Real Analysis

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Chapter 1

Limit Theory for the Real Numbers

PROPOSITION 1.1.
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Suppose that $\lim\limits_{n\in\mathbb{N}} x_n = x_\bullet$ for some $x_\bullet \in \mathbb{R}$. Then

$$\lim_{n\in\mathbb{N}} \overline{x}_n := \lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=1}^{n} x_i = x_\bullet.$$

Chapter 2

Differentiation

2.1 Theory in One Dimension

DEFINITION 2.1 (Differentiability, Derivative)**.**
Let f be a function from Ω to $\mathbb{R}$ where Ω is some open subset of $\mathbb{R}$. Let x be a point in Ω. We say that f is **differentiable** if the limit L given by

$$L := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. In this case, we define the **derivative** of f at point x to be the number L.

PROPOSITION 2.2.
Differentiability implies continuity.

2.2 Theory in Higher Dimensions

DEFINITION 2.3 (Directional Derivative)**.**
Let f be a proper function from $\mathbb{R}^n$ to $\mathbb{R}^*$. Let x_0 be a point in $\mathrm{dom}(f)$. Let d be a point in $\mathbb{R}^n$. We define the **directional derivative** of f at point x_0 in the direction of d, denoted by $f'(x_0; d)$, to be a number given by

$$f'(x_0; d) := \lim_{t \downarrow 0} \frac{f(x_0 + td) - f(x_0)}{t}.$$

Note that this is a single-sided limit.

EXAMPLE 2.4.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) := \|x\|_\infty$. Let $x, d \in \mathbb{R}^n$. Define

$$K_1 := \left\{ i \in \{1, ..., n\} : |x_i| = \max_{j \in \{1,...,n\}} \{|x_j|\} \right\}.$$

Define

$$K_2 := \left\{ i \in K : \text{sign}(x_i) d_i = \max_{j \in K} \{\text{sign}(x_j) d_j\} \right\}$$

where $\text{sign} : \mathbb{R} \to \{\pm 1\}$ is given by $\text{sign}(a) := 1$ if $a \geq 0$ and $\text{sign}(a) := -1$ if $a < 0$. Let $k \in K_2$ be arbitrary. Then the directional derivative of f at point x in direction d is

$$f'(x; d) = \text{sign}(x_k) d_k.$$

DEFINITION 2.5 (Differentiable)**.**
Let f be a proper function from $\mathbb{R}^n$ to $\mathbb{R}^*$. Let x_0 be a point in $\text{dom}(f)$. We say that f is **differentiable** at point x_0 if there exists a linear operator ∇ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$\lim_{\|y\| \to 0} \frac{\big| f(x_0 + y) - f(x_0) - \langle \nabla f(x_0), y \rangle \big|}{\|y\|} = 0.$$

2.3 Properties

PROPOSITION 2.6.
Let f be a proper function from $\mathbb{R}^n$ to $\mathbb{R}^*$. Let x_0 be a point in $\text{dom}(f)$. Let d be a point in $\mathbb{R}^n$. Assume that f is differentiable at point x_0. Then we have

$$f'(x_0; d) = \langle \nabla f(x), d \rangle.$$

2.4 Examples

EXAMPLE 2.7.

$$f(x, y) = (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right)$$

for $(x, y) \neq 0$ and $f(0, 0) = 0$.

2.5 Higher Order Differentiation

THEOREM 2.8 (Hermann Schwarz and Alexis Clairaut)**.**
Let f be a function from some subset Ω of $\mathbb{R}^n$ to $\mathbb{R}^n$. Let p be an interior point of Ω. Then if f has continuous second order partial derivatives at point p, we get

$$\forall i, j \in \{1, ..., n\}, \frac{\partial^2 f}{\partial x_i \partial x_j}(p) = \frac{\partial^2 f}{\partial x_j \partial x_i}(p).$$

2.6 Differentiation w.r.t. Vectors

DEFINITION 2.9.
Let $\vec{x} = (x_1, ..., x_n)$ be a vector. Let $y = f(\vec{x})$. We define

$$\frac{\partial y}{\partial \vec{x}} := \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}.$$

PROPOSITION 2.10.
Quick results:

1. $\dfrac{\partial \left[\vec{a} \cdot \vec{x}\right]}{\partial \vec{x}} = \vec{a}$.
2. $\dfrac{\partial \left[\vec{x}^T A \vec{x}\right]}{\partial \vec{x}} = Ax + A^T x$.

2.7 Inverse Function Theorem

THEOREM 2.11.
Let F be a C^1 function from Ω to $\mathbb{R}^n$ where Ω is some open subset of $\mathbb{R}^n$. Let x be some point in Ω. Then if $|J_F(p)| \neq 0$, F is invertible near x. Further, F^{-1} is C^1 at $F(x)$ and

$$J_{F^{-1}}(F(x)) = (J_F(x))^{-1}.$$

Chapter 3

Scalar Series

3.1 Convergence

DEFINITION 3.1 (Convergence)**.**
Let $S = \sum_{i=1}^{\infty} a_i$ be an infinite series. We say that S **converges** if the limit $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ exists.

DEFINITION 3.2 (Absolute Convergence)**.**
Let $S = \sum_{i=1}^{\infty} a_i$ be an infinite series. We say that S **converges absolutely** if the series $\sum_{i=1}^{\infty} |a_i|$ converges.

DEFINITION 3.3 (Conditional Convergence)**.**
Let $S = \sum_{i=1}^{\infty} a_i$ be an infinite series. We say that S **converges conditionally** if it converges but does not converge absolutely.

3.2 Properties

THEOREM 3.4 (Bernhard Riemann)**.**
If an infinite series of real numbers is conditionally convergent, then its terms can be arranged in a permutation so that the new series converges to an arbitrary real number,

or diverges. i.e., if $S = \sum_{i=1}^{\infty} a_i$ where $a_i \in \mathbb{R}$ converges conditionally, then for any real number l, there exists some permutation σ such that $S_\sigma := \sum_{i=1}^{\infty} a_{\sigma(i)} = l$; and there exists some permutation τ such that $S_\tau := \sum_{i=1}^{\infty} a_{\tau(i)}$ diverges.

PROPOSITION 3.5.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Suppose that the partial sum sequence $\{S_n\}_{n\in\mathbb{N}}$ is bounded. Then $\{x_n\}_{n\in\mathbb{N}}$ must be bounded.

Proof. Assume for the sake of contradiction that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is unbounded. Since the partial sum sequence $\{S_n\}_{n\in\mathbb{N}}$ is bounded, $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, |S_n| \leq M$. $\square$

3.3 Convergence Tests

THEOREM 3.6 (Ernst Kummer)**.**
Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of scalars. Consider the series $\sum_{n=1}^{\infty} a_n$. Let ζ_n be an auxiliary sequence of positive constants. Define

$$\rho_n := \zeta_n \frac{a_n}{a_{n+1}} - \zeta_{n+1}.$$

Then the series

1. converges if $\liminf_{n\to\infty} \rho_n > 0$, and
2. diverges if $\limsup_{n\to\infty} \rho_n < 0$ and $\sum 1/\zeta_n$ diverges.

Chapter 4

Series of Functions

4.1 Power Series

DEFINITION 4.1.
A power series (in one variable) is an infinite series S of the form

$$S = \sum_{i=0}^{\infty} a_i (x - c)^i.$$

PROPOSITION 4.2.
Every power series is the Taylor series of some smooth function.

Chapter 5

Riemann Integration

5.1 Definitions

DEFINITION 5.1 (Riemann Sum)**.**
Let $(X, \|\cdot\|)$ be a Banach space. Let a and b be real numbers with $a < b$. Let f be a function from $[a, b]$ to X. Let $P = \{a = p_0 < p_1 < ... < p_{N-1} < p_N = b\}$ be a partition of the interval $[a, b]$. Let $P^* = \{\xi_i : i = 1..N\}$ be a set of choices of sample points where $\forall i = 1..N$, $\xi_i \in [p_{i-1}, p_i]$. We define the **Riemann sum** of f w.r.t. partition P and sample points P^*, denoted by $S(f, P, P^*)$, to be the vector given by

$$S(f, P, P^*) := \sum_{i=1}^{N} f(\xi_i)(p_i - p_{i-1}).$$

DEFINITION 5.2 (Riemann Integrable)**.**
Let $(X, \|\cdot\|)$ be a Banach space. Let a and b be real numbers with $a < b$. Let f be a function from $[a, b]$ to X. We say that f is **Riemann Integrable** if

$$\exists x_0 \in X, \forall \varepsilon > 0, \exists P \in \mathcal{P}[a, b], \forall Q \supseteq P, \forall Q^*, \quad \|x_0 - S(f, Q, Q^*)\| < \varepsilon.$$

PROPOSITION 5.3.
The vector x_0 in the definition is unique, if it exists.

DEFINITION 5.4 (Riemann Integral)**.**
Let $(X, \|\cdot\|)$ be a Banach space. Let a and b be real numbers with $a < b$. Let f be a Riemann integrable function from $[a, b]$ to X. We define the **Riemann Integral** of f,

denoted by $\int_a^b f$, to be the unique vector x_0. i.e.

$$x_0 = \int_a^b f.$$

5.2 Cauchy Criterion

PROPOSITION 5.5 (Cauchy Criterion)**.**
Let $(X, \|\cdot\|)$ be a Banach space. Let a and b be real numbers with $a < b$. Let f be a function from $[a, b]$ to X. Then f is integrable if and only if

$$\forall \varepsilon > 0,\ \exists P \in \mathcal{P}[a, b],\ \forall R_1, R_2 \supseteq P,\ \forall R_1^*, R_2^*, \quad \|S(f, R_1, R_1^*) - S(f, R_2, R_2^*)\| < \varepsilon.$$

5.3 Properties

PROPOSITION 5.6.
Continuous functions are Riemann integrable.