Order Theory

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Chapter 1

Infimum and Supremum

1.1 Arithmetic Properties

Proposition 1.1.1. *For any non-empty set S and any non-negative real-valued function f defined on S, we have*

$$\big[\inf_{x\in S} f(x)\big]^2 = \inf_{x\in S}\big[f(x)\big]^2.$$

Proof. Let x be an arbitrary element in S. Since $\inf_{x\in S} f(x) \le f(x)$ and $\inf_{x\in S} f(x) \ge 0$ and $f(x) \ge 0$, we get $\big[\inf_{x\in S} f(x)\big]^2 \le f^2(x)$. So the LHS is a lower bound for the set $\big\{f^2(x) : x \in S\big\}$. Since S is non-empty, $\exists M > 0$ such that

$$\big|\inf_{x\in S} f(x)\big| < M.$$

Let ε be an arbitrary positive number. Since $\varepsilon > 0$, $\exists x_0 \in S$ such that

$$\big|f(x_0) - \inf_{x\in S} f(x)\big| < \min\left\{1, \frac{\varepsilon}{2M+1}\right\}. \tag{1}$$

Then

$$\begin{aligned}
&\big|f(x_0) + \inf_{x\in S} f(x)\big| \\
=\ &\big|f(x_0) - \inf_{x\in S} f(x) + 2\inf_{x\in S} f(x)\big| \\
\le\ &\big|f(x_0) - \inf_{x\in S} f(x)\big| + 2\big|\inf_{x\in S} f(x)\big| \\
<\ &1 + 2M.
\end{aligned}$$

That is,

$$\left|f(x_0) + \inf_{x\in S} f(x)\right| \leq 2M + 1. \tag{2}$$

From (1) and (2), we get

$$\begin{aligned}
&\left|f^2(x_0) - \left[\inf_{x\in S} f(x)\right]^2\right| \\
&= \left|f(x_0) - \inf_{x\in S} f(x)\right|\left|f(x_0) + \inf_{x\in S} f(x)\right| \\
&< \frac{\varepsilon}{2M+1}(2M+1) \\
&= \varepsilon.
\end{aligned}$$

That is,

$$\left|f^2(x_0) - \left[\inf_{x\in S} f(x)\right]^2\right| < \varepsilon. \tag{3}$$

That is, $\forall \varepsilon > 0$, $\exists x_0 \in S$ such that (3) holds. So

$$\left[\inf_{x\in S} f(x)\right]^2 = \inf_{x\in S} \left[f(x)\right]^2,$$

as desired. ∎

Chapter 2

Binary Relations

Definition (Binary Relations)**.** *Let X and Y be two sets. We define a **binary relation** over sets X and Y to be a subset of the Cartesian product $X \times Y$.*

2.1 Homogeneous Relations

Definition (Homogeneous Relation)**.** *Let X be a set. We define a **homogeneous relation** over X, denoted by R, to be a binary relation over X and itself. i.e. a subset of the cartesian product $X \times X$.*

Proposition 2.1.1 (Some properties that a homogeneous relation <u>**may**</u> have)**.**
Group 1: Reflexivity.

- *Reflexive.*
$$\forall x \in X, \quad xRx.$$
- *Irreflexive.*
$$\forall x \in X, \quad \neg xRx.$$
- *Coreflexive.*
$$\forall x, y \in X, \quad xRy \implies x = y.$$
- *Left quasi-reflexive.*
- *Right quasi-reflexive.*
- *Quasi-reflexive.*

Group 2: Symmetry.

- *Symmetric.*
- *Antisymmetric.*
- *Asymmetric.*

Group 3: Transitivity.

- *Transitive.*
- *Antitransitive.*
- *Cotransitive.*
- *Quasi-transitive.*

and more...

2.2 Preorder

Definition (Preorder)**.** *Let S be a set. We define a* ***preorder*** *on S, denoted by $\leq$, to be a homogeneous relation that is reflexive and transitive. i.e.*

- *Reflexive:*
$$\forall x \in S, quad x \leq x.$$
- *Transitive:*
$$\forall x, y, z \in S, \quad x \leq y \text{ and } y \leq z \implies x \leq z.$$

Definition (Strict Preorder)**.** *Let S be a set. We define a* ***strict preorder*** *on S, denoted by $<$, to be a homogeneous relation that is irreflexive and transitive. i.e.*

- *Irreflexive:*
$$\forall x \in S, \neg x < x.$$
- *Transitive:*
$$\forall x, y, z \in S, \quad x \leq y \text{ and } y \leq z \implies x \leq z.$$

Chapter 3

Directed Sets

Definition (Directed Set)**.** *Let S be a set. Let $\leq$ be a preorder. We say that S, with $\leq$, is a **directed set** if every pair of elements in S has an upper bound. i.e.*

$$\forall x, y \in S, \quad \exists z \in S, \quad x \leq z \text{ and } y \leq z.$$