Functional Analysis

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Contents

Chapter 1

Balanced Sets

1.1 Definitions

Definition (Balanced Sets)**.** *Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We say that S is **balanced** if*

$$\forall a \in \mathbb{F} : |a| \leq 1, \quad aS \subseteq S.$$

Definition (Balanced Hull)**.** *Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We define the **balanced hull** of S, denoted by* $\operatorname{balhull}(S)$*, to be the smallest balanced set containing S.*

Definition (Balanced Core)**.** *Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We define the **balanced core** of S, denoted by* $\operatorname{balcore}(S)$*, to be the largest balanced set contained in S.*

1.2 Properties

Proposition 1.2.1. *Let X be a vector space over field $\mathbb{F}$. Let B be a balanced subset of X. Then*

$$\forall a, b \in \mathbb{F} : |a| \leq |b|, \quad aB \subseteq bB.$$

Proposition 1.2.2. *Balanced sets are path connected.*

Proposition 1.2.3 (Act on Other Properties)**.**

- *The balanced hull of a compact set is compact.*
- *The balanced hull of a totally bounded set is totally bounded.*
- *The balanced hull of a bounded set is bounded.*

Proposition 1.2.4 (Act on Other Properties).

- *The balanced core of a closed set is closed.*

Proposition 1.2.5. *Let X be a vector space over field $\mathbb{F}$. Let a be a scalar in field $\mathbb{F}$. Then*

$$a \operatorname{balhull}(S) = \operatorname{balhull}(aS).$$

1.3 Stability of Balance

Proposition 1.3.1 (Set Operations).

- *The union of balanced sets is also balanced.*
- *The intersection of balanced sets is also balanced.*

Proposition 1.3.2 (Linear Mappings).

- *The scalar multiple of a balanced set is also balanced.*
- *The (Minkowski) sum of two balanced sets is also balanced.*
- *The image of a balanced set under a linear operator is also balanced.*
- *The inverse image of a balanced set under a linear operator is also balanced.*

Proposition 1.3.3 (Topological Operations). *The closure of a balanced set is also balanced.*

Proposition 1.3.4. *The convex hull of a balanced set is also balanced (and also convex).*

1.4 Absorbing Sets

Definition (Absorbing Sets). *Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We say that S is* ***absorbing*** *if*

$$\forall x \in X, \quad \exists r \in \mathbb{R} : r > 0, \quad \forall c \in \mathbb{F} : |c| \geq r, \quad x \in cA.$$

Proposition 1.4.1. *Every absorbing set contains the origin.*

Chapter 2

Inner Product Spaces

2.1 Inner Products

2.1.1 Definitions

Definition (Inner Product). *Let V be a vector space over field $\mathbb{F}$. We define an* **inner product** *on V, denoted by $\langle \cdot, \cdot \rangle$, to be a scalar-valued function defined on $V \times V$ such that*

(1) Positive Definiteness

$$\begin{aligned} \forall x, y \in V, \quad & \langle x, x \rangle \geq 0, \text{ and} \\ \forall x \in V, \quad & \langle x, x \rangle = 0 \iff x = O_V. \end{aligned}$$

(2) Sesqui-Linearity

$$\begin{aligned} \forall x, y, z, w \in V, \quad & \langle x + y, z + w \rangle = \langle x, z \rangle + \langle y, z \rangle + \langle x, w \rangle + \langle y, w \rangle, \text{ and} \\ \forall a, b \in \mathbb{F}, \forall x, y \in V, \quad & \langle ax, by \rangle = a\bar{b}\langle x, y \rangle. \end{aligned}$$

(3) Conjugate Symmetry

$$\forall x, y \in V, \quad \langle x, y \rangle = \overline{\langle y, x \rangle}.$$

Definition (Norm). *Let V be an inner product space over field $\mathbb{F}$. We define the* ***norm****, denoted by $\| \cdot \|$, to be a function from V to $\mathbb{R}_+$ given by*

$$\|x\| := \sqrt{\langle x, x \rangle}$$

Definition (Orthogonal Vectors). *Let V be an inner product space. Let x and y be vectors in V. We say that x and y are* ***orthogonal*** *if $\langle x, y \rangle = 0$.*

Definition (Orthogonal Sets). *Let S be a subset of V. We say that S is* ***orthogonal*** *if*

$$\forall x, y \in S, \quad \langle x, y \rangle = 0.$$

2.1.2 Examples

Definition (Standard Inner Product)**.** *For $V = \mathbb{F}^n$, we define the* ***standard inner product*** *by*

$$\langle x, y\rangle := \sum_{i=1}^{n} x_i \overline{y_i}.$$

Definition (Frobenius Inner Product)**.** *For $V = \mathbb{F}^{n\times n}$, we define the* ***Frobenius inner product*** *by*

$$\langle M_1, M_2\rangle := \operatorname{tr}(M_2^* M_1).$$

Definition. *Let V be the space of continuous scalar-valued functions on $[0, 2\pi]$. We define the inner product on V by*

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx.$$

2.1.3 Properties

Proposition 2.1.1. *Let V be a finite dimensional inner product space. Let $\mathcal{B}$ be a basis for V. Let x and y be vectors in V. Then*

$$x = y \iff \forall b \in \mathcal{B}, \quad \langle x, b\rangle = \langle y, b\rangle.$$

2.2 Inequalities

Theorem 1 (Minkowski)**.**

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

Proposition 2.2.1 (Cauchy-Schwarz Inequality)**.** *Let V be an inner product space. Then*

$$\forall x, y \in V, \quad |\langle x, y\rangle| \le \|x\| \cdot \|y\|$$

Proposition 2.2.2 (Triangle Inequality)**.** *Let V be an inner product space. Then*

$$\forall x, y \in V, \quad \|x + y\| \le \|x\| + \|y\|$$

Proposition 2.2.3 (Parallelogram Law)**.** *Let V be an inner product space. Then*

$$\forall x, y \in V, \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

2.3 Orthogonality

2.3.1 Orthogonal Sets

Definition (Orthogonality)**.** *Let V be an inner product space. We say that points x and y in V are **orthogonal** if $\langle x, y \rangle = 0$.*

Definition (Orthogonal Sets)**.** *Let V be an inner product space and S be a subset of V. We say that S is **orthogonal** if any two vectors in S are orthogonal.*

Proposition 2.3.1. *Orthogonal sets are linearly independent.*

2.3.2 Orthogonal Bases

Definition (Orthogonal Basis)**.** *Let V be an inner product space and S be a subset of V. We say that S is an **orthogonal basis** for V if it is an ordered basis for V and orthogonal.*

Proposition 2.3.2. *Let V be an inner product space. Let $S = \{v_1, ..., v_n\}$ be an orthogonal subset of V where each v_i is non-zero. Then*

$$\forall y \in \operatorname{span}(S), \quad y = \sum_{i=1}^{n} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

Theorem 2 (Gram-Schmidt Process)**.** *Let V be an inner product space. Let $S = \{v_0, ..., v_n\}$ be a linearly independent subset of V. Then the set $S' = \{v_0', ..., v_n'\}$ given by $v_0' := v_0$ and*

$$\forall i \in \{1, ..., n\}, \quad v_i' := v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, v_j' \rangle}{\|v_j'\|} v_j'$$

is an orthogonal subset of V consisting of non-zero vectors. Furthermore, we have $\operatorname{span}(S') = \operatorname{span}(S)$.

Proposition 2.3.3. *Let V be an inner product space and $S = \{v_0, v_1, \ldots, v_n\}$ be an orthogonal subset of V. Then the set $S^{'}$ derived from the Gram-Schmidt process is exactly S.*

Theorem 3 (Parseval's Identity)**.** *Let V be a finite-dimensional inner product space. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthogonal basis for V. Then*

$$\forall x, y \in V, \quad \langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

Theorem 4 (Bessel's Inequality)**.** *Let V be a finite-dimensional inner product space. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthogonal subset for V. Then*

$$\forall x \in V, \quad \|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$

2.3.3 Orthogonal Complements

Definition (Orthogonal Complement)**.** *Let V be an inner product space and S be a non-empty subset of V. We define the* ***orthogonal complement*** *of S, denoted by $S^\perp$, to be the set of all points in V that are orthogonal to all vectors in S.*

Proposition 2.3.4. *Let V be a finite-dimensional inner product space. Then*

(1) $V^\perp = \{O_V\}$

(2) $\{O_V\}^\perp = V$

Proposition 2.3.5. *Orthogonal complements are always linear subspaces.*

Proposition 2.3.6. *Let V be an inner product space and W be a subspace of V with basis β. Then a vector in V is also in $W^\perp$ if and only if it is orthogonal to all vectors in β.*

Proposition 2.3.7 (Extension)**.** *Let V be an n-dimensional inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal subset of V. Then S can be extended to an orthogonal basis $B = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.*

2.3.4 Properties of the Orthogonal Complement Operator

Proposition 2.3.8. *Let V be an inner product space. Then*

(1) $S \subseteq T$ *implies* $T^\perp \subseteq S^\perp$ *for any subsets S and T of V.*

(2) $S \subseteq (S^\perp)^\perp$ *for any subset S of V.*

Proposition 2.3.9. *Let V be a finite-dimensional inner product space and W be a subspace of V. Then*

(1) $W = (W^\perp)^\perp$

(2) $V = W \oplus W^\perp$

Proposition 2.3.10. *Let V be a finite-dimensional inner product space and W_1 and W_2 be subspaces of V. Then*

(1) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

(2) $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

2.3.5 Orthogonal Projection

Definition (Orthogonal Projection)**.** *Let V be a vector space. Let W be a finite-dimensional subspace of V. Let x be a vector in V. We define the* ***orthogonal projection*** *of x on W, denoted by (x), to be the vector u in W such that $x = u + v$ where v is another vector in $W^\perp$.*

Chapter 3

Normed Linear Spaces

3.1 Definitions

Definition (Seminorm)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **seminorm** on $\mathfrak{X}$, denoted by ν, to be a map from $\mathfrak{X}$ to $\mathbb{R}$ that satisfies the following conditions.*

(1) $\forall x \in \mathfrak{X}, \quad \nu(x) \geq 0.$

(2) $\forall \lambda \in \mathbb{F}, \forall x \in \mathfrak{X}, \quad \nu(\lambda x) = \lambda \nu(x).$

(3) Triangle Inequality.

$$\forall x, y \in \mathfrak{X}, \quad \nu(x+y) \leq \nu(x) + \nu(y).$$

The idea behind the seminorm is that we are trying to give our vector space a notion of "length" of vectors.

Definition (Norm)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **norm** on $\mathfrak{X}$, denoted by ν, to be a seminorm on $\mathfrak{X}$ that satisfies the additional condition:*

$$\forall x \in \mathfrak{X}, \quad \mu(x) = 0 \iff x = 0.$$

3.2 Properties

Proposition 3.2.1. *Let $(V, \|\cdot\|_V)$ be a normed vector space over field $\mathbb{F}$. Then $(V, \|\cdot\|)$ is complete if and only if $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.*

Proof.
For one direction, assume that $(V, \|\cdot\|)$ is complete.
We are to prove that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

Since $(\overline{B(0,1)}, \|\cdot\|_V)$ is a closed subspace of $(V, \|\cdot\|)$ and $(V, \|\cdot\|)$ is complete, $(\overline{B(0,1)}, \|\cdot\|_V)$ is also complete.
For the reverse direction, assume that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.
We are to prove that $(V, \|\cdot\|_V)$ is complete.
Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $(V, \|\cdot\|_V)$.
Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ is bounded in $(V, \|\cdot\|_V)$.
Let λ be a positive upper bound for $\{\|x_i\|_V\}_{i\in\mathbb{N}}$.
Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$.
Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$ and $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$.
Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ converges in $(V, \|\cdot\|_V)$.
Since any Cauchy sequence in $(V, \|\cdot\|_V)$ converges in $(V, \|\cdot\|_V)$, $(V, \|\cdot\|_V)$ is complete.

■

3.3 Equivalence of Norms

Definition (Equivalence of Norms)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are* ***equivalent*** *if*

$$\exists c_1, c_2 > 0, \quad \forall v \in \mathfrak{X}, \quad c_1\|v\|_1 \le \|v\|_2 \le c_2\|v\|_2.$$

Or equivalently,

$$c_1\|v\|_2 \le \|v\|_1 \le c_2\|v\|_2.$$

Proposition 3.3.1. *The equivalence of norms is an equivalence relation.*

Theorem 5. *Let V be a finite dimensional vector space over field $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$. Then any two norms on V are equivalent.*

Proof.
Let $\|\cdot\|_p$ be an arbitrary p-norm on V and $\|\cdot\|$ be an arbitrary norm on V.
Let $\mathcal{B}$ be the standard basis for V. Say $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$.
Let v be an arbitrary vector in V.

$$\begin{aligned}
\|v\| &= \|\sum_{i=1}^{n} v_i e_i\| \\
&\le \sum_{i=1}^{n} |v_i|\|e_i\| \\
&\le \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}}
\end{aligned}$$

$$= \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \|v\|_p$$
$$:= c_1 \|v\|_p.$$

■

Proposition 3.3.2. *Let X be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if they generate the same metric topology.*

Proof. Convergence to 0 is equivalent under either $\|\cdot\|_1$ or $\|\cdot\|_2$. i.e., equivalent norms give rise to the same set of sequences that are convergent. Convergence of sequences defines the topology. ■

Proposition 3.3.3. *Let $\mathfrak{X}$ be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathfrak{X}$. Let ι be the identity map from $(\mathfrak{X}, \|\cdot\|_1)$ to $(\mathfrak{X}, \|\cdot\|_2)$. Then if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, iota is continuous, and in fact, a homeomorphism between $(\mathfrak{X}, \|\cdot\|_1)$ and $(\mathfrak{X}, \|\cdot\|_2)$.*

Proposition 3.3.4. *Let $(\mathfrak{X}, \|\cdot\|_1)$ be a Banach space. Let $S \in \mathcal{B}(\mathfrak{X})$ be a bounded linear map that is invertible. Define a norm $\|\cdot\|_2$ on $\mathfrak{X}$ as*

$$\|x\|_2 := \|Sx\|_1.$$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proof. On one hand, since S is bounded, $\exists c_1$ such that $\forall x \in \mathfrak{X}$, $\|Sx\|_1 \leq c_1\|x\|_1$. That is, $\|x\|_2 \leq c_1\|x\|_1$.
On the other hand, since S is invertible, S^{-1} exists and is also bounded. Since S^{-1} is bounded, $\exists c_2$ such that $\forall x \in \mathfrak{X}$, $\|S^{-1}x\|_1 \leq c_2\|x\|_1$. Consider $x = Sx$, we get $\forall x \in \mathfrak{X}$, $\|S^{-1}Sx\|_1 \leq c_2\|Sx\|_1$. That is, $\|x\|_1 \leq c_2\|x\|_2$.
So $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ■

3.4 Dual Norms

Definition (Dual Norm)**.** *Let $(V, \|\cdot\|)$ be an normed vector space. We define the* ***dual norm*** *of $\|\cdot\|$, denoted by $\|\cdot\|_\circ$, to be a function given by*

$$\|v\|_\circ := \max_{\|w\|=1} v \cdot w = \max_{\|w\|\neq 0} \frac{|v \cdot w|}{\|w\|}.$$

Proposition 3.4.1. *Dual norms of norms are indeed norms.*

Proposition 3.4.2. *Let $(V, \|\cdot\|)$ be a normed vector space. Let v, w be vectors in the space. Then*

$$|v \cdot w| \leq \|v\| \cdot \|w\|_\circ.$$

3.5 p-norms

Definition (p-norm). *Let V be a finite-dimensional normed vector space over field $\mathcal{F}$. Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis for V where $n = \dim(V)$. Let v be a vector in a normed vector space. For $p \in [1, +\infty)$, we define the **p-norm** of v, denoted by $\|v\|_p$, to be the number given by*

$$\|v\|_p = \left(\sum_{i=1}^{n} |(v_{\mathcal{B}})_i|^p \right)^{\frac{1}{p}}.$$

Definition (Infinity Norm - 1). *Let $\mathfrak{X} = \mathbb{K}^n$ where $\mathbb{K}$ is a field and $n \in \mathbb{N}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by*

$$\|v\|_\infty := \max\{|v_i|\}_{i=1}^{n}.$$

Definition (Infinity Norm - 2). *Let $\mathfrak{X} = \mathbb{K}^{\mathbb{N}}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by*

$$\|v\|_\infty := \sup_{i \in \mathbb{N}} |v_i|.$$

Definition (Infinity Norm - 3). *Let $\mathfrak{X} = \mathcal{C}([0,1], \mathbb{C})$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by*

$$\nu(f) := \sup_{x \in [0,1]} |f(x)|.$$

Proposition 3.5.1. *Let $\mathfrak{X} := \mathcal{C}([0,1], \mathbb{C})$. Let x be an arbitrary number in $[0,1]$. Define a function ν_x on $\mathfrak{X}$ by $\nu_x(f) := |f(x)|$. Define a function ν on $\mathfrak{X}$ by $\nu(f) := \sup_{x \in [0,1]} |f(x)|$. Then ν_x is a seminorm on $\mathfrak{X}$ for each x and ν is a norm on $\mathfrak{X}$ and we have $\nu = \sup_{x \in [0,1]} \nu$.*

Proposition 3.5.2. *p-norms are indeed norms.*

Proposition 3.5.3. *For any vector v in $\mathbb{R}^n$, we have*

$$\lim_{p \to \infty} \|v\|_p = \|v\|_\infty.$$

i.e.,

$$\lim_{p \to \infty} \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}} = \max\{|v_i|\}_{i=1}^{n}.$$

Proof. Let p be an arbitrary number in $[1, +\infty)$. Let k be an arbitrary index in $\{1, .., n\}$. Then

$$|v_k| \leq (\sum_{i=1}^{n} |v_k|^p)^{1/p} = \|v\|_p.$$

So

$$\max\{|v_k|\} = \|v\|_\infty \leq \|v\|_p.$$

So

$$\lim_{p\to\infty} \|v\|_p \geq \|v\|_\infty. \tag{1}$$

On the other hand, note that

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p = \sum_{i=1}^{n} (\frac{|v_i|}{\|v\|_\infty})^p$$

decreases as p increases. So it is bounded above. Say

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p \leq C$$

for some $C \in \mathbb{R}$. Then

$$(\sum_{i=1}^{n} |v_i|^p)^{1/p} = \|v\|_p \leq C^{1/p}\|v\|_\infty.$$

So

$$\lim_{p\to\infty} \|v\|_p \leq \lim_{p\to\infty} C^{1/p}\|v\|_\infty = \|v\|_\infty. \tag{2}$$

From (1) and (2) we get

$$\lim_{p\to\infty} \|v\|_p = \|v\|_\infty.$$

■

Proposition 3.5.4. *Let p be an arbitrary number in $[1, +\infty)$. Then the dual norm of the p-norm $\|\cdot\|_p$ is the q-norm $\|\cdot\|_q$ where q is such that satisfies*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proposition 3.5.5. *Let p and q be numbers in $[1, +\infty]$. Let v be a vector in $\mathbb{R}^n$. Then if $p \leq q$,*

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \cdot \|x\|_q.$$

Proposition 3.5.6. *Let w and z be vectors in $\mathbb{E}^d$. Then*

$$\|w + z\|_2^2 + \|w - z\|_2^2 = 2(\|w\|_2^2 + \|z\|_2^2).$$

3.6 Banach Spaces

Definition (Banach Space)**.** *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let d be the metric induced by $\|\cdot\|$. We say that $\mathfrak{X}$ is a **Banach space** if $(\mathfrak{X}, d)$ is a complete metric space. i.e., we define a Banach space to be a complete normed linear space.*

Proposition 3.6.1. *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed vector space over field $\mathbb{F}$. Then $(\mathfrak{X}, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in X is summable.*

Example 3.6.1. *$(\mathcal{C}([0,1],\mathbb{F}),\|\cdot\|_\infty)$ is a Banach space.*

Example 3.6.2 (Disc Algebra)**.** *Define $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$. Define $\mathcal{A}(\mathbb{D}):=\{f\in\mathcal{C}(\overline{\mathbb{D}}):f|_{\mathbb{D}}$ is holomorphic $\}$. Define $\|\cdot\|_\infty$ by $\|f\|_\infty:=\sup_{z\in\overline{\mathbb{D}}}|f(z)|$. Then $(\mathcal{A}(\mathbb{D}),\|\cdot\|_\infty)$ is a Banach space.*

Example 3.6.3. *Let (X,Ω,μ) be a measure space. Let p be a number in $[1,+\infty)$. Define*

$$\mathcal{L}^p(X,\mu):=\operatorname{span}\{f:X\to[0,+\infty]\mid f \text{ is measurable and } \int_X|f|^p<+\infty\}.$$

Define an equivalence relation on $\mathcal{L}^p(X,\mu)$ by $f\equiv g$ if and only if

$$\mu(\{x\in X:f(x)\neq g(x)\})=0.$$

Define a space $L^p(X,\mu):=\mathcal{L}^p(X,\mu)/\equiv$. Then $L^p(X,\mu)$ is a Banach space when equipped with the norm

$$\|[f]\|_p:=\left(\int_X|f|^p\right)^{1/p}.$$

Example 3.6.4. *Let $\mathcal{P}_{\mathbb{C}}[0,1]$ denote the set of all polynomials with complex coefficients. For each $p\in[1,+\infty)$, define a norm*

$$\|f\|_p:=\left(\int_0^1|f|^p\right)^{1/p}.$$

For $p=+\infty$, define a norm

$$\|f\|_\infty:=\sup_{x\in[0,1]}|f(z)|.$$

3.7 Construction of Banach Spaces

Definition. *Let $(\mathfrak{X},\|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y},\|\cdot\|_{\mathfrak{Y}})$ be two Banach spaces over field $\mathbb{K}$. Let $p\in[1,+\infty)$. We define*

$$\mathfrak{X}\oplus_p\mathfrak{Y}:=\{(x,y):x\in\mathfrak{X},y\in\mathfrak{Y}\}$$

and

$$\|(x,y)\|_p:=(\|x\|_{\mathfrak{X}}^p+\|y\|_{\mathfrak{Y}}^p)^{1/p}.$$

For $p=+\infty$, we define

$$\mathfrak{X}\oplus_\infty\mathfrak{Y}:=\{(x,y):x\in\mathfrak{X},y\in\mathfrak{Y}\}$$

and

$$\|(x,y)\|_\infty:=\max(\|x\|_{\mathfrak{X}},\|y\|_{\mathfrak{Y}}).$$

- *Note that the norms behave similarly to what the p norm would do.*

- *We can similarly define the direct sum of finitely many Banach spaces.*

Proposition 3.7.1. $\|\cdot,\cdot\|_p$ *is a norm on* $\mathfrak{X} \oplus_p \mathfrak{Y}$.

Proposition 3.7.2. $\mathfrak{X} \oplus_p \mathfrak{Y}$ *is complete with respect to* $\|\cdot,\cdot\|_p$.

Chapter 4

Quotient Spaces

4.1 Definitions

Definition. *Let $\mathfrak{V}$ be a vector space. Let $\mathfrak{W}$ be a subspace of $\mathfrak{V}$. We define a* ***quotient space****, denoted by $\mathfrak{V}/\mathfrak{W}$, to be a set $\{v + \mathfrak{W} : v \in \mathfrak{V}\}$ with operations*

$$(v_1 + \mathfrak{W}) + (v_2 + \mathfrak{W}) := (v_1 + v_2) + \mathfrak{W} \text{ and}$$

$$\kappa(v + \mathfrak{W}) := (\kappa v) + \mathfrak{W}.$$

Definition (Quotient Map)**.** *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define the* ***quotient map*** *on $\mathfrak{X}$ with respect to $\mathfrak{M}$, denoted by $q_{\mathfrak{M}}$, to be a function from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$ given by*

$$q_{\mathfrak{M}}(x) := x + \mathfrak{M}$$

Proposition 4.1.1. *Quotient maps are contractive. i.e.,*

$$\forall x \in \mathfrak{X}, \quad \|x + \mathfrak{M}\|_{\mathfrak{X}/\mathfrak{M}} \le \|x\|_{\mathfrak{X}}.$$

Proposition 4.1.2. *Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a closed subspace of $\mathfrak{X}$. Let q be the canonical quotient map from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$. Then*

- *q is a continuous map. i.e.,*

$$\forall \text{ open set } W \subseteq \mathfrak{X}/\mathfrak{M}, \quad q^{-1}(W) \text{ is open in } \mathfrak{X}.$$

- *q is an open map. i.e.,*

$$\forall \text{ open set } G \subseteq \mathfrak{X}, \quad q(G) \text{ is open in } \mathfrak{X}/\mathfrak{M}.$$

Proof. Since q is contractive, q is continuous and hence (1). ■

Definition (Seminorm on Quotient Spaces)**.** *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define a* ***seminorm*** *on $\mathfrak{X}/\mathfrak{M}$ to be a function from $\mathfrak{X}/\mathfrak{M}$ to $\mathbb{R}$ given by*

$$p(x + \mathfrak{M}) := \inf\{\|x + m\| : m \in \mathfrak{M}\}.$$

Proposition 4.1.3. *Seminorms on quotient spaces are indeed seminorms.*

Proposition 4.1.4. *A seminorm on a quotient space $\mathfrak{X}/\mathfrak{M}$ is a norm if and only if $\mathfrak{M}$ is closed.*

Chapter 5

Topological Vector Spaces

5.1 Definitions

5.2 Topological Vector Spaces

Definition (Vector Topology). *Let X be a vector space over a topological field $\mathbb{K}$. We define a **vector topology** on X to be a topology on X such that vector addition and scalar multiplication are continuous.*

Proposition 5.2.1 (Stability under Linear Combinations). *Let X be a normed vector space over $\mathbb{F}$. Let K be a compact set in the space. Let C be a closed set in the space. Then $\forall \alpha, \beta \in \mathbb{F}, S := \alpha K + \beta C$ is closed.*

Proof.
The case where $\beta = 0$ is trivial. I will assume $\beta \neq 0$.
Let $\alpha, \beta \in \mathbb{F}$ be arbitrary.
Let $\{s_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in S that converges.
Say the limit is s_∞.
Since $s_i \in S$ for any $i \in \mathbb{N}$ and $S = \alpha K + \beta C$, $s_i = \alpha k_i + \beta c_i$ for some $k_i \in K$ and some $c_i \in C$, for any $i \in \mathbb{N}$.
Since $\{k_i\}_{i \in \mathbb{N}}$ is a sequence in K and K is compact, there exists a convergent subsequence $\{k_i\}_{i \in I}$ of $\{k_i\}_{i \in \mathbb{N}}$ in K.
Say $\{k_i\}_{i \in I}$ converges to $k_\infty \in K$.
Since $\{s_i\}_{i \in \mathbb{N}}$ converges to s_∞, $\{s_i\}_{i \in I}$ also converges to s_∞.
Since $s_i = \alpha k_i + \beta c_i$, $c_i = \beta^{-1}(s_i - \alpha k_i)$.
Define $c_\infty := \beta^{-1}(s_\infty - \alpha k_\infty)$
Since $\{s_i\}_{i \in I}$ converges to s_∞ and $\{k_i\}_{i \in I}$ converges to k_∞ and $c_i = \beta^{-1}(s_i - \alpha k_i)$, $\{c_i\}_{i \in I}$ converges to c_∞.

Since $\{c_i\}_{i\in I}$ is a sequence in C and converges to c_∞ and C is closed, $c_\infty \in C$.
Since $s_\infty = \alpha k_\infty + \beta c_\infty$ and $k_\infty \in K$ and $c_\infty \in C$, $s_\infty \in \alpha K + \beta C$.
Since for any sequence in S that converges, the limit is also in S, S is closed.

■

Remark. *The sum of two closed sets may not be closed.*

Proof.
Counter-example 1
Consider $A := \{n : n \in \mathbb{N}\}$ and $B := \{n + \frac{1}{n} : n \in \mathbb{N}\}$.
(`https://math.stackexchange.com/questions/124130/sum-of-two-closed-sets-in-mathbb-r-is-closed`)
Their sum contains the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ but does not contain 0.
Counter-example 2
Consider $A := \mathbb{R} \times \{0\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x, y > 0, xy \geq 1\}$.
Their sum is $\mathbb{R} \times \mathbb{R}_{++}$.

■

5.3 Neighborhoods

Chapter 6

Sequence Spaces

6.1 ℓ_p Space

Definition ($\ell_p^{(n)}$ Space)**.** *We define the $\ell_p^{(n)}$ space to be the set of all sequences $\{x_i\}_{i=1}^{i=n}$ such that*

Definition (ℓ_p Space)**.** *We define the ℓ_p space to be the set of all sequences x such that $\|x\|_p$ is finite, equipped with the p-norm $\|\cdot\|_p$.*

Proposition 6.1.1. *For $p \in [1, +\infty)$, $(\ell_p, \|\cdot\|_p)$ is complete.*

Proof.
Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in ℓ_p.
Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in ℓ_p, $\forall\varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $\forall m, n > N$, we have $\|x_m - x_n\|_p < \varepsilon$.
Since $\|x_m - x_n\|_p < \varepsilon$ and $|x_m^{(i)} - x_n^{(i)}| \leq \|x_m - x_n\|_p$ for any $i \in \mathbb{N}$, $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$ for any $i \in \mathbb{N}$.
Since for any $i \in \mathbb{N}$ and any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N$, we have $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$, by definition, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$.
Since $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$ and $\mathbb{F}$ is complete, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ converges.
Let $x_0^{(i)} = x_n^{(i)}$. Let $x_0 = \{x_0^{(i)}\}_{i\in\mathbb{N}}$.

$$\|x_0\|_p = (\sum_{i=1}^{\infty} |x_0^{(i)}|^p)^{\frac{1}{p}}$$

■

6.2 c_0 Space and c_{00} Space

Definition (c_0 Space)**.** *We define c_0 to be*

$$c_0 := \big\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\big\}.$$

Definition (c_{00} Space)**.** *We define c_{00} to be*

$$c_{00} := \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}, \forall n > N, x_n = 0\big\}.$$

i.e., the set of all eventually zero sequences of real numbers. i.e., the set of all sequences with finite support.

Proposition 6.2.1. *The c_{00} is not complete in $(\ell_1, \|\cdot\|_1)$.*

Proof. Define a sequence of vectors $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ by $\mathfrak{x}_i^j := \frac{1}{j^2}$ for $j \in \{1..i\}$ and $\mathfrak{x}_i^j := 0$ for $j > i$. Then $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ converges to something that is not in c_{00}. ∎

Proposition 6.2.2. *The closure of c_{00} in the space $(\mathbb{R}^\omega, d_1)$ is ℓ_1.*

Proof. For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq \ell_1$. Let x be an arbitrary element in $\mathrm{cl}(c_{00})$. Since $x \in \mathrm{cl}(c_{00})$, there exists another element $y \in c_{00}$ such that $d_1(x, y) < 1$. Let $N \in \mathbb{N}$ be such that $\forall n > N$, $y_n = 0$. Then

$$\begin{aligned}
& d_1(x,y) < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n - y_n| < 1 \\
\iff & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n - y_n| < 1 \\
\iff & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} \big||x_n| - |y_n|\big| + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} \big(|x_n| - |y_n|\big) + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} |x_n| - \sum_{n=1}^{N} |y_n| + \sum_{n>N} |x_n| < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n| - \sum_{n=1}^{N} |y_n| < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n| < 1 + \sum_{n=1}^{N} |y_n|.
\end{aligned}$$

Since $\sum_{n\in\mathbb{N}} |x_n|$ is bounded, $x \in \ell_1$.
For the reverse direction, we are to prove that $\ell_1 \subseteq \text{cl}(c_{00})$. Let x be an arbitrary element in ℓ_1. For $i \in \mathbb{N}$, define $x^i = \{x^i_j\}_{j\in\mathbb{N}}$ as $x^i_j = x_j$ for $j \leq i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}, x^i \in c_{00}$. Then

$$\begin{aligned}
&\lim_{i\in\mathbb{N}} d_1(x^i, x)\\
=\ &\lim_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |x^i_j - x_j|\\
=\ &\lim_{i\in\mathbb{N}} \sum_{j>i} |x^i_j - x_j|\\
=\ &\lim_{i\in\mathbb{N}} \sum_{j>i} |x_j|\\
=\ &0.
\end{aligned}$$

That is, $\lim_{i\in\mathbb{N}} d_1(x^i, x) = 0$. So $\lim_{i\in\mathbb{N}} x^i = x$. So $x \in \text{cl}(c_{00})$.

■

Proposition 6.2.3. *The closure of c_{00} in the space $(\mathbb{R}^\omega, d_\infty)$ is c_0.*

Proof. For one direction, we are to prove that $\text{cl}(c_{00}) \subseteq c_0$. Let x be an arbitrary element in $\text{cl}(c_{00})$. Let ε be an arbitrary positive real number. Since $x \in \text{cl}(c_{00})$, there exists another element y in c_{00} such that $d_\infty(x, y) < \varepsilon$. That is, $\forall j \in \mathbb{N}, |x_j - y_j| < \varepsilon$. Since $y \in c_{00}$, $\exists N \in \mathbb{N}$ such that $\forall j > N$, $y_j = 0$. So $\forall j > N, |x_j| < \varepsilon$. That is,

$$\forall \varepsilon > 0, \quad , \exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon.$$

By definition of convergence of limits, $\lim_{j\in\mathbb{N}} x_j = 0$. So $x \in c_0$.
For the reverse direction, we are to prove that $c_0 \subseteq \text{cl}(c_{00})$. Let x be an arbitrary element in c_0. For $i \in \mathbb{N}$, define x^i as $x^i_j = x_j$ for $j \leq i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}$, $x^i \in c_{00}$. Let ε be an arbitrary positive real number. Since $x \in c_0$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon/2.$$

Let $i > N$. Then

$$\begin{aligned}
&d_\infty(x^i, x)\\
=\ &\sup_{j\in\mathbb{N}} |x^i_j - x_j|\\
=\ &\sup_{j>i} |x^i_j - x_j|\\
=\ &\sup_{j>i} |x_j|\\
\leq\ &\varepsilon/2 < \varepsilon.
\end{aligned}$$

That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall i > N, \quad d_\infty(x^i, x) < \varepsilon.$$

By definition of convergence of sequences, $\lim_{i\in\mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

■

Proposition 6.2.4. *Let* $A := \left\{\{x_n\}_{n\in\mathbb{N}} \in c_{00} : \sum_{n\in\mathbb{N}} x_n = 0\right\}$*. Then* A *is a subset of* ℓ^1 *and is closed in* (ℓ^1, d_1)*. i.e.* $\mathrm{cl}(A) = A$ *in* (ℓ^1, d_1)*.*

Proof. Let $x = \{x^i\}_{i\in\mathbb{N}}$ be a sequence in ℓ^1, where each $x^i = \{x^i_j\}_{j\in\mathbb{N}}$ is an element in A, that converges in (ℓ^1, d_1). Say $\lim_{i\to\infty} x^i = x^\infty$.
First I claim that $x^\infty \in c_{00}$.
Now I claim that $\sum_{j\in\mathbb{N}} x^\infty_j = 0$. i.e. $x^\infty \in A$. Since $x^\infty \in c_{00}$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad x^\infty_j = 0.$$

Define $y_i := \sum_{j=1}^N x^i_j$. Define $y_\infty := \sum_{j=1}^N x^\infty_j$. It is easy to see that $\lim_{i\in\mathbb{N}} y_i = y_\infty$. Assume for the sake of contradiction that $y_\infty \neq 0$. i.e. $\{y_i\}_{i\in\mathbb{N}}$ does not converge to 0. Then

$$\exists \varepsilon_0 > 0, \quad \forall M \in \mathbb{N}, \quad \exists i_0 > M, \quad |y_{i_0} - 0| = |y_{i_0}| \geq \varepsilon_0. \tag{1}$$

Since $\lim_{i\to\infty} x^i = x^\infty$,

$$\exists M_0 \in \mathbb{N}, \quad \forall i > M_0, \quad d_1(x^i, x^\infty) < \varepsilon_0. \tag{2}$$

Consider statement (1) for a particular M, M_0, we have

$$\exists i_0 > M_0, \quad |y_{i_0}| \geq \varepsilon_0. \tag{3}$$

That is,

$$\Big|\sum_{j=1}^N x^{i_0}_j\Big| \geq \varepsilon_0. \tag{3'}$$

Consider statement (2) for a particular i, i_0, we have

$$d_1(x^{i_0}, x^\infty) < \varepsilon_0. \tag{4}$$

From statement (4) we can derive:

$$\begin{aligned} & d_1(x^{i_0}, x^\infty) < \varepsilon_0 \\ \iff & \sum_{j\in\mathbb{N}} |x^{i_0}_j - x^\infty_j| < \varepsilon_0 \\ \iff & \sum_{j=1}^N |x^{i_0}_j - x^\infty_j| + \sum_{j>N} |x^{i_0}_j - x^\infty_j| < \varepsilon_0 \end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow \sum_{j>N} |x_j^{i_0} - x_j^{\infty}| < \varepsilon_0 \\
&\Longleftrightarrow \sum_{j>N} |x_j^{i_0} - 0| < \varepsilon_0 \\
&\Longleftrightarrow \sum_{j>N} |x_j^{i_0}| < \varepsilon_0 \\
&\Longrightarrow \Big|\sum_{j>N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\Longrightarrow \Big|\sum_{j>N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\Longleftrightarrow \Big|\sum_{j\in\mathbb{N}} x_j^{i_0} - \sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\Longleftrightarrow \Big|0 - \sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\Longleftrightarrow \Big|\sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0.
\end{aligned}
$$

This contradicts to statement (3'). So the original assumption that $y_\infty \neq 0$ is false. i.e. $y_\infty = 0$. It follows that $\sum_{j\in\mathbb{N}} x_j^{\infty} = 0$. This completes the proof.

■

6.3 Hölder's Inequality

Theorem 6 (Hölder's Inequality)**.** *Let $\mathfrak{X} = \mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ be vectors in $\mathfrak{X}$. Then $\forall p, q \in (1, +\infty) : 1/p + 1/q = 1$, $\|xy\|_1 \leq \|x\|_p \|y\|_q$. i.e.,*

$$
\sum_{i=1}^{n} |x_i y_i| \leq (\sum_{i=1}^{n} |x_i|^p)^{1/p} (\sum_{i=1}^{n} |y_i|^q)^{1/q}.
$$

Chapter 7

Function Spaces

7.1 The $\mathcal{L}^p$ Norm

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

the instructors' answer, *where instructors collectively construct a single answer*

In the sup norm, convergence coincides with uniform convergence. Moreover, $C[a, b]$ is complete in this norm. It is not complete in any of the L^p norms for $1 \leq p < \infty$. The completion in these norms is called $L^p(a, b)$.

undo thanks | 1

Updated 1 day ago by Kenneth Davidson

Chapter 8

Hilbert Space

8.1 Hilbert Spaces

Definition (Hilbert Space)**.** *We define a **Hilbert space** to be a complete inner product space.*

Example 8.1.1. ℓ^2 *is a Hilbert space.*

Chapter 9

Equicontinuity in Metric Spaces

9.1 Definitions

Definition ((Pointwise) Equicontinuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{F}$ be a collection of functions from X to Y. Let x_0 be a point in X. We say that $\mathcal{F}$ is* ***(pointwise) equicontinuous*** *at point x_0 if for any positive number ε, there exists some number $\delta(x_0, \varepsilon)$ such that for any function f in $\mathcal{F}$ and any point x in X, we have*

$$d_Y(f(x), f(x_0)) < \varepsilon$$

whenever $d_X(x, x_0) < \delta(x_0, \varepsilon)$ is satisfied.

Definition (Uniform Equicontinuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{F}$ be a collection of functions from X to Y. We say that $\mathcal{F}$ is* ***uniformly equicontinuous*** *if for any positive number ε, there exists some number $\delta(\varepsilon)$ such that for any function f in $\mathcal{F}$ and any points x_1 and x_2 in X, we have*

$$d_Y(f(x_1), f(x_2)) < \varepsilon$$

whenever $d_X(x_1, x_2) < \delta(\varepsilon)$ is satisfied.

9.2 Sufficient Conditions

Proposition 9.2.1. *The closure of an equicontinuous family of functions is equicontinuous.*

Proof.
Let (X, d_X) and (Y, d_Y) be metric spaces.
Let $\mathcal{F}$ be an equicontinuous family of functions from X to Y.
We are to prove that $cl(\mathcal{F})$ is equicontinuous.

Let x_0 be an arbitrary point in X.
Let ε be an arbitrary positive number.
Since $\mathcal{F}$ is equicontinuous at point x_0, there exists some $\delta(x_0, \varepsilon)$ such that for any function f in $\mathcal{F}$ and any point x in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon/3$.
Let f be an arbitrary function in $cl(\mathcal{F})$.
Let x be an arbitrary point in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$.
Since $f \in cl(\mathcal{F})$, there exists some function $f_0 \in \mathcal{F}$ such that $d_\infty(f, f_0) < \varepsilon/3$.
Since $d_\infty(f, f_0) < \varepsilon/3$, $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$.
Since $f_0 \in \mathcal{F}$ and $d_X(x, x_0) < \delta(x_0, \varepsilon)$, $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$.
Since $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$ and $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$, $d_Y(f(x), f(x_0)) < \varepsilon$.
Since for any positive number ε, there exists some $\delta(x_0, \varepsilon)$ such that for any function f in $cl(\mathcal{F})$ and any point x in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, by definition of equicontinuous, $cl(\mathcal{F})$ is equicontinuous at point x_0.
Since $cl(\mathcal{F})$ is equicontinuous at point x_0 for any point x_0 in X, $cl(\mathcal{F})$ is equicontinuous.

■

Chapter 10

Operators

10.1 Bounded Operators

Definition (Bounded Operator). *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let T be a linear map from $\mathfrak{X}$ to $\mathfrak{Y}$. We say that T is a **bounded operator** if*

$$\exists k \in \mathbb{R}, \quad \forall x \in \mathfrak{X}, \quad \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}.$$

Definition (Operator Norm). *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let T be a bounded operator from $\mathfrak{X}$ to $\mathfrak{Y}$. We define the **operator norm** of T, denoted by $\|T\|$, to be the number given by*

$$\|T\| := \inf\{k \in \mathbb{R} : \forall x \in \mathfrak{X}, \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}\}.$$

Proposition 10.1.1.

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{X}, \|x\|_{\mathfrak{X}} = 1\}.$$

Proposition 10.1.2. *Let X and Y be normed linear spaces. Let T be a linear map from X to Y. Then T is bounded if and only if T is continuous.*

Proposition 10.1.3. *Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space. Let S be a map in $\mathcal{B}(\mathfrak{X})$ that is invertible. Then*

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

Proof.

$$\begin{aligned}
\|S^{-1}\| &= \sup\{\|S^{-1}x\| : \|x\| = 1\} \\
&= \sup\{\frac{\|S^{-1}x\|}{\|x\|} : \|x\| \neq 0\} \\
&= \sup\{\frac{\|S^{-1}Sx\|}{\|Sx\|} : \|x\| \neq 0\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup\{\frac{\|x\|}{\|Sx\|} : \|x\| \neq 0\} \\
&= \sup\{\|Sx\|^{-1} : \|x\| = 1\} \\
&= (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.
\end{aligned}
$$

That is,

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

■

Example 10.1.1 (The Multiplication Operator)**.** *Let* $\mathfrak{X} = \big(\mathcal{C}([0,1],\mathbb{C}), \|\cdot\|_\infty\big)$*. Let* f *be a function in* $\mathfrak{X}$*. We define the* ***multiplication operator*** *on* $\mathfrak{X}$*, w.r.t.* f*, denoted by* M_f*, as*

$$M_f(g) = fg.$$

Then M_f *is bounded and* $\|M_f\| = \|f\|_\infty$*.*

Proof. Let g be an arbitrary function in $\mathfrak{X}$. Then

$$
\begin{aligned}
\|M_f g\|_\infty &= \|fg\|_\infty \\
&= \sup_{x\in[0,1]} |f(x)g(x)| \\
&= \sup_{x\in[0,1]} |f(x)||g(x)| \\
&\leq \sup_{x\in[0,1]} |f(x)| \sup_{x\in[0,1]} |g(x)| \\
&= \|f\|_\infty \|g\|_\infty.
\end{aligned}
$$

That is, $\|M_f g\|_\infty \leq \|f\|_\infty \|g\|_\infty$. So $\|f\|_\infty$ is an element of the set $S = \{k \in \mathbb{R} : \forall g \in \mathfrak{X}, \|M_f g\|_{\mathfrak{Y}} \leq k\|g\|_{\mathfrak{X}}\}$. So $\|M_f\| = \inf(S) \leq \|f\|_\infty$. Consider g_0 given by $g_0(x) = 1$. Then g_0 in $\mathfrak{X}$. Then

$$\|M_f g_0\|_\infty = \|fg_0\|_\infty = \|f\|_\infty = \|f\|_\infty \|g_0\|_\infty.$$

Let k be an arbitrary element in S. Assume for the sake of contradiction that $k < \|f\|_\infty$. Then

$$
\begin{aligned}
\|f\|_\infty \|g_0\|_\infty &= \|M_f g_0\|_\infty \\
&\leq k\|g_0\|_\infty \\
&< \|f\|_\infty \|g_0\|_\infty.
\end{aligned}
$$

This leads to a contradiction. So $\forall k \in S$, $k \geq \|f\|_\infty$. So $\|f\|_\infty$ is a lower bound for the set S. So $\|M_f\| = \inf(S) \geq \|f\|_\infty$. Since $\|M_f\| \leq \|f\|_\infty$ and $\|M_f\| \geq \|f\|_\infty$, we get $\|M_f\| = \|f\|_\infty$. ■

Example 10.1.2 (The Volterra Operator)**.** *Let* $\mathfrak{X} = \big(\mathcal{C}([0,1],\mathbb{C}), \|\cdot\|_\infty\big)$*. Define*

$$Vf := x \mapsto \int_0^x f(t)dt.$$

Then the Volterra Operator is bounded and $\|V\| \leq 1$*.*

Proof. Let f be an arbitrary function in $\mathfrak{X}$ with $\|f\|_\infty = 1$. Then $\forall x \in [0,1]$,

$$\begin{aligned}
|Vf(x)| &= \left|\int_0^x f(t)dt\right| \\
&\leq \int_0^x |f(t)|dt \\
&\leq \int_0^x \sup_{t\in[0,1]} |f(t)|dt \\
&= \int_0^x \|f\|_\infty dt \\
&= \int_0^x 1dt \\
&= x.
\end{aligned}$$

That is, $\forall x \in [0,1]$, $|Vf(x)| \leq 1$. So $\|Vf\|_\infty \leq 1$. Since $\forall f \in \mathfrak{X} : \|f\|_\infty = 1$, $\|Vf\|_\infty \leq 1$, we get $\|V\| \leq 1$. ∎

Example 10.1.3 (The Diagonal Operator)**.** *Let* $\mathfrak{X} = \ell^2(\mathbb{N})$*. Let*

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & \ddots \end{bmatrix}.$$

Then D *is bounded if and only if* $(d_i)_{i\in\mathbb{N}}$ *is bounded and* $\|D\| = \|(d_i)_{i\in\mathbb{N}}\|_\infty$*.*

Proof. Case 1.

$$\begin{aligned}
\|Dx\|_2^2 &= \sum_{i\in\mathbb{N}} |d_i x_i|^2 \\
&=\leq \sum_{i\in\mathbb{N}} \|(d_j)_{j\in\mathbb{N}}\|_\infty |x_i|^2 \\
&= \|(d_j)_{j\in\mathbb{N}}\|_\infty \sum_{i\in\mathbb{N}} |x_i|^2 \\
&= \|(d_j)_{j\in\mathbb{N}}\|_\infty \|x\|_2^2.
\end{aligned}$$

Case 2.

If $(d_i)_{i\in\mathbb{N}} \notin \ell^\infty$, $\exists (d_{n_i})_{i\in\mathbb{N}} \to \infty$.

$$\begin{aligned}\|De_{n_i}\|_2 &= \|d_{n_i}e_{n_i}\|_2\\ &= |d_{n_i}|\|e_{n_i}\|_2\\ &= |d_{n_i}|.\end{aligned}$$

So $\|D\| \geq \|De_{n_i}\|_2 \to \infty$. ■

Example 10.1.4 (Weighted Shifts)**.**

- *Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(w_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an* ***unilateral forward weighted shift*** *W on $\mathcal{H}$ as*

  $$W(x_n) := (0, w_1x_1, w_2x_2, w_3x_3, ...).$$

  i.e.,

  $$W = \begin{bmatrix} 0 & & & & \\ w_1 & 0 & & & \\ & w_2 & 0 & & \\ & & w_3 & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.$$

  Then W is bounded and $\|W\| = \sup\{|w_n| : n \in \mathbb{N}\}$.

- *Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(v_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an* ***unilateral backward weighted shift*** *V on $\mathcal{H}$ as*

  $$V(x_n) := (v_1x_2, v_2x_3, v_3x_4, ...).$$

  Then V is bounded and $\|V\| = \sup\{|v_n| : n \in \mathbb{N}\}$.

- *Let $\mathcal{H} = \ell^2_{\mathbb{Z}}$. Let $(u_n)_{n\in\mathbb{Z}} \in \ell^\infty_{\mathbb{Z}}$. We define a* ***bilateral weighted shift*** *U on $\mathcal{H}$ as*

  $$U(x_n) := (u_{n-1}x_{n-1})_{n\in\mathbb{Z}}.$$

  Then U is bounded and $\|U\| = \sup\{|u_n| : n \in \mathbb{Z}\}$.

Example 10.1.5 (The Composition Operators)**.** *Let $\mathfrak{X} = \mathcal{C}([0,1],\mathbb{C})$. Let $\varphi \in \mathcal{C}([0,1],[0,1])$. We define the* ***composition operator*** *on $\mathfrak{X}$, denoted by C_φ as*

$$C_\varphi(f) := f \circ \varphi.$$

Then C_φ is contractive.

Proof.

$$\begin{aligned}\|C_\varphi(f)\| &= \sup_{x\in[0,1]} |(f\circ\varphi)(x)|\\ &\leq \|f\|_\infty.\end{aligned}$$

■

10.2 Space of Bounded Operators

Proposition 10.2.1. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Then $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is a vector space and the operator norm is a norm on $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$.*

Proposition 10.2.2. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Then $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}, \|\cdot\|_1)$ if and only if $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}, \|\cdot\|_2)$.*

Proposition 10.2.3. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ be the space of bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$. Then if $\mathfrak{Y}$ is complete, $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is complete.*

10.3 Dual Spaces

Definition ((Topological) Dual Space)**.** *Let $\mathfrak{X}$ be a normed linear space over field $\mathbb{K}$. We define the **(topological) dual space** of $\mathfrak{X}$, denoted by $\mathfrak{X}^*$, to be the space $\mathcal{B}(\mathfrak{X}, \mathbb{K})$.*

Definition (Linear Functionals)**.** *We call the elements of $\mathfrak{X}^*$ **linear functionals**.*

Proposition 10.3.1. *Let X be a normed linear space. Then there exists a contractive map from X to its double dual X^{**}.*

Chapter 11

Adjoint Operator

11.1 Definitions

Definition (Adjoint Matrix). *Let A be an $m \times n$ matrix. We define the **adjoint** of A, denoted by A^*, to be an $n \times m$ matrix given by*

$$(A^*)_{ij} := \overline{(A)_{ji}}.$$

Definition (Adjoint Operator). *Let V and W be inner product spaces. Let T be a linear map from V to W. We define the **adjoint** of T, denoted by T^*, to be a map from W to V such that*

$$\forall x \in V, \forall y \in W, \quad \langle T(x), y\rangle_W = \langle x, T^*(y)\rangle_V.$$

Proposition 11.1.1 (Existence). *Let V be a finite-dimensional inner product space and T be a linear operator on V. Then the adjoint of T exists.*

Proposition 11.1.2 (Uniqueness). *Let V be an inner product space and T be a linear operator on V. Then the adjoint of T is unique, provided that it exists.*

11.2 Properties of the Adjoint Operator

Proposition 11.2.1. *Let V be an inner product space. Then*

(1) $(I_V)^* = I_V$ *where I_V is the identity operator on V.*

(2) $T^{**} = T$ *for any linear operator T on V.*

Proposition 11.2.2. *Let V be an inner product space and T be a linear operator on V. Then T^* is also linear.*

Proposition 11.2.3. *Let V be an inner product space. Then*

(1) *For any linear operators T and U,*

$$(T+U)^* = T^* + U^*.$$

(2) *For any linear operator T,*

$$(cT)^* = \overline{c} \cdot T^*.$$

(3) *For any linear operator T and U,*

$$(TU)^* = U^* T^*.$$

Proposition 11.2.4. *Let V be a finite-dimensional inner product space and T be a linear operator on V. Then if T is invertible, T^* is also invertible.*

Proposition 11.2.5. *Let V be an inner product space and T be an invertible linear operator on V. Then $(T^{-1})^* = (T^*)^{-1}$.*

11.3 Normal Operators

Definition (Normal)**.** *Let V be an inner product space and T be a linear operator on V. We say that T is **normal** if $TT^* = T^*T$.*

11.4 Self-adjoint

Chapter 12

Convolution

Definition (Convolution)**.** *Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. We define the **convolution** of f and g, denoted by $f * g$, to be a function on $\mathbb{R}$ given by*

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)dt.$$

Chapter 13

Coercive Functions

13.1 Definitions

Definition (Coercive)**.** *Let f be a function from $\mathbb{R}^d$ to $\mathbb{R}^*$. We say that f is* ***coercive*** *if $\lim_{\|x\|\to\infty} f(x) = +\infty$.*

13.2 Properties

Proposition 13.2.1. *Let f be a proper lower semi-continuous function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let K be a compact set in $\mathbb{R}^d$. Assume $K \cap \mathrm{dom}(f) \neq \emptyset$. Then f attains its minimum over K.*

Proof.
Define $m := \inf_{x\in K} f(x)$.
Since $m = \inf_{x\in K} f(x)$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in K such that $\lim_{i\to\infty} f(x_i) = m$.
Since K is compact and $\{x_i\}_{i\in\mathbb{N}} \subseteq K$, there exists a convergent subsequence $\{x_i\}_{i\in I}$ in K where I is an infinite subset of $\mathbb{N}$.
Say the limit is x_∞ where $x_\infty \in K$.
Since $\lim_{i\to\infty} f(x_i) = m$, we get $\lim_{i\in I, i\to\infty} f(x_i) = m$.
Since $\lim_{i\in I, i\to\infty} f(x_i) = m$, we get $\liminf_{i\in I, i\to\infty} f(x_i) = m$.
Since f is lower semi-continuous and $\lim_{i\in I, i\to\infty} x_i = x_\infty$, we get $f(x_\infty) \leq \liminf_{i\in I, i\to\infty} x_i$.
That is, $f(x_\infty) \leq m$.
Since $m = \inf_{x\in K} f(x)$, we have $\forall x \in K, f(x) \geq m$.
In particular, $f(x_\infty) \geq m$.
Since $f(x_\infty) \geq m$ and $f(x_\infty) \leq m$, $f(x_\infty) = m$.
Since f is proper, $f(x_\infty) = m \neq -\infty$.
So f attains its minimum at point x_∞.

■

Proposition 13.2.2. *Let f be a proper, lower semi-continuous, and coercive function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let C be a closed subset of $\mathbb{R}^d$. Assume $C \cap \operatorname{dom}(f) \neq \emptyset$. Then f attains its minimum over C.*

Proof.
Since $C \cap \operatorname{dom}(f) \neq \emptyset$, take $x \in C \cap \operatorname{dom}(f)$.
Since f is coercive, $\exists R$ such that $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$.
Since $x \in C \cap \operatorname{dom}(f)$ and $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$, the set of minimizers of f over C is the same as the set of minimizers of f over $C \cap \operatorname{ball}[0, R]$.
Since C and $\operatorname{ball}[0, R]$ are both closed, $C \cap \operatorname{ball}[0, R]$ is closed.
Since $\operatorname{ball}[0, R]$ is bounded, $C \cap \operatorname{ball}[0, R]$ is bounded.
Since $C \cap \operatorname{ball}[0, R]$ is closed and bounded, by the Heine-Borel Theorem, $C \cap \operatorname{ball}[0, R]$ is compact.
Since f is proper and lower semi-continuous and $C \cap \operatorname{ball}[0, R]$ is compact, f attains its minimum over $C \cap \operatorname{ball}[0, R]$.
So f attains its minimum over C.

∎

Chapter 14

Unclassified Results

Proposition 14.0.1. *Let (X, d) be a compact metric space. Let $L(X)$ be the set of all Lipschitz functions from X to $\mathbb{R}$. Let $C(X)$ be the set of all continuous functions from X to $\mathbb{R}$. Then $L(X)$ is dense in $C(X)$.*

Proposition 14.0.2. *Let $(V, \|\cdot\|)$ be a normed vector space. Let S be a subset of V. Let p be a vector in V. Then we have the followings.*

(1) $p + int(S) = int(p + S)$,

(2) $p + \mathrm{cl}(S) = \mathrm{cl}(p + S)$.

Proof.
Proof of (1).
For one direction, let x be an arbitrary point in the set $(p + int(S))$.
We are to prove that $x \in int(p + S)$.
Since $x \in (p + int(S))$, $(x - p) \in int(S)$.
Since $(x - p) \in int(S)$, by definition of interior, there exists a radius r such that

$$B(x - p, r) \subseteq S.$$

It follows that $B(x, r) \subseteq p + S$.
Since there exists a radius r such that $B(x, r) \subseteq p + S$, by definition of interior,

$$x \in int(p + S).$$

For the reverse direction, let x be an arbitrary point in $int(p + S)$.
We are to prove that $x \in p + int(S)$.
Since $x \in int(p + S)$, by definition of interior, there exists a radius r such that

$$B(x, r) \subseteq (p + S).$$

It follows that $B(x-p,r) \subseteq S$.
Since there exists a radius r such that $B(x-p,r) \subseteq S$, by definition of interior,

$$(x-p) \in int(S).$$

Since $(x-p) \in int(S)$, we get $x \in (p + int(S))$.
Proof of (2).
For one direction, let x be an arbitrary point in the set $(p + cl(S))$.
We are to prove that $x \in cl(p+S)$.
Since $x \in (p + cl(S))$, we get $(x-p) \in cl(S)$.
Since $(x-p) \in cl(S)$, by definition of closure, for any radius r, we have

$$B(x-p,r) \cap S \neq \varnothing.$$

It follows that $B(x,r) \cap (p+S) \neq \varnothing$.
Since for any radius r, $B(x,r) \cap (p+S) \neq \varnothing$, by definition of closure, we get

$$x \in cl(p+S).$$

For the reverse direction, let x be an arbitrary point in $cl(p+S)$.
We are to prove that $x \in (p + cl(S))$.
Since $x \in cl(p+S)$, by definition of closure, for any radius r, we have

$$B(x,r) \cap (p+S) \neq \varnothing.$$

It follows that $B(x-p,r) \cap S \neq \varnothing$.
Since for any radius r, $B(x-p,r) \cap S \neq \varnothing$, by definition of closure, we get

$$(x-p) \in cl(S).$$

Since $(x-p) \in cl(S)$, we get $x \in (p + cl(S))$.

■

Proposition 14.0.3. *Let $(V, \|\cdot\|)$ be a normed vector space. Let S be a subset of V. Let λ be a non-zero real number. Then*

(1) $\lambda int(S) = int(\lambda S)$.

(2) $\lambda\, \mathrm{cl}(S) = \mathrm{cl}(\lambda S)$.

Proof.
Proof of (1).
For one direction, let x be an arbitrary point in $\lambda int(S)$.
We are to prove that $x \in int(\lambda S)$.

Since $x \in \lambda int(S)$, we get $x/\lambda \in int(S)$.
Since $x/\lambda \in int(S)$, by definition of interior, there exists a radius r such that

$$B(x/\lambda, r) \subseteq S.$$

Let y be an arbitrary point in $B(x, \lambda r)$.
Since $y \in B(x, \lambda r)$, we get $\|y - x\| \leq \lambda r$.
Since $\|y - x\| \leq \lambda r$, we get $\|y/\lambda - x/\lambda\| \leq r$.
Since $\|y/\lambda - x/\lambda\| \leq r$, we get $y/\lambda \in B(x/\lambda, r)$.
Since $y/\lambda \in B(x/\lambda, r)$ and $B(x/\lambda, r) \subseteq S$, we get $y/\lambda \in S$.
Since $y/\lambda \in S$, we get $y \in \lambda S$.
Since any point in $B(x, \lambda r)$ is also in λS, we get $B(x, \lambda r) \subseteq \lambda S$.
Since there exists a radius r such that $B(x, \lambda r) \subseteq \lambda S$, by definition of interior, we get

$$x \in int(\lambda S).$$

For the reverse direction,

$\blacksquare$