Variational Analysis

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Chapter 1

Semi-Continuity

1.1 Definitions

DEFINITION (Lower Semi-Continuous - 1)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x_0 be a point in $\mathbb{E}$. We say that f is **lower semi-continuous** at point x_0 if for any sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x_0, we have $f(x) \leq \liminf_{n\to\infty} f(x_i)$.

DEFINITION (Lower Semi-Continuous - 2)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **lower semi-continuous** if $\text{epi}(f)$ is closed.

PROPOSITION 1.1.1. The two definitions of lower semi-continuity are equivalent.

DEFINITION (Upper Semi-Continuous)**.** Let X be a topological space. Let f be a extended real-valued function on X. Let x_0 be a point in X. We say that f is **upper semi-continuous** at point x_0 if for any positive number ε, there exists some neighborhood $\mathcal{N}$ of x_0 such that $f(x) \leq f(x_0) + \varepsilon$ for any $x \in \mathcal{N}$ when $f(x_0) \neq -\infty$; or if $\lim_{x\to x_0} f(x) = -\infty$ when $f(x_0) = -\infty$.

1.2 Properties

PROPOSITION 1.2.1 (Supremum)**.** The supremum of a collection of lower semi-continuous functions is again lower semi-continuous. i.e., Let $\{f_i\}_{i\in I}$ be a collection of lower semi-continuous functions where I is some index set. Then the function F given by $F := \sup_{i\in I} f_i$ is lower semi-continuous.

Proof.

$$\begin{aligned} & (x,\alpha) \in \operatorname{epi}(F) \\ \iff & \sup_{i\in I} f_i(x) \le \alpha \\ \iff & \forall i \in I, f_i(x) \le \alpha \\ \iff & \forall i \in I, (x,\alpha) \in \operatorname{epi}(f_i \\ \iff & (x,\alpha) \in \bigcap_{i\in I} \operatorname{epi}(f_i). \end{aligned}$$

So $\operatorname{epi}(F) = \bigcap_{i\in I} \operatorname{epi}(f_i)$. Since f_i are lower semi-continuous, $\operatorname{epi}(f_i)$ are closed. Since $\operatorname{epi}(f_i)$ are closed, $\bigcap_{i\in I} \operatorname{epi}(f_i)$ is closed. That is, $\operatorname{epi}(F)$ is closed. Since $\operatorname{epi}(F)$ is closed, F is lower semi-continuous.

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PROPOSITION 1.2.2. A function is continuous at a point if and only if it is both upper and lower semi-continuous there.

Chapter 2

Subgradients

2.1 Definitions and Examples

DEFINITION (Sub-Differential)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **sub-differential** of f, denoted by ∂f, to be a set-valued function on $\mathbb{E}$ given by

$$\partial f(x) := \big\{v \in \mathbb{E} : \forall y \in \mathbb{E}, \langle v, y - x\rangle \leq f(y) - f(x)\big\}.$$

DEFINITION (Subdifferentiable)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x be a point in $\mathbb{E}$. We say that f is **subdifferentiable** at point x if $\partial f(x) \neq \emptyset$.

DEFINITION (Subgradient)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **subgradient**s of f to be the elements of $\partial f(x)$.

EXAMPLE 2.1.1. Let C be a non-empty closed convex set in $\mathbb{E}$. Let x be some point in $\mathbb{E}$. Then

$$\partial \delta_C(x) = N_C(x)$$

where δ_C denotes the indicator function of C and N_C denotes the normal cone to C.

Proof. If $x \notin C$, then $\partial \delta_C(x) = N_C(x) = \emptyset$. Else, let u be an arbitrary point in $\mathbb{E}$. Then

$$u \in \partial \delta_C(x)$$

$$\begin{aligned}
&\iff \forall y \in \mathbb{E}, \delta_C(y) - \delta_C(x) \geq \langle u, y - x \rangle \\
&\iff \forall y \in C, \delta_C(y) - \delta_C(x) \geq \langle u, y - x \rangle \\
&\iff \forall y \in C, 0 - 0 \geq \langle u, y - x \rangle \\
&\iff \forall y \in C, \langle u, y - x \rangle \leq 0 \\
&\iff \forall y \in C - x, \langle u, y \rangle \leq 0 \\
&\iff u \in N_C(x).
\end{aligned}$$

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2.2 Basic Properties

PROPOSITION 2.2.1 (Domain of the Subdifferential)**.** Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$.

(1) $\mathrm{dom}(\partial f) \subseteq \mathrm{dom}(f)$.

(2) $\mathrm{ri}(\mathrm{dom}(f)) \subseteq \mathrm{dom}(\partial f)$.

(3) $\mathrm{ri}(\mathrm{dom}(\partial f)) = \mathrm{ri}(\mathrm{dom}(f))$.

(4) $\mathrm{cl}(\mathrm{dom}(\partial f)) = \mathrm{cl}(\mathrm{dom}(f))$.

Proof of (1). Let x be an arbitrary point in $\mathrm{dom}(\partial f)$. We are to prove that $x \in \mathrm{dom}(f)$. Assume for the sake of contradiction that $x \notin \mathrm{dom}(f)$. Since $x \notin \mathrm{dom}(f)$, $f(x) = +\infty$. Since f is proper, $\exists y \in \mathbb{E}$ such that $f(y) < +\infty$. Since $f(y) < +\infty$ and $f(x) = +\infty$, we have

$$\forall u \in \mathbb{E}, \quad f(y) - f(x) < \langle u, y - x \rangle.$$

So $\forall u \in \mathbb{E}$, $u \notin \partial f(x)$. i.e. $\partial f(x) = \emptyset$. So $x \notin \mathrm{dom}(\partial f)$. This contradicts to the assumption that $x \in \mathrm{dom}(\partial f)$. So the assumption that $x \notin \mathrm{dom}(f)$ is false. i.e. $x \in \mathrm{dom}(f)$. Since $\forall x \in \mathrm{dom}(\partial f)$, $x \in \mathrm{dom}(f)$, we get

$$\mathrm{dom}(\partial f) \subseteq \mathrm{dom}(f).$$

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2.3 Calculus of Sub-Differentials

PROPOSITION 2.3.1. Let f and g be proper functions from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x \in \mathbb{E}$, $\partial f(x) + \partial g(x) \subseteq \partial(f+g)(x)$.

Proof.

Let x be an arbitrary point in $\mathbb{E}$.

Let v be an arbitrary point in $\partial f(x) + \partial g(x)$.

Since $v \in \partial f(x) + \partial g(x)$, $\exists u \in \partial f(x)$, $\exists w \in \partial g(x)$ such that $v = u + w$.

Let y be an arbitrary point in $\mathbb{E}$.

Since $u \in \partial f(x)$, $f(y) \geq f(x) + \langle u, y - x \rangle$.

Since $w \in \partial g(x)$, $g(y) \geq g(x) + \langle w, y - x \rangle$.

$$
\begin{aligned}
&(f+g)(y) = f(y) + g(y) \\
&\geq f(x) + \langle u, y - x \rangle + g(x) + \langle w, y - x \rangle \\
&= f(x) + g(x)\langle u + w, y - x \rangle \\
&= (f+g)(x) + \langle v, y - x \rangle.
\end{aligned}
$$

That is, $(f+g)(y) \geq (f+g)(x) + \langle v, y - x \rangle$.

This is true for any $y \in \mathbb{E}$.

So $v \in \partial(f+g)(x)$.

This is true for any $v \in \partial f(x) + \partial g(x)$.

So $\partial f(x) + \partial g(x) \subseteq \partial(f+g)$.

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THEOREM 2.1. Let f and g be proper convex lower semi-continuous functions from $\mathbb{E}$ to $\mathbb{R}^*$. Assume that $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g)) \neq \emptyset$. Then $\partial(f+g) = \partial f + \partial g$.

2.4 Subdifferentiation and Differentiation

THEOREM 2.2. Let f be a proper convex function from $\mathbb{R}^n$ to $\mathbb{R}^*$. Let x_0 be a point in $\mathrm{dom}(f)$. Let u be a point in $\mathbb{R}^n$. Then u is a subgradient of f at point x_0 if and only if

$$\forall d \in \mathbb{R}^n, f'(x_0; d) \geq \langle u, d \rangle.$$

Proof.

$$
\begin{aligned}
& u \in \partial f(x_0) \\
\iff\ & \forall y \in \mathbb{R}^n, && f(y) \geq f(x_0) + \langle u, y - x_0 \rangle \\
\iff\ & \forall d \in \mathbb{R}^n, \forall \lambda > 0, && f(x_0 + \lambda d) \geq f(x_0) + \langle u, x_0 + \lambda d - x_0 \rangle \\
\iff\ & \forall d \in \mathbb{R}^n, \forall \lambda > 0, && \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda} \geq \langle u, d \rangle \\
\iff\ & \forall d \in \mathbb{R}^n, && f'(x_0; d) \geq \langle u, d \rangle.
\end{aligned}
$$

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PROPOSITION 2.4.1. Let f be a proper convex function from $\mathbb{R}^n$ to $\mathbb{R}^*$. Let x_0 be a point in $\mathrm{dom}(f)$. Assume that f is differentiable at point x_0. Then $\nabla f(x_0)$ is the unique subgradient of f at point x_0.

Chapter 3

Quasigradients

3.1 Definitions

DEFINITION (Quasigradients)**.** Let f be a quasiconvex function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x_0 be a point in $\mathbb{E}$. We define the **quasigradients** of f at point x_0 to be the vectors v such that

$$\forall x \in \mathbb{E}, \quad \left\langle v, x - x_0 \right\rangle \geq 0 \implies f(x) - f(x_0) \geq 0.$$