Let $\varepsilon$ be an arbitrary positive real number. Since $x\in c_0$,

$$\exists N\in\mathbb{N},\quad\forall j>N,\quad|x_j|<\varepsilon/2.$$

Let $i>N$. Then

$$d_\infty(x^i,x)$$
$$=\sup_{j\in\mathbb{N}}|x^i_j-x_j|$$
$$=\sup_{j>i}|x^i_j-x_j|$$
$$=\sup_{j>i}|x_j|$$
$$\le\varepsilon/2<\varepsilon.$$

That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall i > N, \quad d_\infty(x^i, x) < \varepsilon.$$

By definition of convergence of sequences, $\lim_{i \in \mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

∎

**Proposition 6.2.4.** *Let $A := \left\{ \{x_n\}_{n \in \mathbb{N}} \in c_{00} : \sum_{n \in \mathbb{N}} x_n = 0 \right\}$. Then $A$ is a subset of $\ell^1$ and is closed in $(\ell^1, d_1)$. i.e. $\mathrm{cl}(A) = A$ in $(\ell^1, d_1)$.*

*Proof.* Let $x = \{x^i\}_{i \in \mathbb{N}}$ be a sequence in $\ell^1$, where each $x^i = \{x^i_j\}_{j \in \mathbb{N}}$ is an element in $A$, that converges in $(\ell^1, d_1)$. Say $\lim_{i \to \infty} x^i = x^\infty$.

First I claim that $x^\infty \in c_{00}$.

Now I claim that $\sum_{j \in \mathbb{N}} x^\infty_j = 0$. i.e. $x^\infty \in A$. Since $x^\infty \in c_{00}$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad x^\infty_j = 0.$$

Define $y_i := \sum_{j=1}^{N} x^i_j$. Define $y_\infty := \sum_{j=1}^{N} x^\infty_j$. It is easy to see that $\lim_{i \in \mathbb{N}} y_i = y_\infty$. Assume for the sake of contradiction that $y_\infty \neq 0$. i.e. $\{y_i\}_{i \in \mathbb{N}}$ does not converge to 0. Then

$$\exists \varepsilon_0 > 0, \quad \forall M \in \mathbb{N}, \quad \exists i_0 > M, \quad |y_{i_0} - 0| = |y_{i_0}| \geq \varepsilon_0. \tag{1}$$

Since $\lim_{i \to \infty} x^i = x^\infty$,

$$\exists M_0 \in \mathbb{N}, \quad \forall i > M_0, \quad d_1(x^i, x^\infty) < \varepsilon_0. \tag{2}$$

Consider statement (1) for a particular $M$, $M_0$, we have

$$\exists i_0 > M_0, \quad |y_{i_0}| \geq \varepsilon_0. \tag{3}$$

That is,

$$\left| \sum_{j=1}^{N} x^{i_0}_j \right| \geq \varepsilon_0. \tag{3'}$$

Consider statement (2) for a particular $i$, $i_0$, we have

$$d_1(x^{i_0}, x^\infty) < \varepsilon_0. \tag{4}$$

From statement (4) we can derive:

$$d_1(x^{i_0}, x^\infty) < \varepsilon_0$$

$$\Longleftrightarrow \quad \sum_{j \in \mathbb{N}} |x_j^{i_0} - x_j^\infty| < \varepsilon_0$$

$$\Longleftrightarrow \quad \sum_{j=1}^{N} |x_j^{i_0} - x_j^\infty| + \sum_{j > N} |x_j^{i_0} - x_j^\infty| < \varepsilon_0$$

$$\Longrightarrow \quad \sum_{j > N} |x_j^{i_0} - x_j^\infty| < \varepsilon_0$$

$$\Longleftrightarrow \quad \sum_{j > N} |x_j^{i_0} - 0| < \varepsilon_0$$

$$\Longleftrightarrow \quad \sum_{j > N} |x_j^{i_0}| < \varepsilon_0$$

$$\Longrightarrow \quad \left| \sum_{j > N} x_j^{i_0} \right| < \varepsilon_0$$

$$\Longrightarrow \quad \left| \sum_{j > N} x_j^{i_0} \right| < \varepsilon_0$$

$$\Longleftrightarrow \quad \left| \sum_{j \in \mathbb{N}} x_j^{i_0} - \sum_{j=1}^{N} x_j^{i_0} \right| < \varepsilon_0$$

$$\Longleftrightarrow \quad \left| 0 - \sum_{j=1}^{N} x_j^{i_0} \right| < \varepsilon_0$$

$$\Longleftrightarrow \quad \left| \sum_{j=1}^{N} x_j^{i_0} \right| < \varepsilon_0.$$

This contradicts to statement (3'). So the original assumption that $y_\infty \neq 0$ is false. i.e. $y_\infty = 0$. It follows that $\sum_{j \in \mathbb{N}} x_j^\infty = 0$. This completes the proof.

∎

# Chapter 7

# Function Spaces

## 7.1 The $\mathcal{L}^p$ Norm

$$\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

the instructors' answer, *where instructors collectively construct a single answer*

In the sup norm, convergence coincides with uniform convergence. Moreover, $C[a, b]$ is complete in this norm. It is not complete in any of the $L^p$ norms for $1 \leq p < \infty$. The completion in these norms is called $L^p(a, b)$.

undo thanks | 1

Updated 1 day ago by Kenneth Davidson

# Chapter 8

# Hilbert Space

## 8.1 Hilbert Spaces

**Definition** (Hilbert Space)**.** *We define a **Hilbert space** to be a complete inner product space.*

**Example 8.1.1.** *$\ell^2$ is a Hilbert space.*

# Chapter 9

# Equicontinuity in Metric Spaces

## 9.1 Definitions

**Definition** ((Pointwise) Equicontinuity)**.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $\mathcal{F}$ be a collection of functions from $X$ to $Y$. Let $x_0$ be a point in $X$. We say that $\mathcal{F}$ is* **(pointwise) equicontinuous** *at point $x_0$ if for any positive number $\varepsilon$, there exists some number $\delta(x_0, \varepsilon)$ such that for any function $f$ in $\mathcal{F}$ and any point $x$ in $X$, we have*

$$d_Y(f(x), f(x_0)) < \varepsilon$$

*whenever $d_X(x, x_0) < \delta(x_0, \varepsilon)$ is satisfied.*

**Definition** (Uniform Equicontinuity)**.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $\mathcal{F}$ be a collection of functions from $X$ to $Y$. We say that $\mathcal{F}$ is* **uniformly equicontinuous** *if for any positive number $\varepsilon$, there exists some number $\delta(\varepsilon)$ such that for any function $f$ in $\mathcal{F}$ and any points $x_1$ and $x_2$ in $X$, we have*

$$d_Y(f(x_1), f(x_2)) < \varepsilon$$

*whenever $d_X(x_1, x_2) < \delta(\varepsilon)$ is satisfied.*

## 9.2 Sufficient Conditions

**Proposition 9.2.1.** *The closure of an equicontinuous family of functions is equicontinuous.*

*Proof.*
Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces.
Let $\mathcal{F}$ be an equicontinuous family of functions from $X$ to $Y$.
We are to prove that $cl(\mathcal{F})$ is equicontinuous.

Let $x_0$ be an arbitrary point in $X$.

Let $\varepsilon$ be an arbitrary positive number.

Since $\mathcal{F}$ is equicontinuous at point $x_0$, there exists some $\delta(x_0, \varepsilon)$ such that for any function $f$ in $\mathcal{F}$ and any point $x$ in $X$ such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon/3$.

Let $f$ be an arbitrary function in $cl(\mathcal{F})$.

Let $x$ be an arbitrary point in $X$ such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$.

Since $f \in cl(\mathcal{F})$, there exists some function $f_0 \in \mathcal{F}$ such that $d_\infty(f, f_0) < \varepsilon/3$.

Since $d_\infty(f, f_0) < \varepsilon/3$, $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$.

Since $f_0 \in \mathcal{F}$ and $d_X(x, x_0) < \delta(x_0, \varepsilon)$, $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$.

Since $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$ and $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$, $d_Y(f(x), f(x_0)) < \varepsilon$.

Since for any positive number $\varepsilon$, there exists some $\delta(x_0, \varepsilon)$ such that for any function $f$ in $cl(\mathcal{F})$ and any point $x$ in $X$ such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, by definition of equicontinuous, $cl(\mathcal{F})$ is equicontinuous at point $x_0$.

Since $cl(\mathcal{F})$ is equicontinuous at point $x_0$ for any point $x_0$ in $X$, $cl(\mathcal{F})$ is equicontinuous.

■

# Chapter 10

# Operators

## 10.1   Bounded Operators

**Definition** (Bounded Operator)**.** *Let $X$ and $Y$ be normed linear spaces. Let $T$ be a linear map from $X$ to $Y$. We say that $T$ is a **bounded operator** if*

$$\exists k \in \mathbb{R}, \quad \forall x \in X, \quad \|Tx\|_Y \leq k\|x\|_X.$$

**Definition** (Operator Norm)**.** *Let $X$ and $Y$ be normed linear spaces. Let $T$ be a bounded operator from $X$ to $Y$. We define the **operator norm** of $T$, denoted by $\|T\|$, to be the number given by*
$$\|T\| := \inf\{k \in \mathbb{R} : \forall x \in X, \|Tx\|_Y \leq k\|x\|_X\}.$$

**Proposition 10.1.1.** *Let $X$ and $Y$ be normed linear spaces. Let $T$ be a linear map from $X$ to $Y$. Then $T$ is bounded if and only if $T$ is continuous.*

**Example 10.1.1** (Multiplication Operator)**.** *Let $X = \big(\mathcal{C}([0,1], \mathbb{C}), \|\cdot\|_\infty\big)$. Let $f$ be a function in $X$. We define the **multiplication operator** on $X$, w.r.t. $f$, denoted by $M_f$, as*

$$M_f(g) = fg.$$

*Then $M_f$ is bounded and $\|M_f\| = \|f\|_\infty$.*

*Proof.* Let $g$ be an arbitrary function in $X$. Then

$$\|M_f g\|_\infty = \|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty.$$

So $\|f\|_\infty$ is an element of the set $S = \{k \in \mathbb{R} : \forall x \in X, \|Tx\|_Y \leq k\|x\|_X\}$. So $\|M_f\| \leq \|f\|_\infty$. Consider $g_0$ given by $g_0(x) = 1$. Then

$$\|M_f g_0\|_\infty = \|fg_0\|_\infty = \|f\|_\infty = \|f\|_\infty \|g\|_\infty.$$

For any $k \in S$, if $k < \|f\|_\infty$, $k \notin S$.  So $\forall k \in S$, $k \geq \|f\|_\infty$.  So $\|f\|_\infty$ is a lower bound for the set $S$.  So $\|M_f\| \geq \|f\|_\infty$.  Since $\|M_f\| \leq \|f\|_\infty$ and $\|M_f\| \geq \|f\|_\infty$, we get $\|M_f\| = \|f\|_\infty$. ∎

**Example 10.1.2** (Weighted Shifts)**.**

- *Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(w_n)_{n \in \mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an **unilateral forward weighted shift** $W$ on $\mathcal{H}$ as*

$$W(x_n) := (0, w_1 x_1, w_2 x_2, w_3 x_3, ...).$$

  *Then $W$ is bounded and $\|W\| = \sup\{|w_n| : n \in \mathbb{N}\}$.*

- *Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(v_n)_{n \in \mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an **unilateral backward weighted shift** $V$ on $\mathcal{H}$ as*

$$V(x_n) := (v_1 x_2, v_2 x_3, v_3 x_4, ...).$$

  *Then $V$ is bounded and $\|V\| = \sup\{|v_n| : n \in \mathbb{N}\}$.*

- *Let $\mathcal{H} = \ell^2_{\mathbb{Z}}$. Let $(u_n)_{n \in \mathbb{Z}} \in \ell^\infty_{\mathbb{Z}}$. We define a **bilateral weighted shift** $U$ on $\mathcal{H}$ as*

$$U(x_n) := (u_{n-1} x_{n-1})_{n \in \mathbb{Z}}.$$

  *Then $U$ is bounded and $\|U\| = \sup\{|u_n| : n \in \mathbb{Z}\}$.*

## 10.2   Space of Bounded Operators

**Proposition 10.2.1.** *Let $X$ and $Y$ be normed linear spaces. Let $\mathcal{B}(X, Y)$ be the space of bounded linear operators from $X$ to $Y$. Then if $Y$ is complete, $\mathcal{B}(X, Y)$ is complete.*

## 10.3   Dual Spaces

**Definition** (Dual Space)**.** *Let $X$ be a normed linear space over field $\mathbb{K}$. We define the **dual** of $X$, denoted by $X^*$, to be the space $\mathcal{B}(X, \mathbb{K})$.*

**Proposition 10.3.1.** *Let $X$ be a normed linear space. Then there exists a contractive map from $X$ to its double dual $X^{**}$.*

# Chapter 11

# Adjoint Operator

## 11.1 Definitions

**Definition** (Adjoint Matrix). *Let $A$ be an $m \times n$ matrix. We define the **adjoint** of $A$, denoted by $A^*$, to be an $n \times m$ matrix given by*

$$(A^*)_{ij} := \overline{(A)_{ji}}.$$

**Definition** (Adjoint Operator). *Let $V$ and $W$ be inner product spaces. Let $T$ be a linear map from $V$ to $W$. We define the **adjoint** of $T$, denoted by $T^*$, to be a map from $W$ to $V$ such that*

$$\forall x \in V, \forall y \in W, \quad \langle T(x), y \rangle_W = \langle x, T^*(y) \rangle_V.$$

**Proposition 11.1.1** (Existence). *Let $V$ be a finite-dimensional inner product space and $T$ be a linear operator on $V$. Then the adjoint of $T$ exists.*

**Proposition 11.1.2** (Uniqueness). *Let $V$ be an inner product space and $T$ be a linear operator on $V$. Then the adjoint of $T$ is unique, provided that it exists.*

## 11.2 Properties of the Adjoint Operator

**Proposition 11.2.1.** *Let $V$ be an inner product space. Then*

*(1) $(I_V)^* = I_V$ where $I_V$ is the identity operator on $V$.*

*(2) $T^{**} = T$ for any linear operator $T$ on $V$.*

**Proposition 11.2.2.** *Let $V$ be an inner product space and $T$ be a linear operator on $V$. Then $T^*$ is also linear.*

**Proposition 11.2.3.** *Let $V$ be an inner product space. Then*

*(1) For any linear operators $T$ and $U$,*

$$(T + U)^* = T^* + U^*.$$

*(2) For any linear operator $T$,*
$$(cT)^* = \overline{c} \cdot T^*.$$

*(3) For any linear operator $T$ and $U$,*

$$(TU)^* = U^* T^*.$$

**Proposition 11.2.4.** *Let $V$ be a finite-dimensional inner product space and $T$ be a linear operator on $V$. Then if $T$ is invertible, $T^*$ is also invertible.*

**Proposition 11.2.5.** *Let $V$ be an inner product space and $T$ be an invertible linear operator on $V$. Then $(T^{-1})^* = (T^*)^{-1}$.*

## 11.3  Normal Operators

**Definition** (Normal). *Let $V$ be an inner product space and $T$ be a linear operator on $V$. We say that $T$ is **normal** if $TT^* = T^*T$.*

## 11.4  Self-adjoint

# Chapter 12

# Convolution

**Definition** (Convolution). *Let $f$ and $g$ be functions from $\mathbb{R}$ to $\mathbb{R}$. We define the **convolution** of $f$ and $g$, denoted by $f * g$, to be a function on $\mathbb{R}$ given by*

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)dt.$$

# Chapter 13

# Coercive Functions

## 13.1   Definitions

**Definition** (Coercive). *Let $f$ be a function from $\mathbb{R}^d$ to $\mathbb{R}^*$. We say that $f$ is **coercive** if $\lim_{\|x\| \to \infty} f(x) = +\infty$.*

## 13.2   Properties

**Proposition 13.2.1.** *Let $f$ be a proper lower semi-continuous function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let $K$ be a compact set in $\mathbb{R}^d$. Assume $K \cap \operatorname{dom}(f) \neq \emptyset$. Then $f$ attains its minimum over $K$.*

*Proof.*

Define $m := \inf_{x \in K} f(x)$.

Since $m = \inf_{x \in K} f(x)$, there exists a sequence $\{x_i\}_{i \in \mathbb{N}}$ in $K$ such that $\lim_{i \to \infty} f(x_i) = m$.

Since $K$ is compact and $\{x_i\}_{i \in \mathbb{N}} \subseteq K$, there exists a convergent subsequence $\{x_i\}_{i \in I}$ in $K$ where $I$ is an infinite subset of $\mathbb{N}$.

Say the limit is $x_\infty$ where $x_\infty \in K$.

Since $\lim_{i \to \infty} f(x_i) = m$, we get $\lim_{i \in I, i \to \infty} f(x_i) = m$.

Since $\lim_{i \in I, i \to \infty} f(x_i) = m$, we get $\liminf_{i \in I, i \to \infty} f(x_i) = m$.

Since $f$ is lower semi-continuous and $\lim_{i \in I, i \to \infty} x_i = x_\infty$, we get $f(x_\infty) \leq \liminf_{i \in I, i \to \infty} x_i$.

That is, $f(x_\infty) \leq m$.

Since $m = \inf_{x \in K} f(x)$, we have $\forall x \in K, f(x) \geq m$.

In particular, $f(x_\infty) \geq m$.

Since $f(x_\infty) \geq m$ and $f(x_\infty) \leq m$, $f(x_\infty) = m$.

Since $f$ is proper, $f(x_\infty) = m \neq -\infty$.

So $f$ attains its minimum at point $x_\infty$.

$\blacksquare$

**Proposition 13.2.2.** *Let $f$ be a proper, lower semi-continuous, and coercive function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let $C$ be a closed subset of $\mathbb{R}^d$. Assume $C \cap \operatorname{dom}(f) \neq \emptyset$. Then $f$ attains its minimum over $C$.*

*Proof.*

Since $C \cap \operatorname{dom}(f) \neq \emptyset$, take $x \in C \cap \operatorname{dom}(f)$.

Since $f$ is coercive, $\exists R$ such that $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$.

Since $x \in C \cap \operatorname{dom}(f)$ and $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$, the set of minimizers of $f$ over $C$ is the same as the set of minimizers of $f$ over $C \cap \operatorname{ball}[0, R]$.

Since $C$ and $\operatorname{ball}[0, R]$ are both closed, $C \cap \operatorname{ball}[0, R]$ is closed.

Since $\operatorname{ball}[0, R]$ is bounded, $C \cap \operatorname{ball}[0, R]$ is bounded.

Since $C \cap \operatorname{ball}[0, R]$ is closed and bounded, by the Heine-Borel Theorem, $C \cap \operatorname{ball}[0, R]$ is compact.

Since $f$ is proper and lower semi-continuous and $C \cap \operatorname{ball}[0, R]$ is compact, $f$ attains its minimum over $C \cap \operatorname{ball}[0, R]$.

So $f$ attains its minimum over $C$.

∎

# Chapter 14

# Unclassified Results

**Proposition 14.0.1.** *Let $(X, d)$ be a compact metric space. Let $L(X)$ be the set of all Lipschitz functions from $X$ to $\mathbb{R}$. Let $C(X)$ be the set of all continuous functions from $X$ to $\mathbb{R}$. Then $L(X)$ is dense in $C(X)$.*

**Proposition 14.0.2.** *Let $(V, \|\cdot\|)$ be a normed vector space. Let $S$ be a subset of $V$. Let $p$ be a vector in $V$. Then we have the followings.*

(1) $p + int(S) = int(p + S)$,

(2) $p + \text{cl}(S) = \text{cl}(p + S)$.

*Proof.*

***Proof of (1).***
For one direction, let $x$ be an arbitrary point in the set $(p + int(S))$.
We are to prove that $x \in int(p + S)$.
Since $x \in (p + int(S))$, $(x - p) \in int(S)$.
Since $(x - p) \in int(S)$, by definition of interior, there exists a radius $r$ such that

$$B(x - p, r) \subseteq S.$$

It follows that $B(x, r) \subseteq p + S$.
Since there exists a radius $r$ such that $B(x, r) \subseteq p + S$, by definition of interior,

$$x \in int(p + S).$$

For the reverse direction, let $x$ be an arbitrary point in $int(p + S)$.
We are to prove that $x \in p + int(S)$.
Since $x \in int(p + S)$, by definition of interior, there exists a radius $r$ such that

$$B(x, r) \subseteq (p + S).$$

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It follows that $B(x - p, r) \subseteq S$.

Since there exists a radius $r$ such that $B(x - p, r) \subseteq S$, by definition of interior,

$$(x - p) \in int(S).$$

Since $(x - p) \in int(S)$, we get $x \in (p + int(S))$.

**Proof of (2).**

For one direction, let $x$ be an arbitrary point in the set $(p + cl(S))$.

We are to prove that $x \in cl(p + S)$.

Since $x \in (p + cl(S))$, we get $(x - p) \in cl(S)$.

Since $(x - p) \in cl(S)$, by definition of closure, for any radius $r$, we have

$$B(x - p, r) \cap S \neq \varnothing.$$

It follows that $B(x, r) \cap (p + S) \neq \varnothing$.

Since for any radius $r$, $B(x, r) \cap (p + S) \neq \varnothing$, by definition of closure, we get

$$x \in cl(p + S).$$

For the reverse direction, let $x$ be an arbitrary point in $cl(p + S)$.

We are to prove that $x \in (p + cl(S))$.

Since $x \in cl(p + S)$, by definition of closure, for any radius $r$, we have

$$B(x, r) \cap (p + S) \neq \varnothing.$$

It follows that $B(x - p, r) \cap S \neq \varnothing$.

Since for any radius $r$, $B(x - p, r) \cap S \neq \varnothing$, by definition of closure, we get

$$(x - p) \in cl(S).$$

Since $(x - p) \in cl(S)$, we get $x \in (p + cl(S))$.

$$\blacksquare$$

**Proposition 14.0.3.** *Let $(V, \|\cdot\|)$ be a normed vector space. Let $S$ be a subset of $V$. Let $\lambda$ be a non-zero real number. Then*

*(1)  $\lambda int(S) = int(\lambda S)$.*

*(2)  $\lambda \, cl(S) = cl(\lambda S)$.*

*Proof.*

**Proof of (1).**

For one direction, let $x$ be an arbitrary point in $\lambda int(S)$.

We are to prove that $x \in int(\lambda S)$.

Since $x \in \lambda int(S)$, we get $x/\lambda \in int(S)$.

Since $x/\lambda \in int(S)$, by definition of interior, there exists a radius $r$ such that

$$B(x/\lambda, r) \subseteq S.$$

Let $y$ be an arbitrary point in $B(x, \lambda r)$.

Since $y \in B(x, \lambda r)$, we get $\|y - x\| \leq \lambda r$.

Since $\|y - x\| \leq \lambda r$, we get $\|y/\lambda - x/\lambda\| \leq r$.

Since $\|y/\lambda - x/\lambda\| \leq r$, we get $y/\lambda \in B(x/\lambda, r)$.

Since $y/\lambda \in B(x/\lambda, r)$ and $B(x/\lambda, r) \subseteq S$, we get $y/\lambda \in S$.

Since $y/\lambda \in S$, we get $y \in \lambda S$.

Since any point in $B(x, \lambda r)$ is also in $\lambda S$, we get $B(x, \lambda r) \subseteq \lambda S$.

Since there exists a radius $r$ such that $B(x, \lambda r) \subseteq \lambda S$, by definition of interior, we get

$$x \in int(\lambda S).$$

For the reverse direction,

$\blacksquare$