

# Matrix Theory

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# Chapter 1

## Fundamentals

### 1.1 Definitions

**DEFINITION** (Column Space). Let  $A$  be an  $m \times n$  matrix. We define the **column space** of  $A$ , denoted by  $\text{col}(A)$ , to be the set given by

$$\text{col}(A) := \{Av : v \in \mathbb{R}^n\}.$$

**DEFINITION** (Row Space). Let  $A$  be an  $m \times n$  matrix. We define the **row space** of  $A$ , denoted by  $\text{row}(A)$ , to be the set given by

$$\text{row}(A) := \{A^\top v : v \in \mathbb{R}^m\}.$$

**DEFINITION** (Nullspace). Let  $A$  be an  $m \times n$  matrix. We define the **nullspace** of  $A$ , denoted by  $\text{null}(A)$ , to be the set given by

$$\text{null}(A) := \{v \in \mathbb{R}^n : Av = \mathbf{0}\}.$$

**DEFINITION** (Left Nullspace). Let  $A$  be an  $m \times n$  matrix. We define the **left**

**nullspace** of  $A$ , denoted by  $\text{null}(A^\top)$ , to be the set given by

$$\text{null}(A^\top) := \{v \in \mathbb{R}^m : A^\top v = \mathbf{0}\}.$$

## 1.2 Main Results

**THEOREM 1.1** (The Fundamental Theorem of Linear Algebra). Let  $A$  be an  $m \times n$  matrix. Then  $\text{col}(A)^\perp = \text{null}(A^\top)$  and  $\text{row}(A)^\perp = \text{null}(A)$ .

## Chapter 2

# Rank

### 2.1 Definitions

**DEFINITION** (Column Rank). Let  $A$  be a matrix. We define the **column rank** of  $A$  to be the dimension of the column space of  $A$ . i.e.

$$\text{colrank}(A) := \dim(\text{col}(A)).$$

**DEFINITION** (Row Rank). Let  $A$  be a matrix. We define the **row rank** of  $A$  to be the dimension of the row space of  $A$ . i.e.

$$\text{rowrank}(A) := \dim(\text{row}(A)).$$

**DEFINITION** (Rank). Let  $A$  be a matrix. Then the column rank and the row rank are the same. We define the **rank** of  $A$  to be this common number.

**DEFINITION** (Full Rank). Let  $A$  be an  $m \times n$  matrix. We say that  $A$  has **full rank** if  $\text{rank}(A) = \min\{m, n\}$ .

### 2.2 Properties

**PROPOSITION 2.2.1.** Let  $A$  be an  $m \times n$  matrix. Then

- $A$  is injective if and only if  $A$  has full column rank. i.e.  $\text{rank}(A) = n$ , and
- $A$  is surjective if and only if  $A$  has full row rank. i.e.  $\text{rank}(A) = m$ .

**PROPOSITION 2.2.2.** Let  $A$  and  $B$  be matrices with appropriate dimensions. Then

$$\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}.$$

**PROPOSITION 2.2.3.** Let  $A$ ,  $B$ , and  $C$  be matrices with appropriate dimensions. Then

- If  $B$  has full row rank, then  $\text{rank}(AB) = \text{rank}(A)$ , and
- If  $C$  has full column rank, then  $\text{rank}(CA) = \text{rank}(A)$ .

**PROPOSITION 2.2.4** (Subadditivity). Let  $A$  and  $B$  be matrices with appropriate dimensions. Then

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

**PROPOSITION 2.2.5.** Let  $A$  be a matrix over  $\mathbb{C}$ . Let  $A^-$  denote the complex conjugate of  $A$ . Let  $A^\top$  denote the transpose of  $A$ . Let  $A^*$  denote the conjugate transpose of  $A$ . Then

$$\text{rank}(A) = \text{rank}(A^-) = \text{rank}(A^\top) = \text{rank}(A^*) = \text{rank}(AA^*) = \text{rank}(A^*A).$$



## Chapter 3

# Matrix Inverse

### 3.1 Definitions

**DEFINITION** (Invertible). Let  $A$  be an  $n \times n$  matrix over  $\mathbb{C}$ . We say that  $A$  is **invertible** if there exists another  $n \times n$  matrix  $B$  over  $\mathbb{C}$  such that  $AB = BA = I_n$ .

**PROPOSITION 3.1.1.** Let  $A$  be an  $n \times n$  invertible matrix over  $\mathbb{C}$ . Then the  $n \times n$  matrix  $B$  over  $\mathbb{C}$  satisfying  $AB = BA = I_n$  is unique.

**DEFINITION** (Inverse). Let  $A$  be an  $n \times n$  matrix over  $\mathbb{C}$ . We define the **inverse** of  $A$ , denoted by  $A^{-1}$ , to be the unique  $n \times n$  matrix over  $\mathbb{C}$  satisfying  $AA^{-1} = A^{-1}A = I_n$ .

**DEFINITION** (Left/Right Inverse). Let  $A$  be an  $m \times n$  matrix over  $\mathbb{C}$ . We define

- the **left inverse** of  $A$ , to be an  $n \times m$  matrix  $B$  over  $\mathbb{C}$  such that  $BA = I_n$ .
- the **right inverse** of  $A$ , to be an  $n \times m$  matrix  $B$  over  $\mathbb{C}$  such that  $AB = I_n$ .

### 3.2 Characterization

**PROPOSITION 3.2.1.** Let  $A$  be an  $n \times n$  matrix over field  $K$ . Then the following statements are equivalent.

- $A$  is invertible.
- $\dim(\text{row}(A)) = n$ .
- $\dim(\text{col}(A)) = n$ .
- $\dim(\text{null}(A)) = 0$ .

**PROPOSITION 3.2.2.** Let  $A$  be an  $n \times n$  matrix over field  $K$ . Then the following statements are equivalent.

- $A$  is invertible.
- $A$  is row-equivalent to  $I_n$ .
- $A$  is column-equivalent to  $I_n$ .
- $A$  can be written as a finite product of elementary matrices.

**PROPOSITION 3.2.3.** Let  $A$  be an  $n \times n$  matrix over field  $K$ . Then  $A$  is invertible if and only if  $\det(A) \neq 0$ .

**PROPOSITION 3.2.4.** Let  $A$  be an  $n \times n$  matrix over field  $K$ . Then  $A$  is invertible if and only if 0 is not an eigenvalue of  $A$ .

### 3.3 Arithmetic Properties

**PROPOSITION 3.3.1.** Let  $A$  be an invertible matrix. Then

- $(A^{-1})^{-1} = A$ .
- $(kA)^{-1} = k^{-1}A^{-1}$ .

- $(AB)^{-1} = B^{-1}A^{-1}$ .
- $(A^T)^{-1} = (A^{-1})^T$ .

### 3.4 Pseudo-Inverse

**DEFINITION** (Moore-Penrose Pseudo-Inverse). Let  $A$  be an  $n \times d$  matrix. We define the **Moore-Penrose pseudo-inverse** of  $A$ , denoted by  $A^\dagger$ , to be a  $d \times n$  matrix  $G$  such that

$$AGA = A, \quad GAG = G, \quad (AG)^\top = AG, \quad (GA)^\top = GA.$$



## Chapter 4

# Determinant

### 4.1 Definitions

**DEFINITION** (Cofactor). Let  $M$  be an  $n \times n$  matrix where  $n \geq 2$ . We define the  $(i, j)^{\text{th}}$  **cofactor** of  $M$ , denoted by  $C_{i,j}(M)$ , to be a number given by

$$C_{i,j}(M) := (-1)^{i+j} \det(M(i, j))$$

where  $M(i, j)$  denotes the submatrix obtained from  $M$  by removing the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column.

**DEFINITION** (Determinant). Let  $M$  be an  $n \times n$  matrix where  $n \geq 2$ . We define the **determinant** of  $M$ , denoted by  $\det(M)$ , to be a number given by

$$\det(M) := \sum_{i=1}^n [M]_{i,j} C_{i,j}(M)$$

where  $j$  can be anything in  $\{1, \dots, n\}$ ,  $[M]_{i,j}$  denotes the  $(i, j)^{\text{th}}$  entry of  $M$ , and  $C_{i,j}(M)$  denotes the  $(i, j)^{\text{th}}$  cofactor of  $M$ . Equivalently,

$$\det(M) := \sum_{j=1}^n [M]_{i,j} C_{i,j}(M)$$

where  $i$  can be anything in  $\{1, \dots, n\}$ ,  $[M]_{i,j}$  denotes the  $(i, j)^{\text{th}}$  entry of  $M$ , and  $C_{i,j}(M)$  denotes the  $(i, j)^{\text{th}}$  cofactor of  $M$ .

We define the determinant of an  $1 \times 1$  matrix to be the number itself.

## 4.2 Properties

**PROPOSITION 4.2.1.** Let  $A$  be a matrix. Then

$$\det(A^\top) = \det(A).$$

**PROPOSITION 4.2.2.** Let  $A$  and  $B$  be positive semi-definite matrices with appropriate dimensions. Then

$$\det(A + B) \geq \det(A) + \det(B).$$

**PROPOSITION 4.2.3.** Let  $A$  be an  $n \times n$  matrix. Let  $c$  be some scalar. Then

$$\det(cA) = c^n \det(A).$$

**PROPOSITION 4.2.4.** Let  $A$  be an invertible matrix. Then

$$\det(A^{-1}) = \det(A)^{-1}.$$

**PROPOSITION 4.2.5.** Let  $A$  and  $B$  be matrices with appropriate dimensions. Then

$$\det(AB) = \det(A) \det(B).$$

**PROPOSITION 4.2.6.** The determinant operator is a multi-linear operator on the rows/columns.

## 4.3 Adjoint of a Matrix

**DEFINITION (Adjoint).** Let  $M$  be an  $n \times n$  matrix. We define the **adjoint** of  $M$ ,

denoted by  $\text{adj}(M)$ , to be an  $n \times n$  matrix given by

$$(\text{adj}(M))_{ij} = C_{ji}(M),$$

for  $i, j = 1, \dots, n$ .

**PROPOSITION 4.3.1.** Let  $M$  be an  $n \times n$  matrix. Then

$$M \text{adj}(M) = \text{adj}(M)M = \det(M)I_n.$$





## Chapter 5

# Trace

**DEFINITION.** Let  $A$  be a square matrix. We define the trace of  $A$ , denoted by  $\text{tr}(A)$ , to be the sum of the entries on the main diagonal of  $A$ .

### 5.1 Properties

**PROPOSITION 5.1.1.** Trace is a linear operator.

**PROPOSITION 5.1.2.** The trace of the transpose of a matrix equals the trace of the matrix itself. i.e. if  $M$  is a square matrix, then

$$\text{tr}(M) = \text{tr}(M^\top).$$

**PROPOSITION 5.1.3.** If  $A \in M_{m \times n}$  and  $B \in M_{n \times m}$ , then

$$\text{tr}(AB) = \text{tr}(BA).$$

**PROPOSITION 5.1.4.** Trace is similarity-invariant. i.e., if  $A$  is similar to  $B$ , then  $\text{tr}(A) = \text{tr}(B)$ .

**PROPOSITION 5.1.5.** The trace of an idempotent matrix is equal to its rank.

**PROPOSITION 5.1.6.** The trace of a matrix equals the sum of its eigenvalues.

## Chapter 6

# Matrix Norm

**DEFINITION.**  $\|A\| := \sup_{\|x\|=1} \|Ax\|$

### 6.1 Properties

**PROPOSITION 6.1.1.** Let  $A$  be an  $n \times n$  matrix. Then if  $A$  is symmetric, we have

$$\|A\| = \max\{\lambda_i\}_{i=1}^n$$

where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$ .



## Chapter 7

# Eigenvalues and Eigenvectors

### 7.1 Definitions

**DEFINITION** (Eigenvalue and Eigenvector). Let  $A$  be a matrix. Let  $x$  be a vector. Let  $\lambda$  be a scalar. We say that  $x$  is an **eigenvector** of  $A$  and that  $\lambda$  is an **eigenvalue** of  $A$  if  $x \neq 0$  and

$$Ax = \lambda x.$$

### 7.2 Properties

**PROPOSITION 7.2.1.** Let  $A$  be an invertible matrix. Let  $\{\lambda_i\}_{i=1}^n$  be the eigenvalues of  $A$ . Then the eigenvalues of  $A^{-1}$  are  $\{\lambda_i^{-1}\}_{i=1}^n$ .

*Proof.*

$$\begin{aligned} Av &= \lambda v \\ \iff A^{-1}Av &= A^{-1}\lambda v \\ \iff v &= \lambda A^{-1}v \\ \iff A^{-1}v &= \lambda^{-1}v. \end{aligned}$$

■

**PROPOSITION 7.2.2.** Let  $A$  be an invertible matrix. Let  $\{x_i\}_{i=1}^n$  be the eigenvectors of  $A$ . Then the eigenvectors of  $A^{-1}$  are also  $\{x_i\}_{i=1}^n$ .

**PROPOSITION 7.2.3.** Let  $A$  be a matrix. Let  $n$  be a positive integer. Let  $(x, \lambda)$  be an eigenpair of  $A$ . Then

$$A^n x = \lambda^n x.$$

*Proof.* I will prove by induction on  $n$ .

Base Case:  $n = 1$ .

This is to prove that  $Ax = \lambda x$ . This holds since  $(x, \lambda)$  is an eigenpair of  $A$ .

Inductive Step:

Assume that  $A^n x = \lambda^n x$  for some  $n \in \mathbb{N}$ . We are to prove that  $A^{n+1}x = \lambda^{n+1}x$ .

$$\begin{aligned} A^{n+1}x &= A^n Ax \\ &= A^n \lambda x \\ &= \lambda A^n x \\ &= \lambda \lambda^n x \text{ by the inductive hypothesis} \\ &= \lambda^{n+1}x. \end{aligned}$$

That is,

$$A^{n+1}x = \lambda^{n+1}x.$$

Summary:

By the principle of mathematical induction,

$$\forall n \in \mathbb{N}, \quad A^n x = \lambda^n x.$$

■

**PROPOSITION 7.2.4.** If a square matrix is idempotent, then its eigenvalues are either 0 or 1.

*Proof.* Since  $A$  is idempotent, by definition,  $A^2 = A$ . Let  $(x, \lambda)$  be an arbitrary eigenpair of  $A$ . Then

$$Ax = \lambda x \text{ and } A^2x = \lambda^2 x.$$

Since  $A^2 = A$  and  $A^2x = \lambda^2 x$ , we get  $Ax = \lambda^2 x$ . Since  $Ax = \lambda x$  and  $Ax = \lambda^2 x$ , we get  $\lambda x = \lambda^2 x$ . Since  $x$  is an eigenvector of  $A$ ,  $x \neq 0$ . Since  $\lambda x = \lambda^2 x$  and  $x \neq 0$ , we get  $\lambda \in \{0, 1\}$ . ■

## 7.3 Eigenspace

**DEFINITION** (Eigenspace). Let  $A$  be an  $m \times n$  matrix over field  $\mathbb{F}$ . Let  $\lambda$  be an eigenvalue of  $A$ . We define the **eigenspace** of  $A$ , associated with  $\lambda$ , denoted by  $E_\lambda$ , to be a set given by

$$E_\lambda := \{v \in \mathbb{F}^n : Av = \lambda v\}.$$

i.e.,  $E_\lambda$  is the set of all eigenvectors of  $A$  with eigenvalue  $\lambda$  and the zero vector.

**PROPOSITION 7.3.1.** Eigenspaces are linear subspaces.





## Chapter 8

# Singular Values and Singular Vectors

**DEFINITION** (Singular Value, Singular Vector). Let  $M$  be an  $m \times n$  real or complex matrix. We define a **singular value** for  $M$  to be a non-negative real number  $\sigma$  such that there exist unit vectors  $u \in \mathbb{F}^m$  and  $v \in \mathbb{F}^n$  such that  $Mv = \sigma u$  and  $M^*u = \sigma v$ . We call  $u$  the **left-singular vector** for  $\sigma$  and  $v$  the **right-singular vector** for  $\sigma$ .



## Chapter 9

# Special Types of Matrices

### 9.1 Elementary Matrices

**PROPOSITION 9.1.1.** The inverse of an elementary matrix can be obtained by multiplying its off-diagonal entries by  $-1$ .

Unconfirmed...

### 9.2 Triangular Matrix

**DEFINITION** (Upper Triangular Matrix).

**DEFINITION** (Lower Triangular Matrix).

**PROPOSITION 9.2.1.** The product of two upper triangular matrices is also upper triangular. i.e. if  $U_1$  and  $U_2$  are upper triangular matrices with appropriate dimensions, then  $U := U_1U_2$  is also upper triangular.

**PROPOSITION 9.2.2.** The inverse of an upper triangular matrix is also upper triangular, if it exists. i.e. if  $U$  is an invertible upper triangular matrix, then  $U^{-1}$  is also upper triangular.

## 9.3 Symmetric and Hermitian Matrices

### 9.3.1 Equivalent Conditions

**DEFINITION** (Symmetric Matrix). Let  $M \in \mathcal{M}_{n \times n}(\mathbb{R})$  (a real square matrix). We say that  $M$  is **symmetric** if

$$M = M^T$$

where  $M^T$  denotes the transpose of  $M$ .

**DEFINITION** (Hermitian Matrix - 1). Let  $M \in \mathcal{M}_{n \times n}(\mathbb{C})$  (a complex square matrix). We say that  $M$  is **Hermitian**, or **self-adjoint**, if

$$M = M^*$$

where  $M^*$  denotes the conjugate transpose of  $M$ .

**DEFINITION** (Hermitian Matrix - 2). Let  $M \in \mathcal{M}_{n \times n}(\mathbb{C})$  (a complex square matrix). We say that  $M$  is **Hermitian**, or **self-adjoint**, if

$$\forall x, y \in \mathbb{C}^n, \quad \langle x, Ay \rangle = \langle Ax, y \rangle.$$

**DEFINITION** (Hermitian Matrix - 3). Let  $M \in \mathcal{M}_{n \times n}(\mathbb{C})$  (a complex square matrix). We say that  $M$  is **Hermitian**, or **self-adjoint**, if

$$\forall x \in \mathbb{C}^n, \quad \langle x, Ax \rangle \in \mathbb{R}.$$

### 9.3.2 Stability of Hermitian Matrices

**PROPOSITION 9.3.1** (Sum of Two Hermitian Matrices). Let  $A$  and  $B$  be Hermitian matrices. Then  $A + B$  is also Hermitian.

**PROPOSITION 9.3.2** (Associative Product). Let  $A$  and  $B$  be Hermitian matrices. Suppose that  $AB = BA$ . Then  $AB$  is also Hermitian.

**PROPOSITION 9.3.3** (Inverse of a Hermitian Matrix). Let  $M$  be a Hermitian matrix. Suppose that  $M$  is invertible. Then  $M^{-1}$  is also Hermitian.

### 9.3.3 Properties of Hermitian Matrices

**PROPOSITION 9.3.4.** Hermitian matrices are normal.

**PROPOSITION 9.3.5.** The determinant of a Hermitian matrix is real.

*Proof.* Let  $M$  be a Hermitian matrix. Then

$$\det(M) = \det(M^*) = \det(\overline{M}^\top) = \det(\overline{M}) = \overline{\det(M)}.$$

That is,  $\det(M) = \overline{\det(M)}$ . So  $\det(M) \in \mathbb{R}$ . ■

**PROPOSITION 9.3.6** (Eigenvalues). The eigenvalues of a Hermitian matrix are all real.

**Proof Approach 1.** Let  $A$  be a Hermitian matrix. Let  $(\lambda, v)$  be an arbitrary eigenpair of  $A$ . Then we have  $Av = \lambda v$  and hence

$$v^*Av = v^*\lambda v = \lambda v^*v. \tag{1}$$

Note that  $v^*Av$  has size  $1 \times 1$ . So  $v^*Av = [a]$  for some  $a \in \mathbb{C}$ .

$$(v^*Av)^* = v^*A^*v^{**} = v^*Av$$

$$\begin{aligned}
&\implies v^*Av \text{ is Hermitian} \iff [a] \text{ is Hermitian} \\
&\implies a = \bar{a} \implies a \in \mathbb{R}.
\end{aligned}$$

That is,

$$v^*Av = a \in \mathbb{R}. \quad (2)$$

Note that  $v^*v$  has size  $1 \times 1$ . So  $v^*v = [b]$  for some  $b \in \mathbb{C}$ .

$$\begin{aligned}
&(v^*v)^* = v^*v^{**} = v^*v \\
&\implies v^*v \text{ is Hermitian} \iff [b] \text{ is Hermitian} \\
&\implies b = \bar{b} \implies b \in \mathbb{R}.
\end{aligned}$$

That is,

$$v^*v = b \in \mathbb{R}. \quad (3)$$

From (1), (2), and (3), we get  $a = \lambda b$ . It follows that  $\lambda \in \mathbb{R}$ . ■

**Proof Approach 2.** Let  $A$  be a Hermitian matrix. Let  $(\lambda, v)$  be an arbitrary eigenpair of  $A$ .

$$\begin{aligned}
&\lambda \langle v, v \rangle \\
&= \langle \lambda v, v \rangle \\
&= \langle Av, v \rangle \\
&= \langle v, A^*v \rangle \\
&= \langle v, Av \rangle \\
&= \langle v, \lambda v \rangle \\
&= \bar{\lambda} \langle v, v \rangle.
\end{aligned}$$

That is,  $\lambda \langle v, v \rangle = \bar{\lambda} \langle v, v \rangle$ . Since  $v$  is an eigenvector,  $v \neq \vec{0}$ . Since  $v \neq \vec{0}$ ,  $\langle v, v \rangle \neq 0$ . Since  $\langle v, v \rangle \neq 0$  and  $\lambda \langle v, v \rangle = \bar{\lambda} \langle v, v \rangle$ ,  $\lambda = \bar{\lambda}$ . Since  $\lambda = \bar{\lambda}$ ,  $\lambda$  is real. ■

## 9.4 Orthogonal and Unitary Matrices

### 9.4.1 Equivalent Conditions

**DEFINITION (Orthogonal).** Let  $U \in \mathcal{M}_{n \times n}(\mathbb{R})$  (a real square matrix). We say that  $U$  is **orthogonal** if

$$UU^\top = U^\top U = I$$

where  $U^\top$  denotes the transpose of  $U$  and  $I$  denotes the  $n \times n$  identity matrix.

**DEFINITION** (Unitary - 1). Let  $U \in \mathcal{M}_{n \times n}(\mathbb{C})$  (a complex square matrix). We say that  $U$  is **unitary** if

$$UU^* = U^*U = I$$

where  $U^*$  denotes the complex conjugate of  $U$  and  $I$  denotes the  $n \times n$  identity matrix.

**DEFINITION** (Unitary - 2). Let  $U \in \mathcal{M}_{n \times n}(\mathbb{C})$  (a complex square matrix). We say that  $U$  is **unitary** if the columns of  $U$  form an orthonormal basis for  $\mathbb{C}^n$ , or equivalently, the rows of  $U$  form an orthonormal basis for  $\mathbb{C}^n$ .

#### 9.4.2 Stability of Unitary Matrices

**PROPOSITION 9.4.1.** The product of two unitary matrices is still unitary.

#### 9.4.3 Properties of Unitary Matrices

**PROPOSITION 9.4.2** (Unitary Matrices Preserve Inner Products). Let  $U$  be a complex square matrix. Then  $U$  is unitary if and only if

$$\forall x, y \in \mathbb{C}^n, \quad \langle Ux, Uy \rangle = \langle x, y \rangle.$$

**PROPOSITION 9.4.3** (Eigenvalues). The eigenvalues of a unitary matrix are all unimodular.

*Proof.* Let  $U$  be a unitary matrix. Let  $(\lambda, v)$  be an arbitrary eigenpair of  $U$ . Since  $U$  is a unitary matrix, we get

$$\langle Uv, Uv \rangle = \langle v, v \rangle.$$

Since  $(\lambda, v)$  is an eigenpair of  $U$ , we get

$$\langle Uv, Uv \rangle = \langle \lambda v, \lambda v \rangle = \lambda^2 \langle v, v \rangle.$$

So  $\langle v, v \rangle = \lambda^2 \langle v, v \rangle$ . Since  $v$  is an eigenvector,  $v \neq 0$  and hence  $\langle v, v \rangle \neq 0$ . So  $\lambda^2 = 1$ . ■

## 9.5 Normal Matrices

### 9.5.1 Equivalent Conditions

**DEFINITION** (Normal Matrix - 1). Let  $M \in \mathcal{M}_{n \times n}(\mathbb{C})$ . We say that  $M$  is **normal** if

$$MM^* = M^*M,$$

where  $M^*$  denotes the conjugate transpose of  $M$ .

**DEFINITION** (Normal Matrix - 2). Let  $M \in \mathcal{M}_{n \times n}(\mathbb{C})$ . We say that  $M$  is **normal** if  $\exists \mathcal{B} \subseteq \mathcal{E}(M)$  such that  $\mathcal{B}$  is a orthonormal basis for  $\mathbb{C}^n$  where  $\mathcal{E}(M)$  denotes the set of eigenvectors of  $M$ .

**PROPOSITION 9.5.1.** Definitions (1) and (2) of normal matrices are equivalent.

*Proof.* Let  $M \in \mathcal{M}_{n \times n}(\mathbb{C})$ .

**Forward Direction** Assume that  $MM^* = M^*M$ . I will show that  $M$  has an orthonormal basis of eigenvectors. ■

**DEFINITION** (Normal Matrix - 3). Let  $M \in \mathcal{M}_{n \times n}(\mathbb{C})$ . We say that  $M$  is **normal** if  $M$  is diagonalizable by a unitary matrix.

### 9.5.2 Stability of Normal Matrices

**PROPOSITION 9.5.2.** Let  $A$  and  $B$  be normal matrices. Suppose that  $AB = BA$ . Then

- (1)  $A + B$  is also normal.



(2)  $AB$  is also normal.

### 9.5.3 Properties of Normal Matrices

**PROPOSITION 9.5.3.** Let  $M$  be a normal matrix. Then if  $M$  is triangular,  $M$  is diagonal.

**PROPOSITION 9.5.4.** Let  $M$  be a normal matrix. Then  $M$  is Hermitian if and only if  $\sigma(M) \subseteq \mathbb{R}$  where  $\sigma(M)$  denotes the set of eigenvalues of  $M$ .

*Proof. Forward Direction* Assume that  $M$  is Hermitian. I will show that  $\sigma(M) \subseteq \mathbb{R}$ . Since  $M$  is Hermitian, we get  $\sigma(M) \subseteq \mathbb{R}$ .

*Backward Direction* Assume that  $\sigma(M) \subseteq \mathbb{R}$ . I will show that  $M$  is Hermitian. Since  $M$  is normal, it is diagonalizable by a unitary matrix. Say  $M = U^*DU$  where  $U$  is unitary and  $D$  is diagonal. Then the diagonal entries of  $D$  are the eigenvalues of  $M$  and hence are real. So  $D^* = D$ . Then

$$M^* = (U^*DU)^* = U^*D^*U^{**} = U^*D^*U = U^*DU = M.$$

So  $M$  is Hermitian. ■

**PROPOSITION 9.5.5.** Let  $M$  be a normal matrix. Then  $M$  is unitary if and only if  $\sigma(M) \subseteq \mathbb{T}$  where  $\sigma(M)$  denotes the set of eigenvalues of  $M$  and  $\mathbb{T}$  denotes the unit circle of the complex plane.

## 9.6 Definite Matrices

**DEFINITION** (Definite Matrices). Let  $M$  be an  $n \times n$  Hermitian matrix. We say that

- $M$  is **positive definite**, denoted by  $M \succ 0$ , if

$$\forall x \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad x^*Mx > 0.$$

- $M$  is **negative definite**, denoted by  $M \prec 0$ , if

$$\forall x \in \mathbb{C}^n \setminus \{0\}, \quad x^* M x < 0.$$

- $M$  is **positive semi-definite**, denoted by  $M \succeq 0$ , if

$$\forall x \in \mathbb{C}^n \setminus \{0\}, \quad x^* M x \geq 0.$$

- $M$  is **negative semi-definite**, denoted by  $M \preceq 0$ , if

$$\forall x \in \mathbb{C}^n \setminus \{0\}, \quad x^* M x \leq 0.$$

**PROPOSITION 9.6.1.** Let  $M$  be an  $n \times n$  Hermitian matrix. Then

- $M$  is positive definite if and only if all of its eigenvalues are positive.
- $M$  is negative definite if and only if all of its eigenvalues are negative.
- $M$  is positive semi-definite if and only if all of its eigenvalues are non-negative.
- $M$  is negative semi-definite if and only if all of its eigenvalues are non-positive.

*Proof.* Assume that  $M$  is positive definite. I will show that the eigenvalues of  $M$  are all positive. Let  $(\lambda, x)$  be an arbitrary eigenpair of  $M$ . Then we have  $Mx = \lambda x$ . Since  $M$  is positive definite, we have  $x^* M x > 0$ . So  $x^* \lambda x = \lambda x^* x > 0$ . Note that  $x^* x \geq 0$ . So  $\lambda > 0$ . ■

**PROPOSITION 9.6.2.** If  $A$  is positive definite, then  $A^{-1}$  exists and is also positive definite.

*Proof Approach 1.* Let  $y$  be an arbitrary vector. Then there exists some  $x$  such that  $y = Ax$  since  $A$  is invertible. Now

$$y^T A^{-1} y \tag{9.1}$$

$$= x^T A^T A^{-1} A x \tag{9.2}$$

$$= x^T A^T x = x^T A x > 0. \tag{9.3}$$

Since  $\forall y, y^T A^{-1} y > 0$ , we get  $A^{-1}$  is positive definite. ■

*Proof Approach 2.* Since  $A$  is positive definite, all its eigenvalues are positive. Eigenvalues of  $A^{-1}$  are reciprocals of eigenvalues of  $A$ . So all eigenvalues of  $A^{-1}$  are positive. So  $A^{-1}$  is positive definite. ■



## Chapter 10

# Matrix Diagonalization

### 10.1 Diagonalization in General

**DEFINITION** (Diagonalizable Matrix). Let  $M \in \mathcal{M}_{n \times n}(\mathbb{C})$ . We say that  $M$  is **diagonalizable** if and only if  $P^{-1}MP = D$  for some invertible matrix  $P \in \mathcal{M}_{n \times n}(\mathbb{C})$  and some diagonal matrix  $D \in \mathcal{M}_{n \times n}(\mathbb{C})$ .

**PROPOSITION 10.1.1.** Let  $M \in \mathcal{M}_{n \times n}(\mathbb{C})$ . Then  $M$  is diagonalizable if and only if  $\exists$  eigenpairs  $((\lambda_i, v_i))_{i=1}^n$  of  $M$  such that the matrix  $P = [v_1, \dots, v_n]$  is invertible. In this case, we have

$$P^{-1}MP = \text{diag}(\lambda_1, \dots, \lambda_n).$$

### 10.2 Unitary Diagonalization

#### 10.2.1 Definitions

**DEFINITION** (Unitarily Similar). Let  $A, B \in \mathcal{M}_{n \times n}(\mathbb{C})$ . We say that  $A$  and  $B$  are **unitarily similar** if there exists a unitary matrix  $U$  such that

$$U^*AU = B.$$

**THEOREM 10.1** (Schur). Any matrix is unitarily similar to an upper triangular matrix.

**DEFINITION** (Unitarily Diagonalizable). Let  $M$  be a complex square matrix. We say that  $M$  is **unitarily diagonalizable** if  $M$  is unitarily similar to a diagonal matrix.

### 10.2.2 Properties

**PROPOSITION 10.2.1.** Unitarily diagonalizable matrices are normal.

## 10.3 Sufficient Conditions

**PROPOSITION 10.3.1.** Hermitian matrices are unitarily diagonalizable.

**PROPOSITION 10.3.2.** Normal matrices are unitarily diagonalizable.

## Chapter 11

# Matrix Decomposition

### 11.1 Lower-Upper Decomposition

**DEFINITION** (Lower-Upper (LU) Decomposition). Let  $A$  be some square matrix. In the following let  $L$  denote lower triangular matrices,  $U$  denote upper triangular matrices,  $P$  denote permutation matrices, and  $D$  denote diagonal matrices. We define the followings:

- **LU decomposition:**

$$A = LU.$$

- **LUP decomposition:**

$$A = LUP.$$

- **PLU decomposition:**

$$A = PLU.$$

- **LDU decomposition:**

$$A = LDU$$

where  $L$  and  $U$  are required to be unitriangular.

**THEOREM 11.1** (Lower-Upper (LU) Decomposition).

- All square matrices admit LUP and PLU decompositions.

LU decomposition can be viewed as the matrix form of Gaussian elimination.

## 11.2 Cholesky Decomposition

**DEFINITION** (Cholesky Decomposition). Let  $A$  be some square matrix. In the following let  $L$  denote lower triangular matrices and  $D$  denote diagonal matrices. We define the followings:

- **Cholesky decomposition:**

$$A = LL^*$$

where the diagonal entries of  $L$  are real.

- **Square-Root-Free Cholesky (LDL) decomposition:**

$$A = LDL$$

where  $L$  is required to be unitriangular.

The diagonal elements of  $L$  are required to be 1 at the cost of introducing an additional diagonal matrix  $D$  in the decomposition.

**THEOREM 11.2** (Existence and Uniqueness).

- All Hermitian positive definite matrices admit a unique Cholesky decomposition and the matrix  $L$  has strictly positive real diagonal entries.
- All Hermitian positive semi-definite matrices admit a Cholesky decomposition and the matrix  $L$  has non-negative real diagonal entries.

## 11.3 Eigenvalue Decomposition

**DEFINITION** (Eigenvalue Decomposition). Let  $A$  be an  $n \times n$  matrix where  $n \in \mathbb{N}$ . Let  $\{(x_i, \lambda_i)\}_{i=1}^n$  be the eigenpairs of  $A$ . We define the **eigenvalue decomposition** of  $A$  to be a factorization of  $A$  given by

$$A = Q\Lambda Q^{-1}$$

where  $Q = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}$  and  $\Lambda = \text{diag}(\{\lambda_i\}_{i=1}^n)$ .



**PROPOSITION 11.3.1.** Let  $A$  be an  $n \times n$  matrix. Then  $A$  can be eigendecomposed if and only if  $A$  has  $n$  linearly independent eigenvectors.

## 11.4 Singular Value Decomposition

**DEFINITION** (Singular Value Decomposition). Let  $M$  be an  $m \times n$  real or complex matrix. We define a **singular value decomposition** to be a factorization of the form  $M = U\Sigma V^*$  where  $U$  is an  $m \times m$  unitary matrix, the columns of  $U$  are the left-singular vectors of  $M$ ;  $V$  is an  $n \times n$  unitary matrix, the columns of  $V$  are the right-singular vectors of  $M$ ;  $\Sigma$  is an  $m \times n$  rectangular diagonal matrix, the diagonal entries of  $\Sigma$  are the singular values of  $M$ .