Graph Theory

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Chapter 1

Graph Basics

1.1 Paths

DEFINITION 1.1 (Vertex-Independent Paths)**.** Let $G = (V, E)$ be a finite undirected graph. Let P and Q be two paths in G. We say that P and Q are **vertex-independent** if and only if they do not have any internal vertex in common.

DEFINITION 1.2 (Edge-Independent Paths)**.** Let $G = (V, E)$ be a finite undirected graph. Let P and Q be two paths in G. We say that P and Q are **edge-independent** if and only if they do not have any internal edge in common.

1.2 Others

DEFINITION 1.3 (Spanning Subgraph)**.** Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G. We say that H is a **spanning** subgraph of G if and only if $W = V$. i.e., if H contains all vertices of G.

1.3 Exercises from [Bollobas]

EXERCISE 1.4. Show that every simple graph G on at least two vertices contains two distinct vertices of equal degree.

Proof. Let $n := |V(G)|$. Then the possible degrees are $\{0, ..., n-1\}$. Assume for the sake of contradiction that the degrees are distinct. Then $\exists x \in V(G)$ and $y \in V(G)$ such that $\deg(x) = 0$ and $\deg(y) = n-1$. So y is connected to all other vertices in the graph. In particular, there is an edge between x and y. So $\deg(x) \geq 1$. This contradicts to the choice of x that $\deg(x) = 0$. So there are two distinct vertices in the graph of the same degree. $\square$

Chapter 2

Graph Connectivity

2.1 Definitions

DEFINITION 2.1 (Vertex Cut)**.** Let $G = (V, E)$ be a finite undirected graph. Let a and b be distinct vertices in G. Let $S \subseteq V \setminus \{a, b\}$. We say that S is a **vertex cut** for a and b if and only if the removal of S from G separates a and b into distinct connected components.

DEFINITION 2.2 (Edge Cut)**.** Let $G = (V, E)$ be a finite undirected graph. Let a and b be nonadjacent vertices in G. Let $S \subseteq E$. We say that S is an **edge cut** for a and b if and only if the removal of S from G separates a and b into distinct connected components.

DEFINITION 2.3 (k-Vertex-Connected Graphs)**.** We say a connected graph is **k-vertex-connected** if and only if it has more than k vertices and remains connected whenever (strictly) fewer than k vertices are removed.

DEFINITION 2.4 (k-Edge-Connected Graphs)**.** We say a connected graph is **k-edge-connected** if and only if it has more than k edges and remains connected whenever (strictly) fewer than k edges are removed.

DEFINITION 2.5 (Cut)**.** A **cut** is a partition of the vertices of a graph into two disjoint subsets.

DEFINITION 2.6 (s-t Cut)**.** Let $G = (V, E)$ be a finite undirected graph. Let s and t be two vertices in G. We define an **s-t cut** to be a cut $C = (S, T)$ of V such that $s \in S$ and $t \in T$.

DEFINITION 2.7 (Size/Value of a Cut)**.** Let $G = (V, E)$ be a finite undirected graph. Let $C = (S, T)$ be a cut of V. We define the **size** of C to be the number of edges crossing the cut. In the case that G is weighted, we define the **value** of C to be the sum of the weights of the edges crossing the cut.

DEFINITION 2.8 (Minimum Cut)**.** We say that a cut is **minimum** if the size/value of the cut is the minimum among all cuts in the graph.

DEFINITION 2.9 (Maximum Cut)**.** We say that a cut is **maximum** if the size/value of the cut is the maximum among all cuts in the graph.

DEFINITION 2.10 (Cut-Set)**.** Let $G = (V, E)$ be a finite undirected graph. Let $C = (S, T)$ be a partition of V. We define the **cut-set** of C to be the set given by

$$\{(u, v) \in E : u \in S, v \in T\}.$$

DEFINITION 2.11 (Cut Space)**.** Let $G = (V, E)$ be a finite undirected graph. Let $\mathcal{C}$ denote the collection of all cut-sets in G. Let $\mathbb{F}_2 := \{0, 1\}$ be a two-element finite field of arithmetic modulo two. We define the addition operation in $\mathcal{C}$, denoted by $+$, to be a function from $\mathcal{C}^2$ to $\mathcal{C}$ given by $C_1 + C_2 := C_1 \Delta C_2$ where Δ denotes the symmetric difference operation. What about scalar multiplication???

2.2 Properties (bug)

PROPOSITION 2.12 (Graph Theory An Introductory Course, Bollobas)**.** The complement of a disconnected graph is connected.

Proof. Let G be a disconnected graph. Let $\bar{G}$ denote the complement of G. Let $x, y \in V(\bar{G})$ be arbitrary. ... not finished □

2.3 Ear Decomposition

DEFINITION 2.13 (Adding a Path)**.** We say that a graph G is obtained from G_0 by **adding a path** P if and only if the following condition hold:

- G_0 and P are subgraphs of G with $G = G_0 \cup P$;
- $E(G_0) \cap E(P) = \varnothing$;
- $|V(G_0) \cap V(P)| = 2$;
- P is a path in G between the two vertices in $V(G_0) \cap V(P)$.

THEOREM 2.14 (Ear Decomposition for 2-Vertex-Connected Graphs)**.** Let G be a loopless graph with $|V(G)| \geq 3$. Then G is 2-vertex-connected if and only if there exist subgraphs G_0, G_1, ..., G_k of G such that

- G_0 is a cycle;
- $G_k = G$;
- $\forall i \in \{1, ..., k\}$, G_i is obtained from G_{i-1} by adding a path.

Such a finite sequence G_0, G_1, ..., G_k is an ear decomposition of G.

DEFINITION 2.15 (Adding a Cycle)**.** We say that a graph G is obtained from G_0 by **adding a cycle** C if and only if the following conditions hold:

- G_0 and C are subgraphs of G with $G = G_0 \cup C$;
- $E(G_0) \cap E(C) = \varnothing$;
- $|V(G_0) \cap V(C)| = 1$;
- C is a cycle in G.

THEOREM 2.16 (Ear Decomposition for 2-Edge-Connected Graphs)**.** Let G be a graph with $|E(G)| \geq 1$. Then G is 2-edge-connected if and only if there exist subgraphs $G_0, G_1, ..., G_k$ of G such that

- G_0 is a cycle;
- $G_k = G$;
- $\forall i \in \{1, ..., k\}$, G_i is obtained from G_{i-1} by adding a path or adding a cycle.

2.4 Menger's Theorem

THEOREM 2.17 (Menger's Theorem - Edge Connectivity)**.** Let G be a finite undirected graph and x and y two distinct vertices. Then the size of the minimum edge cut for x and y is equal to the maximum number of pairwise edge-independent paths from x to y.

THEOREM 2.18 (Menger's Theorem - Vertex Connectivity)**.** Let G be a finite undirected graph and x and y two nonadjacent vertices. Then the size of the minimum vertex cut for x and y is equal to the maximum number of pairwise vertex-independent paths from x to y.

2.5 Blocks of Graphs

DEFINITION 2.19 (Block)**.** Let G be a graph. Let B be a subgraph of G. We say that B is a **block** of G if and only if B is a maximal subgraph of G with the property that either B is 2-connected or $|V(B)| = 1$.

Note that the only single-vertex blocks are isolated vertices.

Chapter 3

Trees

3.1 Definitions

DEFINITION 3.1 (Spanning Tree)**.** Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G. We say that H is a **spanning tree** if H is a spanning subgraph of G and is a tree.

3.2 Properties

PROPOSITION 3.2. A graph is connected if and only if it has a spanning tree.

PROPOSITION 3.3 (Graph Theory An Introductory Course, Bollobas)**.** $d_1 \leq ... \leq d_n$ is the degree sequence of a tree if and only if $d_1 \geq 1$ and $\displaystyle\sum_{i=1}^{n} d_i = 2n - 2$.

PROPOSITION 3.4 (Graph Theory An Introductory Course, Bollobas)**.** Every integer sequence $d_1 \leq ... \leq d_n$ with $d_1 \geq 1$ and $\displaystyle\sum_{i=1}^{n} d_i = 2n - 2k$ is the degree sequence of a forest with k components.

Chapter 4

Graph Isomorphism

4.1 Definitions

DEFINITION 4.1 (Isomorphism)**.** Let G and H be two graphs. We define an **isomorphism** from G to H to be a function f from $V(G)$ to $V(H)$ such that

- f is bijective, and that
- for any pair of vertices $v, w \in V(G)$, $f(v)f(w) \in E(H)$ if and only if $vw \in E(G)$.

i.e., a bijective function that both itself and its inverse preserve adjacency.

DEFINITION 4.2 (Isomorphic)**.** Let G and H be two graphs. We say that G and H are **isomorphic**, denoted by $G \simeq H$, if there exists an isomorphism from G to H.

PROPOSITION 4.3. The relation $\simeq$ of isomorphism is an equivalence relation. That is, it is reflexive, symmetric, and transitive.

4.2 Properties

PROPOSITION 4.4. Let G and H be isomorphic graphs with isomorphism f. Then for any vertex $v \in V(G)$, we have $\deg_G(v) = \deg_H(f(v))$.

Chapter 5

Matchings and Covers

5.1 Matching

DEFINITION 5.1 (Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of E. We say that M is a **matching** in G if every vertex in the spanning subgraph (V, M) has degree at most one.

DEFINITION 5.2 (Saturated)**.** Let $(G = (V, E)$ be a graph. Let M be a subset of E. Let v be a vertex of G. We say that v is M**-saturated** if $\deg(v) = 1$ in (V, M).

DEFINITION 5.3 (Maximal Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximal matching** if it is a matching in G and any other matching is not a superset of it.

DEFINITION 5.4 (Maximum Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximum matching** if it is a matching in G and any other matching contains edges no more than M.

DEFINITION 5.5 (Perfect Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **perfect matching** if it matches all vertices of the graph. i.e., any vertex in G is incident to some edge in M.

PROPOSITION 5.6. Every maximum matching is maximal.

PROPOSITION 5.7. Every perfect matching is maximum.

PROPOSITION 5.8. Let $G = (V, E)$ be a graph. Let A and B be two maximal matchings of G. Then both $|A| \leq 2|B|$ and $|B| \leq 2|A|$.

5.2 Cover

DEFINITION 5.9 (Cover)**.** Let $G = (V, E)$ be a graph. Let C be a subset of V. We say that C is a **cover** of G if any edge has an end in C.

5.3 Relations Between Matchings and Covers

PROPOSITION 5.10. Let $G = (V, E)$ be a graph. Let M be a matching of G. Let C be a cover of G. Then $|M| \leq |C|$.

Chapter 6

Bipartite Graphs

6.1 Definitions

DEFINITION 6.1 (Bipartition)**.** Let $G = (V, E)$ be a graph. Let A and B be two subsets of V. We say the pair (A, B) is a **bipartition** of G if and only if $A \cap B = \varnothing$, $A \cup B = V$, and A and B are both independent.

DEFINITION 6.2 (Bipartite Graph)**.** Let $G = (V, E)$ be a graph. We say that G is **bipartite** if and only if there exists a bipartition of G.

DEFINITION 6.3 (Balanced Bipartite Graph)**.** Let $G = (V, E)$ be a bipartite graph with bipartition (A, B). We say that G is **balanced** if and only if $|A| = |B|$.

6.2 Properties of Bipartite Graphs

PROPOSITION 6.4. Let $G = (V, E)$ be a bipartite graph with bipartition (A, B). Then

$$\sum_{a \in A} \deg(a) = \sum_{b \in B} \deg(b) = |E|.$$

6.3 Characterizations

PROPOSITION 6.5. A graph is bipartite if and only if it has no odd cycles.

PROPOSITION 6.6. A graph is bipartite if and only if it is 2-colorable.

Chapter 7

Graph Planarity

7.1 Definitions

DEFINITION 7.1 (Polygon)**.** Let $P \subseteq \mathbb{R}^2$. We say that P is a **polygon** if and only if P is the union of a finite collection of line segments that is homeomorphic to the unit circle.

DEFINITION 7.2 (Region)**.** Let $O \subseteq \mathbb{R}^2$ be open. Then being linked by an arc in O defines an equivalence relation on O, and the corresponding equivalence classes are open subsets of O. We define the **regions** of O to be the equivalence classes of O under the relation of being linked by an arc.

DEFINITION 7.3 (Plane Embedding)**.** We define a **plane embedding** to be a function $\varphi : V(G) \cup E(G) \to \mathcal{P}(\mathbb{R}^2)$ such that

- $\forall x \in V(G)$, $\varphi(x) \in \mathbb{R}^2$;
- $\forall e \in E(G)$, $\varphi(e)$ is an arc between two vertices;
- different edges have different sets of endpoints;
- the interior of any edge contains no vertex and no point of any other edge.

DEFINITION 7.4 (Planar Graphs (Bollobas, Book))**.** Let G be a graph. Let $\{p_i\}_{i=1}^{|V(G)|}$ be distinct points in $\mathbb{R}^3$ such that no plane in $\mathbb{R}^3$ contains more than 3 of these points. Define the **realization** $R(G) \subseteq \mathbb{R}^3$ of G as

$$R(G) := \bigcup_{x_i x_j \in E(G)} \operatorname{conv}\{p_i, p_j\}.$$

We say that G is **planar** if and only if $R(G)$ is homeomorphic to a subset of $\mathbb{R}^2$.

DEFINITION 7.5 (Homeomorphic Grpahs (Bollobas Book))**.** Let G and H be graphs. We say that G is **homeomorphic** to H if and only if $R(G)$ is homeomorphic to $R(H)$, or equivalently, G and H have isomorphic subdivisions.

7.2 Properties

PROPOSITION 7.6. Every subgraph of a planar graph is planar.

PROPOSITION 7.7. A multi-graph is planar if and only if its simplification is planar.

PROPOSITION 7.8. A graph G is planar if and only if every subdivision of G is planar.

PROPOSITION 7.9. Let G be a multi-graph and e be an edge in G. Then G is planar if and only if $G \bullet e$ is planar.

PROPOSITION 7.10. A graph has an embedding if and only if it has a polygonal embedding.

THEOREM 7.11 (Fáry's Theorem)**.** A simple graph has an embedding if and only if it has a straight-lines embedding. That is, an embedding under which every edge is embedded as a line segment.

Proof. Let G be a simple graph. Let $n := |V(G)|$. It is trivial that if G has a straight-line embedding, then it has an embedding. Now I will show by induction that every simple planar graph has a straight-lines embedding.

Base Case: $n = 3$. Now G has only three vertices. The simple complete graph on three vertices is a triangle and trivially has a straight-lines embedding.

Inductive Step: Assume that any simple planar graph on $< n$ vertices has a straight-lines embedding. Let G be a simple planar graph on n vertices. Then $\exists v \in V(G)$ such that $\deg(v) < 5$. Let G' be obtained by removing v from G and re-triangulating the new face f formed when removing v. Then $|V(G')| = n - 1$. By the inductive hypothesis, G' has a straight-lines embedding. Now the region bounded by f is a polygon P with at most 5 sides. Place v in P and join v to the vertices of P via straight lines. By the art gallery theorem, there is a way to place v so that the straight lines from v to the vertices of P do not cross. So G has a straight-lines embedding. This completes the proof. $\square$

7.3 The Jordan Curve Theorem

7.4 Numerology

DEFINITION 7.12 (Footprint)**.** Let G be a planar graph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define the **footprint** of G, denoted by $fp(G)$, to be the union of the points and curves in $\mathbb{R}^2$ representing the vertices and edges in G.

7.5 Face of Planar Graphs

DEFINITION 7.13 (Face)**.** Let G be a planar graph with plane embedding φ. We define the collection of **faces** of G, denoted by $F(G)$, to be the collection of regions of $\mathbb{R}^2 \setminus \varphi(G)$.

DEFINITION 7.14 (Outer Face, Inner Face)**.** Let $D \subseteq \mathbb{R}^2$ be such that contains $\varphi(G)$. We define the **outer face** of G to be the face that contains the set $\mathbb{R}^2 \setminus D$. We define the **inner faces** of G to be the faces that are not the outer face.

DEFINITION 7.15 (Degree of Face)**.** We define the **degree** of a face f to be the number of edges in $\mathrm{bd}(f)$ with cut edges counted twice.

DEFINITION 7.16 (Face Regular)**.** We say that a planar graph is face regular if and only if all faces have the same degree.

PROPOSITION 7.17. Every planar graph has exactly one unbounded face.

PROPOSITION 7.18. Let G be a planar graph. Let $f \in F(G)$. Let H be a subgraph of G. Then $\exists f' \in F(H)$ such that $f' \supseteq f$. Moreover, if $\mathrm{bd}(f) \subseteq \varphi(H)$, then $f = f'$.

Proof. Since $H \subseteq G$, $\varphi(H) \subseteq \varphi(G)$. So $\mathbb{R}^2 \backslash \varphi(G) \subseteq \mathbb{R}^2 \backslash \varphi(H)$. Since $f \in F(G)$, $f \subseteq \mathbb{R}^2 \backslash \varphi(G)$ and f is connected. So $f \subseteq \mathbb{R}^2 \setminus \varphi(H)$ and f is connected. Let $f' \in F(H)$ be obtained by extending f to the maximal connected subset of $\mathbb{R}^2 \setminus \varphi(H)$.

Now assume further that $\mathrm{bd}(f) \subseteq \varphi(H)$. I will show that $f = f'$. Assume for the sake of contradiction that $f \subsetneq f'$. Since f' is connected, we get $\mathrm{bd}(f) \cap f' \neq \varnothing$. So it is not the case that $\mathrm{bd}(f) \subseteq \varphi(H)$. This contradicts to the assumption that $\mathrm{bd}(f) \subseteq \varphi(H)$. So $f = f'$. This completes the proof. $\square$

PROPOSITION 7.19. Let G be a planar graph with planar embedding φ and let $e \in E(G)$.

1. Let $f \in F(G)$. Then either $\varphi(e) \in \mathrm{bd}(f)$ or $\varphi(e)^\circ \cap \mathrm{bd}(f) = \varnothing$.
2. If e lies in some cycle C of G, then $\varphi(e)$ lies on the boundary of exactly two faces of G, and these two faces are contained in two distinct faces of C.

3. If e does not lie in any cycle of G, then $\varphi(e)$ lies on the boundary of exactly one face of G.

Proof. Let S be some line segment of $\varphi(e)$. Let $x_0 \in S^\circ$ be fixed. Notice that $\varphi(G) \setminus S^\circ$ is a compact subset of $\mathbb{R}^2$. So $\exists r_0 > 0$ such that $\mathrm{ball}(x_0, r_0) \cap \varphi(G) = \mathrm{ball}(x_0, r_0) \cap S$. Let $D_0 := \mathrm{ball}(x_0, r_0)$. Then $D_0 \setminus \varphi(G) = D_0 \setminus S$ is the union of two open half-discs. Notice that each of the two open half-discs is a connected subset of $\mathbb{R}^2 \setminus \varphi(G)$. So $\exists f_1, f_2 \in F(G)$, possibly the same, such that each of the two open half-discs is contained in f_1 or f_2, that $x_0 \in \mathrm{bd}(f_1) \cap \mathrm{bd}(f_2)$, and that $\forall f \in F(G) \setminus \{f_1\} \setminus \{f_2\}$, $x_0 \notin \mathrm{bd}(f)$.

Suppose that e lies in some cycle C of G. By the Polygonal Jordan Curve Theorem, D_0 intersects both faces of C. Since $C \subseteq G$, each of the faces f_1 and f_2 of G is contained in one of the two faces of C. So $f_1 \neq f_2$. So x_0 lies on the boundary of exactly two faces of G.

Suppose that e does not lie in any cycle of G. Then e is a bridge of G and $\exists X_1, X_2 \subseteq \varphi(G)$ such that $\varphi(G) \setminus \varphi(e)^\circ = X_1 \cup X_2$. Notice that $f_1 \cup \varphi(e)^\circ \cup f_2 \subseteq f$ for some $f \in F(G \setminus e)$. Then $f \setminus \varphi(e)^\circ$ is a face of G and $f_1, f_2 \subseteq f \setminus \varphi(e)^\circ$. Since f_1, f_2, and $f \setminus \varphi(e)^\circ$ are all faces of G, we get $f_1 = f_2 = f \setminus \varphi(e)^\circ$. So x_0 lies on the boundary of exactly one face of G.

Let $x_1 \in \varphi(e)^\circ$ be arbitrary. Since $\varphi(G) \setminus \varphi(e)^\circ$ is compact, $\exists r_1 > 0$ such that $\mathrm{ball}(x_1, r_1) \cap \varphi(G) = \mathrm{ball}(x_1, r_1) \cap \varphi(e)$. Let $D_1 := \mathrm{ball}(x_1, r_1)$. Then $\forall b \in D_1 \setminus \varphi(e)$, $\exists a \in D_0 \setminus \varphi(e)$ such that a and b are polygonally connected. So x_1 cannot lie on the boundary of a face of G that is not f_1 or f_2. Now x_1 indeed lies on the boundary of both f_1 and f_2 since any point in $D_0 \setminus \varphi(e)$ can be polygonally connected to some point in $D_1 \setminus \varphi(e)$. This completes the proof. □

COROLLARY 7.20. The frontier of a face is always the point set of a subgraph.

PROPOSITION 7.21. An edge e in a planar multi-graph is a bridge if and only if the two faces on either side of the curve γ_e are the same.

PROPOSITION 7.22. If a plane graph has two different faces with the same boundary, then the graph is a cycle.

Proof. If G is a forest, then G has exactly one face. This contradicts to the assumption. So G contains a cycle C. Let f_1 and f_2 denote the two faces of C. Then $\mathrm{bd}(f_1) = \mathrm{bd}(f_2)$. Assume for the sake of contradiction that $C \subsetneq G$. Then $\varphi(C) \subsetneq \varphi(G)$ and hence $\forall f \in F(G)$, either $f \subseteq f_1$ or $f \subseteq f_2$. So $\forall a, b \in F(G)$ such that $a \cup b \subseteq f_i$, we must have $\mathrm{bd}(a) \neq \mathrm{bd}(b)$.

Let $a, b \in F(G)$ be such that $a \subseteq f_1$ and $b \subseteq f_2$. Since $\varphi(G) \setminus \varphi(C) \neq \varnothing$, $a \subsetneq f_1$ and $b \subsetneq f_2$. So $\mathrm{bd}(a) \neq \mathrm{bd}(f_1)$ and $\mathrm{bd}(b) \neq \mathrm{bd}(f_2)$. Notice that in this case we have $a \cap b = \varnothing$. not finished $\square$

PROPOSITION 7.23. Let G be a planar graph with a cycle. Then the boundary of every face of G contains a cycle of G.

Proof. Let φ be a planar embedding of G. Assume for the sake of contradiction that there is some face f of G such that $\mathrm{bd}(f)$ does not contain a cycle of G. We know that $\mathrm{bd}(f) = \varphi(H)$ for some $H \subseteq G$. So H does not contain a cycle and hence is a forest. We know that each forest has only one face. So f is the only face of H. So $f \cup \varphi(H) = \mathbb{R}^2$. Notice $f \cap \varphi(G) = \varnothing$ and $\varphi(H) \subseteq \varphi(G)$. So $\varphi(G) = \varphi(H)$ and $G = H$. So G is also a forest, contradicting to the assumption that G contains a cycle. So $\forall f \in F(G)$, $\mathrm{bd}(f)$ contains a cycle of G. $\square$

PROPOSITION 7.24. Let G be a 2-connected planar graph on ≥ 3 vertices. Then the boundary of every face of G is a cycle of G.

PROPOSITION 7.25. The class of planar graphs that are both regular and face-regular is

PROPOSITION 7.26. Every planar graph is the union of three forests.

7.6 Euler's Formula

COROLLARY 7.27. Let G be a planar graph with $|V(G)| \geq 3$. Then $|E(G)| \leq 3|V(G)| - 6$. If $\varphi(G)$ equals its triangulation, then $|E(G)| = 3|V(G)| - 6$.

REMARK 7.28. The converse of the above corollary is not true. Consider the graph $G := K_{3,3}$. We have $|E(G)| = 9$ and $|V(G)| = 6$. So G satisfies

$$|E(G)| = 9 \leq 12 = 3|V(G)| - 6.$$

However, we know that $K_{3,3}$ is not planar.

PROPOSITION 7.29. Let G be a planar graph on ≥ 3 vertices with $\chi(G) = 2$. Then $|E(G)| \leq 2|V(G)| - 4$.

PROPOSITION 7.30. Let G be a connected planar graph with girth $g \geq 3$. Then $m \leq \frac{g}{g-2}(|V(G)| - 2)$.

7.7 Vertex Degrees in Planar Graphs

PROPOSITION 7.31. Let G be a planar graph on ≥ 2 vertices. Then $\exists x, y \in V(G)$ with $x \neq y$ such that $\deg(x) \leq 5$ and $\deg(y) \leq 5$.

Proof. Let $X \subseteq V(G)$ denote the set of vertices whose degree is ≤ 5. Then it is equivalent to show that $|X| \geq 2$. Notice that $\forall x \in V(G) \setminus X$, $\deg(x) \geq 6$. Assume for the sake of contradiction that $|X| < 2$. Then by the Degree Sum Formula, we get

$$\begin{aligned} |E(G)| &= \frac{1}{2} \sum_{x \in V(G)} \deg(x) = \frac{1}{2} \left[\sum_{x \in X} \deg(x) + \sum_{x \in V(G) \setminus X} \deg(x) \right] \\ &\geq \frac{1}{2} \sum_{x \in V(G) \setminus X} \deg(x) \geq \frac{1}{2} \cdot |V(G) \setminus X| \cdot 6 = 3|V(G) \setminus X| \\ &> 3(|V(G)| - 2) = 3|V(G)| - 6. \end{aligned}$$

That is, $|E(G)| > 3|V(G)| - 6$. This contradicts to the assumption that G is planar. So $|X| \geq 2$. This completes the proof. $\square$

PROPOSITION 7.32. Let G be a planar graph on < 12 vertices. Then $\exists x \in V(G)$ such that $\deg(x) \leq 4$.

Proof. Assume for the sake of contradiction that $\forall x \in V(G)$, $\deg(x) \geq 5$. Then by the Degree Sum Formula, we get

$$|E| = \frac{1}{2} \sum_{x \in V(G)} \deg(x) \geq \frac{1}{2} \cdot |V(G)| \cdot 5 = \frac{5}{2}|V(G)|. \tag{7.1}$$

On the other hand, since G is planar, we have $|E| \leq 3|V(G)| - 6$. So we must have $\frac{5}{2}|V(G)| \leq 3|V(G)| - 6$. Rearranging the terms we get $|V(G)| \geq 12$. This contradicts to the assumption that $|V(G)| < 12$. So $\exists x \in V(G)$ such that $\deg(x) \leq 4$. $\square$

7.8 Unique Plane Embeddings

PROPOSITION 7.33. A cycle in a simple 3-connected plane graph is a facial cycle if and only if it is non-separating.

Proof. ($\Leftarrow$) Suppose that C is non-separating and induced. Then in any embedding φ of G, any two points x and y in $\varphi(G) \setminus \varphi(C)$ are joined by a polygonal arc in $\varphi(G) \setminus \varphi(C)$. In particular, this arc avoids the polygon $\varphi(C)$. So any two points in $\varphi(G) \setminus \varphi(C)$ lie in the same region of $\mathbb{R}^2 \setminus \varphi(C)$. By the PJCT, the other region of $\mathbb{R}^2 \setminus \varphi(C)$ contains no points of $\varphi(G) \setminus \varphi(C)$ and hence a face of φ bounded by C.

($\Rightarrow$) Suppose that C is a facial cycle of some face $f \in F(G)$. I will show that C is non-separating and induced. Assume for the sake of contradiction that C is not induced. Then $\exists x, y \in V(G)$ that are adjacent in G but not in C. Since $xy \notin E(C)$, we know that $C - \{x, y\}$ has exactly two connected components. Let u and v be vertices in two different components of $C - \{x, y\}$. Since G is 3-connected, $G - \{x, y\}$ is connected. So there is a uv-path Q in $G - \{x, y\}$. Since f is a face, both the arcs $\varphi(xy)$ and $\varphi(Q)$ have their interior in the region of $\mathbb{R}^2 \setminus \varphi(C)$ that is not f. But the paths xy and Q are disjoint in G so $\varphi(xy)$ and $\varphi(Q)$ should be disjoint in $\mathbb{R}^2$. This contradicts the θ-lemma. So C is induced.

Assume for the sake of contradiction that C is separating. Then $G - V(C)$ is disconnected. Let x and y be vertices in two distinct components of $G - V(C)$. By 3-connectedness of G, there are 3 internally disjoint xy-paths P_1, P_2, and P_3 in G. Since C is separating, each P_i must intersect $V(C)$ in one of its internal vertices. By the θ-lemma, the set $\mathbb{R}^2 \setminus (\varphi(P_1) \cup \varphi(P_2) \cup \varphi(P_3))$ has 3 regions, each bounded by the cycle containing two of these paths. Since f is a connected subset of $\mathbb{R}^2 \setminus \varphi(G)$, f must be contained in one of these 3 regions, say the region bounded by $\varphi(P_1) \cup \varphi(P_2)$. Then the boundary of f is contained in the subset of $\varphi(G)$ within this region. In particular, the boundary of f contains no internal vertex in P_3. However, the boundary of f is C. Since x and y are disconnected in $G - V(C)$, the path P_3 does contain an interval vertex in C. This leads to a contradiction. So C is separating. This completes the proof. $\square$

PROPOSITION 7.34. Every simple 3-connected planar graph has a unique planar embedding.

7.9 Graph Minors

PROPOSITION 7.35. Minors of planar graphs are planar.

PROPOSITION 7.36. If a simple graph G does not contain K_3 as subgraph, then $\delta(G)+\Delta(G) \leq |V(G)|$ where $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum vertex degree of G.

Proof. Suppose that G does not contain K_3 as a subgraph. Assume for the sake of contradiction that $\delta(G) + \Delta(G) > |V(G)|$. Let $x \in V(G)$ be such that $\deg(x) = \Delta(G)$. Let $v \in V(G) \setminus \{x\}$ be arbitrary. Then $\deg(v) \geq \delta(G)$ and hence $\deg(v) + \deg(x) > |V(G)|$. Since G is a simple graph, $\forall y \in V(G)$, $\deg(y) = |\mathcal{N}(y)|$. So $\mathcal{N}(v) \cap \mathcal{N}(x) \neq \varnothing$. So there is some $w \in V(G)$ such that $vw, xw \in E(G)$. Since G does not contain K_3 as a subgraph, $xv \notin E(G)$. This holds for any $v \in V(G) \setminus \{x\}$. So $\deg(x) = 0$. That is, $\delta(G) = \Delta(G) = 0$. This contradicts to the assumption that $\delta(G) + \Delta(G) > |V(G)| \geq 0$. So $\delta(G) + \Delta(G) \leq |V(G)|$. □

PROPOSITION 7.37. A simple graph has a K_3-minor if and only if it contains a cycle.

Proof. ($\Rightarrow$) Suppose that G has a K_3-minor. Then $\exists V_1, V_2, V_3 \subseteq V(G)$ such that $\forall i, j \in \{1, 2, 3\}$, $V_i \cap V_j = \varnothing$, not finished

($\Leftarrow$) Suppose that G contains a cycle C. Then we can obtain a K_3-minor by deleting all vertices not in $V(C)$ and all edges not in $E(C)$ to obtain C, and then reduce C to K_3 by contracting edges. So G has a K_3-minor. □

PROPOSITION 7.38. Let F be a graph with $\Delta(F) \leq 3$. Then a graph G has an F-minor if and only if G has contains an F-subdivision.

Proof. The backward direction is trivial. Now suppose that G has an F-minor. I will show that G contains an F-subdivision. □

7.10 Kuratowski's Theorem

LEMMA 7.39. Every planar graph does not contain K_5 or $K_{3,3}$ minors.

LEMMA 7.40. Every edge-maximal graph on at least 4 vertices with no K_5 or $K_{3,3}$ minors is 3-connected.

LEMMA 7.41. Every 3-connected graph with no K_5 or $K_{3,3}$ minors is planar.

THEOREM 7.42 (Kuratowski's Theorem)**.** A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ minors.

7.11 Exercises

EXERCISE 7.43 (Bondy and Murty 2018 Book)**.** Show that any 3-connected cubic plane graph on n vertices, where $n \geq 6$, may be obtained from one on $n - 2$ vertices by subdividing two edges in the boundary of a face and joining the resulting new vertices by an edge subdividing the face.

Chapter 8

Duality

8.1 Definitions

DEFINITION 8.1 (Dual Graph)**.** Let $G = (V, E, B)$ be a multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of G. Let $\mathcal{F}$ be the set of faces of G. We define the **dual graph** of this embedding to be the multigraph $G^* = (V^*, E^*, B^*)$ where $V^* = \mathcal{F}$ and $E^* = \{e^* : e \in E\}$.

PROPOSITION 8.2. Let $G = (V, E, B)$ be a multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of G. Let $G^* = (V^*, E^*, B^*)$ be the dual graph of G. Then for any face $f \in \mathcal{F}$, the degree of f as a face of $\mathcal{P}, \Gamma$ equals the degree of f as a vertex of G^*.

PROPOSITION 8.3. If G is a connected multigraph embedded in the plane, then G^{**} is isomorphic with G.

Chapter 9

Graph Coloring

9.1 Chromatic Number

DEFINITION 9.1 ((Proper) Coloring)**.** Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define a **(proper)** X**-coloring** of G to be a function $f : V \to X$ such that if $vw \in E$, then $f(v) \neq f(w)$.

DEFINITION 9.2 (Chromatic Number)**.** Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define the **chromatic number** of G, denoted by $\chi(G)$, to be the smallest natural number $k \in \mathbb{N}$ for which G has a (proper) k-coloring.

PROPOSITION 9.3. The chromatic number exists and $\chi(G) \leq |V|$.

Proof. Take $X = V$. □

PROPOSITION 9.4. G is complete if and only if $\chi(G) = |V(G)|$.

PROPOSITION 9.5. The only graph with chromatic number zero is the empty graph.

PROPOSITION 9.6. A graph has chromatic number one if and only if it has no edges and at least one vertex.

PROPOSITION 9.7. A graph has chromatic number two if and only if it is bipartite and has at least one edge.

PROPOSITION 9.8. Let G be a graph. Let $d_{max}(G)$ be the maximum degree of a vertex in G. Then $\chi(G) \leq 1 + d_{max}(G)$.

9.2 5-color Theorem

THEOREM 9.9. Every planar graph is 5-colorable.

Proof. (1890)

True for $|V| \leq 5$.

Inductively, suppose the theorem holds for planar graphs on $n-1$ vertices for $n \geq 5$. Suppose G is a planar graph on n vertices.

Let v be a vertex of degree ≤ 5 in G. This exists by a lemma in our lectures.

Since G is a planar, $G-v$ is planar. By the induction hypothesis, $G-v$ has a 5-coloring.

If some color does not appear on any neighbor of v, we can extend the coloring to a coloring of G.

Otherwise, v has exactly 5 neighbors with different colors.

For each pair i, j of colors, let G_{ij} be the subgraph of $G-v$ induced by the vertices colored i or j.

If the component H of G_{ij} containing x_i does not contain x_j, then we can switch the colors of all vertices in H between i and j to get a coloring of $G-v$ that assigns only 4 colors to neighbors of v, and thus extends to a coloring of G.

So G_{ij} contains a path from x_i to x_j.

Because $G_{2,5}$ and $G_{1,4}$ have disjoint vertex sets, this contradicts the planarity of G.

$\square$

DEFINITION 9.10 (Near-triangulation)**.** Planar drawing of G where the infinite face is bounded by a cycle, and every other face is bounded by a triangle

THEOREM 9.11. Every planar near-triangulation has a 5-coloring.

Theorem 9.11 $\implies$ Theorem 9.9.

DEFINITION 9.12 (List Assignment)**.** A **list assignment** L of G is a function that assigns a set $L(v)$ of colors to each $v \in V$.

DEFINITION 9.13 (L-coloring)**.** An L-coloring of G is a choice of a color in $L(v)$ for each $v \in V$ such that adjacent vertices get different colors.

DEFINITION 9.14 (5-list-colorable)**.** A graph is **5-list-colorable** if for every list assignment L of G with $|L(v)| \geq 5$, G is L-colorable.

THEOREM 9.15. Every planar near-triangulation is 5-list-colorable.

Theorem 9.15 $\implies$ Theorem 9.11 because coloring is a special case of list coloring.

THEOREM 9.16 (Carsten Thomassen, 1993)**.** If G is a near-triangulation and L is a list assignment such that

1. $|L(v)| = 5$ for every non-boundary vertex,
2. $|L(v)| = 3$ for every boundary vertex.

Then G has an L-coloring even if two adjacent boundary vertices have their colors arbitrarily decided in advance.

Proof.

Case 1. There is a "chord" between two boundary vertices.

Let G_1 and G_2 be subgraph of G obtained by "cutting" G along the chord, where G_1 contains the pre-colored vertices.

By applying the inductive hypothesis to G_1, and then applying it to G_2 with the two ends of the chord pre-colored according to the coloring of G_1, we get a coloring of G_1.

Case 2. There is no chord.

Let u and u' be the pre-colored vertices.

Let x, y be the next two vertices occuring in order around the boundary.

$\square$

Theorem 9.16 $\implies$ Theorem 9.15.

Chapter 10

Probability and Edge Density

Q: Let G be a graph on n vertices with no triangles. How many edges can G have?

THEOREM 10.1 (Mantel)**.** If G is triangle-free and has n vertices, then

$$|E| \leq \frac{n^2}{4}.$$

Proof. Let $P_{2,1}$ denote the probability that a pair of distinct vertices chosen uniformly at random, are adjacent.

$$P_{2,1} = |E|/\binom{n}{2}.$$

Let $P_{3,2}$ denote the probability that a randomly chosen triple of vertices contains exactly two edges. Let $P_{3,1}$ denote ... one edge. Let $P_{3,0}$ denote ... no edges. Notice $P_{3,2}+P_{3,1}+P_{3,0} = 1$.

Part 1: Show that $P_{2,1} = \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1}$. Notice that the graph is triangle-free. So $P_{3,3} = 0$. Choosing a pair at random is the same as choosing a triple at random, then choosing a pair at random within that triple.

For a fixed vertex v, let $Q_{v,1}$ denote the probability that a randomly chosen vertex $u \neq v$ is adjacent to v.

$$Q_{v,1} = \frac{deg(v)}{n-1}.$$

Let $Q_{v,2}$ denote the probability that two distinct randomly chosen vertices other than v are both adjacent to v.

$$Q_{v,2} = \binom{deg(v)}{2}/\binom{n-1}{2}.$$

Part 2: Show that $Q_{v,1}^2 \approx Q_{v,2}$. Both give (essentially) the probability that a pair x, y of vertices other than v are both adjacent to v. The LHS allows $x = y$. The RHS does not. But $x = y$ occurs with negligible probability.

Part 3: Show that $P_{2,1} = \frac{1}{n}\sum_v Q_{v,1}$. Both the RHS and LHS are just the probability of an edge between two vertices. The RHS calls the first chosen vertex v.

Part 4: Show that $\frac{1}{3}P_{3,2} = \frac{1}{n}\sum_v Q_{v,2}$. Both sides give the probability that, if we choose 3 vertices at random, and then choose one among those 3 and call it v, that v is adjacent to both the others.

Proof of the theorem.

$$\begin{aligned} P_{2,1} &= \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1} \geq \frac{2}{3}P_{3,2} \\ &= 2\left(\frac{1}{n}\sum_v Q_{v,2}\right) \approx 2\left(\frac{1}{n}\sum_v Q_{v,1}^2\right) \\ &\geq 2\left(\frac{1}{n}\sum_v Q_{v,1}\right)^2 = 2P_{2,1}^2. \end{aligned}$$

So $P_{2,1} \leq \frac{1}{2}$. So $|E| \leq \frac{n^2}{4}$. □

Q: If G has n vertices, no K_{t+1}-subgraph, how many edges can G have?

THEOREM 10.2 (Turan)**.** If G is a graph on n vertices with no K_{t+1}-subgraph, then

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{t}\right).$$

THEOREM 10.3 (Erdos-Stone)**.** If H is a graph and G is a graph on n vertices without H as a subgraph, then

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{\chi(H) - 1} + \varepsilon(n)\right)$$

where $\varepsilon(n) \to 0$ as $n \to \infty$ and $\chi(H)$ is the chromatic number of H, the fewest number of colors needed to properly color the vertices of H.

Chapter 11

Weird Stuffs

11.1 Geometric Representation of Graphs

DEFINITION 11.1 (Geometric Representation)**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. We define a **geometric representation** of G to be a map from V to $\mathbb{R}^d$.

DEFINITION 11.2 (Unit Distance Representation)**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \to \mathbb{R}^d$ be a geometric representation of G. We say that u is a **unit distance representation** of G if and only if $\forall \{i, j\} \in E$, $\|u(i) - u(j)\|_2 = 1$.

DEFINITION 11.3 (Orthonormal Representation)**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \to \mathbb{R}^d$ be a geometric representation of G. We say that u is an **orthonormal representation** of G if and only if

- $\forall i \in V$, $\|u(i)\|_2 = 1$; and
- $\forall \{i, j\} \in \overline{E}$, $\langle u(i), u(j) \rangle = 0$ where $\overline{E}$ is the edge set of the complement of G.

DEFINITION 11.4. We define $t_h(G)$ to be the square radius of the smallest hypersphere that contains a unit distance representation of G.

THEOREM 11.5 (CO 471, Spring 2022, Levent Tuncel)**.** Let $G = (V, E)$ be a graph. Then

$$\begin{aligned} t_h(G) \quad = \quad & \min \quad t \\ & \text{subject to:} \quad X_{ii} = t, \forall i \in V \\ & X_{ii} - 2X_{ij} + X_{jj} = 1, \forall \{i, j\} \in E \\ & X \in S_+^V \end{aligned}$$

PROPOSITION 11.6 (CO 471, Spring 2022, Levent Tuncel)**.** Let $G = (V, E)$ be a graph. Then G is bipartite if and only if $t_h(G) \leq \frac{1}{4}$.

Proof. □

PROPOSITION 11.7. Let $n \in \mathbb{Z}_{++}$. Let K_n denote the n-clique. Then $t_h(K_n) =$.

Proof. □

11.2 Stable Sets

DEFINITION 11.8 (Stable Sets)**.** Let $G = (V, E)$ be a graph. Let S be a subset of the vertex set V. We say that S is a **stable set** in G if and only if $\forall \{i, j\} \in E$, at most one of i or j is in S. i.e., S is a set of pairwise non-adjacent vertices.

DEFINITION 11.9 (Stability Number)**.** Let $G = (V, E)$ be a graph. We define the **stability number** of G, denoted by $\alpha(G)$, to be a number given by

$$\alpha(G) := \max\{|S| : S \text{ is stable in } G\}.$$

DEFINITION 11.10 (Stable Set Polytope)**.** Let $G = (V, E)$ be a graph. We define the **stable set polytope** of G, denoted by $\mathrm{STAB}(G)$, to be a subset of $\mathbb{R}^V$ given by

$$\mathrm{STAB}(G) := \mathrm{conv}\left\{x \in \{0,1\}^V : x \text{ is the incidence vector of some stable set in } G\right\}.$$

DEFINITION 11.11 (Fractional Stable Set Polytope)**.** Let $G = (V, E)$ be a graph. We define the **fractional stable set polytope** of G, denoted by $\mathrm{FRAC}(G)$, to be a subset of $\mathbb{R}^V$ given by

$$\mathrm{FRAC}(G) := \left\{x \in [0,1]^V : x_i + x_j \leq 1, \forall \{i,j\} \in E\right\}.$$

PROPOSITION 11.12. Let $G = (V, E)$ be a graph. Then

$$\mathrm{STAB}(G) = \mathrm{conv}(\mathrm{FRAC}(G) \cap \{0,1\}^V).$$

11.3 Clique Polytope

DEFINITION 11.13. Let $A_{clq}(G)$ denote the 0-1 clique-node incidence matrix of G where each row corresponds to a clique.

DEFINITION 11.14 (Clique Polytope)**.** We define the **clique polytope** of G to be

$$\mathrm{CLQ}(G) := \{x \in \mathbb{R}^V_+ : A_{clq}(G)x \leq \bar{e}\}$$

11.4 Theta Bodies

DEFINITION 11.15 (Theta Body)**.** Let $G = (V, E)$ be a graph. We define the

theta body of G, denoted by $\mathrm{TH}(G)$, to be a subset of $\mathbb{R}_+^V$ given by

$$\mathrm{TH}(G) := \left\{ x \in \mathbb{R}_+^V : \sum_{i \in V} (c^\top u(i))^2 x_i \leq 1, \begin{array}{l} \forall c \in \mathbb{R}^V : \|c\|_2 = 1, \\ \forall \text{ orth. rep. } u \text{ of } G \end{array} \right\}.$$

THEOREM 11.16 (CO 471, Spring 2022, Levent Tuncel)**.** For every graph $G = (V, E)$, $\mathrm{TH}(G)$ is a nonempty compact convex set such that

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{CLQ}(G) \subseteq \mathrm{FRAC}(G).$$

Proof. We already observed $\mathrm{CLQ}(G) \subseteq \mathrm{FRAC}(G)$ for all graphs G.
Part 1: Show that $\mathrm{TH}(G) \subseteq \mathrm{CLQ}(G)$. Let $\mathcal{C} \subseteq V$ be a nonempty clique in G. Let $C \in \mathbb{R}^V$ be any vector with $\|c\|_2 = 1$. Let $u(i) := c$, $\forall i \in \mathcal{C}$. For all $i \in V \setminus \mathcal{C}$, choose $u(i)$'s as an orthonormal system in $\{c\}^\perp$. Note: u is an orthonormal representation of G. The corresponding orthonormal representation constraint is

$$1 \geq \sum_{i \in V} (c^\top u(i))^2 x_i = \sum_{i \in \mathcal{C}} \underbrace{(c^\top u(i))^2}_{=1} x_i + \sum_{i \in V \setminus \mathcal{C}} \underbrace{(c^\top u(i))^2}_{=0} x_i = \sum_{i \in \mathcal{C}} x_i.$$

Also, by definition $\mathrm{TH}(G) \subseteq \mathbb{R}_+^V$; therefore, $\mathrm{TH}(G) \subseteq \mathrm{CLQ}(G)$ for all graph G.
Part 2: Show that $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G)$. We will show that incidence vectors of every stable set in G belongs to $\mathrm{TH}(G)$. Since $\mathrm{TH}(G)$ is a convex set and $\mathrm{STAB}(G)$ is the convex hull of these incidence vectors, this will prove $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G)$. Let $\mathcal{S} \subseteq V$ be a stable set in G. Let $\bar{x} \in \{0,1\}^V$ be the incidence vector of $\mathcal{S}$. Clearly $\bar{x} \geq 0$. Let $u : V \to \mathbb{R}^V$ be any orthonormal representation of G. Let $c \in \mathbb{R}^V$ be any vector such that $\|c\|_2 = 1$. We may assume $\mathcal{S} = \{1, ..., k\}$. Define $Q^\top := (u(1), ..., u(k))$. Then by definition of stable sets, $Q^\top$ is orthogonal.

$$\begin{aligned} \sum_{i \in V} (c^\top u(i))^2 \bar{x}_i &= \sum_{i \in \mathcal{S}} (c^\top u(i))^2 \underbrace{\bar{x}_i}_{=1} + \sum_{i \in V \setminus \mathcal{S}} (c^\top u(i))^2 \underbrace{\bar{x}_i}_{=0} = \sum_{i \in \mathcal{S}} (c^\top u(i))^2 \\ &= \|Q^\top c\|_2^2 \leq \|\tilde{Q}^\top c\|_2^2 = \|c\| = 1. \end{aligned}$$

where $\tilde{Q}^\top = [Q^\top |$ complete to an orthonormal basis $]$. Since $0 \in \mathrm{TH}(G)$, $\mathrm{TH}(G) \neq \varnothing$. Since $\mathrm{TH}(G) \subseteq \mathrm{FRAC}(G) \subseteq [0,1]^V$, $\mathrm{TH}(G)$ is bounded. ☐

DEFINITION 11.17 (Lovase Theta Function)**.** Let $G = (V, E)$ be a graph. Let $w \in \mathbb{R}_+^V$. We define the **Lovase Theta function**, denoted by θ, to be a function of G

and w given by

$$\theta(G,w) := \max\{w^\top x : x \in \mathrm{TH}(G)\}.$$

DEFINITION 11.18 (Lovase Theta Number)**.** Let $G=(V,E)$ be a graph. We define the **Lovase Theta number** of G, denoted by $\theta(G)$, to be a number given by

$$\theta(G) := \theta(G,\bar{e}) = \max\{\bar{e}^\top x : x \in \mathrm{TH}(G)\}.$$

THEOREM 11.19 (CO 471, Spring 2022, Levent Tuncel)**.** Let $G=(V,E)$ be a graph. Let $w \in \mathbb{R}^V_+$ be a weight vector. Define a matrix $W \in \mathbb{S}^V$ by $W_{ij} := \sqrt{w_i w_j}$, $\forall i,j \in V$. Then the following quantities are the same:

1. $\theta(G,w)$;
2. If $w_i = 0$, define $\dfrac{w_i}{(c^\top u(i))^2} := 0$,
$$\inf\left\{\max_{i\in V}\left\{\frac{w_i}{(c^\top u(i))^2}\right\} : \begin{array}{l} c \in \mathbb{R}^V, \|c\|_2 = 1, \\ u \text{ is an orth. rep. of } G \end{array}\right\};$$
3. $\min\{\eta \in \mathbb{R} : S \in \mathbb{S}^V, \mathrm{diag}(S) = 0, S_{ij} = 0, \forall\{i,j\} \in \overline{E}, \eta I - S \succeq W\}$;
4. $\max\{\mathrm{tr}(WX) : X_{ij} = 0, \forall\{i,j\} \in E, \mathrm{tr}(X) = 1, X \in \mathbb{S}^V_+\}$.

11.5 Product of Graphs

DEFINITION 11.20 (Strong Product)**.** Let $G=(V,E)$ and $H=(W,F)$ be graphs. We define the **strong product** of G and H, denoted by $G \otimes H$, to be a graph given by $G \otimes H = (V(G\otimes H), E(G \times H))$ where

$$V(G\otimes H) := V \times W \text{ and}$$
$$E(G\otimes H) := \left\{ \Big\{(i,u),(j,v)\Big\} : \begin{array}{l} (\{i,j\} \in E \text{ and } \{u,v\} \in F) \text{ or} \\ (\{i,j\} \in E \text{ and } u = v \in W) \text{ or} \\ (i = j \in V \text{ and } \{u,v\} \in F) \end{array} \right\}.$$

PROPOSITION 11.21. Let $G = (V, E)$ and $H = (W, F)$ be graphs. Then

$$\theta(G \otimes H) \leq \theta(G) \times \theta(H).$$

DEFINITION 11.22 (Shannon Capacity)**.** Let $G = (V, E)$ be a graph. We define the **Shannon capacity** of G, denoted by $\Theta(G)$, to be a number given by

$$\Theta(G) := \limsup_{k \to +\infty} (\alpha(G^{\otimes k}))^{1/k}$$

where $\alpha(G^{\otimes k})$ denotes the stability number of $G^{\otimes k}$.

11.6 Lift-and-Project Operators

DEFINITION 11.23 (Lift-and-Project Operators)**.** Let P be a convex subset of $[0, 1]^d$. Define a subset K_P of $\mathbb{R}^{1+d}$ by $K_P := \text{cone}(1 \oplus P)$. Define a subset $M_+(P)$ of $\mathbb{S}_+^{1+d}$ by

$$M_+(P) := \left\{ Y \in \mathbb{S}_+^{1+d} : \text{diag}(Y) = Ye_0, \begin{array}{r} Ye_i \in K_P, \forall i \in \{1, ..., d\}, \\ Y(e_0 - e_i) \in K_P, \forall i \in \{1, ..., d\} \end{array} \right\}.$$

We define the **lift-and-project operator**, denoted by LS_+, to be a function from $\mathcal{P}(\mathbb{R}^d)$ to $\mathcal{P}(\mathbb{R}^d)$ given by

$$\text{LS}_+(P) := \left\{ x \in \mathbb{R}^d : \begin{pmatrix} 1 \\ x \end{pmatrix} = Ye_0 \text{ for some } Y \in M_+(P) \right\}.$$