

# General Topology

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# 1

## Topological Spaces

### 1.1 Topology

**Definition** (Topology). *Let  $X$  be a set. We define a **topology** on  $X$ , denoted by  $\tau_X$ , to be a collection of subsets of  $X$  that satisfies all of the following conditions.*

- (1)  $\emptyset, X \in \tau_X$ .
- (2)  $\tau_X$  is closed under union.
- (3)  $\tau_X$  is closed under finite intersection.

**Definition** (Finer, Coarser). *Let  $\tau_1$  and  $\tau_2$  be two topologies on  $X$ .*

- We say that  $\tau_1$  is **finer** than  $\tau_2$  if  $\tau_1 \supseteq \tau_2$ .
- We say that  $\tau_1$  is **coarser** than  $\tau_2$  if  $\tau_1 \subseteq \tau_2$ .

**Proposition 1.1.1.** *Let  $X$  be a set. Let  $\{\tau_\lambda\}_{\lambda \in \Lambda}$  be a collection of topologies on  $X$ . Then their intersection  $\bigcap_{\lambda \in \Lambda} \tau_\lambda$  is also a topology on  $X$ .*

**Example 1.1.1** (Trivial Topology). *Let  $X$  be a set. We define the **trivial topology** on  $X$  to be  $\tau := \{\emptyset, X\}$ .*

**Example 1.1.2** (Discrete Topology). *Let  $X$  be a set. We define the **discrete topology** on  $X$  to be  $\tau := \mathcal{P}(X)$  where  $\mathcal{P}(X)$  denotes the power set of  $X$ .*

**Example 1.1.3** (Metric Topology). *Let  $(X, d)$  be a metric space. We define the **metric topology** on  $X$ , induced by the metric  $d$ , to be*

$$\tau := \{G \subseteq X : \forall x \in G, \exists r > 0, \text{ball}(x, r) \subseteq G\}.$$

**Example 1.1.4** (Cofinite Topology). *Let  $X$  be a set. We define the **cofinite topology** on  $X$  to be*

$$\tau := \{\emptyset\} \cup \{Y \subseteq X : X \setminus Y \text{ is finite}\}.$$

## 1.2 Open Sets and Closed Sets

**Definition** (Open Sets). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say that  $S$  is **open** if  $S \in \tau$ .

**Definition** (Closed Sets). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say that  $S$  is **closed** if  $\bar{S} \in \tau$ .

## 1.3 Neighborhood

**Definition** (Neighborhood of a Point). Let  $(X, \tau)$  be a topological space. Let  $x$  be a point in  $X$ . Let  $\mathcal{N}$  be a subset of  $X$ . We say that  $\mathcal{N}$  is a **neighborhood** of point  $x$  if

$$\exists G \in \tau, \quad x \in G \text{ and } G \subseteq \mathcal{N}.$$

**Definition** (Neighborhood of a Set). Let  $(X, \tau)$  be a topological space. Let  $S$  be a subset of  $X$ . Let  $\mathcal{N}$  be a subset of  $X$ . We say that  $\mathcal{N}$  is a **neighborhood** of set  $S$  if

$$\exists G \in \tau, \quad S \subseteq G \text{ and } G \subseteq \mathcal{N}.$$

**Definition** (Neighborhood System). Let  $(X, \tau)$  be a topological space. Let  $x$  be a point in  $X$ . We define a **neighborhood system** at point  $x$ , denoted by  $\mathcal{U}_x$ , to be the set of all neighborhoods of  $x$ .

**Proposition 1.3.1** (Properties of Neighborhood System). Let  $(X, \tau)$  be a topological space. Let  $x$  be a point in the space. Then

- (1) Neighborhood systems are closed under finite intersections. i.e.,

$$\forall U, V \in \mathcal{U}_x, \quad U \cap V \in \mathcal{U}_x.$$

- (2) There always exists smaller neighborhoods. i.e.,

$$\forall U \in \mathcal{U}_x, \exists V \in \mathcal{U}_x, \forall y \in V, U \in \mathcal{U}_y.$$

*Note that here  $V$  is not only a subset of  $U$ , but  $U$  is a neighborhood of  $V$ .*

- (3) Any superset of a neighborhood is also a neighborhood. i.e.,

$$\forall U \in \mathcal{U}_x, \forall V \subseteq X, V \supseteq U \implies V \in \mathcal{U}_x.$$

**Proposition 1.3.2.** Let  $(X, \tau)$  be a topological space. Let  $G$  be a set in the space. Then

$$G \in \tau \iff \forall x \in G, \exists \mathcal{N} \subseteq G, \mathcal{N} \in \mathcal{U}_x.$$



## 1.4 Basis

**Definition** (Basis). Let  $(X, \tau)$  be a topological space. We define a **basis** for  $\tau$ , denoted by  $\mathcal{B}$ , to be a subset of  $\tau$  such that

$$\forall G \in \tau, \quad \exists \mathcal{C} \subseteq \mathcal{B}, \quad G = \bigcup_{B \in \mathcal{C}} B.$$

**Definition** (Subbasis). Let  $(X, \tau)$  be a topological space. We define a **subbasis** for  $\tau$ , denoted by  $\mathcal{S}$ , to be a subset of  $\tau$  such that the collection of all finite intersections of elements of  $\mathcal{S}$  forms a basis for  $\tau$ .

**Definition** (Neighborhood Basis). Let  $(X, \tau)$  be a topological space. Let  $x$  be a point in  $X$ . We define a **neighborhood basis** at point  $x$ , denoted by  $\mathcal{B}_x$ , to be a sub-collection of the neighborhood system  $\mathcal{U}_x$  at  $x$  such that

$$\forall U \in \mathcal{U}_x, \quad \exists B \in \mathcal{B}_x, \quad B \subseteq U.$$

## 1.5 Weak Topology

**Definition** (Weak Topology).



## 2

# Metric Spaces

## 2.1 Metrics

**Definition** (Metric). Let  $X$  be a non-empty set. Let  $d$  be a function from  $X \times X$  to  $\mathbb{R}_+$ . We say that  $d$  is a **metric** (or distance function) on  $X$  if it satisfies all of the following conditions.

(1) *Non-negativity*

$$\forall x, y \in X, \quad d(x, y) \geq 0.$$

(2) *Identity of Indiscernible*

$$\forall x, y \in X, \quad d(x, y) = 0 \iff x = y.$$

(3) *Symmetry*

$$\forall x, y \in X, \quad d(x, y) = d(y, x).$$

(4) *Sub-Additivity (Triangle Inequality)*

$$\forall x, y, z \in X, \quad d(x, y) \leq d(x, z) + d(z, y).$$

**Definition** (Metric Space). Let  $X$  be a non-empty set. Let  $d$  be a metric on  $X$ . We call the pair  $(X, d)$  a **metric space**.

## 2.2 Continuity of Metrics

**Proposition 2.2.1.** Let  $(X, d)$  be a metric space. Let  $x_0$  be a point in  $X$ . Let  $d_{x_0}$  be a function from  $X$  to  $\mathbb{R}$  given by  $d_{x_0}(x) := d(x, x_0)$ . Then  $d_{x_0}$  is continuous.

**Proposition 2.2.2.** *Let  $(X, d)$  be a metric space. Let  $S$  be a subset of  $X$ . Let  $d_S$  be a function from  $X$  to  $\mathbb{R}$  given by  $d_S(x) := d(x, S)$ . Then  $d_S$  is continuous.*

*Proof.*

Let  $x_0$  be an arbitrary point in  $X$ .

Let  $\varepsilon$  be an arbitrary positive number.

Let  $\delta(\varepsilon)$  be a positive number given by  $\delta(\varepsilon)\varepsilon$ .

Let  $x$  be an arbitrary point in  $X$  such that  $d(x, x_0) < \delta(\varepsilon)$ .

Let  $x'$  be an arbitrary point in  $S$ .

Since  $x' \in S$ , by definition of distance from a point to a set,  $d(x_0, S) \leq d(x_0, x')$ .

Since  $d$  is a metric on  $X$  and  $x_0$  and  $x'$  are points in  $X$ , by the triangle inequality,  $d(x_0, x') \leq d(x_0, x) + d(x, x')$ .

Since  $d(x_0, S) \leq d(x_0, x')$  and  $d(x_0, x') \leq d(x_0, x) + d(x, x')$ ,  $d(x_0, S) - d(x_0, x) \leq d(x, x')$ .

Since  $d(x_0, S) - d(x_0, x) \leq d(x, x')$  for any  $x' \in S$ ,  $d(x_0, S) - d(x_0, x)$  is a lower bound for the set  $\{d(x, x') : x' \in S\}$ .

Since  $d(x_0, S) - d(x_0, x)$  is a lower bound for the set  $\{d(x, x') : x' \in S\}$ , by definition of infimum,  $d(x_0, S) - d(x_0, x) \leq \inf\{d(x, x') : x' \in S\} = d(x, S)$ .

Since  $x' \in S$ , by definition of distance from a point to a set,  $d(x, S) \leq d(x, x')$ .

Since  $d$  is a metric on  $X$  and  $x$  and  $x'$  are points in  $X$ , by the triangle inequality,  $d(x, x') \leq d(x, x_0) + d(x_0, x')$ .

Since  $d(x, S) \leq d(x, x')$  and  $d(x, x') \leq d(x, x_0) + d(x_0, x')$ ,  $d(x, S) - d(x, x_0) \leq d(x_0, x')$ .

Since  $d(x, S) - d(x, x_0) \leq d(x_0, x')$  for any  $x' \in S$ ,  $d(x, S) - d(x, x_0)$  is a lower bound for the set  $\{d(x_0, x') : x' \in S\}$ .

Since  $d(x, S) - d(x, x_0)$  is a lower bound for the set  $\{d(x_0, x') : x' \in S\}$ , by definition of infimum,  $d(x, S) - d(x, x_0) \leq \inf\{d(x_0, x') : x' \in S\} = d(x_0, S)$ .

Since  $d(x_0, S) - d(x_0, x) \leq d(x, S)$  and  $d(x, S) - d(x, x_0) \leq d(x_0, S)$ ,  $|d(x, S) - d(x_0, S)| \leq d(x, x_0)$ .

Since  $|d(x, S) - d(x_0, S)| \leq d(x, x_0)$  and  $d(x, x_0) < \delta(\varepsilon) = \varepsilon$ ,  $|d(x, S) - d(x_0, S)| < \varepsilon$ .

Since for any positive number  $\varepsilon$ , there exists a positive number  $\delta(\varepsilon)$  such that if for any point  $x$  in  $X$ , if  $d(x, x_0) < \delta(\varepsilon)$ , then  $|d(x, S) - d(x_0, S)| < \varepsilon$ , by definition of continuity, the function  $x \mapsto d(x, S)$  is continuous.

■

## 2.3 Open Sets and Closed Sets in Metric Spaces

**Definition (Openness).** *Let  $(X, d)$  be a metric space. Let  $S$  be a subset of  $X$ . We say that  $S$  is **open** in  $(X, d)$  if for any sequence  $\{x_k\}_{k=1}^{\infty}$  in  $X$  that converges to some point in  $S$ , there exists an integer  $N$  such that for any index  $k$  with  $k > N$ , we have  $x_k \in S$ .*

**Definition** (Openness). Let  $(X, d)$  be a metric space. Let  $S$  be a subset of  $X$ . We say that  $S$  is **open** in  $(X, d)$  if

$$\forall x \in S, \exists r_0 \in \mathbb{R}_{++}, \text{ball}(x, r_0) \subseteq S.$$

**Proposition 2.3.1.** Two definitions of open-ness are equivalent.

*Proof.*

For one direction, assume that for any sequence  $\{x_k\}_{k=1}^{\infty}$  in  $X$  that converges to some point in  $S$ , there exists an integer  $N$  such that for any index  $k$  with  $k > N$ , we have  $x_k \in S$ .

We are to prove that

$$\forall x \in S, \exists r_0 \in \mathbb{R}_{++}, \text{ball}(x, r_0) \subseteq S.$$

Let  $x$  be an arbitrary point in  $S$ .

Let  $\{x_k\}_{k=1}^{\infty}$  be an arbitrary sequence in  $X$  that converges to  $x$ .

By assumption, there exists an integer  $N$  such that  $x_N \in S$ .

Consider the open ball  $B_0(x, d(x_N, x))$ .

By definition, there exists another integer  $N'$  such that for any index  $k$  with  $k > N'$ , we have  $x_k \in B_0(x, d(x_N, x))$ .

Assume that for any point  $x$  in  $G$ , there exists an open ball  $B(x, r)$  centered at  $x$ , of some radius  $r$ , such that  $B(x, r) \subseteq G$ .

Let  $\{x_k\}_{k=1}^{\infty}$  be an arbitrary sequence in  $X$  that converges to some point  $x_0$  in  $G$ . We are to show that  $\{x_k\}$  is eventually in  $G$ .

By assumption, there exists an open ball  $B_0(x_0, r_0)$  centered at  $x_0$ , of some radius  $r_0$ , such that  $B_0(x_0, r_0) \subseteq G$ .

$$d(x_k, x_0) < r_0$$

By definition of open balls,  $x_k$  is in  $B_0(x_0, r_0)$ . Thus  $x_k$  is also in  $G$ .

i.e., any sequence in  $X$  that converges to some point in  $G$  is eventually in  $G$ . ■

*Proof.* For the reverse direction, assume that any sequence in  $X$  that converges to some point in  $G$  is eventually in  $G$ .

Let  $x_0$  be an arbitrary point in  $G$ .

We are to prove that there exists an open ball  $B_0(x_0, r_0)$  centered at  $x_0$ , of some radius  $r_0$ , such that  $B_0(x_0, r_0) \subseteq G$ .

$$d(x_k, x_0) < 1/k$$

Then  $\{x_k\}$  converges to  $x_0$ .

By assumption, there exists an integer  $N$  such that  $x_N \in G$ .

Take  $r_0 = 1/N$ . Then the open ball  $B_0(x_0, r_0)$  is contained in  $G$ .

i.e., for any point  $x$  in  $G$ , there exists an open ball  $B(x, r)$  centered at  $x$ , of some radius  $r$ , such that  $B(x, r) \subseteq G$ . ■

**Definition** (Closedness). *Let  $(X, d)$  be a metric space. Let  $S$  be a subset of  $X$ . We say that  $S$  is **closed** in  $(X, d)$  if for any sequence in  $S$  that converges to some point  $x_0$  in  $S$ ,  $x_0$  is also in  $S$ .*

**Proposition 2.3.2** (Stability under Set Operations).

- (1) *The complement of an open set is closed.*
- (2) *The complement of a closed set is open.*
- (3) *The union of an arbitrary collection of open sets is open.*
- (4) *The union of a finite collection of closed sets is closed.*
- (5) *The intersection of a finite collection of open sets is open.*
- (6) *The intersection of an arbitrary collection of closed sets is closed.*
- (7) *If  $S_1$  is open and  $S_2$  is closed, then  $S_1 \setminus S_2$  is open.*
- (8) *If  $S_1$  is closed and  $S_2$  is open, then  $S_1 \setminus S_2$  is closed.*

*Proof of (1).* Let  $\{U_\alpha\}$  be a set of open sets.

By Lemma 2.1, each of the open set can be written as a union of open balls.

i.e. for all  $\alpha$ , there exists a set of open balls  $\{B_{\alpha\beta}\}$  such that

$$U_\alpha = \bigcup_{\beta} B_{\alpha\beta}.$$

Then the union of the open sets  $\{U_\alpha\}$  is

$$\bigcup_{\alpha} U_\alpha = \bigcup_{\alpha} \bigcup_{\beta} B_{\alpha\beta}.$$

By Lemma 2.1 again, the union is open. ■

*Proof of (2).* Let  $\{U_k\}_{k=1}^{k=n}$  be a finite sequence of open sets.

Let  $x$  be an arbitrary element in their intersection.

Then  $x$  is in each of  $U_k$ .

By definition of open sets, for each  $U_k$ , there exists an open ball  $B_k(r_k, x)$  such that  $B_k \subseteq U_k$ .

Define  $r = \min\{r_k\}$ .

Consider the open ball  $B(r, x)$ .

Then  $B$  is a subset of each  $U_k$  and hence a subset of their intersection.  
By definition of open sets, the intersection is open. ■

*Proof of (3).* Let  $\{F_\alpha\}$  be a set of closed subsets of  $X$ .

Then each of  $(F_\alpha)^c$  is open.

By Proposition 2.1, we get  $\bigcup_\alpha (F_\alpha)^c$  is open.

By the De Morgan's Laws, we get  $(\bigcap_\alpha F_\alpha)^c$  is open.

It follows that  $\bigcap_\alpha F_\alpha$  is closed. ■

*Proof of (4).* Let  $\{F_\alpha\}$  be a set of closed subsets of  $X$ .

Then each of  $(F_\alpha)^c$  is open.

By Proposition 2.2, we get  $\bigcap_\alpha (F_\alpha)^c$  is open.

By the De Morgan's Laws, we get  $(\bigcup_\alpha F_\alpha)^c$  is open.

It follows that  $\bigcup_\alpha F_\alpha$  is closed. ■

## 2.4 The Discrete Metric

**Definition** (Discrete Metric). *Let  $X$  be a set. We define a **discrete metric** on  $X$  to be a function from  $X$  to  $\mathbb{R}_+$  given by*

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

**Proposition 2.4.1.** *In a discrete metric space, any set is both open and closed.*

**Proposition 2.4.2.** *In a discrete metric space, the unit closed ball is the whole space, and the unit open ball is the singleton set consisting of only the center.*

**Proposition 2.4.3.** *Discrete metric spaces are always bounded.*

*Proof.* Let  $(X, d)$  be a discrete metric space and  $S$  be a subset of  $X$ .

**Case 1.**  $S$  is empty.

Since  $S$  is empty, by definition of boundedness,  $S$  is bounded.

**Case 2.**  $S$  is not empty.

Since  $S$  is not empty, pick a point  $x_0$  in  $S$ .

Since  $(S, d)$  is discrete,  $d(x, x_0) \leq 1$  for any  $x \in S$ .

Since  $d(x, x_0) \leq 1$  for any  $x \in S$ , by definition of boundedness,  $S$  is bounded.

**Summary.**

Since in all cases,  $S$  is bounded,  $S$  is bounded. ■

**Proposition 2.4.4.** *A discrete metric space is totally bounded only if it is empty or it is a singleton set. (unconfirmed)*

**Proposition 2.4.5.** *Discrete metric spaces are always complete.*

*Proof.* Let  $(X, d)$  be a discrete metric space and  $\{x_i\}_{i \in \mathbb{N}}$  be a Cauchy sequence in  $X$ .

We are to prove that  $\{x_i\}_{i \in \mathbb{N}}$  is convergent.

Since  $\{x_i\}_{i \in \mathbb{N}}$  is Cauchy, there exists an integer  $N(1)$  such that for any indices  $m$  and  $n$ , if  $m, n > N(1)$ , then  $d(x_m, x_n) < 1$ .

Since  $d(x_m, x_n) < 1$  and  $d$  is discrete,  $d(x_m, x_n) = 0$ .

Since  $d(x_m, x_n) = 0$  and  $d$  is a metric,  $x_m = x_n$ .

Since  $m, n > N(1)$  implies  $x_m = x_n$  for any indices  $m$  and  $n$ , in particular,  $i > N(1)$  implies  $x_i = x_{N(1)+1}$  for any index  $i$ .

Since  $i > N(1)$  implies  $x_i = x_{N(1)+1}$  for any index  $i$ , for any positive number  $\varepsilon$  and any index  $i$ , if  $i > N(1)$ , then  $d(x_i, x_{N(1)+1}) < \varepsilon$ .

Since for any positive number  $\varepsilon$  and any index  $i$ , if  $i > N(1)$ , then  $d(x_i, x_{N(1)+1}) < \varepsilon$ , by definition of convergence,  $\{x_i\}_{i \in \mathbb{N}}$  converges to  $x_{N(1)+1}$ .

Since any Cauchy sequence in  $(X, d)$  converges in  $(X, d)$ , by definition of completeness,  $(X, d)$  is complete. ■

**Proposition 2.4.6.** *Discrete metric spaces are totally disconnected.*

**Proposition 2.4.7.** *Any function defined on a discrete metric space is uniformly continuous.*

### not sure what the codomain could be.

## 2.5 The Hausdorff Metric

**Proposition 2.5.1.** *Let  $(X, d)$  be a metric space. Let  $(\mathcal{H}(X), d_H)$  be the induced Hausdorff space. Then if  $(X, d)$  is complete,  $(\mathcal{H}(X), d)$  is complete.*

**Proposition 2.5.2.** *Let  $(X, d)$  be a metric space. Let  $(\mathcal{H}(X), d_H)$  be the induced Hausdorff space. Then if  $(X, d)$  is totally bounded,  $(\mathcal{H}(X), d_H)$  is totally bounded.*

*Proof.* Let  $r$  be an arbitrary radius.

Since  $(X, d)$  is totally bounded, there exists a finite collection  $\S = \{x_i\}_{i \in I}$  of points in  $X$  such that  $\{\text{ball}(x_i, r)\}_{i \in I}$  covers  $X$ .



Let  $S$  be an arbitrary set in  $\mathcal{H}(X)$ .

Let  $\S'(S) = \{x_i\}_{i \in I'(S)}$  be the subcollection of  $\S$  of points  $x_i$  such that  $\text{ball}(x_i, r) \cap S \neq \emptyset$ .

Since  $S$  is a nonempty subset of  $X$  and  $\{\text{ball}(x_i, r)\}_{i \in I}$  covers  $X$ ,  $\S'(S)$  is nonempty.

Since  $\S'(S)$  is finite,  $\S'(S)$  is closed and bounded in  $(X, d)$ .

Since  $\S'(S)$  is nonempty, closed, and bounded in  $(X, d)$ ,  $\S'(S) \in \mathcal{H}(X)$ .

Since  $\text{ball}(x_i, r) \cap S \neq \emptyset$  for each  $i \in I'(S)$ ,  $(S)_r \supseteq \S'(S)$ .

Since  $\{\text{ball}(x_i, r)\}_{i \in I}$  covers  $X$  and  $S$  is a subset of  $X$ ,  $\{\text{ball}(x_i, r)\}_{i \in I}$  covers  $S$ .

Since  $\{\text{ball}(x_i, r)\}_{i \in I}$  covers  $S$  and  $\text{ball}(x_i, r) \cap S = \emptyset$  for any  $i \in I \setminus I'(S)$ ,  $\{\text{ball}(x_i, r)\}_{i \in I'(S)}$  covers  $S$ .

Since  $\{\text{ball}(x_i, r)\}_{i \in I'(S)}$  covers  $S$ ,  $(\S'(S))_r \supseteq S$ .

Since  $(S)_r \supseteq \S'(S)$  and  $(\S'(S))_r \supseteq S$ , by definition of Hausdorff metric,  $d_H(S, \S'(S)) < r$ .

Since  $d_H(S, \S'(S)) < r$ ,  $S \in \text{ball}(\S'(S), r)$ .

Since for any  $S$  in  $\mathcal{H}(X)$ , there exists another set  $\S'(S)$  in  $\mathcal{H}(X)$  such that  $S \in \text{ball}(\S'(S), r)$ ,  $\{\text{ball}(\S', r)\}_{\S' \in \mathcal{P}(\S)}$  covers  $\mathcal{H}(X)$ .

Since for any radius  $r$ , there exists a collection  $\mathcal{C}$  of sets in  $\mathcal{H}(X)$  such that  $\{\text{ball}(\S', r)\}_{\S' \in \mathcal{C}}$  covers  $\mathcal{H}(X)$ , by definition of total boundedness,  $\mathcal{H}(X)$  is totally bounded.

■



# 3

## Interior, Closure, and Boundary

### 3.1 Definitions

#### 3.1.1 Interior

**Definition** (Interior Point). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. Let  $x$  be a point in the space. We say that  $x$  is an **interior point** of  $S$  if

$$\exists \text{ neighborhood } \mathcal{N} \text{ of } x, \quad \mathcal{N} \subseteq S.$$

Equivalently,

$$\mathcal{N} \cap X \setminus S = \emptyset.$$

**Definition** (Interior). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in  $(X, \tau)$ . We define the **interior** of  $S$ , denoted by  $\text{int}(S)$ , to be the set of all interior points of  $S$ .

**Definition** (Interior). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in  $(X, \tau)$ . We define the **interior** of  $S$ , denoted by  $\text{int}(S)$ , to be the union of all open subsets of  $S$ . Equivalently, the largest open subset of  $S$ .

**Proposition 3.1.1.** The two definitions of interior are equivalent.

#### 3.1.2 Boundary

**Definition** (Boundary Point). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. Let  $x$  be a point in the space. We say that  $x$  is a **boundary point** of  $S$  if for any neighborhood  $\mathcal{N}(x)$  of  $x$ , we have  $\mathcal{N}(x) \cap S \neq \emptyset$  and  $\mathcal{N}(x) \cap C_X(S) \neq \emptyset$ .

**Definition** (Boundary). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in  $(X, \tau)$ . We define the **boundary** of  $S$ , denoted by  $\text{bd}(S)$ , to be the set of all boundary points of  $S$ .

**Definition** (Boundary). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in  $(X, \tau)$ . We define the **boundary** of  $S$ , denoted by  $bd(S)$ , to be the set given by  $C_X(\text{int}(S) \cup \text{int}(C_X(S)))$ .

**Definition** (Boundary). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in  $(X, \tau)$ . We define the **boundary** of  $S$ , denoted by  $bd(S)$ , to be the set given by  $\text{cl}(S) \cap \text{cl}(C_X(S))$ .

**Proposition 3.1.2.** The three definitions of boundary are equivalent.

### 3.1.3 Closure

**Definition** (Adherent Point). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. Let  $x$  be a point in the space. We say that  $x$  is an **adherent point** of  $S$  if for any neighborhood  $\mathcal{N}(x)$  of  $x$ , we have  $\mathcal{N}(x) \cap S \neq \emptyset$ .

**Definition** (Closure). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in  $(X, \tau)$ . We define the **closure** of  $S$ , denoted by  $\text{cl}(S)$ , to be the set of all adherent points of  $S$ .

**Definition** (Closure). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in  $(X, \tau)$ . We define the **closure** of  $S$ , denoted by  $\text{cl}(S)$ , to be the intersection of all closed superset of  $S$ , or equivalently, the smallest closed superset of  $S$ .

**Proposition 3.1.3.** The two definitions of closure are equivalent.

*Proof.*

For one direction, let  $x$  be an arbitrary point in the intersection of all closed supersets of  $S$ . We are to prove that  $x$  is an adherent point of  $S$ .

Assume for the sake of contradiction that  $x$  is not an adherent point of  $S$ .

Since  $x$  is not an adherent point of  $S$ , there exists some open set  $U$  containing  $x$  such that  $U \cap S = \emptyset$ .

Since  $U$  is open in  $(X, \tau)$ ,  $C_X(U)$  is closed in  $(X, \tau)$ .

Since  $U \cap S = \emptyset$ ,  $C_X(U) \supseteq S$ .

Since  $x \in U$ ,  $x \notin C_X(U)$ .

Since  $C_X(U)$  is a closed superset of  $S$  and  $x \notin C_X(U)$ ,  $x$  is not in the intersection of all closed supersets of  $S$ .

This contradicts to the assumption that  $x$  is in the intersection of all closed supersets of  $S$ .

So the assumption that  $x$  is not an adherent point of  $S$  is false.

i.e.,  $x$  is an adherent point of  $S$ .

For the other direction, let  $x$  be an adherent point of  $S$ .

We are to prove that  $x$  is in the intersection of all closed supersets of  $S$ .

Assume for the sake of contradiction that  $x$  is not in the intersection of all closed supersets of  $S$ .

Since  $x$  is not in the intersection of all closed supersets of  $S$ , there exists some closed superset  $E$  of  $S$  such that  $x \notin E$ .

Since  $E$  is closed in  $(X, \tau)$ ,  $C_X(E)$  is open in  $(X, \tau)$ .

Since  $E \supseteq S$ ,  $C_X(E) \cap S = \emptyset$ .

Since  $x \notin E$ ,  $x \in C_X(E)$ .

Since there exists some open set  $U$  such that  $x \in U$  and  $U \cap S = \emptyset$ ,  $x$  is not an adherent point of  $S$ .

This contradicts to the assumption that  $x$  is an adherent point of  $S$ .

So the assumption that  $x$  is not in the intersection of all closed supersets of  $S$  is false.

i.e.,  $x$  is in the intersection of all closed supersets of  $S$ . ■

## 3.2 Basic Properties

**Proposition 3.2.1.**

- (1) *Interiors are open.*
- (2) *Closures are closed.*
- (3) *Boundaries are closed.*

**Proposition 3.2.2.** *For any set in any topological space, the closure is the disjoint union of the interior and the boundary. i.e., an adherent point is exactly one of an interior point or a boundary point.*

**Proposition 3.2.3.** *Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. Then*

- (1)  $\text{int}(X \setminus S) = X \setminus \text{cl}(S)$ .
- (2)  $\text{cl}(X \setminus S) = X \setminus \text{int}(S)$ .
- (3)  $\text{bd}(X \setminus S) = \text{bd}(S)$ .

*Proof of (1).* For one direction, let  $x$  be an arbitrary point in  $\text{int}(C_X(S))$ .

We are to prove that  $x \notin \text{cl}(S)$ .

Since  $x \in \text{int}(C_X(S))$ , by definition of interior, there exists an open set  $U$  such that  $x \in U$  and  $U \subseteq C_X(S)$ .

Since  $U$  is an open subset of  $S$ ,  $C_X(U)$  is a closed superset of  $S$ .

Since  $x \in U$ ,  $x \notin C_X(U)$ .

Since  $C_X(U)$  a closed superset of  $S$  and  $x \notin C_X(U)$ ,  $x$  is not in the intersection of all closed superset of  $S$ .

Since  $x$  is not in the intersection of all closed superset of  $S$ , by definition of closure,  $x \notin \text{cl}(S)$ .

Since  $x \notin \text{cl}(S)$ , we get  $x \in C_X(\text{cl}(S))$ .

For the reverse direction, let  $x$  be an arbitrary point in  $\text{cl}(S)$ .

We are to prove that  $x \notin \text{int}(C_X(S))$ .

Since  $x \in \text{cl}(S)$ , by definition of closure,  $x$  is in any closed superset  $F$  of  $S$ .

Since  $F$  is a closed superset of  $S$ ,  $C_X(F)$  is an open subset of  $C_X(S)$ .

Since  $x \in F$ ,  $x \notin C_X(F)$ .

Since  $C_X(F)$  is an arbitrary open subset of  $C_X(S)$  and  $x \notin C_X(F)$ ,  $x$  is not in the union of all open subsets of  $C_X(S)$ .

Since  $x$  is not in the union of all open subsets of  $C_X(S)$ , by definition of interior,  $x \notin \text{int}(C_X(S))$ . ■

*Proof of (2).*

$$\text{cl}(S^c) \subseteq (\text{int}(S))^c \# (*) \setminus n(\text{int}(S))^c \subseteq \text{cl}(S^c) \# (**)$$

Let  $x$  be an arbitrary point in  $\text{cl}(S^c)$ .

By definition of closure,  $x$  is in every closed superset of  $S^c$ .

Note that  $x$  is not in any open subset of  $S$ .

By definition, we conclude that  $x \notin \text{int}(S)$  and hence  $x \in (\text{int}(S))^c$ .

$$\text{cl}(S^c) \subseteq (\text{int}(S))^c \# (*)$$

Let  $x$  be an arbitrary point in  $(\text{int}(S))^c$ .

By definition of interior,  $x$  is not in any open subset of  $S$ .

Note that the complement of open subsets of  $S$  are closed supersets of  $S^c$ .

Thus  $x$  is in every closed superset of  $S^c$ .

By definition, we conclude that  $x \in \text{cl}(S^c)$ . ■

$$(\text{int}(S))^c \subseteq \text{cl}(S^c) \# (**)$$

**Proposition 3.2.4.** *Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. Then*

(1)  *$S$  is open if and only if  $S = \text{int}(S)$ .*

(2)  *$S$  is closed if and only if  $S = \text{cl}(S)$ .*

*Proof of (1).* For one direction, assume that  $S$  is open.

We are to prove that  $S = \text{int}(S)$ .

Let  $x$  be an arbitrary point in  $S$ .

Since  $x$  is in  $S$  and  $S$  is open and  $S$  is a subset of  $S$ ,  $x$  is in some open subset of  $S$ .

Since  $x$  is in some open subset of  $S$ ,  $x$  is in the union of all open subsets of  $S$ .

Since  $x$  is in the union of all open subsets of  $S$ , by definition,  $x \in \text{int}(S)$ .

Since for any point  $x \in S$ , we have  $x \in \text{int}(S)$ , we get  $S \subseteq \text{int}(S)$ .

Let  $x$  be an arbitrary point in  $\text{int}(S)$ .

Since  $x \in \text{int}(S)$ , by definition,  $x$  is in the union of all subsets of  $S$ .

Since  $x$  is in the union of all subsets of  $S$ ,  $x$  is in some open subset  $S'$  of  $S$ .

Since  $x \in S'$  and  $S' \subseteq S$ ,  $x \in S$ .

Since for any point  $x \in \text{int}(S)$ , we have  $x \in S$ , we get  $\text{int}(S) \subseteq S$ .

Since  $S \subseteq \text{int}(S)$  and  $\text{int}(S) \subseteq S$ ,  $S = \text{int}(S)$ .

For the reverse direction, assume that  $S = \text{int}(S)$ .

We are to prove that  $S$  is open.

Since  $S = \text{int}(S)$  and  $\text{int}(S)$  is open,  $S$  is open. ■

**Proposition 3.2.5.**

(1) A set  $S$  is open if and only if  $S$  and  $\text{bd}(S)$  are disjoint.

(2) A set  $S$  is closed if and only if  $S$  contains  $\text{bd}(S)$ .

**Proposition 3.2.6.** The boundary of some open set or of some closed set has no interior.

*Proof for Open Sets.* Let  $(X, \tau)$  be a topological space and  $S$  be an open set in  $(X, \tau)$ .

Let  $x$  be an arbitrary point in  $\text{bd}(S)$ .

Let  $\mathcal{N}(x)$  be an arbitrary open neighborhood of  $x$  in  $(X, \tau)$ .

Since  $x$  is a boundary point of  $S$  and  $\mathcal{N}(x)$  is some open neighborhood of  $x$  in  $(X, \tau)$ , by definition of boundary,  $\mathcal{N}(x) \cap S \neq \emptyset$ .

Since  $S$  is open,  $S \cap \text{bd}(S) = \emptyset$ .

Since  $\mathcal{N}(x) \cap S \neq \emptyset$  and  $S \cap \text{bd}(S) = \emptyset$ ,  $\mathcal{N}(x) \not\subseteq \text{bd}(S)$ .

Since  $\mathcal{N}(x) \not\subseteq \text{bd}(S)$  for any open neighborhood of  $x$  in  $(X, \tau)$ , by definition of interior,  $x \notin \text{int}(\text{bd}(S))$ .

Since  $x \notin \text{int}(\text{bd}(S))$  for any  $x \in \text{bd}(S)$ ,  $\text{bd}(S)$  has no interior. ■

*Proof for Closed Sets.* Let  $(X, \tau)$  be a topological space and  $S$  be a closed set in  $(X, \tau)$ .

Let  $x$  be an arbitrary point in  $\text{bd}(S)$ .

Let  $\mathcal{N}(x)$  be an arbitrary open neighborhood of  $x$  in  $(X, \tau)$ .

Since  $x$  is a boundary point of  $S$  and  $\mathcal{N}(x)$  is some open neighborhood of  $x$  in  $(X, \tau)$ , by definition of boundary,  $\mathcal{N}(x) \cap C_X(S) \neq \emptyset$ .

Since  $S$  is closed,  $C_X(S) \cap \text{bd}(S) = \emptyset$ .

Since  $\mathcal{N}(x) \cap C_X(S) \neq \emptyset$  and  $C_X(S) \cap \text{bd}(S) = \emptyset$ ,  $\mathcal{N}(x) \not\subseteq \text{bd}(S)$ .

Since  $\mathcal{N}(x) \not\subseteq \text{bd}(S)$  for any open neighborhood of  $x$  in  $(X, \tau)$ , by definition of interior,  $x \notin \text{int}(\text{bd}(S))$ .

Since  $x \notin \text{int}(bd(S))$  for any  $x \in bd(S)$ ,  $bd(S)$  has no interior. ■

**Proposition 3.2.7.** *The boundary of a set is empty if and only if the set is both open and closed.*

*Proof.*

Let  $(X, d)$  be a metric space and  $S$  be a subset of  $X$ .

For one direction, assume that  $bd(S) = \emptyset$ . We are to prove that  $S$  is both open and closed.

Let  $x$  be an arbitrary point in  $S$ . Then  $x \in \text{int}(S)$ . Thus  $S = \text{int}(S)$ .

Note that  $\text{cl}(S) = \text{int}(S) \cup bd(S) = \text{int}(S)$ . Thus  $S = \text{cl}(S)$ .

By definition, we conclude that  $S$  is both open and closed.

For the reverse direction, assume that  $S$  is both open and closed. We are to prove that  $bd(S) = \emptyset$ .

Since  $S$  is open,  $S = \text{int}(S)$ .

Since  $S$  is closed,  $S = \text{cl}(S)$ .

Since  $S = \text{int}(S)$  and  $S = \text{cl}(S)$ ,  $\text{int}(S) = \text{cl}(S)$ .

Since  $bd(S) = \text{cl}(S) - \text{int}(S)$ ,  $bd(S) = \emptyset$ . ■

### 3.3 As Operators

**Proposition 3.3.1** (The Interior Operator). *In any topological space,*

(1)

$$\forall S \subseteq X, \quad \text{int}(S) \subseteq S.$$

(2) *Monotonic:*

$$\forall S_1, S_2 \subseteq X, \quad S_1 \subseteq S_2 \implies \text{int}(S_1) \subseteq \text{int}(S_2).$$

(3) *Idempotent:*

$$\forall S \subseteq X, \quad \text{int}(S) = \text{int}(\text{int}(S)).$$

*Proof of (2).* Let  $x$  be an arbitrary point in  $\text{int}(S_1)$ .

Since  $x \in \text{int}(S_1)$ , by definition of interior,  $x$  is in some open subset  $S'$  of  $S_1$ .

Since  $S'$  is an open subset of  $S_1$  and  $S_1 \subseteq S_2$ ,  $S'$  is an open subset of  $S_2$ .

Since  $S'$  is an open subset of  $S_2$  and  $x \in S'$ , by definition of interior,  $x \in \text{int}(S_2)$ .

Since for any point in  $\text{int}(S_1)$  is also in  $\text{int}(S_2)$ ,  $\text{int}(S_1) \subseteq \text{int}(S_2)$ . ■

**Proposition 3.3.2** (The Closure Operator). *In any topological space,*



(1)

$$\forall S \subseteq X, \quad S \subseteq \text{cl}(S).$$

(2) *Monotonic:*

$$\forall S_1, S_2 \subseteq X, \quad S_1 \subseteq S_2 \implies \text{cl}(S_1) \subseteq \text{cl}(S_2).$$

(3) *Idempotent:*

$$\forall S \subseteq X, \quad \text{cl}(S) = \text{cl}(\text{cl}(S)).$$

*Proof of (2).* Let  $x$  be an arbitrary point in  $\text{cl}(S_1)$ .

Let  $S'$  be an arbitrary closed superset of  $S_2$ .

Since  $S'$  is a closed superset of  $S_2$  and  $S_1 \subseteq S_2$ ,  $S'$  is a closed superset of  $S_1$ .

Since  $S'$  is a closed superset of  $S_1$  and  $x \in \text{cl}(S_1)$ , by definition of closure,  $x \in S'$ .

Since  $x$  is in any closed superset of  $S_2$ , by definition,  $x \in \text{cl}(S_2)$ .

Since any point in  $\text{cl}(S_1)$  is in  $\text{cl}(S_2)$ ,  $\text{cl}(S_1) \subseteq \text{cl}(S_2)$ . ■

**Proposition 3.3.3** (The Exterior Operator). *In any topological space,*

(1) (Monotonic) *For any sets  $S$  and  $T$ , if  $S \subseteq T$ , then  $\text{ext}(S) \supseteq T$ .*

**Remark.** *The exterior operator is not idempotent.*

**Proposition 3.3.4.** *Let  $(X, \tau)$  be a topological space and  $S$  be a subset of  $X$ . Then*

(1)  $\text{int}(S) \subseteq \text{int}(\text{cl}(S))$ .

(2)  $\text{cl}(S) \supseteq \text{cl}(\text{int}(S))$ .

*Proof.* By the properties of the closure operator,  $S \subseteq \text{cl}(S)$ . Since  $S \subseteq \text{cl}(S)$  and the interior operator is monotonic increasing,  $\text{int}(S) \subseteq \text{int}(\text{cl}(S))$ . By the properties of the interior operator,  $\text{int}(S) \subseteq S$ . Since  $\text{int}(S) \subseteq S$  and the closure operator is monotonic increasing,  $\text{cl}(\text{int}(S)) \subseteq \text{cl}(S)$ . ■

**Remark.** *The point of this proposition is to remind the readers that the equalities might not hold. See section 3.4 for counter examples.*

**Proposition 3.3.5.** *Let  $S$  be a set in some topological space. Then*

(1)  $\text{bd}(\text{int}(S)) \subseteq \text{bd}(S)$ .

(2)  $\text{bd}(\text{cl}(S)) \subseteq \text{bd}(S)$ .

(3)  $\text{bd}(\text{bd}(S)) \subseteq \text{bd}(S)$

*Proof of (1).* Let  $x$  be an arbitrary element in  $bd(\text{int}(S))$ .

We are to prove that  $x \in bd(S)$ .

Since  $x \in bd(\text{int}(S))$ , by definition, for any neighborhood  $N(x)$  of  $x$ , there exist a point  $x_1$  in  $N(x) \cap \text{int}(S)$  and a point  $x_2$  in  $N(x) \cap (\text{int}(S))^c$ .

Since  $x_1 \in \text{int}(S)$  and  $\text{int}(S) \subseteq S$ ,  $x_1 \in S$ .

Since  $x_1 \in S$  and  $x_1 \in N(x)$ ,  $x_1 \in N(x) \cap S$ .

To prove that  $N(x) \cap S^c \neq \emptyset$ , assume for the sake of contradiction that  $N(x) \subseteq S$ .

Since  $x_2 \in N(x)$  and  $N(x) \subseteq S$ , by definition,  $x_2 \in \text{int}(S)$ .

This contradicts to the fact that  $x_2 \in (\text{int}(S))^c$ .

Thus the assumption that  $N(x) \subseteq S$  is false.

i.e., there exists a point  $x'_2$  in  $N(x) \cap S^c$ .

In short, I have proved that for any neighborhood  $N(x)$  of  $x$ , there exist a point  $x_1$  in  $N(x) \cap S$  and a point  $x'_2$  in  $N(x) \cap S^c$ .

By definition, I conclude that  $x \in bd(S)$ . ■

*Proof of (2).* Let  $x$  be an arbitrary element in  $bd(\text{cl}(S))$ .

We are to prove that  $x \in bd(S)$ .

Since  $x \in bd(\text{cl}(S))$ , by definition, for any neighborhood  $N(x)$  of  $x$ , there exist a point  $x_1$  in  $N(x) \cap \text{cl}(S)$  and a point  $x_2$  in  $N(x) \cap (\text{cl}(S))^c$ .

To prove that  $N(x) \cap S \neq \emptyset$ , assume for the sake of contradiction that  $N(x) \subseteq S^c$ .

Since  $x_1 \in N(x)$  and  $N(x) \subseteq S^c$ ,  $x_1 \in \text{ext}(S)$ .

This contradicts to the fact that  $x_1 \in \text{cl}(S)$ .

Thus the assumption that  $N(x) \subseteq S^c$  is false.

i.e., there exists a point  $x'_1$  in  $N(x) \cap S$ .

Since  $x_2 \in (\text{cl}(S))^c$  and  $(\text{cl}(S))^c \subseteq S^c$ ,  $x_2 \in S^c$ .

Since  $x_2 \in S^c$  and  $x_2 \in N(x)$ ,  $x_2 \in N(x) \cap S^c$ .

In short, I have proved that for any neighborhood  $N(x)$  of  $x$ , there exist a point  $x'_1$  in  $N(x) \cap S$  and a point  $x_2$  in  $N(x) \cap S^c$ .

By definition, I conclude that  $x \in bd(S)$ . ■

**Proposition 3.3.6** (Set Operations). *Let  $S_1$  and  $S_2$  be sets in some topological space. Then*

$$(1) \text{ int}(S_1 \cup S_2) \supseteq \text{int}(S_1) \cup \text{int}(S_2).$$

$$(2) \text{ int}(S_1 \cap S_2) = \text{int}(S_1) \cap \text{int}(S_2).$$

$$(3) \text{ cl}(S_1 \cup S_2) = \text{cl}(S_1) \cup \text{cl}(S_2).$$

$$(4) \text{ cl}(S_1 \cap S_2) \subseteq \text{cl}(S_1) \cap \text{cl}(S_2).$$

$$(5) \text{ bd}(S_1 \cup S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2).$$

$$(6) \text{ bd}(S_1 \cap S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2).$$

$$(7) \text{ bd}(S_1 \setminus S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2).$$

**Remark** (Remark on (1).). “ $\text{int}(S_1 \cup S_2) = \text{int}(S_1) \cup \text{int}(S_2)$ ” may not be true. For example, consider  $X = \mathbb{R}$ ,  $S_1 = \mathbb{Q}$ , and  $S_2 = X \setminus \mathbb{Q}$ . Then  $LHS = \text{int}(S_1 \cup S_2) = \mathbb{R}$  and  $RHS = \text{int}(S_1) \cup \text{int}(S_2) = \emptyset$ .

*Proof of (2).*

It is clear that  $\text{int}(S_1 \cap S_2) \subseteq \text{int}(S_1) \cap \text{int}(S_2)$ .

So it suffices to prove that  $\text{int}(S_1 \cap S_2) \supseteq \text{int}(S_1) \cap \text{int}(S_2)$ .

Let  $x$  be an arbitrary point in  $\text{int}(S_1) \cap \text{int}(S_2)$ .

Then  $x \in \text{int}(S_1)$  and  $x \in \text{int}(S_2)$ .

Since  $x \in \text{int}(S_1)$ , there exists some open set  $G_1$  such that  $x \in G_1 \subseteq S_1$ .

Since  $x \in \text{int}(S_2)$ , there exists some open set  $G_2$  such that  $x \in G_2 \subseteq S_2$ .

Since  $x \in G_1$  and  $x \in G_2$ ,  $x \in G_1 \cap G_2$ .

Since  $G_1$  and  $G_2$  are both open,  $G_1 \cap G_2$  is open.

Since  $G_1 \subseteq S_1$  and  $G_2 \subseteq S_2$ ,  $G_1 \cap G_2 \subseteq S_1 \cap S_2$ .

Since  $G_1 \cap G_2$  is open and  $x \in G_1 \cap G_2 \subseteq S_1 \cap S_2$ , by definition of interior points,  $x \in \text{int}(S_1 \cap S_2)$ .

Since

$$\forall x \in \text{int}(S_1) \cap \text{int}(S_2), \quad x \in \text{int}(S_1 \cap S_2),$$

we get  $\text{int}(S_1 \cap S_2) \supseteq \text{int}(S_1) \cap \text{int}(S_2)$ . ■

*Proof of (3).*

$$\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B).$$

Let  $x$  be an arbitrary point in  $\text{cl}(A \cup B)$ .

By definition of closure,  $x$  is in every closed superset of  $(A \cup B)$ .

Note that the union of an arbitrary closed superset of  $A$  and an arbitrary closed superset of  $B$  is a closed superset of  $(A \cup B)$ .

Thus  $x$  is in the union of the intersection of all closed supersets of  $A$  and the intersection of all closed supersets of  $B$ .

By definition again, we conclude that  $x \in \text{cl}(A) \cup \text{cl}(B)$ .

Since  $x$  is an arbitrary point in  $\text{cl}(A \cup B)$ ,  $\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B)$ . ■

**Remark** (Remark on (4).). “ $\text{cl}(S_1 \cap S_2) \subseteq \text{cl}(S_1) \cap \text{cl}(S_2)$ ” may not be true. For example, consider  $X = \mathbb{R}$ ,  $S_1 = \mathbb{Q}$ , and  $S_2 = X \setminus \mathbb{Q}$ . Then  $\text{LHS} = \text{cl}(S_1 \cap S_2) = \emptyset$  and  $\text{RHS} = \text{cl}(S_1) \cap \text{cl}(S_2) = \mathbb{R}$ .

*Proof of (5).*

Let  $x$  be an arbitrary point in  $\text{bd}(S_1 \cup S_2)$ .

Since  $x \in \text{bd}(S_1 \cup S_2)$ , by definition of boundary, for any neighborhood  $\mathcal{N}(x)$  around  $x$ ,  $\mathcal{N}(x) \cap (S_1 \cup S_2) \neq \emptyset$  and  $\mathcal{N}(x) \cap C_X(S_1 \cup S_2) \neq \emptyset$ .

Since  $\mathcal{N}(x) \cap (S_1 \cup S_2) \neq \emptyset$ , either  $\mathcal{N}(x) \cap S_1 \neq \emptyset$  or  $\mathcal{N}(x) \cap S_2 \neq \emptyset$ .

Since  $\mathcal{N}(x) \cap C_X(S_1 \cup S_2) \neq \emptyset$ ,  $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$  and  $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$ .

Case 1.  $\mathcal{N}(x) \cap S_1 \neq \emptyset$ .

Since  $\mathcal{N}(x) \cap S_1 \neq \emptyset$  and  $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$  for any neighborhood  $\mathcal{N}(x)$  around  $x$ , by definition of boundary,  $x \in \text{bd}(S_1)$ .

Since  $x \in \text{bd}(S_1)$ ,  $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$ .

Case 2.  $\mathcal{N}(x) \cap S_2 \neq \emptyset$ .

Since  $\mathcal{N}(x) \cap S_2 \neq \emptyset$  and  $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$  for any neighborhood  $\mathcal{N}(x)$  around  $x$ , by definition of boundary,  $x \in \text{bd}(S_2)$ .

Since  $x \in \text{bd}(S_2)$ ,  $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$ .

Summary.

Since “ $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$ ” holds in all cases, we conclude that it is true.

Since

$$\forall x \in \text{bd}(S_1 \cup S_2), \quad x \in \text{bd}(S_1) \cup \text{bd}(S_2),$$

we get  $\text{bd}(S_1 \cup S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2)$ . ■

*Proof of (6).*

Let  $x$  be an arbitrary point in  $\text{bd}(S_1 \cap S_2)$ .

Since  $x \in \text{bd}(S_1 \cap S_2)$ , by definition of boundary, for any neighborhood  $\mathcal{N}(x)$  around  $x$ ,  $\mathcal{N}(x) \cap (S_1 \cap S_2) \neq \emptyset$  and  $\mathcal{N}(x) \cap C_X(S_1 \cap S_2) \neq \emptyset$ .

Since  $\mathcal{N}(x) \cap (S_1 \cap S_2) \neq \emptyset$ ,  $\mathcal{N}(x) \cap S_1 \neq \emptyset$  and  $\mathcal{N}(x) \cap S_2 \neq \emptyset$ .

Since  $\mathcal{N}(x) \cap C_X(S_1 \cap S_2) \neq \emptyset$ , either  $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$  or  $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$ .

Case 1.  $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$ .

Since  $\mathcal{N}(x) \cap S_1 \neq \emptyset$  and  $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$  for any neighborhood  $\mathcal{N}(x)$  around  $x$ , by definition of boundary,  $x \in \text{bd}(S_1)$ .

Since  $x \in \text{bd}(S_1)$ ,  $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$ .

Case 2.  $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$ .

Since  $\mathcal{N}(x) \cap S_2 \neq \emptyset$  and  $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$  for any neighborhood  $\mathcal{N}(x)$  around  $x$ , by definition of boundary,  $x \in \text{bd}(S_2)$ .

Since  $x \in \text{bd}(S_2)$ ,  $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$ .

Summary.

Since “ $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$ ” holds in all cases, we conclude that it is true.

Since

$$\forall x \in \text{bd}(S_1 \cap S_2), \quad x \in \text{bd}(S_1) \cup \text{bd}(S_2),$$

we get  $\text{bd}(S_1 \cap S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2)$ .

■

### 3.4 Examples

**Example 3.4.1.** *It is **not** true that if  $S$  is open, then  $\text{int}(\text{cl}(S)) = \text{int}(S)$ .*

*Consider the set  $\mathbb{R} \setminus \{0\}$ .*

**Example 3.4.2.** *It is **not** true that if  $S$  is closed, then  $\text{cl}(\text{int}(S)) = \text{cl}(S)$ .*

*Consider the set  $\{0\}$ .*



## 4

# Dense and Nowhere Dense

## 4.1 Dense

### 4.1.1 Definitions

**Definition** (Dense-1). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say that  $S$  is **dense** in the space if any point in the space is an adherent point of  $S$ , or equivalently, the closure of  $S$  equals the whole space.

**Definition** (Dense-2). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say that  $S$  is **dense** in the space if  $S$  has nonempty intersection with any nonempty open set in the space.

**Proposition 4.1.1.** The two definitions of dense sets are equivalent.

*Proof.*

For one direction, assume that  $cl(S) = X$ .

We are to prove that  $\forall$  nonempty open  $\mathcal{O}$ ,  $S \cap \mathcal{O} \neq \emptyset$ .

Let  $\mathcal{O}$  be an arbitrary nonempty open set in the space.

Assume for the sake of contradiction that  $\mathcal{O} \cap S = \emptyset$ .

Define  $\mathcal{C} := X \setminus \mathcal{O}$ .

Since  $\mathcal{O} \cap S = \emptyset$ ,  $\mathcal{C} \supseteq S$ .

Since  $\mathcal{O}$  is open,  $\mathcal{C}$  is closed.

Since  $\mathcal{C}$  is a closed superset of  $S$ ,  $cl(S) \subseteq \mathcal{C}$ .

Since  $cl(S) \subseteq \mathcal{C}$  and  $cl(S) = X$ ,  $\mathcal{C} \supseteq X$ .

Since  $\mathcal{C} = X \setminus \mathcal{O}$ ,  $\mathcal{C} \subseteq X$ .

Since  $\mathcal{C} \subseteq X$  and  $\mathcal{C} \supseteq X$ ,  $\mathcal{C} = X$ .

Since  $\mathcal{C} = X$ ,  $\mathcal{O} = \emptyset$ .

This contradicts to the assumption that  $\mathcal{O}$  is nonempty.

So the assumption that  $\mathcal{O} \cap S = \emptyset$  is false.

i.e.,  $\mathcal{O} \cap S \neq \emptyset$ .

For the reverse direction, assume that  $\forall$  nonempty open  $\mathcal{O}$ ,  $S \cap \mathcal{O} \neq \emptyset$ .

We are to prove that  $cl(S) = X$ .

Let  $\mathcal{C}$  be an arbitrary closed superset of  $S$ .

Define  $\mathcal{O} := X \setminus \mathcal{C}$ .

Since  $\mathcal{C}$  is closed,  $\mathcal{O}$  is open.

Since  $\mathcal{C} \supseteq S$ ,  $\mathcal{O} \cap S = \emptyset$ .

Assume for the sake of contradiction that  $\mathcal{C} \neq X$ .

Since  $\mathcal{C} \neq X$ ,  $\mathcal{O} \neq \emptyset$ .

Since  $\forall$  nonempty open  $\mathcal{O}$ ,  $S \cap \mathcal{O} \neq \emptyset$ , in particular,  $\mathcal{O} \cap S \neq \emptyset$ .

This contradicts to the fact that  $\mathcal{O} \cap S = \emptyset$ .

So the assumption that  $\mathcal{C} \neq X$  is false.

i.e.,  $\mathcal{C} = X$ .

Since  $\mathcal{C} = X$  for any closed superset  $\mathcal{C}$  of  $S$ ,  $cl(S) = X$ .

■

### 4.1.2 Properties

**Proposition 4.1.2** (Transitivity). *Denseness is transitive. i.e.: Let  $(X, \tau)$  be a topological space. Let  $S_1$  and  $S_2$  and  $S_3$  be subsets of  $X$ . Suppose  $S_1$  is dense in  $S_2$  and  $S_2$  is dense in  $S_3$ , then  $S_1$  is dense in  $S_3$ .*

**Proposition 4.1.3.** *A superset of a dense set is dense.*

**Proposition 4.1.4** (Images). *A continuous image of a dense set is dense in the range.*

*Proof.*

Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces.

Let  $f$  be a surjective function from  $X$  to  $Y$ .

Let  $S$  be a dense set in  $(X, \tau_X)$ .

Let  $T$  denote  $f(S)$ .

We are to prove that  $T$  is dense in  $(Y, \tau_Y)$ .

Let  $y$  be an arbitrary point in  $Y$ .

Since  $y \in Y$ , there exists some point  $x$  in  $X$  such that  $f(x) = y$ .

Since  $x \in X$  and  $S$  is dense in  $(X, \tau_X)$ , there exists a sequence  $\xi$  in  $S$  that converges to  $x$  in  $(X, \tau_X)$ .

Let  $\dagger$  denote  $f(\xi)$ .

Since  $\xi \subseteq S$ ,  $\dagger \subseteq T$ .



Since  $\S$  converges to  $x$  in  $(X, \tau_X)$  and  $f$  is continuous,  $\dagger$  converges to  $y$  in  $(Y, \tau_Y)$ .

Since for any point  $y$  in  $Y$ , there exists a sequence  $\dagger$  in  $T$  that converges to  $y$  in  $(Y, \tau_Y)$ , by definition of dense sets,  $T$  is dense in  $(Y, \tau_Y)$ . ■

## 4.2 Nowhere Dense

### 4.2.1 Definitions

**Definition** (Nowhere Dense). *Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say that  $S$  is **nowhere dense** in  $(X, \tau)$  if there is no nonempty open set in  $(X, \tau)$  in which  $S$  is dense.*

**Definition** (Nowhere Dense). *Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say that  $S$  is **nowhere dense** in  $(X, \tau)$  if  $\text{int}(\text{cl}(S)) = \emptyset$ .*

**Proposition 4.2.1.** *The two definitions of nowhere dense sets are equivalent.*

### 4.2.2 Properties

**Proposition 4.2.2** (Subspaces). *Let  $(X, \tau)$  be a topological space. Let  $(Y, \tau)$  be a subspace of  $(X, \tau)$ . Then*

- (1) *Let  $S$  be a nowhere dense set in  $(Y, \tau)$ . Then  $S$  is also nowhere dense in  $(X, \tau)$ .*
- (2) *Let  $S$  be a nowhere dense set in  $(X, \tau)$ . Then if  $Y$  is open in  $(X, \tau)$ ,  $S$  is also nowhere dense in  $(Y, \tau)$ .*

**Proposition 4.2.3.** *The nowhere dense sets in a space form an ideal of sets.*

**Proposition 4.2.4** (Closure). *A set is nowhere dense if and only if its closure is nowhere dense.*

**Proposition 4.2.5** (Exterior). *A set is nowhere dense if and only if its exterior is dense.*

*Proof. Sketch*

Let  $(X, \tau)$  be a topological space.

Let  $S$  be a set in the space.

$S$  is nowhere dense if and only if  $\text{int}(\text{cl}(S)) = \emptyset$ .

$\text{ext}(S)$  is dense if and only if  $\text{cl}(\text{ext}(S)) = X$ .

So it suffices to prove that  $\text{int}(\text{cl}(S)) = \emptyset$  if and only if  $\text{cl}(\text{ext}(S)) = X$ .

It suffices to prove that  $C_X(\text{int}(\text{cl}(S))) = \text{cl}(\text{int}(C_X(S)))$ .

$$\begin{aligned}
\text{int}(C_X(S)) &= C_X(\text{cl}(S)) \\
&\Downarrow \\
\text{cl}(\text{int}(C_X(S))) &= \text{cl}(C_X(\text{cl}(S))) \\
&\Downarrow \\
\text{cl}(\text{int}(C_X(S))) &= C_X(\text{int}(\text{cl}(S)))
\end{aligned}$$

This completes the proof. ■

**Proposition 4.2.6.** *A set is open and dense if and only if its complement is closed and nowhere dense.*

**Proposition 4.2.7.** *The boundary of an open set or of a closed set is nowhere dense.*

**Proposition 4.2.8** (not sure...). *A singleton set is nowhere dense if and only if the point in it is not an isolated point.*

# 5

## Meager and Residual

### 5.1 Definitions

**Definition** (Meager, or First Category). *Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say  $S$  is **meager** if  $S$  is the union of some countable collection of nowhere dense sets in the space.*

**Definition** (Residual). *Let  $(X, \tau)$  be a topological space. Let  $S$  be a subset of  $X$ . We say  $S$  is **residual** if  $S$  is the intersection of some countable collection of sets with dense interior.*

### 5.2 Sufficient Conditions

**Proposition 5.2.1.** *A set is meager if and only if its complement is residual.*

**Proposition 5.2.2** (Set Operations).

- (1) *A subset of a meager set is meager.*
- (2) *A countable union of meager sets is meager.*

*Proof.*

**Proof of (1).**

Let  $(X, \tau)$  be a topological space and  $S_1$  be a meager set in  $(X, \tau)$ .

Let  $S_2$  be a subset of  $S_1$ .

Since  $S_1$  is meager in  $(X, \tau)$ , by definition of meager, there exist a countable collection  $\{A_i\}_{i \in \mathbb{N}}$  of nowhere dense sets in  $(X, \tau)$  such that  $S_1 = \bigcup_{i \in \mathbb{N}} A_i$ .

Define  $B_i A_i \cap S_2$ .

Since  $S_2 \subseteq S_1$  and  $S_1 = \bigcup_{i \in \mathbb{N}} A_i$  and  $B_i = A_i \cap S_2$ ,  $S_2 = \bigcup_{i \in \mathbb{N}} B_i$ .

Since  $B_i \subseteq A_i$  and  $A_i$  is nowhere dense in  $(X, \tau)$ ,  $B_i$  is nowhere dense in  $(X, \tau)$ .

Since  $B_i$  is nowhere dense in  $(X, \tau)$  for any  $i \in \mathbb{N}$  and  $S_2 = \bigcup_{i \in \mathbb{N}} B_i$ , by definition of meager,  $S_2$  is meager in  $(X, \tau)$ . ■

**Proposition 5.2.3** (Set Operations). *A countable intersection of residual sets is residual.*

## 6

# Sequences

**Definition** (Sequence). Let  $(X, \tau)$  be a topological space. We define a **sequence** in  $X$ , denoted by  $\{x_i\}_{i \in I}$  where  $I$  is a subset of the natural numbers, to be a function from  $I$  to  $X$ .

### 6.1 Convergence of Sequences

**Definition** (Convergence). Let  $(X, \tau)$  be a topological space. Let  $\{x_i\}_{i \in \mathbb{N}}$  be a sequence in  $X$ . Let  $x_0$  be a point in  $X$ . We say that  $\{x_i\}_{i \in \mathbb{N}}$  **converges** to  $x_0$  if for any neighborhood  $\mathcal{N}_X(x_0)$  of  $x_0$  in  $X$ , there exists an integer  $N(\mathcal{N}_X(x_0))$  such that for any index  $i$  greater than  $N(\mathcal{N}_X(x_0))$ , we have  $x_i \in \mathcal{N}_X(x_0)$ .

**Proposition 6.1.1** (Uniqueness). Limits are unique in Hausdorff spaces.

*Proof.* (only in a metric space setting)

Let  $(X, d)$  be a metric space.

Let  $\{x_i\}_{i \in \mathbb{N}}$  be a sequence in the space.

Assume that  $\{x_k\}$  converges to two elements  $l_1$  and  $l_2$  in  $X$ .

Let  $\varepsilon$  be an arbitrary positive number.

Since  $\{x_i\}_{i \in \mathbb{N}}$  converges to  $l_1$ , there exists some cutoff  $N_1$  such that  $d(x_i, l_1) < \varepsilon/2$  whenever  $i > N_1$ .

Since  $\{x_i\}_{i \in \mathbb{N}}$  converges to  $l_2$ , there exists some cutoff  $N_2$  such that  $d(x_i, l_2) < \varepsilon/2$  whenever  $i > N_2$ .

Let  $i$  be some fixed index such that  $i > \max\{N_1, N_2\}$ .

Since  $i > N_1$ ,  $d(x_i, l_1) < \varepsilon/2$ .

Since  $i > N_2$ ,  $d(x_i, l_2) < \varepsilon/2$ .

Since  $d(x_i, l_1) < \varepsilon/2$  and  $d(x_i, l_2) < \varepsilon/2$ ,  $d(l_1, l_2) < \varepsilon$ .

Since  $d(l_1, l_2) < \varepsilon$  for any  $\varepsilon > 0$ ,  $d(l_1, l_2) = 0$ .

Since  $d(l_1, l_2) = 0$ ,  $l_1 = l_2$ . ■

**Proposition 6.1.2.** *Let  $(X, \tau)$  be a topological space. Let  $\{x_i\}_{i \in \mathbb{N}}$  be a sequence in  $X$ . Let  $x_0$  be a point in  $X$ . Then if  $\{x_i\}_{i \in \mathbb{N}}$  converges to  $x_0$ , any subsequence of  $\{x_i\}_{i \in \mathbb{N}}$  also converges to  $x_0$ .*

*Proof.*

Let  $\{x_i\}_{i \in I}$  be an arbitrary subsequence of  $\{x_i\}_{i \in \mathbb{N}}$ .

Let  $\mathcal{N}(x_0)$  be an arbitrary neighborhood of  $x_0$ .

Since  $\{x_i\}_{i \in \mathbb{N}}$  converges to  $x_0$ , there exists some cutoff  $N$  such that  $x_i \in \mathcal{N}(x_0)$  whenever  $i > N$ .

Since  $x_i \in \mathcal{N}(x_0)$  whenever  $i > N$ , in particular,  $x_i \in \mathcal{N}(x_0)$  whenever  $i \in I$  and  $i > N$ .

Since for any positive number  $\varepsilon$ , there exists some neighborhood  $\mathcal{N}(x_0)$  such that  $x_i \in \mathcal{N}(x_0)$  whenever  $i \in I$  and  $i > N$ ,  $\{x_i\}_{i \in I}$  converges to  $x_0$ . ■

**Proposition 6.1.3.** *Let  $(X, \tau)$  be a topological space. Let  $\{x_i\}_{i \in \mathbb{N}}$  be a sequence in  $X$ . Let  $x_0$  be a point in  $X$ . Then if any subsequence of  $\{x_i\}_{i \in \mathbb{N}}$  has a subsequence that converges to  $x_0$ ,  $\{x_i\}_{i \in \mathbb{N}}$  also converges to  $x_0$ .*

## 6.2 Cauchyness of Sequences

**Definition** (Cauchy Sequence). *Let  $(X, d)$  be a metric space. Let  $\{x_i\}_{i \in \mathbb{N}}$  be a sequence in  $X$ . We say that  $\{x_i\}_{i \in \mathbb{N}}$  is **Cauchy** if for any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m$  and  $n$  with  $m, n > N(\varepsilon)$ , we have  $d(x_m, x_n) < \varepsilon$ .*

**Proposition 6.2.1.** *Cauchy sequences are bounded.*

*Proof.*

Let  $(X, d)$  be a metric space.

Let  $\{x_i\}_{i \in \mathbb{N}}$  be a Cauchy sequence in  $X$ .

Let  $x_0$  be some fixed point in the space.

Let  $\varepsilon$  be some fixed positive number.

Since  $\{x_i\}_{i \in \mathbb{N}}$  is Cauchy, there exists some cutoff  $N$  such that  $d(x_m, x_n) < \varepsilon$  whenever  $m, n > N$ .

Let  $j$  be some fixed index such that  $j > N$ .

Since  $d(x_m, x_n) < \varepsilon$  for any  $m, n > N$ , in particular,  $d(x_i, x_j) < \varepsilon$  for any  $i > N$ .

Define  $M_j := d(x_j, x_0) + \varepsilon$ .

Define  $M_i := d(x_i, x_0)$  for  $i \in \{1, \dots, N\}$ .

Define  $M := \max\{M_i\}$ ,  $i \in \{1, \dots, N, j\}$ .

Since  $M = \max\{M_i\}$ ,  $i \in \{1, \dots, N, j\}$ ,  $M_i \leq M$  for  $i \in \{1, \dots, N, j\}$ .

Let  $i$  be an arbitrary index.

If  $i \in \{1, \dots, N\}$ , then  $d(x_i, x_0) = M_i \leq M$ .

If  $i > N$ , then  $d(x_i, x_0) \leq d(x_i, x_j) + d(x_j, x_0) < \varepsilon + d(x_j, x_0) = M_j \leq M$ .

Since  $d(x_i, x_0) \leq M$  for any  $i \in \mathbb{N}$ ,  $\{x_i\}_{i \in \mathbb{N}}$  is bounded.

■

**Proposition 6.2.2.** *Let  $(X, d)$  be a metric space. Let  $\{x_k\}$  be a Cauchy sequence in  $X$ . Then if there exists a convergent subsequence,  $\{x_k\}$  also converges to the same limit.*

*Proof.*

Let  $\{x_{n_k}\}_{k=1}^{\infty}$  be a subsequence of  $\{x_k\}$  that converges to point  $x$  in  $X$ .

By definition of convergence, for any positive number  $\varepsilon$ , there exists an integer  $N_1(\varepsilon)$  such that for any index  $k$  with  $k > N_1$ , we have  $d(x_{n_k}, x) < \varepsilon/2$ . (\*\*)

By definition of Cauchy, there exists an integer  $N_2(\varepsilon)$  such that for any indices  $m, n > N_2$ , we have  $d(x_m, x_n) < \varepsilon/2$ . (\*)

Take  $N = \max\{N_1, N_2\} + 1$ . Then we have  $N > N_1$ ,  $N > N_2$  and  $n_N > N_2$ .

Apply statement (\*) with  $k = N > N_1$ , we get  $d(x_{n_N}, x) < \varepsilon/2$ .

Apply statement (\*\*) with  $m > N > N_2$  and  $n = n_N > N_2$ , we get  $d(x_m, x_{n_N}) < \varepsilon/2$ .

Combining the preceding two inequalities, we get  $d(x_k, x) < \varepsilon$  for any  $k > N$ .

In short, for any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any index  $k$  with  $k > N$ , we have  $d(x_k, x) < \varepsilon$ .

By definition, we conclude that  $\{x_k\}$  converges to  $x$ .

■

**Proposition 6.2.3.** *Convergent sequences are Cauchy.*

*Proof.* Let  $(X, d)$  be a metric space and  $\{x_k\}$  be an arbitrary convergent sequence in  $X$ .

By definition of convergence, for all  $\varepsilon > 0$ , there exists an integer  $N$  such that for all  $k > N$ , we have  $d(x_k, x) < \varepsilon/2$ . (\*)

Let  $m, n > N$  be arbitrary.

Apply statement (\*) to  $m$  and  $n$ , we get

$$d(x_m, x) < \varepsilon/2 \#(1)$$

$$d(x_n, x) < \varepsilon/2 \#(2)$$

By the triangle inequality, we get

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) \#(3)$$

From inequations (1) ~ (3), we get

$$d(x_m, x_n) < \varepsilon$$

In short, we have proved that for all  $\varepsilon > 0$ , there exists an integer  $N$  such that for all  $m, n > N$ , we have  $d(x_m, x_n) < \varepsilon$ .

By definition,  $\{x_k\}$  is Cauchy.

■



# 7

## Nets

### 7.1 Definitions

**Definition (Net).** Let  $X$  be a set. Let  $\Lambda$  be a directed set. We define a **net**, denoted by  $\{x_\lambda\}_{\lambda \in \Lambda}$ , to be a function from  $\Lambda$  to  $X$ .

**Definition (Subnet).** Let  $X$  be a set. Let  $\Lambda$  be a directed set. Let  $P : \Lambda \rightarrow X$  be a net. We define a **subnet** of  $P$  to be a composition of a function  $\varphi$  and  $P$  where  $\varphi$  is an increasing and cofinal function from a directed set  $M$  to  $\Lambda$ . That is,

- $\forall \mu_1, \mu_2 \in M, \quad \mu_1 \leq \mu_2 \implies \varphi(\mu_1) \leq \varphi(\mu_2)$ , and
- $\forall \lambda \in \Lambda, \quad \exists \mu \in M, \quad \varphi(\mu) \geq \lambda$ .

### 7.2 Convergence of Nets

**Definition (Convergence).** Let  $(X, \tau)$  be a topological space. Let  $\{x_\lambda\}_{\lambda \in \Lambda}$  be a net in  $X$ . Let  $x$  be a point in  $X$ . We say that the net  $\{x_\lambda\}_{\lambda \in \Lambda}$  **converges** to the point  $x$ , denoted by  $\lim_{\lambda \in \Lambda} x_\lambda = x$ , if

$$\forall U \in \mathcal{U}_x, \quad \exists \lambda_0 \in \Lambda, \quad \lambda \geq \lambda_0 \implies x_\lambda \in U.$$

### 7.3 Cauchyness of Nets

**Proposition 7.3.1.** *Convergent nets are Cauchy.*

*Proof.* Let  $(X, \tau)$  be a topological space. Let  $\{x_\lambda\}_{\lambda \in \Lambda}$  be a convergent net in  $(X, \tau)$  where  $\Lambda$  is some directed index set. Define a point  $x$  in  $X$  as  $x := \lim_{\lambda \in \Lambda} x_\lambda$ . Let  $U$  be an arbitrary neighborhood of  $x$ . Since  $\{x_\lambda\}_{\lambda \in \Lambda}$  converges to  $x$ ,  $\exists \lambda_0 \in \Lambda$  such that  $\forall \lambda \in \Lambda, \quad \lambda \geq \lambda_0 \implies x_\lambda \in U$ . ■



## 8

# Continuous Functions

## 8.1 Continuity

**Definition** (Continuity). Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say that  $f$  is **continuous** if the preimage of any open (closed) set in  $(Y, \tau_Y)$  under  $f$  is open (closed) in  $(X, \tau_X)$ .

**Definition** (Continuity). Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say that  $f$  is **continuous** if  $f$  preserves convergence of nets.

**Proposition 8.1.1.** The two definitions of continuity are equivalent.

**Proposition 8.1.2** (Restriction). Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces. Let  $f$  be a continuous function from  $X$  to  $Y$ . Let  $S$  be a subset of  $X$ . Then the restriction  $f_S$  of  $f$  to  $(S, \tau_X)$  is continuous.

**Proposition 8.1.3** (Composition). Let  $(X, \tau_X)$ ,  $(Y, \tau_Y)$ , and  $(Z, \tau_Z)$  be topological spaces. Let  $f$  be a continuous map from  $X$  to  $Y$ . Let  $g$  be a continuous map from  $Y$  to  $Z$ . Then the composition  $g \circ f$  is a continuous map from  $X$  to  $Z$ .

**Proposition 8.1.4.** The limit, with respect to the infinity norm, of a convergent sequence of continuous functions is continuous.

## 8.2 Sequential Continuity

**Definition** (Sequential Continuity). Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say that  $f$  is **sequentially continuous** at a point  $x_0$  in  $X$  if  $f$  preserves convergence of sequences.

**Proposition 8.2.1.** Continuity implies sequential continuity.

*Proof.*

Let  $U$  be an arbitrary neighborhood of  $f(x_0)$  in  $Y$ .

Since  $U$  is a neighborhood of  $f(x_0)$ , by definition of neighborhood,  $U$  is open.

Since  $f$  is continuous and  $U$  is open, by definition of continuity,  $f^{-1}(U)$  is open in  $X$ .

Since  $U$  is a neighborhood of  $f(x_0)$ , by definition of neighborhood,  $f(x_0) \in U$ .

Since  $f(x_0) \in U$ ,  $x_0 \in f^{-1}(U)$ .

Since  $x_0 \in f^{-1}(U)$  and  $f^{-1}(U)$  is open in  $X$ ,  $f^{-1}(U)$  is a neighborhood of  $x_0$  in  $X$ .

Since  $f^{-1}(U)$  is a neighborhood of  $x_0$  in  $X$  and  $\{x_i\}_{i \in \mathbb{N}}$  converges to  $x_0$ , by definition of convergence, there exists an integer  $N(U)$  such that for any index  $i$  with  $i > N$ , we have  $x_i \in f^{-1}(U)$ .

Since for any index  $i$  with  $i > N$ ,  $x_i \in f^{-1}(U)$ , for any index  $i$  with  $i > N$ , we have  $f(x_i) \in U$ .

In short, we have proved that for any neighborhood  $U$  of  $f(x_0)$  in  $Y$ , there exists an integer  $N(U)$  such that for any index  $i$  with  $i > N$ , we have  $f(x_i) \in U$ .

By definition, we conclude that  $\{f(x_i)\}_{i \in \mathbb{N}}$  converges to  $f(x_0)$ . ■

**Proposition 8.2.2.** *Sequential continuity can imply continuity if the space is first-countable.*

### 8.3 Cauchy Continuity

**Definition** (Cauchy Continuous). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say  $f$  is **Cauchy continuous** if it preserves Cauchy-ness.*

**Proposition 8.3.1.** *Cauchy continuous functions are continuous.*

**Proposition 8.3.2.** *Continuous functions are Cauchy continuous if the domain is complete.*

### 8.4 Uniform Continuity

**Definition** (Uniform Continuity). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say that  $f$  is **uniformly continuous** on  $X$  if for all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all elements  $x$  and  $y$  in  $X$  such that  $d(x, y) < \delta$ , we have  $\rho(f(x), f(y)) < \varepsilon$ .*

**Proposition 8.4.1.** *Uniformly continuous functions are Cauchy continuous.*

*Proof.*

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces.

Let  $f$  be a uniformly continuous function from  $X$  to  $Y$ .

Let  $\{x_i\}_{i \in \mathbb{N}}$  be an arbitrary Cauchy sequence in  $(X, d_X)$ .

Let  $\varepsilon$  be an arbitrary positive number.

Since  $f$  is uniformly continuous, there exists a positive number  $\delta(\varepsilon)$  such that for any points  $x_1$  and  $x_2$  in  $X$ , if  $d_X(x_1, x_2) < \delta(\varepsilon)$ , then  $d_Y(f(x_1), f(x_2)) < \varepsilon$ .

Since  $\{x_i\}_{i \in \mathbb{N}}$  is Cauchy in  $(X, d_X)$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m$  and  $n$ , if  $m, n > N$ , then  $d_X(x_m, x_n) < \delta(\varepsilon)$ .

Since  $d_X(x_m, x_n) < \delta(\varepsilon)$ ,  $d_Y(f(x_m), f(x_n)) < \varepsilon$ .

Since for any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m$  and  $n$  such that  $m, n > N$ ,  $d_Y(f(x_m), f(x_n)) < \varepsilon$ ,  $\{f(x_i)\}_{i \in \mathbb{N}}$  is Cauchy in  $(Y, d_Y)$ . ■

**Proposition 8.4.2.** *Cauchy continuous functions are uniformly continuous if the domain is totally bounded.*

## 8.5 Lipschitz Continuity

**Definition** (Lipschitz). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say that  $f$  is **Lipschitz** on  $X$  if there exists a non-negative constant  $c$  such that for any points  $x_1$  and  $x_2$  in  $X$ , we have*

$$d_Y(f(x_1), f(x_2)) \leq c d_X(x_1, x_2).$$

**Definition** (Bi-Lipschitz). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say that  $f$  is **bi-Lipschitz** on  $X$  if there exist positive constants  $c_1$  and  $c_2$  such that for any points  $x_1$  and  $x_2$  in  $X$ , we have*

$$c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq c_2 d_X(x_1, x_2).$$

**Proposition 8.5.1.** *Lipschitz continuous functions are uniformly continuous.*

*Proof.*

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and  $f$  be a Lipschitz continuous function from  $X$  to  $Y$ .

Since  $f$  is Lipschitz continuous, by definition, there exists a non-negative constant  $c$  such that for any points  $x_1$  and  $x_2$  in  $X$ ,  $d_Y(f(x_1), f(x_2)) \leq c d_X(x_1, x_2)$ .

Since for any points  $x_1$  and  $x_2$  in  $X$ ,  $d_Y(f(x_1), f(x_2)) \leq c d_X(x_1, x_2)$ , for any positive number  $\varepsilon$  and any points  $x_1$  and  $x_2$  in  $X$ , if  $d_X(x_1, x_2) < \varepsilon/c$ , then  $d_Y(f(x_1), f(x_2)) < \varepsilon$ .

Since for any positive number  $\varepsilon$  and any points  $x_1$  and  $x_2$  in  $X$ , if  $d_X(x_1, x_2) < \varepsilon/c$ , then  $d_Y(f(x_1), f(x_2)) < \varepsilon$ , for any positive number  $\varepsilon$ , there exists a positive number  $\delta(\varepsilon)$  such that for any points  $x_1$  and  $x_2$  in  $X$ , if  $d_X(x_1, x_2) < \delta(\varepsilon)$ , then  $d_Y(f(x_1), f(x_2)) < \varepsilon$ .

Since for any positive number  $\varepsilon$ , there exists a positive number  $\delta(\varepsilon)$  such that for any points  $x_1$  and  $x_2$  in  $X$ , if  $d_X(x_1, x_2) < \delta(\varepsilon)$ , then  $d_Y(f(x_1), f(x_2)) < \varepsilon$ , by definition of uniform continuity,  $f$  is uniformly continuous.

■

**Proposition 8.5.2.** *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and let  $\{f_k\}_{k=1}^{\infty}$  be a sequence of Lipschitz continuous functions from  $X$  to  $Y$  with the sequence  $\{K_k\}_{k=1}^{\infty}$  of Lipschitz constants bounded by some non-negative number  $K$ . Then if  $\{f_k\}$  converges to some function  $f$ ,  $f$  is also Lipschitz continuous with Lipschitz constant also bounded by  $K$ .*

**Proposition 8.5.3.** *Bi-Lipschitz functions are injective.*

*Proof.*

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and  $f$  be a bi-Lipschitz function from  $X$  to  $Y$  with Lipschitz constants  $c_1$  and  $c_2$ .

Let  $x_1$  and  $x_2$  be distinct points in  $X$ .

Since  $f$  is bi-Lipschitz, by definition, we have  $c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq c_2 d_X(x_1, x_2)$ .

Assume for the sake of contradiction that  $f(x_1) = f(x_2)$ .

Since  $f(x_1) = f(x_2)$  and  $d_Y$  is a metric, we get  $d_Y(f(x_1), f(x_2)) = 0$ .

Since  $d_Y(f(x_1), f(x_2)) = 0$  and  $c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2))$ , we get  $c_1 d_X(x_1, x_2) = 0$ .

Since  $c_1 d_X(x_1, x_2) = 0$  and  $c_1 > 0$ , we get  $d_X(x_1, x_2) = 0$ .

Since  $d_X(x_1, x_2) = 0$  and  $d_X$  is a metric, by definition,  $x_1 = x_2$ .

This contradicts to the assumption that  $x_1 \neq x_2$ .

■

## 8.6 Other Forms of Continuity

**Definition** (Hölder continuous). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say that  $f$  is **Hölder continuous** with exponent  $\alpha$  on  $X$  if there exists a number  $K$  such that for any points  $x$  and  $y$  in  $X$ , we have  $d_Y(f(x), f(y)) \leq K(d_X(x, y))^\alpha$ .*

## 8.7 Oscillation in Metric Spaces

**Definition** (Oscillation). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f$  be a function from  $X$  to  $Y$ . Let  $x_0$  be a point in  $X$ . We define the **oscillation** of  $f$  at point  $x_0$ , denoted by  $\omega(f, x_0)$ , to be the number given by*

$$\omega(f, x_0) = \inf\{\omega(f, x_0, \delta) : \delta > 0\}$$

where  $\omega(f, x_0, \delta)$  is given by

$$\omega(f, x_0, \delta) = \sup\{d_Y(f(x_1), f(x_2)) : d_X(x_1, x_0) < \delta, d_X(x_2, x_0) < \delta\}.$$

**Proposition 8.7.1.** *A function is continuous at a point if and only if the oscillation at the point is 0.*

*Proof.*

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces.

Let  $f$  be a function from  $X$  to  $Y$ .

Let  $x_0$  be a point in  $X$ .

For one direction, assume that  $f$  is continuous at point  $x_0$ .

We are to prove that  $\omega(f, x_0) = 0$ .

Let  $\varepsilon$  be an arbitrary positive number.

Since  $f$  is continuous at point  $x_0$ , by definition of continuity, there exists some  $\delta(\varepsilon)$  such that  $d_Y(f(x), f(x_0)) < \varepsilon/4$  whenever  $d_X(x, x_0) < \delta(\varepsilon)$ .

Let  $x_1$  and  $x_2$  be arbitrary points such that  $d_X(x_1, x_0) < \delta(\varepsilon)$  and  $d_X(x_2, x_0) < \delta(\varepsilon)$ .

Since  $d_Y(f(x_1), f(x_0)) < \varepsilon/4$  and  $d_Y(f(x_2), f(x_0)) < \varepsilon/4$ , by the triangle inequality,  $d_Y(f(x_1), f(x_2)) < \varepsilon/2$ .

Since  $d_Y(f(x_1), f(x_2)) < \varepsilon/2$  for any  $x_1$  and  $x_2$  such that  $d_X(x_1, x_0) < \delta(\varepsilon)$  and  $d_X(x_2, x_0) < \delta(\varepsilon)$ , by definition of supremum,  $\omega(f, x_0, \delta(\varepsilon)) \leq \varepsilon/2$ .

Since for any positive number  $\varepsilon$ , there exists a  $\delta(\varepsilon)$  such that  $\omega(f, x_0, \delta(\varepsilon)) < \varepsilon$ , by definition of infimum,  $\omega(f, x_0) = 0$ .

For the reverse direction, assume that  $\omega(f, x_0) = 0$ .

We are to prove that  $f$  is continuous at point  $x_0$ .

Let  $\varepsilon$  be an arbitrary positive number.

Since  $\omega(f, x_0) = 0$ , by definition of infimum, there exists some  $\delta(\varepsilon)$  such that  $\omega(f, x_0, \delta(\varepsilon)) < \varepsilon$ .

Since  $\omega(f, x_0, \delta(\varepsilon)) < \varepsilon$ , by definition of supremum,  $d_Y(f(x_1), f(x_2)) < \varepsilon$  for any  $x_1$  and  $x_2$  such that  $d_X(x_1, x_0) < \delta(\varepsilon)$  and  $d_X(x_2, x_0) < \delta(\varepsilon)$ .

Since  $d_Y(f(x_1), f(x_2)) < \varepsilon$  for any  $x_1$  and  $x_2$  such that  $d_X(x_1, x_0) < \delta(\varepsilon)$  and  $d_X(x_2, x_0) < \delta(\varepsilon)$ , in particular,  $d_Y(f(x), f(x_0)) < \varepsilon$  for any point  $x$  such that  $d_X(x, x_0) < \delta(\varepsilon)$ .

Since for any positive number  $\varepsilon$ , there exists some  $\delta(\varepsilon)$  such that for any point  $x$  such that  $d_X(x, x_0) < \delta(\varepsilon)$ ,  $d_Y(f(x), f(x_0)) < \varepsilon$ , by definition of continuity,  $f$  is continuous at point  $x_0$ . ■

## 8.8 Isomorphisms

**Definition (Isometry).** *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say that  $f$  is **isometric** if for any points  $x_1$  and  $x_2$  in  $X$ , we have*

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

**Proposition 8.8.1.** *Isometries are injective.*

## 8.9 Homeomorphisms

### 8.9.1 Definitions

**Definition** (Homeomorphism). *Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces. Let  $f$  be a function from  $X$  to  $Y$ . We say that  $f$  is a **homeomorphism** if  $f$  is a bijection and both  $f$  and  $f^{-1}$  are continuous.*

### 8.9.2 Properties

**Proposition 8.9.1.** *Homeomorphisms preserve interior. i.e., Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces. Let  $f$  be a homeomorphism from  $X$  to  $Y$ . Let  $S$  be a subset of  $X$ . Then  $f(\text{int}(S)) = \text{int}(f(S))$ .*

*Proof.*

If  $S$  is empty, then both sides evaluate to the empty set and the equality holds.

Now assume  $S \neq \emptyset$ .

For one direction, let  $y$  be an arbitrary point in  $f(\text{int}(S))$ .

We are to prove that  $y \in \text{int}(f(S))$ .

Since  $y \in f(\text{int}(S))$ , there exists some point  $x$  in  $\text{int}(S)$  such that  $f(x) = y$ .

Since  $x \in \text{int}(S)$ , by definition of interior, there exists some open set  $U$  in  $(X, \tau_X)$  such that  $x \in U \subseteq S$ .

Let  $V$  denote  $f(U)$ .

Since  $U$  is open in  $(X, \tau_X)$  and  $f$  is a homeomorphism,  $V$  is open in  $(Y, \tau_Y)$ .

Since  $x \in U \subseteq S$ ,  $y \in V \subseteq f(S)$ .

Since there exists some open set  $V$  in  $(Y, \tau_Y)$  such that  $y \in V \subseteq f(S)$ , by definition of interior,  $y \in \text{int}(f(S))$ .

Since  $y \in \text{int}(f(S))$  for any  $y \in f(\text{int}(S))$ ,  $f(\text{int}(S)) \subseteq \text{int}(f(S))$ .

For the reverse direction, let  $y$  be an arbitrary point in  $\text{int}(f(S))$ .

We are to prove that  $y \in f(\text{int}(S))$ .

Let  $x$  denote  $f^{-1}(y)$ .

Since  $y \in \text{int}(f(S))$ , by definition of interior, there exists some open subset  $V$  of  $f(S)$  such that  $y \in V \subseteq f(S)$ .

Let  $U$  denote  $f^{-1}(V)$ .

Since  $V$  is open in  $(Y, \tau_Y)$  and  $f$  is a homeomorphism,  $U$  is open in  $(X, \tau_X)$ .

Since  $f$  is a homeomorphism,  $f^{-1}(f(S)) = S$ .

Since  $y \in V \subseteq f(S)$ ,  $x \in U \subseteq S$ .

Since there exists some open set  $U$  such that  $x \in U \subseteq S$ , by definition of interior,  $x \in \text{int}(S)$ .



Since  $x \in \text{int}(S)$ ,  $y \in f(\text{int}(S))$ .

Since  $y \in f(\text{int}(S))$  for any  $y \in \text{int}(f(S))$ ,  $\text{int}(f(S)) \subseteq f(\text{int}(S))$ . ■

**Proposition 8.9.2.** *Homeomorphisms preserve closure. i.e., Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces. Let  $f$  be a homeomorphism from  $X$  to  $Y$ . Let  $S$  be a subset of  $X$ . Then  $f(\text{cl}(S)) = \text{cl}(f(S))$ .*

*Proof.*

If  $S$  is empty, then both sides evaluate to the empty set and the equality holds.

Now assume  $S \neq \emptyset$ .

For one direction, let  $y$  be an arbitrary point in  $f(\text{cl}(S))$ .

We are to prove that  $y \in \text{cl}(f(S))$ .

Let  $F$  be an arbitrary closed superset of  $f(S)$ .

Let  $E$  denote  $f^{-1}(F)$ .

Since  $y \in f(\text{cl}(S))$ , there exists some point  $x$  in  $\text{cl}(S)$  such that  $f(x) = y$ .

Since  $F$  is closed in  $(Y, \tau_Y)$  and  $f$  is a homeomorphism,  $E$  is closed in  $(X, \tau_X)$ .

Since  $f$  is a homeomorphism,  $f^{-1}(f(S)) = S$ .

Since  $f(S) \subseteq F$  and  $f^{-1}(f(S)) = S$  and  $f^{-1}(F) = E$ , we get  $S \subseteq E$ .

Since  $x \in \text{cl}(S)$  and  $E$  is a closed superset of  $S$ , we get  $x \in E$ .

Since  $x \in E$  and  $f(x) = y$  and  $f(E) = F$ ,  $y \in F$ .

Since  $y \in F$  for any closed superset of  $f(S)$ , by definition of closure,  $y \in \text{cl}(f(S))$ .

For the reverse direction, let  $y$  be an arbitrary point in  $\text{cl}(f(S))$ .

We are to prove that  $y \in f(\text{cl}(S))$ .

Let  $x$  denote  $f^{-1}(y)$ .

Let  $E$  be an arbitrary closed superset of  $S$ .

Let  $F$  denote  $f(E)$ .

Since  $E$  is closed in  $(X, \tau_X)$  and  $f$  is a homeomorphism,  $F$  is closed in  $(Y, \tau_Y)$ .

Since  $S \subseteq E$  and  $f(E) = F$ ,  $f(S) \subseteq F$ .

Since  $F$  is a closed superset of  $f(S)$  and  $y \in \text{cl}(f(S))$ , by definition of closure,  $y \in F$ .

Since  $y \in F$  and  $f^{-1}(y) = x$  and  $f^{-1}(F) = E$ ,  $x \in E$ .

Since  $x \in E$  for any closed superset  $E$  of  $S$ , by definition of closure,  $x \in \text{cl}(S)$ .

Since  $x \in \text{cl}(S)$  and  $y = f(x)$ ,  $y \in f(\text{cl}(S))$ . ■

**Proposition 8.9.3.** *Homeomorphisms preserve nowhere denseness.*

*Proof.*

Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces.

Let  $f$  be a homeomorphism from  $X$  to  $Y$ .

Let  $S$  be a nowhere dense set in  $(X, \tau_X)$ .

We are to prove that  $f(S)$  is nowhere dense in  $(Y, \tau_Y)$ .

Assume for the sake of contradiction that  $f(S)$  is not nowhere dense in  $(Y, \tau_Y)$ .

Since  $f(S)$  is not nowhere dense in  $(Y, \tau_Y)$ , by definition of nowhere dense, there exists some nonempty open set  $V$  in  $(Y, \tau_Y)$  such that  $f(S)$  is dense in  $V$ .

Let  $E$  denote  $f^{-1}(V)$ .

Since  $f(S)$  is dense in  $V$ ,  $V \subseteq cl(f(S))$ .

Since  $V$  is open in  $(Y, \tau_Y)$  and  $f$  is a homeomorphism,  $E$  is open in  $(X, \tau_X)$ .

Since  $V \subseteq cl(f(S))$  and  $f^{-1}(V) = E$ ,  $E \subseteq f^{-1}(cl(f(S)))$ .

Since  $f$  is a homeomorphism,  $f^{-1}$  is also a homeomorphism.

Since  $f^{-1}$  is a homeomorphism,  $f^{-1}(cl(f(S))) = cl(f^{-1}(f(S)))$ .

Since  $f$  is a homeomorphism,  $f^{-1}(f(S)) = S$ .

Since  $E \subseteq f^{-1}(cl(f(S)))$  and  $f^{-1}(cl(f(S))) = cl(f^{-1}(f(S)))$  and  $f^{-1}(f(S)) = S$ ,  $E \subseteq cl(S)$ .

Since  $E \subseteq cl(S)$ , by definition of denseness,  $S$  is dense in  $E$ .

Since  $S$  is dense in  $E$  and  $E$  is open in  $(X, \tau_X)$ , by definition of nowhere dense,  $S$  is not nowhere dense in  $(X, \tau_X)$ .

This contradicts to the assumption that  $S$  is nowhere dense in  $(X, \tau_X)$ .

So the assumption that  $f(S)$  is not nowhere dense in  $(Y, \tau_Y)$  is false.

i.e.,  $f(S)$  is nowhere dense in  $(Y, \tau_Y)$ . ■

**Proposition 8.9.4.** *Homeomorphisms preserve meager sets.*

# 9

## Fixed Point Theory

### 9.1 Definitions

**Definition** (Contraction Maps). *Let  $(X, d_X)$  be a metric space. Let  $f$  be a function from  $X$  to  $X$ . We say that  $f$  is a **contraction map** if  $\exists c \in [0, 1)$  such that*

$$\forall x_1, x_2 \in X, \quad d(f(x_1), f(x_2)) \leq c \cdot d(x_1, x_2).$$

**Definition** (Non-Expansive Operator). *Let  $(X, d)$  be a metric space. Let  $f$  be a function from  $X$  to  $X$ . We say that  $f$  is **non-expansive** if*

$$\forall x_1, x_2 \in X, \quad d(f(x_1), f(x_2)) \leq d(x_1, x_2).$$

*Or equivalently,  $\exists c \in [0, 1]$  such that*

$$\forall x_1, x_2 \in X, \quad d(f(x_1), f(x_2)) \leq c \cdot d(x_1, x_2).$$

**Definition** (Firmly Non-Expansive Operator). *Let  $\mathcal{H}$  be a Hilbert space. Let  $f$  be a function from  $\mathcal{H}$  to  $\mathcal{H}$ . We say that  $f$  is **firmly non-expansive** if*

$$\forall x, y \in \mathcal{H}, \quad \langle f(y) - f(x), f(y) - f(x) \rangle \leq \langle y - x, f(y) - f(x) \rangle.$$

*Or equivalently,*

$$\forall x, y \in \mathcal{H}, \quad \|f(y) - f(x)\|^2 \leq \langle y - x, f(y) - f(x) \rangle.$$

**Definition** (Firmly Non-Expansive Operator). *Let  $\mathcal{H}$  be a Hilbert space. Let  $f$  be a function from  $\mathcal{H}$  to  $\mathcal{H}$ . We say that  $f$  is **firmly non-expansive** if*

$$\forall x, y \in \mathcal{H}, \quad \|f(y) - f(x)\|^2 + \|(I - f)(y) - (I - f)(x)\|^2 \leq \|y - x\|^2.$$

**Definition** (Firmly Non-Expansive Operator). *Let  $\mathcal{H}$  be a Hilbert space. Let  $f$  be a function from  $\mathcal{H}$  to  $\mathcal{H}$ . We say that  $f$  is **firmly non-expansive** if*

$$\forall x, y \in \mathcal{H}, \quad \langle f(y) - f(x), (I - f)(y) - (I - f)(x) \rangle \geq 0.$$

**Definition** (Averaged Non-Expansive Operator). *Let  $\mathcal{H}$  be a Hilbert space. Let  $f$  be a function from  $\mathcal{H}$  to  $\mathcal{H}$ . We say that  $f$  is a  **$\alpha$ -averaged non-expansive operator** if  $\exists \alpha \in (0, 1)$  and  $\exists$  non-expansive operator  $N$  from  $\mathcal{H}$  to  $\mathcal{H}$  such that*

$$f = \alpha N + (1 - \alpha)I$$

where  $I$  denoted the identity operator from  $\mathcal{H}$  to  $\mathcal{H}$ .

## 9.2 Properties

**Proposition 9.2.1.** *Firmly non-expansive operators are special cases of averaged non-expansive operators when  $\alpha = \frac{1}{2}$ .*

**Proposition 9.2.2.** *Averaged non-expansive operators are non-expansive.*

*Proof.* Let  $T$  be an  $\alpha$ -average of a non-expansive operator  $N$ . i.e.  $T = (1 - \alpha)\text{id} + \alpha N$ . Let  $x$  and  $y$  be arbitrary points in  $\mathcal{H}$ . Then

$$\begin{aligned} & \|Tx - Ty\| \\ &= \|[(1 - \alpha)\text{id} + \alpha N]x - [(1 - \alpha)\text{id} + \alpha N]y\| \\ &= \|[(1 - \alpha)x + \alpha Nx] - [(1 - \alpha)y + \alpha Ny]\| \\ &= \|(1 - \alpha)(x - y) + \alpha(Nx - Ny)\| \\ &\leq \|(1 - \alpha)(x - y)\| + \|\alpha(Nx - Ny)\| \\ &= (1 - \alpha)\|x - y\| + \alpha\|Nx - Ny\| \\ &\leq (1 - \alpha)\|x - y\| + \alpha\|x - y\| \\ &\leq \|x - y\|. \end{aligned}$$

That is,

$$\forall x, y \in \mathcal{H}, \quad \|Tx - Ty\| \leq \|x - y\|.$$

So by definition,  $T$  is non-expansive. ■

**Proposition 9.2.3.** *If  $f$  is linear, then  $f$  being firmly non-expansive is equivalent to the followings.*

- $\forall x \in \mathbb{R}^d, \|fx\|^2 \leq \langle x, fx \rangle$ .
- $\forall x \in \mathbb{R}^d, \langle fx, x - fx \rangle \geq 0$ .

### 9.3 Algebra & Stability of Non-Expansiveness

**Proposition 9.3.1.**  *$f$  is firmly non-expansive if and only if  $I - f$  is firmly non-expansive.*

**Proposition 9.3.2.** *Let  $M$  be a linear operator from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ . Then  $M$  is non-expansive if and only if  $M^T$  is non-expansive.*

*Proof.* ■

**Proposition 9.3.3** (Convex Combination). *The class of firmly non-expansive maps is closed under convex combinations.*

**Proposition 9.3.4** (Composition). *Let  $f_1$  be an  $\alpha_1$ -averaged non-expansive operator. Let  $f_2$  be an  $\alpha_2$ -averaged non-expansive operator. Define an operator  $f$  by  $f := f_1 f_2$ . Then  $f$  is also averaged non-expansive with “rate”*

$$\alpha_f = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2}.$$

### 9.4 Fixed Points

**Proposition 9.4.1.** *Contraction maps has at most one fixed point.*

**Proposition 9.4.2.** *Contraction maps on a complete space has a unique fixed point.*

*Proof.*

**Part 1: construction**

Let  $(X, d)$  be a complete metric space and  $f$  be a contraction map on  $X$ .

Let  $x_0$  be a point in  $X$ . Construct a sequence  $\{x_k\}_{k=1}^\infty$  by  $x_k = f(x_{k-1})$ . We are to prove that  $\{x_k\}$  converges to some point  $x^*$  in  $X$ .

$$d(x_m, x_n) \leq \sum_{k=n+1}^m d(x_k, x_{k-1}) \leq \sum_{k=n+1}^m K^{k-1} d(x_1, x_0) = \frac{K^n - K^m}{1 - K} d(x_1, x_0)$$

Since  $K < 1$ , for any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m$  and  $n$  with  $m, n > N$ , we have  $K^n - K^m < \varepsilon(1 - K)/d(x_1, x_0)$ .

Since  $d(x_m, x_n) < (K^n - K^m)d(x_1, x_0)/(1 - K)$  and  $K^n - K^m < \varepsilon(1 - K)/d(x_1, x_0)$ , we get  $d(x_m, x_n) < \varepsilon$ .

In short, we have proved that for any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m$  and  $n$  with  $m, n > N$ , we have  $d(x_m, x_n) < \varepsilon$ .

By definition,  $\{x_k\}$  is Cauchy in  $X$ .

Since  $\{x_k\}$  is Cauchy in  $X$  and  $X$  is complete, by definition of completeness,  $\{x_k\}$  converges to some point  $x^*$  in  $X$ .

**Part 2: existence.**

Since  $f$  is a contraction map,  $f$  is continuous.

Since  $f$  is continuous and  $x_k \rightarrow x^*$ ,  $f(x^*) = f(x_k) = f(x_k)$ .

Since  $x_k \rightarrow x^*$  and  $x_k = f(x_{k-1})$ ,  $f(x_k) = x_k = x^*$ .

Since  $f(x^*) = f(x_k)$  and  $f(x_k) = x^*$ ,  $f(x^*) = x^*$ .

**Part 3: uniqueness.**

Assume that there exists a point  $x'$  in  $X$  such that  $f(x') = x'$ .

Compute  $d(x^*, x') = d(f(x^*), f(x')) \leq Kd(x^*, x')$ .

That is,  $(1 - K)d(x^*, x') \leq 0$ .

Since  $K < 1$  and  $(1 - K)d(x^*, x') \leq 0$ ,  $d(x^*, x') = 0$ .

It follows that  $x^* = x'$ .

Thus  $x^*$  is a unique fixed point. ■

## 9.5 Fejér Monotonic Sequences

**Definition** (Fejér Monotonic Sequences). *Let  $\mathcal{H}$  be a real Hilbert space. Let  $S$  be a non-empty subset of  $\mathcal{H}$ . Let  $\{x_i\}_{i \in \mathbb{N}}$  be a sequence in  $\mathcal{H}$ . We say that  $\{x_i\}_{i \in \mathbb{N}}$  is **Fejér monotonic** with respect to  $S$  if  $\forall p \in S$ , the sequence  $\{\|x_i - p\|\}_{i \in \mathbb{N}}$  is decreasing. i.e.*

$$\forall p \in S, \forall i \in \mathbb{N}, \quad \|x_{i+1} - p\| \leq \|x_i - p\|.$$

**Proposition 9.5.1.** *Fejér monotonic sequences are bounded.*

*Proof.*

Let  $\mathcal{H}$  be a real Hilbert space.

Let  $S$  be a non-empty subset of  $\mathcal{H}$ .

Let  $\mathfrak{x}$  be a Fejér monotonic sequence in  $\mathcal{H}$  with respect to  $S$ .

Let  $p$  be an arbitrary point in  $S$ .

$$\begin{aligned} \|x_i\| &\leq \|p\| + \|x_i - p\| \\ &\leq \|p\| + \|x_{i-1} - p\| \\ &\leq \dots \\ &\leq \|p\| + \|x_1 - p\|. \end{aligned}$$

Since  $\forall i \in \mathbb{N}$ ,  $\|x_i\| \leq \|p\| + \|x_1 - p\|$ , we get  $\{x_i\}_{i \in \mathbb{N}}$  is bounded. ■

**Proposition 9.5.2.** *Let  $\mathcal{H}$  be a real Hilbert space. Let  $S$  be a non-empty subset of  $\mathcal{H}$ . Let  $\{x_i\}_{i \in \mathbb{N}}$  be a Fejér monotonic sequence in  $\mathcal{H}$  with respect to  $S$ . Then  $\forall p \in S$ , the sequence  $\{x_i - p\}_{i \in \mathbb{N}}$  converges.*

*Proof.*  $\|x_i - p\|_{i \in \mathbb{N}}$  is a decreasing sequence bounded below by 0. ■





# 10

## Separation Axioms

### 10.1 Definitions

**Definition** (Topologically Distinguishable Points). *Let  $(X, \tau)$  be a topological space. Let  $x$  and  $y$  be two points in the space. We say that the points  $x$  and  $y$  are **topologically distinguishable** if*

$$(\exists G \in \tau, x \in G \text{ and } y \notin G) \text{ or } (\exists G \in \tau, y \in G \text{ and } x \notin G).$$

**Definition** (Separated Sets). *Let  $(X, \tau)$  be a topological space. Let  $A$  and  $B$  be two sets in the space. We say that  $A$  and  $B$  are **separated** by  $\tau$  if*

$$\exists U, V \in \tau : U \cap V = \emptyset, \quad A \subseteq U \text{ and } B \subseteq V.$$

**Definition** ( $T_0$  Space). *Let  $(X, \tau)$  be a topological space. We say that the space is  **$T_0$** , or **Kolmogorov**, if any two distinct points in the space are topologically distinguishable. i.e.,*

$$\forall x, y \in X : x \neq y, \quad (\exists U \in \tau, x \in U, y \notin U) \text{ or } (\exists V \in \tau, y \in V, x \notin V).$$

**Definition** ( $T_1$  Space). *Let  $(X, \tau)$  be a topological space. We say that the space is  **$T_1$** , or **Fréchet**, if*

$$\forall x, y \in X : x \neq y, \quad (\exists U \in \tau, x \in U, y \notin U) \text{ and } (\exists V \in \tau, y \in V, x \notin V).$$

**Definition** ( $T_2$  Space). *Let  $(X, \tau)$  be a topological space. We say that the space is  **$T_2$** , or **Hausdorff**, if any two distinct points in the space are separated by  $\tau$ . i.e.,*

$$\forall x, y \in X : x \neq y, \quad (\exists U \in \tau, x \in U) \text{ and } (\exists V \in \tau, y \in V) \text{ and } (U \cap V = \emptyset).$$

**Definition** (Regular Space). *Let  $(X, \tau)$  be a topological space. We say that the space is **regular** if any closed set  $F$  and a point  $x \notin F$  are separated by  $\tau$ . i.e.,*

$$\forall F \subseteq X, x \in X : F \text{ is closed and } x \notin F, \quad (\exists U \in \tau, F \subseteq U) \text{ and } (\exists V \in \tau, x \in V) \text{ and } (U \cap V = \emptyset).$$

**Definition** (Normal Space). Let  $(X, \tau)$  be a topological space. We say that the space is **normal** if any two disjoint closed sets in the space are separated by  $\tau$ .

$$\forall E, F \subseteq X : E \text{ and } F \text{ are closed and } E \cap F = \emptyset, \quad (\exists U \in \tau, E \subseteq U) \text{ and } (\exists V \in \tau, F \subseteq V) \text{ and } (U \cap V = \emptyset).$$

**Definition** ( $T_3$  Space). Let  $(X, \tau)$  be a topological space. We say that the space is  **$T_3$**  if it is  $T_1$  and regular.

**Definition** ( $T_4$  Space). Let  $(X, \tau)$  be a topological space. We say that the space is  **$T_4$**  if it is  $T_1$  and normal.

## 10.2 Sufficient Conditions

**Proposition 10.2.1.** Compact  $T_2$  spaces are normal.

*Proof.* Let  $(X, \tau)$  be a topological space. Suppose that the space is compact and  $T_2$ . We are to prove that the space is normal. Let  $E$  and  $F$  be two arbitrary disjoint closed sets in the space. Let  $x$  be an arbitrary point in  $E$ . Let  $y$  be an arbitrary point in  $F$ . Since  $E \cap F = \emptyset$  and  $x \in E$  and  $y \in F$ ,  $x \neq y$ . Since  $(X, \tau)$  is  $T_2$  and  $x \neq y$ ,  $x$  and  $y$  are separated. So  $\exists U, V \in \tau : U \cap V = \emptyset$  such that  $x \in U, y \notin U, y \in V, x \notin V$ . The sets  $\{V(x, y)\}_{y \in F}$  form an open cover of  $F$ . Since the space is compact and  $F$  is closed,  $F$  has a finite subcover. Say  $F'$  is a finite subset of  $F$  such that  $\{V(x, y)\}_{y \in F'}$  is a finite open cover of  $F$ . Define sets  $\mathcal{U}(x)$  and  $\mathcal{V}(x)$  as

$$\mathcal{U}(x) := \bigcap_{y \in F'} U(x, y) \text{ and } \mathcal{V}(x) := \bigcup_{y \in F'} V(x, y).$$

Then  $\mathcal{U}(x)$  and  $\mathcal{V}(x)$  form a separation of  $x$  and  $F$ . The sets  $\{\mathcal{U}(x)\}_{x \in E}$  form an open cover of  $E$ . Since the space is compact and  $E$  is closed,  $E$  has a finite subcover. Say  $E'$  is a finite subset of  $E$  such that  $\{\mathcal{U}(x)\}_{x \in E'}$  is a finite open cover of  $E$ . Define sets  $\mathfrak{U}$  and  $\mathfrak{V}$  as

$$\mathfrak{U} := \bigcup_{x \in E'} \mathcal{U}(x) \text{ and } \mathfrak{V} := \bigcap_{x \in E'} \mathcal{V}(x).$$

Then  $\mathfrak{U}$  and  $\mathfrak{V}$  form a separation of  $E$  and  $F$ . Since any two disjoint closed sets can be separated by  $\tau$ , by definition, the space is normal. ■

## 10.3 Examples

**Example 10.3.1.** Metric spaces are  $T_4$ .

# 11

## Countability

### 11.1 Definitions

**Definition** (First Countability). *We say that a topological space is first-countable if every point has a countable local base.*

**Definition** (Second Countability). *We say that a topological space is second-countable if it has a countable base.*

### 11.2 Properties

**Proposition 11.2.1.** *Second-countable spaces are separable.*

*Proof.* Let  $(X, \tau)$  be a non-empty second-countable topological space.

We are to prove that  $(X, \tau)$  is separable.

Let  $G$  be an arbitrary non-empty open set in  $(X, \tau)$ .

Since  $(X, \tau)$  is second-countable, by definition of second-countability, there exists a countable basis  $\mathcal{B} = \{B_i\}_{i \in I_{\mathcal{B}}} \cup \{\emptyset\}$  for  $(X, \tau)$  where  $I_{\mathcal{B}}$  is a non-empty subset of  $\mathbb{N}$  and each  $B_i$  is not empty.

Let  $x_i$  be some point in  $B_i$ .

Let  $D = \{x_i\}_{i \in I_{\mathcal{B}}}$ .

Since  $\mathbb{N}$  is countable,  $D$  is countable.

Since  $G$  is a non-empty open set in  $(X, \tau)$  and  $\mathcal{B}$  is a basis for  $(X, \tau)$ ,  $G$  can be written as  $G = \bigcup_{i \in I'_{\mathcal{B}}(G)} B_i$  where  $I'_{\mathcal{B}}(G)$  is a non-empty subset of  $I_{\mathcal{B}}$ .

Since  $x_i \in B_i$  for each  $i \in I'_{\mathcal{B}}(G)$  and  $G = \bigcup_{i \in I'_{\mathcal{B}}(G)} B_i$ ,  $x_i \in G$  for each  $i \in I'_{\mathcal{B}}(G)$ .

Since  $x_i \in G, D$  for each  $i \in I'_{\mathcal{B}}(G)$ ,  $G \cap D$  is not empty.

Since  $D$  intersects any non-empty open set in  $(X, \tau)$  non-trivially, by definition of density,  $D$  is dense in  $(X, \tau)$ .

Since  $D$  is countable and dense in  $(X, \tau)$ , by definition of separability,  $(X, \tau)$  is separable. ■

**Proposition 11.2.2.** *Second-countable spaces are Lindelöf.*

*Proof.* Let  $(X, \tau)$  be a second-countable space.

We are to prove that  $(X, \tau)$  is Lindelöf.

Since  $(X, \tau)$  is second-countable, by definition of second-countability, there exists a countable basis  $\mathcal{B} = \{B_i\}_{i \in I_{\mathcal{B}}}$  for  $(X, \tau)$  where  $I_{\mathcal{B}}$  is a non-empty subset of  $\mathbb{N}$ .

Let  $\mathcal{U} = \{U_{\lambda}\}_{\lambda \in \Lambda}$  be an arbitrary open cover of  $(X, \tau)$ .

Since each  $U_{\lambda}$  is open in  $(X, \tau)$  and  $\mathcal{B}$  is a basis for  $(X, \tau)$ , each  $U_{\lambda}$  can be written as  $U_{\lambda} = \bigcup_{i \in I'_{\mathcal{B}}(\lambda)} B_i$  where  $I'_{\mathcal{B}}(\lambda)$  is a non-empty subset of  $I_{\mathcal{B}}$ .

Let  $I = \bigcup_{\lambda \in \Lambda} I'_{\mathcal{B}}(\lambda)$ .

Since  $I'_{\mathcal{B}}(\lambda) \subseteq I_{\mathcal{B}}$  for each  $\lambda \in \Lambda$ ,  $I \subseteq I_{\mathcal{B}}$ .

Since  $I$  is countable,  $\{B_i\}_{i \in I}$  is countable.

Since  $X = \bigcup_{\lambda \in \Lambda} U_{\lambda}$  and  $U_{\lambda} = \bigcup_{i \in I'_{\mathcal{B}}(\lambda)} B_i$  for each  $\lambda \in \Lambda$ ,  $X = \bigcup_{i \in I} B_i$ .

Since any open cover of  $X$  has a countable subcover, by definition of Lindelöf,  $(X, \tau)$  is Lindelöf. ■

### 11.3 Sufficient Conditions

**Proposition 11.3.1.** *Subspaces of a second-countable space are second-countable.*

**Proposition 11.3.2** (Product). *The product of a countable collection of second-countable spaces is second-countable.*

# 12

## Separability

### 12.1 Definitions

**Definition** (Separable). Let  $(X, d)$  be a metric space. Let  $S$  be a subset of  $X$ . We say that  $S$  is **separable** if  $S$  has a countable dense subset.

### 12.2 Properties

**Proposition 12.2.1.** *Separable metric spaces are Lindelöf.*

*Proof.* Let  $(X, d)$  be a metric space and  $S$  be a separable subset of  $X$ .

Let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover of  $S$ .

Since  $S$  is separable,  $S$  has a countable dense subset  $S'$ .

$\{U_\alpha\}$  is an open cover of  $S'$ .

Find for each element in  $S'$  a  $U_\alpha$ . Construct  $\{U_{\alpha_k}\}_{k=1}^\infty$ . Then  $\{U_{\alpha_k}\}$  is an open cover of  $S'$ .

Since  $S'$  is dense, for any  $x \in S$  and any neighborhood  $N(x)$ , there exists  $x_0 \in S' \cap N(x)$ . ■

**Proposition 12.2.2.** *Separable metric spaces are second-countable.*

*Proof.* Let  $(X, d)$  be a non-empty separable metric space.

We are to prove that  $(X, d)$  is second-countable.

Since  $(X, d)$  is separable, there exists a countable dense subset  $D = \{d_i\}_{i \in \mathbb{N}}$  of  $X$ .

Let  $\mathcal{B} = \{\text{ball}(d_i, 1/n) : d_i \in D, n \in \mathbb{N}\}$ .

Since  $D$  and  $\mathbb{N}$  are countable,  $\mathcal{B}$  is countable.

Let  $G$  be an arbitrary non-empty open set in  $(X, d)$ .

Let  $x$  be an arbitrary point in  $G$ .

Since  $x \in G$  and  $G$  is open in  $(X, d)$ , there exists some radius  $r(x)$  of the form  $r(x) = 1/n$  where  $n \in \mathbb{N}$  such that  $\text{ball}(x, 2r(x)) \subseteq G$ .

Since  $\text{ball}(x, r(x))$  is open in  $(X, d)$  and  $D$  is dense in  $(X, d)$ , by definition of density, there exists some  $d_{i_0(x)}$  in  $D$  such that  $d_{i_0(x)} \in \text{ball}(x, r(x))$ .

Since  $d_{i_0(x)} \in \text{ball}(x, r(x))$ ,  $x \in \text{ball}(d_{i_0(x)}, r(x))$ .

Since  $d_{i_0(x)} \in \text{ball}(x, r(x))$ ,  $\text{ball}(d_{i_0(x)}, r(x)) \subseteq \text{ball}(x, 2r(x))$ .

Since  $\text{ball}(d_{i_0(x)}, r(x)) \subseteq \text{ball}(x, 2r(x))$  and  $\text{ball}(x, 2r(x)) \subseteq G$ ,  $\text{ball}(d_{i_0(x)}, r(x)) \subseteq G$ .

Since for any  $x \in G$ , there exists some open ball  $B(x)$  in  $\mathcal{B}$  such that  $x \in B(x) \subseteq G$ ,  $G = \bigcup_{x \in G} B(x)$ .

Since any non-empty open set in  $(X, d)$  can be written as a union of open balls in  $\mathcal{B}$ ,  $\mathcal{B} \cup \{\emptyset\}$  is a basis.

Since  $\mathcal{B} \cup \{\emptyset\}$  is a countable set and  $\mathcal{B} \cup \{\emptyset\}$  is a basis for  $(X, d)$ , by definition of second-countability,  $(X, d)$  is second-countable. ■

## 12.3 Stability of Separability

**Proposition 12.3.1.** *An open subspace of a separable space is separable.*

**Proposition 12.3.2.** *Subsets of a separable metric space are also separable.*

**Proposition 12.3.3** (Images). *A continuous image of a separable space is also separable.*

*Proof.* Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological spaces.

Let  $f$  be a continuous function from  $(X, \tau_X)$  to  $(Y, \tau_Y)$ .

Let  $(S, \tau_X)$  be a separable subspace of  $(X, \tau_X)$ .

Let  $T$  denote  $f(S)$ .

We are to prove that  $(T, \tau_Y)$  is separable.

Since  $(S, \tau_X)$  is separable, by definition, there exists a countable dense set  $S'$  in  $(S, \tau_X)$ .

Let  $T'$  denote  $f(S')$ .

Since  $S'$  is countable,  $T'$  is countable.

Since  $S'$  is dense in  $(S, \tau_X)$ ,  $T'$  is dense in  $(T, \tau_Y)$ .

Since  $T'$  is a countable dense set in  $(T, \tau_Y)$ , by definition,  $(T, \tau_Y)$  is separable. ■

**Proposition 12.3.4** (Set Operations).

(1) *The union of a countable collection of separable spaces is separable.*

## 12.4 Separation Properties

**Proposition 12.4.1.** *Let  $(X, d)$  be a metric space and  $\mathcal{B}$  be a set of open subsets of  $X$ . Then  $\mathcal{B}$  is a basis for  $(X, d)$  if and only if for all  $x \in X$  and all open neighborhood  $U$  of  $x$ , there exists a set  $B_0 \in \mathcal{B}$  such that  $x \in B_0 \subseteq U$ .*

**Proposition 12.4.2.** *Let  $(X, d)$  be a metric space. Let  $D$  be a subset of  $X$ . Then  $D$  is dense if and only if for any element  $x$  in  $X$  and any radius  $r$ ,  $B(x, r) \cap D \neq \emptyset$ .*





# 13

## Totally Bounded Space

### 13.1 Diameter

**Definition (Diameter).** Let  $(X, d)$  be a metric space. Let  $S$  be a set in the space. We define the **diameter** of  $S$ , denoted by  $\text{diam}(S)$ , to be a number given by

$$\text{diam}(S) := \sup\{d(x, y) : x, y \in S\}.$$

**Proposition 13.1.1.** If the diameter of a set is zero, then the set is a singleton set.

*Proof.* Let  $(X, d)$  be a metric space and  $S$  be a subset of  $X$  with  $\text{diam}(S) = 0$ .

Assume for the sake of contradiction that there exist distinct points  $x_1$  and  $x_2$  in  $S$ .

Since  $x_1 \neq x_2$ , by definition of metric,  $d(x_1, x_2) \neq 0$ .

Since  $d(x_1, x_2) \in \{d(x, y) : x, y \in S\}$ , by definition of supremum,  $\sup\{d(x, y) : x, y \in S\} \geq d(x, y)$ .

Since  $\sup\{d(x, y) : x, y \in S\} \geq d(x, y)$ , by definition of diameter,  $\text{diam}(S) \geq d(x, y)$ .

This contradicts to the assumption that  $\text{diam}(S) = 0$ .

Thus  $S$  is a singleton set. ■

**Proposition 13.1.2 (Monotonicity).** The diameter function is increasing.

**Proposition 13.1.3.** The diameter of the closure of a set is equal to the diameter of the set itself.

*Proof.* Let  $(X, d)$  be a metric space and  $S$  be a subset of  $X$ .

Let  $\varepsilon$  be an arbitrary positive number.

Let  $x$  and  $y$  be arbitrary points in  $\text{cl}(S)$ .

Since  $x \in \text{cl}(S)$ , there exists a point  $x'$  in  $S$  such that  $d(x, x') < \varepsilon/2$ .

Since  $y \in \text{cl}(S)$ , there exists a point  $y'$  in  $S$  such that  $d(y, y') < \varepsilon/2$ .

Since  $x', y' \in S$ , by definition of diameter,  $d(x', y') \leq \text{diam}(S)$ .

Since  $d$  is a metric on  $X$ , by the triangle inequality,  $d(x, y) \leq d(x, x') + d(x', y') + d(y', y)$ .

Since  $d(x, y) \leq d(x, x') + d(x', y') + d(y', y)$  and  $d(x, x') < \varepsilon/2$  and  $d(y, y') < \varepsilon/2$  and  $d(x', y') \leq \text{diam}(S)$ ,  $d(x, y) < \text{diam}(S) + \varepsilon$ .

Since for any positive number  $\varepsilon$ ,  $d(x, y) < \text{diam}(S) + \varepsilon$ ,  $d(x, y) \leq \text{diam}(S)$ .

Let  $x_0$  and  $y_0$  be points in  $S$  such that  $d(x_0, y_0) > \text{diam}(S) - \varepsilon$ .

Since  $x_0 \in S$  and  $S \subseteq \text{cl}(S)$ ,  $x_0 \in \text{cl}(S)$ .

Since  $y_0 \in S$  and  $S \subseteq \text{cl}(S)$ ,  $y_0 \in \text{cl}(S)$ .

Since for any points  $x$  and  $y$  in  $\text{cl}(S)$ ,  $d(x, y) \leq \text{diam}(S)$  and for any positive number  $\varepsilon$ , there exists  $x_0, y_0 \in S$  such that  $d(x_0, y_0) > \text{diam}(S) - \varepsilon$ , by definition of supremum,  $\sup\{d(x, y) : x, y \in \text{cl}(S)\} = \text{diam}(S)$ .

Since  $\sup\{d(x, y) : x, y \in \text{cl}(S)\} = \text{diam}(S)$ , by definition of diameter,  $\text{diam}(\text{cl}(S)) = \text{diam}(S)$ . ■

## 13.2 Boundedness

**Definition** (Boundedness). Let  $(X, d)$  be a metric space. Let  $S$  be a subset of  $X$ . We say  $S$  is **bounded** if its diameter is finite.

**Proposition 13.2.1.** The closure of a bounded set is bounded.

## 13.3 Total Boundedness

**Definition** (Total Boundedness). Let  $(X, d)$  be a metric space. We say that the space is **totally bounded** if

$$\forall r > 0, \exists n \in \mathbb{N}, \exists \{x_i\}_{i=1}^n \subseteq X, \bigcup_{i=1}^n \text{ball}(x_i, r) = X.$$

**Proposition 13.3.1.** A space is totally bounded if and only if any sequence in the space has a Cauchy subsequence.

*Proof.* For one direction, assume that for any radius  $r$ , there exists a finite collection  $\mathbf{p}(r) = \{p_i\}_{i=1}^{i=k}$  of points in  $X$  such that  $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$  covers  $S$ .

We are to prove that any sequence in  $(S, d)$  has a Cauchy subsequence in  $(S, d)$ .

Let  $\{x_i\}_{i \in \mathbb{N}}$  be an arbitrary sequence in  $(S, d)$ .

Let  $r$  be an arbitrary radius.

Since  $\{x_i\}_{i \in \mathbb{N}}$  is a sequence in  $(S, d)$  and  $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$  covers  $S$ , there exists some  $p_{i_0}$  among  $\mathbf{p}(r)$  such that  $\text{ball}(p_{i_0}, r)$  contains a subsequence  $\{x_i\}_{i \in I}$  of  $\{x_i\}_{i \in \mathbb{N}}$ .

Since  $\{x_i\}_{i \in I} \subseteq \text{ball}(p_{i_0}, r)$ , for any indices  $m$  and  $n$  in  $I$ ,  $d(x_m, x_n) < r$ .

Since  $d(x_m, x_n) < r$  for any radius  $r$ , by definition of Cauchy-ness,  $\{x_i\}_{i \in I}$  is Cauchy.

Since  $\{x_i\}_{i \in I}$  is Cauchy, any sequence in  $(S, d)$  has a Cauchy subsequence in  $(S, d)$ .

For the reverse direction, assume that any sequence in  $(S, d)$  has a Cauchy sequence in  $(S, d)$ .

We are to prove that for any radius  $r$ , there exists a finite collection  $\mathbf{p}(r) = \{p_i\}_{i=1}^{i=k}$  of points in  $X$  such that  $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$  covers  $S$ .

**Case 1.**  $S$  is empty.

Since  $S$  is empty, for any radius  $r$ , there exists a finite collection  $\mathbf{p}(r) = \{p_i\}_{i=1}^{i=k}$  of points in  $X$  such that  $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$  covers  $S$ .

**Case 2.**  $S$  is not empty.

Assume for the sake of contradiction that there exists some radius  $r_0$  such that for any finite collection  $\mathfrak{s}(r_0) = \{x_i\}_{i=1}^{i=k}$  of points in  $X$ ,  $\{\text{ball}(x_i, r_0)\}_{i=1}^{i=k}$  does not cover  $S$ .

Since  $S$  is not empty, pick  $x_1$  from  $S$ .

Since  $\{\text{ball}(x_i, r_0)\}_{i=1}^{i=1}$  does not cover  $S$ , pick  $x_2$  from  $C_S(\bigcup_{i=1}^{i=1} \text{ball}(x_i, r_0))$ .

In general, since  $\{\text{ball}(x_i, r_0)\}_{i=1}^{i=k}$  does not cover  $S$ , pick  $x_{k+1}$  from  $C_S(\bigcup_{i=1}^{i=k} \text{ball}(x_i, r_0))$ .

Let  $m$  and  $n$  be arbitrary indices such that  $m > n$ .

Since  $x_m \in C_S(\bigcup_{i=1}^{i=m-1} \text{ball}(x_i, r_0))$ ,  $x_m \notin \text{ball}(x_n, r_0)$ .

Since  $x_m \notin \text{ball}(x_n, r_0)$ , by definition of open ball,  $d(x_m, x_n) \geq r_0$ .

Since for any indices  $m$  and  $n$ ,  $d(x_m, x_n) \geq r_0$ , by definition of Cauchy-ness,  $\{x_i\}_{i \in \mathbb{N}}$  does not have a Cauchy subsequence in  $(S, d)$ .

This contradicts to the fact that any sequence in  $(S, d)$  has a Cauchy sequence in  $(S, d)$ .

Thus the assumption that there exists some radius  $r_0$  such that for any finite collection  $\mathfrak{s}(r_0) = \{x_i\}_{i=1}^{i=k}$  of points in  $X$ ,  $\{\text{ball}(x_i, r_0)\}_{i=1}^{i=k}$  does not cover  $S$  is false.

i.e., for any radius  $r$ , there exists a finite collection  $\mathbf{p}(r) = \{p_i\}_{i=1}^{i=k}$  of points in  $X$  such that  $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$  covers  $S$ .

**Summary.**

Since in either cases, for any radius  $r$ , there exists a finite collection  $\mathbf{p}(r) = \{p_i\}_{i=1}^{i=k}$  of points in  $X$  such that  $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$  covers  $S$ , I conclude that for any radius  $r$ , there exists a finite collection  $\mathbf{p}(r) = \{p_i\}_{i=1}^{i=k}$  of points in  $X$  such that  $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$  covers  $S$ . ■

## 13.4 Properties of Totally Bounded Spaces

**Proposition 13.4.1.** *Totally bounded spaces are bounded.*

**Proposition 13.4.2.** *Totally bounded metric spaces are separable.*

*Proof.* Let  $(X, d)$  be a metric space. Suppose that the space is totally bounded. We are to prove that the space has a countable dense subset. Let  $n$  be an arbitrary natural number.

Since the space is totally bounded, there exists a finite collection of points  $\mathbf{p}_n$  such that  $\bigcup_{p \in \mathbf{p}_n} \text{ball}(p, 1/n) = X$ . Define a set  $\mathbf{p}$  by  $\mathbf{p} := \bigcup_{n \in \mathbb{N}} \mathbf{p}_n$ . Then  $\mathbf{p}$  is a countable dense subset of  $X$ . ■

### 13.5 Stability of Total Boundedness

**Proposition 13.5.1.** *The closure of a totally bounded set is totally bounded.*

*Proof.* Let  $(X, d)$  be a metric space and  $S$  be a totally bounded subset of  $X$ .

We are to prove that  $\text{cl}(S)$  is also totally bounded.

Let  $r$  be an arbitrary radius.

Since  $S$  is totally bounded, by definition of total boundedness, there exist a finite collection  $\S(r) = \{x_i\}_{i=1}^{i=k}$  of points in  $X$  such that  $\{\text{ball}(x_i, r/2)\}_{i=1}^{i=k}$  covers  $S$ .

Let  $x$  be an arbitrary point in  $\text{cl}(S)$ .

Since  $x \in \text{cl}(S)$ , by definition of closure, there exists a point  $x'(x)$  in  $S$  such that  $d(x, x'(x)) < r/2$ .

Since  $x'(x) \in S$  and  $\{\text{ball}(x_i, r/2)\}_{i=1}^{i=k}$  covers  $S$ , there exists some  $x_{i_0(x)}$  among  $\{x_i\}_{i=1}^{i=k}$  such that  $x'(x) \in \text{ball}(x_{i_0(x)}, r/2)$ .

Since  $x'(x) \in \text{ball}(x_{i_0(x)}, r/2)$ , by definition of open ball,  $d(x'(x), x_{i_0(x)}) < r/2$ .

Since  $d(x, x'(x)) < r/2$  and  $d(x'(x), x_{i_0(x)}) < r/2$ , by the triangle inequality,  $d(x, x_{i_0(x)}) < r$ .

Since  $d(x, x_{i_0(x)}) < r$ , by definition of open ball,  $x \in \text{ball}(x_{i_0(x)}, r)$ .

Since for any  $x \in \text{cl}(S)$ , there exists a point  $x_{i_0(x)}$  among  $\{x_i\}_{i=1}^{i=k}$  such that  $x \in \text{ball}(x_{i_0(x)}, r)$ ,  $\{\text{ball}(x_i, r)\}_{i=1}^{i=k}$  covers  $\text{cl}(S)$ .

Since for any radius  $r$ , there exists a finite collection  $\S(r) = \{x_i\}_{i=1}^{i=k}$  of points in  $X$  such that  $\{\text{ball}(x_i, r)\}_{i=1}^{i=k}$  covers  $\text{cl}(S)$ , by definition,  $\text{cl}(S)$  is totally bounded. ■

**Proposition 13.5.2** (Set Operations). (1) *A subset of a totally bounded set is totally bounded.*

(2) *The union of a finite collection of totally bounded sets is totally bounded.*

**Proposition 13.5.3** (Images). *A uniformly continuous image of a totally bounded set is totally bounded.*

*Proof.* Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces.

Let  $S$  be a totally bounded set in  $(X, d_X)$ .

Let  $f$  be a uniformly continuous function from  $X$  to  $Y$ .

We are to prove that  $f(S)$  is totally bounded.

Let  $r$  be an arbitrary radius.

Since  $f$  is uniformly continuous on  $S$ , there exists a  $\delta(r)$  such that  $d_Y(f(x_1), f(x_2)) < r$  whenever  $d_X(x_1, x_2) < \delta(r)$ .

Since  $S$  is totally bounded in  $(X, d_X)$ , there exists a finite collection  $\S$  of points in  $X$  such that  $\{(x, \delta(r))\}_{x \in \S}$  covers  $S$ .

Let  $\dagger$  denote  $f(\S)$ .

Let  $y$  be an arbitrary point in  $f(S)$ .

Since  $y \in f(S)$ , there exists a point  $x$  in  $S$  such that  $f(x) = y$ .

Since  $x \in S$  and  $\{(x, \delta(r))\}_{x \in \S}$  covers  $S$ , there exists a point  $x_0 \in \S$  such that  $x \in (x_0, \delta(r))$ .

Let  $y_0$  denote  $f(x_0)$ .

Since  $x \in (x_0, \delta(r))$ , by definition of  $\delta(r)$ ,  $y \in (y_0, r)$ .

Since  $y \in (y_0, r)$  and  $y_0 \in \dagger$ ,  $\{(y, r)\}_{y \in \dagger}$  covers  $y$ .

Since  $\{(y, r)\}_{y \in \dagger}$  covers  $y$  for any  $y \in f(S)$ ,  $\{(y, r)\}_{y \in \dagger}$  covers  $f(S)$ .

Since for any radius  $r$ , there exists a collection  $\dagger$  of points in  $f(S)$  such that  $\{(y, r)\}_{y \in \dagger}$  covers  $f(S)$ , by definition of total boundedness,  $f(S)$  is totally bounded in  $(Y, d_Y)$ .

■



## 14

# Lindelöf Space

### 14.1 Definition

**Definition** (Lindelöf Space). *Let  $(X, \tau)$  be a topological space. We say that the space is **Lindelöf** if any open cover has a countable subcover.*





# 15

## Compact Space

### 15.1 Definitions

**Definition** (Open Cover). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. Let  $\{U_\lambda\}_{\lambda \in \Lambda}$  be a collection of sets in the space. We say that  $\{U_\lambda\}_{\lambda \in \Lambda}$  is an **open cover** of  $S$  if  $U_\lambda$  is open for each  $\lambda \in \Lambda$  and

$$\bigcup_{\lambda \in \Lambda} U_\lambda \supseteq S.$$

**Definition** (Compactness). Let  $(X, \tau)$  be a topological space. We say that the space is **compact** if any open cover has a finite subcover.

**Definition** (Finite Intersection Property). Let  $(X, \tau)$  be a topological space. Let  $\{S_i\}_{i \in I}$  be a collection of sets in the space. We say that  $\{S_i\}_{i \in I}$  has the **finite intersection property** if for any finite subcollection  $\{S_i\}_{i \in I'}$ , where  $I'$  is a finite subset of  $I$ ,

$$\bigcap_{i \in I'} S_i \neq \emptyset.$$

**Definition** (Compactness). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say that  $S$  is **compact** if for any collection  $\{S_i\}_{i \in I}$  of relatively closed subsets of  $S$  with the finite intersection property,

$$\bigcap_{i \in I} S_i \neq \emptyset.$$

**Definition** (Compactness). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say that  $S$  is **compact** if any net on  $S$  has a convergent subnet.

**Definition** (Compactness). Let  $(X, \tau)$  be a topological space. Let  $S$  be a set in the space. We say that  $S$  is **compact** if any filter on  $S$  has a convergent refinement.

**Proposition 15.1.1.** *Definitions 1 and 2 of compactness are equivalent.*

*Proof.* [**Definition 1**]  $\implies$  [**Definition 2**].

For one direction, assume that any open cover of the space has a finite subcover.

We are to prove that any collection of closed sets in the space with the finite intersection property has nonempty intersection.

Let  $\mathcal{F}$  be an arbitrary collection of closed sets in the space with the finite intersection property.

Say  $\mathcal{F} = \{F_\lambda\}_{\lambda \in \Lambda}$  where  $\Lambda$  is an index set and  $F_\lambda$  is a closed set in the space for each  $\lambda \in \Lambda$  and  $\bigcap_{\lambda \in \Lambda'} F_\lambda \neq \emptyset$  for any finite subset  $\Lambda'$  of  $\Lambda$ .

Define  $U_\lambda := X \setminus F_\lambda$ .

Define  $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ .

Since  $F_\lambda$  is closed for each  $\lambda \in \Lambda$ ,  $U_\lambda$  is open for each  $\lambda \in \Lambda$ .

Assume for the sake of contradiction that  $\mathcal{F}$  has empty intersection.

Since  $\bigcap_{\lambda \in \Lambda} F_\lambda = \emptyset$ , by the De Morgan's Law,  $\bigcup_{\lambda \in \Lambda} U_\lambda = X$ .

Since  $\mathcal{U}$  is an open cover of the space and any open cover of the space has a finite subcover, in particular,  $\mathcal{U}$  has a finite subcover  $\mathcal{U}'$ .

Say  $\mathcal{U}' = \{U_\lambda\}_{\lambda \in \Lambda'}$  where  $\Lambda'$  is a finite subset of  $\Lambda$  and  $\bigcup_{\lambda \in \Lambda'} U_\lambda = X$ .

Since  $\bigcup_{\lambda \in \Lambda'} U_\lambda = X$ , by the De Morgan's Law,  $\bigcap_{\lambda \in \Lambda'} F_\lambda = \emptyset$ .

This contradicts to the assumption that  $\mathcal{F}$  has the finite intersection property.

So the assumption that  $\mathcal{F}$  has empty intersection is false.

i.e.,  $\mathcal{F}$  has nonempty intersection. ■

*Proof.* [**Definition 2**]  $\implies$  [**Definition 1**].

For the reverse direction, assume that any collection of closed sets in the space with the finite intersection property has nonempty intersection.

We are to prove that any open cover has a finite subcover.

Let  $\mathcal{U}$  be an arbitrary open cover of the space.

Say  $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$  where  $\Lambda$  is an index set and  $U_\lambda$  is an open set in the space for any  $\lambda \in \Lambda$  and  $\bigcup_{\lambda \in \Lambda} U_\lambda = X$ .

Define  $F_\lambda := X \setminus U_\lambda$  for each  $\lambda \in \Lambda$ .

Define  $\mathcal{F} := \{F_\lambda\}_{\lambda \in \Lambda}$ .

Since  $U_\lambda$  is open for each  $\lambda \in \Lambda$ ,  $F_\lambda$  is closed for each  $\lambda \in \Lambda$ .

Assume for the sake of contradiction that  $\mathcal{U}$  does not have a finite subcover.

Let  $\Lambda'$  be an arbitrary finite subset of  $\Lambda$ .

Define  $\mathcal{U}' := \{U_\lambda\}_{\lambda \in \Lambda'}$ .

Define  $\mathcal{F}' := \{F_\lambda\}_{\lambda \in \Lambda'}$ .

Since  $\mathcal{U}'$  is a finite subcollection of  $\mathcal{U}$  and  $\mathcal{U}$  does not have a finite subcover, in particular,  $\mathcal{U}'$  cannot cover the whole space.

Since  $\bigcup_{\lambda \in \Lambda'} U_\lambda \neq X$ , by the De Morgan's Law,  $\bigcap_{\lambda \in \Lambda'} F_\lambda \neq \emptyset$ .

Since  $\bigcap_{\lambda \in \Lambda'} F_\lambda \neq \emptyset$  for any finite subcollection  $\Lambda'$  of  $\Lambda$ ,  $\mathcal{F}$  has the finite intersection property.

Since  $\mathcal{F}$  has the finite intersection property and any collection of closed sets with the finite intersection property has nonempty intersection, in particular,  $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$ .

Since  $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$ , by the De Morgan's Law,  $\bigcup_{\lambda \in \Lambda} U_\lambda \neq X$ .

This contradicts to the assumption that  $\mathcal{U}$  is an open cover of the space.

So the assumption that  $\mathcal{U}$  does not have a finite subcover is false.

i.e.,  $\mathcal{U}$  has a finite subcover. ■

## 15.2 Properties

**Proposition 15.2.1.** *A compact subspace of a Hausdorff space is closed.*

**Proposition 15.2.2.** *Compact metric spaces are totally bounded.*

*Proof.*

Assume for the sake of contradiction that the space is not totally bounded.

Since the space is not totally bounded, by definition of total boundedness, there exists some radius  $r_0$  such that the space cannot be covered by finitely many open balls of radius  $r_0$ .

Let  $x_1$  be an arbitrary point in the space.

Since  $\{ball(x_i, r_0)\}_{i \leq n}$  cannot cover the space for any  $n \in \mathbb{N}$ , pick  $x_{n+1}$  from  $X \setminus \bigcup_{i \leq n} ball(x_i, r_0)$ .

Define  $\mathfrak{x} := \{x_n\}_{n \in \mathbb{N}}$ .

Let  $m$  and  $n$  be arbitrary indices in  $\mathbb{N}$ .

Assume without loss of generality that  $m > n$ .

Since  $x_m \in X \setminus \bigcup_{i \leq n} ball(x_i, r_0)$ ,  $x_m \notin ball(x_n, r_0)$ .

Since  $x_m \notin ball(x_n, r_0)$ ,  $d(x_m, x_n) \geq r_0$ .

Since  $d(x_m, x_n) \geq r_0$  for any  $m, n \in \mathbb{N}$ ,  $\mathfrak{x}$  has no convergent subsequence.

Since  $\mathfrak{x}$  is a sequence in the space and  $\mathfrak{x}$  has no convergent subsequence, by definition of compactness, the space is not compact.

This contradicts to the assumption that the space is compact.

Thus the assumption that the space is not totally bounded is false.

i.e., the space is totally bounded. ■

**Proposition 15.2.3.** *Compact metric spaces are complete.*

*Proof.* Let  $\mathfrak{x}$  be an arbitrary Cauchy sequence in the space.

Since the space is compact, any sequence has a convergent subsequence.

Since  $\mathfrak{x}$  is a sequence in the space and any sequence has a convergent subsequence,  $\mathfrak{x}$  has a convergent subsequence.

Since  $\mathfrak{x}$  is Cauchy and has a convergent subsequence,  $\mathfrak{x}$  converges.

Since any Cauchy sequence in the space converges, by definition of completeness, the space is complete. ■

**Proposition 15.2.4.** *Compact metric spaces are separable.*

**Theorem 1** (Extension). *Let  $(X, d_X)$  be a compact metric space and  $(Y, d_Y)$  be an arbitrary metric spaces. Let  $f$  be a continuous function from  $X$  to  $Y$ . Then  $f$  is uniformly continuous on  $X$ .*

*Proof.*

Since  $f$  is continuous on  $X$ , for all  $\varepsilon > 0$  and all  $x \in X$ , there exists a  $\delta(x) > 0$  such that for all  $x' \in X$  with  $d(x, x') < \delta(x)$ , we have  $\rho(f(x), f(x')) < \varepsilon/2$ .

Consider the set of open balls  $\mathcal{B} = \{B(\frac{1}{2}\delta(x), x)\}$ .

Then  $\mathcal{B}$  is a cover of  $X$ .

By definition of compactness, there exists a finite set of open balls  $\mathcal{B}_n = \{B_k\}_{k=1}^n$ .

Define  $\delta > 0$  by  $\delta = \min\{\frac{1}{2}\delta(x_k)\}$ .

Let  $x$  and  $y$  be arbitrary elements in  $X$  with  $d(x, y) < \delta$ .

Since  $x \in X$  and  $\mathcal{B}_n$  is a cover of  $X$ , there exists an open ball  $B_0(\frac{1}{2}\delta(x_0), x_0)$  such that  $x \in B_0$ .

By our choice of  $x$ ,  $y$  and  $\delta$ , we have

$$d(y, x) < \delta \leq \frac{1}{2}\delta(x_0) \# (1)$$

Since  $x \in B_0$ , we get

$$d(x, x_0) < \frac{1}{2}\delta(x_0) \# (2)$$

By the triangle inequality, we get

$$d(y, x_0) \leq d(y, x) + d(x, x_0) \# (3)$$

From inequations (1) ~ (3), we get

$$d(y, x_0) < \delta(x_0) \# (4)$$

Since  $f$  is continuous at point  $x_0$  and  $d(x, x_0) < \delta(x_0)$ , we get

$$\rho(f(x), f(x_0)) < \frac{\varepsilon}{2} \# (5)$$

Since  $f$  is continuous at pint  $x_0$  and  $d(y, x_0) < \delta(x_0)$ , we get

$$\rho(f(y), f(x_0)) < \frac{\varepsilon}{2} \#(6)$$

By the triangle inequality again, we get

$$\rho(f(x), f(y)) < \rho(f(x), f(x_0)) + \rho(f(y), f(x_0)) \#(7)$$

From inequations (5) ~ (7), we get

$$\rho(f(x), f(y)) < \varepsilon$$

In short, we have proved that for all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all elements  $x$  and  $y$  in  $X$  with  $d(x, y) < \delta$ , we have  $\rho(f(x), f(y)) < \varepsilon$ .

By definition of uniform continuity,  $f$  is uniformly continuous on  $X$ . ■

## 15.3 Sufficient Conditions

**Proposition 15.3.1.** *A closed subspace of a compact space is compact.*

*Proof.*

Let  $(X, \tau)$  be a compact topological space.

Let  $(S, \tau)$  be a closed subspace of  $(X, \tau)$ .

We are to prove that  $(S, \tau)$  is compact.

Let  $\{U_\lambda\}_{\lambda \in \Lambda}$  be an arbitrary open cover of  $S$ .

Since  $S$  is closed,  $X \setminus S$  is open.

Since  $\{U_\lambda\}_{\lambda \in \Lambda}$  covers  $S$ ,  $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus S\}$  covers  $X$ .

Since each set in  $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus S\}$  is open and  $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus S\}$  covers  $X$ , by definition of open cover,  $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus S\}$  is an open cover of  $X$ .

Since  $(X, \tau)$  is compact, by definition of compactness, any open cover of  $(X, \tau)$  has a finite subcover.

Since  $\mathcal{U} \cup \{X \setminus S\}$  is an open cover of  $(X, \tau)$  and any open cover has a finite subcover, in particular,  $\mathcal{U}$  has a finite subcover  $\mathcal{U}'$ .

Say  $\mathcal{U}' = \{U_\lambda\}_{\lambda \in \Lambda'}$  where  $\Lambda'$  is a finite subset of  $\Lambda$  and  $\bigcup_{\lambda \in \Lambda'} U_\lambda = X$

Since  $\{U_\lambda\}_{\lambda \in \Lambda'} \cup \{X \setminus S\}$  covers  $X$ ,  $\{U_\lambda\}_{\lambda \in \Lambda'}$  covers  $S$ .

Since any open cover of  $S$  has a finite subcover, by definition of compactness,  $(S, \tau)$  is compact. ■

**Proposition 15.3.2** (Set Operations).

(1) *The intersection of any collection of compact spaces is compact. ### got some problem with this! Need the space to be Hausdorff.*

(2) *The union of any finite collection of compact spaces is compact.*

*Proof.*

**Proof of (1).**

Let  $X$  be a topological space and  $\{S_i\}_{i \in I}$  be a collection of compact sets in  $X$ .

Let

**Proof of (1).**

Let  $\{S_\alpha\}_{\alpha \in A}$  be an arbitrary collection of compact sets and let  $S$  be their intersection.

Then there exists a compact set  $S_{\alpha_0}$  such that  $S \subseteq S_{\alpha_0}$ .

Since for any  $\alpha \in A$ ,  $S_\alpha$  is closed,  $S$  is closed.

Since  $S$  is closed and  $S_{\alpha_0}$  is compact,  $S$  also compact.

**Proof of (2).**

Let  $(X, \tau)$  be a topological space and  $\{S_\lambda\}_{\lambda \in \Lambda}$  be a finite collection of compact subsets of  $X$ .

Let  $\{U_\lambda\}$

Let  $\{S_k\}_{k=1}^N$  be an arbitrary finite collection of compact sets and let  $S$  be their union.

Let  $\mathcal{U}$  be an arbitrary open cover of  $S$ .

Since  $\mathcal{U}$  is an open cover of  $S$  and for each  $k$ ,  $S_k \subseteq S$ , for each  $k$ ,  $\mathcal{U}$  is an open cover of  $S_k$ .

Since  $\mathcal{U}$  is an open cover of  $S_k$  and  $S_k$  is compact, by definition, there exists a finite subcover  $\mathcal{U}_k$  of  $S_k$ .

Let  $\mathcal{U}'$  be the union of  $\{\mathcal{U}_k\}_{k=1}^N$ . Then  $\mathcal{U}'$  is finite.

In short, we have proved that any open cover of  $S$  has a finite subcover.

By definition, we conclude that  $S$  is compact. ■

**Proposition 15.3.3** (Product, Tychonoff). *The product of any collection of compact sets is compact.*

**Proposition 15.3.4** (Image). *The continuous image of a compact set is compact.*

*Proof.*

Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be topological space.

Let  $f$  be a continuous function from  $X$  to  $Y$ .

Let  $S$  be a compact set in  $(X, \tau_X)$ .

Define  $T := f(S)$ .

We are to prove that  $T$  is a compact set in  $(Y, \tau_Y)$ .

Let  $\mathcal{V}$  be an arbitrary open cover of  $T$ .

Say  $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$  where  $\Lambda$  is an index set and  $V_\lambda$  is an open set in  $(Y, \tau_Y)$  for each  $\lambda \in \Lambda$  and  $T \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$ .

Define  $U_\lambda := f^{-1}(V_\lambda)$  for each  $\lambda \in \Lambda$ .

Define  $\mathcal{U} := \{U_\lambda\}_{\lambda \in \Lambda}$ .

Since  $V_\lambda \neq \emptyset$  for each  $\lambda \in \Lambda$ ,  $U_\lambda \neq \emptyset$  for each  $\lambda \in \Lambda$ .

Since  $V_\lambda$  is open in  $(Y, \tau_Y)$  for each  $\lambda \in \Lambda$  and  $f$  is a continuous function from  $(X, \tau_X)$  to  $(Y, \tau_Y)$ , by definition of continuity,  $U_\lambda$  is open in  $(X, \tau_X)$  for each  $\lambda \in \Lambda$ .

Since  $T \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$ ,  $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$ .

Since  $U_\lambda$  is a nonempty open set in  $(X, \tau_X)$  for each  $\lambda \in \Lambda$  and  $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$ ,  $\mathcal{U}$  is an open cover of  $S$  in  $(X, \tau_X)$ .

Since  $S$  is a compact set in  $(X, \tau_X)$ , by definition of compactness, any open cover of  $S$  in  $(X, \tau_X)$  has a finite subcover.

Since  $\mathcal{U}$  is an open cover of  $S$  in  $(X, \tau_X)$  and any open cover of  $S$  in  $(X, \tau_X)$  has a finite subcover, in particular,  $\mathcal{U}$  has a finite subcover  $\mathcal{U}'$ .

Say  $\mathcal{U}' = \{U_\lambda\}_{\lambda \in \Lambda'}$  where  $\Lambda'$  is a finite subset of  $\Lambda$  and  $S \subseteq \bigcup_{\lambda \in \Lambda'} U_\lambda$ .

Define  $\mathcal{V}' := \{V_\lambda\}_{\lambda \in \Lambda'}$ .

Since  $S \subseteq \bigcup_{\lambda \in \Lambda'} U_\lambda$ ,  $T \subseteq \bigcup_{\lambda \in \Lambda'} V_\lambda$ .

Since  $V_\lambda$  is a nonempty open set in  $(Y, \tau_Y)$  for each  $\lambda \in \Lambda'$  and  $T \subseteq \bigcup_{\lambda \in \Lambda'} V_\lambda$  and  $\Lambda'$  is finite,  $\mathcal{V}'$  is a finite open cover of  $T$  in  $(Y, \tau_Y)$ .

Since any open cover of  $T$  in  $(Y, \tau_Y)$  has a finite subcover, by definition of compactness,  $T$  is a compact set in  $(Y, \tau_Y)$ . ■

**Proposition 15.3.5.** *Finite spaces are compact.*

*Proof.*

Let  $(X, \tau)$  be a finite topological space.

Say  $X = \{x_i\}_{i \in I}$  where  $I$  is a finite index set.

Let  $\mathcal{U}$  be an arbitrary open cover of the space.

Say  $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$  where  $\Lambda$  is an index set and  $U_\lambda$  is an open set in  $(X, \tau)$  for each  $\lambda \in \Lambda$  and  $\bigcup_{\lambda \in \Lambda} U_\lambda = X$ .

Since  $\mathcal{U}$  covers  $X$  and  $x_i \in X$  for each  $i \in I$ ,  $\mathcal{U}$  covers  $x_i$  for each  $i \in I$ .

Since  $x_i \in \mathcal{U}$  for each  $i \in I$ , there exists some  $\lambda_i$  for each  $i \in I$  such that  $x_i \in U_{\lambda_i}$ .

Define  $\mathcal{U}' := \{U_{\lambda_i}\}_{i \in I}$ .

Since  $x_i \in U_{\lambda_i}$  for each  $i \in I$ ,  $X \subseteq \bigcup_{i \in I} U_{\lambda_i}$ .

Since each  $U_{\lambda_i}$  is open in  $(X, \tau)$  and  $X \subseteq \bigcup_{i \in I} U_{\lambda_i}$  and  $I$  is finite,  $\mathcal{U}'$  is a finite open cover of the space.

Since any open cover of the space has a finite subcover, by definition, of compactness,  $(X, \tau)$  is compact. ■

**Proposition 15.3.6.** *A complete and totally bounded metric space is (sequentially) compact.*

*Proof.* Every sequence in a totally bounded metric space admits Cauchy subsequence. Every Cauchy sequence in a complete metric space converges. So every sequence has a convergent subsequence. ■

## 15.4 Countably Compact

### 15.4.1 Definitions

**Definition** (Countably Compact). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is **countably compact** if every countable open cover has a finite subcover.*

**Definition** (Countably Compact). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is **countably compact** if every infinite set in the space has a  $\omega$ -limit point.*

**Definition** (Weakly Countably Compact). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is **weakly countably compact** if every infinite set in the space has a limit point.*

**Proposition 15.4.1.** *The two definitions of countably compact are equivalent.*

### 15.4.2 Sufficient Conditions

**Proposition 15.4.2** (Image). *A continuous image of a countably compact set is countably compact.*

**Remark.** *Continuous images of weakly countably compact sets may not be weakly countably compact.*

### 15.4.3 Relation to Other Forms of Compactness

**Proposition 15.4.3.** *Compactness implies countable compactness.*

**Proposition 15.4.4.** *Countable compactness can imply compactness if the space is Lindelöf.*

**Proposition 15.4.5.** *Countable compactness implies weak countable compactness.*

**Proposition 15.4.6.** *Weak countable compactness can imply countable compactness if the space is metrizable.*



## 15.5 Sequentially Compact

### 15.5.1 Definitions

**Definition** (Sequentially Compact). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is **sequentially compact** if every sequence in the space has a convergent subsequence.*

### 15.5.2 Relation to Other Forms of Compactness

**Proposition 15.5.1.** *Compactness can imply sequential compactness if the space is first countable.*

*Proof.*

Assume that any open cover has a finite subcover.

We are to prove that any sequence has a convergent subsequence.

Let  $\{x_i\}_{i \in \mathbb{N}}$  be an arbitrary sequence in  $(X, d)$ .

Define  $F_n := cl(\{x_i\}_{i \geq n})$  for each  $n \in \mathbb{N}$ .

Define  $U_n := X \setminus F_n$  for each  $n \in \mathbb{N}$ .

Define  $\mathcal{U} := \{U_n\}_{n \in \mathbb{N}}$ .

Assume for the sake of contradiction that  $\bigcup_{n \in \mathbb{N}} U_n = X$ .

Since  $\{U_n\}_{n \in \mathbb{N}}$  is an open cover of  $X$  and any open cover of  $X$  has a finite subcover, there exists a finite index set  $I$  such that  $\{U_n\}_{n \in I}$  covers  $X$ .

Since  $\{U_n\}_{n \in I}$  covers  $X$ ,  $\bigcap_{n \in I} F_n = \emptyset$ .

Define  $i_0 := \max(I)$ .

Since  $\{F_n\}_{n \in I}$  is decreasing,  $\bigcap_{n \in I} F_n = F_{i_0}$ .

Since  $\bigcap_{n \in I} F_n = \emptyset$  and  $\bigcap_{n \in I} F_n = F_{i_0}$ ,  $F_{i_0} = \emptyset$ .

This contradicts to the assumption that  $F_n = cl(\{x_i\}_{i \geq n})$ .

So the assumption that  $\bigcup_{n \in \mathbb{N}} U_n = X$  is false.

i.e.,  $\bigcup_{n \in \mathbb{N}} U_n \neq X$ .

Since  $U_n \subseteq X$  for any  $n \in \mathbb{N}$ ,  $\bigcup_{n \in \mathbb{N}} U_n \subseteq X$ .

Since  $\bigcup_{n \in \mathbb{N}} U_n \subseteq X$  and  $\bigcup_{n \in \mathbb{N}} U_n \neq X$ , there exists a point  $x_0$  in  $X \setminus \bigcup_{n \in \mathbb{N}} U_n$ .

Since  $x_0 \in X \setminus \bigcup_{n \in \mathbb{N}} U_n$  and  $X \setminus \bigcup_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} F_n$ ,  $x_0 \in \bigcap_{n \in \mathbb{N}} F_n$ .

■

**Remark.** *Compactness does not imply sequential compactness in general.*

**Proposition 15.5.2.** *Sequential compactness can imply compactness if the space is metrizable.*

*Proof.*

Let  $(X, d)$  be a metrizable topological space.

Assume that any sequence has a convergent subsequence.

We are to prove that any open cover has a finite subcover.

Let  $\mathcal{U}$  be an arbitrary open cover of the space.

Say  $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$  where  $\Lambda$  is an index set and  $U_\lambda$  is open for each  $\lambda \in \Lambda$  and  $\bigcup_{\lambda \in \Lambda} U_\lambda = X$ .

Assume for the sake of contradiction that  $\mathcal{U}$  does not have a finite subcover.

Let  $x_1$  be some point in the space.

Since  $\{x_1\}$  is closed,  $X \setminus \{x_1\}$  is open.

Since

■

**Remark.** *I do not know if the statement is still true if the condition "the space is metrizable" is replaced with a weaker one. But this is at least true for metric spaces.*

**Proposition 15.5.3.** *Sequential compactness implies countable compactness.*

*Proof.*

Assume that every sequence has a convergent subsequence.

We are to prove that every countable open cover has a finite subcover.

Let  $\mathcal{U}$  be an arbitrary countable open cover of the space.

Say  $\mathcal{U} = \{U_i\}_{i \in \mathbb{N}}$  where each  $U_i$  is an open set in the space and  $\bigcup_{i \in \mathbb{N}} U_i = X$ .

Assume for the sake of contradiction that  $\mathcal{U}$  has no finite subcover.

Since  $\{U_i\}_{i \leq n}$  is a finite subcollection of  $\mathcal{U}$  and  $\mathcal{U}$  has no finite subcover, in particular,  $\{U_i\}_{i \leq n}$  does not cover the whole space.

Since  $\{U_i\}_{i \leq n}$  does not cover the whole space, take  $x_n \in X \setminus \bigcup_{i \leq n} U_i$ .

Define  $\mathfrak{x} := \{x_n\}_{n \in \mathbb{N}}$ .

Since  $\mathfrak{x}$  is a sequence and any sequence has a convergent subsequence, in particular,  $\mathfrak{x}$  has a convergent subsequence  $\mathfrak{x}'$ .

Say  $\mathfrak{x}$  converges to a point  $x_\infty$ .

Since  $\mathcal{U}$  covers the whole space, there exists an index  $i_\infty \in \mathbb{N}$  such that  $x_\infty \in U_{i_\infty}$ .

Since  $\mathfrak{x}$  converges to  $x_\infty$  and  $x_\infty \in U_{i_\infty}$ ,  $U_{i_\infty}$  contains a tail of  $\mathfrak{x}$ .

Since  $x_n \in X \setminus \bigcup_{i \leq n} U_i$  for all  $n$ ,  $x_n \notin U_{i_\infty}$  for all  $n > i_\infty$ .

This contradicts to the fact that  $U_{i_\infty}$  contains a tail of  $\mathfrak{x}$ .

So the assumption that  $\mathcal{U}$  has no finite subcover is false.

i.e.,  $\mathcal{U}$  has a finite subcover.

Since any countable open cover has a finite subcover, by definition of countable compactness, the space is countably compact.

■

**Proposition 15.5.4.** *Countable compactness can imply sequential compactness if the space is metrizable.*

*Proof.*

Assume that any countable open cover has a finite subcover.

We are to prove that any sequence has a convergent subsequence.

Let  $\mathfrak{x}$  be an arbitrary sequence in the space.

Say  $\mathfrak{x} = \{x_i\}_{i \in \mathbb{N}}$  where  $x_i$  is a point in the space for each  $i \in \mathbb{N}$ .

Define  $U_i := X \setminus \{x_i\}$ .

Define  $\mathcal{U} := \{U_i\}_{i \in \mathbb{N}}$ .

Since  $X$  is open and  $\{x_i\}$  is closed,  $X \setminus \{x_i\}$  is open.

Since  $U_i = X \setminus \{x_i\}$  for each  $i \in \mathbb{N}$ ,  $X = \bigcup_{i \in \mathbb{N}} U_i$ .

Since  $U_i$  is open and  $X = \bigcup_{i \in \mathbb{N}} U_i$  and  $\mathbb{N}$  is countable,  $\mathcal{U}$  is a countable open cover of the space.

Since  $\mathcal{U}$  is a countable open cover of the space and any countable open cover of the space has a finite subcover, in particular,  $\mathcal{U}$  has a finite subcover  $\mathcal{U}'$ .

Say  $\mathcal{U}' = \{U_i\}_{i \in I}$  where  $I$  is a finite subset of  $\mathbb{N}$ .

■



# 16

## Connectedness

### 16.1 Definitions

**Definition** (Separation). Let  $(X, \tau)$  be a topological space. Let  $S$  be a subset of  $X$ . Let  $S_1$  and  $S_2$  be two sets. We say the pair  $(S_1, S_2)$  is a **separation** of  $S$  if all of the following conditions hold.

- (1)  $S_1$  and  $S_2$  are both subsets of  $S$ .
- (2)  $S_1$  and  $S_2$  are both non-empty.
- (3)  $S_1$  and  $S_2$  are both open.
- (4)  $S_1 \cap S_2 = \emptyset$ .
- (5)  $S_1 \cup S_2 = S$ .

**Definition** (Disconnected). Let  $(X, \tau)$  be a topological space. We say  $X$  is **disconnected** if there exists a separation of  $X$ .

**Definition** (Connected). Let  $(X, \tau)$  be a topological space. We say  $X$  is **connected** if it is not disconnected.

### 16.2 Properties

**Proposition 16.2.1.** A space  $X$  is connected if and only if the only subsets of  $X$  that are both open and closed are  $\emptyset$  and  $X$ .

*Proof.* For one direction, assume that  $X$  is connected. We are to prove that the only subsets of  $X$  that are both open and closed are  $\emptyset$  and  $X$ .

Assume for the sake of contradiction that there exists a non-empty proper subset  $S_0$  of  $X$  that is both open and closed.

Since  $S_0$  is a non-empty proper subset of  $X$ , both  $S_0$  and  $C_X(S_0)$  are non-empty.

Since  $S_0$  is closed,  $C_X(S_0)$  is open.

Since  $S_0$  and  $C_X(S_0)$  are both non-empty and open,  $S_0 \cap C_X(S_0) = \emptyset$ , and  $S_0 \cup C_X(S_0) = X$ , we get the pair  $(S_0, C_X(S_0))$  is a separation of  $X$ .

By definition, we conclude that  $X$  is disconnected.

This contradicts to the fact that  $X$  is connected.

Thus the only subsets of  $X$  that are both open and closed are  $\emptyset$  and  $X$ .

For the reverse direction, assume that the only subsets of  $X$  that are both open and closed are  $\emptyset$  and  $X$ . We are to prove that  $X$  is connected.

Assume for the sake of contradiction that there exists a pair  $(S_1, S_2)$  of non-empty proper open subsets of  $X$  such that  $S_1 \cap S_2 = \emptyset$  and  $S_1 \cup S_2 = X$ .

Since  $S_2$  is open,  $S_1 \cap S_2 = \emptyset$ , and  $S_1 \cup S_2 = X$ , we get  $S_1$  is closed.

This contradicts to the fact that the only subsets of  $X$  that are both open and closed are  $\emptyset$  and  $X$ .

Thus  $X$  is connected. ■

**Proposition 16.2.2.** *Let  $(X, d)$  be a connected metric space. The only subsets of  $X$  that has empty boundaries are  $X$  and  $\emptyset$ .*

*Proof.* Assume for the sake of contradiction that there exists a non-empty proper-subset  $S_0$  of  $X$  such that the boundary of  $S_0$  is empty.

Since  $\partial(S_0) = \emptyset$ ,  $S_0$  is both open and closed.

This contradicts to the fact that the only subsets of  $X$  that are both open and closed are  $X$  and  $\emptyset$ .

It follows that the only subsets of  $X$  that has empty boundaries are  $X$  and  $\emptyset$ . ■

**Proposition 16.2.3.** *Let  $X$  be a disconnected topological space. Let  $S_1$  and  $S_2$  be a separation of  $X$ . Let  $Y$  be a connected subset of  $X$ . Then either  $Y \subseteq S_1$  or  $Y \subseteq S_2$ .*

*Proof.* Assume for the sake of contradiction that  $Y \not\subseteq S_1$  and  $Y \not\subseteq S_2$ .

Since  $Y \not\subseteq S_1$ , there exists a point  $x_1$  such that  $x_1 \in Y$  and  $x_1 \notin S_1$ .

Since  $x_1 \notin S_1$ ,  $x_1 \in X$ , and  $S_1 \cup S_2 = X$ , we have  $x_1 \in S_2$ .

Since  $x_1 \in Y$  and  $x_1 \in S_2$ , we have  $x_1 \in Y \cap S_2$ .

Since  $x_1 \in Y \cap S_2$ ,  $Y \cap S_2$  is not empty.

Since  $Y \not\subseteq S_2$ , there exists a point  $x_2$  such that  $x_2 \in Y$  and  $x_2 \notin S_2$ .

Since  $x_2 \notin S_2$ ,  $x_2 \in X$ , and  $S_1 \cup S_2 = X$ , we have  $x_2 \in S_1$ .

Since  $x_2 \in Y$  and  $x_2 \in S_1$ , we have  $x_2 \in Y \cap S_1$ .

Since  $x_2 \in Y \cap S_1$ ,  $Y \cap S_1$  is not empty.

Since  $S_1$  is open in  $X$  and  $Y \subseteq X$ , we have  $Y \cap S_1$  is open in  $Y$ .

Since  $S_2$  is open in  $X$  and  $Y \subseteq X$ , we have  $Y \cap S_2$  is open in  $Y$ .

Since  $S_1 \cap S_2 = \emptyset$ ,  $(Y \cap S_1) \cap (Y \cap S_2) = \emptyset$ .

Since  $S_1 \cup S_2 = X$  and  $Y \subseteq X$ , we have  $(Y \cap S_1) \cup (Y \cap S_2) = Y$ .

Since  $Y \cap S_1$  and  $Y \cap S_2$  are both not empty, open in  $Y$ ,  $(Y \cap S_1) \cap (Y \cap S_2) = \emptyset$ , and  $(Y \cap S_1) \cup (Y \cap S_2) = Y$ , we have  $Y \cap S_1$  and  $Y \cap S_2$  is a separation of  $Y$ .

Since  $Y \cap S_1$  and  $Y \cap S_2$  is a separation of  $Y$ , by definition,  $Y$  is disconnected.

This contradicts to the assumption that  $Y$  is connected.

Thus the assumption that  $Y \not\subseteq S_1$  and  $Y \not\subseteq S_2$  is false.

i.e.,  $Y \subseteq S_1$  or  $Y \subseteq S_2$ .

■

## 16.3 Sufficient Conditions

**Proposition 16.3.1.** *Let  $S$  be a connected topological space. Let  $T$  be a set such that  $S \subseteq T \subseteq \text{cl}(S)$ . Then  $T$  is also connected.*

*Proof.* Assume for the sake of contradiction that there exists a separation  $(T_1, T_2)$  of  $T$ .

Since  $(T_1, T_2)$  is a separation of  $T$  and  $S \subseteq T$  and  $S$  is connected, either  $S \subseteq T_1$  or  $S \subseteq T_2$ .

Assume without loss of generality that  $S \subseteq T_1$ .

Since  $S \subseteq T_1$ , by properties of the closure operator,  $\text{cl}(S) \subseteq \text{cl}(T_1)$ .

Since  $T_1$  and  $T_2$  are both open and  $T_1 \cap T_2 = \emptyset$ ,  $\text{cl}(T_1) \cap T_2 = \emptyset$ .

Since  $\text{cl}(S) \subseteq \text{cl}(T_1)$  and  $\text{cl}(T_1) \cap T_2 = \emptyset$ ,  $\text{cl}(S) \cap T_2 = \emptyset$ .

Since  $T \subseteq \text{cl}(S)$  and  $\text{cl}(S) \cap T_2 = \emptyset$ ,  $T \cap T_2 = \emptyset$ .

This contradicts to the fact that  $T_2$  is non-empty.

Thus  $T$  is connected.

■

**Corollary.** *The closure of a connected space is also connected.*

**Proposition 16.3.2** (Set Operations). *Let  $X$  be a topological space. Let  $\{S_i\}_{i \in \mathbb{N}}$  be a sequence of connected subspaces of  $X$  with  $S_i \cap S_{i+1} \neq \emptyset$  for each  $i$ . Then the set  $\bigcup_{i \in \mathbb{N}} S_i$  is also connected.*

*Proof.* Let  $U$  denote the set  $\bigcup_{i \in \mathbb{N}} S_i$ . We are to prove that  $U$  is connected.

Assume for the sake of contradiction that there exists a separation  $(U_1, U_2)$  of  $U$ .

Since each  $S_i$  is a subset of  $U$  and  $U$  is separated by  $(U_1, U_2)$ , either  $S_i \subseteq U_1$  or  $S_i \subseteq U_2$ .

Assume for the sake of contradiction that either for any index  $i$  in  $\mathbb{N}$ , both  $S_i$  and  $S_{i+1}$  lie in  $U_1$ , or for any  $i \in \mathbb{N}$ , both  $S_i$  and  $S_{i+1}$  lie in  $U_2$ .

Assume without loss of generality that for any index  $i$  in  $\mathbb{N}$ , both  $S_i$  and  $S_{i+1}$  lie in  $U_1$ .

Since for any index  $i$  in  $\mathbb{N}$ , both  $S_i$  and  $S_{i+1}$  lie in  $U_1$ , all of  $\{S_i\}_{i \in \mathbb{N}}$  lie in  $U_1$ .

Since all of  $\{S_i\}_{i \in \mathbb{N}}$  lie in  $U_1$ ,  $U \subseteq U_1$ .

Since  $U \subseteq U_1$  and  $U_1 \subseteq U$ ,  $U_1 = U$ .

Since  $U_1 = U$  and  $U_1 \cap U_2 = \emptyset$  and  $U_1 \cup U_2 = U$ ,  $U_2 = \emptyset$ .

This contradicts to the fact that  $U_1$  is non-empty.

Thus there exists an index  $i_0$  in  $\mathbb{N}$  such that either  $S_{i_0} \subseteq U_1$  and  $S_{i_0+1} \subseteq U_2$  or  $S_{i_0} \subseteq U_2$  and  $S_{i_0+1} \subseteq U_1$ .

Assume without loss of generality that  $S_{i_0} \subseteq U_1$  and  $S_{i_0+1} \subseteq U_2$ .

Since  $S_{i_0} \subseteq U_1$  and  $S_{i_0+1} \subseteq U_2$  and  $U_1 \cap U_2 = \emptyset$ ,  $S_{i_0} \cap S_{i_0+1} = \emptyset$ .

This contradicts to the fact that for any index  $i$  in  $\mathbb{N}$ ,  $S_i \cap S_{i+1} = \emptyset$ .

Thus  $U$  is connected. ■

**Proposition 16.3.3** (Set Operations). *Let  $X$  be a topological space. Let  $\{S_i\}_{i \in I}$  where  $I$  is an index set be a collection of connected subspaces of  $X$  such that there exists a point  $x_0$  in  $X$  such that  $x_0 \in S_i$  for each  $i$ . Then the set  $\bigcup_{i \in I} S_i$  is also connected.*

*Proof.* Let  $U$  denote the set  $\bigcup_{i \in I} S_i$ . We are to prove that  $U$  is also connected.

Assume for the sake of contradiction that there exists a separation  $(U_1, U_2)$  of  $U$ .

Since  $x_0 \in U$  and  $U = U_1 \cup U_2$ , either  $x_0 \in U_1$  or  $x_0 \in U_2$ .

Assume without loss of generality that  $x_0 \in U_1$ .

Since  $x_0 \in U_1$  and  $U_1 \cap U_2 = \emptyset$ ,  $x_0 \notin U_2$ .

Since each  $S_i$  is a subset of  $U$  and  $U$  is separated by  $(U_1, U_2)$ , either  $S_i \subseteq U_1$  or  $S_i \subseteq U_2$ .

Since  $U_2$  is non-empty, there exists a set  $S_0$  in  $\{S_i\}_{i \in I}$  such that  $S_0 \subseteq U_2$ .

Since  $x_0 \notin U_2$  and  $S_0 \subseteq U_2$ ,  $x_0 \notin S_0$ .

This contradicts to the assumption that  $x_0 \in S_0$ .

Thus  $U$  is connected. ■

**Proposition 16.3.4** (Set Operations). *Let  $X$  be a topological space. Let  $\{S_i\}_{i \in I}$  where  $I$  is an index set be a collection of connected sets such that there exists a set  $S$  such that  $S \cap S_i \neq \emptyset$  for any  $i$ . Then the set  $S \cup (\bigcup_{i \in I} S_i)$  is also connected.*

**Proposition 16.3.5** (Continuous Maps). *The continuous image of a connected set is also connected.*

**Comment.**

Continuity only guarantees open-ness. The other parts follows from basic set theory and are trivial.



*Proof.* Let  $X$  and  $Y$  be topological spaces and  $f$  be a function from  $X$  to  $Y$ .

Let  $S$  be a connected subset of  $X$ . We are to prove that  $f(S)$  is also connected.

Assume for the sake of contradiction that  $f(S)$  is disconnected.

Since  $f(S)$  is disconnected, there exists a separation  $(T_1, T_2)$  of  $f(S)$ .

Since  $T_1$  is non-empty,  $f^{-1}(T_1)$  is non-empty.

Since  $T_2$  is non-empty,  $f^{-1}(T_2)$  is non-empty.

Since  $T_1$  is open and  $f$  is continuous,  $f^{-1}(T_1)$  is open.

Since  $T_2$  is open and  $f$  is continuous,  $f^{-1}(T_2)$  is open.

Since  $T_1 \cap T_2 = \emptyset$ , from set theory we know that  $f^{-1}(T_1) \cap f^{-1}(T_2) = \emptyset$ .

Since  $f(S) \subseteq T_1 \cup T_2$ , from set theory we know that  $S \subseteq f^{-1}(T_1) \cup f^{-1}(T_2)$ .

Since  $f^{-1}(T_1)$  and  $f^{-1}(T_2)$  are both non-empty and open,  $f^{-1}(T_1) \cap f^{-1}(T_2) = \emptyset$ , and  $S \subseteq f^{-1}(T_1) \cup f^{-1}(T_2)$ , we get  $(f^{-1}(T_1), f^{-1}(T_2))$  is a separation of  $S$ .

Since  $(f^{-1}(T_1), f^{-1}(T_2))$  is a separation of  $S$ , by definition,  $S$  is disconnected.

This contradicts to the fact that  $S$  is connected.

Thus  $f(S)$  is connected. ■

**Proposition 16.3.6** (Cartesian Product). *The cartesian product of a finite collection of connected sets is also connected.*

## 16.4 Connected Components

**Definition** (Connected Component). *Let  $(X, \tau)$  be a topological space. We define the **connected components** of  $(X, \tau)$  to be the maximal connected subsets of  $(X, \tau)$ .*

**Proposition 16.4.1.** *The connected components of a space form a partition of it.*

**Proposition 16.4.2.** *Connected components are closed.*

**Proposition 16.4.3.** *If the number of connected components of a space is finite, then each component is open.*

## 16.5 Connectedness on the Real Line

**Proposition 16.5.1.** *Intervals are connected.*

**Theorem 2** (Intermediate Value Theorem). *Let  $f : X \rightarrow Y$  be a continuous map, where  $X$  is a connected space and  $Y$  is an ordered set in the order topology. If  $a$  and  $b$  are two points of  $X$  and if  $r$  is a point of  $Y$  lying between  $f(a)$  and  $f(b)$ , then there exists a point  $c$  of  $X$  such that  $f(c) = r$ .*

## 16.6 Pathwise Connectedness

**Definition** (Pathwise Connectedness). *Let  $(X, d)$  be a metric space. We say that  $X$  is **pathwise connected** if for any points  $x$  and  $y$  in  $X$ , there exists a continuous function  $f$  from  $[0, 1]$  to  $X$  such that  $f(0) = x$  and  $f(1) = y$ .*

**Proposition 16.6.1.** *Pathwise connected spaces are connected.*

**Claim 1.** *The combination of two paths is also a path. i.e., if there exists a path  $p_1$  from  $x$  to  $y$  and another path  $p_2$  from  $y$  to  $z$ , then there exists a path  $p$  from  $x$  to  $z$ .*

*Proof.* By definition of path, we have  $p_1(0) = x$ ,  $p_1(1) = y$ ,  $p_2(0) = y$ , and  $p_2(1) = z$ . Define  $p$  by  $p(x) = p_1(2x)$  if  $0 \leq x \leq 1/2$  and  $p(x) = p_2(2x - 1)$  if  $1/2 \leq x \leq 1$ . Notice that at point  $x = 1/2$ ,  $p_1(2x) = p_1(1) = y = p_2(2x - 1) = p_2(0) = y$ . Since  $\varphi_1$  given by  $\varphi_1(x) = 2x$  is continuous,  $\varphi_2$  given by  $\varphi_2(x) = 2x - 1$  is continuous, and  $p_1$  and  $p_2$  are continuous,  $p$  is continuous on both  $[0, 1/2]$  and  $[1/2, 1]$ . Since the value of  $p$  given by the two pieces agree at the overlapping point,  $p$  is continuous. ■

**Claim 2.** *If there exists a path from  $x$  to  $y$ , then there exists a path from  $y$  to  $x$ .*

*Proof.* Say  $p$  is the path from  $x$  to  $y$ . i.e., we have  $p$  is continuous and  $p(0) = x$  and  $p(1) = y$ . Define  $p'$  by  $p'(x) = p(1 - x)$ . Since the function  $\varphi$  given by  $\varphi(x) = 1 - x$  is continuous and  $p$  is continuous,  $p'(x)$  is continuous. Further,  $p'(0) = p(1) = y$  and  $p'(1) = p(0) = x$ . So  $p'$  is a path from  $y$  to  $x$ . ■

## 16.7 Totally Disconnectedness

**Definition** (Totally Disconnected). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is **totally disconnected** if all connected components of  $(X, \tau)$  are singleton sets.*

**Proposition 16.7.1.** *Subspaces of a totally disconnected space are totally disconnected.*

**Proposition 16.7.2.** *Cartesian products of totally disconnected spaces are totally disconnected.*

**Proposition 16.7.3.** *Countable metric spaces are totally disconnected.*

# 17

## Cauchy Completeness

### 17.1 Definitions

**Definition** (Cauchy Completeness). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is **cauchy complete** if every Cauchy net in  $(X, \tau)$  converges to some point in  $X$ .*



# 18

## Completeness in Metric Spaces

### 18.1 Definitions

**Definition** (Completeness). *Let  $(X, d)$  be a metric space. Let  $S$  be a subset of  $X$ . We say that  $S$  is **complete** if any Cauchy sequence in  $S$  converges in  $S$ .*

**Definition** (Completeness). *Let  $(X, d)$  be a metric space. Let  $S$  be a subset of  $X$ . We say that  $S$  is **complete** if any infinite totally bounded subset of  $S$  has an accumulation point in  $S$ .*

**Proposition 18.1.1.** *The two definitions of completeness are equivalent.*

### 18.2 Properties

**Proposition 18.2.1.** *Complete metric spaces are closed.*

*Proof.* Let  $(X, d)$  be a metric space and  $S$  be a complete subspace of  $X$ . We are to prove that  $S$  is closed.

Let  $\{x_k\}$  be a convergent sequence in  $S$ . Then  $\{x_k\}$  is Cauchy.

By definition of completeness,  $\{x_k\}$  converges to some point in  $S$ .

By definition, we conclude that  $S$  is closed.

■

### 18.3 Sufficient Conditions

**Proposition 18.3.1.** *Closed subspaces of a complete space are also complete.*

*Proof.* Let  $(X, d)$  be a complete metric space and  $S$  be a closed subset of  $X$ .

Let  $\{x_k\}_{k=1}^{\infty}$  be an arbitrary Cauchy sequence in  $S$ . We are to prove that  $\{x_k\}$  converges to some point in  $S$ .

Since  $\{x_k\}$  is Cauchy in  $X$  and  $X$  is complete, by definition,  $\{x_k\}$  converges to some point in  $X$ .

Since  $\{x_k\}$  converges in  $X$  and  $S$  is closed, by definition,  $\{x_k\}$  converges to some point in  $S$ .

In short, we have proved that any Cauchy sequence in  $S$  converges to some point in  $S$ .

By definition, we conclude that  $S$  is complete. ■

**Proposition 18.3.2** (Cartesian Product). *Let  $(X_1, d_1)$  and  $(X_2, d_2)$  be metric spaces. Then the product space  $(X_1 \times X_2, D)$  is complete if and only if both  $(X_1, d_1)$  and  $(X_2, d_2)$  are complete.*

*Proof.* For one direction, assume  $(X_1 \times X_2, D)$  is complete.

We are to prove that  $(X_1, d_1)$  and  $(X_2, d_2)$  are complete.

Let  $\{x_i\}_{i \in \mathbb{N}}$  be an arbitrary Cauchy sequence in  $(X_1, d_1)$  and  $c$  be an arbitrary point in  $X_2$ .

Let  $\varepsilon$  be an arbitrary positive number.

Since  $\{x_i\}_{i \in \mathbb{N}}$  is Cauchy in  $(X_1, d_1)$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m, n > N(\varepsilon)$ , we have  $d_1(x_m, x_n) < \varepsilon$ .

Since  $d_1(x_m, x_n) < \varepsilon$  and  $d_2(c, c) = 0$ , by definition of  $D$ ,  $D((x_m, c), (x_n, c)) < \varepsilon$ .

Since for any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m, n > N(\varepsilon)$ , we have  $D((x_m, c), (x_n, c)) < \varepsilon$ ,  $\{(x_i, c)\}_{i \in \mathbb{N}}$  is Cauchy in  $(X_1 \times X_2, D)$ .

Since  $\{(x_i, c)\}_{i \in \mathbb{N}}$  is Cauchy in  $(X_1 \times X_2, D)$  and  $(X_1 \times X_2, D)$  is complete,  $\{(x_i, c)\}_{i \in \mathbb{N}}$  converges in  $(X_1 \times X_2, D)$ .

Since  $\{(x_i, c)\}_{i \in \mathbb{N}}$  converges in  $(X_1 \times X_2, D)$ ,  $\{x_i\}_{i \in \mathbb{N}}$  converges in  $(X_1, d_1)$ .

Since any Cauchy sequence in  $(X_1, d_1)$  converges in  $(X_1, d_1)$ ,  $(X_1, d_1)$  is complete.

Similarly,  $(X_2, d_2)$  is also complete.

For the reverse direction, assume that both  $(X_1, d_1)$  and  $(X_2, d_2)$  are complete.

We are to prove that  $(X_1 \times X_2, D)$  is complete.

Let  $\{(x_1^{(i)}, x_2^{(i)})\}_{i \in \mathbb{N}}$  be an arbitrary Cauchy sequence in  $(X_1 \times X_2, D)$ .

Since  $\{(x_1^{(i)}, x_2^{(i)})\}_{i \in \mathbb{N}}$  is Cauchy in  $(X_1 \times X_2, D)$ ,  $\{x_1^{(i)}\}_{i \in \mathbb{N}}$  is Cauchy in  $(X_1, d_1)$  and  $\{x_2^{(i)}\}_{i \in \mathbb{N}}$  is Cauchy in  $(X_2, d_2)$ .

Since  $\{x_1^{(i)}\}_{i \in \mathbb{N}}$  is Cauchy in  $(X_1, d_1)$  and  $(X_1, d_1)$  is complete,  $\{x_1^{(i)}\}_{i \in \mathbb{N}}$  converges in  $(X_1, d_1)$ .

Since  $\{x_2^{(i)}\}_{i \in \mathbb{N}}$  is Cauchy in  $(X_2, d_2)$  and  $(X_2, d_2)$  is complete,  $\{x_2^{(i)}\}_{i \in \mathbb{N}}$  converges in  $(X_2, d_2)$ .

Since  $\{x_1^{(i)}\}_{i \in \mathbb{N}}$  converges in  $(X_1, d_1)$  and  $\{x_2^{(i)}\}_{i \in \mathbb{N}}$  converges in  $(X_2, d_2)$ ,  $\{(x_1^{(i)}, x_2^{(i)})\}_{i \in \mathbb{N}}$  converges in  $(X_1 \times X_2, D)$ .

Since any Cauchy sequence in  $(X_1 \times X_2, D)$  converges in  $(X_1 \times X_2, D)$ ,  $(X_1 \times X_2, D)$  is complete. ■

**Theorem 3** (Cantor's Intersection Theorem). *Let  $(X, d)$  be a metric space. Then  $(X, d)$  is complete if and only if for any decreasing sequence  $\{S_i\}_{i \in \mathbb{N}}$  of non-empty closed sets with  $\text{diam}(S_i) = 0$ ,  $\bigcap_{i \in \mathbb{N}} S_i$  is a singleton set.*

**###question. How does this theorem relate to compactness?**

*Proof.* For one direction, assume that  $(X, d)$  is complete.

Since  $S_i$  is non-empty for all  $i \in \mathbb{N}$ , there exists a point  $x_i$  in  $S_i$  for each  $i \in \mathbb{N}$ .

Let  $\varepsilon$  be an arbitrary positive number.

Since  $\text{diam}(S_i) = 0$ , there exists an integer  $N(\varepsilon)$  such that  $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$ .

Since  $\{S_i\}_{i \in \mathbb{N}}$  is decreasing, for any indices  $m$  and  $n$ , if  $m, n > N(\varepsilon)$ , we have  $x_m, x_n \in S_{N(\varepsilon)}$ .

Since  $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$ , by definition of diameter,  $d(x_m, x_n) < \varepsilon$ .

Since for any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m$  and  $n$ , if  $m, n > N(\varepsilon)$ , then  $d(x_m, x_n) < \varepsilon$ , by definition of Cauchy-ness,  $\{x_i\}_{i \in \mathbb{N}}$  is Cauchy in  $(X, d)$ .

Since  $\{x_i\}_{i \in \mathbb{N}}$  is Cauchy in  $(X, d)$  and  $(X, d)$  is complete, by definition of completeness,  $\{x_i\}_{i \in \mathbb{N}}$  is convergent in  $(X, d)$ .

Let  $x_0$  be the limit of  $\{x_i\}_{i \in \mathbb{N}}$ .

Let  $S_{i_0}$  be an arbitrary set in  $\{S_i\}_{i \in \mathbb{N}}$ .

Since  $\{S_i\}_{i \in \mathbb{N}}$  is decreasing, by construction of  $\{x_i\}_{i \in \mathbb{N}}$ ,  $\{x_i\}_{i \geq i_0}$  is a sequence in  $S_{i_0}$ .

Since  $\{x_i\}_{i \in \mathbb{N}}$  converges to  $x_0$  in  $(X, d)$ ,  $\{x_i\}_{i \geq i_0}$  also converges to  $x_0$  in  $(X, d)$ .

Since  $\{x_i\}_{i \geq i_0}$  is a sequence in  $S_{i_0}$  and converges to  $x_0$  in  $(X, d)$  and  $S_{i_0}$  is closed,  $x_0 \in S_{i_0}$ .

Since  $x_0$  is in any set in  $\{S_i\}_{i \in \mathbb{N}}$ ,  $x_0 \in \bigcap_{i \in \mathbb{N}} S_i$ .

Since  $\text{diam}(S_i) = 0$ ,  $\text{diam}(\bigcap_{i \in \mathbb{N}} S_i) = 0$ .

Since  $\text{diam}(\bigcap_{i \in \mathbb{N}} S_i) = 0$  and  $x_0 \in \bigcap_{i \in \mathbb{N}} S_i$ ,  $\bigcap_{i \in \mathbb{N}} S_i$  is a singleton set consisting of only  $x_0$ .

For the reverse direction, assume that for any decreasing sequence  $\{S_i\}_{i \in \mathbb{N}}$  of non-empty closed sets with  $\text{diam}(S_i) = 0$ ,  $\bigcap_{i \in \mathbb{N}} S_i$  is a singleton set.

We are to prove that  $(X, d)$  is complete.

Let  $\{x_i\}_{i \in \mathbb{N}}$  be an arbitrary Cauchy sequence in  $(X, d)$ .

Let  $S_k = \text{cl}(\{x_i\}_{i \geq k})$ .

Since  $S_k = \text{cl}(\{x_i\}_{i \geq k})$ ,  $\{S_k\}_{k \in \mathbb{N}}$  is a decreasing sequence of non-empty closed sets in  $(X, d)$ .

Let  $\varepsilon$  be an arbitrary positive number.

Since  $\{x_i\}_{i \in \mathbb{N}}$  is Cauchy in  $(X, d)$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m$  and  $n$ , if  $m, n > N(\varepsilon)$ , then  $d(x_m, x_n) < \varepsilon/2$ .

Since for any indices  $m$  and  $n$ , if  $m, n > N$ , then  $d(x_m, x_n) < \varepsilon/2$ ,  $\text{diam}(\{x_i\}_{i \geq N(\varepsilon)}) < \varepsilon$ .

Since  $\text{diam}(\{x_i\}_{i \geq N(\varepsilon)}) < \varepsilon$ , and  $\text{diam}(\{x_i\}_{i \geq N(\varepsilon)}) = \text{diam}(S_{N(\varepsilon)})$ ,  $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$ .

Since for any index  $i$ , if  $i > N(\varepsilon)$ , then  $\text{diam}(S_i) \leq \text{diam}(S_{N(\varepsilon)})$ , and  $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$ , for any index  $i$ , if  $i > N(\varepsilon)$ , then  $\text{diam}(S_i) < \varepsilon$ .

Since for any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any index  $i$ , if  $i > N(\varepsilon)$ , then  $\text{diam}(S_i) < \varepsilon$ , by definition of limits,  $\text{diam}(S_i) = 0$ .

Since  $\{S_i\}_{i \in \mathbb{N}}$  is a decreasing sequence of non-empty closed sets with  $\text{diam}(S_i) = 0$ , by assumption,  $\bigcap_{i \in \mathbb{N}} S_i$  is a singleton set.

Let  $x_0 \in \bigcap_{i \in \mathbb{N}} S_i$ .

Let  $\varepsilon$  be an arbitrary positive number.

Since  $\text{diam}(S_i) = 0$ , there exists an integer  $N(\varepsilon)$  such that  $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$ .

Since for any index  $i$ , if  $i > N(\varepsilon)$ , then  $x_i \in S_{N(\varepsilon)}$ , and  $x_0 \in S_{N(\varepsilon)}$ , by definition of diameter, for any index  $i$ , if  $i > N(\varepsilon)$ ,  $d(x_i, x_0) < \text{diam}(S_{N(\varepsilon)})$ .

Since  $d(x_i, x_0) < \text{diam}(S_{N(\varepsilon)})$  and  $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$ ,  $d(x_i, x_0) < \varepsilon$ .

Since for any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any index  $i$ , if  $i > N(\varepsilon)$ , then  $d(x_i, x_0) < \varepsilon$ , by definition of limits,  $\{x_i\}_{i \in \mathbb{N}}$  converges to  $x_0$ .

Since any Cauchy sequence in  $(X, d)$  converges,  $(X, d)$  is complete. ■

## 18.4 Metric Completion

$$Cd_1(x, y) \leq d_2(x, y) \leq Dd_1(x, y)$$

**Proposition 18.4.1.** *Let  $(X, d_1)$  and  $(X, d_2)$  be equivalent metric spaces. Then they have the same class of convergent sequences and Cauchy sequences.*



## 19

# Baire Space

### 19.1 Definitions

**Definition** (Baire Space). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is a **Baire Space** if any nonempty open set in the space is not meager.*

**Definition** (Baire Space). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is a **Baire Space** if any comeager set in the space is dense.*

**Definition** (Baire Space). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is a **Baire Space** if the intersection of any countable collection of dense open sets is again dense.*

**Definition** (Baire Space). *Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is a **Baire Space** if the union of any countable collection of closed nowhere dense sets has empty interior.*

**Proposition 19.1.1.** *The four definitions of Baire space are equivalent.*

### 19.2 Properties

**Proposition 19.2.1.** *An open subspace of a Baire space is again Baire.*

**Remark.** *A closed subspace of a Baire space is not necessarily Baire.*



# Uniform Spaces

## 20.1 Entourages

**Definition** (Uniform Structure, Entourages). *Let  $X$  be a set. Let  $\Phi$  be a collection of subsets of  $X \times X$ . We say that  $\Phi$  is a **uniform structure** and the elements of  $\Phi$  are **entourages** if  $\Phi$  satisfies all of the following conditions.*

- *If  $U \in \Phi$ , then  $\Delta \subseteq U$  where  $\Delta := \{(x, x) : x \in X\}$  is the diagonal on  $X \times X$ .*
- *If  $U \in \Phi$  and  $V$  is a set such that  $U \subseteq V \subseteq X \times X$ , then  $V \in \Phi$ .*
- *$\forall U, V \in \Phi, \quad U \cap V \in \Phi$ .*
- *$\forall U \in \Phi, \quad \exists V \in \Phi, \quad V \circ V \subseteq U$  where  $\circ$  is a function given by*

$$S \circ T := \{(x, z) : \exists y \in X \text{ such that } (x, y) \in T \text{ and } (y, z) \in S\}.$$

- *$\forall U \in \Phi, \quad U^{-1} \in \Phi$  where  $U^{-1} := \{(y, x) : (x, y) \in U\}$ .*