Stochastic Process

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Chapter 1

Stochastic Process

1.1 Definitions

Definition (Stochastic Process)**.** *Let $\mathcal{T}$ be an index set. Let $X(t)$ be a random variable. We define a* ***stochastic process*** *to be the net $(X(t))_{t\in\mathcal{T}}$.*

Definition (Discrete-Time Stochastic Process)**.** *Let $(X(t))_{t\in\mathcal{T}}$ be a stochastic process. We say that it is a* ***discrete-time stochastic process*** *if the index set $\mathcal{T}$ is countable.*

Definition (Markov Property)**.** *Let $\mathcal{S}$ be a state space. Let $(X_n)_{n\in\mathbb{N}}$ be a discrete-time stochastic process. We say that is has the* ***Markov property*** *if*

$$\forall n \in \mathbb{N}, \forall x_0..x_{n+1} \in \mathcal{S}, \quad \Pr(X_{n+1} = x_{n+1} \mid (X_n)_{n=0}^{n} = (x_n)_{n=0}^{n}) = \Pr(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

This property states that the conditional distribution of any future state X_{n+1} given the past states $X_0, ..., X_{n-1}$ and the present state X_n is independent of the past states.

i.e., if we know the value taken by te process at a given time, we will not get any additional information about the future behavior of the process by gathering more knowledge about the past.

Definition (Markov Chain)**.** *We define a* ***Markov chain*** *to be a discrete-time stochastic process with the Markov property.*

Proposition 1.1.1.

$$\forall n \in \mathbb{N}, \forall j \in \{0..n{-}1\}, \forall x_0..x_{n+1} \in \mathcal{S}, \quad \Pr(X_{n+1} = x_{n+1} \mid X_n = x_n, (X_i)_{i=1}^{j-1} = (x_i)_{i=1}^{j-1}, (X_i)_{i=j+1}^{n-1} = (x_i)_{i=j+1}^{n-1}) = \Pr$$

Proof.

$$\Pr(X_{n+1} = x_{n+1} \mid X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j}) \tag{1.1}$$

$$
\begin{aligned}
&= \frac{\Pr(X_{n+1} = x_{n+1}, X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j}}{\Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j})} && (1.2)\\
&= \frac{\sum_{x_j=0}^{\infty} \Pr(X_{n+1} = x_{n+1}, X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j}, X_j = x_j)}{\Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j})} && (1.3)\\
&= \frac{\sum_{x_j=0}^{\infty} \Pr(X_{n+1} = x_{n+1} \mid X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j}, X_j = x_j) \Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j}, X_j = x_j)}{\Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j})} && (1.4)\\
&= \frac{\sum_{x_j=0}^{\infty} \Pr(X_{n+1} = x_{n+1} \mid X_n = x_n) \Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j}, X_j = x_j)}{\Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j})} && (1.5)\\
&= \Pr(X_{n+1} = x_{n+1} \mid X_n = x_n) \frac{\sum_{x_j=0}^{\infty} \Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j}, X_j = x_j)}{\Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j})} && (1.6)\\
&= \Pr(X_{n+1} = x_{n+1} \mid X_n = x_n) \frac{\Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j})}{\Pr(X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j})} && (1.7)\\
&= \Pr(X_{n+1} = x_{n+1} \mid X_n = x_n). && (1.8)
\end{aligned}
$$

That is,

$$\Pr(X_{n+1} = x_{n+1} \mid X_n = x_n, (X_i)_{i\neq j} = (x_i)_{i\neq j}) = \Pr(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

■

Definition (Transition Probability)**.** *Let i and j be a pair of states. Let n be some time step. We define the* ***transition probability*** *from state i at time n to state j at time $n+1$, denoted by $P_{n,i,j}$. to be the conditional probability given by*

$$P_{n,i,j} = \Pr(X_{n+1} = j \mid X_n = i).$$

Definition (Stationary / Homogeneous)**.** *We say that a discrete-time Markov chain is* ***stationary*** *or* ***homogeneous*** *if $\forall i, j \in \mathcal{S}$, $\forall n \in \mathbb{N}$, $P_{n,i,j} = P_{i,j}$ for some $P_{i,j}$.*

Theorem 1 (Chapman-Kolmogorov Equations)**.**

$$P^{(n)} = P^{(m)} P^{(n-m)}.$$

1.2 Accessibility and Communication

Definition (Accessible)**.** *Let i and j be two states. We say that state j is* ***accessible*** *from state i if $\exists n \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$.*

Definition (Communicate)**.** *Let i and j be two states. We say that state i and state j* ***communicate*** *if i and j are accessible from each other.*

Proposition 1.2.1. *The communication relation is an equivalence relation. i.e., it is reflexive, symmetric, and transitive.*

Proof. **Transitivity**:

Let i, j, k be states. Assume that $i \leftrightarrow j$ and $j \leftrightarrow k$. We are to prove that $i \leftrightarrow k$. Since $i \leftrightarrow j$, $\exists n \in \mathbb{N}$ such that $P_{i,j}^{(n)} > 0$. Since $j \leftrightarrow k$, $\exists m \in \mathbb{N}$ such that $P_{i,j}^{(m)} > 0$. By the Chapman-Kolmogorov equation, we get

$$P_{i,k}^{(n+m)} = \sum_{l=0}^{\infty} P_{i,l}^{(n)} P_{l,k}^{(m)} \geq P_{i,j}^{(n)} P_{j,k}^{(m)} > 0.$$

That is, $P_{i,k}^{(n+m)} > 0$. So $i \to k$. Similarly, we can show that $k \to i$. So $i \leftrightarrow k$. ■

Proposition 1.2.2. *Let i and j be two states. If state j is not accessible from state i, then*

$$\Pr(\textit{DTMC ever exists state } j \mid X_0 = i) = 0.$$

Proof. Since state j is not accessible from state i, we have $\forall n \in \mathbb{N}$, $P_{i,j}^{(n)} = 0$.

$$\begin{aligned} & \Pr(\text{DTMC ever exists state } j \mid X_0 = i) \\ = \ & \Pr(\bigcup_{n=0}^{\infty} \{X_n = j\} \mid X_0 = i) \leq \sum_{n=0}^{\infty} \Pr(X_n = j \mid X_0 = i) \\ = \ & \sum_{n=0}^{\infty} P_{i,j}^{(n)} = 0. \end{aligned}$$

That is,

$$\Pr(\text{DTMC ever exists state } j \mid X_0 = i) = 0.$$

■

Definition (Communication Class)**.** *We define a **communication class** to the set of states that communicate with each other.*

Definition (Irreducible, Reducible)**.** *We say that a discrete-time Markov chain is **irreducible** if it has only one communication class; and we say that it is **reducible** otherwise.*

1.3 Periodicity

Definition (Period)**.** *Let i be a state. We define the **period** of i, denoted by $d(i)$, to be the number given by*

$$d(i) := \gcd\{n \in \mathbb{Z}_+ : P_{i,i}^{(n)} > 0\}.$$

Definition (Aperiodic)**.** *We say that a state i is **aperiodic** if $d(i) = 1$. We say that a discrete-time Markov chain is **aperiodic** if $d(i) = 1$ for all state i.*

Proposition 1.3.1. *Let i and j be two states. If $i \leftrightarrow j$, then $d(i) = d(j)$.*

Proof. Since $i \leftrightarrow j$, $\exists n \in \mathbb{Z}_+$ such that $P_{i,j}^{(n)} > 0$; $\exists m \in \mathbb{Z}_+$ such that $P_{j,i}^{(m)} > 0$; and $\exists s \in \mathbb{Z}_+$ such that $P_{j,j}^{(s)} > 0$. Note that

$$P_{i,i}^{(n+m)} \geq P_{i,j}^{(n)} P_{j,i}^{(m)} > 0.$$

and

$$P_{i,i}^{(n+s+m)} \geq P_{i,j}^{(n)} P_{j,j}^{(s)} P_{j,i}^{(m)} > 0.$$

So $d(i) \mid (n+m)$ and $d(i) \mid (n+s+m)$. So $d(i) \mid ((n+s+m)-(n+m)) = s$. Since $\forall s \in \mathbb{Z}_+ : P_{j,j}^{(s)} > 0$, $d(i) \mid s$, we get $d(i) \mid d(j)$. Similarly, we have $d(j) \mid d(i)$. So $d(i) = d(j)$. ■

1.4 Transience and Recurrence

1.4.1 Preliminaries

Notation (First Visit Probability)**.** *Let i and j be two states. Let $n \in \mathbb{Z}_+$ be a time step. We define the **first visit probability** to state j, starting from state i, occurs at time step n, denoted by $f_{i,j}^{(n)}$, to be the probability given by*

$$f_{i,j}^{(n)} = \Pr(X_n = j, X_{n-1}..X_1 \neq j \mid X_0 = j).$$

Proposition 1.4.1.

$$P_{i,j}^{(n)} = \sum_{k=1}^{n} P_{j,j}^{(n-k)} f_{i,j}^{(k)}.$$

Proof.

$$\begin{aligned}
P_{i,j}^{(n)} &= \Pr(X_n = j \mid X_0 = i) \\
&= \sum_{k=1}^{n} \Pr(X_n = j, \text{ first visit occurs at time } k \mid X_0 = i) \\
&= \sum_{k=1}^{n} \Pr(X_n = j, X_k = j, X_{k-1}..X_1 \neq j \mid X_0 = i) \\
&= \sum_{k=1}^{n} \Pr(X_n = j \mid X_k = j, X_{k-1}..X_1 \neq j, X_0 = i) \Pr(X_k = j, X_{k-1}..X_1 \neq j \mid X_0 = i) \\
&= \sum_{k=1}^{n} \Pr(X_n = j \mid X_k = j) \Pr(X_k = j, X_{k-1}..X_1 \neq j \mid X_0 = i), \text{ by the Markov property} \\
&= \sum_{k=1}^{n} P_{j,j}^{(n-k)} f_{i,j}^{(k)}.
\end{aligned}$$

That is,

$$P_{i,j}^{(n)} = \sum_{k=1}^{n} P_{j,j}^{(n-k)} f_{i,j}^{(k)}.$$

■

Proposition 1.4.2.

$$f_{i,j}^{(n)} = P_{i,j}^{(n)} - \sum_{k=1}^{n-1} f_{i,j}^{(k)} P_{j,j}^{(n-k)}.$$

Proof.

$$\begin{aligned} f_{i,j}^{(n)} &= f_{i,j}^{(n)} \cdot 1 = f_{i,j}^{(n)} P_{j,j}^{(0)} = f_{i,j}^{(n)} P_{j,j}^{(n-n)} \\ &= \sum_{k=1}^{n} f_{i,j}^{(k)} P_{j,j}^{(n-k)} - \sum_{k=1}^{n-1} f_{i,j}^{(k)} P_{j,j}^{(n-k)} \\ &= P_{i,j}^{(n)} - \sum_{k=1}^{n-1} f_{i,j}^{(k)} P_{j,j}^{(n-k)}. \end{aligned}$$

That is,

$$f_{i,j}^{(n)} = P_{i,j}^{(n)} - \sum_{k=1}^{n-1} f_{i,j}^{(k)} P_{j,j}^{(n-k)}.$$

■

The above proposition gives a recursive means to compute $f_{i,j}^{(n)}$ for $n \geq 2$.

Notation $(f_{i,j})$**.** *Let i and j be two states.*

$$f_{i,j} := \Pr(\text{ DTMC ever makes a future visit to state } j \mid X_0 = i) = \sum_{k=1}^{\infty} f_{i,j}^{(k)}.$$

Note that $f_{i,j}$ is an infinite sum of probabilities and that $f_{i,j}$ itself is defined to be a probability. So the infinite sum of probabilities is ≤ 1.

1.4.2 Definitions

Definition (Transient and Recurrent - 1)**.** *Let i be a state. We say that state i is **transient** if $f_{i,i} < 1$; and we say that state i is **recurrent** if $f_{i,i} = 1$.*

Definition (Transient and Recurrent - 2)**.** *Let i be a state. We say that state i is **transient** if $\mathbb{E}[M_i \mid X_0 = i] < \infty$; and we say that state i is **transient** if $\mathbb{E}[M_i \mid X_0 = i] = \infty$.*

Definition (Transient and Recurrent - 3)**.** *Let i be a state. We say that state i is **transient** if $\sum_{n=1}^{\infty} P_{i,i}^{(n)} < \infty$; and we say that state i is **transient** if $\sum_{n=1}^{\infty} P_{i,i}^{(n)} = \infty$.*

Proposition 1.4.3. *The three definitions of transient and recurrent are equivalent.*

Proposition 1.4.4. *Let i be a state. Let M_i be a random variable that denotes the number of future visits to state i. Then $\Pr(M_i = k \mid X_0 = i) = f_{i,i}^{k}(1 - f_{i,i})$.*

Proposition 1.4.5. *$M_i \sim GEO_f(1 - f_{i,i})$ and hence $\mathbb{E}[M_i \mid X_0 = i] = \frac{f_{i,i}}{1-f_{i,i}}$.*

Proposition 1.4.6.

$$\mathbb{E}[M_i \mid X_0 = i] = \sum_{n=1}^{\infty} P_{i,i}^{(n)}.$$

Proof. Define a random variable A_n as

$$A_n := \begin{cases} 0, & \text{if } X_n \neq i \\ 1, & \text{if } X_n = i. \end{cases}$$

$$\begin{aligned} \mathbb{E}[M_i \mid X_0 = i] = \mathbb{E}[\sum_{n=1}^{\infty} A_n \mid X_0 = i] &= \sum_{n=1}^{\infty} \mathbb{E}[A_n \mid X_0 = i] \\ &= \sum_{n=1}^{\infty} \Big[0 \cdot \Pr(A_n = 0 \mid X_0 = i) + 1 \cdot \Pr(A_n = 1 \mid X_0 = i) \Big] \\ &= \sum_{n=1}^{\infty} \Pr(A_n = 1 \mid X_0 = i) = \sum_{n=1}^{\infty} \Pr(X_n = i \mid X_0 = i) \\ &= \sum_{n=1}^{\infty} P_{i,i}^{(n)}. \end{aligned}$$

That is,

$$\mathbb{E}[M_i \mid X_0 = i] = \sum_{n=1}^{\infty} P_{i,i}^{(n)}.$$

■

1.4.3 Properties

Proposition 1.4.7. *Let i and j be two states. If $i \leftrightarrow j$, i is recurrent (transient) if and only if j is recurrent (transient).*

Proof. It suffices to show that i is recurrent implies j is recurrent. Since $i \leftrightarrow j$, $\exists m \in \mathbb{Z}_+$ such that $P_{i,j}^{(m)} > 0$, and $\exists n \in \mathbb{Z}_+$ such that $P_{j,i}^{(n)} > 0$. Since i is recurrent, we have $\sum_{l=1}^{\infty} P_{i,i}^{(l)} = +\infty$. Let $s \in \mathbb{Z}_+$. Then $P_{j,j}^{(n+s+m)} \geq P_{j,i}^{(n)} P_{i,i}^{(s)} P_{i,j}^{(m)}$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} P_{j,j}^{(k)} &\geq \sum_{k=n+s+m}^{\infty} P_{j,j}^{(k)} \\ &= \sum_{s=1}^{\infty} P_{j,j}^{(n+s+m)} \\ &\geq \sum_{s=1}^{\infty} P_{j,i}^{(n)} P_{i,i}^{(s)} P_{i,j}^{(m)} \end{aligned}$$

$$= \underbrace{P^{(n)}_{j,i}}_{>0} \underbrace{P^{(m)}_{i,j}}_{>0} \underbrace{\sum_{s=1}^{\infty} P^{(s)}_{i,i}}_{=+\infty} = +\infty.$$

That is, $\sum_{k=1}^{\infty} P^{(k)}_{j,j} = +\infty$. So j is recurrent. ■

Proposition 1.4.8. *If $i \leftrightarrow j$ and i is recurrent, then $f_{i,j} = 1$.*

Proof. Since $i \leftrightarrow j$ and i is recurrent, we know that j is also recurrent. Since j is recurrent, we have $f_{j,j} = 1$. Assume for the sake of contradiction that $f_{i,j} < 1$. Since $i \leftrightarrow j$, $\exists n \in \mathbb{Z}_+$ such that $P^{(n)}_{j,i} > 0$. Let n_i be the smallest of such. So if the DTMC reaches state i from state j in n_i time steps, there is no state j in between. Then

$$\begin{aligned} 1 - f_{j,j} &= \Pr(\text{ DTMC never makes a future visit to state } j \mid X_0 = j) \\ &\geq \underbrace{P^{(n_i)}_{j,i}}_{>0} \underbrace{(1 - f_{i,j})}_{>0} > 0. \end{aligned}$$

That is, $f_{j,j} < 1$. This contradicts to the fact that $f_{j,j} = 1$. So $f_{i,j} = 1$. ■

Based on the above result, we know that starting from any state of a recurrent class, a DTMC will visit each state of the class infinitely many times.

Theorem 2. *A finite-state discrete-time Markov chain has at least one recurrent state.*

Proof. Let $\mathcal{S} = \{1..N\}$ be a state space where $N \in \mathbb{N}$. Assume for the sake of contradiction that all states are transient. Then after some finite amount of time T, all states will never be visited again. However, this is not possible. So there must be some state that is recurrent. ■

Corollary. *An irreducible, finite-state discrete-time Markov chain must be recurrent.*

Proposition 1.4.9. *If i is recurrent and $i \nleftrightarrow j$, then $\forall k \in \mathbb{Z}_+$, $P^{(k)}_{i,j} = 0$.*

Proof. Assume for the sake of contradiction that $\exists k \in \mathbb{Z}_+$ such that $P^{(k)}_{i,j} > 0$. i.e., $i \to j$. Let k_i be the smallest of such. Since $i \nleftrightarrow j$ and $i \to j$, $\forall n \in \mathbb{Z}_+$, $P^{(n)}_{j,i} = 0$. So there is a probability of at least $P^{(k_i)}_{i,j}$ that the DTMC starts from state i and never returns to state i. This contradicts to the assumption that i is recurrent. So $\forall k \in \mathbb{Z}_+$, we have $P^{(k)}_{i,j} = 0$. ■

Once a process enters a recurrent class of states, it can never leave that class. For this reason, a recurrent class if often referred to as a **closed class**.

Corollary. *If $P^{(k)}_{i,j} > 0$ for some k and $i \nleftrightarrow j$, then i is transient.*

Proof. Notice this statement is equivalent to the last via some simple logical operations. ■

1.5 Random Walk

Definition (Random Walk)**.** *Let $\mathcal{S} = \mathbb{Z}$ be a state space. Let $(X_n)_{n\in\mathbb{N}}$ be a discrete-time Markov chain with state space $\mathcal{S}$. We say that $(X_n)_{n\in\mathbb{N}}$ is a* ***random walk*** *if it satisfies the property that*

$$\forall i \in \mathbb{Z}, \quad P_{i,i+1} = p \text{ and } P_{i,i-1} = 1-p, \text{ for some } p \in (0,1).$$

Proposition 1.5.1. *Random walks are irreducible.*

Proposition 1.5.2. *The period of all states of a random walk is 2. i.e., $\forall i \in \mathbb{Z}$, $d(i) = 2$.*

Proof. Note that $P_{i,i}^{(2n-1)} = 0$ for all $n \in \mathbb{Z}_+$ and $P_{i,i}^{(2n)} > 0$ for all $n \in \mathbb{Z}_+$. ■

Proposition 1.5.3. *If $p = 0.5$, then the DTMC is recurrent. If $p \neq 0.5$, then the DTMC is transient.*

Proof.

$$\begin{aligned}
f_{0,0} &= \Pr(\text{ DTMC ever makes a future visit to state } 0 \mid X_0 = 0) \\
&= \Pr(X_1 = 1 \mid X_0 = 0)\Pr(\text{ DTMC ever makes a future visit to state } 0 \mid X_1 = 1, X_0 = 0) \\
&\quad + \Pr(X_1 = -1 \mid X_0 = 0)\Pr(\text{ DTMC ever makes a future visit to state } 0 \mid X_1 = -1, X_0 = 0) \\
&= \Pr(X_1 = 1 \mid X_0 = 0)\Pr(\text{ DTMC ever makes a future visit to state } 0 \mid X_1 = 1) \\
&\quad + \Pr(X_1 = -1 \mid X_0 = 0)\Pr(\text{ DTMC ever makes a future visit to state } 0 \mid X_1 = -1), \text{ by the Markov property} \\
&= pf_{1,0} + (1-p)f_{-1,0}.
\end{aligned}$$

That is,

$$f_{0,0} = pf_{1,0} + (1-p)f_{-1,0}. \tag{*}$$

Let $\mathfrak{F}_0$ denote the event that the DTMC ever makes a future visit to state 0. Then $\mathfrak{F}_0 = \bigcup_{i=1}^{\infty}\{X_i = 0\}$. So

$$\begin{aligned}
f_{1,0} &= \Pr(\mathfrak{F}_0 \mid X_0 = 1) \\
&= \Pr(\mathfrak{F}_0 \cap \{X_1 = 0\} \mid X_0 = 1) + \Pr(\mathfrak{F}_0 \cap \{X_1 = 2\} \mid X_0 = 1) \\
&= \Pr(\mathfrak{F}_0 \mid X_1 = 0, X_0 = 1)\Pr(X_1 = 0 \mid X_0 = 1) + \Pr(\mathfrak{F}_0 \mid X_1 = 2, X_0 = 1)\Pr(X_1 = 2 \mid X_0 = 1) \\
&= 1 \cdot \Pr(X_1 = 0 \mid X_0 = 1) + \Pr(\mathfrak{F}_0 \mid X_1 = 2, X_0 = 1)\Pr(X_1 = 2 \mid X_0 = 1) \\
&= \Pr(X_1 = 0 \mid X_0 = 1) + \Pr(\mathfrak{F}_0 \mid X_1 = 2, X_0 = 1)\Pr(X_1 = 2 \mid X_0 = 1) \\
&= (1-p) + \Pr(\mathfrak{F}_0 \mid X_1 = 2, X_0 = 1)\Pr(X_1 = 2 \mid X_0 = 1) \\
&= (1-p) + \Pr(\mathfrak{F}_0 \mid X_1 = 2, X_0 = 1) \cdot p \\
&= (1-p) + p\Pr(\mathfrak{F}_0 \mid X_1 = 2, X_0 = 1) \\
&= (1-p) + p\Pr(\mathfrak{F}_0 \mid X_1 = 2), \text{ by the Markov property}
\end{aligned}$$

$$
\begin{aligned}
&= (1-p) + p\,\Pr\Big(\bigcup_{i=2}^{\infty}\{X_i = 0\} \cup \{X_1 = 0\} \mid X_1 = 2\Big) \\
&= (1-p) + p\,\Pr\Big(\bigcup_{i=2}^{\infty}\{X_i = 0\} \mid X_1 = 2\Big) \\
&= (1-p) + p\,\Pr\Big(\bigcup_{i=1}^{\infty}\{X_i = 0\} \mid X_0 = 2\Big), \text{ by the stationary assumption} \\
&= (1-p) + p\,\Pr(\mathfrak{F}_0 \mid X_0 = 2) \\
&= (1-p) + p f_{2,0} \\
&= (1-p) + p \cdot f_{2,1} \cdot f_{1,0} \\
&= (1-p) + p \cdot f_{1,0} \cdot f_{1,0} \\
&= (1-p) + p f_{1,0}^2.
\end{aligned}
$$

That is, $f_{1,0} = (1-p) + p f_{1,0}^2$. Solving for $f_{1,0}$ gives $f_{1,0} = \frac{1 \pm |p-q|}{2p}$ where $q := 1-p$. Let $r_1 = \frac{1+|p-q|}{2p}$ and $r_2 = \frac{1-|p-q|}{2p}$.

Case 1: $p = 0.5$.

Then we have $r_1 = r_2 = 1$. So $f_{1,0} = 1$. Similarly, we have $f_{-1,0} = 1$. So $f_{0,0} = p f_{1,0} + (1-p) f_{-1,0} = p + (1-p) = 1$. So the state 0 is recurrent and hence the entire DTMC is recurrent.

Case 2: $p \neq 0.5$.

We can show that if $p < 0.5$, $f_{1,0} = 1$ and $f_{-1,0} = \frac{p}{1-p} < 1$. If $p > 0.5$, $f_{-1,0} = 1$ and $f_{1,0} = \frac{1-p}{p} < 1$. So

$$
f_{0,0} = p f_{1,0} + (1-p) f_{-1,0} = \begin{cases} 2p, & \text{if } p < 0.5 \\ 2(1-p), & \text{if } p > 0.5. \end{cases}
$$

So $f_{0,0} < 1$ always.

Summary:

So

$$
f_{0,0} = 2\min\{p, 1-p\} \begin{cases} = 1, & \text{if } p = 0.5 \\ < 1, & \text{if } p \neq 0.5. \end{cases}
$$

■

1.6 Limiting Behaviors

Proposition 1.6.1. *Let $\mathcal{S}$ be a state space. Let $(X_n)_{n\in\mathbb{N}}$ be a discrete-time Markov chain with state space $\mathcal{S}$. Let i and j be two states in $\mathcal{S}$. Suppose that state j is transient. Then* $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$.

Proof.

$$
\begin{aligned}
\sum_{n=1}^{\infty} P_{i,j}^{(n)} &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{i,j}^{(k)} P_{j,j}^{(n-k)} = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} f_{i,j}^{(k)} P_{j,j}^{(n-k)} \\
&= \sum_{k=1}^{\infty} f_{i,j}^{(k)} \sum_{n=k}^{\infty} P_{j,j}^{(n-k)} = \sum_{k=1}^{\infty} f_{i,j}^{(k)} \sum_{l=0}^{\infty} P_{j,j}^{(l)} \\
&= f_{i,j} \sum_{l=0}^{\infty} P_{j,j}^{(l)} = f_{i,j}\left(1+\sum_{l=1}^{\infty} P_{j,j}^{(l)}\right) \\
&\leq 1\cdot\left(1+\sum_{l=1}^{\infty} P_{j,j}^{(l)}\right) = 1+\sum_{l=1}^{\infty} P_{j,j}^{(l)} \\
&< 1+\infty, \text{ since state } j \text{ is transient.} \\
&= +\infty.
\end{aligned}
$$

That is, $\sum_{n=1}^{\infty} P_{i,j}^{(n)} < +\infty$. So $\lim_{n\to\infty} P_{i,j}^{(n)} = 0$. ■

1.6.1 Mean Recurrent Time

Notation.

$$N_i := \min\{n \in \mathbb{Z}_+ : X_n = i\}.$$

Definition (Mean Recurrent Time)**.** *Let i be a recurrent state of a discrete-time Markov chain. We define the **mean recurrent time** of state i, denoted by m_i, to be the condition mean given by*

$$m_i := \mathbb{E}[N_i \mid X_0 = i].$$

In words, m_i represents the average time it takes the discrete-time Markov chain to make successive visits to state i.

Definition (Positive Recurrent and Null Recurrent)**.** *Let i be a recurrent state. We say that i is **positive recurrent** if $m_i < +\infty$; and we say that i is **null recurrent** if $m_i = +\infty$.*

Proposition 1.6.2. *Let i and j be two states. Suppose $i \leftrightarrow j$ and i. Then i is positive recurrent (null recurrent) if and only if j is positive recurrent (null recurrent).*

Proposition 1.6.3. *A finite-state discrete-time Markov chain has no null recurrent states.*

Definition (Ergodic)**.** *We say a state is **ergodic** if it is positive recurrent and aperiodic.*

1.6.2 Stationary Distribution

Definition (Stationary Distribution)**.** *Let $\{p_i\}_{i=0}^{\infty}$ be a probability distribution over $\mathbb{Z}_+$. We say that it is **stationary** if $\sum_{i=1}^{\infty} p_i = 1$ and $p = pP$ where P is the transition probability matrix.*

If a discrete-time Markov chain started according to a stationary distribution, then the probability of being in a given state remains unchanged over time.

Proposition 1.6.4. *If all the states of a discrete-time Markov chain are either null recurrent or transient, then there is no stationary distribution.*

Proposition 1.6.5. *An irreducible discrete-time Markov chain is positive recurrent if and only if a stationary distribution exists.*

1.6.3 Doubly Stochastic Transition Probability Matrix

Definition (Doubly Stochastic)**.** *Let P be a transition probability matrix. We say that P is **doubly stochastic** if the column sums are all equal to 1.*

1.7 Limiting Behaviors

Theorem 3 (Basic Limit Theorem)**.** *Let $\mathcal{S} = \mathbb{N}$ be a state space. Consider a irreducible, recurrent, and aperiodic discrete-time Markov chain with state space $\mathcal{S}$.*

$$\forall i, j \in \mathcal{S}, \quad \lim_{n \to \infty} P_{i,j}^{(n)} = \frac{1}{m_j},$$

where m_j is the mean recurrent time of state j. Moreover, if the discrete-time Markov chain is positive recurrent, then $\{\frac{1}{m_j}\}_{j \in \mathcal{S}}$ is the unique stationary distribution over $\mathcal{S}$.

Notation.

$$\pi_j := \lim_{n \to \infty} P_{i,j}^{(n)}.$$

Theorem 4. *Consider a finite-state, irreducible, and aperiodic discrete-time Markov chain. Suppose that it is doubly stochastic. Then the limiting probabilities are $\pi_i = \frac{1}{|\mathcal{S}|}$ for each $i \in \mathcal{S}$, where $\mathcal{S}$ is the state space.*

Proof. Since the DTMC has finite state, it must have a positive recurrent state. Since the DTMC is irreducible and has a positive recurrent state, the entire DTMC is positive recurrent. Since the DTMC is irreducible, positive recurrent, and aperiodic, by the Basic Limit Theorem, we know that the limiting probabilities $\{\pi_j\}_{j \in \mathcal{S}}$ are $\frac{1}{m_j}$ and that $\{\frac{1}{m_j}\}_{j \in \mathcal{S}}$ is the unique stationary distribution over $\mathcal{S}$. So it remains to show that the probability distribution given by $\{\frac{1}{|\mathcal{S}|}\}_{j \in \mathcal{S}}$ is stationary. Note that

$$\begin{aligned} \sum_{k \in \mathcal{S}} \frac{1}{|\mathcal{S}|} P_{k,j} &= \frac{1}{|\mathcal{S}|} \sum_{k \in \mathcal{S}} P_{k,j} \\ &= \frac{1}{|\mathcal{S}|} \cdot 1, \text{ since } P \text{ is doubly stochastic} \end{aligned}$$

$$= \frac{1}{|\mathcal{S}|}.$$

So $\{\frac{1}{|\mathcal{S}|}\}_{j\in\mathcal{S}}$ is stationary. This completes the proof. ■

Proposition 1.7.1. *Let j be a state. Define a random variable A_k as $A_k := \mathbb{I}[X_k = j]$. Then*

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} A_k \,\middle|\, X_0 = i\right] = \pi_j.$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} A_k \,\middle|\, X_0 = i\right] &= \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[A_k \mid X_0 = i] \\
&= \frac{1}{n}\sum_{k=1}^{n} \Big[0 \cdot \Pr(A_k = 0 \mid X_0 = i) + 1 \cdot \Pr(A_k = 1 \mid X_0 = i)\Big] \\
&= \frac{1}{n}\sum_{k=1}^{n} \Pr(A_k = 1 \mid X_0 = i) = \frac{1}{n}\sum_{k=1}^{n} \Pr(X_k = j \mid X_0 = i) \\
&= \frac{1}{n}\sum_{k=1}^{n} P_{i,j}^{(k)}.
\end{aligned}$$

That is,

$$\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} A_k \,\middle|\, X_0 = i\right] = \frac{1}{n}\sum_{k=1}^{n} P_{i,j}^{(k)}.$$

So

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} A_k \,\middle|\, X_0 = i\right] = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{i,j}^{(k)} = \lim_{n\to\infty} P_{i,j}^{(n)} = \pi_j.$$

That is,

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} A_k \,\middle|\, X_0 = i\right] = \pi_j,$$

as desired. ■

The above proposition is essentially saying that $\pi_j := \lim_{n\to\infty} P_{i,j}^{(n)}$ is the long-run mean fraction of time that the process spends in state j.

Chapter 2

Convergence of Random Variables

2.1 Definitions

Definition (Convergence in Distribution)**.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let F_n be the cumulative distribution function of X_n. Let X be a random variable. Let F_X be the cumulative distribution function of X. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$* ***converges in distribution*** *to X, denoted by $X_n \xrightarrow{d} X$, if $\forall x$ at which F is continuous,*

$$\lim_{n\to\infty} F_n(x) = F_X(x).$$

In this case, we say F_X is the asymptotic distribution of $\{X_n\}_{n\in\mathbb{N}}$.

Definition (Convergence in Probability)**.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let X be a random variable. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$* ***converges in probability*** *to X, denoted by $X_n \xrightarrow{p}$, if*

$$\forall\varepsilon > 0, \quad \lim_{n\to\infty} P\big(|X_n - X| \geq \varepsilon\big) = 0.$$

Or equivalently,

$$\forall\varepsilon > 0, \quad \lim_{n\to\infty} P\big(|X_n - X| < \varepsilon\big) = 1.$$

Definition (Almost Sure Convergence)**.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let X be a random variable. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$* ***converges almost surely*** *to X if*

$$P(\lim_{n\to\infty} X_n = X) = 1.$$

Definition (Sure Convergence)**.** *Let Ω be a sample space of the underlying probability space. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let X be a random variable. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$ **converges surely** to X if*

$$\forall \omega \in \Omega, \quad \lim_{n\to\infty} X_n(\omega) = X(\omega).$$

Definition (Convergence in Mean)**.** *Let $r \geq 1$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let X be a random variable. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$ **converges in the** r^{th} **mean** to X, denoted by $X_n \xrightarrow{L^r} X$, if the r^{th} absolute moments $\mathbb{E}[|X_n^r|]$ and $\mathbb{E}[|X|^r]$ of X_n and X exists and*

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^r] = 0.$$

2.2 Markov's Inequality

Theorem 5 (Markov's Inequality)**.** *Let X be a random variable. Let k and c be arbitrary positive numbers. Then*

$$P\big(|X| \geq c\big) \leq \frac{\mathbb{E}\big[|X|^k\big]}{c^k}.$$

Corollary.

$$P\Big(|X - \mathbb{E}[X]| > k\sqrt{\mathrm{var}[X]}\Big) \leq \frac{1}{k^2}.$$

2.3 Properties

Proposition 2.3.1. *Convergence in probability implies convergence in distribution.*

Proposition 2.3.2. *Almost sure convergence implies convergence in probability.*

Proposition 2.3.3. *Convergence in the r^{th} mean for $r \geq 1$ implies convergence in probability.*

Proposition 2.3.4. *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of random variables. Let c be a constant. Then $\{X_i\}_{i\in\mathbb{N}}$ converges to c in distribution if and only if $\{X_i\}_{i\in\mathbb{N}}$ converges to c in probability.*

Sketch Proof.

$$\begin{aligned} P\big(|X_i - c| \geq \varepsilon\big) &= P\big(X_i \geq c + \varepsilon\big) + P\big(X_i \leq c - \varepsilon\big) \\ &= 1 - P\big(X_i < c + \varepsilon\big) + F_i(c - \varepsilon) \\ &\leq 1 - P\big(X_i \leq c + \varepsilon/2\big) + F_i(c - \varepsilon) \\ &= 1 - F_i(c + \varepsilon/2) + F_i(c - \varepsilon) \end{aligned}$$

$$
\begin{aligned}
&\lim_{i\to\infty}\left[1 - F_i(c+\varepsilon/2) + F_i(c-\varepsilon)\right] \\
&= 1 - F(c+\varepsilon/2) + F(c-\varepsilon) \\
&= 1 - 1 + 0 \\
&= 0.
\end{aligned}
$$

■

Proposition 2.3.5 (Continuous Map). *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of random variables. Let g be a continuous function on the X_i's. Then*

(1) *if $X_i \xrightarrow{d} X$, we have $g(X_i) \xrightarrow{d} g(X)$.*

(2) *if $X_i \xrightarrow{p} c$, we have $g(X_i) \xrightarrow{p} g(c)$.*

Proposition 2.3.6 (Slutsky's Theorem). *Let $\{X_i\}_{i\in\mathbb{N}}$ and $\{Y_i\}_{i\in\mathbb{N}}$ be sequences of random variables. Suppose $X_i \xrightarrow{d} X$ for some random variable X and $Y_i \xrightarrow{p} c$ for some constant c. Then*

(1) $X_i + Y_i \xrightarrow{d} X + c$.

(2) $X_i Y_i \xrightarrow{d} cX$.

(3) $X_i / Y_i \xrightarrow{d} X/c$.

2.4 Law of Large Numbers

Theorem 6 (Strong Law of Large Nubmers). *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables. Suppose that $\mathbb{E}[X_i] = \mu$ for some $\mu \in \mathbb{R}$ for all $i \in \mathbb{N}$. Then their cumulative average $\bar{X}_n$ converges almost surely to μ. That is,*

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \overset{almost surely}{\longrightarrow} \mu.$$

Chapter 3

Markov Decision Process

Chapter 4

Poisson Process

4.1 Homogeneous Poisson Process

4.1.1 Definitions

Definition (Homogeneous Poisson Process)**.** *We say a counting process is a* ***homogeneous Poisson counting process*** *with rate* $\lambda > 0$ *if it has the following three properties:*

- $N(0) = 0$*;*
- *it has independent increments; and*
- *the number of events in any interval of length* t *is a Poisson random variable with parameter* λt*.*

Definition (Homogeneous Poisson Process)**.** *We say a point process is a* ***homogeneous Poisson point process*** *with rate* $\lambda > 0$ *if the following two conditions hold:*

- *The probability* $\mathbb{P}\{N(a,b] = n\}$ *of the number* $N(a,b]$ *of points of the process in the interval* $(a,b]$ *being equal to some counting number* n *is given by*

$$\mathbb{P}\{N(a,b] = n\} = \frac{[\lambda(b-a)]^n}{n!} e^{-\lambda(b-a)}.$$

  i.e. the number of arrivals in each finite interval has a Poisson distribution.

- *For any positive integer* k *and non-overlapping intervals* $(a_1, b_1], ..., (a_k, b_k]$*,*

$$\mathbb{P}\left\{\bigwedge_{i=1}^{k} N(a_i, b_i] = n_i\right\} = \prod_{i=1}^{k} \mathbb{P}\{N(a_i, b_i] = n_i\}.$$

  i.e. the number of arrivals in disjoint intervals are independent random variables.