Graph Theory

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Chapter 1

Graph Basics

Definition (Spanning Subgraph)**.** *Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G. We say that H is **spanning** if $W = V$. i.e., if H contains all vertices of G.*

Chapter 2

Trees

2.1 Definitions

Definition (Spanning Tree)**.** *Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G. We say that H is a* ***spanning tree*** *if H is a spanning subgraph of G and is a tree.*

2.2 Properties

Proposition 2.2.1. *A graph is connected if and only if it has a spanning tree.*

Chapter 3

Graph Isomorphism

3.1 Definitions

Definition (Isomorphism)**.** *Let G and H be two graphs. We define an **isomorphism** from G to H to be a function f from $V(G)$ to $V(H)$ such that*

- *f is bijective, and that*
- *for any pair of vertices $v, w \in V(G)$, $f(v)f(w) \in E(H)$ if and only if $vw \in E(G)$.*

i.e., a bijective function that both itself and its inverse preserve adjacency.

Definition (Isomorphic)**.** *Let G and H be two graphs. We say that G and H are **isomorphic**, denoted by $G \simeq H$, if there exists an isomorphism from G to H.*

Proposition 3.1.1. *The relation simeq of isomorphism is an equivalence relation. That is, it is reflexive, symmetric, and transitive.*

3.2 Properties

Proposition 3.2.1. *Let G and H be isomorphic graphs with isomorphism f. Then for any vertex $v \in V(G)$, we have $\deg_G(v) = \deg_H(f(v))$.*

Chapter 4

Matchings and Covers

4.1 Matching

Definition (Matching)**.** *Let $G = (V, E)$ be a graph. Let M be a subset of E. We say that M is a* ***matching*** *in G if every vertex in the spanning subgraph (V, M) has degree at most one.*

Definition (Saturated)**.** *Let $(G = (V, E)$ be a graph. Let M be a subset of E. Let v be a vertex of G. We say that v is M**-saturated** if $\deg(v) = 1$ in (V, M).*

Definition (Maximal Matching)**.** *Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a* ***maximal matching*** *if it is a matching in G and any other matching is not a superset of it.*

Definition (Maximum Matching)**.** *Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a* ***maximum matching*** *if it is a matching in G and any other matching contains edges no more than M.*

Definition (Perfect Matching)**.** *Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a* ***perfect matching*** *if it matches all vertices of the graph. i.e., any vertex in G is incident to some edge in M.*

Proposition 4.1.1. *Every maximum matching is maximal.*

Proposition 4.1.2. *Every perfect matching is maximum.*

Proposition 4.1.3. *Let $G = (V, E)$ be a graph. Let A and B be two maximal matchings of G. Then both $|A| \leq 2|B|$ and $|B| \leq 2|A|$.*

4.2 Cover

Definition (Cover)**.** *Let $G = (V, E)$ be a graph. Let C be a subset of V. We say that C is a* ***cover*** *of G if any edge has an end in C.*

4.3 Relations Between Matchings and Covers

Proposition 4.3.1. *Let $G = (V, E)$ be a graph. Let M be a matching of G. Let C be a cover of G. Then $|M| \leq |C|$.*

Chapter 5

Bipartite Graphs

5.1 Definitions

Definition (bipartition)**.** *Let $G = (V, E)$ be a graph. Let A and B be two subsets of V. We say the pair (A, B) is a **bipartition** if*

Definition (Bipartite)**.** *Let $G = (V, E)$ be a graph. We say that G is bipartite if there exists a bipartition of G.*

5.2 Characterizations

Proposition 5.2.1. *A graph is bipartite if and only if it has no odd cycles.*

Chapter 6

Planar Graphs

6.1 Definitions

Definition (Plane Embedding)**.** *Let $G(V, E, B)$ be an undirected multigraph. A* ***plane embedding*** *of G is a pair of sets $(\mathcal{P}, \Gamma)$ such that*

6.2 Properties

Proposition 6.2.1. *Every subgraph of a planar graph is planar.*

Proposition 6.2.2. *A multigraph is planar if and only if its simplification is planar.*

Proposition 6.2.3. *Let G be a multigraph and e be an edge in G. Then G is planar if and only if $G \bullet e$ is planar.*

Theorem 1. *A multigraph is planar if and only if it does not contain a repeated subdivision of K_5 or $K_{3,3}$ as a subgrph.*

6.3 Numerology

Definition (Footprint)**.** *Let $G(V, E, B)$ be a planar multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define the* ***footprint*** *of G, denoted by $fp(G)$, to be the union of the points and curves in $\mathbb{R}^2$ representing the vertices and edges in G.*

Definition (Face)**.** *Let $G(V, E, B)$ be a planar multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define a* ***face*** *of $(\mathcal{P}, \Gamma)$ to be a connected component of the set $\mathbb{R}^2 \setminus fp(G)$.*

Definition (Degree)**.** *Let $G(V, E, B)$ be a planar multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define the* ***degree*** *of a face to be the sum of the number of edges and the number of bridges in its boundary.*

Proposition 6.3.1. *An edge e in a planar multigraph is a bridge if and only if the two faces on either side of the curve γ_e are the same.*

Chapter 7

Duality

7.1 Definitions

Definition (Dual Graph)**.** *Let $G = (V, E, B)$ be a multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of G. Let $\mathcal{F}$ be the set of faces of G. We define the* ***dual graph*** *of this embedding to be the multigraph $G^* = (V^*, E^*, B^*)$ where $V^* = \mathcal{F}$ and $E^* = \{e^* : e \in E\}$.*

Proposition 7.1.1. *Let $G = (V, E, B)$ be a multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of G. Let $(G^* = (V^*, E^*, B^*)$ be the dual graph of G. Then for any face $f \in \mathcal{F}$, the degree of f as a face of $\mathcal{P}, \Gamma$ equals the degree of f as a vertex of G^*.*

Proposition 7.1.2. *If G is a connected multigraph embedded in the plane, then G^{**} is isomorphic with G.*

Chapter 8

Graph Coloring

8.1 Chromatic Number

Definition ((Proper) Coloring)**.** *Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define a **(proper) X-coloring** of G to be a function $f : V \to X$ such that if $vw \in E$, then $f(v) \neq f(w)$.*

Definition (Chromatic Number)**.** *Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define the **chromatic number** of G, denoted by $\chi(G)$, to be the smallest natural number $k \in \mathbb{N}$ for which G has a (proper) k-coloring.*

Proposition 8.1.1. *The chromatic number exists and $\chi(G) \leq |V|$.*

Proof. Take $X = V$. ■

Proposition 8.1.2. *G is complete if and only if $\chi(G) = |V(G)|$.*

Proposition 8.1.3. *The only graph with chromatic number zero is the empty graph.*

Proposition 8.1.4. *A graph has chromatic number one if and only if it has no edges and at least one vertex.*

Proposition 8.1.5. *A graph has chromatic number two if and only if it is bipartite and has at least one edge.*

Proposition 8.1.6. *Let G be a graph. Let $d_{max}(G)$ be the maximum degree of a vertex in G. Then $\chi(G) \leq 1 + d_{max}(G)$.*

8.2 5-color Theorem

Theorem 2. *Every planar graph is 5-colorable.*

Proof. (1890)
True for $|V| \leq 5$.
Inductively, suppose the theorem holds for planar graphs on $n-1$ vertices for $n \geq 5$. Suppose G is a planar graph on n vertices.
Let v be a vertex of degree ≤ 5 in G. This exists by a lemma in our lectures.
Since G is a planar, $G-v$ is planar. By the induction hypothesis, $G-v$ has a 5-coloring.
If some color does not appear on any neighbor of v, we can extend the coloring to a coloring of G.
Otherwise, v has exactly 5 neighbors with different colors.
For each pair i, j of colors, let G_{ij} be the subgraph of $G-v$ induced by the vertices colored i or j.
If the component H of G_{ij} containing x_i does not contain x_j, then we can switch the colors of all vertices in H between i and j to get a coloring of $G-v$ that assigns only 4 colors to neighbors of v, and thus extends to a coloring of G.
So G_{ij} contains a path from x_i to x_j.
Because $G_{2,5}$ and $G_{1,4}$ have disjoint vertex sets, this contradicts the planarity of G.

■

Definition (Near-triangulation)**.** *Planar drawing of G where the infinite face is bounded by a cycle, and every other face is bounded by a triangle*

Theorem 3. *Every planar near-triangulation has a 5-coloring.*

Theorem 3 $\implies$ Theorem 2.

Definition (List Assignment)**.** *A **list assignment** L of G is a function that assigns a set $L(v)$ of colors to each $v \in V$.*

Definition (L-coloring)**.** *An L-coloring of G is a choice of a color in $L(v)$ for each $v \in V$ such that adjacent vertices get different colors.*

Definition (5-list-colorable)**.** *A graph is **5-list-colorable** if for every list assignment L of G with $|L(v)| \geq 5$, G is L-colorable.*

Theorem 4. *Every planar near-triangulation is 5-list-colorable.*

Theorem 4 $\implies$ Theorem 3 because coloring is a special case of list coloring.

Theorem 5 (Carsten Thomassen, 1993)**.** *If G is a near-triangulation and L is a list assignment such that*

(1) $|L(v)| = 5$ for every non-boundary vertex,

(2) $|L(v)| = 3$ for every boundary vertex.

Then G has an L-coloring even if two adjacent boundary vertices have their colors arbitrarily decided in advance.

Proof.
Case 1. There is a "chord" between two boundary vertices.
Let G_1 and G_2 be subgraph of G obtained by "cutting" G along the chord, where G_1 contains the pre-colored vertices.
By applying the inductive hypothesis to G_1, and then applying it to G_2 with the two ends of the chord pre-colored according to the coloring of G_1, we get a coloring of G_1.
Case 2. There is no chord.
Let u and u' be the pre-colored vertices.
Let x, y be the next two vertices occuring in order around the boundary.

■

Theorem 5 $\implies$ Theorem 4.

Chapter 9

Probability and Edge Density

Q: Let G be a graph on n vertices with no triangles. How many edges can G have?

Theorem 6 (Mantel)**.** *If G is triangle-free and has n vertices, then*

$$|E| \leq \frac{n^2}{4}.$$

Proof.
Let $P_{2,1}$ denote the probability that a pair of distinct vertices chosen uniformly at random, are adjacent.

$$P_{2,1} = |E| / \binom{n}{2}.$$

Let $P_{3,2}$ denote the probability that a randomly chosen triple of vertices contains exactly two edges.
Let $P_{3,1}$ denote ... one edge.
Let $P_{3,0}$ denote ... no edges.
Notice $P_{3,2} + P_{3,1} + P_{3,0} = 1$.

claim (1)**.**

$$P_{2,1} = \frac{2}{3} P_{3,2} + \frac{1}{3} P_{3,1}.$$

Proof. Notice that the graph is triangle-free. So $P_{3,3} = 0$. Choosing a pair at random is the same as choosing a triple at random, then choosing a pair at random within that triple. ∎

For a fixed vertex v, let $Q_{v,1}$ denote the probability that a randomly chosen vertex $u \neq v$ is adjacent to v.

$$Q_{v,1} = \frac{deg(v)}{n-1}.$$

Let $Q_{v,2}$ denote the probability that two distinct randomly chosen vertices other than v are both adjacent to v.

$$Q_{v,2} = \binom{deg(v)}{2} / \binom{n-1}{2}.$$

claim (2)**.**

$$Q_{v,1}^2 \approx Q_{v,2}.$$

Proof. Both give (essentially) the probability that a pair x, y of vertices other than v are both adjacent to v. The LHS allows $x = y$. The RHS does not. But $x = y$ occurs with negligible probability. ∎

claim (3)**.**

$$P_{2,1} = \frac{1}{n} \sum_v Q_{v,1}.$$

Proof. Both the RHS and LHS are just the probability of an edge between two vertices. The RHS calls the first chosen vertex v. ∎

claim (4)**.**

$$\frac{1}{3} P_{3,2} = \frac{1}{n} \sum_v Q_{v,2}.$$

Proof. Both sides give the probability that, if we choose 3 vertices at random, and then choose one among those 3 and call it v, that v is adjacent to both the others. ∎

claim (Cauchy–Schwarz Inequality)**.**

$$\langle x, y \rangle \leq \|x\| \|y\|.$$

Proof of the theorem.
Now

$$\begin{aligned} P_{2,1} &= \frac{2}{3} P_{3,2} + \frac{1}{3} P_{3,1} \geq \frac{2}{3} P_{3,2} \\ &= 2 \left(\frac{1}{n} \sum_v Q_{v,2} \right) \approx 2 \left(\frac{1}{n} \sum_v Q_{v,1}^2 \right) \\ &\geq 2 \left(\frac{1}{n} \sum_v Q_{v,1} \right)^2 = 2 P_{2,1}^2. \end{aligned}$$

So $P_{2,1} \leq \frac{1}{2}$. So $|E| \leq \frac{n^2}{4}$.

∎

Q: If G has n vertices, no K_{t+1}-subgraph, how many edges can G have?

Theorem 7 (Turan)**.** *If G is a graph on n vertices with no K_{t+1}-subgraph, then*

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{t}\right).$$

Theorem 8 (Erdos-Stone)**.** *If H is a graph and G is a graph on n vertices without H as a subgraph, then*

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{\chi(H) - 1} + \varepsilon(n)\right)$$

where $\varepsilon(n) \to 0$ as $n \to \infty$ and $\chi(H)$ is the chromatic number of H, the fewest number of colors needed to properly color the vertices of H.