Convex Analysis

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Chapter 1

Affine Sets

1.1 Definitions

DEFINITION 1.1 (Affine Combination)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define an **affine combination** of S to be a point x in the space of the form

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where $n \in \mathbb{N}$, $v_i \in S$, $\forall i \in [n]$, $\lambda_i \in \mathbb{R}$, $\forall i \in [n]$, and $\sum_{i=1}^{n} \lambda_i = 1$.

DEFINITION 1.2 (Affine Span)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define the **affine span** of S, denoted by $\text{affspan}(S)$, to be the set of all affine combinations of S.

DEFINITION 1.3 (Affine Set)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We say that S is an **affine set** if and only if $S = \text{affspan}(S)$.

DEFINITION 1.4 (Affine Hull)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define the **affine hull** of S, denoted by $\operatorname{affhull}(S)$, to be the smallest affine set containing S.

THEOREM 1.5. Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. Then

$$\operatorname{affspan}(S) = \operatorname{affhull}(S).$$

Chapter 2

Generalized Interiors

2.1 Definitions

DEFINITION 2.1 (Relative Interior - 1)**.** Let $\mathcal{V}$ be a normed linear space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define the **relative interior** of S, denoted by $\mathrm{ri}(S)$, to be the interior of S for the topology relative to the affine hull $\mathrm{aff}(S)$. i.e., the set given by

$$\mathrm{ri}(S) := \left\{ x \in \mathrm{aff}(S) : \exists r > 0, \mathrm{ball}(x, r) \cap \mathrm{aff}(S) \subseteq S \right\}.$$

DEFINITION 2.2 (Relative Interior - 2)**.** Let $\mathcal{V}$ be a normed linear space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define the **relative interior** of S, denoted by $\mathrm{ri}(S)$, to be a subset of S given by

$$\mathrm{ri}(S) := \left\{ x \in S : \mathrm{cone}(S - x) = \mathrm{span}(S - x) \right\}.$$

DEFINITION 2.3 (Strong Relative Interior)**.** Let $\mathcal{V}$ be a normed linear space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define the **strong relative interior** of S, denoted by $\mathrm{sri}(S)$, to be a subset of S given by

$$\mathrm{sri}(S) := \left\{ x \in S : \mathrm{cone}(S - x) = \overline{\mathrm{span}}(S - x) \right\}.$$

DEFINITION 2.4 (Quasi-Relative Interior)**.** Let $\mathcal{V}$ be a normed linear space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define the **quasi-relative interior** of S, denoted by $\mathrm{qri}(S)$, to be a subset of S given by

$$\mathrm{qri}(S) := \left\{ x \in S : \overline{\mathrm{cone}}(S - x) = \overline{\mathrm{span}}(S - x) \right\}.$$

2.2 Basic Properties

PROPOSITION 2.5. For a singleton set S, $\mathrm{ri}(S) = S = \mathrm{cl}(S)$.

PROPOSITION 2.6. For any set S, we have $\mathrm{ri}(S) \subseteq S$.

REMARK 2.7. The relative interior operator is not monotonic. Consider $\mathbb{R}$ with the usual topology and sets $\{0\}$ and $[0,1]$. Then $\mathrm{ri}(\{0\}) = \{0\}$ and $\mathrm{ri}([0,1]) = (0,1)$.

PROPOSITION 2.8. Let $\mathcal{V}$ be a normed linear space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. Then if $\mathrm{int}(S) \neq \emptyset$ we have $\mathrm{ri}(S) = \mathrm{int}(S)$.

Proof. It suffices to show that $\mathrm{aff}(S) = \mathcal{V}$. Since $\mathrm{int}(S) \neq \emptyset$, we can take $x \in \mathrm{int}(S)$. Then $\exists r > 0, \mathrm{ball}(x, r) \subseteq S$. Then

$$\mathcal{V} = \mathrm{aff}(\mathrm{ball}(x, r)) \subseteq \mathrm{aff}(S) \subseteq \mathcal{V}.$$

So $\mathrm{aff}(S) = \mathcal{V}$. □

2.3 Arithmetic Properties

PROPOSITION 2.9 (Linearity)**.** Let $\mathcal{V}$ be a normed linear space over $\mathbb{R}$. Let C_1

and C_2 be convex subsets of $\mathbb{R}$. Let $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$\operatorname{ri}(\lambda_1 C_1 + \lambda_2 C_2) = \lambda_1 \operatorname{ri}(C_1) + \lambda_2 \operatorname{ri}(C_2).$$

PROPOSITION 2.10. Let C_1 be a convex set in $\mathbb{E}_1$. Let C_2 be a convex set in $\mathbb{E}_2$. Then

$$\operatorname{ri}(C_1 \oplus C_2) = \operatorname{ri}(C_1) \oplus \operatorname{ri}(C_2).$$

Chapter 3

Convex Sets

3.1 Definitions (bug)

DEFINITION 3.1 (Convex Combination)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define a **convex combination** of S to be a point x in $\mathcal{V}$ of the form

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where (1) $n \in \mathbb{N}$, (2) $v_1, ..., v_n \in S$, (3) $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$, and (4) $\sum_{i=1}^{n} \lambda_i = 1$.

DEFINITION 3.2 (Convex Span)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define a **convex span** of S, denoted by $\operatorname{convspan}(S)$, to be the set of all possible convex combinations of S.

DEFINITION 3.3 (Convex)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathbb{E}$. We say that S is **convex** if $S = \operatorname{convspan}(S)$, or equivalently, if

$$\forall x, y \in S, \forall \alpha, \beta \in [0, 1] : \alpha + \beta = 1, \quad \alpha x + \beta y \in S.$$

DEFINITION 3.4 (Convex Hull)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define the **convex hull** of S, denoted by $\operatorname{convhull}(S)$, to be the smallest convex set containing S.

PROPOSITION 3.5. Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. For any subset S of $\mathcal{V}$, we have $\operatorname{convspan}(S) = \operatorname{convhull}(S)$. They will both be denoted by $\operatorname{conv}(S)$ from now on.

Proof. **Forward Inclusion**: Let x be an arbitrary element of $\operatorname{convspan}(S)$. I will show that $x \in \operatorname{convhull}(S)$. Let C be an arbitrary convex set containing S. Since x is a convex combination of elements in S, x is also a convex combination of elements in C. So $x \in C$. This holds for any convex set in $\mathcal{V}$ containing S. So $x \in \operatorname{convhull}(S)$. So $\operatorname{convspan}(S) \subseteq \operatorname{convhull}(S)$.

Backward Inclusion: I will show that $\operatorname{convhull}(S) \subseteq \operatorname{convspan}(S)$. □

not finished

DEFINITION 3.6 (Pointed)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We say that S is **pointed** if and only if S contains no line.

3.2 Arithmetic Properties of Convex Sets

PROPOSITION 3.7. Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let C be a convex subset of $\mathcal{V}$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$. Then

$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C.$$

Proof. The case where $\lambda_1 = 0$ or $\lambda_2 = 0$ is trivial. Now suppose that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

Forward Inclusion: Let x be an arbitrary element of $(\lambda_1 + \lambda_2)C$. I will show that $x \in \lambda_1 C + \lambda_2 C$. Since $x \in (\lambda_1 + \lambda_2)C$, $x = (\lambda_1 + \lambda_2)c$ for some $c \in C$. Since $\mathcal{V}$ is a vector space over $\mathbb{R}$, $c \in C \subseteq \mathcal{V}$, and $\lambda_1, \lambda_2 \in \mathbb{R}_+ \subseteq \mathbb{R}$, we get $x = (\lambda_1 + \lambda_2)c = \lambda_1 c + \lambda_2 c$. Notice $\lambda_1 c \in \lambda_1 C$, and $\lambda_2 c \in \lambda_2 C$. So $x \in \lambda_1 C + \lambda_2 C$. So $(\lambda_1 + \lambda_2)C \subseteq \lambda_1 C + \lambda_2 C$.

Backward Inclusion: Let x be an arbitrary element of $\lambda_1 C + \lambda_2 C$. I will show that

$x \in (\lambda_1 + \lambda_2)C$. Since $x \in \lambda_1 C + \lambda_2 C$, $x = \lambda_1 c_1 + \lambda_2 c_2$ for some $c_1, c_2 \in C$. Notice

$$x = \lambda_1 c_1 + \lambda_2 c_2 = (\lambda_1 + \lambda_2)\left(\underbrace{\frac{\lambda_1}{\lambda_1 + \lambda_2}}_{\in[0,1]} \underbrace{c_1}_{\in C} + \underbrace{\frac{\lambda_2}{\lambda_1 + \lambda_2}}_{\in[0,1]} \underbrace{c_2}_{\in C} \right).$$

Notice the second term is a convex combination of two points in C and hence is in C. So $x \in (\lambda_1 + \lambda_2)C$. So $\lambda_1 C + \lambda_2 C \subseteq (\lambda_1 + \lambda_2)C$. $\square$

3.3 The Convex Hull Operator

PROPOSITION 3.8 (The Convex Hull Operator)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$.

1. Expansive
$$\forall S \subseteq \mathcal{V}, \quad S \subseteq \operatorname{conv}(S).$$
2. Monotonic Increasing
$$\forall S_1, S_2 \subseteq \mathcal{V} : S_1 \subseteq S_2, \quad \operatorname{conv}(S_1) \subseteq \operatorname{conv}(S_2).$$
3. Idempotent
$$\forall S \subseteq \mathcal{V}, \quad \operatorname{conv}(\operatorname{conv}(S)) = \operatorname{conv}(S).$$

PROPOSITION 3.9 (Bounded)**.** The convex hull of a bounded set is bounded.

Proof. Let $\mathcal{V}$ be a normed linear space over $\mathbb{R}$. Let C be a bounded subset of $\mathcal{V}$. Then $\exists R > 0$ such that $\forall c \in C$, $\|c\| < R$. Let x be an arbitrary element of $\operatorname{conv}(C)$. Then $\exists n \in \mathbb{Z}_{++}$, $\exists \lambda_1, ..., \lambda_n \in [0,1]$, $\exists c_1, ..., c_n \in C$ such that $\sum_{i\in[n]} \lambda_i = 1$ and $x = \sum_{i\in[n]} \lambda_i c_i$. Then by the triangle inequality of $\|\cdot\|$, we get

$$\|x\| = \left\| \sum_{i\in[n]} \lambda_i c_i \right\| \leq \sum_{i\in[n]} \lambda_i \|c_i\| < \sum_{i\in[n]} \lambda_i R = 1 \cdot R = R.$$

So $\forall x \in \operatorname{conv}(C)$, $\|x\| < R$. So $\operatorname{conv}(C)$ is bounded. $\square$

PROPOSITION 3.10 (Open)**.** The convex hull of an open set is open.

Proof Approach (1). Let $\mathcal{V}$ be a topological vector space. Let G be an open subset of $\mathcal{V}$. I will show that $\operatorname{conv}(G)$ is open. Let $x \in \operatorname{conv}(G)$ be arbitrary. Then $x = \sum_{i \in [n]} \lambda_i g_i$ for some $n \in \mathbb{Z}_{++}$, $\lambda_i \in [0,1]$, $\forall i \in [n]$, and $g_i \in G$, $\forall i \in [n]$. Since G is an open set and $g_i \in G$, $\forall i \in [n]$, there exist neighborhoods $\mathcal{N}_1, ..., \mathcal{N}_n$ of $g_1, ..., g_n$, respectively, such that $g_i \in \mathcal{N}_i \subseteq G$, $\forall i \in [n]$. Define $\mathcal{N} := \sum_{i \in [n]} \lambda_i \mathcal{N}_i$. Then $\mathcal{N}$ is a neighborhood of g, and $\mathcal{N} \subseteq \operatorname{conv}(G)$. So $g \in \operatorname{int}(\operatorname{conv}(G))$. So $\operatorname{conv}(G) \subseteq \operatorname{int}(\operatorname{conv}(G))$. So $\operatorname{conv}(G) = \operatorname{int}(\operatorname{conv}(G))$. So $\operatorname{conv}(G)$ is open. □

Proof Approach (2). Let $\mathcal{V}$ be a topological vector space. Let G be an open subset of $\mathcal{V}$. I will show that $\operatorname{conv}(G)$ is open. Let $x \in \operatorname{conv}(G)$ be arbitrary. Then $x = \sum_{i \in [n]} \lambda_i g_i$ for some $n \in \mathbb{Z}_{++}$, $\lambda_i \in [0,1]$, $\forall i \in [n]$, and $g_i \in G$, $\forall i \in [n]$. Let $i_0 \in [n]$ be such that $\lambda_{i_0} \neq 0$. Then

$$x = \sum_{i=1}^{n} \lambda_i g_i = \left(\sum_{i \neq i_0} \lambda_i g_i \right) + \lambda_{i_0} g_{i_0} \in \left(\sum_{i \neq i_0} \lambda_i g_i \right) + i_0 G \subseteq \operatorname{conv}(G).$$

So

$$\operatorname{conv}(G) = \bigcup_{x \in \operatorname{conv}(G)} \left\{ \sum_{i \neq i_0} \lambda_i g_i + i_0 G \right\}.$$

Note that for each $x \in \operatorname{conv}(G)$, the function $f_x : \mathcal{V} \to \mathcal{V}$ given by $f_x(v) := \sum_{i \neq i_0} \lambda_i g_i + i_0 v$ is a homeomorphism. So

$$\operatorname{conv}(G) = \bigcup_{x \in \operatorname{conv}(G)} f_x(G)$$

is the union of a collection of open sets and hence is open. □

REMARK 3.11 (Closed)**.** The convex hull of a closed set need not be closed.

- Example in $\mathbb{R}^2$: The set $S := \{(x,y) \in \mathbb{R}^2 : y \geq \frac{1}{1+x^2}\}$ is closed. However, $\operatorname{conv}(S) = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ is open.
- Example in ℓ^∞: Define for each $n \in \mathbb{Z}_{++}$ a sequence x_n by $x_n^{(n)} := \frac{1}{n}$ and $x_n^{(i)} := 0$, $\forall i \neq n$. Consider the set $S := \{x_n\}_{n \in \mathbb{N}} \cup \{0\}$. Then S is a compact subset of ℓ^∞. However, $\operatorname{conv}(S)$ contains the elements $\sum_{n=1}^{k} 2^{-n} x_n$ for $k \in \mathbb{Z}_{++}$. Notice they converge to the sequence $\sum_{n=1}^{\infty} 2^{-n} x_n$, which is not in $\operatorname{conv}(K)$ (it has infinitely many non-zero entries).

PROPOSITION 3.12 (Compact in $\mathbb{R}^n$)**.** Let K be a compact subset of $\mathbb{R}^n$. Then $\operatorname{conv}(K)$ is also compact.

PROPOSITION 3.13. Let $\mathcal{V}$ be a normed linear space. Let K be a compact subset of $\mathcal{V}$. Then $\overline{\operatorname{conv}}(K)$ is pre-compact (totally bounded). Moreover, if $\mathcal{V}$ is complete, then $\overline{\operatorname{conv}}(K)$ is also compact.

3.4 The Closed Convex Hull Operator

DEFINITION 3.14 (Closed Convex Hull)**.** Let S be a set in some Euclidean space. We define the **closed convex hull** of S, denoted by $\overline{\operatorname{conv}}(S)$, to be the smallest <u>closed</u> convex containing S.

PROPOSITION 3.15. The closed convex hull is the closure of the convex hull.

PROPOSITION 3.16. A closed convex hull does not distinguish a set from its closure. i.e., for any set S, we have $\overline{\operatorname{conv}}(S) = \overline{\operatorname{conv}}(\operatorname{cl}(S))$.

PROPOSITION 3.17. If S is bounded, then the closure operation and the convex hull operation commute. i.e., $\operatorname{conv}(\operatorname{cl}(S)) = \operatorname{cl}(\operatorname{conv}(S))$.

REMARK 3.18. The closure operation and the convex hull operation do not commute in general.

3.5 Stability of Convexity

PROPOSITION 3.19 (Intersection)**.** Convexity is stable under intersection. i.e., the intersection of any collection of convex sets is convex.

Proof. Let $\{C_i\}_{i\in I}$ be an arbitrary collection of convex sets where I is an index set and C_i is convex for any $i \in I$. Let C denote their intersection. If $C = \emptyset$, then we are done. Else, let x and y be two arbitrary points in C. Let λ be an arbitrary number in $(0, 1)$. Define a point $z := \lambda x + (1 - \lambda)y$. Since $x \in C$ and $C = \bigcap_{i\in I} C_i$, we get $x \in C_i$ for any $i \in I$. Since $y \in C$ and $C = \bigcap_{i\in I} C_i$, we get $y \in C_i$ for any $i \in I$. Let i be an arbitrary index in I. Since $x \in C_i$ and $y \in C_i$ and $\lambda \in (0, 1)$ and C_i is convex and $z = \lambda x + (1 - \lambda)y$, we get $z \in C_i$. Since $z \in C_i$ for any $i \in I$ and $C = \bigcap_{i\in I} C_i$, we get $z \in C$. Since

$$\forall x, y \in C, \forall \lambda \in (0, 1), \quad \lambda x + (1 - \lambda)y \in C,$$

by definition of convex sets, we get C is convex. □

PROPOSITION 3.20 (Affine Map)**.** Convexity is stable under affine mapping. i.e., the affine image of a convex set is convex.

PROPOSITION 3.21 (Linear Combinations)**.** Convexity is stable under linear combinations. i.e., if C_1 and C_2 are convex sets and λ_1 and λ_2 are real numbers, then the set C defined as

$$C := \lambda_1 C_1 + \lambda_2 C_2$$

is convex.

Proof. If $C_1 = \emptyset$ or $C_2 = \emptyset$, then $\lambda_1 C_1 + \lambda_2 C_2 = \emptyset$ and we are done. Now assume that $C_1, C_2 \neq \emptyset$. Then $C = \lambda_1 C_1 + \lambda_2 C_2 \neq \emptyset$. Let x and y be arbitrary points in C.
Since $x \in C$, $\exists x_1 \in C_1, x_2 \in C_2$ such that $x = \lambda_1 x_1 + \lambda_2 x_2$.
Since $y \in C$, $\exists y_1 \in C_1, y_2 \in C_2$ such that $y = \lambda_1 y_1 + \lambda_2 y_2$.
Let $\lambda \in [0, 1]$ be arbitrary. Define a point z as $z := \lambda x + (1 - \lambda)y$. Then

$$\begin{aligned}
z &= \lambda x + (1 - \lambda)y \\
&= \lambda\big(\lambda_1 x_1 + \lambda_2 x_2\big) + (1 - \lambda)\big(\lambda_1 y_1 + \lambda_2 y_2\big) \\
&= \lambda_1\big(\lambda x_1 + (1 - \lambda)y_1\big) + \lambda_2\big(\lambda x_2 + (1 - \lambda)y_2\big).
\end{aligned}$$

Since $x_1, y_1 \in C_1$, $\lambda \in [0,1]$ and C_1 is convex, we get $\lambda x_1 + (1-\lambda)y_1 \in C_1$.
Since $x_2, y_2 \in C_2$, $\lambda \in [0,1]$ and C_2 is convex, we get $\lambda x_2 + (1-\lambda)y_2 \in C_2$.
So $z = \lambda_1(\lambda x_1 + (1-\lambda)y_1) + \lambda_2(\lambda x_2 + (1-\lambda)y_2) \in \lambda_1 C_1 + \lambda_2 C_2$.
That is, $\forall x \in C$, $\forall y \in C$, $\forall \lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in C$.
So by definition, C is convex. $\square$

COROLLARY 3.22. The Minkowski sum of two convex sets is convex.

LEMMA 3.23. Let $\mathcal{V}$ be a normed linear space. Let C be a convex subset of $\mathcal{V}$. Let $x \in \operatorname{int}(C)$. Let $y \in \operatorname{cl}(C)$. Then

$$\forall t \in (0,1], \quad tx + (1-t)y \in C.$$

Proof Approach (1). Let $t \in (0,1]$ be arbitrary. Define a point $z \in \mathcal{V}$ by $z := tx + (1-t)y$. If $t = 1$, then $z = x \in \operatorname{int}(C) \subseteq C$ and we are done. Otherwise, $t \in (0,1)$. Since $x \in \operatorname{int}(C)$, $\exists r_x > 0$ such that $\operatorname{ball}(x, r_x) \subseteq C$. Define $r_y := \dfrac{t}{1-t} r_x$. Since $y \in \operatorname{cl}(C)$, $\exists y' \in \operatorname{ball}(y, r_y) \cap C$. Define a point $z' \in \mathcal{V}$ by $z' := tx + (1-t)y'$. Since $x, y' \in C$, $t \in (0,1)$, and C is convex, we get $z' \in C$. Define a point $x' \in \mathcal{V}$ by $x' := \dfrac{1}{t}(z - (1-t)y')$ so that $z = tx' + (1-t)y'$. Notice $x = \dfrac{1}{t}(z' - (1-t)y')$. So

$$\begin{aligned}
\|x - x'\| = \|x - \frac{1}{t}(z - (1-t)y')\| &= \left\| \frac{1}{t}(z' - (1-t)y') - \frac{1}{t}(z - (1-t)y') \right\| \\
&= \frac{1}{t}\|z - z'\| = \frac{1}{t}\left\| tx + (1-t)y - tx - (1-t)y' \right\| \\
&= \frac{1-t}{t}\|y - y'\| \leq \frac{1-t}{t} r_y = \frac{1-t}{t}\frac{t}{1-t} r_x = r_x.
\end{aligned}$$

That is, $\|x - x'\| \leq r_x$. So $x' \in \operatorname{ball}(x, r_x) \subseteq C$. Since $x', y' \in C$, $t \in (0,1)$, and C is convex, we get $z \in C$. $\square$

Proof Approach (2). Let $t \in (0,1]$ be arbitrary. If $t = 1$, then $tx + (1-t)y = x \in \operatorname{int}(C) \subseteq C$ and we are done. Otherwise, $t \in (0,1)$. Define $B := \operatorname{ball}(0,1)$. Then for some small enough $\varepsilon > 0$, we have

$$\begin{aligned}
tx + (1-t)y + \varepsilon B &\subseteq tx + (1-t)(C + \varepsilon B) + \varepsilon B, \text{ since } y \in \operatorname{cl}(C) \\
&= tx + (1-t)C + (1-t)\varepsilon B + \varepsilon B \\
&= tx + (2-t)\varepsilon B + (1-t)C
\end{aligned}$$

$$
\begin{aligned}
&= t(x + \frac{2-t}{t}\varepsilon B) + (1-t)C \\
&\subseteq tC + (1-t)C, \text{ since } x \in \operatorname{int}(C) \\
&= C.
\end{aligned}
$$

□

LEMMA 3.24. Let C be a convex set in $\mathbb{E}$. Let $x \in \operatorname{ri}(C)$. Let $y \in \operatorname{cl}(C)$. Then

$$\forall \lambda \in (0, 1], \quad \lambda x + (1-\lambda) y \in C.$$

Proof.

Case 1. $int(C) \neq \emptyset$.

Then $int(C) = \operatorname{ri}(C)$.

Since $x \in int(C)$ and $y \in \operatorname{cl}(C)$, $\forall t \in (0, 1], z := tx + (1-t)y \in C$.

Case 2. $int(C) = \emptyset$.

Now $\dim(C) < d$.

Say $\dim(C) = l$.

Apply case 1 in $\mathbb{R}^l$.

□

PROPOSITION 3.25 (Interior)**.** Convexity is stable under interior. i.e., the interior of a convex set is convex.

Proof. Let $\mathcal{V}$ be a normed linear space. Let S be a convex subset of $\mathcal{V}$. If $int(S) = \emptyset$, then we are done. Else: let x and y be two arbitrary points in $int(S)$. Let λ be an arbitrary number in $(0, 1)$. Define a point z by $z := \lambda x + (1-\lambda)y$. Since $x, y \in int(S)$ and $\lambda \in (0, 1)$, by the lemma, we get $z \in int(S)$. Since

$$\forall x, y \in int(S), \forall \lambda \in (0, 1), \quad \lambda x + (1-\lambda)y \in \operatorname{int}(S),$$

we get $\operatorname{int}(S)$ is convex. □

PROPOSITION 3.26 (Relative Interior)**.** Convexity is stable under relative interior. i.e., the relative interior of a convex set is convex.

Proof. Let $\mathcal{V}$ be a normed linear space. Let S be a convex subset of $\mathcal{V}$. If $\operatorname{ri}(S) = \emptyset$, then we are done. Otherwise, let x and y be two arbitrary points in $\operatorname{ri}(S)$. Let λ be an arbitrary number in $(0, 1)$. Define a point z by $z := \lambda x + (1 - \lambda)y$. Since $x, y \in \operatorname{ri}(S)$ and $\lambda \in (0, 1)$, by the lemma, we get $z \in \operatorname{ri}(S)$. Since

$$\forall x, y \in \operatorname{ri}(S), \forall \lambda \in (0, 1), \quad \lambda x + (1 - \lambda)y \in \operatorname{ri}(S),$$

we get $\operatorname{ri}(S)$ is convex. □

PROPOSITION 3.27 (Closure)**.** Convexity is stable under closure. i.e., the closure of a convex set is convex.

Proof Approach 1.

Let $x, y \in \operatorname{cl}(C)$.

Let $t \in [0, 1]$.

Since $x \in \operatorname{cl}(C)$, $\exists \{x_i\}_{i \in \mathbb{N}} \subseteq C$, $\lim\limits_{i \to \infty} x_i = x$.

Since $y \in \operatorname{cl}(C)$, $\exists \{y_i\}_{i \in \mathbb{N}} \subseteq C$, $\lim\limits_{i \to \infty} y_i = y$.

Since $\lim\limits_{i \to \infty} x_i = x$ and $\lim\limits_{i \to \infty} y_i = y$, $\lim\limits_{i \to \infty} \big(tx_i + (1 - t)y_i\big) = tx + (1 - t)y$.

Since $x_i, y_i \in C$ and C is convex, $tx_i + (1 - t)y_i \in C$.

Since $tx_i + (1 - t)y_i \in C$ $\lim\limits_{i \to \infty} \big(tx_i + (1 - t)y_i\big) = tx + (1 - t)y$, $tx + (1 - t)y \in \operatorname{cl}(C)$.

Since $\forall x, y \in \operatorname{cl}(C), \forall t \in [0, 1], tx + (1 - t)y \in \operatorname{cl}(C)$, we get $\operatorname{cl}(C)$ is convex.

□

Proof Approach 2.

$\operatorname{cl}(C) = \bigcap\limits_{\varepsilon > 0} \big[C + \varepsilon \operatorname{ball}(0, 1)\big]$. This is an intersection of linear combinations of convex sets and hence convex.

□

PROPOSITION 3.28 (Conical Hull)**.** Convexity is stable under conical hull. i.e., if C is convex, then $\operatorname{cone}(C)$ is convex.

Proof.

Let x and y be arbitrary points in $\operatorname{cone}(C)$.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda x + (1 - \lambda)y$.

Since $x \in \operatorname{cone}(C)$, $\exists x' \in C$ and $\exists \alpha > 0$ such that $x = \alpha x'$.

Since $y \in \operatorname{cone}(C)$, $\exists y' \in C$ and $\exists \beta > 0$ such that $y = \beta y'$.

Define point z' as $z' := \dfrac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta}x' + \dfrac{(1-\lambda)\beta}{\lambda\alpha + (1-\lambda)\beta}y'$.

Since $x', y' \in C$ and $\dfrac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta} \in (0,1)$ and $\dfrac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta} + \dfrac{(1-\lambda)\beta}{\lambda\alpha + (1-\lambda)\beta} = 1$ and C is convex and $z' := \dfrac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta}x' + \dfrac{(1-\lambda)\beta}{\lambda\alpha + (1-\lambda)\beta}y'$, we get $z' \in C$.

Since $z' \in C$ and $z = (\lambda\alpha + (1-\lambda)\beta)z'$, $z \in \text{cone}(C)$.

That is, $\lambda x + (1-\lambda)y \in \text{cone}(C)$.

Since $\forall x, y \in \text{cone}(C)$, $\forall \lambda \in (0,1)$, $\lambda x + (1-\lambda)y \in \text{cone}(C)$, we get cone$(C)$ is convex.

□

3.6 Topological Properties of Convex Sets

THEOREM 3.29. Let $\mathcal{V}$ be a topological vector space. Let C be a convex subset of $\mathcal{V}$ with $\text{int}(C) \neq \emptyset$. Then

1. $\text{int}(C) = \text{int}(\text{cl}(C))$, and
2. $\text{cl}(C) = \text{cl}(\text{int}(C))$.

Proof of 1. Since $C \subseteq \text{cl}(C)$, we have $\text{int}(C) \subseteq \text{int}(\text{cl}(C))$. So there remains only to show that $\text{int}(\text{cl}(C)) \subseteq \text{int}(C)$. Let x be an arbitrary element of $\text{int}(\text{cl}(C))$. Then $\exists \varepsilon > 0$ such that $\text{ball}(x, \varepsilon) \subseteq \text{cl}(C)$. □

Proof of (1). $int(C) \subseteq int(\text{cl}(C))$ is clear. For $int(\text{cl}(C)) \subseteq int(C)$, let x be an arbitrary point in $int(\text{cl}(C))$.

Since $x \in int(\text{cl}(C))$,

$$\exists r > 0 \text{ such that } \text{ball}(x, r) \subseteq \text{cl}(C).$$

Since $int(C) \neq \emptyset$, pick $y \in int(C)$.

Define a scalar λ by

$$\lambda := \frac{r}{2\|x-y\|}.$$

Define a point z by

$$z := x + \lambda(x-y).$$

Since $\lambda = \dfrac{r}{2\|x-y\|}$ and $z = x + \lambda(x-y)$,

$$\begin{aligned} &\|z - x\| \\ &= \big\|x + \lambda(x-y) - x\big\| \end{aligned}$$

$$
\begin{aligned}
&= \big\|\lambda(x-y)\big\| \\
&= \lambda\|x-y\| \\
&= \frac{r}{2\|x-y\|}\|x-y\| \\
&= \frac{r}{2} \\
&< r.
\end{aligned}
$$

That is,

$$\|z-x\| < r.$$

So $z \in \text{ball}(x,r)$. It follows that $z \in \text{cl}(C)$.

Since $z = x + \lambda(x-y)$, rearranging this yields

$$x = \frac{1}{1+\lambda}z + \frac{\lambda}{1+\lambda}y.$$

$$\text{Since} \begin{cases} x = \dfrac{1}{1+\lambda}z + \dfrac{\lambda}{1+\lambda}y \\ z \in \text{cl}(C) \\ y \in int(C) \\ \dfrac{1}{1+\lambda}, \dfrac{\lambda}{1+\lambda} \in (0,1) \\ \dfrac{1}{1+\lambda} + \dfrac{\lambda}{1+\lambda} = 1 \end{cases} \text{, by the lemma, we get}$$

$$x \in int(C).$$

Since $\forall x \in int(\text{cl}(C)), x \in int(C)$, we get $int(\text{cl}(C)) \subseteq int(C)$.

□

Proof of (2). $\text{cl}(int(C)) \subseteq \text{cl}(C)$ is clear. For $\text{cl}(C) \subseteq \text{cl}(int(C))$, let x be an arbitrary point in $\text{cl}(C)$.

Since $int(C) \neq \emptyset$, pick $y \in int(C)$.

Let $\lambda \in [0,1)$.

Define a point z by

$$z(\lambda) := \lambda x + (1-\lambda)y.$$

$$\text{Since} \begin{cases} z(\lambda) := \lambda x + (1-\lambda)y \\ x \in \text{cl}(C) \\ y \in int(C) \\ \lambda \in [0,1) \end{cases} \text{, by the lemma, we get}$$

$$z(\lambda) \in int(C).$$

Since $\begin{cases} z(\lambda) \in int(C) \\ \lim_{\lambda \to 1} z(\lambda) = x \end{cases}$, we get

$$x \in \mathrm{cl}(int(C)).$$

Since $\forall x \in \mathrm{cl}(C), x \in \mathrm{cl}(int(C))$, we get $\mathrm{cl}(C) \subseteq \mathrm{cl}(int(C))$.

□

PROPOSITION 3.30. Let C be a convex set. Then

1. $\mathrm{aff}(\mathrm{ri}(C)) = \mathrm{aff}(C) = \mathrm{aff}(\mathrm{cl}(C))$,
2. $\mathrm{ri}(\mathrm{ri}(C)) = \mathrm{ri}(C) = \mathrm{ri}(\mathrm{cl}(C))$, and
3. $\mathrm{cl}(\mathrm{ri}(C)) = \mathrm{cl}(C) = \mathrm{cl}(\mathrm{cl}(C))$.

PROPOSITION 3.31. Let C be a convex set. Then

$$C \neq \emptyset \iff \mathrm{ri}(C) \neq \emptyset.$$

Proof. **Forward Direction**: Assume that $C \neq \emptyset$. I will show that $\mathrm{ri}(C) \neq \emptyset$. Since $C \neq \emptyset$, $\mathrm{aff}(C) \neq \emptyset$. Since C is convex, $\mathrm{aff}(C) = \mathrm{aff}(\mathrm{ri}(C))$. Since $\begin{cases} \mathrm{aff}(C) \neq \emptyset \\ \mathrm{aff}(C) = \mathrm{aff}(\mathrm{ri}(C)) \end{cases}$, we get

$$\mathrm{aff}(\mathrm{ri}(C)) \neq \emptyset.$$

Since $\mathrm{aff}(\mathrm{ri}(C)) \neq \emptyset$, we get $\mathrm{ri}(C) \neq \emptyset$.

Backward Direction: Assume that $\mathrm{ri}(C) \neq \emptyset$. I will show that $C \neq \emptyset$. Since $\mathrm{ri}(C) \neq \emptyset$ and $\mathrm{ri}(C) \subseteq C$, we get $C \neq \emptyset$. □

3.7 Examples of Convex Sets

EXAMPLE 3.32. Let I be an index set. Let b_i for $i \in I$ be vectors in $\mathbb{E}$. Let β_i for $i \in I$ be reals. Then the set C given by

$$C := \{x \in \mathbb{E} : \forall i \in I, \langle x, b_i \rangle \leq \beta_i\}$$

is convex.

Proof.

Each of $C_i := \{x \in \mathbb{E} : \langle x, b_i \rangle \leq \beta_i\}$ is convex and $C = \bigcap_{i \in I} C_i$.

$$\begin{aligned} \langle z, b_i \rangle &= \langle \lambda x + (1-\lambda) y, b_i \rangle \\ &= \lambda \langle x, b_i \rangle + (1-\lambda) \langle y, b_i \rangle \\ &\leq \lambda \beta_i + (1-\lambda)\beta_i \\ &= \beta_i. \end{aligned}$$

□

3.8 The Carathéodory Theorem

THEOREM 3.33 (Carathéodory)**.** Let S be a subset of $\mathbb{R}^n$. Let x be some point in $\operatorname{conv}(S)$. Then x can be represented as a convex combination of at most $n+1$ points in S. i.e., x lies in some r-simplex with vertices in S, where $r \leq n$.

Chapter 4

Geometric Objects

4.1 Definitions

DEFINITION 4.1 (Hyperplane)**.** Let $\mathbb{E}$ be a Euclidean space over $\mathbb{R}$. Let H be a subset of $\mathbb{E}$. We say that H is a **hyperplane** if and only if H can be expressed as

$$H = \{x \in \mathbb{E} : a^\top x = b\}$$

for some $a \in \mathbb{E} \setminus \{0\}$ and $b \in \mathbb{R}$.

DEFINITION 4.2 (Closed Half-Space)**.** Let $\mathbb{E}$ be a Euclidean space over $\mathbb{R}$. Let P be a subset of $\mathbb{E}$. We say that P is a **closed half-space** if and only if P can be expressed as

$$P = \{x \in \mathbb{E} : a^\top x \leq b\}$$

for some $a \in \mathbb{E} \setminus \{0\}$ and $b \in \mathbb{R}$.

DEFINITION 4.3 (Polyhedron)**.** Let $\mathbb{E}$ be a Euclidean space over $\mathbb{R}$. Let P be a subset of $\mathbb{E}$. We say that P is a **polyhedron** if and only if P can be expressed as the intersection of finitely many closed half-spaces in $\mathbb{E}$.

4.2 Properties

PROPOSITION 4.4. Polyhedrons are convex.

Chapter 5

Cones

5.1 Definitions

DEFINITION 5.1 (Conical Combination)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define a **conical combination** of S to be a point x in the space of the form

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where (1) $n \in \mathbb{N}$, (2) $v_i \in S$ for all i, and (3) $\lambda_i \in \mathbb{R}_{++}$ for all i.

DEFINITION 5.2 (Cone)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We say that S is a **cone** if and only if $S = \mathbb{R}_{++}S$.

DEFINITION 5.3 (Conical Hull)**.** Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. We define the **conical hull** of S, denoted by $\text{cone}(S)$, to be the intersection of all cones containing S.

PROPOSITION 5.4. Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. Let S be a subset of $\mathcal{V}$. Then $\text{cone}(S) = \mathbb{R}_{++}S$.

Proof. **Forward Direction**: I will show that $\text{cone}(S) \subseteq \mathbb{R}_{++}S$. Since $\mathbb{R}_{++}\mathbb{R}_{++}S = \mathbb{R}_{++}S$, $\mathbb{R}_{++}S$ is a cone. Since $1 \in \mathbb{R}_{++}$, $S \subseteq \mathbb{R}_{++}S$. Since $\mathbb{R}_{++}S$ is a cone containing S and $\text{cone}(S)$ is the smallest cone containing S, we get

$$\text{cone}(S) \subseteq \mathbb{R}_{++}S.$$

Backward Direction: I will show that $\mathbb{R}_{++}S \subseteq \text{cone}(S)$. Let C be an arbitrary cone containing S. Since $S \subseteq C$, $\mathbb{R}_{++}S \subseteq \mathbb{R}_{++}C$. Since C is a cone, $\mathbb{R}_{++}C = C$. So $\mathbb{R}_{++}S \subseteq C$. Since $\mathbb{R}_{++}S \subseteq C$ for any cone C containing S, we get

$$\mathbb{R}_{++}S \subseteq \text{cone}(S).$$

□

5.2 Stability of the Cone Structure

PROPOSITION 5.5. The closure of a cone is a cone.

Proof. Let $\mathcal{V}$ be a normed linear space. Let C be a cone in $\mathcal{V}$. I will show that $\text{cl}(C)$ is also a cone. Let x be an arbitrary element of $\text{cl}(C)$. Then $\exists (x_n)_{n\in\mathbb{N}} \subseteq C$ such that $\lim_{n\in\mathbb{N}} x_n = x$. Let $\lambda \in \mathbb{R}_{++}$ be arbitrary. Since C is a cone and $x_n \in C$, $\forall n \in \mathbb{N}$, we have $\lambda x_n \in C$. So $(\lambda x_n)_{n\in\mathbb{N}} \subseteq C$. Moreover, notice $\lim_{n\in\mathbb{N}} \lambda x_n = \lambda \lim_{n\in\mathbb{N}} x_n = \lambda x$. So $\lambda x \in \text{cl}(C)$. This holds for any $\lambda \in \mathbb{R}_{++}$. So $\mathbb{R}_{++}\,\text{cl}(C) \subseteq \text{cl}(C)$. It is clear that $\text{cl}(C) \subseteq \mathbb{R}_{++}\,\text{cl}(C)$. So we have $\text{cl}(C) = \mathbb{R}_{++}\,\text{cl}(C)$. So $\text{cl}(C)$ is a cone. □

5.3 Other Properties

PROPOSITION 5.6. Let C be a convex set in $\mathbb{E}$. Assume $\text{int}(C) \neq \emptyset$ and $0 \in C$. Then $\text{int}(\text{cone}(C)) = \text{cone}(\text{int}(C))$.

Proof.

For one direction, let x be an arbitrary point in $int(\text{cone}(C))$. We are to prove that $x \in \text{cone}(int(C))$.

Since $x \in int(\text{cone}(C))$, $\exists r$ such that $\text{ball}(x, r) \subseteq \text{cone}(C)$.

Since $x \in int(\text{cone}(C))$, $x \in \text{cone}(C)$.

Since $x \in \text{cone}(C)$, $\exists n \in \mathbb{N}$, $\exists \lambda_1, ..., \lambda_n > 0$, $\exists v_1, ..., v_n \in C$ such that $x = \sum_{i=1}^{n} \lambda_i v_i$.

Assume for the sake of contradiction that $\exists k \in \{1, ..., n\}$ such that $\forall r_k > 0$, $\text{ball}(v_k, r_k) \cap \mathbb{E} \setminus C \neq \emptyset$.

not finished

For the reverse direction, let x be an arbitrary point in $\text{cone}(int(C))$. We are to prove that $x \in int(\text{cone}(C))$.

Since $x \in \text{cone}(int(C))$, $\exists n \in \mathbb{N}$, $\exists \lambda_1, ..., \lambda_n > 0$, $\exists v_1, ..., v_n \in int(C)$ such that $x = \sum_{i=1}^{n} \lambda_i v_i$.

Since $v_i \in int(C)$ for each $i \in \{1, ..., n\}$, $\exists r_i$ such that $\text{ball}(v_i, r_i) \subseteq C$.

Define $R := \min\{\lambda_i r_i\}_{i=1}^{n}$.

Say $R = \lambda_k r_k$ for some $k \in \{1, ..., n\}$.

Let y be an arbitrary point in $\text{ball}(x, R)$.

Since $y \in \text{ball}(x, R)$, $\exists w$ such that $\|w\| < R$ and $y = x + w$.

$$\begin{aligned} y &= \sum_{i=1}^{n} \lambda_i v_i + w \\ &= \sum_{i \neq k} \lambda_i v_i + \lambda_k v_k + w \\ &= \sum_{i \neq k} \lambda_i v_i + \lambda_k (v_k + w/\lambda_k). \end{aligned}$$

Since $\|w\| < R$, $\|w/\lambda_k\| < R/\lambda_k = r_k$.

Since $\|w/\lambda_k\| < r_k$, $v_k + w/\lambda_k \in \text{ball}(v_k, r_k)$.

So $v_k + w/\lambda_k \in C$.

So $y \in \text{cone}(C)$.

Since $\forall y \in \text{ball}(x, R)$, $y \in \text{cone}(C)$, $\text{ball}(x, R) \subseteq \text{cone}(C)$.

Since $\exists r$ such that $\text{ball}(x, r) \subseteq \text{cone}(C)$, $x \in int(\text{cone}(C))$.

This proves $\text{cone}(int(C)) \subseteq int(\text{cone}(C))$.

□

PROPOSITION 5.7. Let C be a convex set in $\mathbb{E}$. Assume $\text{int}(C) \neq \emptyset$ and $0 \in C$. Then

$$0 \in \text{int}(C) \iff \text{cone}(C) = \mathbb{E}.$$

Proof. For one direction, assume that $0 \in int(C)$. We are to prove that $\text{cone}(C) = \mathbb{E}$. Clearly

$$\text{cone}(C) \subseteq \mathbb{E}.$$

Since $0 \in int(C)$, $\exists r > 0$ such that $\text{ball}(0, r) \subseteq C$. Since $\text{ball}(0, r) \subseteq C$, $\text{cone}(\text{ball}(0, r)) \subseteq \text{cone}(C)$. Since $\text{cone}(\text{ball}(0, r)) = \mathbb{E}$ and $\text{cone}(\text{ball}(0, r)) \subseteq \text{cone}(C)$, we get

$$\mathbb{E} \subseteq \text{cone}(C).$$

For the reverse direction, assume that $\text{cone}(C) = \mathbb{E}$. We are to prove that $0 \in int(C)$.

$$\mathbb{E} = int(\mathbb{E}) = int(\text{cone}(C)) = \text{cone}(int(C)).$$

If $0 \notin int(C)$, then $0 \notin \text{cone}(int(C))$. So $0 \in int(C)$.

$\square$

5.4 Closed Conical Hull

DEFINITION 5.8 (Closed Conical Hull)**.** Let $\mathcal{V}$ be a normed linear space. Let S be a subset of $\mathcal{V}$. We define the **closed conical hull** of S, denoted by $\overline{\text{cone}}(S)$, to be the intersection of all closed cones containing C.

PROPOSITION 5.9. Let $\mathcal{V}$ be a normed linear space. Let S be a subset of $\mathcal{V}$. Then

$$\overline{\text{cone}}(S) = \text{cl}(\text{cone}(S)).$$

Proof. **Forward Direction**: I will show that $\overline{\text{cone}}(S) \subseteq \text{cl}(\text{cone}(S))$. By Proposition 5.5, $\text{cl}(\text{cone}(S))$ is a closed cone containing S. Since $\overline{\text{cone}}(S)$ is the smallest closed cone containing S, we have $\overline{\text{cone}}(S) \subseteq \text{cl}(\text{cone}(S))$.

Backward Direction: I will show that $\text{cl}(\text{cone}(S)) \subseteq \overline{\text{cone}}(S)$.

$$\begin{aligned} & S \subseteq \overline{\text{cone}}(S) \\ \implies & \text{cone}(S) \subseteq \text{cone}(\overline{\text{cone}}(S)), \text{ since cone is monotonic increasing} \\ \implies & \text{cl}(\text{cone}(S)) \subseteq \text{cl}(\text{cone}(\overline{\text{cone}}(S))), \text{ since cl is monotonic increasing} \\ \iff & \text{cl}(\text{cone}(S)) \subseteq \overline{\text{cone}}(S), \text{ since } \overline{\text{cone}}(S) \text{ is a closed cone.} \end{aligned}$$

This completes the proof. $\square$

5.5 The cone and $\overline{\text{cone}}$ Operators

PROPOSITION 5.10 (The cone Operator)**.** The cone operator has the following properties.

1. Expansive: $\forall S \subseteq \mathbb{E}$,
$$S \subseteq \text{cone}(S).$$

2. Monotone: $\forall S_1, S_2 \subseteq \mathbb{E}$,
$$S_1 \subseteq S_2 \implies \text{cone}(S_1) \subseteq \text{cone}(S_2).$$

3. Idempotence: $\forall S \subseteq \mathbb{E}$,
$$\text{cone}(\text{cone}(S)) = \text{cone}(S).$$

PROPOSITION 5.11 (Bauschke-Combettes, 2017 Book)**.** Let $\mathcal{V}$ be a normed linear space. Let S be a subset of $\mathcal{V}$. Then

$$\text{cone}(\text{conv}(S)) = \text{conv}(\text{cone}(S)).$$

This is the smallest convex cone containing S.

Proof.

For $\text{cone}(\text{conv}(S)) \subseteq \text{conv}(\text{cone}(S))$, let x be an arbitrary point in $\text{cone}(\text{conv}(S))$.

Since $x \in \text{cone}(\text{conv}(S))$, we get $\exists \lambda \in \mathbb{R}_+$, $\exists n \in \mathbb{N}$, $\exists v_1, ..., v_n \in S$, $\exists \mu_1, ..., \mu_n \in [0,1]$, $\sum_{i=1}^{n} \mu_i = 1$ such that $x = \lambda \sum_{i=1}^{n} \mu_i v_i$.

Since $x = \lambda \sum_{i=1}^{n} \mu_i v_i$, $x = \sum_{i=1}^{n} \mu_i (\lambda v_i)$.

Since $\lambda \in \mathbb{R}_+$ and $v_i \in S$, $\lambda v_i \in \text{cone}(S)$.

Since $\lambda v_i \in \text{cone}(S)$ and $\mu_i \in [0,1]$, $\sum_{i=1}^{n} \mu_i = 1$, $\sum_{i=1}^{n} \mu_i (\lambda v_i) \in \text{conv}(\text{cone}(S))$.

Since $\forall x \in \text{cone}(\text{conv}(S)), x \in \text{conv}(\text{cone}(S))$, $\text{cone}(\text{conv}(S)) \subseteq \text{conv}(\text{cone}(S))$.

For $\text{conv}(\text{cone}(S)) \subseteq \text{cone}(\text{conv}(S))$, let x be an arbitrary point in $\text{conv}(\text{cone}(S))$.

Since $x \in \text{conv}(\text{cone}(S))$, $\exists n \in \mathbb{N}$, $\exists \lambda_i \in [0,1]$, $\sum_{i=1}^{n} \lambda_i = 1$, $\exists \mu_i \in \mathbb{R}_+$, $\exists v_i \in S$ such that $x = \sum_{i=1}^{n} \lambda_i \mu_i v_i$.

Define $\alpha := \sum_{i=1}^{n} \lambda_i \mu_i$.

Define $\beta_i := \lambda_i \mu_i / \alpha$.

Then $\alpha \in \mathbb{R}_+$ and $\beta_i \in [0,1]$ and $\sum_{i=1}^{n} \beta_i = 1$ and $x = \alpha \sum_{i=1}^{n} \beta_i v_i$.

Since $\beta_i \in [0,1]$ and $\sum_{i=1}^{n} \beta_i = 1$ and $v_i \in S$, we get $\sum_{i=1}^{n} \beta_i v_i \in \operatorname{conv}(S)$.

Since $\alpha \in \mathbb{R}_+$ and $\sum_{i=1}^{n} \beta_i v_i \in \operatorname{conv}(S)$ and $x = \alpha \sum_{i=1}^{n} \beta_i v_i$, we get $x \in \operatorname{cone}(\operatorname{conv}(S))$.

Since $\forall x \in \operatorname{conv}(\operatorname{cone}(S)), x \in \operatorname{cone}(\operatorname{conv}(S))$, we get $\operatorname{conv}(\operatorname{cone}(S)) \subseteq \operatorname{cone}(\operatorname{conv}(S))$.

Since $\operatorname{cone}(\operatorname{conv}(S)) \subseteq \operatorname{conv}(\operatorname{cone}(S))$ and $\operatorname{conv}(\operatorname{cone}(S)) \subseteq \operatorname{cone}(\operatorname{conv}(S))$, we get $\operatorname{conv}(\operatorname{cone}(S)) = \operatorname{cone}(\operatorname{conv}(S))$.

$\square$

PROPOSITION 5.12 (Bauschke-Combettes, 2017 Book)**.** Let $\mathcal{V}$ be a normed linear space. Let S be a subset of $\mathcal{V}$. Then

$$\overline{\operatorname{cone}}(\operatorname{conv}(S)) = \overline{\operatorname{conv}}(\operatorname{cone}(S)).$$

This is the smallest closed convex cone containing S.

5.6 Dual Cone

DEFINITION 5.13 (Dual of a Convex Cone)**.** Let $\mathfrak{X}$ be vector space over $\mathbb{R}$. Let C be a subset of $\mathfrak{X}$. We define the **dual cone** of C, denoted by C^*, to be the subset of $\mathfrak{X}$ given by

$$C^* := \{x \in \mathfrak{X} : \forall y \in C, \langle x, y \rangle \geq 0\}.$$

PROPOSITION 5.14. The dual of a convex cone is always a closed convex cone.

PROPOSITION 5.15. Let $\mathbb{E}$ be a Euclidean space. Let K be a convex cone in $\mathbb{E}$. Then $K^{**} = \operatorname{cl}(K)$.

PROPOSITION 5.16. Let $\mathbb{E}$ be a Euclidean space. Let K be a pointed, closed convex cone with nonempty interior. Then so is K^*.

PROPOSITION 5.17. Let $\mathbb{E}$ be a Euclidean space. Let K_1 and K_2 be nonempty convex cones. Then

1. $(K_1 + K_2)^* = K_1^* \cap K_2^*$.
2. $(\mathrm{cl}(K_1) \cap \mathrm{cl}(K_2))^* = \mathrm{cl}(K_1^* + K_2^*)$.
3. If K_1 and K_2 are closed and $\mathrm{ri}(K_1) \cap \mathrm{ri}(K_2) \neq \emptyset$, then $(K_1 \cap K_2)^* = K_1^* + K_2^*$.

Proof of (1). **Forward Direction:** Let x be an arbitrary element of $(K_1 + K_2)^*$. I will show that $x \in K_1^* \cap K_2^*$. Since $x \in (K_1 + K_2)^*$, $\forall k \in K_1 + K_2$, we have $\langle x, k \rangle \geq 0$. Let k_1 be an arbitrary element of K_1. Let k_2 be an arbitrary element of K_2. Then

$$\begin{aligned}
\langle x, k_1 \rangle &= \left\langle x, \lim_{n\to\infty} (k_1 + \frac{1}{n} k_2) \right\rangle \\
&= \lim_{n\to\infty} \left\langle x, k_1 + \frac{1}{n} k_2 \right\rangle, \text{ since } \langle x, \cdot \rangle \text{ is continuous} \\
&\geq \lim_{n\to\infty} 0, \text{ since } k_1 + \frac{1}{n} k_2 \in K_1 + K_2 \\
&= 0.
\end{aligned}$$

That is, $\langle x, k_1 \rangle \geq 0$. A similar argument can show that $\langle x, k_2 \rangle \geq 0$. So $x \in K_1^*$ and $x \in K_2^*$. So $x \in K_1^* \cap K_2^*$.

Backward Direction: Let x be an arbitrary element of $K_1^* \cap K_2^*$. I will show that $x \in (K_1 + K_2)^*$. Let k be an arbitrary element of $K_1 + K_2$. Then k can be written as $k = k_1 + k_2$ where $k_1 \in K_1$ and $k_2 \in K_2$. Since $x \in K_1^* \cap K_2^*$, $x \in K_1^*$. Since $x \in K_1^*$ and $k_1 \in K_1$, we get $\langle x, k_1 \rangle \geq 0$. A similar argument can show that $\langle x, k_2 \rangle \geq 0$. So

$$\langle x, k \rangle = \langle x, k_1 + k_2 \rangle = \langle x, k_1 \rangle + \langle x, k_2 \rangle \geq 0 + 0 = 0.$$

That is, $\langle x, k \rangle \geq 0$. So $x \in (K_1 + K_2)^*$. □

5.7 Polar Cone

DEFINITION 5.18 (Polar Cone)**.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$. Let S be a subset of $\mathcal{H}$. We define the **polar cone** of S, denoted by S°, to be a subset of $\mathcal{H}$ given by

$$S^\circ := \{x \in \mathcal{H} : \forall y \in S, \langle x, y \rangle \leq 0\}.$$

PROPOSITION 5.19 (Bauschke-Combettes, 2017 Book)**.** The polar cone of any set is a closed convex cone.

PROPOSITION 5.20 (Bauschke-Combettes, 2017 Book)**.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$. Let $A, B \subseteq \mathcal{H}$. Then $A \subseteq B \implies B^\circ \subseteq A^\circ$.

PROPOSITION 5.21. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$. Let C be a subset of $\mathcal{H}$. Then $C^\circ = -C^*$.

PROPOSITION 5.22. If S is a linear subspace of some Euclidean space $\mathbb{E}$, then $S^\circ = S^\perp$.

5.8 Extreme Rays

DEFINITION 5.23 (Rays)**.** Let $\mathcal{V}$ be a vector space. Let R be a subset of $\mathcal{V}$. We say that R is a **ray** if and only if R can be expressed as

$$R = \{\alpha v : \alpha \in \mathbb{R}_+\}$$

for some $v \in \mathbb{E} \setminus \{0\}$.

DEFINITION 5.24 (Extreme Rays)**.** Let $\mathcal{V}$ be a vector space. Let K be a convex cone in $\mathcal{V}$. Let R be a ray in K. We say that R is an **extreme ray** in K if and only if for any pair of rays R_1 and R_2 in K such that $R_1 + R_2 \supseteq R$, we have either $R_1 = R$ or $R_2 = R$ (or both).

Chapter 6

Tangent Cones and Normal Cones

6.1 Definitions

DEFINITION 6.1 (Tangent Cones)**.** Let C be a non-empty convex set in $\mathbb{R}^n$. Let x be a point in $\mathbb{R}^n$. We define the **tangent cone** to C at point x, denoted by $T_C(x)$, to be the subset of $\mathbb{R}^n$ given by

$$T_C(x) := \begin{cases} \overline{\text{cone}}(C - x), & \text{if } x \in C \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

DEFINITION 6.2 (Normal Cones)**.** Let C be a non-empty convex set in $\mathbb{R}^n$. Let x be a point in $\mathbb{R}^n$. We define the **normal cone** to C at point x, denoted by $N_C(x)$, to be the subset of $\mathbb{R}^n$ given by

$$N_C(x) := \begin{cases} \{v \in \mathbb{R}^n : \forall y \in C - x, \langle y, v \rangle \leq 0\}, & \text{if } x \in C \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

6.2 Basic Properties

PROPOSITION 6.3. Let C be a closed convex set in $\mathbb{E}$. Let x be a point in $\mathbb{E}$. Then $T_C(x)$ and $N_C(x)$ are closed convex cones.

Proof.

If $C = \emptyset$, then $T_C(x) = N_C(x) = \emptyset$.

If $C \neq \emptyset$ and $x \notin C$, then $T_C(x) = N_C(x) = \emptyset$.

So now I assume that $C \neq \emptyset$ and $x \in C$.

Tangent Cone is Closed:

By definition, $T_C(x) = \overline{\text{cone}}(C - x)$. So $T_C(x)$ is a closed.

Tangent Cone is Convex:

$$\begin{array}{cl} C \text{ is convex} & \\ \Downarrow & \text{since affine mapping preserves convexity} \\ C - x \text{ is convex} & \\ \Downarrow & \text{since the cone operator preserves convexity} \\ \text{cone}(C - x) \text{ is convex} & \\ \Downarrow & \text{since the cl operator preserves convexity} \\ \text{cl}(\text{cone}(C - x)) \text{ is convex} & \\ \Downarrow & \text{since cl} \circ \text{cone} = \overline{\text{cone}} \\ \overline{\text{cone}}(C - x) \text{ is convex} & \end{array}$$

That is, $T_C(x)$ is convex.

Tangent Cone is a Cone

By definition, $T_C(x) = \overline{\text{cone}}(C - x)$. So $T_C(x)$ is a cone.

Normal Cone is Closed:

Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in $N_C(x)$ that converges to some point in $\mathbb{E}$.

Say $x_i \to x_\infty$.

Let y be an arbitrary point in $C - x$.

Since $x_i \in N_C(x)$ and $y \in C - x$, by definition of $N_C(x)$, we get $\langle x_i, y \rangle \leq 0$.

Since $\langle x_i, y \rangle \leq 0$ for any $i \in \mathbb{N}$ and $x_i \to x_\infty$, we get $\langle x_\infty, y \rangle \leq 0$.

Since $\forall y \in C - x$, $\langle x_\infty, y \rangle \leq 0$, by definition of $N_C(x)$, we get $x_\infty \in N_C(x)$.

Since any convergent sequence whose terms are in $N_C(x)$ has its limit also in $N_C(x)$, $N_C(x)$ is closed.

Normal Cone is Convex:

Let u and v be arbitrary points in $N_C(x)$.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda u + (1 - \lambda)v$.

Let y be an arbitrary point in $C - x$.
Since $u \in N_C(x)$, $\langle u, y\rangle \leq 0$.
Since $v \in N_C(x)$, $\langle v, y\rangle \leq 0$.

$$\begin{aligned}&\langle z, y\rangle \\ =\ &\langle \lambda u + (1-\lambda)v, y\rangle \\ =\ &\lambda\langle u, y\rangle + (1-\lambda)\langle v, y\rangle \\ \leq\ &\lambda 0 + (1-\lambda)0 \\ =\ &0.\end{aligned}$$

That is, $\langle z, y\rangle \leq 0$.
Since $\forall y \in C - x$, $\langle z, y\rangle \leq 0$, we get $z \in N_C(x)$.
That is, $\lambda u + (1-\lambda)v \in N_C(x)$.
Since $\forall u, v \in N_C(x)$, $\forall \lambda \in (0, 1)$, $\lambda u + (1-\lambda)v \in N_C(x)$, we get $N_C(x)$ is convex.
Normal Cone is a Cone:
Let v be an arbitrary point in $N_C(x)$.
Let λ be an arbitrary number such that $\lambda > 0$.
Let y be an arbitrary point in $C - x$.
Since $v \in N_C(x)$, $\langle v, y\rangle \leq 0$.
Since $\langle v, y\rangle \leq 0$ and $\lambda > 0$, $\langle \lambda v, y\rangle \leq 0$.
Since $\forall y \in C - x$, $\langle \lambda v, y\rangle \leq 0$, we get $\lambda v \in N_C(x)$.
Since $\forall v \in N_C(x)$, $\forall \lambda > 0$, $\lambda v \in N_C(x)$, we get $N_C(x)$ is a cone.

□

PROPOSITION 6.4. Let C be a non-empty closed convex set in $\mathbb{E}$. Let x be a point in C. Let n be a point in $\mathbb{E}$. Then

$$n \in N_C(x) \iff \forall t \in T_C(x), \langle n, t\rangle \leq 0.$$

Proof.
For one direction, assume that $n \in N_C(x)$.
We are to prove that

$$\forall t \in T_C(x), \quad \langle n, t\rangle \leq 0.$$

Let t be an arbitrary point in $T_C(x)$.
Since $t \in T_C(x) = \mathrm{cl}(\mathrm{cone}(C - x))$,

$$\exists \{t_i\}_{i\in\mathbb{N}} \subseteq \mathrm{cone}(C - x), \text{ such that } t_i \to t. \tag{1}$$

Since $t_i \in \operatorname{cone}(C - x)$,

$$\forall i \in \mathbb{N}, \exists \lambda_i \in \mathbb{R}_{++}, \exists c_i \in C \text{ such that } t_i = \lambda_i(c_i - x). \tag{2}$$

Since $n \in N_C(x)$ and $c_i \in C$,

$$\langle n, c_i - x \rangle \leq 0. \tag{3}$$

Now using (2) and (3), we have

$$\begin{aligned}
&\langle n, t_i \rangle \\
&= \langle n, \lambda_i(c_i - x) \rangle, && \text{since } t_i = \lambda_i(c_i - x)s \\
&= \lambda_i \langle n, c_i - x \rangle \\
&\leq \lambda_i \cdot 0, && \text{since } \langle n, c_i - x \rangle \leq 0 \\
&= 0.
\end{aligned}$$

That is,

$$\forall i \in \mathbb{N}, \quad \langle n, t_i \rangle \leq 0.$$

Since $\langle n, t_i \rangle \leq 0$ for each $i \in \mathbb{N}$ and $t_i \to t$, we get

$$\langle n, t \rangle \leq 0.$$

For the reverse direction, assume that n is a vector such that

$$\forall t \in T_C(x), \quad \langle n, t \rangle \leq 0.$$

We are to prove that $n \in N_C(x)$.
Let y be an arbitrary point in $C - x$.
Since $C - x \subseteq \overline{\operatorname{cone}}(C - x) = T_C(x)$ and $y \in C - x$, we get $y \in T_C(x)$.
Since $y \in T_C(x)$ and $\forall t \in T_C(x), \langle n, t \rangle \leq 0$, we get $\langle n, y \rangle \leq 0$.
Since $\forall y \in C - x, \langle n, y \rangle \leq 0$, we get $n \in N_C(x)$.

□

THEOREM 6.5. Let C be a closed convex set in $\mathbb{E}$ such that $\operatorname{int}(C) \neq \emptyset$. Let x be a point in $\mathbb{E}$. Then

$$x \in \operatorname{int}(C) \iff T_C(x) = \mathbb{E} \iff N_C(x) = \{0\}.$$

Proof.

Part 1.

$x \in \operatorname{int}(C)$ if and only if $0 \in \operatorname{int}(C - x)$, if and only if $\overline{\operatorname{cone}}(C - x) = \mathbb{E}$.

Part 2.

For one direction, assume that $T_C(x) = \mathbb{E}$.

We are to prove that $N_C(x) = \{0\}$.

Consider $n = 0$.

Since

$$\forall t \in T_C(x), \quad \langle 0, t \rangle = 0 \leq 0,$$

we get $0 \in N_C(x)$.

Let n be an arbitrary vector in $N_C(x)$.

By another proposition, we have

$$\begin{aligned} & n \in N_C(x) \\ \iff & \forall t \in T_C(x) = \mathbb{E}, \langle n, t \rangle \leq 0 \\ \implies & \text{for } t = n, \langle n, t \rangle = \langle n, n \rangle \leq 0 \\ \implies & n = 0. \end{aligned}$$

That is, $n \in N_C(x) \implies n = 0$.

So $N_C(x) = \{0\}$.

For the reverse direction, assume that $N_C(x) = \{0\}$.

We are to prove that $T_C(x) = \mathbb{E}$.

Clearly $T_C(x) \subseteq \mathbb{E}$.

For $\mathbb{E} \subseteq T_C(x)$, let x be an arbitrary point in $\mathbb{E}$.

Define $p := \text{proj}_{T_C(x)}(x)$.

Since $p = \text{proj}_{T_C(x)}(x)$,

$$\forall y \in T_C(x), \quad \langle x - p, y - p \rangle \leq 0. \tag{1}$$

Since $p = \text{proj}_{T_C(x)}(x)$, $p \in T_C(x)$.

Since $p \in T_C(x)$ and $T_C(x)$ is a cone,

$$2p \in T_C(x). \tag{2}$$

Apply (1) to $y = 2p$, we get

$$\langle x - p, 2p - p \rangle = \langle x - p, p \rangle \leq 0. \tag{3}$$

Since $T_C(x)$ is a closed cone,

$$0 \in T_C(x). \tag{4}$$

Apply (1) to $y = 0$, we get

$$\langle x - p, 0 - p \rangle = \langle x - p, -p \rangle \leq 0. \tag{5}$$

From (3) and (5), we get

$$\langle x - p, p \rangle = 0.$$

So (1) becomes

$$\forall y \in T_C(x), \quad \langle x - p, y \rangle \leq 0.$$

So $x - p \in N_C(x)$.
So $x - p = 0$.
So $x = p$.
So $x \in T_C(x)$.
Since $\forall x \in \mathbb{E}, x \in T_C(x)$, we get

$$\mathbb{E} \subseteq T_C(x).$$

□

6.3 Arithmetic Properties

PROPOSITION 6.6. Let C and D be convex subsets of $\mathbb{E}$. Let x be a point in $\mathbb{E}$. Then

$$N_C(x) + N_D(x) \subseteq N_{C \cap D}(x).$$

Proof.
If C or D is empty, then $N_C(x) + N_D(x) = N_{C \cap D}(x) = \emptyset$.
So now I assume that $C, D \neq \emptyset$.
If $x \notin C \cap D$, then $N_C(x) + N_D(x) = N_{C \cap D}(x) = \emptyset$.
So now I assume that $x \in C \cap D$.
Let v be an arbitrary point in $N_C(x) + N_D(x)$.
Since $v \in N_C(x) + N_D(x)$, $\exists u \in N_C(x)$, $\exists w \in N_D(x)$ such that $v = u + w$.
Since $u \in N_C(x)$, $\forall y \in C - x$, $\langle u, y \rangle \leq 0$.
Since $w \in N_D(x)$, $\forall y \in D - x$, $\langle w, y \rangle \leq 0$.
Let y be an arbitrary point in $C \cap D - x$.
Since $y \in C \cap D - x$, we get $y \in C - x$ and $y \in D - x$.

$$\begin{aligned} &\langle v, y \rangle \\ &= \langle u + w, y \rangle \\ &= \langle u, y \rangle + \langle w, y \rangle \\ &\leq 0 + 0 = 0. \end{aligned}$$

This is true for any $y \in C \cap D - x$.
So $v \in N_{C\cap D}(x)$.
This is true for any $v \in N_C(x) + N_D(x)$.
So $N_C(x) + N_D(x) \subseteq N_{C\cap D}(x)$.

□

THEOREM 6.7. Let C and D be convex sets in $\mathbb{E}$. Assume that $\mathrm{ri}(C) \cap \mathrm{ri}(D) \neq \emptyset$. Let x be a point in $C \cap D$. Then

$$N_{C\cap D}(x) = N_C(x) + N_C(x).$$

6.4 Other Properties

PROPOSITION 6.8. Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x be a point in $\mathrm{dom}(f)$. Let u be a point in $\mathbb{E}$. Then $u \in \partial f(x)$ if and only if $(u, -1) \in N_{\mathrm{epi}(f)}(x, f(x))$.

Proof.

$$\begin{aligned}
& u \in \partial f(x) \\
\iff & \forall y \in \mathbb{E}, f(y) \geq f(x) + \langle u, y - x\rangle \\
\iff & \forall y \in \mathrm{dom}(f), f(y) \geq f(x) + \langle u, y - x\rangle \\
\iff & \forall (y, \beta) \in \mathrm{epi}(f), f(x) + \langle u, y - x\rangle \leq \beta \\
\iff & \forall (y, \beta) \in \mathrm{epi}(f), \big\langle (u, -1), (y - x, \beta - f(x))\big\rangle \leq 0 \\
\iff & \forall (y, \beta) \in \mathrm{epi}(f), \big\langle (u, -1), (y, \beta) - (x, f(x))\big\rangle \leq 0 \\
\iff & (u, -1) \in N_{\mathrm{epi}(f)}(x, f(x)).
\end{aligned}$$

□

Chapter 7

Extreme Points and Faces

7.1 Extreme Points

DEFINITION 7.1 (Extreme Points - 1)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex subset of $\mathcal{V}$. Let z be some point in C. We say that z is an **extreme point** of C if and only if it does not lie between any two distinct points in C. i.e.,

$$\forall x, y \in C, \forall t \in (0,1), \quad tx + (1-t)y = z \implies x = y = z.$$

DEFINITION 7.2 (Extreme Points - 2)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex subset of $\mathcal{V}$. Let x be some point in C. We say that x is an **extreme point** of C if and only if $C \setminus \{x\}$ is still convex.

PROPOSITION 7.3. The two definitions of extreme point are equivalent.

Proof. **Forward Direction**: Assume that x does not lie between any two distinct points in C. I will show that $C \setminus \{x\}$ is convex. Let x_1 and x_2 be two arbitrary distinct points in $C \setminus \{x\}$. Let λ be an arbitrary number in $(0,1)$. Define a point y as $y := \lambda x_1 + (1-\lambda)x_2$. Since C is convex, $x_1, x_2 \in C$, and $\lambda \in (0,1)$, we get $y \in C$. Since x does not lie between any two distinct points in C, $y \neq x$. So $y \in C \setminus \{x\}$. That is, I have proved that

$$\forall x_1, x_2 \in C \setminus \{x\}, \forall \lambda \in (0,1), \quad y = \lambda x_1 + (1-\lambda)x_2 \in C \setminus \{x\}.$$

By definition, $C \setminus \{x\}$ is convex.

Backward Direction: Assume that $C \setminus \{x\}$ is convex. I will show that x does not lie between any two distinct points in C. Assume for the sake of contradiction that x does lie between two distinct points in C. Say $x = \lambda x_1 + (1-\lambda)x_2$ where $x_1, x_2 \in C$, $x_1 \neq x_2$, and $\lambda \in (0,1)$. Clearly $x \neq x_1$ and $x \neq x_2$. So $x_1, x_2 \in C \setminus \{x\}$. Since $C \setminus \{x\}$ is convex, $x_1, x_2 \in C \setminus \{x\}$, and $\lambda \in (0,1)$, we get $x = \lambda x_1 + (1-\lambda)x_2 \in C \setminus \{x\}$. This leads to a contradiction. So the assumption that x lies between two distinct points in C does not hold. i.e. x does not lie between any two distinct points in C. $\square$

PROPOSITION 7.4. If C is nonempty, convex, and compact, then $\mathrm{Ext}(C) \neq \emptyset$.

PROPOSITION 7.5. Let $\mathcal{V}$ be a locally convex space. Let K be a nonempty, compact, and convex set in $\mathcal{V}$. Then $\mathrm{Ext}(K) \neq \emptyset$.

7.2 Faces

DEFINITION 7.6 (Faces - 1)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex subset of $\mathcal{V}$. Let $F \subseteq \mathcal{V}$. We say that F is a **face** of C, denoted by $F \trianglelefteq C$, if and only if F is a nonempty convex subset of C such that

$$\forall x, y \in C, \forall t \in (0,1), \quad tx + (1-t)y \in F \implies x, y \in F.$$

DEFINITION 7.7 (Faces - 2)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex subset of $\mathcal{V}$. Let $F \subseteq \mathcal{V}$. We say that F is a **face** of C, denoted by $F \trianglelefteq C$, if and only if F is a nonempty convex subset of C such that

$$\forall n \in \mathbb{N}, \forall x \in C^n, \forall t \in (0,1)^n : \sum_{i=1}^{n} t_i = 1, \quad \sum_{i=1}^{n} t_i x_i \in F \implies x \in F^n.$$

PROPOSITION 7.8. The two definitions of faces above are equivalent.

Proof. **Forward Direction**: Suppose that $\forall x, y \in C$, $\forall t \in (0,1)$ such that $tx+(1-t)y \in F$, we have $x, y \in F$. Let $n \in \mathbb{N}$, $x \in C^n$, $t \in (0,1)^n$ be arbitrary such that $\sum_{i=1}^{n} t_i = 1$. Define a point z by $z := \sum_{i=1}^{n} t_i x_i$. Suppose that $z \in F$. I will show that $x \in F^n$. Note that $\forall i \in \{1, ..., n\}$, we have

$$z = \sum_{j=1}^{n} t_j x_j = t_i x_i + (1-t_i) \sum_{j \neq i} \frac{t_j}{1-t_i} x_j.$$

Consider the point $z_i := \sum_{j \neq i} \frac{t_j}{1-t_i} x_j$. Note that $\forall j \neq i$, $\frac{t_j}{1-t_i} \in (0,1)$ and that $\sum_{j \neq i} \frac{t_j}{1-t_i} = 1$. So since $\forall j \neq i$, $x_j \in C$ and C is convex, we get $z_i \in C$. By assumption, we get $x_i, z_i \in F$. In particular, $x_i \in F$. So $x \in F^n$.

Backward Direction: Suppose that $\forall n \in \mathbb{N}$, $\forall x \in C^n$, $\forall t \in (0,1)^n$ such that $\sum_{i=1}^{n} t_i = 1$ and that $\sum_{i=1}^{n} t_i x_i \in F$, we have $x \in F^n$. Take $n := 2$, then $\forall x, y \in C$, $\forall t \in (0,1)$ such that $tx + (1-t)y \in F$, we have $x, y \in F$. $\square$

Faces are generalizations of extreme points.

PROPOSITION 7.9 (Transitivity)**.** Let $\mathcal{V}$ be a vector space. Let A, B, and C be nonempty convex subsets of $\mathcal{V}$. Suppose that $A \trianglelefteq B$ and $B \trianglelefteq C$. Then $A \trianglelefteq C$.

Proof. Let x and y be two arbitrary elements of C. Let t be an arbitrary element of $(0,1)$. Define a point z by $z := tx + (1-t)y$. Suppose that $z \in A$. I will show that $x, y \in A$. Note that since $A \trianglelefteq B$, we have $A \subseteq B$. So $z \in A \subseteq B$. Since $x, y \in C$, $t \in (0,1)$, $z \in B$, and $B \trianglelefteq C$, we get $x, y \in B$. Since $x, y \in B$, $t \in (0,1)$, $z \in A$, and $A \trianglelefteq B$, we get $x, y \in A$. So $A \trianglelefteq C$. $\square$

PROPOSITION 7.10 (Intersection)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex subset of $\mathcal{V}$. Let $A, B \trianglelefteq C$. Then $(A \cap B) \trianglelefteq C$.

Proof. Let x and y be two arbitrary elements of C. Let t be an arbitrary element of $(0,1)$. Define a point z by $z := tx+(1-t)y$. Suppose that $z \in A \cap B$. I will show that $x, y \in A \cap B$. Since $A \trianglelefteq C$, $x, y \in C$, $t \in (0,1)$, and $z \in A \cap B \subseteq A$, we get $x, y \in A$. Similarly, we get $x, y \in B$. So $x, y \in A \cap B$. So $(A \cap B) \trianglelefteq C$. $\square$

7.2.1 Exposed Faces

DEFINITION 7.11 (Exposed Face of a Convex Cone)**.** Let $\mathcal{V}$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Let C be a nonempty convex conic subset of $\mathcal{V}$. Let $F \trianglelefteq C$. We say that F is **exposed** if and only if $\exists a \in \mathcal{V} \setminus \{0\}$ such that

$$F = \{x \in C : \langle a, x \rangle = 0\} \text{ and } C \subseteq \{x \in \mathcal{V} : \langle a, x \rangle \leq 0\}.$$

PROPOSITION 7.12. Let $\mathcal{V}$ be a vector space. Let C be nonempty convex subset of $\mathcal{V}$. Then every face of C is contained in some exposed face of C.

7.2.2 Relation Between Extreme Points and Faces

DEFINITION 7.13 (Extreme Points - 3)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex subset of $\mathcal{V}$. Let x be some point in C. We say that x is an **extreme point** of C if and only if $\{x\}$ is a face of C.

PROPOSITION 7.14. This definition of extreme points is equivalent to the previous two.

PROPOSITION 7.15. If F is a face of C, then $\text{Ext}(F) \subseteq \text{Ext}(C)$.

7.3 The Krein-Milman Theorem

LEMMA 7.16. Let $\mathcal{V}$ be a locally convex space. Let K be a nonempty compact convex subset of $\mathcal{V}$. Let $\rho \in \mathcal{V}^*$. Define $r := \sup\{\Re\rho(x) : x \in K\}$. Define $F := \{x \in K : \Re\rho(x) = r\}$. Then F is a nonempty compact face of K.

Proof. **Nonempty**: Since $\Re\rho$ is continuous and K is compact, $\{\Re\rho(x) : x \in K\}$ is a compact set in $\mathbb{R}$. So $r = \sup\{\Re\rho(x) : x \in K\}$ is attained. So $F \neq \emptyset$.

Compact: Notice $F = (\Re\rho)^{-1}(\{r\})$. Since $\Re\rho$ is continuous and $\{r\} \subseteq \mathbb{R}$ is closed, F is closed. Since F is a closed subset of K and K is compact, F is compact.

Convex: Let x and y be arbitrary elements of F. Let $t \in (0, 1)$. Since $x, y \in F$, we have $\Re\rho(x) = \Re\rho(y) = r$. So

$$\Re\rho(tx + (1-t)y) = t\Re\rho(x) + (1-t)\Re\rho(y) = tr + (1-t)r = r.$$

So $tx + (1-t)y \in F$. So F is convex.

Face: Let x and y be arbitrary elements of K. Let $t \in (0, 1)$. Suppose that $tx+(1-t)y \in F$. Since $x, y \in K$, we have $\Re\rho(x) \le r$ and $\Re\rho(y) \le r$. Since $tx + (1-t)y \in F$, we have

$$t\Re\rho(x) + (1-t)\Re\rho(y) = \Re\rho(tx + (1-t)y) = r.$$

So we must have $\Re\rho(x) = \Re\rho(y) = r$. So $x, y \in F$. So F is a face of K.

□

THEOREM 7.17 (Krein-Milman Theorem)**.** A compact convex set in a locally convex space is the closed convex hull of its extreme points.

Proof. Let $\mathcal{V}$ be a locally convex space. Let K be a nonempty, compact, and convex set in $\mathcal{V}$.

Forward Direction: Show that $K \subseteq \overline{\text{conv}}(\text{Ext}(K))$. Let m be an arbitrary element of K. Assume for the sake of contradiction that $m \notin \overline{\text{conv}}(\text{Ext}(K))$. By the Hahn-Banach Theorem, there is some $\tau \in \mathcal{V}^*$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha > \beta$ and

$$\forall b \in \overline{\text{conv}}(\text{Ext}(K)), \quad \Re\tau(m) \ge \alpha > \beta \ge \Re\tau(b).$$

Define $s := \sup\{\Re\tau(w) : w \in K\}$. Define $L := \{z \in K : \Re\tau(z) = s\}$. Then L is a nonempty compact face of K. So $\text{Ext}(L) \ne \emptyset$. Let e be an element of $\text{Ext}(L)$. Then $e \in \text{Ext}(L) \subseteq L$. So $\Re\tau(e) = s$. So

$$\forall b \in \overline{\text{conv}}(\text{Ext}(K)), \quad \Re\tau(e) = s \ge \Re\tau(m) \ge \alpha > \beta \ge \Re\tau(b).$$

That is, $\Re\tau(e) > \Re\tau(b)$. Since L is a face of K, $\text{Ext}(L) \subseteq \text{Ext}(K)$. Notice $e \in \text{Ext}(L) \subseteq \text{Ext}(K) \subseteq \overline{\text{conv}}(\text{Ext}(K))$. So in particular, $\Re\tau(e) > \Re\tau(e)$. This is a contradiction. So $m \in \overline{\text{conv}}(\text{Ext}(K))$. So $K \subseteq \overline{\text{conv}}(\text{Ext}(K))$.

Backward Direction: Show that $\overline{\text{conv}}(\text{Ext}(K)) \subseteq K$. Note that $\text{Ext}(K) \subseteq K$. Since K is closed and convex and $\text{Ext}(K) \subseteq K$, we get $\overline{\text{conv}}(\text{Ext}(K)) \subseteq K$.

□

PROPOSITION 7.18. Let $\mathcal{V}$ be a vector space. Let K be a nonempty compact convex subset of $\mathcal{V}$. Let $\mathcal{F}(K)$ denote the set of faces of K, partially ordered by inclusion. Then the minimal proper faces in $\mathcal{F}(K)$ are the extreme points of K.

PROPOSITION 7.19. Let $\mathcal{V}$ be a vector space. Let K be a nonempty compact convex subset of $\mathcal{V}$. Let $\mathcal{F}(K)$ denote the set of faces of K, partially ordered by inclusion. Then the maximal proper faces in $\mathcal{F}(K)$ are exposed.

Chapter 8

Projection Operators

8.1 Definitions

DEFINITION 8.1 (Projection)**.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$. Let S be a non-empty set in the space. Let x be a point in the space. We define the **projection** of x onto S, denoted by $\text{proj}_S(x)$, to be a point given by

$$\text{proj}_S(x) := \text{argmin}_{p \in S} \|p - x\|.$$

i.e., $\text{proj}_S(x)$ is the closest point in S to x.

PROPOSITION 8.2 (Existence)**.** If S is non-empty and closed, then the projection $\text{proj}_S(x)$ exists.

Proof. Define for an $n \in \mathbb{N}$ a point c_m to be a point in S that satisfies

$$\lim_{i \in \mathbb{N}} \|c_i - x\| = d_S(x) \text{ where } d_S(x) = \inf_{p \in S} \|p - x\|.$$

Since $\mathcal{H}$ is a Hilbert space, the norm $\|\cdot\|$ on $\mathcal{H}$ satisfies the Parallelogram Law. So

$$\begin{aligned}
\|c_m - c_n\|^2 &= 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - \|c_m + c_n - 2x\|^2 \\
&= 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - 4\left\|\frac{c_m + c_n}{2} - x\right\|^2 \\
&\leq 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - 4d_S(x) \\
&\to 2d_S(x) + 2d_S(x) - 4d_S(x) = 0.
\end{aligned}$$

So the sequence $(c_i)_{i\in\mathbb{N}}$ is Cauchy. Since $\mathcal{H}$ is a Hilbert space, it is complete. So $(c_i)_{i\in\mathbb{N}}$ converges. Since S is closed, and $(c_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in S, $p := \lim_{i\in\mathbb{N}} c_i \in S$. So $\|p - x\| = \|\lim_{i\in\mathbb{N}} c_i - x\| = \lim_{i\in\mathbb{N}} \|c_i - x\| = d_S(x)$. So p is the minimizer of the distance to the point x over S. So $p = \text{proj}_S(x)$. □

PROPOSITION 8.3 (Uniqueness)**.** If S is non-empty, closed, and convex, then the projection $\text{proj}_S(x)$ is unique.

Proof. Let p denote $\text{proj}_S(x)$. Then $\|p - x\| = d_S(x)$. Let q be a point in S such that $\|q - x\| = d_S(x)$. Then by the Parallelogram Law,

$$\begin{aligned}
0 \le \|p - q\|^2 &= 2\|x - p\|^2 + 2\|q - x\| - 4\left\|x - \frac{1}{2}(p + q)\right\|^2 \\
&\le 2d_S^2(x) + 2d_S^2(x) - 4d_S^2(x) \\
&= 0.
\end{aligned}$$

This shows $\|p - q\| = 0$ and hence $p = q$. Thus the projection is unique. □

8.2 Properties

PROPOSITION 8.4 (Idempotent)**.** The projection operator is idempotent. i.e., if C is a nonempty closed convex set in $\mathbb{E}$, then $\text{proj}_C = \text{proj}_C \text{proj}_C$.

Proof. Let x be an arbitrary point in $\mathbb{E}$. By definition, $\text{proj}_C(x) \in C$. Since $\text{proj}_C(x) \in C$, the closest point in C to $\text{proj}_C(x)$ is $\text{proj}_C(x)$. So $\text{proj}_C \text{proj}_C(x) = \text{proj}_C(x)$. This is true for any $x \in \mathbb{E}$. So $\text{proj}_C = \text{proj}_C \text{proj}_C$. □

PROPOSITION 8.5. Let C be a nonempty closed convex set in $\mathbb{E}$. Then the set of fixed points of the operator proj_C is C.

Proof. For one direction, let x be an arbitrary fixed point of proj_C. We are to prove that $x \in C$. Since x is a fixed point of proj_C, $x = \text{proj}_C(x)$. By definition of projection, $\text{proj}_C(x) \in C$. So $x = \text{proj}_C(x) \in C$.

For the reverse direction, let x be an arbitrary point in C. We are to prove that x is a fixed point of C. Since $x \in C$, the closest point in C to x is x. So $x = \text{proj}_C(x)$. So x is a fixed point of proj_C. □

PROPOSITION 8.6 (Linearity)**.** Let C be a nonempty closed convex set in $\mathbb{E}$. Then the operator proj_C is linear if and only if C is a linear subspace.

PROPOSITION 8.7 (Non-expansive)**.** The projection operator is non-expansive. i.e., if C is a nonempty closed convex set in $\mathbb{E}$, then $\|\operatorname{proj} C(x)\| \leq \|x\|$ for any $x \in \mathbb{E}$.

this is not true. I guess it will be true when C is a linear subspace.

PROPOSITION 8.8. Let C be a nonempty closed convex set in $\mathbb{E}$. Then proj_C is Lipschitz with constant 1.

Proof. Let x and y be two arbitrary points in $\mathbb{E}$. If $\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\| = 0$, then $\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\| \leq \|x - y\|$. Otherwise,

$$\begin{aligned}
&\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|^2 \\
&= \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), \operatorname{proj}_C(x) - \operatorname{proj}_C(y)\rangle \\
&= \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), \operatorname{proj}_C(x) - x\rangle \\
&+ \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&+ \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), y - \operatorname{proj}_C(y)\rangle \\
&= \langle x - \operatorname{proj}_C(x), \operatorname{proj}_C(y) - \operatorname{proj}_C(x)\rangle \\
&+ \langle y - \operatorname{proj}_C(y), \operatorname{proj}_C(x) - \operatorname{proj}_C(y)\rangle \\
&+ \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&\leq 0 + 0 + \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&= \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&\leq \|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|\|x - y\|.
\end{aligned}$$

That is,

$$\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|^2 \leq \|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|\|x - y\|.$$

Dividing both sides by $\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|$ gives

$$\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\| \leq \|x - y\|.$$

So proj_C is Lipschitz with constant 1. $\square$

PROPOSITION 8.9 (Firmly Non-expansive)**.** Let C be a nonempty closed convex set in $\mathbb{E}$. Then proj_C is firmly non-expansive.

Proof. This is to prove.

$$\forall x, y \in \mathbb{E}, \quad \left\| \mathrm{proj}_C(y) - \mathrm{proj}_C(x) \right\|^2 \leq \left\langle \mathrm{proj}_C(y) - \mathrm{proj}_C(x), y - x \right\rangle.$$

$$\begin{aligned}
&\left\| \mathrm{proj}_C(y) - \mathrm{proj}_C(x) \right\|^2 \\
&= \left\langle \mathrm{proj}_C(y) - \mathrm{proj}_C(x), \mathrm{proj}_C(y) - \mathrm{proj}_C(x) \right\rangle \\
&= \left\langle \mathrm{proj}_C(y) - \mathrm{proj}_C(x), \mathrm{proj}_C(y) - y \right\rangle \\
&+ \left\langle \mathrm{proj}_C(y) - \mathrm{proj}_C(x), y - x \right\rangle \\
&+ \left\langle \mathrm{proj}_C(y) - \mathrm{proj}_C(x), x - \mathrm{proj}_C(x) \right\rangle \\
&\leq 0 + \left\langle \mathrm{proj}_C(y) - \mathrm{proj}_C(x), y - x \right\rangle + 0 \\
&= \left\langle \mathrm{proj}_C(y) - \mathrm{proj}_C(x), y - x \right\rangle.
\end{aligned}$$

□

8.3 Examples

EXAMPLE 8.10 (Projection onto Half Spaces, Bauschke-Combettes, 2017 Book)**.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$. Let $C \subseteq \mathcal{H}$ be given by

$$C := \{x \in \mathcal{H} : \langle x, u \rangle \leq \eta\}$$

where $u \in \mathcal{H}$ and $\eta \in \mathbb{R}$ are constants. Then if $u = 0$ and $\eta \geq 0$, $C = \mathcal{H}$ and $\mathrm{proj}_C = \mathrm{Id}$; if $u = 0$ and $\eta < 0$, $C = \emptyset$; if $u \neq 0$, we have

$$\mathrm{proj}_C(x) = \begin{cases} x, & \text{if } \langle x, u \rangle \leq \eta \\ x + \dfrac{\eta - \langle x, u \rangle}{\|u\|^2} u, & \text{if } \langle x, u \rangle > \eta. \end{cases}$$

8.4 Characterizations

THEOREM 8.11 (Projection Theorem)**.** Let C be a nonempty closed convex set in

$\mathbb{E}$. Let x and p be points in $\mathbb{E}$. Then $p = \mathrm{proj}_C(x)$ if and only if

$$\forall y \in C, \quad \langle y - p, x - p \rangle \leq 0.$$

Proof. Let y be an arbitrary point in C. Let α be an arbitrary number in $[0, 1]$. Define $y_\alpha := \alpha y + (1 - \alpha)p$. Now

$$\begin{aligned}
& p = \mathrm{proj}_C(x) \\
\iff & \forall y \in C, \forall \alpha \in [0, 1], \|x - p\|^2 \leq \|x - y_\alpha\|^2 \\
\iff & \forall y \in C, \forall \alpha \in [0, 1], \|x - p\|^2 \leq \|x - p - \alpha(y - p)\|^2 \\
\iff & \forall y \in C, \langle x - p, y - p \rangle \leq 0.
\end{aligned}$$

□

THEOREM 8.12 (Bauschke-Combettes, 2017 Book)**.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$. Let K be a nonempty closed convex cone in $\mathcal{H}$. Let $x, p \in \mathcal{H}$. Then $p = \mathrm{proj}_K(x)$ if and only if $p \in K$, $\langle x - p, p \rangle = 0$, and $x - p \in K^\circ$.

Chapter 9

Separation

9.1 Definitions

DEFINITION 9.1 (Separated)**.** Let S_1 and S_2 be two sets in $\mathbb{E}$. We say that S_1 and S_2 are **separated** if $\exists b \in \mathbb{E} \setminus \{\vec{0}\}$ such that

$$\sup_{s_1 \in S_1} \langle s_1, b \rangle \leq \inf_{s_2 \in S_2} \langle s_2, b \rangle.$$

DEFINITION 9.2 (Strongly Separated)**.** Let S_1 and S_2 be two sets in $\mathbb{E}$. We say that they are **strongly separated** if the inequality holds strictly.

DEFINITION 9.3 (Properly Separated)**.** Let S_1 and S_2 be two sets in $\mathbb{E}$. We say that S_1 and S_2 are **properly separated** if $\exists b \in \mathbb{E}$ such that

$$\sup_{x \in S_1} \langle x, b \rangle \leq \inf_{y \in S_2} \langle y, b \rangle, \text{ and}$$

$$\inf_{x \in S_1} \langle x, b \rangle > \sup_{y \in S_2} \langle y, b \rangle.$$

9.2 Main Results

PROPOSITION 9.4. Let C be a nonempty closed convex set in $\mathbb{E}$. Let x be a point in $\mathbb{E}$ such that $x \notin C$. Then x and C are strongly separated.

Proof. Define a point p by

$$p := \operatorname{proj}_C(x).$$

Define a point a by

$$a := x - p.$$

To prove that x is strongly separated from C, it suffices to prove that

$$\forall y \in C, \quad \langle y, a\rangle < \langle x, a\rangle.$$

Since $x \notin C$ and C is closed,

$$a \neq 0. \tag{1}$$

Let y be an arbitrary point in C. Since $p = \operatorname{proj}_C(x)$ and $y \in C$,

$$\langle y - p, x - p\rangle \leq 0. \tag{2}$$

$$\begin{aligned}
&\langle y, a\rangle \\
&< \langle y, a\rangle + \langle a, a\rangle, \text{ since } a \neq 0 \\
&= \langle y + a, a\rangle \\
&= \langle y + x - p, x - p\rangle, \text{ substitute } a = x - p \\
&= \langle y - p, x - p\rangle + \langle x, x - p\rangle \\
&\leq 0 + \langle x, x - p\rangle, \text{ since } \langle y - p, x - p\rangle \leq 0 \\
&= \langle x, x - p\rangle \\
&= \langle x, a\rangle.
\end{aligned}$$

That is,

$$\forall y \in C, \quad \langle y, a\rangle < \langle x, a\rangle.$$

So x is strongly separated from C. $\square$

PROPOSITION 9.5. Let C_1 be a non-empty closed convex set in $\mathbb{E}$. Let C_2 be a non-empty compact convex set in $\mathbb{E}$. Assume that C_1 and C_2 are disjoint. Then C_1 and C_2 are strongly separated.

Proof. Since C_1 is non-empty closed and convex and C_2 is non-empty compact and convex, we get $C_1 - C_2$ is non-empty closed and convex. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C_1 - C_2$. Since $C_1 - C_2$ is non-empty closed and convex and $0 \in C_1 - C_2$, 0 and $C_1 - C_2$ are strongly separated. Since 0 is strongly separated from $C_1 - C_2$,

$$\exists a \neq 0 \text{ such that } \forall c_1 \in C_1, c_2 \in C_2, \quad \langle c_1 - c_2, a \rangle < \langle 0, a \rangle.$$

That is,

$$\langle c_1, a \rangle < \langle c_2, a \rangle.$$

So C_1 and C_2 are strongly separated. □

THEOREM 9.6. Let C_1 and C_2 be non-empty closed convex sets in $\mathbb{E}$. Assume that C_1 and C_2 are disjoint. Then C_1 and C_2 are separated.

Proof. For $n \in \mathbb{N}$, define

$$D_n := C_2 \cap \text{ball}(0, n).$$

Then D_n is compact for any $n \in \mathbb{N}$. Since $\left\{ C_1 \text{ is non-empty closed and convex } D_n \text{ is non-empty compact and convex} \right.$ we get C_1 and D_n are strongly separated for any $n \in \mathbb{N}$. So

$$\forall n \in \mathbb{N}, \exists a_n \in \mathbb{E}, \|a_n\| = 1 \text{ such that } \forall c_1 \in C_1, \forall d_2 \in D_n, \quad \langle c_1, a_n \rangle < \langle d_2, a_n \rangle.$$

Since $\|a_n\| = 1$ for any $n \in \mathbb{N}$, there exists a subsequence $\{a_n\}_{n \in I}$ where I is some infinite subset of $\mathbb{N}$ such that $\{a_n\}_{n \in I}$ converges to some point $a \in \mathbb{E}$. Let x be an arbitrary point in C_1. Let y be an arbitrary point in C_2. For large enough n, $y \in D_n$. Since
$$\begin{cases} \langle x, a_n \rangle < \langle y, a_n \rangle \text{ for large enough } n \\ \lim\limits_{n \in I, n \to \infty} \langle x, a_n \rangle = \langle x, a \rangle \\ \lim\limits_{n \in I, n \to \infty} \langle y, a_n \rangle = \langle y, a \rangle \end{cases}, \text{ we get}$$

$$\langle x, a \rangle \leq \langle y, a \rangle.$$

Since

$$\exists a \neq 0 \text{ such that } \forall x \in C_1, \forall y \in C_2, \quad \langle x, a \rangle \leq \langle y, a \rangle,$$

by definition of separated, C_1 and C_2 are separated. □

PROPOSITION 9.7. Let C_1 and C_2 be non-empty convex subsets of $\mathbb{E}$. Then C_1

and C_2 are properly separated if and only if

$$\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset.$$

Chapter 10

Convex Functions

10.1 Preliminaries

DEFINITION 10.1 (Epigraph). Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **epigraph** of f, denoted by $\operatorname{epi}(f)$, to be the set given by

$$\operatorname{epi}(f) := \big\{(x, \alpha) \in \mathbb{E} \times \mathbb{R} : f(x) \leq \alpha\big\}.$$

DEFINITION 10.2 (Domain). Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **domain** of f, denoted by $\operatorname{dom}(f)$, to be a set given by

$$\operatorname{dom}(f) := \big\{x \in \mathbb{E} : f(x) < +\infty\big\}.$$

DEFINITION 10.3 (Proper). Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **proper** if

$$\begin{aligned} &\exists x \in \mathbb{E}, \quad f(x) \neq +\infty, \text{ and} \\ &\forall x \in \mathbb{E}, \quad f(x) \neq -\infty \end{aligned}$$

10.2 The Indicator Function

DEFINITION 10.4 (The Indicator Function)**.** Let S be a subset of $\mathbb{E}$. We define the **indicator function** of S, denoted by δ_S, to be a function from $\mathbb{E}$ to $\mathbb{R}^*$ given by

$$\delta_S(x) = \begin{cases} 0, & \text{if } x \in S \\ +\infty, & \text{if } x \notin S. \end{cases}$$

PROPOSITION 10.5. Let S be a subset of $\mathbb{E}$. Then

1. S is non-empty if and only if δ_S is proper.
2. S is convex if and only if δ_S is convex.
3. S is closed if and only if δ_S is lower semi-continuous.

Proof of (1).

For one direction, assume that S is not empty.
We are to prove that δ_S is proper.
Since $S \neq \emptyset$, pick $p \in S$.
Since $p \in S$, $\delta_S(p) = 0$.
Since $\delta_S(p) = 0$, $\exists x_0 \in \mathbb{E}$ such that $\delta_S(x_0) \neq +\infty$.
By definition of the indicator function, it never takes $-\infty$.
Since $\exists x_0 \in \mathbb{E}$ such that $\delta_S(x_0) \neq +\infty$ and $\forall x \in \mathbb{E}$, $\delta_S(x) \neq -\infty$, we get δ_S is proper.
For the reverse direction, assume that δ_S is proper.
We are to prove that S is non-empty.
Assume for the sake of contradiction that S is empty.
Let x be an arbitrary point in $\mathbb{E}$.
Since $S = \emptyset$, $x \notin S$.
Since $x \notin S$, $\delta_S(x) = +\infty$.
Since $\forall x \in \mathbb{E}$, $\delta_S(x) = +\infty$, by definition of proper function, δ_S is not proper.
This contradicts to the assumption that δ_S is proper.
So the assumption that $S = \emptyset$ is false.
i.e., S is non-empty.

□

Proof of (2).

For one direction, assume that S is convex.
We are to prove that δ_S is convex.

Let x and y be arbitrary points in $\mathrm{dom}(\delta_S)$.

By definition of indicator functions, $\mathrm{dom}(\delta_S) = S$.

So $x, y \in S$.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda x + (1 - \lambda)y$.

Since $x, y \in S$ and $\lambda \in (0, 1)$ and S is convex and $z = \lambda x + (1 - \lambda)y$, we get $z \in S$.

Since $z \in S$, $\delta_S(z) = 0$.

Since $\lambda \in (0, 1)$ and $\mathrm{range}(\delta_S) = \{0, +\infty\}$, we get $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) \geq 0$.

Since $\delta_S(z) = 0$ and $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) \geq 0$, we get $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$.

That is, $\delta_S(\lambda x + (1 - \lambda)y) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$.

Since $\forall x, y \in \mathrm{dom}(\delta_S)$, $\forall \lambda \in (0, 1)$, $\delta_S(\lambda x + (1 - \lambda)y) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$, we get δ_S is convex.

For the reverse direction, assume that δ_S is convex.

We are to prove that S is convex.

The case where S is empty is trivial.

So now I assume $S \neq \emptyset$.

Let x and y be arbitrary points in S.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda x + (1 - \lambda)y$.

Since $x \in S$, $\delta_S(x) = 0$.

Since $y \in S$, $\delta_S(y) = 0$.

Since $\delta_S(x) = \delta_S(y) = 0$, we get $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) = 0$.

Since $\lambda \in (0, 1)$ and δ_S is convex, $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$.

Since $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$ and $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) = 0$, we get $\delta_S(z) \leq 0$.

By definition of the indicator function, $\delta_S(z) \geq 0$.

Since $\delta_S(z) \leq 0$ and $\delta_S(z) \geq 0$, we get $\delta_S(z) = 0$.

Since $\delta_S(z) = 0$, $z \in S$.

That is, $\lambda x + (1 - \lambda)y \in S$.

Since $\forall x, y \in S$, $\forall \lambda \in (0, 1)$, $\lambda x + (1 - \lambda)y \in S$, we get S is convex.

□

Proof of (3).

For one direction, assume that S is closed.

We are to prove that δ_S is lower semi-continuous.

Let $\{(x_i, \alpha_i)\}_{i \in \mathbb{N}}$ be an arbitrary sequence in $\mathrm{epi}(\delta_S)$ that converges.

Say its limit is $(x_\infty, \alpha_\infty)$.

Since $(x_i, \alpha_i) \to (x_\infty, \alpha_\infty)$, $x_i \to x_\infty$.

Since $(x_i, \alpha_i) \in \mathrm{epi}(\delta_S)$, $\delta_S(x_i) \leq \alpha_i$.

Since $\delta_S(x_i) \leq \alpha_i$ and $\alpha_i \in \mathbb{R}$, we get $\delta_S(x_i) \neq +\infty$.

Since $\delta_S(x_i) \neq +\infty$, $x_i \in S$.

Since $x_i \in S$ and $x_i \to x_\infty$ and S is closed, $x_\infty \in S$.

Since $x_\infty \in S$, $\delta_S(x_\infty) = 0$.

Since $x_i \in S$, $\delta_S(x_i) = 0$.

Since $\delta_S(x_i) = 0$ and $\delta_S(x_i) \leq \alpha_i$, $\alpha_i \geq 0$.

Since $(x_i, \alpha_i) \to (x_\infty, \alpha_\infty)$, $\alpha_i \to \alpha_\infty$.

Since $\alpha_i \geq 0$ and $\alpha \to \alpha_\infty$, $\alpha_\infty \geq 0$.

Since $\delta_S(x_\infty) = 0$ and $\alpha_\infty \geq 0$, $\delta_S(x_\infty) \leq \alpha_\infty$.

Since $\delta_S(x_\infty) \leq \alpha_\infty$, $(x_\infty, \alpha_\infty) \in \operatorname{epi}(\delta_S)$.

Since for any convergent sequence in $\operatorname{epi}(\delta_S)$, its limit is also in $\operatorname{epi}(\delta_S)$, we get $\operatorname{epi}(\delta_S)$ is closed.

For the reverse direction, assume that δ_S is lower semi-continuous.

We are to prove that S is closed.

Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in S that converges.

Say its limit is x_∞.

Since $x_i \in S$, $\delta_S(x_i) = 0$.

Since $\delta_S(x_i) = 0$, $(x_i, 0) \in \operatorname{epi}(\delta_S)$.

Since $x_i \to x_\infty$, $(x_i, 0) \to (x_\infty, 0)$.

Since $(x_i, 0) \in \operatorname{epi}(\delta_S)$ and $(x_i, 0) \to (x_\infty, 0)$, $(x_\infty, 0) \in \operatorname{epi}(\delta_S)$.

Since $(x_\infty, 0) \in \operatorname{epi}(\delta_S)$, $\delta_S(x_\infty) \leq 0$.

By definition of the indicator function, $\delta_S(x_\infty) \geq 0$.

Since $\delta_S(x_\infty) \leq 0$ and $\delta_S(x_\infty) \geq 0$, we get $\delta_S(x_\infty) = 0$.

Since $\delta_S(x_\infty) = 0$, $x_\infty \in S$.

Since for any convergent sequence in S, its limit is also in S, we get S is closed.

$\square$

10.3 Definitions

DEFINITION 10.6 (Convex Function)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **convex** if

$$\forall x, y \in \operatorname{dom}(f), \forall \lambda \in [0,1], \quad f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y).$$

DEFINITION 10.7 (Convex Function)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **convex** if the epigraph of f is convex.

PROPOSITION 10.8. The two definitions of convexity of functions are equivalent.

Proof.

The case where $\mathrm{dom}(f), \mathrm{epi}(f) = \emptyset$ is trivial.

So now I assume that $\mathrm{dom}(f), \mathrm{epi}(f) \neq \emptyset$.

For one direction, assume that $\forall x, y \in \mathrm{dom}(f)$, $\forall \lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

We are to prove that the epigraph of f is convex.

Let (x, α) and (y, β) be two arbitrary points in $\mathrm{epi}(f)$.

Since $(x, \alpha), (y, \beta) \in \mathrm{epi}(f)$, $x, y \in \mathrm{dom}(f)$.

Let λ be an arbitrary number in $[0, 1]$.

Define a point $(z, \gamma) := \lambda(x, \alpha) + (1-\lambda)(y, \beta)$.

Then $z = \lambda x + (1-\lambda)y$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$.

Since $x, y \in \mathrm{dom}(f)$, $\lambda \in [0,1]$, we get $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

Since $(x, \alpha) \in \mathrm{epi}(f)$, $f(x) \leq \alpha$.

Since $(y, \beta) \in \mathrm{epi}(f)$, $f(y) \leq \beta$.

Since $f(x) \leq \alpha$ and $f(y) \leq \beta$ and $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, we get $f(\lambda x + (1-\lambda)y) \leq \lambda\alpha + (1-\lambda)\beta$.

Since $z = \lambda x + (1-\lambda)y$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$ and $f(\lambda x + (1-\lambda)y) \leq \lambda\alpha + (1-\lambda)\beta$, we get $f(z) \leq \gamma$.

Since $f(z) \leq \gamma$, $(z, \gamma) \in \mathrm{epi}(f)$.

For the reverse direction, assume that $\mathrm{epi}(f)$ is convex.

We are to prove that $\forall x, y \in \mathrm{dom}(f)$, $\forall \lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

Let x and y be two arbitrary points in $\mathrm{dom}(f)$.

Let λ be an arbitrary number in $[0, 1]$.

Define $z := \lambda x + (1-\lambda)y$.

Define $\gamma := \lambda f(x) + (1-\lambda)f(y)$.

Since $(x, f(x)) \in \mathrm{epi}(f)$ and $(y, f(y)) \in \mathrm{epi}(f)$ and $\lambda \in [0,1]$ and $\mathrm{epi}(f)$ is convex, we get $\lambda(x, f(x)) + (1-\lambda)(y, f(y)) \in \mathrm{epi}(f)$.

Since $z = \lambda x + (1-\lambda)y$ and $\gamma = \lambda f(x) + (1-\lambda)f(y)$ and $\lambda(x, f(x)) + (1-\lambda)(y, f(y)) \in \mathrm{epi}(f)$, we get $(z, \gamma) \in \mathrm{epi}(f)$.

Since $(z, \gamma) \in \mathrm{epi}(f)$, $f(z) \leq \gamma$.

That is, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

□

10.4 Basic Properties

PROPOSITION 10.9 (Necessary Condition)**.** The domain of a convex function is convex.

Proof. Follows from the fact that convexity is stable under affine transformations. Define $A((x, \alpha)) := x$. Then $\operatorname{dom}(f) = A(\operatorname{epi}(f))$. □

PROPOSITION 10.10. The level sets of a convex function are convex.

PROPOSITION 10.11 (Restriction to a Line)**.** A function $f : \mathbb{E} \to \mathbb{R}$ is convex if and only if $\forall x \in \operatorname{dom}(f), \forall v \in \mathbb{E}$, the function $g_{x,v} : \mathbb{R} \to \mathbb{R}$ given by

$$g_{x,v}(t) = f(x + tv)$$

is convex.

10.5 Differentiable Convex Functions

PROPOSITION 10.12. Let f be a proper convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Let $x \in \operatorname{dom}(f)$. If f is differentiable at point x, then $\nabla(f)(x)$ is the unique subgradient of f at point x. i.e., $\partial(f)(x) = \{\nabla(f)(x)\}$. Conversely, if the subgradient $\partial(f)(x)$ of f at point x is a singleton set $\{v\}$, then f is differentiable at point x and $\nabla(f)(x) = v$.

Proof. □

PROPOSITION 10.13 (First-Order Condition)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Assume that $\operatorname{dom}(f)$ is convex and open and that f is differentiable on $\operatorname{dom}(f)$. Then f is convex if and only if

$$\forall x, y \in \operatorname{dom}(f), \quad f(y) - f(x) \geq \big\langle \nabla(f)(x), y - x \big\rangle.$$

i.e., the first-order approximation of f is a global under-estimator.

Proof.

Part 1.

For one direction, assume that f is convex. We are to prove that

$$\forall x, y \in \operatorname{dom}(f), \quad f(y) - f(x) \geq \left\langle \nabla(f)(x), y - x \right\rangle.$$

Let x and y be arbitrary points in $\operatorname{dom}(f)$. Since f is convex and differentiable at point x, $\nabla(f)(x) = \partial(f)(x)$. So $\nabla(f)(x)$ satisfies the subgradient inequality. That is,

$$f(y) - f(x) \geq \left\langle \nabla(f)(x), y - x \right\rangle.$$

Part 2.

For the reverse direction, assume that

$$\forall x, y \in \operatorname{dom}(f), \quad f(y) - f(x) \geq \left\langle \nabla(f)(x), y - x \right\rangle.$$

We are to prove that f is convex.

Not Finished.

□

PROPOSITION 10.14. Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Assume that $\operatorname{dom}(f)$ is convex and open and that f is differentiable on $\operatorname{dom}(f)$. Then f is convex if and only if

$$\forall x, y \in \operatorname{dom}(f), \quad \left\langle \nabla(f)(x) - \nabla(f)(y), x - y \right\rangle \geq 0.$$

Proof.

Part 1.

For one direction, assume that f is convex. We are to prove that

$$\forall x, y \in \operatorname{dom}(f), \quad \left\langle \nabla(f)(x) - \nabla(f)(y), x - y \right\rangle \geq 0.$$

Let x and y be arbitrary points in $\operatorname{dom}(f)$. Since f is convex and differentiable at point x, $\nabla(f)(x) = \partial(f)(x)$. So $\nabla(f)(x)$ satisfies the subgradient inequality. That is,

$$f(y) - f(x) \geq \left\langle \nabla(f)(x), y - x \right\rangle. \tag{1}$$

Since f is convex and differentiable at point y, $\nabla(f)(y) = \partial(f)(y)$. So $\nabla(f)(y)$ satisfies the subgradient inequality. That is,

$$f(x) - f(y) \geq \left\langle \nabla(f)(y), x - y \right\rangle. \tag{2}$$

Take the sum of inequalities (1) and (2), we get

$$(f(y) - f(x)) + (f(x) - f(y)) \geq \left\langle \nabla(f)(x), y - x \right\rangle + \left\langle \nabla(f)(y), x - y \right\rangle$$

$$\begin{aligned}
\implies & \qquad 0 \geq -\big\langle \nabla(f)(x), x-y \big\rangle + \big\langle \nabla(f)(y), x-y \big\rangle \\
\implies & \quad \big\langle \nabla(f)(x), x-y \big\rangle - \big\langle \nabla(f)(y), x-y \big\rangle \geq 0 \\
\implies & \qquad \big\langle \nabla(f)(x) - \nabla(f)(x), x-y \big\rangle \geq 0.
\end{aligned}$$

Part 2.

For the reverse direction, assume that

$$\forall x, y \in \operatorname{dom}(f), \quad \big\langle \nabla(f)(x) - \nabla(f)(y), x-y \big\rangle \geq 0.$$

We are to prove that f is convex. Let x and y be arbitrary points in $\operatorname{dom}(f)$. Define a function φ on $(0,1)$ by

$$\varphi(\lambda) := f\big(\lambda x + (1-\lambda)y\big).$$

Notice φ is differentiable and

$$\varphi'(\lambda) = \big\langle \nabla(f)\big(\lambda x + (1-\lambda)y\big), x-y \big\rangle.$$

Let α and β be arbitrary numbers in $(0,1)$. Assume that $\alpha < \beta$. Define two points z_α and z_β by $z_\alpha := \alpha x + (1-\alpha)y$ and $z_\beta := \beta x + (1-\beta)y$. Then

$$\begin{aligned}
& \varphi'(\beta) - \varphi'(\alpha) \\
={}& \big\langle \nabla(f)(\beta x + (1-\beta)y), x-y \big\rangle - \big\langle \nabla(f)(\alpha x + (1-\alpha)y), x-y \big\rangle \\
={}& \big\langle \nabla(f)(z_\beta), x-y \big\rangle - \big\langle \nabla(f)(z_\alpha), x-y \big\rangle \\
={}& \big\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), x-y \big\rangle \\
={}& \left\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), \frac{z_\beta - z_\alpha}{\beta - \alpha} \right\rangle \\
={}& \frac{1}{\beta-\alpha} \big\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), z_\beta - z_\alpha \big\rangle \\
\geq{}& \frac{1}{\beta-\alpha} \cdot 0, \text{ by assumption} \\
={}& 0.
\end{aligned}$$

That is,

$$\forall \alpha, \beta \in (0,1), \quad \beta > \alpha \implies \varphi'(\beta) - \varphi'(\alpha) \geq 0.$$

So φ' is increasing. So φ is convex. So

$$\varphi(\lambda) \leq \lambda\varphi(1) + (1-\lambda)\varphi(0).$$

That is,

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y).$$

By definition, f is convex.

$\square$

PROPOSITION 10.15 (Second-Order Condition)**.** A twice continuously differentiable real-valued function f defined on a convex set is convex if and only if

$$\forall x \in \mathrm{dom}(f), \quad \nabla^2 f(x) \geq 0$$

where $\nabla^2 f(x) = \begin{bmatrix} \dfrac{\partial^2 f(x)}{\partial x_1 \partial x_1} & \cdots & \dfrac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f(x)}{\partial x_n \partial x_n} \end{bmatrix}$ denotes the Hessian matrix of f at x_0.

PROPOSITION 10.16. Let f be a twice continuously differentiable function from $\mathbb{E}$ to $\mathbb{R}$. Then f is convex if and only if $\forall x \in \mathbb{E}$, $\nabla^2 f(x)$ is positive semi-definite.

10.6 Convexity and Lipschitz-ness

THEOREM 10.17. Let f be a differentiable convex function from $\mathbb{E}$ to $\mathbb{R}$. Then the following statements are equivalent.

1. ∇f is Lipschitz with constant L.
2. $\forall x, y \in \mathbb{E}$, we have
$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2.$$
3. $\forall x, y \in \mathbb{E}$, we have
$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2.$$
4. $\forall x, y \in \mathbb{E}$, we have
$$L\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \|\nabla f(y) - \nabla f(x)\|^2.$$

(1) $\implies$ *(2).*

Assume that ∇f is Lipschitz with constant 1.

Let x and y be two arbitrary points in $\mathbb{E}$.

$$\begin{aligned}
& f(y) - f(x) \\
= & \int_0^1 \langle \nabla f(x + t(y-x)), y - x \rangle dt \\
= & \langle \nabla f(x), y - x \rangle + \int_0^1 \langle \nabla f(x + t(y-x)) - \nabla f(x), y - x \rangle dt \\
\leq & \langle \nabla f(x), y - x \rangle + \int_0^1 \| \langle \nabla f(x + t(y-x)) - \nabla f(x) \rangle \| \| y - x \| dt \\
\leq & \langle \nabla f(x), y - x \rangle + \int_0^1 L \| x + t(y-x) - x \| \| y - x \| dt \\
= & \langle \nabla f(x), y - x \rangle + L \| y - x \|^2 \int_0^1 t dt \\
= & \langle \nabla f(x), y - x \rangle + \frac{L}{2} \| y - x \|^2
\end{aligned}$$

That is,

$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle + \frac{L}{2} \| y - x \|^2.$$

□

THEOREM 10.18. Let f be a twice continuously differentiable function from $\mathbb{E}$ to $\mathbb{R}$. Let L be some non-negative number. Then the following statements are equivalent.

1. ∇f is L-Lipschitz.
2. $\forall x \in \mathbb{E}$, $\left\| \nabla^2 f(x) \right\| \leq L$.

10.7 Stability of Convexity

PROPOSITION 10.19 (Non-Negative Linear Combination)**.** A non-negative linear combination of proper convex functions is again convex.

Proof. It suffices to prove that non-negative scalar multiples of convex functions are convex and sums of two convex functions are convex.
Part 1.

Let f be a proper convex function. Let $\alpha \geq 0$ be an arbitrary scalar. We are to prove that αf is convex. Notice $\mathrm{dom}(f) = \mathrm{dom}(\alpha f)$. Since f is proper, $\mathrm{dom}(f) \neq \emptyset$. So $\mathrm{dom}(\alpha f) \neq \emptyset$.

Let x and y be two arbitrary points in $\mathrm{dom}(\alpha f)$. Let λ be an arbitrary number in $(0,1)$. Define a point z as $z := \lambda x + (1-\lambda)y$. Then

$$\begin{aligned}(\alpha f)(\lambda x + (1-\lambda)y) &= \alpha f(\lambda x + (1-\lambda)y)\\ &\leq \alpha(\lambda f(x) + (1-\lambda)f(y))\\ &= \lambda \alpha f(x) + (1-\lambda)\alpha f(y)\\ &= \lambda(\alpha f)(x) + (1-\lambda)(\alpha f)(y).\end{aligned}$$

That is,

$$\forall x, y \in \mathrm{dom}(\alpha f), \forall \lambda \in (0,1), \quad (\alpha f)(\lambda x + (1-\lambda)y) \leq \lambda(\alpha f)(x) + (1-\lambda)(\alpha f)(y).$$

So by definition, αf is convex.
Part 2.

Let f and g be proper convex functions. We are to prove that $f+g$ is convex. Notice $\mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$. Since f is proper, $\mathrm{dom}(f) \neq \emptyset$. Since g is proper, $\mathrm{dom}(g) \neq \emptyset$. So $\mathrm{dom}(f+g) \neq \emptyset$. Let x and y be two arbitrary points in $\mathrm{dom}(f+g)$. Let λ be an arbitrary number in $(0,1)$. Define a point z as $z := \lambda x + (1-\lambda)y$. Then

$$\begin{aligned}(f+g)(\lambda x + (1-\lambda)y) &= f(\lambda x + (1-\lambda)y) + g(\lambda x + (1-\lambda)y)\\ &\leq \lambda f(x) + (1-\lambda)f(y) + \lambda g(x) + (1-\lambda)g(y)\\ &= \lambda(f(x)+g(x)) + (1-\lambda)(f(y)+g(y))\\ &= \lambda(f+g)(x) + (1-\lambda)(f+g)(y).\end{aligned}$$

That is,

$$\forall x, y \in \mathrm{dom}(f+g), \forall \lambda \in (0,1), \quad (f+g)(\lambda x + (1-\lambda)y) = \lambda(f+g)(x) + (1-\lambda)(f+g)(y).$$

So by definition, $f+g$ is convex.

□

PROPOSITION 10.20 (Direct Sum)**.** Direct sums of convex functions are convex.

Proof. Let z and w be two arbitrary points in $\mathrm{dom}(f \oplus g)$. Let $\lambda \in (0,1)$ be arbitrary. Say $z = x \oplus y$ and $w = u \oplus v$ where $x, u \in \mathbb{R}^m$ and $y, v \in \mathbb{R}^p$. Since $z \in \mathrm{dom}(f \oplus g)$, $(f \oplus g)(z) \neq +\infty$. That is, $f(x) + g(y) \neq +\infty$. So neither $f(x)$ nor $g(y)$ is $+\infty$. So both $x \in \mathrm{dom}(f)$ and $y \in \mathrm{dom}(g)$. Similarly, we have $u \in \mathrm{dom}(f)$ and $v \in \mathrm{dom}(g)$. Consider the point

$$\lambda z + (1-\lambda)w$$

$$\begin{aligned}&= \lambda x \oplus y + (1-\lambda)u \oplus v\\&= \big(\lambda x + (1-\lambda)u\big) \oplus \big(\lambda y + (1-\lambda)v\big).\end{aligned}$$

Apply $f \oplus g$ to both sides, we get

$$\begin{aligned}&(f \oplus g)(\lambda z + (1-\lambda)w)\\&= (f \oplus g)\left[\big(\lambda x + (1-\lambda)u\big) \oplus \big(\lambda y + (1-\lambda)v\big)\right]\\&= f(\lambda x + (1-\lambda)u) + g(\lambda y + (1-\lambda)v).\end{aligned}$$

Since f and g are convex, we get

$$\begin{aligned}&f(\lambda x + (1-\lambda)u) \le \lambda f(x) + (1-\lambda)f(u), \text{ and}\\&g(\lambda y + (1-\lambda)v) \le \lambda g(y) + (1-\lambda)g(v).\end{aligned}$$

So

$$\begin{aligned}&(f \oplus g)(\lambda z + (1-\lambda)w)\\&\le \lambda f(x) + (1-\lambda)f(u) + \lambda g(y) + (1-\lambda)g(v)\\&= \lambda(f(x) + g(y)) + (1-\lambda)(f(u) + g(v))\\&= \lambda(f \oplus g)(x \oplus y) + (1-\lambda)(f \oplus g)(u \oplus v)\\&= \lambda(f \oplus g)(z) + (1-\lambda)(f \oplus g)(w).\end{aligned}$$

That is,

$$(f \oplus g)(\lambda z + (1-\lambda)w) \le \lambda(f \oplus g)(z) + (1-\lambda)(f \oplus g)(w).$$

This holds for any $z, w \in \operatorname{dom}(f \oplus g)$ and any $\lambda \in (0,1)$. So $(f \oplus g)$ is convex.

□

PROPOSITION 10.21 (Composition)**.** The composition of a convex function with an affine function is convex. i.e., if f is convex, then $f(Ax+b)$ is convex.

Proof. Let x an y be arbitrary points in $\mathbb{E}$. Let λ be an arbitrary number in $(0,1)$. Define a point z by $z := \lambda x + (1-\lambda)y$.

$$\begin{aligned}&g\big(\lambda x + (1-\lambda y)\\&= f\big(A(\lambda x + (1-\lambda)y) + b\big)\\&= f\big(\lambda Ax + (1-\lambda)Ay + b\big), && \text{by linearity of } A\\&= f\big(\lambda Ax + (1-\lambda)Ay + \lambda b + (1-\lambda)b\big), && \text{decomposite } b\\&= f\big(\lambda(Ax+b) + (1-\lambda)(Ay+b)\big)\end{aligned}$$

$$\begin{aligned} &\leq \lambda f(Ax+b) + (1-\lambda) f(Ay+b), && \text{by convexity of } f \\ &= \lambda g(x) + (1-\lambda) g(y). \end{aligned}$$

That is,

$$\forall x, y \in \mathbb{E}, \forall \lambda \in (0,1), \quad g\big(\lambda x + (1-\lambda) y\big) \leq \lambda g(x) + (1-\lambda) g(y).$$

So g is convex. □

PROPOSITION 10.22 (Supremum)**.** The supremum of a collection of convex functions is again convex. i.e., Let $\{f_i\}_{i \in I}$ be a collection of convex functions where I is some index set. Then the function F given by $F := \sup\limits_{i \in I} f_i$ is convex.

Proof.

$$\begin{aligned} & (x, \alpha) \in \operatorname{epi}(F) \\ \iff & \sup_{i \in I} f_i(x) \leq \alpha \\ \iff & \forall i \in I, f_i(x) \leq \alpha \\ \iff & \forall i \in I, (x, \alpha) \in \operatorname{epi}(f_i \\ \iff & (x, \alpha) \in \bigcap_{i \in I} \operatorname{epi}(f_i). \end{aligned}$$

So $\operatorname{epi}(F) = \bigcap\limits_{i \in I} \operatorname{epi}(f_i)$. Since f_i are convex, $\operatorname{epi}(f_i)$ are convex. Since $\operatorname{epi}(f_i)$ are convex, $\bigcap\limits_{i \in I} \operatorname{epi}(f_i)$ is convex. That is, $\operatorname{epi}(F)$ is convex. Since $\operatorname{epi}(F)$ is convex, F is convex.

□

PROPOSITION 10.23 (Pointwise Supremum)**.** If $f(x, y)$ is convex in x for each y in some set $\mathcal{A}$, then the function g given by

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex.

10.8 Examples

EXAMPLE 10.24. Affine functions are convex.

EXAMPLE 10.25. Norms are convex.

Proof.

$$\begin{aligned}
&\|\alpha x + \beta y\| \\
\leq{}& \|\alpha x\| + \|\beta y\| \\
={}& |\alpha|\|x\| + |\beta|\|y\| \\
={}& \alpha\|x\| + \beta\|y\|.
\end{aligned}$$

□

EXAMPLE 10.26. Square norms are convex.

Proof Approach 1. Notice $\|\cdot\|^2$ is the direct sum of m squares and squares are convex. So by CO 463 Assignment 2 Problem 3, $\|\cdot\|^2$ is convex. □

Proof Approach 2. The domain is $\mathbb{E}$. Let x and y be two points in $\mathbb{E}$. Let λ be an arbitrary number in $(0,1)$. Define a point z as $z := \lambda x + (1-\lambda)y$.

$$\begin{aligned}
&\|\lambda x + (1-\lambda)y\|^2 \\
={}& \|\lambda x\|^2 + \|(1-\lambda)y\|^2 + 2\langle \lambda x, (1-\lambda)y\rangle \\
={}& \lambda^2\|x\|^2 + (1-\lambda)^2\|y\|^2 + 2\lambda(1-\lambda)\langle x, y\rangle \\
\leq{}& \lambda^2\|x\|^2 + (1-\lambda)^2\|y\|^2 + 2\lambda(1-\lambda)\|x\|\|y\| \\
\leq{}& \lambda(\lambda-1)\|x\|^2 + \lambda(\lambda-1)\|y\|^2 + 2\lambda(1-\lambda)\|x\|\|y\| \\
&\quad + \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
={}& \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
&\quad + \lambda(\lambda-1)\left[\|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\right] \\
\leq{}& \lambda\|x\|^2 + (1-\lambda)\|y\|^2
\end{aligned}$$

That is,

$$\forall x, y \in \mathbb{E}, \forall \lambda \in (0,1), \quad \|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2.$$

So by definition, $\|\cdot\|^2$ is convex. □

EXAMPLE 10.27. The distance function to a convex set is convex.

EXAMPLE 10.28. The perspective of a convex function is convex. i.e., if $f : \mathbb{E} \to \mathbb{R}$

Chapter 11

More Convex Functions

11.1 Strictly Convex

DEFINITION 11.1 (Strictly Convex)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **strictly convex** if $\forall x, y \in \mathrm{dom}(f)$, $\forall \lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$, except when $\lambda x + (1-\lambda)y = x$ or y.

PROPOSITION 11.2. Strictly convex functions are convex.

11.2 Strongly Convex

DEFINITION 11.3 (Strongly Convex)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Let β be a positive constant. We say that f is β **-strongly convex** if $\forall x, y \in \mathrm{dom}(f)$, $\forall \lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) - \frac{\beta}{2}\lambda(1-\lambda)\|x-y\|^2.$$

PROPOSITION 11.4. Strongly convex functions are strictly convex.

PROPOSITION 11.5. Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. Then f is β-strongly convex if and only if $f - \frac{\beta}{2}\|\cdot\|^2$ is convex.

Proof. Let β be a positive constant. Let g denote $f - \frac{\beta}{2}\|\cdot\|^2$. Let x and y be arbitrary elements of $\mathbb{E}$. Let $\lambda \in (0,1)$ be arbitrary.

$$
\begin{aligned}
& f \text{ is } \beta\text{-strongly convex} \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \\
& \qquad - \frac{\beta}{2}\lambda(1-\lambda)\|x-y\|^2 \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \\
& \qquad - \frac{\beta}{2}\lambda(1-\lambda)\big(\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle\big) \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \\
& \qquad - \lambda\frac{\beta}{2}\|x\|^2 + \frac{\beta}{2}\lambda^2\|x\|^2 \\
& \qquad - (1-\lambda)\frac{\beta}{2}\|y\|^2 + \frac{\beta}{2}(1-\lambda)^2\|y\|^2 \\
& \qquad + \beta\lambda(1-\lambda)\langle x, y\rangle \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \\
& \qquad - \lambda\frac{\beta}{2}\|x\|^2 - (1-\lambda)\frac{\beta}{2}\|y\|^2 \\
& \qquad + \frac{\beta}{2}\|\lambda x\|^2 + \frac{\beta}{2}\|(1-\lambda)y\|^2 + \beta\langle \lambda x, (1-\lambda)y\rangle \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \\
& \qquad - \lambda\frac{\beta}{2}\|x\|^2 - (1-\lambda)\frac{\beta}{2}\|y\|^2 \\
& \qquad + \frac{\beta}{2}\|\lambda x + (1-\lambda)y\|^2 \\
\iff\ & g\big(\lambda x + (1-\lambda)y\big) \le \lambda g(x) + (1-\lambda) g(y) \\
\iff\ & f - \frac{\beta}{2}\|\cdot\|^2 \text{ is } \beta \text{ convex.}
\end{aligned}
$$

□

Question: Can we allow f to take on $-\infty$? Do we need f to be proper?

PROPOSITION 11.6. Let f and g be functions from $\mathbb{E}$ to $\mathbb{R}^*$. Suppose f is β-

strongly convex for some positive constant β and g is convex. Then $f+g$ is also β-strongly convex.

Question: Can we allow f or g to take on $-\infty$? Do we need f and g to be proper?

Proof.

$$\begin{aligned} & f \text{ is } \beta\text{-strongly convex} \\ \implies & f - \frac{\beta}{2}\|\cdot\|^2 \text{ is convex} \\ \implies & f + g - \frac{\beta}{2}\|\cdot\|^2 \text{ is convex} \\ \implies & f + g \text{ is } \beta\text{-strongly convex.} \end{aligned}$$

□

11.3 Quasiconvex

DEFINITION 11.7 (Quasiconvex)**.** Let $f : \mathbb{E} \to \mathbb{R}$ be a function with convex domain. We say that f is **quasiconvex** if any level set of f is convex.

PROPOSITION 11.8 (Jensen's Inequality for Quasiconvex Functions)**.** Let f be a quasiconvex function. Then $\forall x, y \in \mathrm{dom}(f)$, $\forall \alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$,

$$f(\alpha x + \beta y) \le \max\{f(x), f(y)\}.$$

PROPOSITION 11.9. A differentiable real-valued function f with convex domain is convex if and only if $\forall x, y \in \mathrm{dom}(f)$,

$$f(y) \le f(x) \implies \nabla f(x) \cdot (y - x) \le 0. \quad ???$$

Not sure where did this come from but I don't think this is correct.

Chapter 12

Support

12.1 Definitions

DEFINITION 12.1 (Support Function)**.** Let S be a subset of $\mathbb{E}$. We define the **support function** of S, denoted by σ_S, to be a function from $\mathbb{E}$ to $\mathbb{R}^*$ given by

$$\sigma_S(x) = \sup_{s \in S} \langle x, s \rangle.$$

DEFINITION 12.2 (Supporting Hyperplane)**.** Let S be a set in $\mathbb{E}$ with nonempty boundary. Let x_0 be a point in the boundary of S. We define a **supporting hyperplane** H to set S at point x_0 to be a set of the form

$$H = \left\{ x \in \mathbb{E} : a^T x = a^T x_0 \right\},$$

such that $a \in \mathbb{E}$ and $a \neq \vec{0}$ and $\forall x \in S, a^T x \leq a^T x_0$.

12.2 Properties

PROPOSITION 12.3. The support function of a non-empty set S is proper, convex, and lower semi-continuous.

Proof.

Part 1. Proper.

Define f_s to be a function from $\mathbb{E}$ to $\mathbb{R}$ by $f_s(x) = \langle s, x\rangle$.

These functions are linear and hence proper, convex, and lower semi-continuous.

Notice $\sigma_S = \sup_{s\in S} f_s$.

So σ_S is convex and lower semi-continuous.

Since $\sigma_S(0) = \sup_{s\in S}\langle 0, s\rangle = 0$, $\exists x_0 \in \mathbb{E}, \sigma_S(x) \neq +\infty$.

Since $\sigma_S(x) = \sup_{s\in S}\langle x, s\rangle \geq \langle x, s\rangle \neq -\infty$, $\forall x \in \mathbb{E}, \sigma_S(x) \neq -\infty$.

Since $\exists x_0 \in \mathbb{E}, \sigma_S(x) \neq +\infty$ and $\forall x \in \mathbb{E}, \sigma_S(x) \neq -\infty$, by definition, σ_S is proper.

□

PROPOSITION 12.4. The support function of a non-empty and bounded set is continuous.

Proof.

Let x_0 be an arbitrary point in $\mathbb{E}$. Let ε be an arbitrary positive number. Define $M := \sup_{y\in C}\|y\| + 1$. Since C is bounded, M is finite. Define $\delta := \varepsilon/M$. Let x be an arbitrary point such that $\|x - x_0\| < \delta$. Let y be an arbitrary point in $\mathbb{E}$. Then by the Cauchy Schwarz inequality, we have

$$\langle x - x_0, y\rangle \leq \|x - x_0\|\|y\|.$$

That is,

$$\langle x, y\rangle \leq \|x - x_0\|\|y\| + \langle x_0, y\rangle.$$

It follows that

$$\begin{aligned}\sup_{y\in C}\langle x, y\rangle &\leq \sup_{y\in C}\big(\|x - x_0\|\|y\| + \langle x_0, y\rangle\big)\\ &\leq \|x - x_0\|\sup_{y\in C}\|y\| + \sup_{y\in C}\langle x_0, y\rangle.\end{aligned}$$

That is,

$$\sigma_C(x) \leq \sigma_C(x_0) + \|x - x_0\|\sup_{y\in C}\|y\|.$$

By definition of δ and M,

$$\sigma_C(x) < \sigma_C(x_0) + \varepsilon. \tag{1}$$

Similarly, reversing the role of x and x_0, we can prove that

$$\sigma_C(x_0) < \sigma_C(x) + \varepsilon. \tag{2}$$

From (1) and (2) we get

$$|\sigma_C(x) - \sigma_C(x_0)| < \varepsilon.$$

Since $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|\sigma_C(x) - \sigma_C(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \delta$, by definition, δ_C is continuous.

$\square$

PROPOSITION 12.5. Let S be a subset of $\mathbb{E}$. Then $\sigma_S = \sigma_{\mathrm{conv}(S)} = \sigma_{\overline{\mathrm{conv}}(S)}$.

Proof.

Let x be an arbitrary point in $\mathbb{E}$.

$$\begin{aligned}\sigma_S(x) &= \sup\left\{\langle x, s\rangle : s \in S\right\}\\ \sigma_{\mathrm{conv}(S)}(x) &= \sup\left\{\langle x, s\rangle : s \in \mathrm{conv}(S)\right\}\\ \sigma_{\overline{\mathrm{conv}}(S)}(x) &= \sup\left\{\langle x, s\rangle : s \in \overline{\mathrm{conv}}(S)\right\}.\end{aligned}$$

It is easy to see that by the linearity of inner products,

$$\mathrm{conv}\left\{\langle x, s\rangle : s \in S\right\} = \left\{\langle x, s\rangle : s \in \mathrm{conv}(S)\right\}.$$

It is easy to see that by the linearity and the continuity of inner products,

$$\overline{\mathrm{conv}}\left\{\langle x, s\rangle : s \in S\right\} = \left\{\langle x, s\rangle : s \in \overline{\mathrm{conv}}(S)\right\}.$$

It is also easy to see that for any subset A of the reals,

$$\sup(A) = \sup(\mathrm{conv}(A)),$$

and

$$\sup(A) = \sup(\mathrm{cl}(A)).$$

So

$$\begin{aligned}&\sigma_S(x)\\ =\ &\sup\left\{\langle x, s\rangle : s \in S\right\}\\ =\ &\sup\mathrm{conv}\left\{\langle x, s\rangle : s \in S\right\}\\ =\ &\sup\left\{\langle x, s\rangle : s \in \mathrm{conv}(S)\right\}\\ =\ &\sigma_{\mathrm{conv}(S)}(x).\end{aligned}$$

That is, $\sigma_S(x) = \sigma_{\mathrm{conv}(S)}(x)$.

$$\begin{aligned}&\sigma_S(x)\\ =\ &\sup\left\{\langle x, s\rangle : s \in S\right\}\end{aligned}$$

$$
\begin{aligned}
&= \sup \operatorname{conv} \big\{ \langle x, s \rangle : s \in S \big\} \\
&= \sup \big\{ \langle x, s \rangle : s \in \operatorname{conv}(S) \big\} \\
&= \sup \operatorname{cl} \big\{ \langle x, s \rangle : s \in \operatorname{conv}(S) \big\} \\
&= \sup \big\{ \langle x, s \rangle : s \in \operatorname{cl}(\operatorname{conv}(S)) \big\} \\
&= \sup \big\{ \langle x, s \rangle : s \in \overline{\operatorname{conv}}(S) \big\} \\
&= \sigma_{\overline{\operatorname{conv}}(S)}(x).
\end{aligned}
$$

That is, $\sigma_S(x) = \sigma_{\overline{\operatorname{conv}}(S)}(x)$.

□

12.3 Supporting Hyperplane

THEOREM 12.6 (Supporting Hyperplane Theorem)**.** For any boundary point x_0 of a convex set C, there exists a supporting hyperplane to C at x_0.

Chapter 13

Conjugacy

13.1 Definition and Examples

DEFINITION 13.1 (Convex Conjugate (Legendre–Fenchel Transformation))**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **convex conjugate** of f, denoted by f^*, to be a function also from $\mathbb{E}$ to $\mathbb{R}^*$ given by

$$f^*(x) := \sup_{y \in \mathbb{E}} \big\{ \langle y, x \rangle - f(y) \big\}.$$

EXAMPLE 13.2. Let S be a subset of $\mathbb{E}$. Then $\delta_S^* = \sigma_S$.

Proof. Recall that

$$\delta_S(x) = \begin{cases} 0, & x \in S, \\ +\infty, & x \notin S, \end{cases}$$
$$\sigma_S(x) = \sup_{y \in S} \langle x, y \rangle.$$

Now for any $x \in \mathbb{E}$,

$$\begin{aligned} & \delta_S^*(x) \\ = \ & \sup_{y \in S} \big(\langle x, y \rangle - \delta_S(y) \big) \\ = \ & \sup_{y \in S} \big(\langle x, y \rangle - 0 \big) \end{aligned}$$

$$
\begin{aligned}
&= \sup_{y \in S} \langle x, y \rangle \\
&= \sigma_S(x).
\end{aligned}
$$

So $\delta_S^* = \sigma_S$. □

13.2 Basic Properties

PROPOSITION 13.3. The convex conjugate function is convex.

Proof. If $\operatorname{dom}(f) = \emptyset$, then one can see that $f^* \equiv -\infty$. It is a pointwise supremum of affine functions. □

PROPOSITION 13.4. The convex conjugate function is lower semi-continuous.

13.3 Double Conjugate

PROPOSITION 13.5. Let f be any function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $f^{**} \leq f$.

Proof. Let x be an arbitrary point in $\mathbb{E}$.

$$
\begin{aligned}
& f^{**}(x) \\
= & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - f^*(y) \right\} \\
= & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \sup_{z \in \mathbb{E}} \left\{ \langle z, y \rangle - f(z) \right\} \right\} \\
\leq & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \left\{ \langle x, y \rangle - f(x) \right\} \right\} \\
= & \sup_{y \in \mathbb{E}} \left\{ f(x) \right\} \\
= & f(x).
\end{aligned}
$$

That is, $f^{**}(x) \leq f(x)$. Since $\forall x \in \mathbb{E}$, $f^{**}(x) \leq f(x)$, we get $f^{**} \leq f$. □

PROPOSITION 13.6. Let f be a proper function. Then f is convex and lower semi-continuous if and only if

$$f^{**} = f.$$

PROPOSITION 13.7. Let f and g be functions from $\mathbb{E}$ to $\mathbb{R}^*$. Then $f \leq g$ implies $f^* \geq g^*$ and $f^{**} \leq g^{**}$.

Proof. Let x be an arbitrary point in $\mathbb{E}$.

$$\begin{aligned}
& f^*(x) \\
= & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - f(y) \right\} \\
\geq & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - g(y) \right\} \\
= & \, g^*(x).
\end{aligned}$$

That is, $f^*(x) \geq g^*(x)$. Since $\forall x \in \mathbb{E}$, $f^*(x) \geq g^*(x)$, we get $f^* \geq g^*$.

Let x be an arbitrary point in $\mathbb{E}$.

$$\begin{aligned}
& f^{**}(x) \\
= & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - f^*(y) \right\} \\
= & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \sup_{z \in \mathbb{E}} \left\{ \langle z, y \rangle - f(z) \right\} \right\} \\
\leq & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \sup_{z \in \mathbb{E}} \left\{ \langle z, y \rangle - g(z) \right\} \right\} \\
= & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - g^*(y) \right\} \\
= & \, g^{**}(x).
\end{aligned}$$

That is, $f^{**}(x) \leq g^{**}(x)$. Since $\forall x \in \mathbb{E}$, $f^{**}(x) \leq g^{**}(x)$, we get $f^{**} \leq g^{**}$. □

PROPOSITION 13.8.

$$\mathrm{epi}(f^{**}) = \mathrm{conv}(\mathrm{epi}(f)).$$

13.4 Conjugates and Sub-Differentials

THEOREM 13.9 (Fenchel-Young)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x, y \in \mathbb{E}$, we have

$$f(x) + f^*(y) \geq \langle x, y \rangle.$$

PROPOSITION 13.10. Let f be a proper closed convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x, y \in \mathbb{E}$,

$$y \in \partial f(x) \iff x \in \partial f^*(y) \iff f(x) + f^*(y) = \langle x, y \rangle.$$

Proof of $y \in \partial f(x) \iff x \in \partial f^*(y)$. For one direction, assume that $y \in \partial f(x)$. We are to prove that $x \in \partial f^*(y)$. Consider an arbitrary point $z \in \mathbb{E}$. Since $y \in \partial f(x)$, we get

$$\forall u \in \mathbb{E}, \quad \langle y, u - x \rangle \leq f(u) - f(x).$$

Rearranging yields

$$\forall u \in \mathbb{E}, \quad \langle y, u \rangle - f(u) \leq \langle y, x \rangle - f(x).$$

It follows that

$$\sup_{u \in \mathbb{E}} \big(\langle y, u \rangle - f(u) \big) \leq \langle y, x \rangle - f(x). \tag{1}$$

By definition of supremum, we have

$$\sup_{u \in \mathbb{E}} \big(\langle y, u \rangle - f(u) \big) \geq \langle y, x \rangle - f(x). \tag{2}$$

From (1) and (2), we get

$$\sup_{u \in \mathbb{E}} \big(\langle y, u \rangle - f(u) \big) = \langle y, x \rangle - f(x).$$

That is,

$$f^*(y) = \langle y, x \rangle - f(x).$$

Then

$$\begin{aligned}
& f^*(z) - f^*(y) \\
= \;& \sup_{u \in \mathbb{E}} \big(\langle z, u \rangle - f(u) \big) - \sup_{u \in \mathbb{E}} \big(\langle y, u \rangle \rangle - f(u) \big) \\
= \;& \sup_{u \in \mathbb{E}} \big(\langle z, u \rangle - f(u) \big) - \langle y, x \rangle + f(x) \\
\geq \;& \langle z, x \rangle - f(x) - \langle y, x \rangle + f(x)
\end{aligned}$$

$$= \langle z - y, x \rangle.$$

That is,

$$\langle x, z - y \rangle \leq f^*(z) - f^*(y).$$

So $x \in \partial f^*(y)$. This proves

$$y \in \partial f(x) \implies x \in \partial f^*(y).$$

Since $f^{**} = f$, similarly, we can prove that

$$x \in \partial f^*(y) \implies y \in \partial f(x).$$

□

PROPOSITION 13.11. Let f be a proper convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x be a point in $\mathbb{E}$. Assume that $\partial f(x) \neq \emptyset$. Then $f^{**}(x) = f(x)$.

Chapter 14

The Proximal Operator

14.1 Definitions

DEFINITION 14.1 (Proximal Operator)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **proximal operator** of f, denoted by prox_f, to be a function from $\mathbb{E}$ to $\mathcal{P}(\mathbb{E})$ given by

$$\operatorname{prox}_f(x) := \operatorname*{argmin}_{y \in \mathbb{E}} \big\{ f(y) + \frac{1}{2}\|y - x\|^2 \big\}.$$

14.2 Examples

EXAMPLE 14.2 (Soft Threshold)**.** Let $\lambda \geq 0$. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) := \lambda|x|$. Then

$$\operatorname{prox}_f(x) = \begin{cases} x + \lambda, & \text{if } x < -\lambda \\ 0, & \text{if } -\lambda \leq x \leq \lambda \\ x - \lambda, & \text{if } x > \lambda. \end{cases}$$

14.3 Basic Properties

(bug)

PROPOSITION 14.3. Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x \in \mathbb{E}$, $\operatorname{prox}_f(x)$ is a singleton set.

Proof. Let x be an arbitrary element of $\mathbb{E}$. Define a function $h : \mathbb{E} \to \mathbb{R}^*$ by $h(y) := \frac{1}{2}\|y-x\|^2$. Define a function $g : \mathbb{E} \to \mathbb{R}^*$ by $g(y) := f(y) + h(y)$. Then $\operatorname{prox}_f(x) = \operatorname*{argmin}_{y\in\mathbb{E}} g(y)$. Note that h is proper, lower semi-continuous, and β-strongly convex for any $\beta \in (0,1)$. Since f and h are proper, g is proper (why?). Since f and h are lower semi-continuous, g is lower semi-continuous. Since f is convex and h is β-strongly convex, g is β-strongly convex. Since g is proper, lower semi-continuous, and strongly convex, g has a unique minimizer (why?). So $\operatorname{prox}_f(x)$ is a singleton set. $\square$

not fully understood

PROPOSITION 14.4. Let C be a nonempty closed convex subset of $\mathbb{E}$. Then $\operatorname{prox}_{\delta_C}$ and proj_C are both singleton and $\operatorname{prox}_{\delta_C} = \operatorname{proj}_C$.

Proof. Since C is nonempty, convex, and closed, δ_C is proper, convex, and lower semi-continuous and hence $\operatorname{prox}_{\delta_C}$ is singleton. Since C is nonempty, convex, and closed, proj_C is singleton. Let x and p be arbitrary elements of $\mathbb{E}$. Then

$$\begin{aligned}
&p \in \operatorname{prox}_{\delta_C}(x)\\
\iff& p \in \operatorname*{argmin}_{y\in\mathbb{E}}\{\delta_C(y) + \frac{1}{2}\|y-x\|^2\}\\
\iff& \forall y \in \mathbb{E}, \delta_C(y) + \frac{1}{2}\|y-x\|^2 \geq \delta_C(p) + \frac{1}{2}\|p-x\|^2\\
\iff& p \in C \text{ and } \forall y \in C, \frac{1}{2}\|y-x\|^2 \geq \frac{1}{2}\|p-x\|^2\\
\iff& p \in C \text{ and } \forall y \in C, \|y-x\| \geq \|p-x\|\\
\iff& p \in \operatorname*{argmin}_{y\in C}\|y-x\|\\
\iff& p \in \operatorname{proj}_C(x).
\end{aligned}$$

$\square$

PROPOSITION 14.5 (Firmly Non-Expansive)**.** Let f be a proper closed convex function. Then prox_f is firmly non-expansive.

14.4 Prox Calculus Rules

PROPOSITION 14.6 (Scaling and Translation)**.**

THEOREM 14.7 (Norm Composition)**.**

PROPOSITION 14.8. Let $f_1, \ldots, f_m$ be proper, convex, and lower semi-continuous functions from $\mathbb{R}$ to $\mathbb{R}^*$. Define a function $f : \mathbb{R}^m \to \mathbb{R}^*$ by $f((x_i)_{i=1}^m) := \sum_{i=1}^{m} f_i(x_i)$. Then

$$\operatorname{prox}_f((x_i)_{i=1}^m) = (\operatorname{prox}_{f_i}(x_i))_{i=1}^m.$$

Proof. Since each f_i is proper, convex, and lower semi-continuous, f is proper, convex, and lower semi-continuous. Let $(x_i)_{i=1}^m$ and $(p_i)_{i=1}^m$ be arbitrary elements of $\mathbb{R}^m$. Then

$$(p_i)_{i=1}^m = \operatorname{prox}_f((x_i)_{i=1}^m)$$
$$\iff$$

□

14.5 The Second Prox Theorem

PROPOSITION 14.9. Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x and p be points in $\mathbb{E}$. Then $p = \operatorname{prox}_f(x)$ if and only if

$$x - p \in \partial f(p).$$

PROPOSITION 14.10. Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x and p be elements of $\mathbb{E}$. Then $p = \operatorname{prox}_f(x)$ if and only if

$$\forall y \in \mathbb{E}, \quad \langle y - p, x - p \rangle \leq f(y) - f(p).$$

Proof. **Forward Direction**:

Assume that $p = \operatorname{prox}_f(x)$. I will show that $\forall y \in \mathbb{E}$, $\langle y - p, x - p\rangle \leq f(y) - f(p)$. Let y be an arbitrary element of $\mathbb{E}$. Define for each $\lambda \in (0,1)$ a point p_λ by $p_\lambda := \lambda y + (1-\lambda)p$. Then

$$\begin{aligned}
& p = \operatorname{prox}_f(x) \\
\implies & f(p) + \frac{1}{2}\|x-p\|^2 \leq f(p_\lambda) + \frac{1}{2}\|x-p_\lambda\|^2 \\
\iff & f(p) \leq f(p_\lambda) + \frac{1}{2}\|x-p_\lambda\|^2 - \frac{1}{2}\|x-p\|^2 \\
\iff & f(p) \leq f(p_\lambda) + \frac{1}{2}\left\langle \Big[(x-p_\lambda)+(x-p)\Big], \Big[(x-p_\lambda)-(x-p)\Big] \right\rangle \\
\iff & f(p) \leq f(p_\lambda) + \frac{1}{2}\left\langle \Big[2x-\lambda y-(1-\lambda)p-p\Big], \Big[p-\lambda y-(1-\lambda)p\Big] \right\rangle \\
\iff & f(p) \leq f(p_\lambda) + \frac{1}{2}\left\langle \Big[2(x-p)+\lambda(p-y))\Big], \Big[\lambda(p-y)\Big] \right\rangle \\
\iff & f(p) \leq f(p_\lambda) + \lambda\langle x-p, p-y\rangle + \frac{1}{2}\lambda^2\|p-y\|^2 \\
\iff & f(p) \leq f(\lambda y + (1-\lambda)p) + \lambda\langle x-p, p-y\rangle + \frac{1}{2}\lambda^2\|p-y\|^2 \\
\implies & f(p) \leq \lambda f(y) + (1-\lambda)f(p) + \lambda\langle x-p, p-y\rangle + \frac{1}{2}\lambda^2\|p-y\|^2 \\
\iff & \lambda\langle y-p, x-p\rangle \leq \lambda f(y) - \lambda f(p) + \frac{1}{2}\lambda^2\|p-y\|^2 \\
\iff & \langle y-p, x-p\rangle \leq f(y) - f(p) + \frac{1}{2}\lambda\|p-y\|^2 \\
\iff & \langle y-p, x-p\rangle \leq f(y) - f(p).
\end{aligned}$$

Backward Direction:

Assume that $\forall y \in \mathbb{E}$, $\langle y - p, x - p\rangle \leq f(y) - f(p)$. I will show that $p = \operatorname{prox}_f(x)$. Let y be an arbitrary element of $\mathbb{E}$. Then

$$\begin{aligned}
& \langle y-p, x-p\rangle \leq f(y) - f(p) \\
\iff & f(p) + \frac{1}{2}\|x-p\|^2 \leq f(y) + \langle x-p, p-y\rangle + \frac{1}{2}\|x-p\|^2 \\
\implies & f(p) + \frac{1}{2}\|x-p\|^2 \leq f(y) + \langle x-p, p-y\rangle + \frac{1}{2}\|x-p\|^2 + \frac{1}{2}\|p-y\|^2 \\
\iff & f(p) + \frac{1}{2}\|x-p\|^2 \leq f(y) + \frac{1}{2}\|(x-p)+(p-y)\|^2 \\
\iff & f(p) + \frac{1}{2}\|x-p\|^2 \leq f(y) + \frac{1}{2}\|x-y\|^2 \\
\implies & p = \operatorname{prox}_f(x).
\end{aligned}$$

This completes the proof.

$\square$

14.6 Moreau Decomposition

THEOREM 14.11 (Moreau Decomposition)**.** Let f be a proper closed convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Then

$$\operatorname{prox}_f + \operatorname{prox}_{f^*} = \operatorname{Id}.$$

Proof. Let x be an arbitrary point in $\mathbb{E}$. We are to prove that

$$\operatorname{prox}_f(x) + \operatorname{prox}_{f^*}(x) = x.$$

Let p denote $\operatorname{prox}_f(x)$. Since f is proper convex and lower semi-continuous and $p = \operatorname{prox}_f(x)$, we get

$$x - p \in \partial f(p).$$

Since $x - p \in \partial f(p)$, we get $p \in \partial f^*(x-p)$. It follows that $x - p = \operatorname{prox}_{f^*}(x)$. Substitute $p = \operatorname{prox}_f(x)$ and rearrange the equation, we get

$$\operatorname{prox}_f(x) + \operatorname{prox}_{f^*}(x) = x.$$

□

Chapter 15

Ellipsoids

DEFINITION 15.1 (Ellipsoid)**.** Let v be a point in some Euclidean space $\mathbb{E}$. We define an **ellipsoid**, centered at point v, to be a set of the form

$$\{x \in \mathbb{E} : (x - v)^T A(x - v) = 1\}$$

where A is some d by d positive definite matrix.

15.1 Properties

PROPOSITION 15.2. The eigenvectors of A define the principal axes of the ellipsoid.