

General Topology

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Chapter 1

Topological Spaces

1.1 Topology

DEFINITION 1.1 (Topology). Let X be a set. We define a **topology** on X , denoted by τ_X , to be a collection of subsets of X that satisfies all of the following conditions.

1. $\emptyset, X \in \tau_X$.
2. τ_X is closed under union.
3. τ_X is closed under finite intersection.

DEFINITION 1.2 (Finer, Coarser). Let τ_1 and τ_2 be two topologies on X .

- We say that τ_1 is **finer** than τ_2 if $\tau_1 \supseteq \tau_2$.
- We say that τ_1 is **coarser** than τ_2 if $\tau_1 \subseteq \tau_2$.

PROPOSITION 1.3. Let X be a set. Let $\{\tau_\lambda\}_{\lambda \in \Lambda}$ be a collection of topologies on X . Then their intersection $\bigcap_{\lambda \in \Lambda} \tau_\lambda$ is also a topology on X .

1.2 Examples of Topology

EXAMPLE 1.4 (Trivial Topology). Let X be a set. We define the **trivial topology** on X to be $\tau := \{\emptyset, X\}$.

EXAMPLE 1.5 (Discrete Topology). Let X be a set. We define the **discrete topology** on X to be $\tau := \mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the power set of X .

EXAMPLE 1.6 (Metric Topology). Let (X, d) be a metric space. We define the **metric topology** on X , induced by the metric d , to be

$$\tau := \{G \subseteq X : \forall x \in G, \exists r > 0, \text{ball}(x, r) \subseteq G\}.$$

EXAMPLE 1.7 (Cofinite Topology). Let X be a set. We define the **cofinite topology** on X to be

$$\tau := \{\emptyset\} \cup \{Y \subseteq X : X \setminus Y \text{ is finite}\}.$$

1.3 Open Sets and Closed Sets

DEFINITION 1.8 (Open Sets). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We say that S is **open** if and only if $S \in \tau$.

DEFINITION 1.9 (Closed Sets). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We say that S is **closed** if and only if $\bar{S} \in \tau$.

1.4 Neighborhood

DEFINITION 1.10 (Neighborhood of a Point). Let (\mathfrak{X}, τ) be a topological space. Let x be a point in \mathfrak{X} . Let \mathcal{N} be a subset of \mathfrak{X} . We say that \mathcal{N} is a **neighborhood** of

the point x if and only if

$$\exists G \in \tau \text{ such that } x \in G \text{ and } G \subseteq \mathcal{N}.$$

DEFINITION 1.11 (Neighborhood of a Set). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . Let \mathcal{N} be a subset of \mathfrak{X} . We say that \mathcal{N} is a **neighborhood** of the set S if and only if

$$\exists G \in \tau \text{ such that } S \subseteq G \text{ and } G \subseteq \mathcal{N}.$$

PROPOSITION 1.12. Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . Let \mathcal{N} be a subset of \mathfrak{X} . Then \mathcal{N} is a neighborhood of S if and only if \mathcal{N} is a neighborhood of every point in S .

Proof. Forward Direction: Assume that \mathcal{N} is a neighborhood of S . I will show that \mathcal{N} is a neighborhood of every point in S . Let x be an arbitrary point in S . Since \mathcal{N} is a neighborhood of S , $\exists G \in \tau$ such that $S \subseteq G \subseteq \mathcal{N}$. So $x \in S \subseteq G \subseteq \mathcal{N}$. So \mathcal{N} is a neighborhood of x .

Backward Direction: Assume that \mathcal{N} is a neighborhood of every point in S . I will show that \mathcal{N} is a neighborhood of S . Let x be an arbitrary point in S . Then \mathcal{N} is a neighborhood of x . So $\exists G_x \in \tau$ such that $x \in G_x \subseteq \mathcal{N}$. Define a set G by $G := \bigcup_{x \in S} G_x$. Since $\forall x \in S$, $G_x \in \tau$, we have $G = \bigcup_{x \in S} G_x \in \tau$. Since $\forall x \in S$, $x \in G_x$, we have $S \subseteq \bigcup_{x \in S} G_x = G$. Since $\forall x \in S$, $G_x \subseteq \mathcal{N}$, we have $G = \bigcup_{x \in S} G_x \subseteq \mathcal{N}$. Since $G \in \tau$ and $S \subseteq G \subseteq \mathcal{N}$, by definition of neighborhoods, \mathcal{N} is a neighborhood of S . This completes the proof. \square

DEFINITION 1.13 (Neighborhood System). Let (\mathfrak{X}, τ) be a topological space. Let x be a point in \mathfrak{X} . We define a **neighborhood system** at point x , denoted by \mathcal{U}_x , to be the set of all neighborhoods of x .

PROPOSITION 1.14 (Properties of Neighborhood System). Let (\mathfrak{X}, τ) be a topological space. Let x be a point in \mathfrak{X} . Then

1. Neighborhood systems are closed under finite intersections. i.e.,

$$\forall U, V \in \mathcal{U}_x, \quad U \cap V \in \mathcal{U}_x.$$

2. Neighborhoods are neighborhoods of neighborhoods. i.e.,

$$\forall U \in \mathcal{U}_x, \quad \exists V \in \mathcal{U}_x, \quad \forall y \in V, \quad U \in \mathcal{U}_y.$$

Note that here U is a neighborhood of V .

3. Any superset of a neighborhood is also a neighborhood. i.e.,

$$\forall U \in \mathcal{U}_x, \forall V \subseteq X, V \supseteq U \implies V \in \mathcal{U}_x.$$

PROPOSITION 1.15. Let (\mathfrak{X}, τ) be a topological space. Let G be a subset of \mathfrak{X} . Then

$$G \in \tau \iff \forall x \in G, \exists \mathcal{N} \subseteq \mathcal{U}_x, \mathcal{N} \subseteq G.$$

1.5 Base for Topologies

DEFINITION 1.16 (Base). Let (\mathfrak{X}, τ) be a topological space. We define a **base** for τ , denoted by \mathcal{B} , to be a subset of τ such that any set in τ can be written as a union of elements of \mathcal{B} . i.e.,

$$\forall G \in \tau, \quad \exists \mathcal{C} \subseteq \mathcal{B}, \quad G = \bigcup_{B \in \mathcal{C}} B.$$

DEFINITION 1.17 (Subbase). Let (\mathfrak{X}, τ) be a topological space. We define a **subbase** for τ , denoted by \mathcal{S} , to be a subset of τ such that the collection of all finite intersections of elements of \mathcal{S} forms a base for τ .

DEFINITION 1.18 (Neighborhood Base). Let (\mathfrak{X}, τ) be a topological space. Let x be a point in \mathfrak{X} . We define a **neighborhood base** at point x , denoted by \mathcal{B}_x , to be a

sub-collection of the neighborhood system \mathcal{U}_x at x such that

$$\forall U \in \mathcal{U}_x, \quad \exists B \in \mathcal{B}_x, \quad B \subseteq U.$$

That is, a neighborhood base at a point is a sub-collection of all the neighborhoods that are “small”.

PROPOSITION 1.19 (Base and Neighborhood Base). Let (X, τ) be a topological space. Let $\mathcal{B} \subseteq \tau$. Define for each point x in the space a set \mathcal{B}_x by $\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\}$. Then \mathcal{B} is a base for the topology if and only if $\forall x \in X$, \mathcal{B}_x is a neighborhood base at point x .

Proof. Forward Direction: Assume that \mathcal{B} is a base for the topology. I will show that $\forall x \in X$, \mathcal{B}_x is a neighborhood base at point x . Let x be an arbitrary point in X .

- First of all we need to show

$$\mathcal{B}_x \subseteq \mathcal{U}_x.$$

Let B be an arbitrary element in \mathcal{B}_x . Notice $\mathcal{B}_x \subseteq \mathcal{B} \subseteq \tau$. So $B \in \tau$. Since $B \in \mathcal{B}_x$, $x \in B$. Since $x \in B$ and $B \in \tau$, $B \in \mathcal{U}_x$. So $\mathcal{B}_x \subseteq \mathcal{U}_x$.

- Secondly, we need to show

$$\forall N_x \in \mathcal{U}_x, \quad \exists B_0 \in \mathcal{B}_x, \quad B_0 \subseteq N_x.$$

Let N_x be an arbitrary element in \mathcal{U}_x . Then $\exists G \in \tau$ such that $x \in G \subseteq N_x$. Since $G \in \tau$ and \mathcal{B} is a base for τ , $\exists \mathcal{B}' \subseteq \mathcal{B}$ such that $G = \bigcup_{B \in \mathcal{B}'} B$. Since $x \in G$ and $G = \bigcup_{B \in \mathcal{B}'} B$, $\exists B_0 \in \mathcal{B}'$ such that $x \in B_0$. Notice $B_0 \subseteq \bigcup_{B \in \mathcal{B}'} B = G \subseteq N_x$. So $x \in B_0 \subseteq N_x$. Notice $B_0 \in \mathcal{B}' \subseteq \mathcal{B}$. So $B_0 \in \mathcal{B}_x$. That is,

$$\forall N_x \in \mathcal{U}_x, \quad \exists B_0 \in \mathcal{B}_x, \quad B_0 \subseteq N_x.$$

So \mathcal{B}_x is indeed a neighborhood base at point x .

Backward Direction: Assume that $\forall x \in X$, \mathcal{B}_x is a neighborhood base at point x . I will show that \mathcal{B} is a base for the topology. We are already given that $\mathcal{B} \subseteq \tau$. Let G be an arbitrary set in τ . If $G = \emptyset$, then we are done. Otherwise, let x be an arbitrary point in G . Since $G \in \tau$ and $x \in G$, $G \in \mathcal{U}_x$. Since $G \in \mathcal{U}_x$ and \mathcal{B}_x is a neighborhood base at point x , $\exists B(x) \in \mathcal{B}_x$ such that $B(x) \subseteq G$. Since $\forall x \in G$, $x \in B(x)$, we get $\bigcup_{x \in G} B(x) \supseteq G$. Since

$\forall x \in G, B(x) \subseteq G$, we get $\bigcup_{x \in G} B(x) \subseteq G$. Since $\bigcup_{x \in G} B(x) \supseteq G$ and $\bigcup_{x \in G} B(x) \subseteq G$, we get $\bigcup_{x \in G} B(x) = G$. So G can be written as a union of elements of \mathcal{B} . Since any open set can be written as a union of elements of \mathcal{B} , \mathcal{B} is a base for the space. □

1.6 Examples of Base

EXAMPLE 1.20. Consider the set of real numbers with the usual topology. The collection

$$\mathcal{B} := \{(a, b) : a, b \in \mathbb{R}, a < b\}$$

is a base for the space.

EXAMPLE 1.21. Consider the set of real numbers with the usual topology. The collection

$$\mathcal{S} := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$$

is a subbase for the space.

1.7 Generating Topology

PROPOSITION 1.22. Let X be a non-empty set. Let \mathcal{B} be a collection of subsets of X . Then \mathcal{B} is a base for some topology on X if and only if

- $X = \bigcup_{B \in \mathcal{B}} B$, and
- $\forall B_1, B_2 \in \mathcal{B}, \forall p \in B_1 \cap B_2, \exists B_3 \in \mathcal{B}$ such that $p \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

Proof. Forward Direction: Assume that \mathcal{B} is a base for some topology τ on X . I will show that the two conditions hold. Let τ denote the topology. Since \mathcal{B} is a base for τ and $X \in \tau$, $\exists \mathcal{B}' \subseteq \mathcal{B}$ such that $X = \bigcup_{B \in \mathcal{B}'} B$. So $X = \bigcup_{B \in \mathcal{B}} B$. Let B_1 and B_2 be arbitrary sets in \mathcal{B} . If $B_1 \cap B_2 = \emptyset$, then we are done. Otherwise, let p be an arbitrary point in $B_1 \cap B_2$. Since \mathcal{B} is a base, $\mathcal{B} \subseteq \tau$. Since $B_1, B_2 \in \mathcal{B}$ and $\mathcal{B} \subseteq \tau$, $B_1, B_2 \in \tau$. Since $B_1, B_2 \in \tau$ and τ is a topology, we get $B_1 \cap B_2 \in \tau$. Since $B_1 \cap B_2 \in \tau$ and \mathcal{B} is a base, $\exists \mathcal{B}'' \subseteq \mathcal{B}$ such that

$B_1 \cap B_2 = \bigcup_{B \in \mathcal{B}''} B$. Since $p \in B_1 \cap B_2 = \bigcup_{B \in \mathcal{B}''} B$, $\exists B_3 \in \mathcal{B}'' \subseteq \mathcal{B}$ such that $p \in B_3$. Notice $B_3 \subseteq \bigcup_{B \in \mathcal{B}''} B = B_1 \cap B_2$. That is, $B_3 \subseteq B_1 \cap B_2$.

Backward Direction: Assume that the two conditions hold. I will show that \mathcal{B} is a base for some topology on X . Define a topology τ on X by

$$\tau := \left\{ \bigcup_{B \in \mathcal{B}'} B : \mathcal{B}' \subseteq \mathcal{B} \right\}.$$

Then I claim that τ is indeed a topology on X .

- Since $\emptyset \subseteq \mathcal{B}$, $\emptyset \in \tau$. Since $\mathcal{B} \subseteq \mathcal{B}$ and $X = \bigcup_{B \in \mathcal{B}} B$, $X \in \tau$.
- Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be a subset of τ for some index set Λ . Then $\forall \lambda \in \Lambda$, $\exists \mathcal{B}_\lambda \subseteq \mathcal{B}$ such that $G_\lambda = \bigcup_{B \in \mathcal{B}_\lambda} B$. So $\bigcup_{\lambda \in \Lambda} G_\lambda = \bigcup_{\lambda \in \Lambda} \bigcup_{B \in \mathcal{B}_\lambda} B = \bigcup_{B \in \mathcal{C}} B$ where $\mathcal{C} = \bigcup_{\lambda \in \Lambda} \mathcal{B}_\lambda$. Notice $\mathcal{C} \subseteq \mathcal{B}$. So $\bigcup_{\lambda \in \Lambda} G_\lambda \in \tau$.
- Let H_1 and H_2 be subsets of τ . Then $\exists \mathcal{B}_1 \subseteq \mathcal{B}$ such that $H_1 = \bigcup_{B \in \mathcal{B}_1} B$ and $\exists \mathcal{B}_2 \subseteq \mathcal{B}$ such that $H_2 = \bigcup_{B \in \mathcal{B}_2} B$. Let p be an arbitrary point in $H_1 \cap H_2$. Since $p \in H_1 \cap H_2$, $\exists B_1 \in \mathcal{B}_1 \subseteq \mathcal{B}$ and $\exists B_2 \in \mathcal{B}_2 \subseteq \mathcal{B}$ such that $p \in B_1$ and $p \in B_2$. Since $B_1, B_2 \subseteq \mathcal{B}$ and $p \in B_1 \cap B_2$, by assumption, $\exists B_3 \in \mathcal{B}$ such that $p \in B_3$ and $B_3 \subseteq B_1 \cap B_2$. So $p \in B_3$ and $B_3 \subseteq H_1 \cap H_2$. Since $\forall p \in H_1 \cap H_2$, $\exists B_3 \in \mathcal{B}$ such that $p \in B_3 \subseteq H_1 \cap H_2$, we get $H_1 \cap H_2 = \bigcup_{p \in H_1 \cap H_2} B_3(p)$. So $H_1 \cap H_2 \in \tau$.

So τ is a topology on X . Since τ is defined to be $\tau = \left\{ \bigcup_{B \in \mathcal{B}'} B : \mathcal{B}' \subseteq \mathcal{B} \right\}$, \mathcal{B} is clearly a base for τ . This completes the proof. \square

PROPOSITION 1.23. Let X be a set. Let \mathcal{S} be a collection of subsets of X . Then \mathcal{S} is a subbase for some topology on X .

Proof. Let \mathcal{B} be the collection of all finite intersections of elements of \mathcal{S} . Notice $X = \bigcap_{S \in \emptyset} S$. So $X \in \mathcal{B}$. So $X = \bigcup_{B \in \mathcal{B}} B$. Let B_1 and B_2 be arbitrary elements of \mathcal{B} . If $B_1 \cap B_2 = \emptyset$, then we are done. Otherwise, let p be an arbitrary point in $B_1 \cap B_2$. Since $B_1 \in \mathcal{B}$, $\exists \mathcal{S}_1 \subseteq \mathcal{S}$: \mathcal{S}_1 is finite such that $B_1 = \bigcap_{S \in \mathcal{S}_1} S$. Since $B_2 \in \mathcal{B}$, $\exists \mathcal{S}_2 \subseteq \mathcal{S}$: \mathcal{S}_2 is finite such that $B_2 = \bigcap_{S \in \mathcal{S}_2} S$.

So $B_1 \cap B_2 = \bigcap_{S \in \mathcal{S}_1 \cup \mathcal{S}_2} S$. Define $B_3 := B_1 \cap B_2$. Then $p \in B_3$ and $B_3 \subseteq B_1 \cap B_2$. Since \mathcal{S}_1 and \mathcal{S}_2 are both finite, $\mathcal{S}_1 \cup \mathcal{S}_2$ is also finite. So $B_3 \in \mathcal{B}$. By the previous proposition, \mathcal{B} is a base for some topology on X . So \mathcal{S} is a subbase for some topology on X . \square

DEFINITION 1.24 (Generated Topology). Let X be a set. Let \mathcal{B} be a collection of subsets of X . We define the **topology generated by \mathcal{B}** to be the smallest topology on X for which \mathcal{B} is a base.

1.8 Weak Topology

DEFINITION 1.25 (Weak Topology). Let X be a non-empty set. Let $\{(Y_\gamma, \tau_\gamma)\}_{\gamma \in \Gamma}$ be a collection of topological spaces where Γ is some index set. Let $\mathcal{F} = \{f_\gamma\}_{\gamma \in \Gamma}$ be a collection of functions where each f_γ is a function from X to Y_γ . Define a set \mathcal{S} as

$$\mathcal{S} := \{f_\gamma^{-1}(G_\gamma) : \gamma \in \Gamma, G_\gamma \in \tau_\gamma\}.$$

We define the **weak topology** on X induced by \mathcal{F} , denoted by $\sigma(X, \mathcal{F})$, to be the topology generated by \mathcal{S} .

PROPOSITION 1.26. Let X be a set. Let $\{(Y_\gamma, \tau_\gamma)\}_{\gamma \in \Gamma}$ be a collection of topological spaces where Γ is some index set. Let $\mathcal{F} = \{f_\gamma\}_{\gamma \in \Gamma}$ be a collection of functions where each f_γ is a function from X to Y_γ . Then for any topology τ on X , if f_γ is continuous for all $\gamma \in \Gamma$, we get $\sigma(X, \mathcal{F}) \subseteq \tau$. i.e., $\sigma(X, \mathcal{F})$ is the weakest topology on X under which all f_γ 's are continuous.

Proof. Let τ be an arbitrary topology on X . Since $\forall \gamma \in \Gamma$, f_γ is continuous, we get $\forall \gamma \in \Gamma$, $\forall G_\gamma \in \tau_\gamma$, $f_\gamma^{-1}(G_\gamma) \in \tau$. So $\mathcal{S} \subseteq \tau$. So the finite intersections of elements of \mathcal{S} are in τ . So the unions of finite intersections of elements of \mathcal{S} are in τ . That is, $\sigma(X, \mathcal{F}) \subseteq \tau$. \square

PROPOSITION 1.27. Let X be a non-empty set. Let $\{(Y_\gamma, \tau_\gamma)\}_{\gamma \in \Gamma}$ be a collection of topological spaces where Γ is some index set. Let $\mathcal{F} = \{f_\gamma\}_{\gamma \in \Gamma}$ be a collection of functions where each f_γ is a function from X to Y_γ . Let (Z, τ_Z) be a topological space.

Let g be a function from (Z, τ_Z) to $(X, \sigma(X, \mathcal{F}))$. Then g is continuous if and only if $\forall \gamma \in \Gamma, f_\gamma \circ g : Z \rightarrow Y_\gamma$ is continuous.

Proof. Forward Direction: Assume that g is continuous. I will show that $\forall \gamma \in \Gamma, f_\gamma \circ g : Z \rightarrow Y_\gamma$ is continuous. Let γ be an arbitrary index in Γ . Let G_γ be an arbitrary element in τ_γ . Notice $f_\gamma^{-1}(G_\gamma) \in \sigma(X, \mathcal{F})$. Since $f_\gamma^{-1}(G_\gamma) \in \sigma(X, \mathcal{F})$ and $g : Z \rightarrow X$ is continuous, $g^{-1}(f_\gamma^{-1}(G_\gamma)) \in \tau_Z$. That is, $\forall \gamma \in \Gamma, \forall G_\gamma \in \tau_\gamma$, we have $g^{-1}(f_\gamma^{-1}(G_\gamma)) \in \tau_Z$. So $f_\gamma \circ g$ is continuous from Z to Y_γ .

Backward Direction: Assume that $\forall \gamma \in \Gamma, f_\gamma \circ g : Z \rightarrow Y_\gamma$ is continuous. I will show that g is continuous. Let G be an arbitrary element in $\sigma(X, \mathcal{F})$. Then $G = \bigcup_{\lambda \in \Lambda} \bigcap_{i=1}^{n_\lambda} S_{\lambda,i}$ where each $S_{\lambda,i} \in \mathcal{S}$. Since $S_{\lambda,i} \in \mathcal{S}, \exists \gamma_0 \in \Gamma$ and $G_{\lambda,i} \in \tau_{\gamma_0}$ such that $S_{\lambda,i} = f_{\gamma_0}^{-1}(G_{\lambda,i})$. So

$$\begin{aligned} g^{-1}(G) &= g^{-1}\left(\bigcup_{\lambda \in \Lambda} \bigcap_{i=1}^{n_\lambda} S_{\lambda,i}\right) \\ &= g^{-1}\left(\bigcup_{\lambda \in \Lambda} \bigcap_{i=1}^{n_\lambda} f_{\gamma_0}^{-1}(G_{\lambda,i})\right) \\ &= \bigcup_{\lambda \in \Lambda} \bigcap_{i=1}^{n_\lambda} g^{-1}(f_{\gamma_0}^{-1}(G_{\lambda,i})). \end{aligned}$$

Since $f_{\gamma_0} \circ g$ is continuous from Z to Y_{γ_0} and $G_{\lambda,i} \in \tau_{\gamma_0}$, we get $g^{-1}(f_{\gamma_0}^{-1}(G_{\lambda,i})) \in \tau_Z$. So $g^{-1}(G) \in \tau_Z$. That is, $\forall G \in \sigma(X, \mathcal{F})$, we have $g^{-1}(G) \in \tau_Z$. So $g : Z \rightarrow X$ is continuous. \square

PROPOSITION 1.28 (Convergence of nets in the weak topology can be characterized by pointwise convergence). Let X be a set. Let (Y, τ_Y) be a topological space. Let $\mathcal{F} = \{f_\gamma\}_{\gamma \in \Gamma}$ be a collection of function from X to Y where Γ is some index set. Let $\sigma(X, \mathcal{F})$ be the weak topology on X induced by \mathcal{F} . Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in $(X, \sigma(X, \mathcal{F}))$. Let x be a point in X . Then $\lim_{\lambda \in \Lambda} x_\lambda = x$ in $(X, \sigma(X, \mathcal{F}))$ if and only if

$$\forall \gamma \in \Gamma, \lim_{\lambda \in \Lambda} f_\gamma(x_\lambda) = f_\gamma(x) \text{ in } (Y, \tau_Y).$$

Proof. Forward Direction: Assume that $\lim_{\lambda \in \Lambda} x_\lambda = x$ in (X, σ) . I will show that $\forall \gamma \in \Gamma, \lim_{\lambda \in \Lambda} f_\gamma(x_\lambda) = f_\gamma(x)$. Let γ be an arbitrary index in Γ . Since $\sigma(X, \mathcal{F})$ is the topology induced by \mathcal{F} , f_γ is continuous from X to Y . Since f_γ is continuous and $\lim_{\lambda \in \Lambda} x_\lambda = x$ in (X, σ) , $\lim_{\lambda \in \Lambda} f_\gamma(x_\lambda) = f_\gamma(x)$.

Backward Direction: Assume that $\forall \gamma \in \Gamma, \lim_{\lambda \in \Lambda} f_\gamma(x_\lambda) = f_\gamma(x)$. I will show that $\lim_{\lambda \in \Lambda} x_\lambda = x$ in (X, σ) . Let U be an arbitrary element in $\mathcal{U}_x^{(X, \sigma)}$. Then $\exists B \in \mathcal{B}_x^{(X, \sigma)}$ such that

$B \subseteq U$. Since $B \in \mathcal{B}_x^{(X,\sigma)} \subseteq \mathcal{B}$, $\exists n \in \mathbb{N}$, $\exists \{S_i\}_{i=1}^n \subseteq \mathcal{S}$ such that $B = \bigcap_{i=1}^n S_i$. Since $S_i \in \mathcal{S}$, $\exists G_i \in \tau_Y$, $\exists f_i \in \mathcal{F}$ such that $S_i = f_i^{-1}(G_i)$. So $B = \bigcap_{i=1}^n f_i^{-1}(G_i)$. Since $B \in \mathcal{B}_x^{(X,\sigma)}$, $x \in B$. So $\forall i = 1..n$, $x \in f_i^{-1}(G_i)$. So $\forall i = 1..n$, $f_i(x) \in G_i$. Since $\lim_{\lambda \in \Lambda} f_i(x_\lambda) = f_i(x)$, $\exists \lambda_i \in \Lambda$ such that $\forall \lambda \geq \lambda_i$, $f_i(x_\lambda) \in G_i$. Let λ be an arbitrary index such that $\lambda \geq \max\{\lambda_i\}_{i=1}^n$. Then $\forall i = 1..n$, $f_i(x_\lambda) \in G_i$. So $\forall i = 1..n$, $x_\lambda \in f_i^{-1}(G_i)$. So $x_\lambda \in \bigcap_{i=1}^n f_i^{-1}(G_i)$. That is, $x_\lambda \in B$. Recall that $B \subseteq U$. So $x_\lambda \in U$. That is, $\forall U \in \mathcal{U}_x^{(X,\sigma)}$, $\exists \max\{\lambda_i\}_{i=1}^n$ such that $\forall \lambda \geq \max\{\lambda_i\}_{i=1}^n$, we get $x_\lambda \in U$. So $\lim_{\lambda \in \Lambda} x_\lambda = x$. This completes the proof. \square

1.9 Product Topology

DEFINITION 1.29 (Projection Map). Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda \in \Lambda}$ be a collection of topological spaces. Let β be an index in Λ . We define the β^{th} **projection map**, denoted by π_β , to be a function from $\prod_{\lambda \in \Lambda} X_\lambda$ to X_β given by

$$\pi_\beta(x) = \pi_\beta((x_\lambda)_{\lambda \in \Lambda}) := x_\beta \in X_\beta.$$

DEFINITION 1.30 (Product Topology). Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda \in \Lambda}$ be a collection of topological spaces where Λ is an index set. We define a **product topology** on $\prod_{\lambda \in \Lambda} X_\lambda$ to be the weak topology on $\prod_{\lambda \in \Lambda} X_\lambda$ induced by $\{\pi_\lambda\}_{\lambda \in \Lambda}$.

PROPOSITION 1.31. The product topology on X is the $\sigma(X, \{\pi_\lambda\}_{\lambda \in \Lambda})$ topology. i.e., the weak topology generated by the projection maps.

PROPOSITION 1.32. A basic neighborhood of a point $x = (x_\lambda)_{\lambda \in \Lambda} \in X$ is of the form $\prod_{\lambda \in \Lambda} U_\lambda$ where each U_λ is a neighborhood of x_λ and for all but finitely many λ 's, we have $U_\lambda = X_\lambda$.

THEOREM 1.33 (Tychonoff's Theorem). Let $(X_\lambda, \mathcal{T}_\lambda)_{\lambda \in \Lambda}$ be a non-empty collection of compact topological spaces. Then $X := \prod_{\lambda \in \Lambda} X_\lambda$ is compact in the product topology.

Chapter 2

Metric Spaces

2.1 Metrics

DEFINITION 2.1 (Metric). Let X be a non-empty set. Let d be a function from $X \times X$ to \mathbb{R}_+ . We say that d is a **metric** (or distance function) on X if it satisfies all of the following conditions.

1. Non-negativity

$$\forall x, y \in X, \quad d(x, y) \geq 0.$$

2. Identity of Indiscernible

$$\forall x, y \in X, \quad d(x, y) = 0 \iff x = y.$$

3. Symmetry

$$\forall x, y \in X, \quad d(x, y) = d(y, x).$$

4. Sub-Additivity (Triangle Inequality)

$$\forall x, y, z \in X, \quad d(x, y) \leq d(x, z) + d(z, y).$$

DEFINITION 2.2 (Metric Space). Let X be a non-empty set. Let d be a metric on X . We call the pair (X, d) a **metric space**.

2.2 Continuity of Metrics

PROPOSITION 2.3. Let (X, d) be a metric space. Let x_0 be a point in X . Let d_{x_0} be a function from X to \mathbb{R} given by $d_{x_0}(x) := d(x, x_0)$. Then d_{x_0} is continuous.

PROPOSITION 2.4. Let (X, d) be a metric space. Let S be a subset of X . Let d_S be a function from X to \mathbb{R} given by $d_S(x) := d(x, S)$. Then d_S is continuous.

Proof.

Let x_0 be an arbitrary point in X .

Let ε be an arbitrary positive number.

Let $\delta(\varepsilon)$ be a positive number given by $\delta(\varepsilon)\varepsilon$.

Let x be an arbitrary point in X such that $d(x, x_0) < \delta(\varepsilon)$.

Let x' be an arbitrary point in S .

Since $x' \in S$, by definition of distance from a point to a set, $d(x_0, S) \leq d(x_0, x')$.

Since d is a metric on X and x_0 and x' are points in X , by the triangle inequality, $d(x_0, x') \leq d(x_0, x) + d(x, x')$.

Since $d(x_0, S) \leq d(x_0, x')$ and $d(x_0, x') \leq d(x_0, x) + d(x, x')$, $d(x_0, S) - d(x_0, x) \leq d(x, x')$.

Since $d(x_0, S) - d(x_0, x) \leq d(x, x')$ for any $x' \in S$, $d(x_0, S) - d(x_0, x)$ is a lower bound for the set $\{d(x, x') : x' \in S\}$.

Since $d(x_0, S) - d(x_0, x)$ is a lower bound for the set $\{d(x, x') : x' \in S\}$, by definition of infimum, $d(x_0, S) - d(x_0, x) \leq \inf\{d(x, x') : x' \in S\} = d(x, S)$.

Since $x' \in S$, by definition of distance from a point to a set, $d(x, S) \leq d(x, x')$.

Since d is a metric on X and x and x' are points in X , by the triangle inequality, $d(x, x') \leq d(x, x_0) + d(x_0, x')$.

Since $d(x, S) \leq d(x, x')$ and $d(x, x') \leq d(x, x_0) + d(x_0, x')$, $d(x, S) - d(x, x_0) \leq d(x_0, x')$.

Since $d(x, S) - d(x, x_0) \leq d(x_0, x')$ for any $x' \in S$, $d(x, S) - d(x, x_0)$ is a lower bound for the set $\{d(x_0, x') : x' \in S\}$.

Since $d(x, S) - d(x, x_0)$ is a lower bound for the set $\{d(x_0, x') : x' \in S\}$, by definition of infimum, $d(x, S) - d(x, x_0) \leq \inf\{d(x_0, x') : x' \in S\} = d(x_0, S)$.

Since $d(x_0, S) - d(x_0, x) \leq d(x, S)$ and $d(x, S) - d(x, x_0) \leq d(x_0, S)$, $|d(x, S) - d(x_0, S)| \leq d(x, x_0)$.

Since $|d(x, S) - d(x_0, S)| \leq d(x, x_0)$ and $d(x, x_0) < \delta(\varepsilon) = \varepsilon$, $|d(x, S) - d(x_0, S)| < \varepsilon$.

Since for any positive number ε , there exists a positive number $\delta(\varepsilon)$ such that if for any point x in X , if $d(x, x_0) < \delta(\varepsilon)$, then $|d(x, S) - d(x_0, S)| < \varepsilon$, by definition of continuity, the function $x \mapsto d(x, S)$ is continuous.

□

2.3 Open Sets and Closed Sets in Metric Spaces

DEFINITION 2.5 (Openness). Let (X, d) be a metric space. Let S be a subset of X . We say that S is **open** in (X, d) if for any sequence $\{x_k\}_{k=1}^{\infty}$ in X that converges to some point in S , there exists an integer N such that for any index k with $k > N$, we have $x_k \in S$.

DEFINITION 2.6 (Openness). Let (X, d) be a metric space. Let S be a subset of X . We say that S is **open** in (X, d) if

$$\forall x \in S, \exists r_0 \in \mathbb{R}_{++}, \text{ball}(x, r_0) \subseteq S.$$

PROPOSITION 2.7. Two definitions of open-ness are equivalent.

Proof.

For one direction, assume that for any sequence $\{x_k\}_{k=1}^{\infty}$ in X that converges to some point in S , there exists an integer N such that for any index k with $k > N$, we have $x_k \in S$.

We are to prove that

$$\forall x \in S, \exists r_0 \in \mathbb{R}_{++}, \text{ball}(x, r_0) \subseteq S.$$

Let x be an arbitrary point in S .

Let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary sequence in X that converges to x .

By assumption, there exists an integer N such that $x_N \in S$.

Consider the open ball $B_0(x, d(x_N, x))$.

By definition, there exists another integer N' such that for any index k with $k > N'$, we have $x_k \in B_0(x, d(x_N, x))$.

Assume that for any point x in G , there exists an open ball $B(x, r)$ centered at x , of some radius r , such that $B(x, r) \subseteq G$.

Let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary sequence in X that converges to some point x_0 in G . We are to show that $\{x_k\}$ is eventually in G .

By assumption, there exists an open ball $B_0(x_0, r_0)$ centered at x_0 , of some radius r_0 , such that $B_0(x_0, r_0) \subseteq G$.

$$d(x_k, x_0) < r_0$$

By definition of open balls, x_k is in $B_0(x_0, r_0)$. Thus x_k is also in G .
i.e., any sequence in X that converges to some point in G is eventually in G . □

Proof. For the reverse direction, assume that any sequence in X that converges to some point in G is eventually in G .

Let x_0 be an arbitrary point in G .

We are to prove that there exists an open ball $B_0(x_0, r_0)$ centered at x_0 , of some radius r_0 , such that $B_0(x_0, r_0) \subseteq G$.

$$d(x_k, x_0) < 1/k$$

Then $\{x_k\}$ converges to x_0 .

By assumption, there exists an integer N such that $x_N \in G$.

Take $r_0 = 1/N$. Then the open ball $B_0(x_0, r_0)$ is contained in G .

i.e., for any point x in G , there exists an open ball $B(x, r)$ centered at x , of some radius r , such that $B(x, r) \subseteq G$. □

DEFINITION 2.8 (Closedness). Let (X, d) be a metric space. Let S be a subset of X . We say that S is **closed** in (X, d) if for any sequence in S that converges to some point x_0 in S , x_0 is also in S .

PROPOSITION 2.9 (Stability under Set Operations).

1. The complement of an open set is closed.
2. The complement of a closed set is open.
3. The union of an arbitrary collection of open sets is open.
4. The union of a finite collection of closed sets is closed.
5. The intersection of a finite collection of open sets is open.
6. The intersection of an arbitrary collection of closed sets is closed.
7. If S_1 is open and S_2 is closed, then $S_1 \setminus S_2$ is open.

8. If S_1 is closed and S_2 is open, then $S_1 \setminus S_2$ is closed.

Proof of (1). Let $\{U_\alpha\}$ be a set of open sets.

By Lemma 2.1, each of the open set can be written as a union of open balls.

i.e. for all α , there exists a set of open balls $\{B_{\alpha\beta}\}$ such that

$$U_\alpha = \bigcup_{\beta} B_{\alpha\beta}.$$

Then the union of the open sets $\{U_\alpha\}$ is

$$\bigcup_{\alpha} U_\alpha = \bigcup_{\alpha} \bigcup_{\beta} B_{\alpha\beta}.$$

By Lemma 2.1 again, the union is open.

□

Proof of (2). Let $\{U_k\}_{k=1}^{k=n}$ be a finite sequence of open sets.

Let x be an arbitrary element in their intersection.

Then x is in each of U_k .

By definition of open sets, for each U_k , there exists an open ball $B_k(r_k, x)$ such that $B_k \subseteq U_k$

Define $r = \min\{r_k\}$.

Consider the open ball $B(r, x)$.

Then B is a subset of each U_k and hence a subset of their intersection.

By definition of open sets, the intersection is open.

□

Proof of (3). Let $\{F_\alpha\}$ be a set of closed subsets of X .

Then each of $(F_\alpha)^c$ is open.

By Proposition 2.1, we get $\bigcup_{\alpha} (F_\alpha)^c$ is open.

By the De Morgan's Laws, we get $(\bigcap_{\alpha} F_\alpha)^c$ is open.

It follows that $\bigcap_{\alpha} F_\alpha$ is closed.

□

Proof of (4). Let $\{F_\alpha\}$ be a set of closed subsets of X .

Then each of $(F_\alpha)^c$ is open.

By Proposition 2.2, we get $\bigcap_{\alpha} (F_\alpha)^c$ is open.

By the De Morgan's Laws, we get $(\bigcup_{\alpha} F_\alpha)^c$ is open.

It follows that $\bigcup_{\alpha} F_{\alpha}$ is closed.

□

2.4 Convex Metric Spaces

DEFINITION 2.10 (Convex Metric Space). Let \mathfrak{X} be a metric space. We say that \mathfrak{X} is **convex** if and only if

$$\forall x, y \in \mathfrak{X} : x \neq y, \exists z \in \mathfrak{X}, \quad d(x, y) = d(x, z) + d(z, y).$$

That is, for any two distinct points in \mathfrak{X} , there is a third point lying between them.

2.5 Injective Metric Spaces

2.6 Examples of Metrics

2.6.1 The Discrete Metric

DEFINITION 2.11 (Discrete Metric). Let X be a set. We define a **discrete metric** on X to be a function from X to \mathbb{R}_+ given by

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

PROPOSITION 2.12. In a discrete metric space, any set is both open and closed.

PROPOSITION 2.13. In a discrete metric space, the unit closed ball is the whole space, and the unit open ball is the singleton set consisting of only the center.

PROPOSITION 2.14. Discrete metric spaces are always bounded.

Proof. Let (X, d) be a discrete metric space and S be a subset of X .

Case 1. S is empty.

Since S is empty, by definition of boundedness, S is bounded.

Case 2. S is not empty.

Since S is not empty, pick a point x_0 in S .

Since (S, d) is discrete, $d(x, x_0) \leq 1$ for any $x \in S$.

Since $d(x, x_0) \leq 1$ for any $x \in S$, by definition of boundedness, S is bounded.

Summary.

Since in all cases, S is bounded, S is bounded.

□

PROPOSITION 2.15. A discrete metric space is totally bounded only if it is empty or it is a singleton set. (**unconfirmed**)

PROPOSITION 2.16. Discrete metric spaces are always complete.

Proof. Let (X, d) be a discrete metric space and $\{x_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence in X .

We are to prove that $\{x_i\}_{i \in \mathbb{N}}$ is convergent.

Since $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy, there exists an integer $N(1)$ such that for any indices m and n , if $m, n > N(1)$, then $d(x_m, x_n) < 1$.

Since $d(x_m, x_n) < 1$ and d is discrete, $d(x_m, x_n) = 0$.

Since $d(x_m, x_n) = 0$ and d is a metric, $x_m = x_n$.

Since $m, n > N(1)$ implies $x_m = x_n$ for any indices m and n , in particular, $i > N(1)$ implies $x_i = x_{N(1)+1}$ for any index i .

Since $i > N(1)$ implies $x_i = x_{N(1)+1}$ for any index i , for any positive number ε and any index i , if $i > N(1)$, then $d(x_i, x_{N(1)+1}) < \varepsilon$.

Since for any positive number ε and any index i , if $i > N(1)$, then $d(x_i, x_{N(1)+1}) < \varepsilon$, by definition of convergence, $\{x_i\}_{i \in \mathbb{N}}$ converges to $x_{N(1)+1}$.

Since any Cauchy sequence in (X, d) converges in (X, d) , by definition of completeness, (X, d) is complete.

□

PROPOSITION 2.17. Discrete metric spaces are totally disconnected.

PROPOSITION 2.18. Any function defined on a discrete metric space is uniformly continuous.

not sure what the codomain could be.

2.6.2 The Hausdorff Metric

PROPOSITION 2.19. Let (X, d) be a metric space. Let $(\mathcal{H}(X), d_H)$ be the induced Hausdorff space. Then if (X, d) is complete, $(\mathcal{H}(X), d)$ is complete.

PROPOSITION 2.20. Let (X, d) be a metric space. Let $(\mathcal{H}(X), d_H)$ be the induced Hausdorff space. Then if (X, d) is totally bounded, $(\mathcal{H}(X), d_H)$ is totally bounded.

Proof. Let r be an arbitrary radius.

Since (X, d) is totally bounded, there exists a finite collection $\mathfrak{S} = \{x_i\}_{i \in I}$ of points in X such that $\{\text{ball}(x_i, r)\}_{i \in I}$ covers X .

Let S be an arbitrary set in $\mathcal{H}(X)$.

Let $\mathfrak{S}'(S) = \{x_i\}_{i \in I'(S)}$ be the subcollection of \mathfrak{S} of points x_i such that $\text{ball}(x_i, r) \cap S \neq \emptyset$.

Since S is a nonempty subset of X and $\{\text{ball}(x_i, r)\}_{i \in I}$ covers X , $\mathfrak{S}'(S)$ is nonempty.

Since $\mathfrak{S}'(S)$ is finite, $\mathfrak{S}'(S)$ is closed and bounded in (X, d) .

Since $\mathfrak{S}'(S)$ is nonempty, closed, and bounded in (X, d) , $\mathfrak{S}'(S) \in \mathcal{H}(X)$.

Since $\text{ball}(x_i, r) \cap S \neq \emptyset$ for each $i \in I'(S)$, $(S)_r \supseteq \mathfrak{S}'(S)$.

Since $\{\text{ball}(x_i, r)\}_{i \in I}$ covers X and S is a subset of X , $\{\text{ball}(x_i, r)\}_{i \in I}$ covers S .

Since $\{\text{ball}(x_i, r)\}_{i \in I}$ covers S and $\text{ball}(x_i, r) \cap S = \emptyset$ for any $i \in I \setminus I'(S)$, $\{\text{ball}(x_i, r)\}_{i \in I'(S)}$ covers S .

Since $\{\text{ball}(x_i, r)\}_{i \in I'(S)}$ covers S , $(\mathfrak{S}'(S))_r \supseteq S$.

Since $(S)_r \supseteq \mathfrak{S}'(S)$ and $(\mathfrak{S}'(S))_r \supseteq S$, by definition of Hausdorff metric, $d_H(S, \mathfrak{S}'(S)) < r$.

Since $d_H(S, \mathfrak{S}'(S)) < r$, $S \in \text{ball}(\mathfrak{S}'(S), r)$.

Since for any S in $\mathcal{H}(X)$, there exists another set $\mathfrak{S}'(S)$ in $\mathcal{H}(X)$ such that $S \in \text{ball}(\mathfrak{S}'(S), r)$, $\{\text{ball}(\mathfrak{S}', r)\}_{\mathfrak{S}' \in \mathcal{P}(\mathfrak{S})}$ covers $\mathcal{H}(X)$.

Since for any radius r , there exists a collection \mathcal{C} of sets in $\mathcal{H}(X)$ such that $\{\text{ball}(\mathfrak{S}', r)\}_{\mathfrak{S}' \in \mathcal{C}}$ covers $\mathcal{H}(X)$, by definition of total boundedness, $\mathcal{H}(X)$ is totally bounded.

□

Chapter 3

Interior, Closure, and Boundary

3.1 Interior

DEFINITION 3.1 (Interior Point). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . Let x be a point in \mathfrak{X} . We say that x is an **interior point** of S if and only if there exists some neighborhood \mathcal{N}_x of x such that $\mathcal{N} \subseteq S$.

DEFINITION 3.2 (Interior). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We define the **interior** of S , denoted by $\text{int}(S)$, to be the set of all interior points of S .

DEFINITION 3.3 (Interior). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We define the **interior** of S , denoted by $\text{int}(S)$, to be the union of all open subsets of S . Equivalently, the largest open subset of S .

PROPOSITION 3.4. The two definitions of interior are equivalent.

3.2 Boundary

DEFINITION 3.5 (Boundary Point). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . Let x be a point in the space. We say that x is a **boundary point** of S if and only if for any neighborhood \mathcal{N}_x of x , we have $\mathcal{N}_x \cap S \neq \emptyset$ and $\mathcal{N}_x \cap (\mathfrak{X} \setminus S) \neq \emptyset$.

DEFINITION 3.6 (Boundary). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We define the **boundary** of S , denoted by $bd(S)$, to be the set of all boundary points of S .

DEFINITION 3.7 (Boundary). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We define the **boundary** of S , denoted by $bd(S)$, to be the set given by $C_X(\text{int}(S)) \cup \text{int}(C_X(S))$.

DEFINITION 3.8 (Boundary). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We define the **boundary** of S , denoted by $bd(S)$, to be the set given by $\text{cl}(S) \cap \text{cl}(C_X(S))$.

PROPOSITION 3.9. The three definitions of boundary are equivalent.

3.3 Closure

DEFINITION 3.10 (Adherent Point). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . Let x be a point in the space. We say that x is an **adherent point** of S if and only if for any neighborhood \mathcal{N}_x of x , we have $\mathcal{N}_x \cap S \neq \emptyset$.

DEFINITION 3.11 (Closure). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We define the **closure** of S , denoted by $\text{cl}(S)$, to be the set of all adherent points of S .

DEFINITION 3.12 (Closure). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We define the **closure** of S , denoted by $\text{cl}(S)$, to be the intersection of all closed superset of S , or equivalently, the smallest closed superset of S .

PROPOSITION 3.13. The two definitions of closure are equivalent.

Proof.

For one direction, let x be an arbitrary point in the intersection of all closed supersets of S .

We are to prove that x is an adherent point of S .

Assume for the sake of contradiction that x is not an adherent point of S .

Since x is not an adherent point of S , there exists some open set U containing x such that $U \cap S = \emptyset$.

Since U is open in (X, τ) , $C_X(U)$ is closed in (X, τ) .

Since $U \cap S = \emptyset$, $C_X(U) \supseteq S$.

Since $x \in U$, $x \notin C_X(U)$.

Since $C_X(U)$ is a closed superset of S and $x \notin C_X(U)$, x is not in the intersection of all closed supersets of S .

This contradicts to the assumption that x is in the intersection of all closed supersets of S .

So the assumption that x is not an adherent point of S is false.

i.e., x is an adherent point of S .

For the other direction, let x be an adherent point of S .

We are to prove that x is in the intersection of all closed supersets of S .

Assume for the sake of contradiction that x is not in the intersection of all closed supersets of S .

Since x is not in the intersection of all closed supersets of S , there exists some closed superset E of S such that $x \notin E$.

Since E is closed in (X, τ) , $C_X(E)$ is open in (X, τ) .

Since $E \supseteq S$, $C_X(E) \cap S = \emptyset$.

Since $x \notin E$, $x \in C_X(E)$.

Since there exists some open set U such that $x \in U$ and $U \cap S = \emptyset$, x is not an adherent point of S .

This contradicts to the assumption that x is an adherent point of S .

So the assumption that x is not in the intersection of all closed supersets of S is false.

i.e., x is in the intersection of all closed supersets of S .

□

3.4 Basic Properties

PROPOSITION 3.14.

1. Interiors are open.
2. Closures are closed.
3. Boundaries are closed.

PROPOSITION 3.15. For any set in any topological space, the closure is the disjoint union of the interior and the boundary. i.e., an adherent point is exactly one of an interior point or a boundary point.

PROPOSITION 3.16. Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . Then

1. $\text{int}(X \setminus S) = X \setminus \text{cl}(S)$.
2. $\text{cl}(X \setminus S) = X \setminus \text{int}(S)$.
3. $\text{bd}(X \setminus S) = \text{bd}(S)$.

Proof of (1). For one direction, let x be an arbitrary point in $\text{int}(C_X(S))$.

We are to prove that $x \notin \text{cl}(S)$.

Since $x \in \text{int}(C_X(S))$, by definition of interior, there exists an open set U such that $x \in U$ and $U \subseteq C_X(S)$.

Since U is an open subset of S , $C_X(U)$ is a closed superset of S .

Since $x \in U$, $x \notin C_X(U)$.

Since $C_X(U)$ a closed superset of S and $x \notin C_X(U)$, x is not in the intersection of all closed superset of S .

Since x is not in the intersection of all closed superset of S , by definition of closure, $x \notin \text{cl}(S)$.

Since $x \notin \text{cl}(S)$, we get $x \in C_X(\text{cl}(S))$.

For the reverse direction, let x be an arbitrary point in $\text{cl}(S)$.

We are to prove that $x \notin \text{int}(C_X(S))$.

Since $x \in \text{cl}(S)$, by definition of closure, x is in any closed superset F of S .

Since F is a closed superset of S , $C_X(F)$ is an open subset of $C_X(S)$.

Since $x \in F$, $x \notin C_X(F)$.

Since $C_X(F)$ is an arbitrary open subset of $C_X(S)$ and $x \notin C_X(F)$, x is not in the union of all open subsets of $C_X(S)$.

Since x is not in the union of all open subsets of $C_X(S)$, by definition of interior, $x \notin \text{int}(C_X(S))$.

□

Proof of (2).

$$\text{cl}(S^c) \subseteq (\text{int}(S))^c \# (*) \setminus n(\text{int}(S))^c \subseteq \text{cl}(S^c) \# (**)$$

Let x be an arbitrary point in $\text{cl}(S^c)$.

By definition of closure, x is in every closed superset of S^c .

Note that x is not in any open subset of S .

By definition, we conclude that $x \notin \text{int}(S)$ and hence $x \in (\text{int}(S))^c$.

$$\text{cl}(S^c) \subseteq (\text{int}(S))^c \# (*)$$

Let x be an arbitrary point in $(\text{int}(S))^c$.

By definition of interior, x is not in any open subset of S .

Note that the complement of open subsets of S are closed supersets of S^c .

Thus x is in every closed superset of S^c .

By definition, we conclude that $x \in \text{cl}(S^c)$.

$$(\text{int}(S))^c \subseteq \text{cl}(S^c) \# (**)$$

□

PROPOSITION 3.17 (Characterizations). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . Then

1. S is open if and only if $S = \text{int}(S)$.
2. S is closed if and only if $S = \text{cl}(S)$.

Proof of 1. Forward Direction: Suppose that S is open. I will show that $S = \text{int}(S)$. By Proposition 3.21 part 1, we have $\text{int}(S) \subseteq S$. Let x be an arbitrary element of S . Since $x \in S$ and S is an open subset of S , we get $x \in \text{int}(S)$. So $S \subseteq \text{int}(S)$. So $S = \text{int}(S)$.

Backward Direction: Suppose that $S = \text{int}(S)$. By Proposition 3.14 part 1, we know that $\text{int}(S)$ is open. So S is also open. \square

Proof of 2. Forward Direction: Suppose that S is closed. I will show that $S = \text{cl}(S)$. By Proposition 3.22 part 1, we have $S \subseteq \text{cl}(S)$. Let x be an arbitrary element of $\text{cl}(S)$. Since $x \in \text{cl}(S)$ and S is a closed superset of S , we get $x \in S$. So $\text{cl}(S) \subseteq S$. So $S = \text{cl}(S)$.

Backward Direction: Suppose that $S = \text{cl}(S)$. By Proposition 3.14 part 2, we know that $\text{cl}(S)$ is closed. So S is also closed. \square

PROPOSITION 3.18 (Characterizations). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . Then

1. S is open if and only if $S \cap \text{bd}(S) = \emptyset$.
2. S is closed if and only if $\text{bd}(S) \subseteq S$.

PROPOSITION 3.19. The boundary of some open set or of some closed set has no interior.

Proof for Open Sets. Let (X, τ) be a topological space and S be an open set in (X, τ) .

Let x be an arbitrary point in $\text{bd}(S)$.

Let $\mathcal{N}(x)$ be an arbitrary open neighborhood of x in (X, τ) .

Since x is a boundary point of S and $\mathcal{N}(x)$ is some open neighborhood of x in (X, τ) , by definition of boundary, $\mathcal{N}(x) \cap S \neq \emptyset$.

Since S is open, $S \cap \text{bd}(S) = \emptyset$.

Since $\mathcal{N}(x) \cap S \neq \emptyset$ and $S \cap \text{bd}(S) = \emptyset$, $\mathcal{N}(x) \not\subseteq \text{bd}(S)$.

Since $\mathcal{N}(x) \not\subseteq \text{bd}(S)$ for any open neighborhood of x in (X, τ) , by definition of interior, $x \notin \text{int}(\text{bd}(S))$.

Since $x \notin \text{int}(\text{bd}(S))$ for any $x \in \text{bd}(S)$, $\text{bd}(S)$ has no interior. \square

Proof for Closed Sets. Let (X, τ) be a topological space and S be a closed set in (X, τ) .

Let x be an arbitrary point in $\text{bd}(S)$.

Let $\mathcal{N}(x)$ be an arbitrary open neighborhood of x in (X, τ) .

Since x is a boundary point of S and $\mathcal{N}(x)$ is some open neighborhood of x in (X, τ) , by definition of boundary, $\mathcal{N}(x) \cap C_X(S) \neq \emptyset$.

Since S is closed, $C_X(S) \cap \text{bd}(S) = \emptyset$.

Since $\mathcal{N}(x) \cap C_X(S) \neq \emptyset$ and $C_X(S) \cap \text{bd}(S) = \emptyset$, $\mathcal{N}(x) \not\subseteq \text{bd}(S)$.

Since $\mathcal{N}(x) \not\subseteq bd(S)$ for any open neighborhood of x in (X, τ) , by definition of interior, $x \notin \text{int}(bd(S))$.

Since $x \notin \text{int}(bd(S))$ for any $x \in bd(S)$, $bd(S)$ has no interior.

□

PROPOSITION 3.20. The boundary of a set is empty if and only if the set is both open and closed.

Proof.

Let (X, d) be a metric space and S be a subset of X .

For one direction, assume that $bd(S) = \emptyset$. We are to prove that S is both open and closed.

Let x be an arbitrary point in S . Then $x \in \text{int}(S)$. Thus $S = \text{int}(S)$.

Note that $\text{cl}(S) = \text{int}(S) \cup bd(S) = \text{int}(S)$. Thus $S = \text{cl}(S)$.

By definition, we conclude that S is both open and closed.

For the reverse direction, assume that S is both open and closed. We are to prove that $bd(S) = \emptyset$.

Since S is open, $S = \text{int}(S)$.

Since S is closed, $S = \text{cl}(S)$.

Since $S = \text{int}(S)$ and $S = \text{cl}(S)$, $\text{int}(S) = \text{cl}(S)$.

Since $bd(S) = \text{cl}(S) - \text{int}(S)$, $bd(S) = \emptyset$.

□

3.5 As Operators

PROPOSITION 3.21 (The Interior Operator). In any topological space,

1. Non-expansive:

$$\forall S \subseteq X, \quad \text{int}(S) \subseteq S.$$

2. Monotonic:

$$\forall S_1, S_2 \subseteq X, \quad S_1 \subseteq S_2 \implies \text{int}(S_1) \subseteq \text{int}(S_2).$$

3. Idempotent:

$$\forall S \subseteq X, \quad \text{int}(S) = \text{int}(\text{int}(S)).$$

Proof of (2). Let x be an arbitrary point in $\text{int}(S_1)$.

Since $x \in \text{int}(S_1)$, by definition of interior, x is in some open subset S' of S_1 .

Since S' is an open subset of S_1 and $S_1 \subseteq S_2$, S' is an open subset of S_2 .

Since S' is an open subset of S_2 and $x \in S'$, by definition of interior, $x \in \text{int}(S_2)$.

Since for any point in $\text{int}(S_1)$ is also in $\text{int}(S_2)$, $\text{int}(S_1) \subseteq \text{int}(S_2)$.

□

PROPOSITION 3.22 (The Closure Operator). In any topological space,

1.

$$\forall S \subseteq X, \quad S \subseteq \text{cl}(S).$$

2. Monotonic:

$$\forall S_1, S_2 \subseteq X, \quad S_1 \subseteq S_2 \implies \text{cl}(S_1) \subseteq \text{cl}(S_2).$$

3. Idempotent:

$$\forall S \subseteq X, \quad \text{cl}(S) = \text{cl}(\text{cl}(S)).$$

Proof of (2). Let x be an arbitrary point in $\text{cl}(S_1)$.

Let S' be an arbitrary closed superset of S_2 .

Since S' is a closed superset of S_2 and $S_1 \subseteq S_2$, S' is a closed superset of S_1 .

Since S' is a closed superset of S_1 and $x \in \text{cl}(S_1)$, by definition of closure, $x \in S'$.

Since x is in any closed superset of S_2 , by definition, $x \in \text{cl}(S_2)$.

Since any point in $\text{cl}(S_1)$ is in $\text{cl}(S_2)$, $\text{cl}(S_1) \subseteq \text{cl}(S_2)$.

□

PROPOSITION 3.23 (The Exterior Operator). In any topological space,

1. (Monotonic) For any sets S and T , if $S \subseteq T$, then $\text{ext}(S) \supseteq T$.

REMARK 3.24. The exterior operator is not idempotent.

PROPOSITION 3.25. Let (X, τ) be a topological space and S be a subset of X . Then

1. $\text{int}(S) \subseteq \text{int}(\text{cl}(S))$.

2. $\text{cl}(S) \supseteq \text{cl}(\text{int}(S))$.

Proof. By the properties of the closure operator, $S \subseteq \text{cl}(S)$. Since $S \subseteq \text{cl}(S)$ and the interior operator is monotonic increasing, $\text{int}(S) \subseteq \text{int}(\text{cl}(S))$. By the properties of the interior operator, $\text{int}(S) \subseteq S$. Since $\text{int}(S) \subseteq S$ and the closure operator is monotonic increasing, $\text{cl}(\text{int}(S)) \subseteq \text{cl}(S)$. \square

REMARK 3.26. The point of this proposition is to remind the readers that the equalities might not hold. See section 3.6 for counter examples.

PROPOSITION 3.27. Let S be a set in some topological space. Then

1. $\text{bd}(\text{int}(S)) \subseteq \text{bd}(S)$.
2. $\text{bd}(\text{cl}(S)) \subseteq \text{bd}(S)$.
3. $\text{bd}(\text{bd}(S)) \subseteq \text{bd}(S)$

Proof of (1). Let x be an arbitrary element in $\text{bd}(\text{int}(S))$.

We are to prove that $x \in \text{bd}(S)$.

Since $x \in \text{bd}(\text{int}(S))$, by definition, for any neighborhood $N(x)$ of x , there exist a point x_1 in $N(x) \cap \text{int}(S)$ and a point x_2 in $N(x) \cap (\text{int}(S))^c$.

Since $x_1 \in \text{int}(S)$ and $\text{int}(S) \subseteq S$, $x_1 \in S$.

Since $x_1 \in S$ and $x_1 \in N(x)$, $x_1 \in N(x) \cap S$.

To prove that $N(x) \cap S^c \neq \emptyset$, assume for the sake of contradiction that $N(x) \subseteq S$.

Since $x_2 \in N(x)$ and $N(x) \subseteq S$, by definition, $x_2 \in \text{int}(S)$.

This contradicts to the fact that $x_2 \in (\text{int}(S))^c$.

Thus the assumption that $N(x) \subseteq S$ is false.

i.e., there exists a point x'_2 in $N(x) \cap S^c$.

In short, I have proved that for any neighborhood $N(x)$ of x , there exist a point x_1 in $N(x) \cap S$ and a point x'_2 in $N(x) \cap S^c$.

By definition, I conclude that $x \in \text{bd}(S)$. \square

Proof of (2). Let x be an arbitrary element in $\text{bd}(\text{cl}(S))$.

We are to prove that $x \in \text{bd}(S)$.

Since $x \in \text{bd}(\text{cl}(S))$, by definition, for any neighborhood $N(x)$ of x , there exist a point x_1 in $N(x) \cap \text{cl}(S)$ and a point x_2 in $N(x) \cap (\text{cl}(S))^c$.

To prove that $N(x) \cap S \neq \emptyset$, assume for the sake of contradiction that $N(x) \subseteq S^c$.

Since $x_1 \in N(x)$ and $N(x) \subseteq S^c$, $x_1 \in \text{ext}(S)$.

This contradicts to the fact that $x_1 \in \text{cl}(S)$.

Thus the assumption that $N(x) \subseteq S^c$ is false.

i.e., there exists a point x'_1 in $N(x) \cap S$.

Since $x_2 \in (\text{cl}(S))^c$ and $(\text{cl}(S))^c \subseteq S^c$, $x_2 \in S^c$.

Since $x_2 \in S^c$ and $x_2 \in N(x)$, $x_2 \in N(x) \cap S^c$.

In short, I have proved that for any neighborhood $N(x)$ of x , there exist a point x_1 in $N(x) \cap S$ and a point $x_2 \in N(x) \cap S^c$.

By definition, I conclude that $x \in \text{bd}(S)$.

□

PROPOSITION 3.28 (Set Operations). Let S_1 and S_2 be sets in some topological space. Then

1. $\text{int}(S_1 \cup S_2) \supseteq \text{int}(S_1) \cup \text{int}(S_2)$.
2. $\text{int}(S_1 \cap S_2) = \text{int}(S_1) \cap \text{int}(S_2)$.
3. $\text{cl}(S_1 \cup S_2) = \text{cl}(S_1) \cup \text{cl}(S_2)$.
4. $\text{cl}(S_1 \cap S_2) \subseteq \text{cl}(S_1) \cap \text{cl}(S_2)$.
5. $\text{bd}(S_1 \cup S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2)$.
6. $\text{bd}(S_1 \cap S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2)$.
7. $\text{bd}(S_1 \setminus S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2)$.

Proof of (2).

It is clear that $\text{int}(S_1 \cap S_2) \subseteq \text{int}(S_1) \cap \text{int}(S_2)$.

So it suffices to prove that $\text{int}(S_1 \cap S_2) \supseteq \text{int}(S_1) \cap \text{int}(S_2)$.

Let x be an arbitrary point in $\text{int}(S_1) \cap \text{int}(S_2)$.

Then $x \in \text{int}(S_1)$ and $x \in \text{int}(S_2)$.

Since $x \in \text{int}(S_1)$, there exists some open set G_1 such that $x \in G_1 \subseteq S_1$.

Since $x \in \text{int}(S_2)$, there exists some open set G_2 such that $x \in G_2 \subseteq S_2$.

Since $x \in G_1$ and $x \in G_2$, $x \in G_1 \cap G_2$.

Since G_1 and G_2 are both open, $G_1 \cap G_2$ is open.

Since $G_1 \subseteq S_1$ and $G_2 \subseteq S_2$, $G_1 \cap G_2 \subseteq S_1 \cap S_2$.

Since $G_1 \cap G_2$ is open and $x \in G_1 \cap G_2 \subseteq S_1 \cap S_2$, by definition of interior points, $x \in \text{int}(S_1 \cap S_2)$.

Since

$$\forall x \in \text{int}(S_1) \cap \text{int}(S_2), \quad x \in \text{int}(S_1 \cap S_2),$$

we get $\text{int}(S_1 \cap S_2) \supseteq \text{int}(S_1) \cap \text{int}(S_2)$.

□

Proof of (3).

$$\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B).$$

Let x be an arbitrary point in $\text{cl}(A \cup B)$.

By definition of closure, x is in every closed superset of $(A \cup B)$.

Note that the union of an arbitrary closed superset of A and an arbitrary closed superset of B is a closed superset of $(A \cup B)$.

Thus x is in the union of the intersection of all closed supersets of A and the intersection of all closed supersets of B .

By definition again, we conclude that $x \in \text{cl}(A) \cup \text{cl}(B)$.

Since x is an arbitrary point in $\text{cl}(A \cup B)$, $\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B)$.

□

Proof of (5).

Let x be an arbitrary point in $\text{bd}(S_1 \cup S_2)$.

Since $x \in \text{bd}(S_1 \cup S_2)$, by definition of boundary, for any neighborhood $\mathcal{N}(x)$ around x , $\mathcal{N}(x) \cap (S_1 \cup S_2) \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_1 \cup S_2) \neq \emptyset$.

Since $\mathcal{N}(x) \cap (S_1 \cup S_2) \neq \emptyset$, either $\mathcal{N}(x) \cap S_1 \neq \emptyset$ or $\mathcal{N}(x) \cap S_2 \neq \emptyset$.

Since $\mathcal{N}(x) \cap C_X(S_1 \cup S_2) \neq \emptyset$, $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$.

Case 1. $\mathcal{N}(x) \cap S_1 \neq \emptyset$.

Since $\mathcal{N}(x) \cap S_1 \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$ for any neighborhood $\mathcal{N}(x)$ around x , by definition of boundary, $x \in \text{bd}(S_1)$.

Since $x \in \text{bd}(S_1)$, $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$.

Case 2. $\mathcal{N}(x) \cap S_2 \neq \emptyset$.

Since $\mathcal{N}(x) \cap S_2 \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$ for any neighborhood $\mathcal{N}(x)$ around x , by definition of boundary, $x \in \text{bd}(S_2)$.

Since $x \in \text{bd}(S_2)$, $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$.

Summary.

Since “ $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$ ” holds in all cases, we conclude that it is true.

Since

$$\forall x \in \text{bd}(S_1 \cup S_2), \quad x \in \text{bd}(S_1) \cup \text{bd}(S_2),$$

we get $\text{bd}(S_1 \cup S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2)$.

□

Proof of (6).

Let x be an arbitrary point in $\text{bd}(S_1 \cap S_2)$.

Since $x \in \text{bd}(S_1 \cap S_2)$, by definition of boundary, for any neighborhood $\mathcal{N}(x)$ around x , $\mathcal{N}(x) \cap (S_1 \cap S_2) \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_1 \cap S_2) \neq \emptyset$.

Since $\mathcal{N}(x) \cap (S_1 \cap S_2) \neq \emptyset$, $\mathcal{N}(x) \cap S_1 \neq \emptyset$ and $\mathcal{N}(x) \cap S_2 \neq \emptyset$.

Since $\mathcal{N}(x) \cap C_X(S_1 \cap S_2) \neq \emptyset$, either $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$ or $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$.

Case 1. $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$.

Since $\mathcal{N}(x) \cap S_1 \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$ for any neighborhood $\mathcal{N}(x)$ around x , by definition of boundary, $x \in \text{bd}(S_1)$.

Since $x \in \text{bd}(S_1)$, $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$.

Case 2. $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$.

Since $\mathcal{N}(x) \cap S_2 \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$ for any neighborhood $\mathcal{N}(x)$ around x , by definition of boundary, $x \in \text{bd}(S_2)$.

Since $x \in \text{bd}(S_2)$, $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$.

Summary.

Since “ $x \in \text{bd}(S_1) \cup \text{bd}(S_2)$ ” holds in all cases, we conclude that it is true.

Since

$$\forall x \in \text{bd}(S_1 \cap S_2), \quad x \in \text{bd}(S_1) \cup \text{bd}(S_2),$$

we get $\text{bd}(S_1 \cap S_2) \subseteq \text{bd}(S_1) \cup \text{bd}(S_2)$.

□

REMARK 3.29 (Remark on 1). “ $\text{int}(S_1 \cup S_2) = \text{int}(S_1) \cup \text{int}(S_2)$ ” may not be true. For example, consider $X = \mathbb{R}$, $S_1 = \mathbb{Q}$, and $S_2 = X \setminus \mathbb{Q}$. Then $LHS = \text{int}(S_1 \cup S_2) = \mathbb{R}$ and $RHS = \text{int}(S_1) \cup \text{int}(S_2) = \emptyset$.

REMARK 3.30 (Remark on 4). “ $\text{cl}(S_1 \cap S_2) \subseteq \text{cl}(S_1) \cap \text{cl}(S_2)$ ” may not be true. For example, consider $X = \mathbb{R}$, $S_1 = \mathbb{Q}$, and $S_2 = X \setminus \mathbb{Q}$. Then $LHS = \text{cl}(S_1 \cap S_2) = \emptyset$ and $RHS = \text{cl}(S_1) \cap \text{cl}(S_2) = \mathbb{R}$.

3.6 Examples

EXAMPLE 3.31. It is **not** true that if S is open, then $\text{int}(\text{cl}(S)) = \text{int}(S)$. Consider the set $\mathbb{R} \setminus \{0\}$.

EXAMPLE 3.32. It is **not** true that if S is closed, then $\text{cl}(\text{int}(S)) = \text{cl}(S)$. Consider the set $\{0\}$.

Chapter 4

Dense and Nowhere Dense

4.1 Dense

4.1.1 Definitions

DEFINITION 4.1 (Dense-1). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We say that S is **dense** in the space if any point in the space is an adherent point of S , or equivalently, the closure of S equals the whole space.

DEFINITION 4.2 (Dense-2). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We say that S is **dense** in the space if S has nonempty intersection with any nonempty open set in the space.

PROPOSITION 4.3. The two definitions of dense sets are equivalent.

Proof.

For one direction, assume that $cl(S) = X$.

We are to prove that \forall nonempty open \mathcal{O} , $S \cap \mathcal{O} \neq \emptyset$.

Let \mathcal{O} be an arbitrary nonempty open set in the space.

Assume for the sake of contradiction that $\mathcal{O} \cap S = \emptyset$.

Define $\mathcal{C} := X \setminus \mathcal{O}$.

Since $\mathcal{O} \cap S = \emptyset$, $\mathcal{C} \supseteq S$.

Since \mathcal{O} is open, \mathcal{C} is closed.

Since \mathcal{C} is a closed superset of S , $cl(S) \subseteq \mathcal{C}$.

Since $cl(S) \subseteq \mathcal{C}$ and $cl(S) = X$, $\mathcal{C} \supseteq X$.

Since $\mathcal{C} = X \setminus \mathcal{O}$, $\mathcal{C} \subseteq X$.

Since $\mathcal{C} \subseteq X$ and $\mathcal{C} \supseteq X$, $\mathcal{C} = X$.

Since $\mathcal{C} = X$, $\mathcal{O} = \emptyset$.

This contradicts to the assumption that \mathcal{O} is nonempty.

So the assumption that $\mathcal{O} \cap S = \emptyset$ is false.

i.e., $\mathcal{O} \cap S \neq \emptyset$.

For the reverse direction, assume that \forall nonempty open \mathcal{O} , $S \cap \mathcal{O} \neq \emptyset$.

We are to prove that $cl(S) = X$.

Let \mathcal{C} be an arbitrary closed superset of S .

Define $\mathcal{O} := X \setminus \mathcal{C}$.

Since \mathcal{C} is closed, \mathcal{O} is open.

Since $\mathcal{C} \supseteq S$, $\mathcal{O} \cap S = \emptyset$.

Assume for the sake of contradiction that $\mathcal{C} \neq X$.

Since $\mathcal{C} \neq X$, $\mathcal{O} \neq \emptyset$.

Since \forall nonempty open \mathcal{O} , $S \cap \mathcal{O} \neq \emptyset$, in particular, $\mathcal{O} \cap S \neq \emptyset$.

This contradicts to the fact that $\mathcal{O} \cap S = \emptyset$.

So the assumption that $\mathcal{C} \neq X$ is false.

i.e., $\mathcal{C} = X$.

Since $\mathcal{C} = X$ for any closed superset \mathcal{C} of S , $cl(S) = X$.

□

4.1.2 Properties

PROPOSITION 4.4 (Transitivity). Let (\mathfrak{X}, τ) be a topological space. Let S_1 and S_2 and S_3 be subsets of X . Suppose S_1 is dense in S_2 and S_2 is dense in S_3 , then S_1 is dense in S_3 .

PROPOSITION 4.5. A superset of a dense set is dense.

PROPOSITION 4.6 (Images). A continuous image of a dense set is dense in the range.

Proof. Let $(\mathfrak{X}, \tau_{\mathfrak{X}})$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}})$ be topological spaces. Let f be a continuous function from \mathfrak{X} to \mathfrak{Y} . Assume without loss of generality that f is surjective so that $f(\mathfrak{X}) = \mathfrak{Y}$. Let S be a dense set in $(\mathfrak{X}, \tau_{\mathfrak{X}})$. I will show that $f(S)$ is dense in $(\mathfrak{Y}, \tau_{\mathfrak{Y}})$. It is clear that $\text{cl}(f(S)) \subseteq \mathfrak{Y}$. Let y be an arbitrary point in \mathfrak{Y} . Since $y \in \mathfrak{Y}$, $\exists x \in \mathfrak{X}$ such that $f(x) = y$. Since $x \in \mathfrak{X}$ and S is dense in $(\mathfrak{X}, \tau_{\mathfrak{X}})$, there exists a net $(x_{\lambda})_{\lambda \in \Lambda} \subseteq S$ that converges to x in $(\mathfrak{X}, \tau_{\mathfrak{X}})$. Since f is continuous from \mathfrak{X} to \mathfrak{Y} and $\lim_{\lambda \in \Lambda} x_{\lambda} = x$, we get $\lim_{\lambda \in \Lambda} f(x_{\lambda}) = f(x) = y$. Notice $(f(x_{\lambda}))_{\lambda \in \Lambda} \subseteq f(S)$. So $y \in \text{cl}(f(S))$. So $\mathfrak{Y} \subseteq \text{cl}(f(S))$. So $\text{cl}(f(S)) = \mathfrak{Y}$. So $f(S)$ is dense in $(\mathfrak{Y}, \tau_{\mathfrak{Y}})$. \square

4.2 Nowhere Dense

4.2.1 Definitions

DEFINITION 4.7 (Nowhere Dense). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We say that S is **nowhere dense** in (\mathfrak{X}, τ) if and only if there is no nonempty open set in (\mathfrak{X}, τ) in which S is dense.

DEFINITION 4.8 (Nowhere Dense). Let (\mathfrak{X}, τ) be a topological space. Let S be a subset of \mathfrak{X} . We say that S is **nowhere dense** in (\mathfrak{X}, τ) if and only if $\text{int}(\text{cl}(S)) = \emptyset$.

PROPOSITION 4.9. The two definitions of nowhere dense sets are equivalent.

4.2.2 Properties

PROPOSITION 4.10 (Subspaces). Let (\mathfrak{X}, τ) be a topological space. Let (\mathfrak{Y}, τ) be a subspace of (\mathfrak{X}, τ) . Then

1. Let S be a nowhere dense set in (\mathfrak{Y}, τ) . Then S is also nowhere dense in (\mathfrak{X}, τ) .
2. Let S be a nowhere dense set in (\mathfrak{X}, τ) . Then if Y is open in (\mathfrak{X}, τ) , S is also nowhere dense in (\mathfrak{Y}, τ) .

PROPOSITION 4.11. The nowhere dense sets in a space form an ideal of sets.

PROPOSITION 4.12 (Closure). A set is nowhere dense if and only if its closure is nowhere dense.

PROPOSITION 4.13 (Exterior). A set is nowhere dense if and only if its exterior is dense.

Proof. Sketch

Let (\mathfrak{X}, τ) be a topological space.

Let S be a subset of \mathfrak{X} .

S is nowhere dense if and only if $\text{int}(cl(S)) = \emptyset$.

$\text{ext}(S)$ is dense if and only if $cl(\text{ext}(S)) = X$.

So it suffices to prove that $\text{int}(cl(S)) = \emptyset$ if and only if $cl(\text{ext}(S)) = X$.

It suffices to prove that $C_X(\text{int}(cl(S))) = cl(\text{int}(C_X(S)))$.

$$\begin{aligned} \text{int}(C_X(S)) &= C_X(cl(S)) \\ &\Downarrow \\ cl(\text{int}(C_X(S))) &= cl(C_X(cl(S))) \\ &\Downarrow \\ cl(\text{int}(C_X(S))) &= C_X(\text{int}(cl(S))) \end{aligned}$$

This completes the proof. □

PROPOSITION 4.14. A set is open and dense if and only if its complement is closed and nowhere dense.

PROPOSITION 4.15. The boundary of an open set or of a closed set is nowhere dense.

PROPOSITION 4.16 (not sure...). A singleton set is nowhere dense if and only if the point in it is not an isolated point.

Chapter 5

Meager and Residual

5.1 Definitions

DEFINITION 5.1 (Meager, or First Category). Let (X, τ) be a topological space. Let S be a set in the space. We say S is **meager** if S is the union of some countable collection of nowhere dense sets in the space.

DEFINITION 5.2 (Residual). Let (X, τ) be a topological space. Let S be a subset of X . We say S is **residual** if S is the intersection of some countable collection of sets with dense interior.

5.2 Sufficient Conditions

PROPOSITION 5.3. A set is meager if and only if its complement is residual.

PROPOSITION 5.4 (Set Operations).

1. A subset of a meager set is meager.
2. A countable union of meager sets is meager.

Proof.

Proof of (1).

Let (X, τ) be a topological space and S_1 be a meager set in (X, τ) .

Let S_2 be a subset of S_1 .

Since S_1 is meager in (X, τ) , by definition of meager, there exist a countable collection $\{A_i\}_{i \in \mathbb{N}}$ of nowhere dense sets in (X, τ) such that $S_1 = \bigcup_{i \in \mathbb{N}} A_i$.

Define $B_i = A_i \cap S_2$.

Since $S_2 \subseteq S_1$ and $S_1 = \bigcup_{i \in \mathbb{N}} A_i$ and $B_i = A_i \cap S_2$, $S_2 = \bigcup_{i \in \mathbb{N}} B_i$.

Since $B_i \subseteq A_i$ and A_i is nowhere dense in (X, τ) , B_i is nowhere dense in (X, τ) .

Since B_i is nowhere dense in (X, τ) for any $i \in \mathbb{N}$ and $S_2 = \bigcup_{i \in \mathbb{N}} B_i$, by definition of meager,

S_2 is meager in (X, τ) .

□

PROPOSITION 5.5 (Set Operations). A countable intersection of residual sets is residual.

Chapter 6

Sequences

6.1 Definitions

DEFINITION 6.1 (Sequence). Let (X, τ) be a topological space. We define a **sequence** in X , denoted by $\{x_i\}_{i \in I}$ where I is a subset of the natural numbers, to be a function from I to X .

6.2 Convergence of Sequences

DEFINITION 6.2 (Convergence). Let (X, τ) be a topological space. Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence in X . Let x_0 be a point in X . We say that $\{x_i\}_{i \in \mathbb{N}}$ **converges** to x_0 if for any neighborhood $\mathcal{N}_X(x_0)$ of x_0 in X , there exists an integer $N(\mathcal{N}_X(x_0))$ such that for any index i greater than $N(\mathcal{N}_X(x_0))$, we have $x_i \in \mathcal{N}_X(x_0)$.

PROPOSITION 6.3. Let (X, τ) be a topological space. Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence in X . Let x_0 be a point in X . Then if $\{x_i\}_{i \in \mathbb{N}}$ converges to x_0 , any subsequence of $\{x_i\}_{i \in \mathbb{N}}$ also converges to x_0 .

Proof. Let $\{x_i\}_{i \in I}$ be an arbitrary subsequence of $\{x_i\}_{i \in \mathbb{N}}$. Let $\mathcal{N}(x_0)$ be an arbitrary neighborhood of x_0 . Since $\{x_i\}_{i \in \mathbb{N}}$ converges to x_0 , there exists some cutoff N such that $x_i \in \mathcal{N}(x_0)$ whenever $i > N$. Since $x_i \in \mathcal{N}(x_0)$ whenever $i > N$, in particular, $x_i \in \mathcal{N}(x_0)$

whenever $i \in I$ and $i > N$. Since for any positive number ε , there exists some neighborhood $\mathcal{N}(x_0)$ such that $x_i \in \mathcal{N}(x_0)$ whenever $i \in I$ and $i > N$, $\{x_i\}_{i \in I}$ converges to x_0 . \square

PROPOSITION 6.4. Let (X, τ) be a topological space. Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence in X . Let x_0 be a point in X . Then if any subsequence of $\{x_i\}_{i \in \mathbb{N}}$ has a subsequence that converges to x_0 , $\{x_i\}_{i \in \mathbb{N}}$ also converges to x_0 .

6.3 Cauchyness of Sequences

DEFINITION 6.5 (Cauchy Sequence). Let (X, d) be a metric space. Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence in X . We say that $\{x_i\}_{i \in \mathbb{N}}$ is **Cauchy** if for any positive number ε , there exists an integer $N(\varepsilon)$ such that for any indices m and n with $m, n > N(\varepsilon)$, we have $d(x_m, x_n) < \varepsilon$.

PROPOSITION 6.6. Cauchy sequences are bounded.

Proof.

Let (X, d) be a metric space.

Let $\{x_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence in X .

Let x_0 be some fixed point in the space.

Let ε be some fixed positive number.

Since $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy, there exists some cutoff N such that $d(x_m, x_n) < \varepsilon$ whenever $m, n > N$.

Let j be some fixed index such that $j > N$.

Since $d(x_m, x_n) < \varepsilon$ for any $m, n > N$, in particular, $d(x_i, x_j) < \varepsilon$ for any $i > N$.

Define $M_j := d(x_j, x_0) + \varepsilon$.

Define $M_i := d(x_i, x_0)$ for $i \in \{1, \dots, N\}$.

Define $M := \max\{M_i\}, i \in \{1, \dots, N, j\}$.

Since $M = \max\{M_i\}, i \in \{1, \dots, N, j\}$, $M_i \leq M$ for $i \in \{1, \dots, N, j\}$.

Let i be an arbitrary index.

If $i \in \{1, \dots, N\}$, then $d(x_i, x_0) = M_i \leq M$.

If $i > N$, then $d(x_i, x_0) \leq d(x_i, x_j) + d(x_j, x_0) < \varepsilon + d(x_j, x_0) = M_j \leq M$.

Since $d(x_i, x_0) \leq M$ for any $i \in \mathbb{N}$, $\{x_i\}_{i \in \mathbb{N}}$ is bounded. \square

PROPOSITION 6.7. Let (X, d) be a metric space. Let $\{x_k\}$ be a Cauchy sequence in X . Then if there exists a convergent subsequence, $\{x_k\}$ also converges to the same limit.

Proof.

Let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_k\}$ that converges to point x in X .

By definition of convergence, for any positive number ε , there exists an integer $N_1(\varepsilon)$ such that for any index k with $k > N_1$, we have $d(x_{n_k}, x) < \varepsilon/2$. (**)

By definition of Cauchy, there exists an integer $N_2(\varepsilon)$ such that for any indices $m, n > N_2$, we have $d(x_m, x_n) < \varepsilon/2$. (*)

Take $N = \max\{N_1, N_2\} + 1$. Then we have $N > N_1$, $N > N_2$ and $n_N > N_2$.

Apply statement (*) with $k = N > N_1$, we get $d(x_{n_N}, x) < \varepsilon/2$.

Apply statement (**) with $m > N > N_2$ and $n = n_N > N_2$, we get $d(x_m, x_{n_N}) < \varepsilon/2$.

Combining the preceding two inequalities, we get $d(x_k, x) < \varepsilon$ for any $k > N$.

In short, for any positive number ε , there exists an integer $N(\varepsilon)$ such that for any index k with $k > N$, we have $d(x_k, x) < \varepsilon$.

By definition, we conclude that $\{x_k\}$ converges to x .

□

PROPOSITION 6.8. Convergent sequences are Cauchy.

Proof. Let (X, d) be a metric space and $\{x_k\}$ be an arbitrary convergent sequence in X .

By definition of convergence, for all $\varepsilon > 0$, there exists an integer N such that for all $k > N$, we have $d(x_k, x) < \varepsilon/2$. (*)

Let $m, n > N$ be arbitrary.

Apply statement (*) to m and n , we get

$$d(x_m, x) < \varepsilon/2 \#(1)$$

$$d(x_n, x) < \varepsilon/2 \#(2)$$

By the triangle inequality, we get

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) \#(3)$$

From inequations (1) ~ (3), we get

$$d(x_m, x_n) < \varepsilon$$

In short, we have proved that for all $\varepsilon > 0$, there exists an integer N such that for all $m, n > N$, we have $d(x_m, x_n) < \varepsilon$.

By definition, $\{x_k\}$ is Cauchy.

□

Chapter 7

Nets

7.1 Definitions

DEFINITION 7.1 (Directed Set). Let Λ be a set. Let \leq be a binary relation on Λ . We say that Λ is a **directed set** if and only if

1. Reflexive: $\forall \lambda \in \Lambda, \lambda \leq \lambda$.
2. Transitive: $\forall \lambda_1, \lambda_2, \lambda_3 \in \Lambda$, if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$.
3. Bounded: $\forall \lambda_1, \lambda_2 \in \Lambda, \exists \lambda_3 \in \Lambda$ such that both $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

DEFINITION 7.2 (Net). Let X be a set. Let Λ be a directed set. We define a **net**, denoted by $(x_\lambda)_{\lambda \in \Lambda}$, to be a function from Λ to X .

DEFINITION 7.3 (Subnet). Let X be a set. Let Λ be a directed set. Let $P : \Lambda \rightarrow X$ be a net. Let M be another directed set. Let φ be an increasing and cofinal function from M to Λ . We define a **subnet** of P , denoted by $(x_{\lambda_\mu})_{\mu \in M}$, to be a composition of the functions φ and P .

7.2 Convergence of Nets

DEFINITION 7.4 (Convergence). Let (X, τ) be a topological space. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X . Let x be a point in X . We say that the net $(x_\lambda)_{\lambda \in \Lambda}$ **converges** to the point x , denoted by $\lim_{\lambda \in \Lambda} x_\lambda = x$, if

$$\forall U \in \mathcal{U}_x, \quad \exists \lambda_0 \in \Lambda, \quad \forall \lambda \geq \lambda_0, \quad x_\lambda \in U.$$

PROPOSITION 7.5. Let (X, τ) be a topological space. Then the space is Hausdorff if and only if the limits of nets in the space are unique.

Proof. Forward Direction: Assume that the space is Hausdorff. I will show that limits of nets in the space are unique. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net where Λ is a directed set. Let x_1 and x_2 be points in the space. Suppose $\lim_{\lambda \in \Lambda} x_\lambda = x_1$ and $\lim_{\lambda \in \Lambda} x_\lambda = x_2$. Let \mathcal{N}_1 be an arbitrary neighborhood of x_1 . Let \mathcal{N}_2 be an arbitrary neighborhood of x_2 . Then $\exists \lambda_1 \in \Lambda$ such that $\forall \lambda \geq \lambda_1, x_\lambda \in \mathcal{N}_1$; and $\exists \lambda_2 \in \Lambda$ such that $\forall \lambda \geq \lambda_2, x_\lambda \in \mathcal{N}_2$. Let λ_3 be an index such that $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$. Then $x_{\lambda_3} \in \mathcal{N}_1$ and $x_{\lambda_3} \in \mathcal{N}_2$. So \mathcal{N}_1 and \mathcal{N}_2 are not disjoint. So x_1 and x_2 are not separated. Since the space is Hausdorff and x_1 and x_2 are not separated, x_1 and x_2 are not distinct. So the limit of $(x_\lambda)_{\lambda \in \Lambda}$ is unique.

Backward Direction: Assume that the limits of nets in the space are unique. I will show that the space is Hausdorff. Assume for the sake of contradiction that the space is not Hausdorff. Then $\exists x, y \in X : x \neq y$ such that $\forall \mathcal{N}_x \in \mathcal{U}_x, \forall \mathcal{N}_y \in \mathcal{U}_y$, we have $\mathcal{N}_x \cap \mathcal{N}_y \neq \emptyset$. Define a directed set Λ by $\Lambda := \{(\mathcal{N}_x, \mathcal{N}_y)\}_{\mathcal{N}_x \in \mathcal{U}_x, \mathcal{N}_y \in \mathcal{U}_y}$. Define a partial order on Λ by $(\mathcal{N}_x, \mathcal{N}_y) \leq (\mathcal{M}_x, \mathcal{M}_y)$ if and only if both $\mathcal{M}_x \subseteq \mathcal{N}_x$ and $\mathcal{M}_y \subseteq \mathcal{N}_y$. Define a net $(x_\lambda)_{\lambda \in \Lambda}$ by $x_\lambda = x_{(\mathcal{N}_x, \mathcal{N}_y)} \in \mathcal{N}_x \cap \mathcal{N}_y$. Then

$$\forall \mathcal{N}_x \in \mathcal{U}_x, \quad \forall (\mathcal{M}_x, \mathcal{M}_y) \geq (\mathcal{N}_x, X), \quad x_{(\mathcal{M}_x, \mathcal{M}_y)} \in \mathcal{M}_x \cap \mathcal{M}_y \subseteq \mathcal{M}_x \subseteq \mathcal{N}_x.$$

So $\lim_{\lambda \in \Lambda} x_\lambda = x$. Similarly, we can get $\lim_{\lambda \in \Lambda} x_\lambda = y$. Recall that $x \neq y$. So limits of nets in the space are not unique. This leads to a contradiction. So the space must be Hausdorff. This completes the proof. □

7.3 Cauchyness of Nets

PROPOSITION 7.6. Convergent nets are Cauchy.

Proof. Let (X, τ) be a topological space. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a convergent net in (X, τ) where Λ is some directed index set. Define a point x in X as $x := \lim_{\lambda \in \Lambda} x_\lambda$. Let U be an arbitrary neighborhood of x . Since $(x_\lambda)_{\lambda \in \Lambda}$ converges to x , $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda \in \Lambda, \lambda \geq \lambda_0 \implies x_\lambda \in U$. \square

7.4 Examples of Nets

EXAMPLE 7.7. Let (X, τ) be a topological space. Let x be a point in the space. Let \mathcal{U}_x be the neighborhood system at point x . Define a relation \leq on \mathcal{U}_x by $U < V$ if $V \subseteq U$. Then (\mathcal{U}_x, \leq) forms a directed set. Define a mapping x from \mathcal{U}_x to X by x_U is a point in U . Then $(x_U)_{U \in \mathcal{U}_x}$ is a net and

$$\lim_{U \in \mathcal{U}_x} x_U = x.$$

EXAMPLE 7.8. Let \mathcal{P} denote the set of all finite partitions of $[0, 1]$, partially ordered by inclusion. Let f be a continuous function on $[0, 1]$. Define a mapping x from \mathcal{P} to \mathbb{R} by

$$x_P := L(P, f) = \sum_{i=1}^n f(t_{i-1})(t_i - t_{i-1}).$$

Then $(x_P)_{P \in \mathcal{P}}$ is a net and

$$\lim_{P \in \mathcal{P}} x(P) = \int_0^1 f(x) dx.$$

EXAMPLE 7.9. Let $V = \{f : [0, 1] \rightarrow \mathbb{R}\}$. Define for each $x \in [0, 1]$ a function $p_x : V \rightarrow \mathbb{R}$ given by

$$p_x(g) := g(x).$$

Then $\sigma(V, \{p_x\}_{x \in [0, 1]})$ is a topology on V . Let g denote the constant 1 function. i.e., $\forall x \in [0, 1], g(x) := 1$. Let $\mathfrak{F} := \{F \subseteq [0, 1] : F \text{ is finite}\}$, partially ordered by inclusion.

Define for each $F \in \mathfrak{F}$ a function g_F given by

$$g_F(x) := \begin{cases} 0, & \text{if } x \notin F \\ 1, & \text{if } x \in F. \end{cases}$$

Then $\lim_{F \in \mathfrak{F}} g_F = g$.

Proof.

□

Chapter 8

Continuous Functions

8.1 Continuity in General (bug)

DEFINITION 8.1 (Continuity - Topology). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y . We say that f is **continuous** if and only if

$$\forall G \in \tau_Y, \quad f^{-1}(G) \in \tau_X.$$

DEFINITION 8.2 (Continuity - Nets). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y . We say that f is **continuous** if and only if f preserves convergence of nets.

PROPOSITION 8.3. The [Topology] definition and the [Nets] definition are equivalent.

Proof. For one direction, assume that $\forall G \in \tau_Y, f^{-1}(G) \in \tau_X$. We are to prove that f preserves convergence of nets. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X where Λ is a directed index set. Suppose $(x_\lambda)_{\lambda \in \Lambda}$ converges. Define a point x_0 in X by $x_0 := \lim_{\lambda \in \Lambda} x_\lambda$. Define a net $(y_\lambda)_{\lambda \in \Lambda}$ in Y by $y_\lambda := f(x_\lambda)$. Define a point y_0 in Y by $y_0 := f(x_0)$. Let V be an arbitrary element in $\mathcal{U}_{y_0}^{(Y, \tau_Y)}$. Then $\exists G \in \tau_Y$ such that $y_0 \in G \subseteq V$. By assumption, $f^{-1}(G) \in \tau_X$. Since $f(x_0) = y_0$ and $y_0 \in G$, $x_0 \in f^{-1}(G)$. So $f^{-1}(G) \in \mathcal{U}_{x_0}^{(X, \tau_X)}$. Since $\lim_{\lambda \in \Lambda} x_\lambda = x_0$ and $f^{-1}(G) \in \mathcal{U}_{x_0}^{(X, \tau_X)}$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0, x_\lambda \in f^{-1}(G)$. So $\forall \lambda \geq \lambda_0$,

$y_\lambda = f(x_\lambda) \in G \subseteq V$. That is, $y_\lambda \in V$. So $\forall V \in \mathcal{U}_{y_0}^{(Y, \tau_Y)}$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0$, $y_\lambda \in V$. So $\lim_{\lambda \in \Lambda} y_\lambda = y_0$. So f preserves convergence of nets.

For the reverse direction, assume that f preserves convergence of nets. We are to prove that $\forall G \in \tau_Y$, $f^{-1}(G) \in \tau_X$. Let G be an arbitrary element in τ_Y . If $G = \emptyset$, then $f^{-1}(G) = \emptyset$ and we are done. Otherwise, let x be an arbitrary point in $f^{-1}(G)$. Let \mathcal{U}_x denote the neighborhood system at point x , consisting of only elements of τ_X , partially ordered by set inclusion. Let $(x_U)_{U \in \mathcal{U}_x}$ be an arbitrary net such that $\forall U \in \mathcal{U}_x$, $x_U \in U$. Then $\lim_{U \in \mathcal{U}_x} x_U = x$. Define a point y in Y by $y := f(x)$. Define a net $(y_U)_{U \in \mathcal{U}_x}$ in Y by $y_U := f(x_U)$. Then by assumption, $\lim_{U \in \mathcal{U}_x} y_U = y$. Since $\lim_{U \in \mathcal{U}_x} y_U = y$ and $G \in \mathcal{U}_y$, $\exists U_0 \in \mathcal{U}_x$ such that $\forall U \geq U_0$, $y_U \in G$. So $\forall U \geq U_0$, $x_U \in f^{-1}(G)$. So $U_0 \subseteq f^{-1}(G)$. That is, $\forall x \in f^{-1}(G)$, $\exists U_0 \in \tau_X$ such that $x \in U_0$ and $U_0 \subseteq f^{-1}(G)$. So $f^{-1}(G) \in \tau_X$. \square

DEFINITION 8.4 (Continuity - Base). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let \mathcal{B} be a base for (Y, τ_Y) . Let f be a function from X to Y . We say that f is **continuous** if

$$\forall B \in \mathcal{B}, \quad f^{-1}(B) \in \tau_X.$$

DEFINITION 8.5 (Continuity - Subbase). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let \mathcal{S}_Y be a subbase for (Y, τ_Y) . Let f be a function from X to Y . We say that f is **continuous** if

$$\forall S \in \mathcal{S}_Y, \quad f^{-1}(S) \in \tau_X.$$

PROPOSITION 8.6. The [Base] definition and the [Subbase] definition are equivalent.

Proof. For one direction, assume that $\forall B \in \mathcal{B}_Y$, $f^{-1}(B) \in \tau_X$. We are to prove that $\forall S \in \mathcal{S}_Y$, $f^{-1}(S) \in \tau_X$. Let S be an arbitrary element in \mathcal{S}_Y . Then $S \in \mathcal{S}_Y \subseteq \mathcal{B}_Y$. Since $S \in \mathcal{B}_Y$, by assumption, $f^{-1}(S) \in \tau_X$.

For the reverse direction, assume that $\forall S \in \mathcal{S}_Y$, $f^{-1}(S) \in \tau_X$. We are to prove that $\forall B \in \mathcal{B}_Y$, $f^{-1}(B) \in \tau_X$. Let B be an arbitrary element in \mathcal{B}_Y . Then $\exists S_1 \dots S_n \in \mathcal{S}_Y$ such that $B = \bigcap_{i=1}^n S_i$. Since $B = \bigcap_{i=1}^n S_i$, $f^{-1}(B) = \bigcap_{i=1}^n f^{-1}(S_i)$. Since $S_i \in \mathcal{S}_Y$, by assumption, $f^{-1}(S_i) \in \tau_X$. So $f^{-1}(B) = \bigcap_{i=1}^n f^{-1}(S_i) \in \tau_X$. This completes the proof. \square

PROPOSITION 8.7. The [Topology] definition and the [Base] definition are equivalent.

Proof. For one direction, assume that $\forall G \in \tau_Y, f^{-1}(G) \in \tau_X$. We are to prove that $\forall B \in \mathcal{B}_Y, f^{-1}(B) \in \tau_X$. Let B be an arbitrary element in \mathcal{B}_Y . Then $B \in \mathcal{B}_Y \subseteq \tau_Y$. Since $B \in \tau_Y$, by assumption, $f^{-1}(B) \in \tau_X$.

For the reverse direction, assume that $\forall B \in \mathcal{B}_Y, f^{-1}(B) \in \tau_X$. We are to prove that $\forall G \in \tau_Y, f^{-1}(G) \in \tau_X$. Let G be an arbitrary element in τ_Y . Then $\exists \mathcal{B}'_Y \subseteq \mathcal{B}_Y$ such that $G = \bigcup_{B \in \mathcal{B}'_Y} B$. Since $G = \bigcup_{B \in \mathcal{B}'_Y} B, f^{-1}(G) = \bigcup_{B \in \mathcal{B}'_Y} f^{-1}(B)$. Since $B \in \mathcal{B}'_Y \subseteq \mathcal{B}_Y$, by assumption, $f^{-1}(B) \in \tau_X$. So $f^{-1}(G) = \bigcup_{B \in \mathcal{B}'_Y} B \in \tau_X$. This completes the proof. \square

DEFINITION 8.8 (Continuity - Neighborhood Systems). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y . Let x be a point in X . Define a point y in Y by $y := f(x)$. We say that f is continuous at point x if

$$\forall V \in \mathcal{U}_y^{(Y, \tau_Y)}, \quad \exists U \in \mathcal{U}_x^{(X, \tau_X)}, \quad f(U) \subseteq V.$$

DEFINITION 8.9 (Continuity - Neighborhood Bases). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y . Let x be a point in X . Define a point y in Y by $y := f(x)$. We say that f is continuous at point x if

$$\forall Z \in \mathcal{B}_y^{(Y, \tau_Y)}, \quad \exists W \in \mathcal{B}_x^{(X, \tau_X)}, \quad f(W) \subseteq Z.$$

PROPOSITION 8.10. The [Neighborhood Systems] definition and the [Neighborhood Bases] definition are equivalent.

Proof. For one direction, assume that $\forall V \in \mathcal{U}_y^{(Y, \tau_Y)}, \exists U \in \mathcal{U}_x^{(X, \tau_X)}$ such that $f(U) \subseteq V$. We are to prove that $\forall Z \in \mathcal{B}_y^{(Y, \tau_Y)}, \exists W \in \mathcal{B}_x^{(X, \tau_X)}$ such that $f(W) \subseteq Z$. Let Z be an arbitrary element in $\mathcal{B}_y^{(Y, \tau_Y)}$. Since $Z \in \mathcal{B}_y^{(Y, \tau_Y)}$ and $\mathcal{B}_y^{(Y, \tau_Y)}$, we get $Z \in \mathcal{U}_y^{(Y, \tau_Y)}$. So by assumption, $\exists U \in \mathcal{U}_x^{(X, \tau_X)}$ such that $f(U) \subseteq Z$.

For the reverse direction, assume that $\forall Z \in \mathcal{B}_y^{(Y, \tau_Y)}, \exists W \in \mathcal{B}_x^{(X, \tau_X)}$ such that $f(W) \subseteq Z$. We are to prove that $\forall V \in \mathcal{U}_y^{(Y, \tau_Y)}, \exists U \in \mathcal{U}_x^{(X, \tau_X)}$ such that $f(U) \subseteq V$. Let V be an arbitrary

element in $\mathcal{U}_y^{(Y, \tau_Y)}$. Since V is a neighborhood of y in (Y, τ_Y) and $\mathcal{B}_y^{(Y, \tau_Y)}$ is a neighborhood base in (Y, τ_Y) , $\exists Z \in \mathcal{B}_y^{(Y, \tau_Y)}$ such that $Z \subseteq V$. By assumption, $\exists W \in \mathcal{B}_x^{(X, \tau_X)}$ such that $f(W) \subseteq Z$. Notice $W \in \mathcal{B}_x^{(X, \tau_X)} \subseteq \mathcal{U}_x^{(X, \tau_X)}$ and $f(W) \subseteq Z \subseteq V$. That is, $W \in \mathcal{U}_x^{(X, \tau_X)}$ and $f(W) \subseteq V$. This completes the proof. \square

PROPOSITION 8.11. The [Topology] definition and the [Neighborhood Systems] definition are equivalent.

Proof. For one direction, assume that $\forall G \in \tau_Y, f^{-1}(G) \in \tau_X$. We are to prove that $\forall x \in X, \forall V \in \mathcal{U}_y^{(Y, \tau_Y)}, \exists U \in \mathcal{U}_x^{(X, \tau_X)}$ such that $f(U) \subseteq V$. Let x be an arbitrary point in the space. Let V be an arbitrary element in $\mathcal{U}_y^{(Y, \tau_Y)}$. Since V is a neighborhood of y in (Y, τ_Y) , $\exists G \in \tau_Y$ such that $y \in G$ and $G \subseteq V$. So by assumption, $f^{-1}(G) \in \tau_X$. Since $y \in G, x \in f^{-1}(G)$. Since $x \in f^{-1}(G)$ and $f^{-1}(G) \in \tau_X, f^{-1}(G) \in \mathcal{U}_x^{(X, \tau_X)}$. Notice $f(f^{-1}(G)) = G \subseteq V$.

For the reverse direction, assume that $\forall x \in X, \forall V \in \mathcal{U}_y^{(Y, \tau_Y)}, \exists U \in \mathcal{U}_x^{(X, \tau_X)}$ such that $f(U) \subseteq V$. We are to prove that $\forall G \in \tau_Y, f^{-1}(G) \in \tau_X$. Let G be an arbitrary set in τ_Y . Let x be an arbitrary point in $f^{-1}(G)$. Since $x \in f^{-1}(G), y \in G$. Since $y \in G$ and $G \in \tau_Y, G \in \mathcal{U}_y^{(Y, \tau_Y)}$. So by assumption, $\exists U \in \mathcal{U}_x^{(X, \tau_X)}$ such that $f(U) \subseteq G$. Since $f(U) \subseteq G, U \subseteq f^{-1}(G)$. Since $U \in \mathcal{U}_x^{(X, \tau_X)}, \exists G_x \in \tau_X$ such that $x \in G_x$ and $G_x \subseteq U$. That is, $\forall x \in f^{-1}(G), \exists G_x$ such that $x \in G_x$ and $G_x \subseteq f^{-1}(G)$. So $f^{-1}(G) = \bigcup_{x \in f^{-1}(G)} G_x$.

Notice $\forall x \in f^{-1}(G), G_x \in \tau_X$. So $f^{-1}(G)$ can be written as a union of elements of τ_X . So $f^{-1}(G) \in \tau_X$. This completes the proof. \square

THEOREM 8.12 (Closed Graph Theorem). Let \mathfrak{X} be a topological space. Let \mathfrak{Y} be a Hausdorff compact topological spaces. Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$. Define the graph $G(f)$ to be the set

$$G(f) := \{(x, f(x)) : x \in \mathfrak{X}\}.$$

Then f is continuous if and only if $G(f) \subseteq \mathfrak{X} \times \mathfrak{Y}$ is closed under the product topology.

8.2 Properties of Continuous Functions

PROPOSITION 8.13 (Restriction). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a continuous function from X to Y . Let S be a subset of X . Then the restriction f_S of f to (S, τ_X) is continuous.

PROPOSITION 8.14 (Composition). Let (X, τ_X) , (Y, τ_Y) , and (Z, τ_Z) be topological spaces. Let f be a continuous map from X to Y . Let g be a continuous map from Y to Z . Then the composition $g \circ f$ is a continuous map from X to Z .

PROPOSITION 8.15. The limit, with respect to the infinity norm, of a convergent sequence of continuous functions is continuous.

8.3 Sequential Continuity (bug)

DEFINITION 8.16 (Sequential Continuity). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y . We say that f is **sequentially continuous** at a point x_0 in X if f preserves convergence of sequences.

PROPOSITION 8.17. Continuity implies sequential continuity.

Proof. Continuous functions preserve convergence of nets and hence preserve convergence of sequences. \square

PROPOSITION 8.18. Sequential continuity can imply continuity if the space is first-countable.

Proof. Let (X, τ) be a topological space. Suppose the space is first-countable. \square

8.4 Homeomorphism

DEFINITION 8.19 (Homeomorphism). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y . We say that f is a **homeomorphism** if f is a bijection and both f and f^{-1} are continuous.

PROPOSITION 8.20 (Homeomorphisms Preserve Interior). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a homeomorphism from X to Y . Let S be a subset of X . Then $f(\text{int}(S)) = \text{int}(f(S))$.

Proof. If S is empty, then both sides evaluate to the empty set and the equality holds. Now assume $S \neq \emptyset$.

Forward Direction: Let y be an arbitrary point in $f(\text{int}(S))$. We are to prove that $y \in \text{int}(f(S))$. Since $y \in f(\text{int}(S))$, there exists some point x in $\text{int}(S)$ such that $f(x) = y$. Since $x \in \text{int}(S)$, by definition of interior, there exists some open set U in (X, τ_X) such that $x \in U \subseteq S$. Let V denote $f(U)$. Since U is open in (X, τ_X) and f is a homeomorphism, V is open in (Y, τ_Y) . Since $x \in U \subseteq S$, $y \in V \subseteq f(S)$. Since there exists some open set V in (Y, τ_Y) such that $y \in V \subseteq f(S)$, by definition of interior, $y \in \text{int}(f(S))$. Since $y \in \text{int}(f(S))$ for any $y \in f(\text{int}(S))$, $f(\text{int}(S)) \subseteq \text{int}(f(S))$.

Forward Direction: Let y be an arbitrary point in $\text{int}(f(S))$. We are to prove that $y \in f(\text{int}(S))$. Let x denote $f^{-1}(y)$. Since $y \in \text{int}(f(S))$, by definition of interior, there exists some open subset V of $f(S)$ such that $y \in V \subseteq f(S)$. Let U denote $f^{-1}(V)$. Since V is open in (Y, τ_Y) and f is a homeomorphism, U is open in (X, τ_X) . Since f is a homeomorphism, $f^{-1}(f(S)) = S$. Since $y \in V \subseteq f(S)$, $x \in U \subseteq S$. Since there exists some open set U such that $x \in U \subseteq S$, by definition of interior, $x \in \text{int}(S)$. Since $x \in \text{int}(S)$, $y \in f(\text{int}(S))$. Since $y \in f(\text{int}(S))$ for any $y \in \text{int}(f(S))$, $\text{int}(f(S)) \subseteq f(\text{int}(S))$. \square

PROPOSITION 8.21 (Homeomorphisms Preserve Closure). Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a homeomorphism from X to Y . Let S be a subset of X . Then $f(\text{cl}(S)) = \text{cl}(f(S))$.

Proof.

If S is empty, then both sides evaluate to the empty set and the equality holds.

Now assume $S \neq \emptyset$.

For one direction, let y be an arbitrary point in $f(\text{cl}(S))$.

We are to prove that $y \in \text{cl}(f(S))$.

Let F be an arbitrary closed superset of $f(S)$.

Let E denote $f^{-1}(F)$.

Since $y \in f(cl(S))$, there exists some point x in $cl(S)$ such that $f(x) = y$.

Since F is closed in (Y, τ_Y) and f is a homeomorphism, E is closed in (X, τ_X) .

Since f is a homeomorphism, $f^{-1}(f(S)) = S$.

Since $f(S) \subseteq F$ and $f^{-1}(f(S)) = S$ and $f^{-1}(F) = E$, we get $S \subseteq E$.

Since $x \in cl(S)$ and E is a closed superset of S , we get $x \in E$.

Since $x \in E$ and $f(x) = y$ and $f(E) = F$, $y \in F$.

Since $y \in F$ for any closed superset of $f(S)$, by definition of closure, $y \in cl(f(S))$.

For the reverse direction, let y be an arbitrary point in $cl(f(S))$.

We are to prove that $y \in f(cl(S))$.

Let x denote $f^{-1}(y)$.

Let E be an arbitrary closed superset of S .

Let F denote $f(E)$.

Since E is closed in (X, τ_X) and f is a homeomorphism, F is closed in (Y, τ_Y) .

Since $S \subseteq E$ and $f(E) = F$, $f(S) \subseteq F$.

Since F is a closed superset of $f(S)$ and $y \in cl(f(S))$, by definition of closure, $y \in F$.

Since $y \in F$ and $f^{-1}(y) = x$ and $f^{-1}(F) = E$, $x \in E$.

Since $x \in E$ for any closed superset E of S , by definition of closure, $x \in cl(S)$.

Since $x \in cl(S)$ and $y = f(x)$, $y \in f(cl(S))$.

□

PROPOSITION 8.22. Homeomorphisms preserve nowhere denseness.

Proof.

Let (X, τ_X) and (Y, τ_Y) be topological spaces.

Let f be a homeomorphism from X to Y .

Let S be a nowhere dense set in (X, τ_X) .

We are to prove that $f(S)$ is nowhere dense in (Y, τ_Y) .

Assume for the sake of contradiction that $f(S)$ is not nowhere dense in (Y, τ_Y) .

Since $f(S)$ is not nowhere dense in (Y, τ_Y) , by definition of nowhere dense, there exists some nonempty open set V in (Y, τ_Y) such that $f(S)$ is dense in V .

Let E denote $f^{-1}(V)$.

Since $f(S)$ is dense in V , $V \subseteq cl(f(S))$.

Since V is open in (Y, τ_Y) and f is a homeomorphism, E is open in (X, τ_X) .

Since $V \subseteq cl(f(S))$ and $f^{-1}(V) = E$, $E \subseteq f^{-1}(cl(f(S)))$.

Since f is a homeomorphism, f^{-1} is also a homeomorphism.

Since f^{-1} is a homeomorphism, $f^{-1}(cl(f(S))) = cl(f^{-1}(f(S)))$.

Since f is a homeomorphism, $f^{-1}(f(S)) = S$.

Since $E \subseteq f^{-1}(cl(f(S)))$ and $f^{-1}(cl(f(S))) = cl(f^{-1}(f(S)))$ and $f^{-1}(f(S)) = S$, $E \subseteq cl(S)$.

Since $E \subseteq cl(S)$, by definition of denseness, S is dense in E .

Since S is dense in E and E is open in (X, τ_X) , by definition of nowhere dense, S is not nowhere dense in (X, τ_X) .

This contradicts to the assumption that S is nowhere dense in (X, τ_X) .

So the assumption that $f(S)$ is not nowhere dense in (Y, τ_Y) is false.

i.e., $f(S)$ is nowhere dense in (Y, τ_Y) .

□

PROPOSITION 8.23. Homeomorphisms preserve meager sets.

Chapter 9

Continuous Functions Between Metric Spaces

9.1 Cauchy Continuity

DEFINITION 9.1 (Cauchy Continuous). Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y . We say f is **Cauchy continuous** if it preserves Cauchyness of sequences.

PROPOSITION 9.2. Cauchy continuous functions are continuous.

PROPOSITION 9.3. Continuous functions are Cauchy continuous if the domain is complete.

9.2 Uniform Continuity

DEFINITION 9.4 (Uniform Continuity). Let (\mathfrak{X}, d) and (\mathfrak{Y}, ρ) be metric spaces. Let f be a function from \mathfrak{X} to \mathfrak{Y} . We say that f is **uniformly continuous** on \mathfrak{X} if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in \mathfrak{X} : d(x, y) < \delta$, we have $\rho(f(x), f(y)) < \varepsilon$.

PROPOSITION 9.5. Uniformly continuous functions are Cauchy continuous.

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a uniformly continuous function from X to Y . Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary Cauchy sequence in (X, d_X) . Let ε be an arbitrary positive number. Since f is uniformly continuous, $\exists \delta(\varepsilon) > 0$ such that $\forall x_1, x_2 \in X$, if $d_X(x_1, x_2) < \delta(\varepsilon)$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$. Since $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy in (X, d_X) , $\exists N(\varepsilon) \in \mathbb{N}$ such that for any indices m and n , if $m, n > N$, then $d_X(x_m, x_n) < \delta(\varepsilon)$. Since $d_X(x_m, x_n) < \delta(\varepsilon)$, $d_Y(f(x_m), f(x_n)) < \varepsilon$. Since for any positive number ε , there exists an integer $N(\varepsilon)$ such that for any indices m and n such that $m, n > N$, $d_Y(f(x_m), f(x_n)) < \varepsilon$, $\{f(x_i)\}_{i \in \mathbb{N}}$ is Cauchy in (Y, d_Y) . \square

PROPOSITION 9.6. Cauchy continuous functions are uniformly continuous if the domain is totally bounded.

9.3 Lipschitz Continuity

DEFINITION 9.7 (Lipschitz). Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y . We say that f is **Lipschitz** on X if there exists a non-negative constant c such that for any points x_1 and x_2 in X , we have

$$d_Y(f(x_1), f(x_2)) \leq c d_X(x_1, x_2).$$

DEFINITION 9.8 (Bi-Lipschitz). Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y . We say that f is **bi-Lipschitz** on X if there exist positive constants c_1 and c_2 such that for any points x_1 and x_2 in X , we have

$$c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq c_2 d_X(x_1, x_2).$$

PROPOSITION 9.9. Lipschitz continuous functions are uniformly continuous.

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces and f be a Lipschitz continuous function from X to Y . Since f is Lipschitz continuous, by definition, there exists a non-negative

constant c such that for any points x_1 and x_2 in X , $d_Y(f(x_1), f(x_2)) \leq cd_X(x_1, x_2)$. Since for any points x_1 and x_2 in X , $d_Y(f(x_1), f(x_2)) \leq cd_X(x_1, x_2)$, for any positive number ε and any points x_1 and x_2 in X , if $d_X(x_1, x_2) < \varepsilon/c$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$. Since for any positive number ε and any points x_1 and x_2 in X , if $d_X(x_1, x_2) < \varepsilon/c$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$, for any positive number ε , there exists a positive number $\delta(\varepsilon)$ such that for any points x_1 and x_2 in X , if $d_X(x_1, x_2) < \delta(\varepsilon)$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$. Since for any positive number ε , there exists a positive number $\delta(\varepsilon)$ such that for any points x_1 and x_2 in X , if $d_X(x_1, x_2) < \delta(\varepsilon)$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$, by definition of uniform continuity, f is uniformly continuous. \square

PROPOSITION 9.10. Let (X, d_X) and (Y, d_Y) be metric spaces and let $\{f_k\}_{k=1}^\infty$ be a sequence of Lipschitz continuous functions from X to Y with the sequence $\{K_k\}_{k=1}^\infty$ of Lipschitz constants bounded by some non-negative number K . Then if $\{f_k\}$ converges to some function f , f is also Lipschitz continuous with Lipschitz constant also bounded by K .

PROPOSITION 9.11. Bi-Lipschitz functions are injective.

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces and f be a bi-Lipschitz function from X to Y with Lipschitz constants c_1 and c_2 . Let x_1 and x_2 be distinct points in X . Since f is bi-Lipschitz, by definition, we have $c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq c_2 d_X(x_1, x_2)$. Assume for the sake of contradiction that $f(x_1) = f(x_2)$. Since $f(x_1) = f(x_2)$ and d_Y is a metric, we get $d_Y(f(x_1), f(x_2)) = 0$. Since $d_Y(f(x_1), f(x_2)) = 0$ and $c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2))$, we get $c_1 d_X(x_1, x_2) = 0$. Since $c_1 d_X(x_1, x_2) = 0$ and $c_1 > 0$, we get $d_X(x_1, x_2) = 0$. Since $d_X(x_1, x_2) = 0$ and d_X is a metric, by definition, $x_1 = x_2$. This contradicts to the assumption that $x_1 \neq x_2$. \square

9.4 Hölder Continuity

DEFINITION 9.12 (Hölder continuous). Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y . We say that f is **Hölder continuous** with exponent α on X if there exists a number K such that for any points x and y in X , we have $d_Y(f(x), f(y)) \leq K(d_X(x, y))^\alpha$.

9.5 Oscillation

DEFINITION 9.13 (Oscillation). Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y . Let x_0 be a point in X . We define the **oscillation** of f at point x_0 , denoted by $\omega(f, x_0)$, to be the number given by

$$\omega(f, x_0) := \inf\{\omega(f, x_0, \delta) : \delta > 0\}$$

where $\omega(f, x_0, \delta)$ is given by

$$\omega(f, x_0, \delta) := \sup\{d_Y(f(x_1), f(x_2)) : d_X(x_1, x_0) < \delta, d_X(x_2, x_0) < \delta\}.$$

PROPOSITION 9.14. A function is continuous at a point if and only if its oscillation at the point is 0.

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y . Let x_0 be a point in X . For one direction, assume that f is continuous at point x_0 . We are to prove that $\omega(f, x_0) = 0$. Let ε be an arbitrary positive number. Since f is continuous at point x_0 , by definition of continuity, there exists some $\delta(\varepsilon)$ such that $d_Y(f(x), f(x_0)) < \varepsilon/4$ whenever $d_X(x, x_0) < \delta(\varepsilon)$. Let x_1 and x_2 be arbitrary points such that $d_X(x_1, x_0) < \delta(\varepsilon)$ and $d_X(x_2, x_0) < \delta(\varepsilon)$. Since $d_Y(f(x_1), f(x_0)) < \varepsilon/4$ and $d_Y(f(x_2), f(x_0)) < \varepsilon/4$, by the triangle inequality, $d_Y(f(x_1), f(x_2)) < \varepsilon/2$. Since $d_Y(f(x_1), f(x_2)) < \varepsilon/2$ for any x_1 and x_2 such that $d_X(x_1, x_0) < \delta(\varepsilon)$ and $d_X(x_2, x_0) < \delta(\varepsilon)$, by definition of supremum, $\omega(f, x_0, \delta(\varepsilon)) \leq \varepsilon/2$. Since for any positive number ε , there exists a $\delta(\varepsilon)$ such that $\omega(f, x_0, \delta(\varepsilon)) < \varepsilon$, by definition of infimum, $\omega(f, x_0) = 0$. For the reverse direction, assume that $\omega(f, x_0) = 0$. We are to prove that f is continuous at point x_0 . Let ε be an arbitrary positive number. Since $\omega(f, x_0) = 0$, by definition of infimum, there exists some $\delta(\varepsilon)$ such that $\omega(f, x_0, \delta(\varepsilon)) < \varepsilon$. Since $\omega(f, x_0, \delta(\varepsilon)) < \varepsilon$, by definition of supremum, $d_Y(f(x_1), f(x_2)) < \varepsilon$ for any x_1 and x_2 such that $d_X(x_1, x_0) < \delta(\varepsilon)$ and $d_X(x_2, x_0) < \delta(\varepsilon)$. Since $d_Y(f(x_1), f(x_2)) < \varepsilon$ for any x_1 and x_2 such that $d_X(x_1, x_0) < \delta(\varepsilon)$ and $d_X(x_2, x_0) < \delta(\varepsilon)$, in particular, $d_Y(f(x), f(x_0)) < \varepsilon$ for any point x such that $d_X(x, x_0) < \delta(\varepsilon)$. Since for any positive number ε , there exists some $\delta(\varepsilon)$ such that for any point x such that $d_X(x, x_0) < \delta(\varepsilon)$, $d_Y(f(x), f(x_0)) < \varepsilon$, by definition of continuity, f is continuous at point x_0 . \square

9.6 Isometry

DEFINITION 9.15 (Isometry). Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y . We say that f is an **isometry** if

$$\forall x_1, x_2 \in X, \quad d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

DEFINITION 9.16 (Isometric Isomorphism). We define a **isometric isomorphism** to be a bijective isometry.

PROPOSITION 9.17. Isometries are injective.

Chapter 10

Separation Axioms

10.1 Definitions

DEFINITION 10.1 (Topologically Distinguishable Points). Let (X, τ) be a topological space. Let x and y be two points in the space. We say that the points x and y are **topologically distinguishable** if

$$(\exists G \in \tau, x \in G \text{ and } y \notin G) \text{ or } (\exists G \in \tau, y \in G \text{ and } x \notin G).$$

DEFINITION 10.2 (Separated Sets). Let (X, τ) be a topological space. Let A and B be two sets in the space. We say that A and B are **separated** by τ if

$$\exists U, V \in \tau : U \cap V = \emptyset, \quad A \subseteq U \text{ and } B \subseteq V.$$

DEFINITION 10.3 (T_0 Space). Let (X, τ) be a topological space. We say that the space is **T_0** , or **Kolmogorov**, if any two distinct points in the space are topologically distinguishable. i.e.,

$$\forall x, y \in X : x \neq y, \quad (\exists U \in \tau, x \in U, y \notin U) \text{ or } (\exists V \in \tau, y \in V, x \notin V).$$

DEFINITION 10.4 (T_1 Space). Let (X, τ) be a topological space. We say that the

space is **T_1** , or **Fréchet**, if

$$\forall x, y \in X : x \neq y, \quad (\exists U \in \tau, x \in U, y \notin U) \text{ and } (\exists V \in \tau, y \in V, x \notin V).$$

DEFINITION 10.5 (T_2 Space). Let (X, τ) be a topological space. We say that the space is **T_2** , or **Hausdorff**, if any two distinct points in the space are separated by τ . i.e.,

$$\forall x, y \in X : x \neq y, \quad (\exists U \in \tau, x \in U) \text{ and } (\exists V \in \tau, y \in V) \text{ and } (U \cap V = \emptyset).$$

DEFINITION 10.6 (Regular Space). Let (X, τ) be a topological space. We say that the space is **regular** if any closed set F and a point x such that $x \notin F$ are separated by τ . i.e.,

$$\forall F \subseteq X, x \in X : F \text{ is closed and } x \notin F, \quad (\exists U \in \tau, F \subseteq U) \text{ and } (\exists V \in \tau, x \in V) \text{ and } (U \cap V = \emptyset).$$

DEFINITION 10.7 (Normal Space). Let (X, τ) be a topological space. We say that the space is **normal** if any two disjoint closed sets in the space are separated by τ .

$$\forall E, F \subseteq X : E \text{ and } F \text{ are closed and } E \cap F = \emptyset, \quad (\exists U \in \tau, E \subseteq U) \text{ and } (\exists V \in \tau, F \subseteq V) \text{ and } (U \cap V = \emptyset).$$

DEFINITION 10.8 (T_3 Space). Let (X, τ) be a topological space. We say that the space is **T_3** if it is T_1 and regular.

DEFINITION 10.9 (T_4 Space). Let (X, τ) be a topological space. We say that the space is **T_4** if it is T_1 and normal.

10.2 Sufficient Conditions

PROPOSITION 10.10. Compact T_2 spaces are normal.

Proof. Let (X, τ) be a topological space. Suppose that the space is compact and T_2 . We are to prove that the space is normal. Let E and F be two arbitrary disjoint closed sets in the space. Let x be an arbitrary point in E . Let y be an arbitrary point in F . Since $E \cap F = \emptyset$ and $x \in E$ and $y \in F$, $x \neq y$. Since (X, τ) is T_2 and $x \neq y$, x and y are separated. So $\exists U, V \in \tau : U \cap V = \emptyset$ such that $x \in U, y \notin U, y \in V, x \notin V$. The sets $\{V(x, y)\}_{y \in F}$ form an open cover of F . Since the space is compact and F is closed, F has a finite subcover. Say F' is a finite subset of F such that $\{V(x, y)\}_{y \in F'}$ is a finite open cover of F . Define sets $\mathcal{U}(x)$ and $\mathcal{V}(x)$ as

$$\mathcal{U}(x) := \bigcap_{y \in F'} U(x, y) \text{ and } \mathcal{V}(x) := \bigcup_{y \in F'} V(x, y).$$

Then $\mathcal{U}(x)$ and $\mathcal{V}(x)$ form a separation of x and F . The sets $\{\mathcal{U}(x)\}_{x \in E}$ form an open cover of E . Since the space is compact and E is closed, E has a finite subcover. Say E' is a finite subset of E such that $\{\mathcal{U}(x)\}_{x \in E'}$ is a finite open cover of E . Define sets \mathfrak{U} and \mathfrak{V} as

$$\mathfrak{U} := \bigcup_{x \in E'} \mathcal{U}(x) \text{ and } \mathfrak{V} := \bigcap_{x \in E'} \mathcal{V}(x).$$

Then \mathfrak{U} and \mathfrak{V} form a separation of E and F . Since any two disjoint closed sets can be separated by τ , by definition, the space is normal. \square

10.3 Examples

EXAMPLE 10.11. Metric spaces are T_4 .

Chapter 11

Countability

11.1 Definitions

DEFINITION 11.1 (First Countability). We say that a topological space is first-countable if every point has a countable local base.

DEFINITION 11.2 (Second Countability). We say that a topological space is second-countable if it has a countable base.

11.2 Properties

PROPOSITION 11.3. Second-countable spaces are separable.

Proof. Let (X, τ) be a non-empty second-countable topological space.

We are to prove that (X, τ) is separable.

Let G be an arbitrary non-empty open set in (X, τ) .

Since (X, τ) is second-countable, by definition of second-countability, there exists a countable basis $\mathcal{B} = \{B_i\}_{i \in I_{\mathcal{B}}} \cup \{\emptyset\}$ for (X, τ) where $I_{\mathcal{B}}$ is a non-empty subset of \mathbb{N} and each B_i is not empty.

Let x_i be some point in B_i .

Let $D = \{x_i\}_{i \in I_{\mathcal{B}}}$.

Since \mathbb{N} is countable, D is countable.

Since G is a non-empty open set in (X, τ) and \mathcal{B} is a basis for (X, τ) , G can be written as $G = \bigcup_{i \in I'_\mathcal{B}(G)} B_i$ where $I'_\mathcal{B}(G)$ is a non-empty subset of $I_\mathcal{B}$.

Since $x_i \in B_i$ for each $i \in I'_\mathcal{B}(G)$ and $G = \bigcup_{i \in I'_\mathcal{B}(G)} B_i$, $x_i \in G$ for each $i \in I'_\mathcal{B}(G)$.

Since $x_i \in G, D$ for each $i \in I'_\mathcal{B}(G)$, $G \cap D$ is not empty.

Since D intersects any non-empty open set in (X, τ) non-trivially, by definition of density, D is dense in (X, τ) .

Since D is countable and dense in (X, τ) , by definition of separability, (X, τ) is separable. \square

PROPOSITION 11.4. Second-countable spaces are Lindelöf.

Proof. Let (X, τ) be a second-countable space.

We are to prove that (X, τ) is Lindelöf.

Since (X, τ) is second-countable, by definition of second-countability, there exists a countable basis $\mathcal{B} = \{B_i\}_{i \in I_\mathcal{B}}$ for (X, τ) where $I_\mathcal{B}$ is a non-empty subset of \mathbb{N} .

Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary open cover of (X, τ) .

Since each U_λ is open in (X, τ) and \mathcal{B} is a basis for (X, τ) , each U_λ can be written as $U_\lambda = \bigcup_{i \in I'_\mathcal{B}(\lambda)} B_i$ where $I'_\mathcal{B}(\lambda)$ is a non-empty subset of $I_\mathcal{B}$.

Let $I = \bigcup_{\lambda \in \Lambda} I'_\mathcal{B}(\lambda)$.

Since $I'_\mathcal{B}(\lambda) \subseteq I_\mathcal{B}$ for each $\lambda \in \Lambda$, $I \subseteq I_\mathcal{B}$.

Since I is countable, $\{B_i\}_{i \in I}$ is countable.

Since $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ and $U_\lambda = \bigcup_{i \in I'_\mathcal{B}(\lambda)} B_i$ for each $\lambda \in \Lambda$, $X = \bigcup_{i \in I} B_i$.

Since any open cover of X has a countable subcover, by definition of Lindelöf, (X, τ) is Lindelöf. \square

11.3 Sufficient Conditions

PROPOSITION 11.5. Subspaces of a second-countable space are second-countable.

PROPOSITION 11.6 (Product). The product of a countable collection of second-countable spaces is second-countable.

Chapter 12

Separability

12.1 Definitions

DEFINITION 12.1 (Separable). Let \mathfrak{X} be a topological space. We say that \mathfrak{X} is **separable** if \mathfrak{X} has a countable dense subset.

12.2 Properties

PROPOSITION 12.2. Separable metric spaces are Lindelöf.

Proof. Let (X, d) be a metric space and S be a separable subset of X .

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of S .

Since S is separable, S has a countable dense subset S' .

$\{U_\alpha\}$ is an open cover of S' .

Find for each element in S' a U_α . Construct $\{U_{\alpha_k}\}_{k=1}^\infty$. Then $\{U_{\alpha_k}\}$ is an open cover of S' .

Since S' is dense, for any $x \in S$ and any neighborhood $N(x)$, there exists $x_0 \in S' \cap N(x)$.

□

PROPOSITION 12.3. Separable metric spaces are second-countable.

Proof. Let (X, d) be a non-empty separable metric space.

We are to prove that (X, d) is second-countable.

Since (X, d) is separable, there exists a countable dense subset $D = \{d_i\}_{i \in \mathbb{N}}$ of X .

Let $\mathcal{B} = \{\text{ball}(d_i, 1/n) : d_i \in D, n \in \mathbb{N}\}$.

Since D and \mathbb{N} are countable, \mathcal{B} is countable.

Let G be an arbitrary non-empty open set in (X, d) .

Let x be an arbitrary point in G .

Since $x \in G$ and G is open in (X, d) , there exists some radius $r(x)$ of the form $r(x) = 1/n$ where $n \in \mathbb{N}$ such that $\text{ball}(x, 2r(x)) \subseteq G$.

Since $\text{ball}(x, r(x))$ is open in (X, d) and D is dense in (X, d) , by definition of density, there exists some $d_{i_0(x)}$ in D such that $d_{i_0(x)} \in \text{ball}(x, r(x))$.

Since $d_{i_0(x)} \in \text{ball}(x, r(x))$, $x \in \text{ball}(d_{i_0(x)}, r(x))$.

Since $d_{i_0(x)} \in \text{ball}(x, r(x))$, $\text{ball}(d_{i_0(x)}, r(x)) \subseteq \text{ball}(x, 2r(x))$.

Since $\text{ball}(d_{i_0(x)}, r(x)) \subseteq \text{ball}(x, 2r(x))$ and $\text{ball}(x, 2r(x)) \subseteq G$, $\text{ball}(d_{i_0(x)}, r(x)) \subseteq G$.

Since for any $x \in G$, there exists some open ball $B(x)$ in \mathcal{B} such that $x \in B(x) \subseteq G$,
 $G = \bigcup_{x \in G} B(x)$.

Since any non-empty open set in (X, d) can be written as a union of open balls in \mathcal{B} , $\mathcal{B} \cup \{\emptyset\}$ is a basis.

Since $\mathcal{B} \cup \{\emptyset\}$ is a countable set and $\mathcal{B} \cup \{\emptyset\}$ is a basis for (X, d) , by definition of second-countability, (X, d) is second-countable. □

12.3 Stability of Separability

PROPOSITION 12.4 (Subspace). An open subspace of a separable space is separable.

PROPOSITION 12.5 (Subspace). A subspace of a separable metric space is separable.

Proof. Let (\mathfrak{X}, d) be a separable metric space. Let S be a subset of \mathfrak{X} . I will show that the subspace (S, d) is separable. Since \mathfrak{X} is separable, there is some countable dense subset $\mathfrak{D} = \{x_i\}_{i \in \mathbb{N}}$ of \mathfrak{X} . Define $\Delta := \{(i, j) \in \mathbb{N}^2 : S \cap \text{ball}(x_i, 1/j) \neq \emptyset\}$. Clearly $\Delta \neq \emptyset$. Define for each $(i, j) \in \Delta$ a point $y_{i,j} \in S \cap \text{ball}(x_i, 1/j)$. Clearly $\{y_{i,j}\}_{(i,j) \in \Delta}$ is a countable subset of S . There remains only to show that $\{y_{i,j}\}_{(i,j) \in \Delta}$ is dense in S . Let x be an arbitrary element of S . Let ε be an arbitrary element of \mathbb{R}_+ . Then there is some $j \in \mathbb{N}$

such that $1/j < \varepsilon/2$ (1). Since \mathfrak{D} is dense in \mathfrak{X} , $\exists i \in \mathbb{N}$ such that $d(x, x_i) < 1/j$ (2). So $x \in \text{ball}(x_i, 1/j)$. So $S \cap \text{ball}(x_i, 1/j) \neq \emptyset$. So $(i, j) \in \Delta$. So $y_{i,j}$ is defined and by definition, $y_{i,j} \in S \cap \text{ball}(x_i, 1/j)$. So $d(y_{i,j}, x_i) < 1/j$ (3). From (1), (2), and (3), we get

$$d(x, y_{i,j}) \leq d(x, x_i) + d(x_i, y_{i,j}) < 1/j + 1/j < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

That is, $d(x, y_{i,j}) < \varepsilon$. So $\{y_{i,j}\}_{(i,j) \in \Delta}$ is dense in S . □

PROPOSITION 12.6 (Set Operations).

1. The union of a countable collection of separable spaces is separable.

PROPOSITION 12.7 (Images). A continuous image of a separable space is also separable.

Proof. Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a continuous function from (X, τ_X) to (Y, τ_Y) . Let (S, τ_X) be a separable subspace of (X, τ_X) . Let T denote $f(S)$. We are to prove that (T, τ_Y) is separable. Since (S, τ_X) is separable, by definition, there exists a countable dense set S' in (S, τ_X) . Let T' denote $f(S')$. Since S' is countable, T' is countable. Since S' is dense in (S, τ_X) , T' is dense in (T, τ_Y) . Since T' is a countable dense set in (T, τ_Y) , (T, τ_Y) is separable. □

12.4 Separation Properties

PROPOSITION 12.8. Let (X, d) be a metric space and \mathcal{B} be a set of open subsets of X . Then \mathcal{B} is a basis for (X, d) if and only if for all $x \in X$ and all open neighborhood U of x , there exists a set $B_0 \in \mathcal{B}$ such that $x \in B_0 \subseteq U$.

PROPOSITION 12.9. Let (X, d) be a metric space. Let D be a subset of X . Then D is dense if and only if for any element x in X and any radius r , $B(x, r) \cap D \neq \emptyset$.

Chapter 13

Cauchy Completeness

13.1 Definitions

DEFINITION 13.1 (Cauchy Completeness). Let (X, τ) be a topological space. We say that (X, τ) is **cauchy complete** if every Cauchy net in (X, τ) converges to some point in X .

Chapter 14

Completeness in Metric Spaces

14.1 Definitions

DEFINITION 14.1 (Completeness). Let (X, d) be a metric space. Let S be a subset of X . We say that S is **complete** if any Cauchy sequence in S converges in S .

DEFINITION 14.2 (Completeness). Let (X, d) be a metric space. Let S be a subset of X . We say that S is **complete** if any infinite totally bounded subset of S has an accumulation point in S .

PROPOSITION 14.3. The two definitions of completeness are equivalent.

14.2 Properties

PROPOSITION 14.4. Complete metric spaces are closed.

Proof. Let (X, d) be a metric space and S be a complete subspace of X . We are to prove that S is closed.

Let $\{x_k\}$ be a convergent sequence in S . Then $\{x_k\}$ is Cauchy.

By definition of completeness, $\{x_k\}$ converges to some point in S .

By definition, we conclude that S is closed.

□

14.3 Sufficient Conditions

PROPOSITION 14.5. Closed subspaces of a complete space are also complete.

Proof. Let (X, d) be a complete metric space and S be a closed subset of X .

Let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary Cauchy sequence in S . We are to prove that $\{x_k\}$ converges to some point in S .

Since $\{x_k\}$ is Cauchy in X and X is complete, by definition, $\{x_k\}$ converges to some point in X .

Since $\{x_k\}$ converges in X and S is closed, by definition, $\{x_k\}$ converges to some point in S .

In short, we have proved that any Cauchy sequence in S converges to some point in S .

By definition, we conclude that S is complete.

□

PROPOSITION 14.6 (Cartesian Product). Let (X_1, d_1) and (X_2, d_2) be metric spaces. Then the product space $(X_1 \times X_2, D)$ is complete if and only if both (X_1, d_1) and (X_2, d_2) are complete.

Proof. For one direction, assume $(X_1 \times X_2, D)$ is complete.

We are to prove that (X_1, d_1) and (X_2, d_2) are complete.

Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary Cauchy sequence in (X_1, d_1) and c be an arbitrary point in X_2 .

Let ε be an arbitrary positive number.

Since $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy in (X_1, d_1) , there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N(\varepsilon)$, we have $d_1(x_m, x_n) < \varepsilon$.

Since $d_1(x_m, x_n) < \varepsilon$ and $d_2(c, c) = 0$, by definition of D , $D((x_m, c), (x_n, c)) < \varepsilon$.

Since for any positive number ε , there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N(\varepsilon)$, we have $D((x_m, c), (x_n, c)) < \varepsilon$, $\{(x_i, c)\}_{i \in \mathbb{N}}$ is Cauchy in $(X_1 \times X_2, D)$.

Since $\{(x_i, c)\}_{i \in \mathbb{N}}$ is Cauchy in $(X_1 \times X_2, D)$ and $(X_1 \times X_2, D)$ is complete, $\{(x_i, c)\}_{i \in \mathbb{N}}$ converges in $(X_1 \times X_2, D)$.

Since $\{(x_i, c)\}_{i \in \mathbb{N}}$ converges in $(X_1 \times X_2, D)$, $\{x_i\}_{i \in \mathbb{N}}$ converges in (X_1, d_1) .

Since any Cauchy sequence in (X_1, d_1) converges in (X_1, d_1) , (X_1, d_1) is complete.

Similarly, (X_2, d_2) is also complete.

For the reverse direction, assume that both (X_1, d_1) and (X_2, d_2) are complete.

We are to prove that $(X_1 \times X_2, D)$ is complete.

Let $\{(x_1^{(i)}, x_2^{(i)})\}_{i \in \mathbb{N}}$ be an arbitrary Cauchy sequence in $(X_1 \times X_2, D)$.

Since $\{(x_1^{(i)}, x_2^{(i)})\}_{i \in \mathbb{N}}$ is Cauchy in $(X_1 \times X_2, D)$, $\{x_1^{(i)}\}_{i \in \mathbb{N}}$ is Cauchy in (X_1, d_1) and $\{x_2^{(i)}\}_{i \in \mathbb{N}}$ is Cauchy in (X_2, d_2) .

Since $\{x_1^{(i)}\}_{i \in \mathbb{N}}$ is Cauchy in (X_1, d_1) and (X_1, d_1) is complete, $\{x_1^{(i)}\}_{i \in \mathbb{N}}$ converges in (X_1, d_1) .

Since $\{x_2^{(i)}\}_{i \in \mathbb{N}}$ is Cauchy in (X_2, d_2) and (X_2, d_2) is complete, $\{x_2^{(i)}\}_{i \in \mathbb{N}}$ converges in (X_2, d_2) .

Since $\{x_1^{(i)}\}_{i \in \mathbb{N}}$ converges in (X_1, d_1) and $\{x_2^{(i)}\}_{i \in \mathbb{N}}$ converges in (X_2, d_2) , $\{(x_1^{(i)}, x_2^{(i)})\}_{i \in \mathbb{N}}$ converges in $(X_1 \times X_2, D)$.

Since any Cauchy sequence in $(X_1 \times X_2, D)$ converges in $(X_1 \times X_2, D)$, $(X_1 \times X_2, D)$ is complete. □

THEOREM 14.7 (Cantor's Intersection Theorem). Let (X, d) be a metric space. Then (X, d) is complete if and only if for any decreasing sequence $\{S_i\}_{i \in \mathbb{N}}$ of non-empty closed sets with $\text{diam}(S_i) = 0$, $\bigcap_{i \in \mathbb{N}} S_i$ is a singleton set.

###question. How does this theorem relate to compactness?

Proof. For one direction, assume that (X, d) is complete.

Since S_i is non-empty for all $i \in \mathbb{N}$, there exists a point x_i in S_i for each $i \in \mathbb{N}$.

Let ε be an arbitrary positive number.

Since $\text{diam}(S_i) = 0$, there exists an integer $N(\varepsilon)$ such that $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$.

Since $\{S_i\}_{i \in \mathbb{N}}$ is decreasing, for any indices m and n , if $m, n > N(\varepsilon)$, we have $x_m, x_n \in S_{N(\varepsilon)}$.

Since $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$, by definition of diameter, $d(x_m, x_n) < \varepsilon$.

Since for any positive number ε , there exists an integer $N(\varepsilon)$ such that for any indices m and n , if $m, n > N(\varepsilon)$, then $d(x_m, x_n) < \varepsilon$, by definition of Cauchy-ness, $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy in (X, d) .

Since $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy in (X, d) and (X, d) is complete, by definition of completeness, $\{x_i\}_{i \in \mathbb{N}}$ is convergent in (X, d) .

Let x_0 be the limit of $\{x_i\}_{i \in \mathbb{N}}$.

Let S_{i_0} be an arbitrary set in $\{S_i\}_{i \in \mathbb{N}}$.

Since $\{S_i\}_{i \in \mathbb{N}}$ is decreasing, by construction of $\{x_i\}_{i \in \mathbb{N}}$, $\{x_i\}_{i \geq i_0}$ is a sequence in S_{i_0} .

Since $\{x_i\}_{i \in \mathbb{N}}$ converges to x_0 in (X, d) , $\{x_i\}_{i \geq i_0}$ also converges to x_0 in (X, d) .

Since $\{x_i\}_{i \geq i_0}$ is a sequence in S_{i_0} and converges to x_0 in (X, d) and S_{i_0} is closed, $x_0 \in S_{i_0}$.

Since x_0 is in any set in $\{S_i\}_{i \in \mathbb{N}}$, $x_0 \in \bigcap_{i \in \mathbb{N}} S_i$.

Since $\text{diam}(S_i) = 0$, $\text{diam}(\bigcap_{i \in \mathbb{N}} S_i) = 0$.

Since $\text{diam}(\bigcap_{i \in \mathbb{N}} S_i) = 0$ and $x_0 \in \bigcap_{i \in \mathbb{N}} S_i$, $\bigcap_{i \in \mathbb{N}} S_i$ is a singleton set consisting of only x_0 .

For the reverse direction, assume that for any decreasing sequence $\{S_i\}_{i \in \mathbb{N}}$ of non-empty closed sets with $\text{diam}(S_i) = 0$, $\bigcap_{i \in \mathbb{N}} S_i$ is a singleton set.

We are to prove that (X, d) is complete.

Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary Cauchy sequence in (X, d) .

Let $S_k = \text{cl}(\{x_i\}_{i \geq k})$.

Since $S_k = \text{cl}(\{x_i\}_{i \geq k})$, $\{S_k\}_{k \in \mathbb{N}}$ is a decreasing sequence of non-empty closed sets in (X, d) .

Let ε be an arbitrary positive number.

Since $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy in (X, d) , there exists an integer $N(\varepsilon)$ such that for any indices m and n , if $m, n > N(\varepsilon)$, then $d(x_m, x_n) < \varepsilon/2$.

Since for any indices m and n , if $m, n > N$, then $d(x_m, x_n) < \varepsilon/2$, $\text{diam}(\{x_i\}_{i \geq N(\varepsilon)}) < \varepsilon$.

Since $\text{diam}(\{x_i\}_{i \geq N(\varepsilon)}) < \varepsilon$, and $\text{diam}(\{x_i\}_{i \geq N(\varepsilon)}) = \text{diam}(S_{N(\varepsilon)})$, $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$.

Since for any index i , if $i > N(\varepsilon)$, then $\text{diam}(S_i) \leq \text{diam}(S_{N(\varepsilon)})$, and $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$, for any index i , if $i > N(\varepsilon)$, then $\text{diam}(S_i) < \varepsilon$.

Since for any positive number ε , there exists an integer $N(\varepsilon)$ such that for any index i , if $i > N(\varepsilon)$, then $\text{diam}(S_i) < \varepsilon$, by definition of limits, $\text{diam}(S_i) = 0$.

Since $\{S_i\}_{i \in \mathbb{N}}$ is a decreasing sequence of non-empty closed sets with $\text{diam}(S_i) = 0$, by assumption, $\bigcap_{i \in \mathbb{N}} S_i$ is a singleton set.

Let $x_0 \in \bigcap_{i \in \mathbb{N}} S_i$.

Let ε be an arbitrary positive number.

Since $\text{diam}(S_i) = 0$, there exists an integer $N(\varepsilon)$ such that $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$.

Since for any index i , if $i > N(\varepsilon)$, then $x_i \in S_{N(\varepsilon)}$, and $x_0 \in S_{N(\varepsilon)}$, by definition of diameter, for any index i , if $i > N(\varepsilon)$, $d(x_i, x_0) < \text{diam}(S_{N(\varepsilon)})$.

Since $d(x_i, x_0) < \text{diam}(S_{N(\varepsilon)})$ and $\text{diam}(S_{N(\varepsilon)}) < \varepsilon$, $d(x_i, x_0) < \varepsilon$.

Since for any positive number ε , there exists an integer $N(\varepsilon)$ such that for any index i , if $i > N(\varepsilon)$, then $d(x_i, x_0) < \varepsilon$, by definition of limits, $\{x_i\}_{i \in \mathbb{N}}$ converges to x_0 .

Since any Cauchy sequence in (X, d) converges, (X, d) is complete.

□

14.4 Metric Completion

$$Cd_1(x, y) \leq d_2(x, y) \leq Dd_1(x, y)$$

PROPOSITION 14.8. Let (X, d_1) and (X, d_2) be equivalent metric spaces. Then they have the same class of convergent sequences and Cauchy sequences.

Chapter 15

Totally Bounded Space

15.1 Preliminaries: Diameter

DEFINITION 15.1 (Diameter). Let (\mathfrak{X}, d) be a metric space. Let S be a set in (\mathfrak{X}, d) . We define the **diameter** of S , denoted by $\text{diam}(S)$, to be a number given by

$$\text{diam}(S) := \sup\{d(x, y) : x, y \in S\}.$$

PROPOSITION 15.2. If the diameter of a set is zero, then the set is a singleton set.

Proof. Let (X, d) be a metric space and S be a subset of X with $\text{diam}(S) = 0$. Assume for the sake of contradiction that there exist distinct points x_1 and x_2 in S . Since $x_1 \neq x_2$, by definition of metric, $d(x_1, x_2) \neq 0$. Since $d(x_1, x_2) \in \{d(x, y) : x, y \in S\}$, by definition of supremum, $\sup\{d(x, y) : x, y \in S\} \geq d(x, y)$. Since $\sup\{d(x, y) : x, y \in S\} \geq d(x, y)$, by definition of diameter, $\text{diam}(S) \geq d(x, y)$. This contradicts to the assumption that $\text{diam}(S) = 0$. Thus S is a singleton set. \square

PROPOSITION 15.3 (Monotonicity). The diameter function is increasing.

PROPOSITION 15.4. For any set S in a metric space, $\text{diam}(\text{cl}(S)) = \text{diam}(S)$.

Proof. Let (\mathfrak{X}, d) be a metric space. Let S be a subset of \mathfrak{X} .

Part 1: Show that $\text{diam}(\text{cl}(S)) \geq \text{diam}(S)$.

Since $S \subseteq \text{cl}(S)$ and the diam operator is monotonically increasing, we have

$$\text{diam}(\text{cl}(S)) \geq \text{diam}(S).$$

Part 2: Show that $\text{diam}(\text{cl}(S)) \leq \text{diam}(S)$.

Let x and y be arbitrary elements of $\text{cl}(S)$. Let $\varepsilon > 0$ be arbitrary. Since $x \in \text{cl}(S)$, $\exists x' \in S$ such that $d(x, x') < \varepsilon/2$. Since $y \in \text{cl}(S)$, $\exists y' \in S$ such that $d(y, y') < \varepsilon/2$. Since $x', y' \in S$, $d(x', y') \leq \text{diam}(S)$. Now we have

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) < \varepsilon/2 + \text{diam}(S) + \varepsilon/2 = \text{diam}(S) + \varepsilon.$$

That is, $d(x, y) < \text{diam}(S) + \varepsilon$. This holds for any $\varepsilon > 0$. So $d(x, y) \leq \text{diam}(S)$. This holds for any $x, y \in \text{cl}(S)$. So $\text{diam}(\text{cl}(S))$ is an upper bound for the set $\{d(x, y) : x, y \in \text{cl}(S)\}$. So

$$\text{diam}(\text{cl}(S)) \leq \text{diam}(S).$$

This completes the proof. □

15.2 Bounded Set

DEFINITION 15.5 (Boundedness). Let (\mathfrak{X}, d) be a metric space. Let S be a set in (\mathfrak{X}, d) . We say S is **bounded** if S has finite diameter.

PROPOSITION 15.6. The closure of a bounded set is bounded.

15.3 Totally Bounded Set

DEFINITION 15.7 (Total Boundedness). Let (\mathfrak{X}, d) be a metric space. Let S be a set in (\mathfrak{X}, d) . We say that S is **totally bounded** if

$$\forall r > 0, \exists n \in \mathbb{N}, \exists x_1, \dots, x_n \in \mathfrak{X}, \text{ such that } S \subseteq \bigcup_{i=1}^n \text{ball}(x_i, r).$$

PROPOSITION 15.8. A set is totally bounded if and only if any sequence in the set admits a Cauchy subsequence.

Proof. Let (\mathfrak{X}, d) be a metric space. Let S be a subset of \mathfrak{X} .

Forward Direction: Assume that S is totally bounded. I will show that any sequence in S admits a Cauchy subsequence. Let $(x_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of elements of S . Since S is totally bounded, $(x_n)_{n \in \mathbb{N}}$ has a subsequence $(x_n^{(1)})_{n \in \mathbb{N}}$ that is contained in $\text{ball}(p_1, 1)$ for some $p_1 \in \mathfrak{X}$. Since S is totally bounded, $(x_n^{(1)})_{n \in \mathbb{N}}$ has a subsequence $(x_n^{(2)})_{n \in \mathbb{N}}$ that is contained in $\text{ball}(p_2, 1/2)$ for some $p_2 \in \mathfrak{X}$. In general, for any $m \in \mathbb{N}$, since S is totally bounded, $(x_n^{(m)})_{n \in \mathbb{N}}$ has a subsequence $(x_n^{(m+1)})_{n \in \mathbb{N}}$ that is contained in $\text{ball}(p_{m+1}, 1/(m+1))$ for some $p_{m+1} \in \mathfrak{X}$. Consider the subsequence $(x_n^{(n)})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$. I claim that it is Cauchy. Let $\varepsilon > 0$ be arbitrary. Then $\exists m_0 \in \mathbb{N}$ such that $1/m_0 < \varepsilon$. Let $i, j > m_0$ be arbitrary. Then $x_i^{(i)}$ and $x_j^{(j)}$ are terms of $(x_n^{(m_0)})_{n \in \mathbb{N}}$. So $d(x_i^{(i)}, x_j^{(j)}) < 1/m_0 < \varepsilon$. So $(x_n^{(n)})_{n \in \mathbb{N}}$ is Cauchy.

Backward Direction: Assume that any sequence in S admits a Cauchy sequence. I will show that S is totally bounded. If S is empty, then we are done. Otherwise, $S \neq \emptyset$. Assume for the sake of contradiction that S is not totally bounded. Then $\exists r_0 > 0$ such that for any finite collection $\{x_i\}_{i=1}^n \subseteq \mathfrak{X}$, $S \not\subseteq \bigcup_{i=1}^n \text{ball}(x_i, r_0)$. Since S is not empty, take $x_1 \in S$. Since $\{x_1\}$ is finite, $S \not\subseteq \text{ball}(x_1, r_0)$. So $\exists x_2 \in S \setminus \text{ball}(x_1, r_0)$. Since $\{x_1, x_2\}$ is finite, $S \not\subseteq \text{ball}(x_1, r_0) \cup \text{ball}(x_2, r_0)$. So $\exists x_3 \in S \setminus \text{ball}(x_1, r_0) \cup \text{ball}(x_2, r_0)$. In general, since $\{x_i\}_{i=1}^n$ is finite, $S \not\subseteq \bigcup_{i=1}^n \text{ball}(x_i, r_0)$. So $\exists x_{n+1} \in S \setminus \bigcup_{i=1}^n \text{ball}(x_i, r_0)$. Consider the sequence $(x_n)_{n \in \mathbb{N}}$. We have $d(x_i, x_j) > r_0$ for any $i, j \in \mathbb{N}$. So $(x_n)_{n \in \mathbb{N}}$ does not have a Cauchy subsequence. This contradicts to the assumption that any sequence in S admits a Cauchy sequence. So the assumption that S is not totally bounded does not hold. i.e., S is totally bounded. □

15.4 Properties of Totally Bounded Spaces

PROPOSITION 15.9. Totally bounded sets are bounded.

PROPOSITION 15.10. Totally bounded metric spaces are separable.

Proof. Let (X, d) be a metric space. Suppose that the space is totally bounded. We are to prove that the space has a countable dense subset. Let n be an arbitrary natural number. Since the space is totally bounded, there exists a finite collection of points \mathbf{p}_n such that $\bigcup_{p \in \mathbf{p}_n} \text{ball}(p, 1/n) = X$. Define a set \mathbf{p} by $\mathbf{p} := \bigcup_{n \in \mathbb{N}} \mathbf{p}_n$. Then \mathbf{p} is a countable dense subset of X . \square

15.5 Stability of Total Boundedness

PROPOSITION 15.11. The closure of a totally bounded set is totally bounded.

Proof. Let (\mathfrak{X}, d) be a metric space. Let S be a totally bounded set in (\mathfrak{X}, d) . I will show that $\text{cl}(S)$ is also totally bounded. Let $r > 0$ be arbitrary. Since S is totally bounded, $\exists n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathfrak{X}$ such that $S \subseteq \bigcup_{i=1}^n \text{ball}(x_i, r/2)$. Let x be an arbitrary point in $\text{cl}(S)$. Since $x \in \text{cl}(S)$, $\exists x' \in S$ such that $d(x, x') < r/2$. Since $x' \in S$ and $S \subseteq \bigcup_{i=1}^n \text{ball}(x_i, r/2)$, $\exists i_0 \in \{1..n\}$ such that $x' \in \text{ball}(x_{i_0}, r/2)$. So $d(x', x_{i_0}) < r/2$. So

$$d(x, x_{i_0}) \leq d(x, x') + d(x', x_{i_0}) < r/2 + r/2 = r.$$

So $x \in \text{ball}(x_{i_0}, r)$. So $x \in \bigcup_{i=1}^n \text{ball}(x_i, r)$. So $\text{cl}(S) \subseteq \bigcup_{i=1}^n \text{ball}(x_i, r)$. So $\text{cl}(S)$ is totally bounded. \square

PROPOSITION 15.12 (Set Operations).

1. A subset of a totally bounded set is totally bounded.
2. The union of a finite collection of totally bounded sets is totally bounded.

PROPOSITION 15.13 (Images). A uniformly continuous image of a totally bounded set is totally bounded.

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces. Let S be a totally bounded set in (X, d_X) . Let f be a uniformly continuous function from X to Y . We are to prove that $f(S)$ is totally bounded. Let r be an arbitrary radius. Since f is uniformly continuous on S , there exists

a δ such that $d_Y(f(x_1), f(x_2)) < r$ whenever $d_X(x_1, x_2) < \delta$. Since S is totally bounded in (X, d_X) , there exists a finite collection \S of points in X such that $\{\text{ball}(x, \delta)\}_{x \in \S}$ covers S . Let \dagger denote $f(\S)$. Let y be an arbitrary point in $f(S)$. Since $y \in f(S)$, there exists a point x in S such that $f(x) = y$. Since $x \in S$ and $\{\text{ball}(x, \delta)\}_{x \in \S}$ covers S , there exists a point $x_0 \in \S$ such that $x \in \text{ball}(x_0, \delta)$. Let y_0 denote $f(x_0)$. Since $x \in \text{ball}(x_0, \delta)$, by definition of δ , $y \in \text{ball}(y_0, r)$. Since $y \in \text{ball}(y_0, r)$ and $y_0 \in \dagger$, $\{\text{ball}(y, r)\}_{y \in \dagger}$ covers y . Since $\{\text{ball}(y, r)\}_{y \in \dagger}$ covers y for any $y \in f(S)$, $\{\text{ball}(y, r)\}_{y \in \dagger}$ covers $f(S)$. Since for any radius r , there exists a collection \dagger of points in $f(S)$ such that $\{\text{ball}(y, r)\}_{y \in \dagger}$ covers $f(S)$, by definition of total boundedness, $f(S)$ is totally bounded in (Y, d_Y) . \square

Chapter 16

Lindelöf Space

16.1 Definition

DEFINITION 16.1 (Lindelöf Space). Let (X, τ) be a topological space. We say that the space is **Lindelöf** if any open cover has a countable subcover.

Chapter 17

Compact Space

17.1 Definitions

DEFINITION 17.1 (Open Cover). Let (X, τ) be a topological space. Let S be a set in the space. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a collection of sets in the space. We say that $\{U_\lambda\}_{\lambda \in \Lambda}$ is an **open cover** of S if U_λ is open for each $\lambda \in \Lambda$ and

$$\bigcup_{\lambda \in \Lambda} U_\lambda \supseteq S.$$

DEFINITION 17.2 (Compactness - 1). Let (X, τ) be a topological space. We say that the space is **compact** if any open cover has a finite subcover.

DEFINITION 17.3 (Finite Intersection Property). Let (X, τ) be a topological space. Let $\{S_i\}_{i \in I}$ be a collection of sets in the space. We say that $\{S_i\}_{i \in I}$ has the **finite intersection property** if for any finite subcollection $\{S_i\}_{i \in I'}$, where I' is a finite subset of I ,

$$\bigcap_{i \in I'} S_i \neq \emptyset.$$

DEFINITION 17.4 (Compactness - 2). Let (X, τ) be a topological space. Let S be a set in the space. We say that S is **compact** if for any collection $\{S_i\}_{i \in I}$ of relatively

closed subsets of S with the finite intersection property,

$$\bigcap_{i \in I} S_i \neq \emptyset.$$

DEFINITION 17.5 (Compactness - 3). Let (X, τ) be a topological space. Let S be a set in the space. We say that S is **compact** if any net on S has a convergent subnet.

DEFINITION 17.6 (Compactness - 4). Let (X, τ) be a topological space. Let S be a set in the space. We say that S is **compact** if any filter on S has a convergent refinement.

PROPOSITION 17.7. Definitions 1 and 2 of compactness are equivalent.

Proof. [**Definition 1**] \implies [**Definition 2**]. For one direction, assume that any open cover of the space has a finite subcover. We are to prove that any collection of closed sets in the space with the finite intersection property has nonempty intersection. Let \mathcal{F} be an arbitrary collection of closed sets in the space with the finite intersection property. Say $\mathcal{F} = \{F_\lambda\}_{\lambda \in \Lambda}$ where Λ is an index set and F_λ is a closed set in the space for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda'} F_\lambda \neq \emptyset$ for any finite subset Λ' of Λ . Define $U_\lambda := X \setminus F_\lambda$. Define $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$. Since F_λ is closed for each $\lambda \in \Lambda$, U_λ is open for each $\lambda \in \Lambda$. Assume for the sake of contradiction that \mathcal{F} has empty intersection. Since $\bigcap_{\lambda \in \Lambda} F_\lambda = \emptyset$, by the De Morgan's Law, $\bigcup_{\lambda \in \Lambda} U_\lambda = X$. Since \mathcal{U} is an open cover of the space and any open cover of the space has a finite subcover, in particular, \mathcal{U} has a finite subcover \mathcal{U}' . Say $\mathcal{U}' = \{U_\lambda\}_{\lambda \in \Lambda'}$ where Λ' is a finite subset of Λ and $\bigcup_{\lambda \in \Lambda'} U_\lambda = X$. Since $\bigcup_{\lambda \in \Lambda'} U_\lambda = X$, by the De Morgan's Law, $\bigcap_{\lambda \in \Lambda'} F_\lambda = \emptyset$. This contradicts to the assumption that \mathcal{F} has the finite intersection property. So the assumption that \mathcal{F} has empty intersection is false. i.e., \mathcal{F} has nonempty intersection. \square

Proof. [**Definition 2**] \implies [**Definition 1**]. For the reverse direction, assume that any collection of closed sets in the space with the finite intersection property has nonempty intersection. We are to prove that any open cover has a finite subcover. Let \mathcal{U} be an arbitrary open cover of the space. Say $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ where Λ is an index set and U_λ is an open set in the space for any $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} U_\lambda = X$. Define $F_\lambda := X \setminus U_\lambda$ for each $\lambda \in \Lambda$.

Define $\mathcal{F} := \{F_\lambda\}_{\lambda \in \Lambda}$. Since U_λ is open for each $\lambda \in \Lambda$, F_λ is closed for each $\lambda \in \Lambda$. Assume for the sake of contradiction that \mathcal{U} does not have a finite subcover. Let Λ' be an arbitrary finite subset of Λ . Define $\mathcal{U}' := \{U_\lambda\}_{\lambda \in \Lambda'}$. Define $\mathcal{F}' := \{F_\lambda\}_{\lambda \in \Lambda'}$. Since \mathcal{U}' is a finite subcollection of \mathcal{U} and \mathcal{U} does not have a finite subcover, in particular, \mathcal{U}' cannot cover the whole space. Since $\bigcup_{\lambda \in \Lambda'} U_\lambda \neq X$, by the De Morgan's Law, $\bigcap_{\lambda \in \Lambda'} F_\lambda \neq \emptyset$. Since $\bigcap_{\lambda \in \Lambda'} F_\lambda \neq \emptyset$ for any finite subcollection Λ' of Λ , \mathcal{F} has the finite intersection property. Since \mathcal{F} has the finite intersection property and any collection of closed sets with the finite intersection property has nonempty intersection, in particular, $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$. Since $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$, by the De Morgan's Law, $\bigcup_{\lambda \in \Lambda} U_\lambda \neq X$. This contradicts to the assumption that \mathcal{U} is an open cover of the space. So the assumption that \mathcal{U} does not have a finite subcover is false. i.e., \mathcal{U} has a finite subcover. \square

17.2 Properties of Compact sets

REMARK 17.8. Compact sets need not to be closed in general.

EXAMPLE 17.9 (Sierpiński Space). Let $X = \{a, b\}$. Let $\tau = \{\emptyset, \{a\}, \{a, b\}\}$. Then (X, τ) is a topological space and $\{a\}$ is a compact set in X . However, $\{a\}$ is not closed.

PROPOSITION 17.10. Compact sets are closed if the space is Hausdorff.

Proof. Let (X, τ) be a Hausdorff space. Let K be a compact set in X . We are to prove that $X \setminus K$ is open. Let x be an arbitrary point in $X \setminus K$. Let y be an arbitrary point in K . Then $x \neq y$. Since $x \neq y$ and (X, τ) is Hausdorff, $\exists U(x, y), V(x, y) \in \tau$ such that $x \in U(x, y)$ and $y \in V(x, y)$. Then $\{V(x, y)\}_{y \in K}$ is an open cover of K . Since $\{V(x, y)\}_{y \in K}$ is an open cover of K and K is compact, it has a finite subcover. Say F is a finite subset of K such that $\{V(x, y)\}_{y \in F}$ suffices to form an open cover of K . Define a set $\mathcal{U}(x)$ as $\mathcal{U}(x) := \bigcap_{y \in F} U(x, y)$. Then $\mathcal{U}(x)$ is an open neighborhood of x and disjoint from K . Since $\forall x \in X \setminus K$, $\exists \mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $\mathcal{U} \subseteq X \setminus K$, we get $X \setminus K$ is open. So K is closed. \square

17.3 Sufficient Conditions for Compactness

PROPOSITION 17.11 (Restriction). Let (X, τ) be a topological space. Let Y be a subset of X . Let $(Y, \tau|_Y)$ be the topological space restricted to Y . Let K be a compact set in $(Y, \tau|_Y)$. Then K is also compact in (X, τ) .

Proof. Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary open cover of K in (X, τ) . Then $\{Y \cap G_\lambda\}_{\lambda \in \Lambda}$ is an open cover of K in $(Y, \tau|_Y)$. Since K is compact in $(Y, \tau|_Y)$, it has a finite subcover. Say $\{Y \cap G_\lambda\}_{\lambda \in \Lambda'}$ is a finite subcover of $\{Y \cap G_\lambda\}_{\lambda \in \Lambda}$ where Λ' is some finite subset of Λ . Then $\{G_\lambda\}_{\lambda \in \Lambda'}$ is a finite subcover of $\{G_\lambda\}_{\lambda \in \Lambda}$ in (X, τ) . Since any open cover of K in (X, τ) has a finite subcover, K is compact in (X, τ) . \square

PROPOSITION 17.12. A closed subspace of a compact space is compact.

Proof. Let (X, τ) be a compact topological space. Let (S, τ) be a closed subspace of (X, τ) . We are to prove that (S, τ) is compact. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary open cover of S . Since S is closed, $X \setminus S$ is open. Since $\{U_\lambda\}_{\lambda \in \Lambda}$ covers S , $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus S\}$ covers X . Since each set in $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus S\}$ is open and $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus S\}$ covers X , by definition of open cover, $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus S\}$ is an open cover of X . Since (X, τ) is compact, by definition of compactness, any open cover of (X, τ) has a finite subcover. Since $\mathcal{U} \cup \{X \setminus S\}$ is an open cover of (X, τ) and any open cover has a finite subcover, in particular, \mathcal{U} has a finite subcover \mathcal{U}' . Say $\mathcal{U}' = \{U_\lambda\}_{\lambda \in \Lambda'}$ where Λ' is a finite subset of Λ and $\bigcup_{\lambda \in \Lambda'} U_\lambda = X$. Since $\{U_\lambda\}_{\lambda \in \Lambda'} \cup \{X \setminus S\}$ covers X , $\{U_\lambda\}_{\lambda \in \Lambda'}$ covers S . Since any open cover of S has a finite subcover, by definition of compactness, (S, τ) is compact. \square

PROPOSITION 17.13 (Set Operations).

1. In a Hausdorff space, the intersection of any non-empty collection of compact sets is compact.
2. The union of any finite collection of compact sets is compact.

Proof of (1). Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary non-empty collection of compact sets where Λ is a non-empty index set. Define a set S as $S := \bigcap_{\lambda \in \Lambda} S_\lambda$. Since the space is Hausdorff and S_λ is compact, S_λ is closed. So $S = \bigcap_{\lambda \in \Lambda} S_\lambda$ is closed. Then S is a closed subset of compact sets and hence compact. \square

Proof of (2). Let $\{K_i\}_{i=1}^n$ be a collection of compact sets where $n \in \mathbb{N}$. Define a set K as $K := \bigcup_{i=1}^n K_i$. We are to prove that K is compact. Let \mathcal{U} be an arbitrary open cover of K . Since \mathcal{U} is an open cover of K , it is an open cover of K_i for each i . Since \mathcal{U} is an open cover of K_i , $\exists \mathcal{U}'_i \subseteq \mathcal{U}$ that is finite and that is an open cover of K_i . Define a set \mathcal{U}' as $\mathcal{U}' := \bigcup_{i=1}^n \mathcal{U}'_i$. Then $\mathcal{U}' \subseteq \mathcal{U}$ and it is a finite open cover of K . Since any open cover of K has a finite subcover, K is compact. \square

PROPOSITION 17.14 (Product, Tychonoff). The product of any collection of compact sets is compact.

PROPOSITION 17.15 (Image). The continuous image of a compact set is compact.

Proof. Let (X, τ_X) and (Y, τ_Y) be topological space. Let f be a continuous function from X to Y . Let S be a compact set in (X, τ_X) . Define $T := f(S)$. We are to prove that T is a compact set in (Y, τ_Y) . Let \mathcal{V} be an arbitrary open cover of T . Say $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ where Λ is an index set and V_λ is an open set in (Y, τ_Y) for each $\lambda \in \Lambda$ and $T \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$. Define $U_\lambda := f^{-1}(V_\lambda)$ for each $\lambda \in \Lambda$. Define $\mathcal{U} := \{U_\lambda\}_{\lambda \in \Lambda}$. Since $V_\lambda \neq \emptyset$ for each $\lambda \in \Lambda$, $U_\lambda \neq \emptyset$ for each $\lambda \in \Lambda$. Since V_λ is open in (Y, τ_Y) for each $\lambda \in \Lambda$ and f is a continuous function from (X, τ_X) to (Y, τ_Y) , by definition of continuity, U_λ is open in (X, τ_X) for each $\lambda \in \Lambda$. Since $T \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$, $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$. Since U_λ is a nonempty open set in (X, τ_X) for each $\lambda \in \Lambda$ and $S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$, \mathcal{U} is an open cover of S in (X, τ_X) . Since S is a compact set in (X, τ_X) , by definition of compactness, any open cover of S in (X, τ_X) has a finite subcover. Since \mathcal{U} is an open cover of S in (X, τ_X) and any open cover of S in (X, τ_X) has a finite subcover, in particular, \mathcal{U} has a finite subcover \mathcal{U}' . Say $\mathcal{U}' = \{U_\lambda\}_{\lambda \in \Lambda'}$ where Λ' is a finite subset of Λ and $S \subseteq \bigcup_{\lambda \in \Lambda'} U_\lambda$. Define $\mathcal{V}' := \{V_\lambda\}_{\lambda \in \Lambda'}$. Since $S \subseteq \bigcup_{\lambda \in \Lambda'} U_\lambda$, $T \subseteq \bigcup_{\lambda \in \Lambda'} V_\lambda$. Since V_λ is a nonempty open set in (Y, τ_Y) for each $\lambda \in \Lambda'$ and $T \subseteq \bigcup_{\lambda \in \Lambda'} V_\lambda$ and Λ' is finite, \mathcal{V}' is a finite open cover of T in (Y, τ_Y) . Since any open cover of T in (Y, τ_Y) has a finite subcover, by definition of compactness, T is a compact set in (Y, τ_Y) . \square

PROPOSITION 17.16. Finite spaces are compact.

Proof.

Let (X, τ) be a finite topological space.

Say $X = \{x_i\}_{i \in I}$ where I is a finite index set.

Let \mathcal{U} be an arbitrary open cover of the space.

Say $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ where Λ is an index set and U_λ is an open set in (X, τ_X) for each $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} U_\lambda = X$.

Since \mathcal{U} covers X and $x_i \in X$ for each $i \in I$, \mathcal{U} covers x_i for each $i \in I$.

Since $x_i \in \mathcal{U}$ for each $i \in I$, there exists some λ_i for each $i \in I$ such that $x_i \in U_{\lambda_i}$.

Define $\mathcal{U}' := \{U_{\lambda_i}\}_{i \in I}$.

Since $x_i \in U_{\lambda_i}$ for each $i \in I$, $X \subseteq \bigcup_{i \in I} U_{\lambda_i}$.

Since each U_{λ_i} is open in (X, τ) and $X \subseteq \bigcup_{i \in I} U_{\lambda_i}$ and I is finite, \mathcal{U}' is a finite open cover of the space.

Since any open cover of the space has a finite subcover, by definition, of compactness, (X, τ) is compact.

□

17.4 Compact Metric Spaces

PROPOSITION 17.17. Compact metric spaces are totally bounded.

Proof. Assume for the sake of contradiction that the space is not totally bounded. Since the space is not totally bounded, by definition of total boundedness, there exists some radius r_0 such that the space cannot be covered by finitely many open balls of radius r_0 . Let x_1 be an arbitrary point in the space. Since $\{ball(x_i, r_0)\}_{i \leq n}$ cannot cover the space for any $n \in \mathbb{N}$, pick x_{n+1} from $X \setminus \bigcup_{i \leq n} ball(x_i, r_0)$. Define $\mathfrak{x} := \{x_n\}_{n \in \mathbb{N}}$. Let m and n be arbitrary

indices in \mathbb{N} . Assume without loss of generality that $m > n$. Since $x_m \in X \setminus \bigcup_{i \leq m} ball(x_i, r_0)$, $x_m \notin ball(x_n, r_0)$. Since $x_m \notin ball(x_n, r_0)$, $d(x_m, x_n) \geq r_0$. Since $d(x_m, x_n) \geq r_0$ for any $m, n \in \mathbb{N}$, \mathfrak{x} has no convergent subsequence. Since \mathfrak{x} is a sequence in the space and \mathfrak{x} has no convergent subsequence, by definition of compactness, the space is not compact. This contradicts to the assumption that the space is compact. Thus the assumption that the space is not totally bounded is false. i.e., the space is totally bounded. □

PROPOSITION 17.18. Compact metric spaces are complete.

Proof. Let \mathfrak{x} be an arbitrary Cauchy sequence in the space. Since the space is compact, any sequence has a convergent subsequence. Since \mathfrak{x} is a sequence in the space and any sequence has a convergent subsequence, \mathfrak{x} has a convergent subsequence. Since \mathfrak{x} is Cauchy and has a convergent subsequence, \mathfrak{x} converges. Since any Cauchy sequence in the space converges, by definition of completeness, the space is complete. \square

PROPOSITION 17.19. Complete and totally bounded metric spaces are compact.

Proof. Every sequence in a totally bounded metric space admits Cauchy subsequence. Every Cauchy sequence in a complete metric space converges. So every sequence has a convergent subsequence. \square

PROPOSITION 17.20. Compact metric spaces are separable.

Proof. Let (X, d) be a compact metric space. Define for each $n \in \mathbb{N}$ an open cover \mathcal{G}_n of X by $\mathcal{G}_n := \{\text{ball}(x, 1/n) : x \in X\}$. Since X is compact and \mathcal{G}_n is an open cover of X , there is a finite subcover \mathcal{G}'_n of \mathcal{G}_n . Say $\mathcal{G}'_n = \{\text{ball}(x_j^{(n)}, 1/n) : j = 1..k_n\}$ for some $k_n \in \mathbb{N}$. Define a subset \mathcal{D} of X by $\mathcal{D} := \bigcup_{n \in \mathbb{N}} \{x_j^{(n)} : j = 1..k_n\}$. Then \mathcal{D} is countable. Now I claim that \mathcal{D} is dense in X . Let x be an arbitrary element of X . Let n be an arbitrary element of \mathbb{N} . Since \mathcal{G}'_n is an open cover of X , there is some $j_n \in \{1..k_n\}$ such that $x \in \text{ball}(x_{j_n}^{(n)}, 1/n)$. So $d(x, x_{j_n}^{(n)}) < 1/n$. Define a sequence x_\bullet by $x_\bullet := (x_{j_n}^{(n)})_{n \in \mathbb{N}}$. Then x_\bullet converges to x . Note that $x_\bullet \subseteq \mathcal{D}$. So $x \in \text{cl}(\mathcal{D})$. So $X \subseteq \text{cl}(\mathcal{D})$. So \mathcal{D} is dense in X . \square

THEOREM 17.21. Let (X, d_X) be a compact metric space and (Y, d_Y) be an arbitrary metric space. Let f be a continuous function from X to Y . Then f is uniformly continuous on X .

Proof. Since f is continuous on X , for all $\varepsilon > 0$ and all $x \in X$, there exists a $\delta(x) > 0$ such that for all $x' \in X$ with $d(x, x') < \delta(x)$, we have $\rho(f(x), f(x')) < \varepsilon/2$. Consider the set of open balls $\mathcal{B} = \{B(\frac{1}{2}\delta(x), x)\}$. Then \mathcal{B} is a cover of X . By definition of compactness, there exists a finite set of open balls $\mathcal{B}_n = \{B_k\}_{k=1}^{k=n}$. Define $\delta > 0$ by $\delta = \min\{\frac{1}{2}\delta(x_k)\}$. Let x and y be arbitrary elements in X with $d(x, y) < \delta$. Since $x \in X$ and \mathcal{B}_n is a cover of X , there exists an open ball $B_0(\frac{1}{2}\delta(x_0), x_0)$ such that $x \in B_0$. By our choice of x, y and δ , we have

$$d(y, x) < \delta \leq \frac{1}{2}\delta(x_0)\#(1)$$

Since $x \in B_0$, we get

$$d(x, x_0) < \frac{1}{2}\delta(x_0) \# (2)$$

By the triangle inequality, we get

$$d(y, x_0) \leq d(y, x) + d(x, x_0) \# (3)$$

From inequations (1) ~ (3), we get

$$d(y, x_0) < \delta(x_0) \# (4)$$

Since f is continuous at point x_0 and $d(x, x_0) < \delta(x_0)$, we get

$$\rho(f(x), f(x_0)) < \frac{\varepsilon}{2} \# (5)$$

Since f is continuous at pint x_0 and $d(y, x_0) < \delta(x_0)$, we get

$$\rho(f(y), f(x_0)) < \frac{\varepsilon}{2} \# (6)$$

By the triangle inequality again, we get

$$\rho(f(x), f(y)) < \rho(f(x), f(x_0)) + \rho(f(y), f(x_0)) \quad (7)$$

From (5), (6), and (7), we get

$$\rho(f(x), f(y)) < \varepsilon$$

In short, we have proved that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all elements x and y in X with $d(x, y) < \delta$, we have $\rho(f(x), f(y)) < \varepsilon$. By definition of uniform continuity, f is uniformly continuous on X . \square

17.5 Countably Compact

DEFINITION 17.22 (Countably Compact). Let (X, τ) be a topological space. We say that (X, τ) is **countably compact** if every countable open cover has a finite subcover.

DEFINITION 17.23 (Countably Compact). Let (X, τ) be a topological space. We say that (X, τ) is **countably compact** if every infinite set in the space has a ω -limit point.

DEFINITION 17.24 (Weakly Countably Compact). Let (X, τ) be a topological space. We say that (X, τ) is **weakly countably compact** if every infinite set in the space has a limit point.

PROPOSITION 17.25. The two definitions of countably compact are equivalent.

PROPOSITION 17.26 (Image). A continuous image of a countably compact set is countably compact.

REMARK 17.27. Continuous images of weakly countably compact sets may not be weakly countably compact.

PROPOSITION 17.28. Compactness implies countable compactness.

PROPOSITION 17.29. Countable compactness can imply compactness if the space is Lindelöf.

PROPOSITION 17.30. Countable compactness implies weak countable compactness.

PROPOSITION 17.31. Weak countable compactness can imply countable compactness if the space is metrizable.

17.6 Sequentially Compact

DEFINITION 17.32 (Sequentially Compact). Let (X, τ) be a topological space. We say that (X, τ) is **sequentially compact** if every sequence in the space has a convergent subsequence.

PROPOSITION 17.33. Compactness can imply sequential compactness if the space is first countable.

Proof.

Assume that any open cover has a finite subcover.

We are to prove that any sequence has a convergent subsequence.

Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in (X, d) .

Define $F_n := cl(\{x_i\}_{i \geq n})$ for each $n \in \mathbb{N}$.

Define $U_n := X \setminus F_n$ for each $n \in \mathbb{N}$.

Define $\mathcal{U} := \{U_n\}_{n \in \mathbb{N}}$.

Assume for the sake of contradiction that $\bigcup_{n \in \mathbb{N}} U_n = X$.

Since $\{U_n\}_{n \in \mathbb{N}}$ is an open cover of X and any open cover of X has a finite subcover, there exists a finite index set I such that $\{U_n\}_{n \in I}$ covers X .

Since $\{U_n\}_{n \in I}$ covers X , $\bigcap_{n \in I} F_n = \emptyset$.

Define $i_0 := \max(I)$.

Since $\{F_n\}_{n \in I}$ is decreasing, $\bigcap_{n \in I} F_n = F_{i_0}$.

Since $\bigcap_{n \in I} F_n = \emptyset$ and $\bigcap_{n \in I} F_n = F_{i_0}$, $F_{i_0} = \emptyset$.

This contradicts to the assumption that $F_n = cl(\{x_i\}_{i \geq n})$.

So the assumption that $\bigcup_{n \in \mathbb{N}} U_n = X$ is false.

i.e., $\bigcup_{n \in \mathbb{N}} U_n \neq X$.

Since $U_n \subseteq X$ for any $n \in \mathbb{N}$, $\bigcup_{n \in \mathbb{N}} U_n \subseteq X$.

Since $\bigcup_{n \in \mathbb{N}} U_n \subseteq X$ and $\bigcup_{n \in \mathbb{N}} U_n \neq X$, there exists a point x_0 in $X \setminus \bigcup_{n \in \mathbb{N}} U_n$.

Since $x_0 \in X \setminus \bigcup_{n \in \mathbb{N}} U_n$ and $X \setminus \bigcup_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} F_n$, $x_0 \in \bigcap_{n \in \mathbb{N}} F_n$.

□

REMARK 17.34. Compactness does not imply sequential compactness in general.

PROPOSITION 17.35. Sequential compactness can imply compactness if the space is metrizable.

Proof.

Let (X, d) be a metrizable topological space.

Assume that any sequence has a convergent subsequence.

We are to prove that any open cover has a finite subcover.

Let \mathcal{U} be an arbitrary open cover of the space.

Say $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ where Λ is an index set and U_λ is open for each $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} U_\lambda = X$.

Assume for the sake of contradiction that \mathcal{U} does not have a finite subcover.

Let x_1 be some point in the space.

Since $\{x_1\}$ is closed, $X \setminus \{x_1\}$ is open.

Since

□

REMARK 17.36. I do not know if the statement is still true if the condition "the space is metrizable" is replaced with a weaker one. But this is at least true for metric spaces.

PROPOSITION 17.37. Sequential compactness implies countable compactness.

Proof.

Assume that every sequence has a convergent subsequence.

We are to prove that every countable open cover has a finite subcover.

Let \mathcal{U} be an arbitrary countable open cover of the space.

Say $\mathcal{U} = \{U_i\}_{i \in \mathbb{N}}$ where each U_i is an open set in the space and $\bigcup_{i \in \mathbb{N}} U_i = X$.

Assume for the sake of contradiction that \mathcal{U} has no finite subcover.

Since $\{U_i\}_{i \leq n}$ is a finite subcollection of \mathcal{U} and \mathcal{U} has no finite subcover, in particular, $\{U_i\}_{i \leq n}$ does not cover the whole space.

Since $\{U_i\}_{i \leq n}$ does not cover the whole space, take $x_n \in X \setminus \bigcup_{i \leq n} U_i$.

Define $\mathfrak{x} := \{x_n\}_{n \in \mathbb{N}}$.

Since \mathfrak{x} is a sequence and any sequence has a convergent subsequence, in particular, \mathfrak{x} has a convergent subsequence \mathfrak{x}' .

Say \mathfrak{x} converges to a point x_∞ .

Since \mathcal{U} covers the whole space, there exists an index $i_\infty \in \mathbb{N}$ such that $x_\infty \in U_{i_\infty}$.

Since \mathfrak{x} converges to x_∞ and $x_\infty \in U_{i_\infty}$, U_{i_∞} contains a tail of \mathfrak{x} .

Since $x_n \in X \setminus \bigcup_{i \leq n} U_i$ for all n , $x_n \notin U_{i_\infty}$ for all $n > i_\infty$.

This contradicts to the fact that U_{i_∞} contains a tail of \mathfrak{x} .

So the assumption that \mathcal{U} has no finite subcover is false.

i.e., \mathcal{U} has a finite subcover.

Since any countable open cover has a finite subcover, by definition of countable compactness, the space is countably compact.

□

PROPOSITION 17.38. Countable compactness can imply sequential compactness if the space is metrizable.

Proof.

Assume that any countable open cover has a finite subcover.

We are to prove that any sequence has a convergent subsequence.

Let \mathfrak{x} be an arbitrary sequence in the space.

Say $\mathfrak{x} = \{x_i\}_{i \in \mathbb{N}}$ where x_i is a point in the space for each $i \in \mathbb{N}$.

Define $U_i := X \setminus \{x_i\}$.

Define $\mathcal{U} := \{U_i\}_{i \in \mathbb{N}}$.

Since X is open and $\{x_i\}$ is closed, $X \setminus \{x_i\}$ is open.

Since $U_i = X \setminus \{x_i\}$ for each $i \in \mathbb{N}$, $X = \bigcup_{i \in \mathbb{N}} U_i$.

Since U_i is open and $X = \bigcup_{i \in \mathbb{N}} U_i$ and \mathbb{N} is countable, \mathcal{U} is a countable open cover of the space.

Since \mathcal{U} is a countable open cover of the space and any countable open cover of the space has a finite subcover, in particular, \mathcal{U} has a finite subcover \mathcal{U}' .

Say $\mathcal{U}' = \{U_i\}_{i \in I}$ where I is a finite subset of \mathbb{N} .

□

17.7 Locally Compact

DEFINITION 17.39 (Locally Compact). Let (\mathfrak{X}, τ) be a topological space. We say that \mathfrak{X} is **locally compact** if each point in \mathfrak{X} admits a neighborhood base consisting of only compact sets.

Chapter 18

Connectedness

18.1 Definitions

DEFINITION 18.1 (Separation). Let (X, τ) be a topological space. Let S be a subset of X . Let S_1 and S_2 be two sets. We say the pair (S_1, S_2) is a **separation** of S if all of the following conditions hold.

1. S_1 and S_2 are both subsets of S .
2. S_1 and S_2 are both non-empty.
3. S_1 and S_2 are both open.
4. $S_1 \cap S_2 = \emptyset$.
5. $S_1 \cup S_2 = S$.

DEFINITION 18.2 (Disconnected). Let (X, τ) be a topological space. We say X is **disconnected** if there exists a separation of X .

DEFINITION 18.3 (Connected). Let (X, τ) be a topological space. We say X is **connected** if it is not disconnected.

18.2 Properties

PROPOSITION 18.4. A space X is connected if and only if the only subsets of X that are both open and closed are \emptyset and X .

Proof. For one direction, assume that X is connected. We are to prove that the only subsets of X that are both open and closed are \emptyset and X .

Assume for the sake of contradiction that there exists a non-empty proper subset S_0 of X that is both open and closed.

Since S_0 is a non-empty proper subset of X , both S_0 and $C_X(S_0)$ are non-empty.

Since S_0 is closed, $C_X(S_0)$ is open.

Since S_0 and $C_X(S_0)$ are both non-empty and open, $S_0 \cap C_X(S_0) = \emptyset$, and $S_0 \cup C_X(S_0) = X$, we get the pair $(S_0, C_X(S_0))$ is a separation of X .

By definition, we conclude that X is disconnected.

This contradicts to the fact that X is connected.

Thus the only subsets of X that are both open and closed are \emptyset and X .

For the reverse direction, assume that the only subsets of X that are both open and closed are \emptyset and X . We are to prove that X is connected.

Assume for the sake of contradiction that there exists a pair (S_1, S_2) of non-empty proper open subsets of X such that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = X$.

Since S_2 is open, $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2 = X$, we get S_1 is closed.

This contradicts to the fact that the only subsets of X that are both open and closed are \emptyset and X .

Thus X is connected. □

PROPOSITION 18.5. Let (X, d) be a connected metric space. The only subsets of X that has empty boundaries are X and \emptyset .

Proof. Assume for the sake of contradiction that there exists a non-empty proper-subset S_0 of X such that the boundary of S_0 is empty.

Since $\partial(S_0) = \emptyset$, S_0 is both open and closed.

This contradicts to the fact that the only subsets of X that are both open and closed are X and \emptyset .

It follows that the only subsets of X that has empty boundaries are X and \emptyset . □

PROPOSITION 18.6. Let X be a disconnected topological space. Let S_1 and S_2 be a separation of X . Let Y be a connected subset of X . Then either $Y \subseteq S_1$ or $Y \subseteq S_2$.

Proof. Assume for the sake of contradiction that $Y \not\subseteq S_1$ and $Y \not\subseteq S_2$.

Since $Y \not\subseteq S_1$, there exists a point x_1 such that $x_1 \in Y$ and $x_1 \notin S_1$.

Since $x_1 \notin S_1$, $x_1 \in X$, and $S_1 \cup S_2 = X$, we have $x_1 \in S_2$.

Since $x_1 \in Y$ and $x_1 \in S_2$, we have $x_1 \in Y \cap S_2$.

Since $x_1 \in Y \cap S_2$, $Y \cap S_2$ is not empty.

Since $Y \not\subseteq S_2$, there exists a point x_2 such that $x_2 \in Y$ and $x_2 \notin S_2$.

Since $x_2 \notin S_2$, $x_2 \in X$, and $S_1 \cup S_2 = X$, we have $x_2 \in S_1$.

Since $x_2 \in Y$ and $x_2 \in S_1$, we have $x_2 \in Y \cap S_1$.

Since $x_2 \in Y \cap S_1$, $Y \cap S_1$ is not empty.

Since S_1 is open in X and $Y \subseteq X$, we have $Y \cap S_1$ is open in Y .

Since S_2 is open in X and $Y \subseteq X$, we have $Y \cap S_2$ is open in Y .

Since $S_1 \cap S_2 = \emptyset$, $(Y \cap S_1) \cap (Y \cap S_2) = \emptyset$.

Since $S_1 \cup S_2 = X$ and $Y \subseteq X$, we have $(Y \cap S_1) \cup (Y \cap S_2) = Y$.

Since $Y \cap S_1$ and $Y \cap S_2$ are both not empty, open in Y , $(Y \cap S_1) \cap (Y \cap S_2) = \emptyset$, and $(Y \cap S_1) \cup (Y \cap S_2) = Y$, we have $Y \cap S_1$ and $Y \cap S_2$ is a separation of Y .

Since $Y \cap S_1$ and $Y \cap S_2$ is a separation of Y , by definition, Y is disconnected.

This contradicts to the assumption that Y is connected.

Thus the assumption that $Y \not\subseteq S_1$ and $Y \not\subseteq S_2$ is false.

i.e., $Y \subseteq S_1$ or $Y \subseteq S_2$.

□

18.3 Sufficient Conditions

PROPOSITION 18.7. Let S be a connected topological space. Let T be a set such that $S \subseteq T \subseteq \text{cl}(S)$. Then T is also connected.

Proof. Assume for the sake of contradiction that there exists a separation (T_1, T_2) of T .

Since (T_1, T_2) is a separation of T and $S \subseteq T$ and S is connected, either $S \subseteq T_1$ or $S \subseteq T_2$.

Assume without loss of generality that $S \subseteq T_1$.

Since $S \subseteq T_1$, by properties of the closure operator, $\text{cl}(S) \subseteq \text{cl}(T_1)$.

Since T_1 and T_2 are both open and $T_1 \cap T_2 = \emptyset$, $\text{cl}(T_1) \cap T_2 = \emptyset$.

Since $\text{cl}(S) \subseteq \text{cl}(T_1)$ and $\text{cl}(T_1) \cap T_2 = \emptyset$, $\text{cl}(S) \cap T_2 = \emptyset$.

Since $T \subseteq \text{cl}(S)$ and $\text{cl}(S) \cap T_2 = \emptyset$, $T \cap T_2 = \emptyset$.

This contradicts to the fact that T_2 is non-empty.

Thus T is connected.

□

COROLLARY 18.8. The closure of a connected space is also connected.

PROPOSITION 18.9 (Set Operations). Let X be a topological space. Let $\{S_i\}_{i \in \mathbb{N}}$ be a sequence of connected subspaces of X with $S_i \cap S_{i+1} \neq \emptyset$ for each i . Then the set $\bigcup_{i \in \mathbb{N}} S_i$ is also connected.

Proof. Let U denote the set $\bigcup_{i \in \mathbb{N}} S_i$. We are to prove that U is connected.

Assume for the sake of contradiction that there exists a separation (U_1, U_2) of U .

Since each S_i is a subset of U and U is separated by (U_1, U_2) , either $S_i \subseteq U_1$ or $S_i \subseteq U_2$.

Assume for the sake of contradiction that either for any index i in \mathbb{N} , both S_i and S_{i+1} lie in U_1 , or for any $i \in \mathbb{N}$, both S_i and S_{i+1} lie in U_2 .

Assume without loss of generality that for any index i in \mathbb{N} , both S_i and S_{i+1} lie in U_1 .

Since for any index i in \mathbb{N} , both S_i and S_{i+1} lie in U_1 , all of $\{S_i\}_{i \in \mathbb{N}}$ lie in U_1 .

Since all of $\{S_i\}_{i \in \mathbb{N}}$ lie in U_1 , $U \subseteq U_1$.

Since $U \subseteq U_1$ and $U_1 \subseteq U$, $U_1 = U$.

Since $U_1 = U$ and $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = U$, $U_2 = \emptyset$.

This contradicts to the fact that U_1 is non-empty.

Thus there exists an index i_0 in \mathbb{N} such that either $S_{i_0} \subseteq U_1$ and $S_{i_0+1} \subseteq U_2$ or $S_{i_0} \subseteq U_2$ and $S_{i_0+1} \subseteq U_1$.

Assume without loss of generality that $S_{i_0} \subseteq U_1$ and $S_{i_0+1} \subseteq U_2$.

Since $S_{i_0} \subseteq U_1$ and $S_{i_0+1} \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$, $S_{i_0} \cap S_{i_0+1} = \emptyset$.

This contradicts to the fact that for any index i in \mathbb{N} , $S_i \cap S_{i+1} \neq \emptyset$.

Thus U is connected.

□

PROPOSITION 18.10 (Set Operations). Let X be a topological space. Let $\{S_i\}_{i \in I}$ where I is an index set be a collection of connected subspaces of X such that there

exists a point x_0 in X such that $x_0 \in S_i$ for each i . Then the set $\bigcup_{i \in I} S_i$ is also connected.

Proof. Let U denote the set $\bigcup_{i \in I} S_i$. We are to prove that U is also connected.

Assume for the sake of contradiction that there exists a separation (U_1, U_2) of U .

Since $x_0 \in U$ and $U = U_1 \cup U_2$, either $x_0 \in U_1$ or $x_0 \in U_2$.

Assume without loss of generality that $x_0 \in U_1$.

Since $x_0 \in U_1$ and $U_1 \cap U_2 = \emptyset$, $x_0 \notin U_2$.

Since each S_i is a subset of U and U is separated by (U_1, U_2) , either $S_i \subseteq U_1$ or $S_i \subseteq U_2$.

Since U_2 is non-empty, there exists a set S_0 in $\{S_i\}_{i \in I}$ such that $S_0 \subseteq U_2$.

Since $x_0 \notin U_2$ and $S_0 \subseteq U_2$, $x_0 \notin S_0$.

This contradicts to the assumption that $x_0 \in S_0$.

Thus U is connected.

□

PROPOSITION 18.11 (Set Operations). Let X be a topological space. Let $\{S_i\}_{i \in I}$ where I is an index set be a collection of connected sets such that there exists a set S such that $S \cap S_i \neq \emptyset$ for any i . Then the set $S \cup (\bigcup_{i \in I} S_i)$ is also connected.

PROPOSITION 18.12 (Continuous Maps). The continuous image of a connected set is also connected.

Comment.

Continuity only guarantees open-ness. The other parts follows from basic set theory and are trivial.

Proof. Let X and Y be topological spaces and f be a function from X to Y .

Let S be a connected subset of X . We are to prove that $f(S)$ is also connected.

Assume for the sake of contradiction that $f(S)$ is disconnected.

Since $f(S)$ is disconnected, there exists a separation (T_1, T_2) of $f(S)$.

Since T_1 is non-empty, $f^{-1}(T_1)$ is non-empty.

Since T_2 is non-empty, $f^{-1}(T_2)$ is non-empty.

Since T_1 is open and f is continuous, $f^{-1}(T_1)$ is open.

Since T_2 is open and f is continuous, $f^{-1}(T_2)$ is open.

Since $T_1 \cap T_2 \neq \emptyset$, from set theory we know that $f^{-1}(T_1) \cap f^{-1}(T_2) \neq \emptyset$.

Since $f(S) \subseteq T_1 \cup T_2$, from set theory we know that $S \subseteq f^{-1}(T_1) \cup f^{-1}(T_2)$.

Since $f^{-1}(T_1)$ and $f^{-1}(T_2)$ are both non-empty and open, $f^{-1}(T_1) \cap f^{-1}(T_2) = \emptyset$, and $S \subseteq f^{-1}(T_1) \cup f^{-1}(T_2)$, we get $(f^{-1}(T_1), f^{-1}(T_2))$ is a separation of S .

Since $(f^{-1}(T_1), f^{-1}(T_2))$ is a separation of S , by definition, S is disconnected.

This contradicts to the fact that S is connected.

Thus $f(S)$ is connected.

□

PROPOSITION 18.13 (Cartesian Product). The cartesian product of a finite collection of connected sets is also connected.

18.4 Connected Components

DEFINITION 18.14 (Connected Component). Let (X, τ) be a topological space. We define the **connected components** of (X, τ) to be the maximal connected subsets of (X, τ) .

PROPOSITION 18.15. The connected components of a space form a partition of it.

PROPOSITION 18.16. Connected components are closed.

PROPOSITION 18.17. If the number of connected components of a space is finite, then each component is open.

18.5 Connectedness on the Real Line

PROPOSITION 18.18. Intervals are connected.

THEOREM 18.19 (Intermediate Value Theorem). Let $f : X \rightarrow Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

18.6 Pathwise Connectedness

DEFINITION 18.20 (Pathwise Connectedness). Let (X, d) be a metric space. We say that X is **pathwise connected** if and only if $\forall x, y \in X, \exists f : [0, 1] \rightarrow X$ continuous such that $f(0) = x$ and $f(1) = y$.

PROPOSITION 18.21. Pathwise connected spaces are connected.

PROPOSITION 18.22 (Symmetry). If there exists a path from x to y , then there exists a path from y to x .

Proof. Say p is the path from x to y . i.e., we have p is continuous and $p(0) = x$ and $p(1) = y$. Define p' by $p'(x) = p(1 - x)$. Since the function φ given by $\varphi(x) = 1 - x$ is continuous and p is continuous, $p'(x)$ is continuous. Further, $p'(0) = p(1) = y$ and $p'(1) = p(0) = x$. So p' is a path from y to x . \square

PROPOSITION 18.23 (Transitivity). The combination of two paths is also a path. i.e., if there exists a path p_1 from x to y and another path p_2 from y to z , then there exists a path p from x to z .

Proof. By definition of path, we have $p_1(0) = x, p_1(1) = y, p_2(0) = y$, and $p_2(1) = z$. Define p by $p(x) = p_1(2x)$ if $0 \leq x \leq 1/2$ and $p(x) = p_2(2x - 1)$ if $1/2 \leq x \leq 1$. Notice that at point $x = 1/2$, $p_1(2x) = p_1(1) = y = p_2(2x - 1) = p_2(0) = y$. Since φ_1 given by $\varphi_1(x) = 2x$ is continuous, φ_2 given by $\varphi_2(x) = 2x - 1$ is continuous, and p_1 and p_2 are continuous, p is continuous on both $[0, 1/2]$ and $[1/2, 1]$. Since the value of p given by the two pieces agree at the overlapping point, p is continuous. \square

18.7 Totally Disconnectedness

DEFINITION 18.24 (Totally Disconnected). Let (X, τ) be a topological space. We say that (X, τ) is **totally disconnected** if all connected components of (X, τ) are singleton sets.

PROPOSITION 18.25. Subspaces of a totally disconnected space are totally disconnected.

PROPOSITION 18.26. Cartesian products of totally disconnected spaces are totally disconnected.

PROPOSITION 18.27. Countable metric spaces are totally disconnected.

Chapter 19

Baire Space

19.1 Definitions

DEFINITION 19.1 (Baire Space). Let (X, τ) be a topological space. We say that (X, τ) is a **Baire Space** if any nonempty open set in the space is not meager.

DEFINITION 19.2 (Baire Space). Let (X, τ) be a topological space. We say that (X, τ) is a **Baire Space** if any comeager set in the space is dense.

DEFINITION 19.3 (Baire Space). Let (X, τ) be a topological space. We say that (X, τ) is a **Baire Space** if the intersection of any countable collection of dense open sets is again dense.

DEFINITION 19.4 (Baire Space). Let (X, τ) be a topological space. We say that (X, τ) is a **Baire Space** if the union of any countable collection of closed nowhere dense sets has empty interior.

PROPOSITION 19.5. The four definitions of Baire space are equivalent.

19.2 Properties

PROPOSITION 19.6. An open subspace of a Baire space is again Baire.

REMARK 19.7. A closed subspace of a Baire space is not necessarily Baire.

Chapter 20

Uniform Spaces

20.1 Entourages

DEFINITION 20.1 (Uniform Structure, Entourages). Let X be a set. Let Φ be a collection of subsets of $X \times X$. We say that Φ is a **uniform structure** and the elements of Φ are **entourages** if Φ satisfies all of the following conditions.

- If $U \in \Phi$, then $\Delta \subseteq U$ where $\Delta := \{(x, x) : x \in X\}$ is the diagonal on $X \times X$.
- If $U \in \Phi$ and V is a set such that $U \subseteq V \subseteq X \times X$, then $V \in \Phi$.
- $\forall U, V \in \Phi, \quad U \cap V \in \Phi$.
- $\forall U \in \Phi, \quad \exists V \in \Phi, \quad V \circ V \subseteq U$ where \circ is a function given by

$$S \circ T := \{(x, z) : \exists y \in X \text{ such that } (x, z) \in T \text{ and } (z, y) \in S\}.$$

- $\forall U \in \Phi, \quad U^{-1} \in \Phi$ where $U^{-1} := \{(y, x) : (x, y) \in U\}$.

Chapter 21

Uniform Boundedness

21.1 The Uniform Boundedness Principle

THEOREM 21.1 (The Uniform Boundedness Principle - Metric Space). Let (X, d) be a complete metric space. Let H be a non-empty family of continuous functionals on X . Suppose that $\forall x \in X, \sup_{h \in H} |h(x)| < +\infty$. Then there is some open set G and some constant M such that

$$\forall x \in G, \quad \sup_{h \in H} |h(x)| \leq M.$$

Chapter 22

Equicontinuity in Metric Spaces

22.1 Definitions

DEFINITION 22.1 ((Pointwise) Equicontinuity). Let (X, d_X) and (Y, d_Y) be metric spaces. Let \mathcal{F} be a collection of functions from X to Y . Let x_0 be a point in X . We say that \mathcal{F} is *(pointwise) equicontinuous* at point x_0 if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall f \in \mathcal{F}, \forall x \in X$ such that $d_X(x, x_0) < \delta$, we have $d_Y(f(x), f(x_0)) < \varepsilon$.

DEFINITION 22.2 (Uniform Equicontinuity). Let (X, d_X) and (Y, d_Y) be metric spaces. Let \mathcal{F} be a collection of functions from X to Y . We say that \mathcal{F} is *uniformly equicontinuous* if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall f \in \mathcal{F}, \forall x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2)) < \varepsilon$.

22.2 Sufficient Conditions

PROPOSITION 22.3. The closure of an equicontinuous family of functions is equicontinuous.

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces. Let \mathcal{F} be an equicontinuous family of functions from X to Y . We are to prove that $cl(\mathcal{F})$ is equicontinuous. Let x_0 be an arbitrary point in X . Let ε be an arbitrary positive number. Since \mathcal{F} is equicontinuous at point x_0 , there exists some $\delta(x_0, \varepsilon)$ such that for any function f in \mathcal{F} and any point x in

X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon/3$. Let f be an arbitrary function in $cl(\mathcal{F})$. Let x be an arbitrary point in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$. Since $f \in cl(\mathcal{F})$, there exists some function $f_0 \in \mathcal{F}$ such that $d_\infty(f, f_0) < \varepsilon/3$. Since $d_\infty(f, f_0) < \varepsilon/3$, $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$. Since $f_0 \in \mathcal{F}$ and $d_X(x, x_0) < \delta(x_0, \varepsilon)$, $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$. Since $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$ and $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$, $d_Y(f(x), f(x_0)) < \varepsilon$. Since for any positive number ε , there exists some $\delta(x_0, \varepsilon)$ such that for any function f in $cl(\mathcal{F})$ and any point x in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, by definition of equicontinuous, $cl(\mathcal{F})$ is equicontinuous at point x_0 . Since $cl(\mathcal{F})$ is equicontinuous at point x_0 for any point x_0 in X , $cl(\mathcal{F})$ is equicontinuous. \square

Chapter 23

Fixed Point Theory

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23.1 Definitions

DEFINITION 23.1 (Contraction Maps). Let (\mathfrak{X}, d) be a metric space. Let $f : \mathfrak{X} \rightarrow \mathfrak{X}$. We say that f is a **contraction map** if and only if $\exists c \in [0, 1)$ such that

$$\forall x_1, x_2 \in \mathfrak{X}, \quad d(f(x_1), f(x_2)) \leq c \cdot d(x_1, x_2).$$

DEFINITION 23.2 (Non-Expansive Operator). Let (\mathfrak{X}, d) be a metric space. Let $f : \mathfrak{X} \rightarrow \mathfrak{X}$. We say that f is **non-expansive** if and only if

$$\forall x_1, x_2 \in \mathfrak{X}, \quad d(f(x_1), f(x_2)) \leq d(x_1, x_2).$$

Or equivalently, $\exists c \in [0, 1]$ such that

$$\forall x_1, x_2 \in \mathfrak{X}, \quad d(f(x_1), f(x_2)) \leq c \cdot d(x_1, x_2).$$

DEFINITION 23.3 (Firmly Non-Expansive Operator). Let \mathcal{H} be a Hilbert space. Let f be a function from \mathcal{H} to \mathcal{H} . We say that f is **firmly non-expansive** if

$$\forall x, y \in \mathcal{H}, \quad \langle f(y) - f(x), f(y) - f(x) \rangle \leq \langle y - x, f(y) - f(x) \rangle.$$

Or equivalently,

$$\forall x, y \in \mathcal{H}, \quad \|f(y) - f(x)\|^2 \leq \langle y - x, f(y) - f(x) \rangle.$$

DEFINITION 23.4 (Firmly Non-Expansive Operator). Let \mathcal{H} be a Hilbert space. Let f be a function from \mathcal{H} to \mathcal{H} . We say that f is **firmly non-expansive** if

$$\forall x, y \in \mathcal{H}, \quad \|f(y) - f(x)\|^2 + \|(I - f)(y) - (I - f)(x)\|^2 \leq \|y - x\|^2.$$

DEFINITION 23.5 (Firmly Non-Expansive Operator). Let \mathcal{H} be a Hilbert space. Let f be a function from \mathcal{H} to \mathcal{H} . We say that f is **firmly non-expansive** if

$$\forall x, y \in \mathcal{H}, \quad \langle f(y) - f(x), (I - f)(y) - (I - f)(x) \rangle \geq 0.$$

DEFINITION 23.6 (Averaged Non-Expansive Operator). Let \mathcal{H} be a Hilbert space. Let f be a function from \mathcal{H} to \mathcal{H} . We say that f is a **α -averaged non-expansive operator** if $\exists \alpha \in (0, 1)$ and \exists non-expansive operator N from \mathcal{H} to \mathcal{H} such that

$$f = \alpha N + (1 - \alpha)I$$

where I denoted the identity operator from \mathcal{H} to \mathcal{H} .

23.2 Properties

PROPOSITION 23.7. Contraction mappings are uniformly continuous.

PROPOSITION 23.8. Firmly non-expansive operators are special cases of averaged non-expansive operators when $\alpha = \frac{1}{2}$.

PROPOSITION 23.9. Averaged non-expansive operators are non-expansive.

Proof. Let T be an α -average of a non-expansive operator N . i.e. $T = (1 - \alpha)\text{Id} + \alpha N$. Let x and y be arbitrary points in \mathcal{H} . Then

$$\begin{aligned}
 & \|Tx - Ty\| \\
 &= \|[(1 - \alpha)\text{Id} + \alpha N]x - [(1 - \alpha)\text{Id} + \alpha N]y\| \\
 &= \|[(1 - \alpha)x + \alpha Nx] - [(1 - \alpha)y + \alpha Ny]\| \\
 &= \|(1 - \alpha)(x - y) + \alpha(Nx - Ny)\| \\
 &\leq \|(1 - \alpha)(x - y)\| + \|\alpha(Nx - Ny)\| \\
 &= (1 - \alpha)\|x - y\| + \alpha\|Nx - Ny\| \\
 &\leq (1 - \alpha)\|x - y\| + \alpha\|x - y\| \\
 &\leq \|x - y\|.
 \end{aligned}$$

That is,

$$\forall x, y \in \mathcal{H}, \quad \|Tx - Ty\| \leq \|x - y\|.$$

So by definition, T is non-expansive. □

PROPOSITION 23.10. If f is linear, then f being firmly non-expansive is equivalent to the followings.

- $\forall x \in \mathbb{R}^d, \|fx\|^2 \leq \langle x, fx \rangle$.
- $\forall x \in \mathbb{R}^d, \langle fx, x - fx \rangle \geq 0$.

23.3 Algebra & Stability of Non-Expansiveness

PROPOSITION 23.11. f is firmly non-expansive if and only if $I - f$ is firmly non-expansive.

PROPOSITION 23.12. Let M be a linear operator from \mathbb{R}^d to \mathbb{R}^d . Then M is non-expansive if and only if M^T is non-expansive.

PROPOSITION 23.13 (Convex Combination). The class of firmly non-expansive maps is closed under convex combinations.

PROPOSITION 23.14 (Composition). Let f_1 be an α_1 -averaged non-expansive operator. Let f_2 be an α_2 -averaged non-expansive operator. Define an operator f by $f := f_1 f_2$. Then f is also averaged non-expansive with “rate”

$$\alpha_f = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2}.$$

23.4 Fixed Point Theorems

THEOREM 23.15 (Banach Fixed Point Theorem (1922)). Let (\mathfrak{X}, d) be a complete metric space. Let \mathcal{U} be a nonempty closed subset of \mathfrak{X} . Let $f : \mathcal{U} \rightarrow \mathcal{U}$ be a contraction mapping with constant $\lambda \in [0, 1)$. Then f has a unique fixed point.

Proof. Take $x_0 \in \mathcal{U}$. Define for each $i \in \mathbb{Z}_+$ a point $x_{i+1} \in \mathcal{U}$ by $x_{i+1} := f(x_i)$. Consider the sequence $(x_i)_{i \in \mathbb{Z}_{++}}$.

Now I claim that $(x_i)_{i \in \mathbb{Z}_{++}}$ is Cauchy and hence converges in \mathcal{U} . Notice $\forall i \in \mathbb{Z}_+$, we have

$$\begin{aligned} d(x_i, x_{i+1}) &= d(f(x_{i-1}), f(x_i)) \leq \lambda d(x_{i-1}, x_i), \text{ since } f \text{ is a contraction} \\ &\leq \lambda^i d(x_0, x_1), \text{ by induction on } i. \end{aligned}$$

Consider arbitrary $m, n \in \mathbb{Z}_{++}$ with $n > m$. Then we have

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=m}^{n-1} d(x_i, x_{i+1}), \text{ by the Triangle Inequality} \\ &\leq \sum_{i=m}^{n-1} \lambda^i d(x_0, x_1), \text{ by above} \\ &= \frac{\lambda^m(1 - \lambda^{m-n})}{1 - \lambda} d(x_0, x_1) = \frac{\lambda^m - \lambda^n}{1 - \lambda} d(x_0, x_1) \\ &\leq \lambda^m \left[\frac{1}{1 - \lambda} d(x_0, x_1) \right]. \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Then $\exists N \in \mathbb{Z}_+$ such that $\forall m > N$, we have $\lambda^m \left[\frac{1}{1 - \lambda} d(x_0, x_1) \right] < \varepsilon$. So $\forall m, n > N$ with $n > m$, we have $d(x_m, x_n) < \varepsilon$. So $(x_i)_{i \in \mathbb{Z}_{++}}$ is Cauchy. Since $(x_i)_{i \in \mathbb{Z}_{++}}$ is Cauchy and (\mathfrak{X}, d) is complete, we have that $(x_i)_{i \in \mathbb{Z}_{++}}$ converges. Let $\bar{x} := \lim_{i \in \mathbb{Z}_+} x_i$. Since $(x_i)_{i \in \mathbb{Z}_+} \subseteq \mathcal{U}$ and $\lim_{i \in \mathbb{Z}_+} x_i = \bar{x}$ and \mathcal{U} is closed, we have that $\bar{x} \in \mathcal{U}$.

Now I claim that \bar{x} is a fixed point of f . Consider an arbitrary $i \in \mathbb{Z}_+$. Then

$$\begin{aligned} d(f(\bar{x}), \bar{x}) &\leq d(f(\bar{x}), x_{i+1}) + d(x_{i+1}, \bar{x}), \text{ by the Triangle Inequality} \\ &= d(f(\bar{x}), f(x_i)) + d(x_{i+1}, \bar{x}) \\ &\leq \lambda d(\bar{x}, x_i) + d(x_{i+1}, \bar{x}), \text{ since } f \text{ is a contraction.} \end{aligned}$$

Note that since $\lim_{i \in \mathbb{Z}_+} x_i = \bar{x}$, the RHS converges to 0 as $i \rightarrow +\infty$. So $d(f(\bar{x}), \bar{x}) = 0$. So $f(\bar{x}) = \bar{x}$. i.e., \bar{x} is a fixed point of f .

Now I claim that \bar{x} is unique. Suppose that $\bar{y} \in \mathcal{U}$ is such that $f(\bar{y}) = \bar{y}$. Then since f is a contraction, we have

$$d(\bar{x}, \bar{y}) = d(f(\bar{x}), f(\bar{y})) \leq \lambda d(\bar{x}, \bar{y}).$$

So $(1 - \lambda)d(\bar{x}, \bar{y}) \leq 0$. Notice $\lambda \in [0, 1)$. So this forces $d(\bar{x}, \bar{y}) = 0$. So $\bar{x} = \bar{y}$. So \bar{x} is unique. This completes the proof. \square

COROLLARY 23.16. Let $\text{ball}(0, r)$ denote an open ball of radius r in \mathbb{R}^n . Let $g : \text{ball}(0, r) \rightarrow \mathbb{R}^n$ be a map. Suppose that $g(0) = 0$ and $\exists c \in (0, 1)$ such that

$$\forall x, y \in \text{ball}(0, r), \quad \|g(y) - g(x)\| \leq c\|y - x\|.$$

Then $f := g + I$ is injective on $\text{ball}(0, r)$ and

$$\text{ball}(0, (1 - c)r) \subseteq f(\text{ball}(0, r)) \subseteq \text{ball}(0, (1 + c)r).$$

Proof. Part 1: Show that f is injective on $\text{ball}(0, r)$.

Let $x, y \in \text{ball}(0, r)$ be arbitrary. Suppose that $f(x) = f(y)$. That is, $g(x) + x = g(y) + y$. So $g(y) - g(x) = x - y$. So $\|g(y) - g(x)\| = \|y - x\|$. Since g is a contraction map with constant $c \in (0, 1)$, we must have $\|g(y) - g(x)\| = \|y - x\| = 0$. So $x = y$. So f is injective.

Part 2: Show that $\text{ball}(0, (1 - c)r) \subseteq f(\text{ball}(0, r))$. \square

THEOREM 23.17 (Brouwer's Fixed Point Theorem (1910)). Let \mathfrak{X} be a Banach space. Let \mathfrak{U} be a nonempty compact convex subset of \mathfrak{X} . Let $f : \mathfrak{U} \rightarrow \mathfrak{U}$ be a continuous surjective function. Then f has a fixed point.

23.5 Fejér Monotonic Sequences

DEFINITION 23.18 (Fejér Monotonic Sequences). Let \mathcal{H} be a real Hilbert space. Let S be a non-empty subset of \mathcal{H} . Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence in \mathcal{H} . We say that $\{x_i\}_{i \in \mathbb{N}}$ is **Fejér monotonic** with respect to S if $\forall p \in S$, the sequence $\{\|x_i - p\|\}_{i \in \mathbb{N}}$ is decreasing. i.e.

$$\forall p \in S, \forall i \in \mathbb{N}, \quad \|x_{i+1} - p\| \leq \|x_i - p\|.$$

PROPOSITION 23.19. Fejér monotonic sequences are bounded.

Proof.

Let \mathcal{H} be a real Hilbert space.

Let S be a non-empty subset of \mathcal{H} .

Let \mathfrak{x} be a Fejér monotonic sequence in \mathcal{H} with respect to S .

Let p be an arbitrary point in S .

$$\begin{aligned} \|x_i\| &\leq \|p\| + \|x_i - p\| \\ &\leq \|p\| + \|x_{i-1} - p\| \\ &\leq \dots \\ &\leq \|p\| + \|x_1 - p\|. \end{aligned}$$

Since $\forall i \in \mathbb{N}$, $\|x_i\| \leq \|p\| + \|x_1 - p\|$, we get $\{x_i\}_{i \in \mathbb{N}}$ is bounded. □

PROPOSITION 23.20. Let \mathcal{H} be a real Hilbert space. Let S be a non-empty subset of \mathcal{H} . Let $\{x_i\}_{i \in \mathbb{N}}$ be a Fejér monotonic sequence in \mathcal{H} with respect to S . Then $\forall p \in S$, the sequence $\{x_i - p\}_{i \in \mathbb{N}}$ converges.

Proof. $\|x_i - p\|_{i \in \mathbb{N}}$ is a decreasing sequence bounded below by 0. □