Chapter 1

Experimental Design

Contents

1.1 Completely Random Design - Model 1

DEFINITION 1.1 (Completely Random Design - Model 1)**.** Let k denote the number of treatments. Let n_i denote the number of units that receive the i-th treatment. We model the observations as

$$y_{ij} = \mu_i + e_{ij}, \text{ for } i \in \{1, ..., k\} \text{ and } j \in \{1, ..., n_i\}$$

where $e_{ij} \sim \mathcal{N}(0, \sigma^2)$ are assumed to be independent. The total number of parameters in this model is $k + 1$

1.1.1 Estimation of Mean

PROPOSITION 1.2. Let y_{ij} for $i \in \{1, ..., k\}$ and $j \in \{1, ..., n_i\}$ be given. Consider the following optimization problem:

$$\begin{aligned} \text{(P)} \qquad \min \qquad & f(\mu) := \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 \\ \text{subject to:} \quad & \mu \in \mathbb{R}^k. \end{aligned}$$

Then the minimizer $\hat{\mu} \in \mathbb{R}^k$ of (P) is given by

$$\hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}, \text{ for } i \in \{1, ..., k\}.$$

Proof. Let $p \in \{1, ..., k\}$ be arbitrary. Then

$$\frac{\partial}{\partial \mu_p} f(\mu) = \frac{\partial}{\partial \mu_p} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 = \sum_{j=1}^{n_p} \frac{\partial}{\partial \mu_p} (y_{pj} - \mu_p)^2 = -2 \sum_{j=1}^{n_p} (y_{pj} - \mu_p), \text{ and}$$

$$\frac{\partial^2}{\partial \mu_p^2} f(\mu) = \frac{\partial}{\partial \mu_p} \left[-2 \sum_{j=1}^{n_p} (y_{pj} - \mu_p) \right] = 2n_p > 0.$$

Suppose $\hat{\mu} \in \mathbb{R}^k$ is a minimizer of f. Then we have $\nabla f(\hat{\mu}) = \mathbb{0} \in \mathbb{R}^k$. So

$$\frac{\partial}{\partial \mu_i} f(\hat{\mu}) = 0 \implies \hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}.$$

Testing the Hessian of f at point $\hat{\mu} \in \mathbb{R}^k$ confirms that it is indeed a minimizer of f. $\square$

PROPOSITION 1.3 (Mean of the Mean Estimator)**.** We have

$$\mathbb{E}(\hat{\mu}) = \mu \in \mathbb{R}^k.$$

i.e., $\hat{\mu}$ is an unbiased estimator for μ.

Proof. Recall that $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, we have $y_{ij} \sim \mathcal{N}(\mu_i, \sigma^2)$. So $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, $\mathbb{E}(y_{ij}) = \mu_i$. Now we can compute

$$\mathbb{E}(\hat{\mu}_i) = \mathbb{E}(\frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}) = \frac{1}{n_i}\sum_{j=1}^{n_i}\mathbb{E}(y_{ij}) = \frac{1}{n_i}\sum_{j=1}^{n_i}\mu_i = \mu_i.$$

That is, $\mathbb{E}(\hat{\mu}) = \mu$, as desired. $\square$

PROPOSITION 1.4 (Variance of the Mean Estimator)**.** We have

$$\mathbb{V}(\hat{\mu}) = \text{Diag}(\frac{\sigma^2}{n_i})_{i=1}^k \in \mathbb{S}_+^k.$$

Proof. Recall that $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, we have $y_{ij} \sim \mathcal{N}(\mu_i, \sigma^2)$. So $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, $\mathbb{V}(y_{ij}) = \sigma^2$. Now we can compute

$$\mathbb{V}(\hat{\mu}_i) = \mathbb{V}(\frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}) = \sum_{j=1}^{n_i}\frac{1}{n_i^2}\mathbb{V}(y_{ij}) = \sum_{j=1}^{n_i}\frac{1}{n_i^2}\sigma^2 = \frac{\sigma^2}{n_i}, \quad \forall i, \text{ and}$$

$$\begin{aligned}
\text{cov}(\hat{\mu}_p, \hat{\mu}_q) = \mathbb{E}(\hat{\mu}_p\hat{\mu}_q) - \mathbb{E}(\hat{\mu}_p)\mathbb{E}(\hat{\mu}_q) &= \mathbb{E}((\frac{1}{n_p}\sum_{j=1}^{n_p} y_{pj})(\frac{1}{n_q}\sum_{j=1}^{n_q} y_{qj})) - \mathbb{E}(\hat{\mu}_i)\mathbb{E}(\hat{\mu}_j) \\
&= (\frac{1}{n_p}\sum_{j=1}^{n_p}\mathbb{E}(y_{pj}))(\frac{1}{n_q}\sum_{j=1}^{n_q}\mathbb{E}(y_{qj})) - \mathbb{E}(\hat{\mu}_p)\mathbb{E}(\hat{\mu}_q), \text{ by independence} \\
&= (\frac{1}{n_p}\sum_{j=1}^{n_p}\mu_p)(\frac{1}{n_q}\sum_{j=1}^{n_q}\mu_q) - \mu_p\mu_q, \text{ by above} \\
&= \mu_p\mu_q - \mu_p\mu_q = 0, \quad \forall p, q \in \{1, ..., k\} : p \neq q.
\end{aligned}$$

$\square$

1.1.2 Estimation of Variance

In this subsection, we assume that $\forall i \in \{1, ..., k\}$, $n_i = n$ for some $n \in \mathbb{Z}_{++}$.

DEFINITION 1.5 (Sum of Squares)**.** We define the following terms:

$$\begin{aligned}
\mathrm{SS}_{\mathrm{trt}} &:= n\sum_{i=1}^{k}(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2,\\
\mathrm{SS}_{\mathrm{err}} &:= \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}-\bar{y}_{i\cdot})^2,\\
\mathrm{SS}_{\mathrm{tot}} &:= \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}-\bar{y}_{\cdot\cdot})^2.
\end{aligned}$$

PROPOSITION 1.6 (Decomposition of $\mathrm{SS}_{\mathrm{tot}}$)**.** We have

$$\mathrm{SS}_{\mathrm{tot}} = \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{err}}.$$

Proof.

$$\begin{aligned}
\mathrm{SS}_{\mathrm{tot}} &= \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}-\bar{y}_{\cdot\cdot})^2 = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}-\bar{y}_{i\cdot}+\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2\\
&= \sum_{i=1}^{k}\sum_{j=1}^{n}\left[(y_{ij}-\bar{y}_{i\cdot})^2+(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2+2(y_{ij}-\bar{y}_{i\cdot})(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})\right]\\
&= \mathrm{SS}_{\mathrm{trt}}+\mathrm{SS}_{\mathrm{err}}+\sum_{i=1}^{k}\sum_{j=1}^{n}2(y_{ij}-\bar{y}_{i\cdot})(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})\\
&= \mathrm{SS}_{\mathrm{trt}}+\mathrm{SS}_{\mathrm{err}}+2\sum_{i=1}^{k}\sum_{j=1}^{n}y_{ij}\bar{y}_{i\cdot}-2\sum_{i=1}^{k}\sum_{j=1}^{n}y_{ij}\bar{y}_{\cdot\cdot}-2\sum_{i=1}^{k}\sum_{j=1}^{n}\bar{y}_{i\cdot}^2+2\sum_{i=1}^{k}\sum_{j=1}^{n}\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot}\\
&= \mathrm{SS}_{\mathrm{trt}}+\mathrm{SS}_{\mathrm{err}}+2n\sum_{i=1}^{k}\bar{y}_{i\cdot}^2-2n\bar{y}_{\cdot\cdot}\sum_{i=1}^{k}\bar{y}_{i\cdot}-2n\sum_{i=1}^{k}\bar{y}_{i\cdot}^2+2n\bar{y}_{\cdot\cdot}\sum_{i=1}^{k}\bar{y}_{i\cdot}\\
&= \mathrm{SS}_{\mathrm{trt}}+\mathrm{SS}_{\mathrm{err}}+0 = \mathrm{SS}_{\mathrm{trt}}+\mathrm{SS}_{\mathrm{err}}.
\end{aligned}$$

□

DEFINITION 1.7 (Mean Squares)**.** We define the following estimators for the variance σ^2.

$$\begin{aligned}\mathrm{MS}_{\mathrm{trt}} &:= \mathrm{SS}_{\mathrm{trt}}/(k-1),\\ \mathrm{MS}_{\mathrm{err}} &:= \mathrm{SS}_{\mathrm{err}}/(k(n-1)).\end{aligned}$$

REMARK 1.8. In the case of $k = 2$, $\mathrm{MS}_{\mathrm{err}}$ reduces to

$$\mathrm{MS}_{\mathrm{err}} = \frac{1}{2n-2}\left[\sum_{j=1}^{n}(y_{1j} - \bar{y}_{1\cdot})^2 + \sum_{j=1}^{n}(y_{2j} - \bar{y}_{2\cdot})^2\right],$$

which is also called the pooled variance and is denoted by s_p^2.

LEMMA 1.9. We have

$$\begin{aligned}\mathbb{E}(y_{ij}^2) &= \sigma^2 + \mu_i^2, \quad \forall i, j,\\ \mathbb{E}(\bar{y}_{i\cdot}^2) &= \frac{\sigma^2}{n} + \mu_i^2, \quad \forall i, \text{ and}\\ \mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \frac{\sigma^2}{kn} + (\frac{1}{k}\sum_{i=1}^{k}\mu_i)^2.\end{aligned}$$

Proof. Recall that $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n\}$, we have $y_{ij} \sim \mathcal{N}(\mu_i, \sigma^2)$. So

$$\begin{aligned}\bar{y}_{i\cdot} &= \frac{1}{n}\sum_{j=1}^{n} y_{ij} \sim \mathcal{N}(\mu_i, \frac{\sigma^2}{n}), \quad \forall i, \text{ and}\\ \bar{y}_{\cdot\cdot} &= \frac{1}{kn}\sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij} \sim \mathcal{N}(\frac{1}{k}\sum_{i=1}^{k}\mu_i, \frac{\sigma^2}{kn}).\end{aligned}$$

So

$$\begin{aligned}\mathbb{E}(y_{ij}^2) &= \mathbb{V}(y_{ij}) + \mathbb{E}^2(y_{ij}) = \sigma^2 + \mu_i^2, \quad \forall i, j,\\ \mathbb{E}(\bar{y}_{i\cdot}^2) &= \mathbb{V}(\bar{y}_{i\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot}) = \frac{\sigma^2}{n} + \mu_i^2, \quad \forall i, \text{ and}\\ \mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \mathbb{V}(\bar{y}_{\cdot\cdot}) + \mathbb{E}^2(\bar{y}_{\cdot\cdot}) = \frac{\sigma^2}{kn} + (\frac{1}{k}\sum_{i=1}^{k}\mu_i)^2.\end{aligned}$$

□

PROPOSITION 1.10 (Mean of $\mathrm{MS}_{\mathrm{err}}$)**.** We have

$$\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \sigma^2.$$

i.e., $\mathrm{MS}_{\mathrm{err}}$ is an unbiased estimator for σ^2.

Proof.

$$\begin{aligned}
\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \mathbb{E}(\mathrm{SS}_{\mathrm{err}}/(k(n-1))) &= \mathbb{E}(\frac{1}{k(n-1)}\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}-\bar{y}_{i\cdot})^2) \\
&= \mathbb{E}(\frac{1}{k(n-1)}\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}^2+\bar{y}_{i\cdot}^2-2y_{ij}\bar{y}_{i\cdot})),\ \text{expand} \\
&= \mathbb{E}(\frac{1}{k(n-1)}\left[\sum_{i=1}^{k}\sum_{j=1}^{n}y_{ij}^2+n\sum_{i=1}^{k}\bar{y}_{i\cdot}^2-2\sum_{i=1}^{k}\bar{y}_{i\cdot}\sum_{j=1}^{n}y_{ij}\right]),\ \text{separate} \\
&= \mathbb{E}(\frac{1}{k(n-1)}\left[\sum_{i=1}^{k}\sum_{j=1}^{n}y_{ij}^2+n\sum_{i=1}^{k}\bar{y}_{i\cdot}^2-2n\sum_{i=1}^{k}\bar{y}_{i\cdot}^2\right]),\ \text{reduce} \\
&= \mathbb{E}(\frac{1}{k(n-1)}\left[\sum_{i=1}^{k}\sum_{j=1}^{n}y_{ij}^2-n\sum_{i=1}^{k}\bar{y}_{i\cdot}^2\right]),\ \text{combine} \\
&= \frac{1}{k(n-1)}\left[\sum_{i=1}^{k}\sum_{j=1}^{n}\mathbb{E}(y_{ij}^2)-n\sum_{i=1}^{k}\mathbb{E}(\bar{y}_{i\cdot}^2)\right],\ \text{by linearity} \\
&= \frac{1}{k(n-1)}\left[\sum_{i=1}^{k}\sum_{j=1}^{n}(\sigma^2+\mu_i^2)-n\sum_{i=1}^{k}(\frac{\sigma^2}{n}+\mu_i^2)\right],\ \text{by Lemma 1.9} \\
&= \frac{1}{k(n-1)}\left[(kn-k)\sigma^2+n\sum_{i=1}^{k}(\mu_i^2-\mu_i^2)\right] = \sigma^2.
\end{aligned}$$

That is, $\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \sigma^2$, as desired. □

PROPOSITION 1.11 (Mean of $\mathrm{MS}_{\mathrm{trt}}$)**.** We have

$$\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) \geq \sigma^2$$

with equality holds if and only if $\mu = \mathbb{1}\mu_0 \in \mathbb{R}^k$ for some $\mu_0 \in \mathbb{R}$. i.e., $\mathrm{MS}_{\mathrm{trt}}$ is an unbiased estimator for σ^2 given that $\mu = \mathbb{1}\mu_0$ for some μ_0.

Proof.

$$
\begin{aligned}
\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) = \mathbb{E}(\mathrm{SS}_{\mathrm{trt}}/(k-1)) &= \mathbb{E}(\frac{n}{k-1}\sum_{i=1}^{k}(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2) \\
&= \mathbb{E}(\frac{n}{k-1}\sum_{i=1}^{k}(\bar{y}_{i\cdot}^2+\bar{y}_{\cdot\cdot}^2-2\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot})),\ \text{expand} \\
&= \mathbb{E}(\frac{n}{k-1}\left[\sum_{i=1}^{k}\bar{y}_{i\cdot}^2+k\bar{y}_{\cdot\cdot}^2-2\bar{y}_{\cdot\cdot}\sum_{i=1}^{k}\bar{y}_{i\cdot}\right]),\ \text{separate} \\
&= \mathbb{E}(\frac{n}{k-1}\left[\sum_{i=1}^{k}\bar{y}_{i\cdot}^2+k\bar{y}_{\cdot\cdot}^2-2k\bar{y}_{\cdot\cdot}^2\right]),\ \text{reduce} \\
&= \mathbb{E}(\frac{n}{k-1}\left[\sum_{i=1}^{k}\bar{y}_{i\cdot}^2-k\bar{y}_{\cdot\cdot}^2\right]),\ \text{combine} \\
&= \frac{n}{k-1}\left[\sum_{i=1}^{k}\mathbb{E}(\bar{y}_{i\cdot}^2)-k\mathbb{E}(\bar{y}_{\cdot\cdot}^2)\right],\ \text{by linearity} \\
&= \frac{n}{k-1}\left[\sum_{i=1}^{k}(\frac{\sigma^2}{n}+\mu_i^2)-k(\frac{\sigma^2}{kn}+(\frac{1}{k}\sum_{i=1}^{k}\mu_i)^2)\right],\ \text{by Lemma 1.9} \\
&= \frac{n}{k-1}\left[(\frac{k}{n}-\frac{1}{n})\sigma^2+\frac{1}{2k}\sum_{i,j=1}^{k}(\mu_i-\mu_j)^2\right]\geq\sigma^2.
\end{aligned}
$$

That is, $\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) \geq \sigma^2$. From the above derivation we can see that equality holds if and only if $\mu = \mathbb{1}\mu_0 \in \mathbb{R}^k$ for some $\mu_0 \in \mathbb{R}$. □

1.1.3 Hypothesis Testing for Completely Randomized Design

PROPOSITION 1.12. Consider the cases where $k = 2$. We are interested in testing the following hypothesis

$$H_0 : \mu_1 = \mu_2 \text{ vs } H_1 : \mu_1 \neq \mu_2.$$

The T-statistics are:

$$T_0 := \frac{\bar{y}_{1\cdot}-\bar{y}_{2\cdot}}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim \mathcal{N}(0,1)$$

in the case where σ^2 is known and reject the null if $|T_0| > z_{\alpha/2}$, or

$$T_0 := \frac{\bar{y}_{1\cdot}-\bar{y}_{2\cdot}}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim \mathcal{T}(n_1+n_2-2)$$

in the case where σ^2 is unknown and is estimated by s_p^2 and reject the null if $|T_0| > \mathcal{T}_{\alpha/2}(n_1 - n_2 - 2)$.

PROPOSITION 1.13. Consider the cases where $k = 2$. We are interested in testing the following hypothesis

$$H_0 : \mu_1 = \mu_2 \text{ vs } H_1 : \mu_1 > \mu_2.$$

We reject the null if $T_0 > +\mathcal{T}_\alpha(n_1 + n_2 - 2)$.

PROPOSITION 1.14. Consider the cases where $k = 2$. We are interested in testing the following hypothesis

$$H_0 : \mu_1 = \mu_2 \text{ vs } H_1 : \mu_1 < \mu_2.$$

We reject the null if $T_0 < -\mathcal{T}_\alpha(n_1 + n_2 - 2)$.

PROPOSITION 1.15. Consider the cases where $k = 2$. We are interested in testing the following hypothesis

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ vs } H_1 : \sigma_1^2 \neq \sigma_2^2.$$

The F-statistics is:

$$F_0 := s_1^2/s_2^2 \sim \mathcal{F}(n_1 - 1, n_2 - 1).$$

We reject the null if any of the following conditions hold:

- $F_0 > \mathcal{F}_{\alpha/2}(n_1 - 1, n_2 - 1)$ or $F_0 < \mathcal{F}_{1-\alpha/2}(n_1 - 1, n_2 - 1)$;

DEFINITION 1.16 (ANOVA Table)**.**

Table 1.1: ANOVA Table for Completely Randomized Design

	Sum of Squares	Degree of Freedom	Mean Squares	F_0
Treatment	$\mathrm{SS}_{\mathrm{trt}}$	$k-1$	$\mathrm{MS}_{\mathrm{trt}}$	$\mathrm{MS}_{\mathrm{trt}}/\mathrm{MS}_{\mathrm{err}}$
Error	$\mathrm{SS}_{\mathrm{err}}$	$k(n-1)$	$\mathrm{MS}_{\mathrm{err}}$	
Total	$\mathrm{SS}_{\mathrm{tot}}$	$kn-1$		

PROPOSITION 1.17. We are interested in testing the following hypothesis

$$H_0 : \forall i, j \in \{1, ..., k\}, \mu_i = \mu_j \text{ vs } H_1 : \exists i, j \in \{1, ..., k\}, \mu_i \neq \mu_j.$$

The F-statistics is

$$F_0 := \mathrm{MS}_{\mathrm{trt}}/\mathrm{MS}_{\mathrm{err}} \sim \mathcal{F}(k-1, k(n-1)).$$

We reject the null if any of the following conditions hold:

- $F_0 > \mathcal{F}_\alpha(k-1, k(n-1))$;
- $\Pr(\mathcal{F}(k-1, k(n-1)) > F_0) < \alpha$.

1.2 Completely Randomized Design - Model 2

DEFINITION 1.18 (Completely Randomized Desing - Model 2)**.** We model the observations as

$$y_{ij} = \mu + \alpha_i + e_{ij}, \text{ for } i \in \{1, ..., k\} \text{ and } j \in \{1, ..., n\}$$

with constraint $\mathbb{1}^\top \alpha = 0$ and $e_{ij} \sim \mathcal{N}(0, \sigma^2)$ are assumed to be independent. The number of parameters in the model is $2 + k$.

1.3 Randomized Block Design - Model 1

In the case where the number of treatments equals 2, this reduces to paired comparison design.

DEFINITION 1.19 (Randomized Block Design - Model 1)**.** Let $a \in \mathbb{Z}_{++}$ denote the number of treatments. Let $b \in \mathbb{Z}_{++}$ denote the number of blocks. We model the observations as

$$y_{ij} = \mu_i + \beta_j + e_{ij}, \text{ for } i \in \{1, ..., a\} \text{ and } j \in \{1, ..., b\}$$

where $e_{ij} \sim \mathcal{N}(0, \sigma^2)$ are assumed to be independent. The total number of parameters in this model is $a + b + 1$.

1.3.1 Hypothesis Testing

PROPOSITION 1.20. Consider the case where $k = 2$. We are interested in testing the following hypothesis

$$H_0 : \mu_1 = \mu_2 \text{ vs } H_1 : \mu_1 \neq \mu_2.$$

The T-statistics is

$$T_0 := \frac{\bar{y}_{1\cdot} - \bar{y}_{2\cdot}}{s_d/\sqrt{n}} \sim \mathcal{T}(n-1)$$

where

$$s_d^2 := \frac{1}{n-1} \sum_{j=1}^{n} \Big[(y_{2j} - \bar{y}_{2\cdot}) - (y_{1j} - \bar{y}_{1\cdot}) \Big]^2.$$

We reject the null if any of the following conditions hold

- $|T_0| > \mathcal{T}_{\alpha/2}(n-1)$;
- $\Pr(\mathcal{T}(n-1) > |T_0|) < \alpha/2$;
- $0 \notin \text{CI} = \Big[\bar{y}_{2\cdot} - \bar{y}_{1\cdot} \pm \mathcal{T}_{\alpha/2}(n-1) s_d/\sqrt{n} \Big]$.

1.4 Randomized Block Design - Model 2

DEFINITION 1.21 (Randomized Block Design - Model 2)**.** Let $a \in \mathbb{Z}_{++}$ denote the number of treatments. Let $b \in \mathbb{Z}_{++}$ denote the number of blocks. We model the observations as

$$y_{ij} = \mu + \alpha_i + \beta_j + e_{ij}, \text{ for } i \in \{1, ..., a\} \text{ and } j \in \{1, ..., b\}$$

with constraints $\mathbb{1}^\top \alpha = 0$ and $\mathbb{1}^\top \beta = 0$, and $e_{ij} \sim \mathcal{N}(0, \sigma^2)$ are assumed to be independent. The total number of parameters in this model is $2 + a + b$.

LEMMA 1.22. We have

$$\begin{aligned}
\mathbb{E}(y_{ij}^2) &= \sigma^2 + (\mu + \alpha_i + \beta_j)^2, \quad \forall i, j, \\
\mathbb{E}(\bar{y}_{i\cdot}^2) &= \frac{\sigma^2}{b} + (\mu + \alpha_i)^2, \quad \forall i, \\
\mathbb{E}(\bar{y}_{\cdot j}^2) &= \frac{\sigma^2}{a} + (\mu + \beta_j)^2, \quad \forall j, \text{ and} \\
\mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \frac{\sigma^2}{ab} + \mu^2.
\end{aligned}$$

Proof. Recall that $\forall i \in \{1, ..., a\}$, $\forall j \in \{1, ..., b\}$, we have $y_{ij} \sim \mathcal{N}(\mu + \alpha_i + \beta_j, \sigma^2)$. So

$$\begin{aligned}
\bar{y}_{i\cdot} &= \frac{1}{b} \sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu + \alpha_i, \frac{\sigma^2}{b}), \quad \forall i \in \{1, ..., a\}, \\
\bar{y}_{\cdot j} &= \frac{1}{a} \sum_{i=1}^{a} y_{ij} \sim \mathcal{N}(\mu + \beta_j, \frac{\sigma^2}{a}), \quad \forall j \in \{1, ..., b\}, \text{ and} \\
\bar{y}_{\cdot\cdot} &= \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu, \frac{\sigma^2}{ab}).
\end{aligned}$$

So

$$\begin{aligned}
\mathbb{E}(y_{ij}^2) &= \mathbb{V}(y_{ij}) + \mathbb{E}^2(y_{ij}) = \sigma^2 + (\mu + \alpha_i + \beta_j)^2, \quad \forall i, j, \\
\mathbb{E}(\bar{y}_{i\cdot}^2) &= \mathbb{V}(\bar{y}_{i\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot}) = \frac{\sigma^2}{b} + (\mu + \alpha_i)^2, \quad \forall i, \\
\mathbb{E}(\bar{y}_{\cdot j}^2) &= \mathbb{V}(\bar{y}_{\cdot j}) + \mathbb{E}^2(\bar{y}_{\cdot j}) = \frac{\sigma^2}{a} + (\mu + \beta_j)^2, \quad \forall j, \text{ and} \\
\mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \mathbb{V}(\bar{y}_{\cdot\cdot}) + \mathbb{E}^2(\bar{y}_{\cdot\cdot}) = \frac{\sigma^2}{ab} + \mu^2.
\end{aligned}$$

□

1.4.1 Estimation of Mean

PROPOSITION 1.23. Let y_{ij} for $i \in \{1, ..., a\}$ and $j \in \{1, ..., b\}$ be given. Consider the following optimization problem:

$$\begin{array}{lll} \text{(P)} & \min & f(\mu, \alpha, \beta) := \displaystyle\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \mu - \alpha_i - \beta_j)^2 \\ & \text{subject to:} & \mu \in \mathbb{R}, \alpha \in \mathbb{R}^a, \beta \in \mathbb{R}^b, \\ & & \mathbb{1}^\top \alpha = 0, \mathbb{1}^\top \beta = 0. \end{array}$$

Then the minimizer $(\hat{\mu}, \hat{\alpha}, \hat{\beta}) \in \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b$ of (P) is given by

$$\begin{aligned} \hat{\mu} &= \bar{y}_{\cdot\cdot} = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}, \\ \hat{\alpha}_i &= \bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot} = \frac{1}{b}\sum_{j=1}^{b} y_{ij} - \bar{y}_{\cdot\cdot}, \text{ for } i \in \{1, ..., a\}, \\ \hat{\beta}_j &= \bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot} = \frac{1}{a}\sum_{i=1}^{a} y_{ij} - \bar{y}_{\cdot\cdot}, \text{ for } j \in \{1, ..., b\}. \end{aligned}$$

Proof. Form the Lagrangian function $\mathcal{L} : \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b \oplus \mathbb{R} \oplus \mathbb{R}$

$$\mathcal{L}(\mu, \alpha, \beta, \xi, \eta) := f(\mu, \alpha, \beta) - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta.$$

Compute the derivatives:

$$\begin{aligned} \frac{\partial}{\partial \mu}\mathcal{L}(\mu, \alpha, \beta, \xi, \eta) &= \frac{\partial}{\partial \mu}\left[f(\mu, \alpha, \beta) - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta\right] \\ &= \frac{\partial}{\partial \mu}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \mu - \alpha_i - \beta_j)^2 - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta\right] \\ &= \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\partial}{\partial \mu}(y_{ij} - \mu - \alpha_i - \beta_j)^2 \\ &= -2\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \mu - \alpha_i - \beta_j), \\ \frac{\partial}{\partial \alpha_p}\mathcal{L}(\mu, \alpha, \beta, \xi, \eta) &= \frac{\partial}{\partial \alpha_p}\left[f(\mu, \alpha, \beta) - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta\right] \\ &= \frac{\partial}{\partial \alpha_p}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \mu - \alpha_i - \beta_j)^2 - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta\right] \\ &= \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\partial}{\partial \alpha_p}(y_{ij} - \mu - \alpha_i - \beta_j)^2 - \xi \frac{\partial}{\partial \alpha_p}\mathbb{1}^\top \alpha \end{aligned}$$

$$
\begin{aligned}
&= -2\sum_{j=1}^{b}(y_{pj} - \mu - \alpha_p - \beta_j) - \xi, \\
\frac{\partial}{\partial \beta_q}\mathcal{L}(\mu,\alpha,\beta,\xi,\eta) &= \frac{\partial}{\partial \beta_q}\left[f(\mu,\alpha,\beta) - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta\right] \\
&= \frac{\partial}{\partial \beta_q}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \mu - \alpha_i - \beta_j)^2 - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta\right] \\
&= \sum_{i=1}^{a}\frac{\partial}{\partial \beta_q}(y_{ij} - \mu - \alpha_i - \beta_j)^2 - \eta \frac{\partial}{\partial \beta_q}\mathbb{1}^\top \beta \\
&= -2\sum_{i=1}^{a}(y_{iq} - \mu - \alpha_i - \beta_q) - \eta, \\
\frac{\partial}{\partial \xi}\mathcal{L}(\mu,\alpha,\beta,\xi,\eta) &= \frac{\partial}{\partial \xi}\left[f(\mu,\alpha,\beta) - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta\right] = -\mathbb{1}^\top \alpha, \\
\frac{\partial}{\partial \eta}\mathcal{L}(\mu,\alpha,\beta,\xi,\eta) &= \frac{\partial}{\partial \eta}\left[f(\mu,\alpha,\beta) - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta\right] = -\mathbb{1}^\top \beta.
\end{aligned}
$$

Let $(\hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\xi}, \hat{\eta}) \in \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b \oplus \mathbb{R} \oplus \mathbb{R}$ be such that $\nabla \mathcal{L}(\hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\xi}, \hat{\eta}) = \mathbb{0} \in \mathbb{R}^{a+b+3}$. Then we get the following system of equations:

$$
\begin{cases}
-2\displaystyle\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j) = 0 \\
-2\displaystyle\sum_{j=1}^{b}(y_{pj} - \hat{\mu} - \hat{\alpha}_p - \hat{\beta}_j) - \hat{\xi} = 0, \forall p \\
-2\displaystyle\sum_{i=1}^{a}(y_{iq} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_q) - \hat{\eta} = 0, \forall q \\
-\mathbb{1}^\top \hat{\alpha} = 0 \\
-\mathbb{1}^\top \hat{\beta} = 0
\end{cases}
\implies
\begin{cases}
\hat{\mu} = \dfrac{1}{ab}\displaystyle\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij} \\
\hat{\alpha}_i = \bar{y}_{i\cdot} - \hat{\mu}, \forall i \\
\hat{\beta}_j = \bar{y}_{\cdot j} - \hat{\mu}, \forall j \\
\hat{\xi} = 0 \\
\hat{\eta} = 0.
\end{cases}
$$

Testing the Hessian of $\mathcal{L}$ at point $(\hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\xi}, \hat{\eta}) \in \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b \oplus \mathbb{R} \oplus \mathbb{R}$ confirms that it is indeed a minimizer of $\mathcal{L}$. □

PROPOSITION 1.24 (Mean of the Mean Estimator)**.** We have

$$\mathbb{E}(\hat{\mu}) = \mu \in \mathbb{R}, \ \mathbb{E}(\hat{\alpha}) = \alpha \in \mathbb{R}^a, \ \text{and } \mathbb{E}(\hat{\beta}) = \beta \in \mathbb{R}^b.$$

i.e., $\hat{\mu}$ is an unbiased estimator for μ, $\hat{\alpha} \in \mathbb{R}^a$ is an unbiased estimator for $\alpha \in \mathbb{R}^a$, and $\hat{\beta} \in \mathbb{R}^b$ is an unbiased estimator for $\beta \in \mathbb{R}^b$.

Proof. Recall that $\forall i \in \{1, ..., a\}$, $\forall j \in \{1, ..., b\}$, we have $y_{ij} \sim \mathcal{N}(\mu + \alpha_i + \beta_j, \sigma^2)$. So

$$\begin{aligned}
\bar{y}_{i\cdot} &= \frac{1}{b}\sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu + \alpha_i, \frac{\sigma^2}{b}), \quad \forall i \in \{1, ..., a\}, \\
\bar{y}_{\cdot j} &= \frac{1}{a}\sum_{i=1}^{a} y_{ij} \sim \mathcal{N}(\mu + \beta_j, \frac{\sigma^2}{a}), \quad \forall j \in \{1, ..., b\}, \\
\bar{y}_{\cdot\cdot} &= \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu, \frac{\sigma^2}{ab}).
\end{aligned}$$

Now we can compute

$$\begin{aligned}
\mathbb{E}(\hat{\mu}) &= \mathbb{E}(\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}) = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{E}(y_{ij}) = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}(\mu + \alpha_i + \beta_j) \\
&= \mu + \frac{1}{a}\sum_{i=1}^{a}\alpha_i + \frac{1}{b}\sum_{j=1}^{b}\beta_j = \mu, \\
\mathbb{E}(\hat{\alpha}_i) &= \mathbb{E}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) = \mathbb{E}(\bar{y}_{i\cdot}) - \mathbb{E}(\bar{y}_{\cdot\cdot}) = \mu + \alpha_i - \mu = \alpha_i, \quad \forall i, \\
\mathbb{E}(\hat{\beta}_j) &= \mathbb{E}(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) = \mathbb{E}(\bar{y}_{\cdot j}) - \mathbb{E}(\bar{y}_{\cdot\cdot}) = \mu + \beta_j - \mu = \beta_j, \quad \forall j.
\end{aligned}$$

That is, $\mathbb{E}(\hat{\mu}) = \mu$, $\mathbb{E}(\hat{\alpha}) = \alpha$, and $\mathbb{E}(\hat{\beta}) = \beta$, as desired. □

PROPOSITION 1.25 (Variance of the Mean Estimator)**.**

$$\begin{aligned}
\mathbb{V}(\hat{\mu}) &= \frac{\sigma^2}{ab}, \\
\mathbb{V}(\hat{\alpha}_i) &= \frac{a-1}{ab}\sigma^2, \quad \forall i \in \{1, ..., a\}, \text{ and} \\
\mathbb{V}(\hat{\beta}_j) &= \frac{b-1}{ab}\sigma^2, \quad \forall j \in \{1, ..., b\}.
\end{aligned}$$

Proof.

$$\begin{aligned}
\mathbb{V}(\hat{\mu}) &= \mathbb{V}(\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}) = \frac{1}{a^2b^2}\sum_{i=1}^{a}\sum_{j=1}^{b}\mathbb{V}(y_{ij}) = \frac{1}{a^2b^2}\sum_{i=1}^{a}\sum_{j=1}^{b}\sigma^2 = \frac{\sigma^2}{ab}, \\
\mathbb{V}(\hat{\alpha}_p) &= \mathbb{V}(\bar{y}_{p\cdot} - \bar{y}_{\cdot\cdot}) = \mathbb{V}(\bar{y}_{p\cdot} - \frac{1}{a}\sum_{i=1}^{a}\bar{y}_{i\cdot}) = \mathbb{V}((1 - \frac{1}{a})\bar{y}_{p\cdot} - \frac{1}{a}\sum_{\substack{i=1 \\ i\neq p}}^{a}\bar{y}_{i\cdot}) \\
&= (1 - \frac{1}{a})^2\mathbb{V}(\bar{y}_{p\cdot}) + \frac{1}{a^2}\sum_{\substack{i=1 \\ i\neq p}}^{a}\mathbb{V}(\bar{y}_{i\cdot}) = (1 - \frac{1}{a})^2\frac{\sigma^2}{b} + \frac{1}{a^2}\sum_{\substack{i=1 \\ i\neq p}}^{a}\frac{\sigma^2}{b} \\
&= \frac{a-1}{ab}\sigma^2, \quad \forall p \in \{1, ..., a\}, \text{ and}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{cov}(\hat{\alpha}_p, \hat{\alpha}_q) &= \mathbb{E}(\hat{\alpha}_p \hat{\alpha}_q) - \mathbb{E}(\hat{\alpha}_p)\mathbb{E}(\hat{\alpha}_q) = \mathbb{E}((\bar{y}_{p\cdot} - \bar{y}_{\cdot\cdot})(\bar{y}_{q\cdot} - \bar{y}_{\cdot\cdot})) - \alpha_p \alpha_q \\
&= \mathbb{E}(\bar{y}_{p\cdot}\bar{y}_{q\cdot}) - \mathbb{E}(\bar{y}_{p\cdot}\bar{y}_{\cdot\cdot}) - \mathbb{E}(\bar{y}_{q\cdot}\bar{y}_{\cdot\cdot}) + \mathbb{E}(\bar{y}_{\cdot\cdot}\bar{y}_{\cdot\cdot}) - \alpha_p \alpha_q \\
&= \mathbb{E}((\frac{1}{b}\sum_{j=1}^{b} y_{pj})(\frac{1}{b}\sum_{j=1}^{b} y_{qj})) + \mathbb{E}((\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij})(\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij})) \\
&- \mathbb{E}((\frac{1}{b}\sum_{j=1}^{b} y_{pj})(\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij})) - \mathbb{E}((\frac{1}{b}\sum_{j=1}^{b} y_{qj})(\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij})) - \alpha_p \alpha_q \\
&= (\frac{1}{b}\sum_{j=1}^{b} \mathbb{E}(y_{pj}))(\frac{1}{b}\sum_{j=1}^{b} \mathbb{E}(y_{qj})) - \alpha_p \alpha_q \\
&+ \frac{1}{a^2b^2}\sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{E}(y_{ij}^2) + (\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{E}(y_{ij}))(\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{E}(y_{ij})) - \frac{1}{a^2b^2}\sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{E}^2(y_{ij}) \\
&- \frac{1}{ab^2}\sum_{j=1}^{b} \mathbb{E}(y_{pj}^2) - (\frac{1}{b}\sum_{j=1}^{b} \mathbb{E}(y_{pj}))(\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{E}(y_{ij})) + \frac{1}{ab^2}\sum_{j=1}^{b} \mathbb{E}^2(y_{pj}) \\
&- \frac{1}{ab^2}\sum_{j=1}^{b} \mathbb{E}(y_{qj}^2) - (\frac{1}{b}\sum_{j=1}^{b} \mathbb{E}(y_{qj}))(\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{E}(y_{ij})) + \frac{1}{ab^2}\sum_{j=1}^{b} \mathbb{E}^2(y_{qj}) \\
&= (\mu + \alpha_p)(\mu + \alpha_q) - \alpha_p \alpha_q + \sigma^2 + \mu^2 - (\mu + \alpha_p)\mu - \frac{1}{ab}\sigma^2 - (\mu + \alpha_q)\mu - \frac{1}{ab}\sigma^2 \\
&= (1 - \frac{2}{ab})\sigma^2, \quad \forall p, q \in \{1, ..., a\} : p \neq q, \\
\mathbb{V}(\hat{\beta}_r) &= \mathbb{V}(\bar{y}_{\cdot r} - \bar{y}_{\cdot\cdot}) = \mathbb{V}(\bar{y}_{\cdot r} - \frac{1}{b}\sum_{j=1}^{b} \bar{y}_{\cdot j}) = \mathbb{V}((1 - \frac{1}{b})\bar{y}_{\cdot r} - \frac{1}{b}\sum_{\substack{j=1 \\ j \neq r}}^{b} \bar{y}_{\cdot j}) \\
&= (1 - \frac{1}{b})^2 \mathbb{V}(\bar{y}_{\cdot r}) + \frac{1}{b^2}\sum_{\substack{j=1 \\ j \neq r}}^{b} \mathbb{V}(\bar{y}_{\cdot j}) = (1 - \frac{1}{b})^2 \frac{\sigma^2}{a} + \frac{1}{b^2}\sum_{\substack{j=1 \\ j \neq r}}^{b} \frac{\sigma^2}{a} \\
&= \frac{b-1}{ab}\sigma^2, \quad \forall r \in \{1, ..., b\}.
\end{aligned}
$$

$\square$

1.4.2 Estimation of Variance

DEFINITION 1.26 (Sum of Squares)**.** We define the following terms:

$$\mathrm{SS}_{\mathrm{trt}} := b\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2,$$

$$\mathrm{SS}_{\mathrm{blk}} := a\sum_{j=1}^{b}(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})^2,$$

$$\mathrm{SS}_{\mathrm{err}} := \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot})^2,$$
$$\mathrm{SS}_{\mathrm{tot}} := \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{\cdot\cdot})^2.$$

PROPOSITION 1.27 (Decomposition of $\mathrm{SS}_{\mathrm{tot}}$)**.** We have

$$\mathrm{SS}_{\mathrm{tot}} = \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{blk}} + \mathrm{SS}_{\mathrm{err}}.$$

Proof.

$$\begin{aligned}
\mathrm{SS}_{\mathrm{tot}} &= \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{\cdot\cdot})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\left[(y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot}) + (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) + (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})\right]^2 \\
&= \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{blk}} + \mathrm{SS}_{\mathrm{err}} + 2\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) \\
&+ 2\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot})(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) + 2\sum_{j=1}^{b}\sum_{i=1}^{a}(y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot})(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) \\
&= \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{blk}} + \mathrm{SS}_{\mathrm{err}} + 2\left[\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})\right]\left[\sum_{j=1}^{b}(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})\right] \\
&+ 2\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})\sum_{j=1}^{b}(y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot}) + 2\sum_{j=1}^{b}(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})\sum_{i=1}^{a}(y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot}) \\
&= \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{blk}} + \mathrm{SS}_{\mathrm{err}} + 2(a\bar{y}_{\cdot\cdot} - a\bar{y}_{\cdot\cdot})(b\bar{y}_{\cdot\cdot} - b\bar{y}_{\cdot\cdot}) \\
&+ 2\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})(b\bar{y}_{i\cdot} - b\bar{y}_{i\cdot} - b\bar{y}_{\cdot\cdot} - b\bar{y}_{\cdot\cdot}) + 2\sum_{j=1}^{b}(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})(a\bar{y}_{\cdot j} - a\bar{y}_{\cdot\cdot} - a\bar{y}_{\cdot j} + a\bar{y}_{\cdot\cdot}) \\
&= \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{blk}} + \mathrm{SS}_{\mathrm{err}} + 2\cdot 0\cdot 0 + 2\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})\cdot 0 + 2\sum_{j=1}^{b}(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})\cdot 0 \\
&= \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{blk}} + \mathrm{SS}_{\mathrm{err}}.
\end{aligned}$$

$\square$

DEFINITION 1.28 (Mean Squares). We define the following estimators for the variance σ^2.

$$\mathrm{MS}_{\mathrm{trt}} := \mathrm{SS}_{\mathrm{trt}}/(a-1),$$

$$\mathrm{MS}_{\mathrm{err}} := \mathrm{SS}_{\mathrm{err}}/((a-1)(b-1)).$$

PROPOSITION 1.29. We have

$$\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \sigma^2.$$

i.e., $\mathrm{MS}_{\mathrm{err}}$ is an unbiased estimator for σ^2.

Proof.

$$\begin{aligned}
\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) &= \mathbb{E}(\mathrm{SS}_{\mathrm{err}}/((a-1)(b-1))) = \mathbb{E}(\frac{1}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i\cdot}-\bar{y}_{\cdot j}+\bar{y}_{\cdot\cdot})^2) \\
&= \mathbb{E}\Bigg(\frac{1}{(a-1)(b-1)}\Bigg[\sum_{i=1}^{a}\sum_{j=1}^{b}\begin{array}{llll}
+y_{ij}^2 & -y_{ij}\bar{y}_{i\cdot} & -y_{ij}\bar{y}_{\cdot j} & +y_{ij}\bar{y}_{\cdot\cdot} \\
-\bar{y}_{i\cdot}y_{ij} & +\bar{y}_{i\cdot}^2 & +\bar{y}_{i\cdot}\bar{y}_{\cdot j} & -\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot} \\
-\bar{y}_{\cdot j}y_{ij} & +\bar{y}_{\cdot j}\bar{y}_{i\cdot} & +\bar{y}_{\cdot j}^2 & -\bar{y}_{\cdot j}\bar{y}_{\cdot\cdot} \\
+\bar{y}_{\cdot\cdot}y_{ij} & -\bar{y}_{\cdot\cdot}\bar{y}_{i\cdot} & -\bar{y}_{\cdot\cdot}\bar{y}_{\cdot j} & +\bar{y}_{\cdot\cdot}^2
\end{array}\Bigg]\Bigg) \\
&= \mathbb{E}\Bigg(\frac{1}{(a-1)(b-1)}\Bigg[\begin{array}{llll}
+\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}^2 & -\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}\bar{y}_{i\cdot} & -\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}\bar{y}_{\cdot j} & +\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}\bar{y}_{\cdot\cdot} \\
-\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{i\cdot}y_{ij} & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{i\cdot}^2 & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{i\cdot}\bar{y}_{\cdot j} & -\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot} \\
-\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot j}y_{ij} & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot j}\bar{y}_{i\cdot} & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 & -\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot j}\bar{y}_{\cdot\cdot} \\
+\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot\cdot}y_{ij} & -\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot\cdot}\bar{y}_{i\cdot} & -\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot\cdot}\bar{y}_{\cdot j} & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot\cdot}^2
\end{array}\Bigg]\Bigg) \\
&= \mathbb{E}\Bigg(\frac{1}{(a-1)(b-1)}\Bigg[\begin{array}{llll}
+\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}^2 & -b\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 & -a\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 & +ab\bar{y}_{\cdot\cdot}^2 \\
-b\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 & +b\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 & +ab\bar{y}_{\cdot\cdot}^2 & -ab\bar{y}_{\cdot\cdot}^2 \\
-a\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 & +ab\bar{y}_{\cdot\cdot}^2 & +a\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 & -ab\bar{y}_{\cdot\cdot}^2 \\
+ab\bar{y}_{\cdot\cdot}^2 & -ab\bar{y}_{\cdot\cdot}^2 & -ab\bar{y}_{\cdot\cdot}^2 & +ab\bar{y}_{\cdot\cdot}^2
\end{array}\Bigg]\Bigg) \\
&= \frac{1}{(a-1)(b-1)}\mathbb{E}\Bigg(\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}^2 - a\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 - b\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 + ab\bar{y}_{\cdot\cdot}^2\Bigg), \text{ combine}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(a-1)(b-1)}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\mathbb{E}(y_{ij}^2) - a\sum_{j=1}^{b}\mathbb{E}(\bar{y}_{\cdot j}^2) - b\sum_{i=1}^{a}\mathbb{E}(\bar{y}_{i\cdot}^2) + ab\mathbb{E}(\bar{y}_{\cdot\cdot}^2)\right], \text{ by linearity} \\
&= \frac{1}{(a-1)(b-1)}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(\sigma^2 + (\mu+\alpha_i+\beta_j)^2) + ab(\frac{\sigma^2}{ab}+\mu^2)\right. \\
&\qquad \left. - a\sum_{j=1}^{b}(\frac{\sigma^2}{a} + (\mu+\beta_j)^2) - b\sum_{i=1}^{a}(\frac{\sigma^2}{b} + (\mu+\alpha_i)^2)\right], \text{ by Lemma 1.22} \\
&= \frac{1}{(a-1)(b-1)}\left[(ab+1-a-b)\sigma^2 + (ab+ab-ab-ab)\mu^2\right. \\
&\qquad \left. + 0\mu + (b-b)\sum_{i=1}^{a}\alpha_i^2 + (a-a)\sum_{j=1}^{b}\beta_j^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i\beta_j\right] \\
&= \frac{1}{(a-1)(b-1)}(ab+1-a-b)\sigma^2 = \sigma^2.
\end{aligned}
$$

That is, $\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \sigma^2$, as desired. □

PROPOSITION 1.30 (Mean of $\mathrm{MS}_{\mathrm{trt}}$)**.** We have

$$\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) \geq \sigma^2$$

with equality holds if and only if $\alpha = \mathbb{0} \in \mathbb{R}^a$. i.e., $\mathrm{MS}_{\mathrm{trt}}$ is an unbiased estimator for σ^2 given that $\alpha = \mathbb{0}$.

Proof.

$$
\begin{aligned}
\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) &= \mathbb{E}(\mathrm{SS}_{\mathrm{trt}}/(a-1)) = \mathbb{E}(\frac{b}{a-1}\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2) \\
&= \mathbb{E}(\frac{b}{a-1}\sum_{i=1}^{a}(\bar{y}_{i\cdot}^2 + \bar{y}_{\cdot\cdot}^2 - 2\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot})), \text{ expand} \\
&= \mathbb{E}(\frac{b}{a-1}\left[\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 + a\bar{y}_{\cdot\cdot}^2 - 2\bar{y}_{\cdot\cdot}\sum_{i=1}^{a}\bar{y}_{i\cdot}\right]), \text{ separate} \\
&= \mathbb{E}(\frac{b}{a-1}\left[\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 + a\bar{y}_{\cdot\cdot}^2 - 2a\bar{y}_{\cdot\cdot}^2\right]), \text{ reduce} \\
&= \mathbb{E}(\frac{b}{a-1}\left[\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 - a\bar{y}_{\cdot\cdot}^2\right]), \text{ combine} \\
&= \frac{b}{a-1}\left[\sum_{i=1}^{a}\mathbb{E}(\bar{y}_{i\cdot}^2) - a\mathbb{E}(\bar{y}_{\cdot\cdot}^2)\right], \text{ by linearity}
\end{aligned}
$$

$$= \frac{b}{a-1}\left[\sum_{i=1}^{a}(\frac{\sigma^2}{b} + (\mu+\alpha_i)^2) - a(\frac{\sigma^2}{ab} + \mu^2)\right], \text{ by Lemma 1.22}$$
$$= \frac{b}{a-1}\left[(\frac{a}{b} - \frac{1}{b})\sigma^2 + \sum_{i=1}^{a}\alpha_i^2\right] \geq \sigma^2.$$

That is, $\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) \geq \sigma^2$. From the above derivation we can see that equality holds if and only if $\alpha = \mathbb{0} \in \mathbb{R}^a$. □

1.4.3 Hypothesis Testing for Randomized Block Design

DEFINITION 1.31 (ANOVA Table)**.**

Table 1.2: ANOVA Table for Randomized Block Design

	Sum of Squares	Degrees of Freedom	Mean Squares	F_0
Treatment	$\mathrm{SS}_{\mathrm{trt}}$	$a-1$	$\mathrm{MS}_{\mathrm{trt}}$	$\mathrm{MS}_{\mathrm{trt}}/\mathrm{MS}_{\mathrm{err}}$
Block	$\mathrm{SS}_{\mathrm{blk}}$	$b-1$	$\mathrm{MS}_{\mathrm{blk}}$	
Error	$\mathrm{SS}_{\mathrm{err}}$	$(a-1)(b-1)$	$\mathrm{MS}_{\mathrm{err}}$	
Total	$\mathrm{SS}_{\mathrm{tot}}$	$ab-1$		

PROPOSITION 1.32. We are interested in testing the following hypothesis:

$$H_0 : \alpha = \mathbb{0} \in \mathbb{R}^a \text{ vs } H_1 : \alpha \neq \mathbb{0} \in \mathbb{R}^a.$$

The F-statistics is

$$F_0 := \mathrm{MS}_{\mathrm{trt}}/\mathrm{MS}_{\mathrm{err}} \sim \mathcal{F}(a-1, (a-1)(b-1)).$$

We reject the null if $F_0 > \mathcal{F}_\alpha(a-1, (a-1)(b-1))$.

1.5 Two-Way Factorial Design

DEFINITION 1.33. Let $a \in \mathbb{Z}_{++}$ denote the number of treatments of factor A. Let $b \in \mathbb{Z}_{++}$ denote the number of treatments of factor B. Let $n \in \mathbb{Z}_{++}$ denote the number of repetitions for each combination of treatments. We model the observations as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}, \text{ for } i \in \{1, ..., a\}, j \in \{1, ..., b\}, k \in \{1, ..., n\}$$

with constraints $\mathbb{1}^\top \alpha = 0$, $\mathbb{1}^\top \beta = 0$, $\gamma^\top \mathbb{1} = \mathbb{0}$, and $\gamma \mathbb{1} = \mathbb{0}$, and $e_{ijk} \sim \mathcal{N}(0, \sigma^2)$ are assumed to be independent. The total number of parameters in this model is $2 + a + b + ab$.

1.5.1 Estimation of Mean

PROPOSITION 1.34.

$$\begin{aligned}
\hat{\mu} = \bar{y}_{\cdots} &= \frac{1}{abn} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk}, \\
\hat{\alpha}_i &= \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}, \\
\hat{\beta}_j &= \bar{y}_{\cdot j \cdot} - \bar{y}_{\cdots}, \\
\hat{\gamma}_{ij} &= \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots}.
\end{aligned}$$

PROPOSITION 1.35. We have

$$\mathbb{E}(\hat{\mu}) = \mu \in \mathbb{R}, \ \mathbb{E}(\hat{\alpha}) = \alpha \in \mathbb{R}^a, \ \mathbb{E}(\hat{\beta}) = \beta \in \mathbb{R}^b, \text{ and } \mathbb{E}(\hat{\gamma}) = \gamma \in \mathbb{R}^{a \times b}.$$

i.e., ...

Proof.

$$\begin{aligned}
\mathbb{E}(\hat{\mu}) &= ... \\
\mathbb{E}(\hat{\gamma}_{ij}) &= \mathbb{E}(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots}) = \mathbb{E}(\bar{y}_{ij\cdot}) - \mathbb{E}(\bar{y}_{i\cdot\cdot}) - \mathbb{E}(\bar{y}_{\cdot j\cdot}) + \mathbb{E}(\bar{y}_{\cdots}) \\
&= (\mu + \alpha_i + \beta_j + \gamma_{ij}) - (\mu + \alpha_i) - (\mu + \beta_j) + (\mu) = \gamma_{ij},
\end{aligned}$$

□

1.5.2 Estimation of Variance

DEFINITION 1.36 (Sum of Squares)**.** We define the following terms:

$$\begin{aligned}
\mathrm{SS_A} &:= bn\sum_{i=1}^{a}(\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot\cdot\cdot})^2\\
\mathrm{SS_B} &:= an\sum_{j=1}^{b}(\bar{y}_{\cdot j\cdot}-\bar{y}_{\cdot\cdot\cdot})^2\\
\mathrm{SS_{AB}} &:= n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij\cdot}-\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot j\cdot}+\bar{y}_{\cdot\cdot\cdot})^2\\
\mathrm{SS_{err}} &:= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij\cdot})^2\\
\mathrm{SS_{tot}} &:= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{\cdot\cdot\cdot})^2.
\end{aligned}$$

REMARK 1.37. Using triple indexing for vectors in $\mathbb{R}^{abn}$, we define vectors x_{A}, x_{B}, x_{AB}, x_{err}, $x_{\mathrm{tot}} \in \mathbb{R}^{abn}$ by

$$(x_{\mathrm{A}})_{i\cdot\cdot} := \bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot\cdot\cdot},\quad (x_{\mathrm{B}})_{\cdot j\cdot} := \bar{y}_{\cdot j\cdot}-\bar{y}_{\cdot\cdot\cdot},\quad (x_{\mathrm{AB}})_{ij\cdot} := \bar{y}_{ij\cdot}-\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot j\cdot}+\bar{y}_{\cdot\cdot\cdot},$$
$$(x_{\mathrm{err}})_{ijk} := y_{ijk}-\bar{y}_{ij\cdot},\ \text{and}\ (x_{\mathrm{tot}})_{ijk} := y_{ijk}-\bar{y}_{\cdot\cdot\cdot}.$$

Then $\forall I \in \{\mathrm{A},\mathrm{B},\mathrm{AB},\mathrm{err},\mathrm{tot}\}$, we have $\mathrm{SS}_I = \|x_I\|_2^2$; and $\forall I,J \in \{\mathrm{A},\mathrm{B},\mathrm{AB},\mathrm{err}\}$, we have $\langle x_I, x_J\rangle = 0$; and $x_{\mathrm{tot}} = x_{\mathrm{A}}+x_{\mathrm{B}}+x_{\mathrm{AB}}+x_{\mathrm{err}}$.

PROPOSITION 1.38 (Decomposition of $\mathrm{SS_{tot}}$)**.** We have

$$\mathrm{SS_{tot}} = \mathrm{SS_A}+\mathrm{SS_B}+\mathrm{SS_{AB}}+\mathrm{SS_{err}}.$$

Proof.

$$\begin{aligned}
\mathrm{SS_{tot}} &= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{\cdot\cdot\cdot})^2\\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\Big[(\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot\cdot\cdot})+(\bar{y}_{\cdot j\cdot}-\bar{y}_{\cdot\cdot\cdot})+(\bar{y}_{ij\cdot}-\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot j\cdot}+\bar{y}_{\cdot\cdot\cdot})+(y_{ijk}-\bar{y}_{ij\cdot})\Big]^2
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{SS_A} + \mathrm{SS_B} + \mathrm{SS_{AB}} + \mathrm{SS_{err}} + 2n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot}) \\
&+ 2\sum_{i=1}^{a}\sum_{j=1}^{b}\Big[(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot}) + (\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot}) + (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})\Big]\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{ij\cdot}) \\
&+ 2n\sum_{i=1}^{a}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})\sum_{j=1}^{b}(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot}) \\
&+ 2n\sum_{j=1}^{b}(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})\sum_{i=1}^{a}(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot}) \\
&= \mathrm{SS_A} + \mathrm{SS_B} + \mathrm{SS_{AB}} + \mathrm{SS_{err}} + 2(a\bar{y}_{\cdot\cdot\cdot} - a\bar{y}_{\cdot\cdot\cdot})(b\bar{y}_{\cdot\cdot\cdot} - b\bar{y}_{\cdot\cdot\cdot}) \\
&+ 2\sum_{i=1}^{a}\sum_{j=1}^{b}\Big[(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot}) + (\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot}) + (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})\Big](n\bar{y}_{ij\cdot} - n\bar{y}_{ij\cdot}) \\
&+ 2\sum_{k=1}^{n}\sum_{i=1}^{a}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})(b\bar{y}_{i\cdot\cdot} - b\bar{y}_{i\cdot\cdot} - b\bar{y}_{\cdot\cdot\cdot} + b\bar{y}_{\cdot\cdot\cdot}) \\
&+ 2\sum_{k=1}^{n}\sum_{j=1}^{b}(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})(a\bar{y}_{\cdot j\cdot} - a\bar{y}_{\cdot\cdot\cdot} - a\bar{y}_{\cdot j\cdot} + a\bar{y}_{\cdot\cdot\cdot}) \\
&= \mathrm{SS_A} + \mathrm{SS_B} + \mathrm{SS_{AB}} + 0 = \mathrm{SS_A} + \mathrm{SS_B} + \mathrm{SS_{AB}}.
\end{aligned}
$$

□

DEFINITION 1.39 (Mean Squares)**.** We define the following estimators for the variance σ^2.

$$
\begin{aligned}
\mathrm{MS_A} &:= \mathrm{SS_A}/(a-1), \\
\mathrm{MS_B} &:= \mathrm{SS_B}/(b-1), \\
\mathrm{MS_{AB}} &:= \mathrm{SS_{AB}}/((a-1)(b-1)), \\
\mathrm{MS_{err}} &:= \mathrm{SS_{err}}/(ab(n-1)).
\end{aligned}
$$

LEMMA 1.40. We have

$$
\begin{aligned}
\mathbb{E}(y_{ijk}^2) &= \sigma^2 + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2, \quad \forall i, j, k \\
\mathbb{E}(\bar{y}_{ij\cdot}^2) &= \frac{\sigma^2}{n} + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2, \quad \forall i, j, \\
\mathbb{E}(\bar{y}_{i\cdot\cdot}^2) &= \frac{\sigma^2}{bn} + \mu^2, \quad \forall i, \text{ and}
\end{aligned}
$$

$$\mathbb{E}(\bar{y}_{\cdots}^2) = \frac{\sigma^2}{abn} + \mu^2.$$

Proof. Recall that $\forall i \in \{1, ..., a\}$, $\forall j \in \{1, ..., b\}$, $\forall k \in \{1, ..., n\}$, we have $y_{ijk} \sim \mathcal{N}(\mu + \alpha_i + \beta_j + \gamma_{ij}, \sigma^2)$. So

$$\begin{aligned}
\bar{y}_{ij\cdot} &= \frac{1}{n}\sum_{k=1}^{n} y_{ijk} \sim \mathcal{N}(\mu + \alpha_i + \beta_j + \gamma_{ij}, \frac{\sigma^2}{n}), \quad \forall i, j, \\
\bar{y}_{i\cdot\cdot} &= \frac{1}{b}\sum_{j=1}^{b} \bar{y}_{ij\cdot} \sim \mathcal{N}(\mu + \alpha_i, \frac{\sigma^2}{bn}), \quad \forall i, \\
\bar{y}_{\cdot j\cdot} &= \frac{1}{a}\sum_{i=1}^{a} \bar{y}_{ij\cdot} \sim \mathcal{N}(\mu + \beta_j, \frac{\sigma^2}{an}), \quad \forall j, \text{ and} \\
\bar{y}_{\cdots} &= \frac{1}{a}\sum_{i=1}^{a} \bar{y}_{i\cdot\cdot} \sim \mathcal{N}(\mu, \frac{\sigma^2}{abn}).
\end{aligned}$$

So

$$\begin{aligned}
&\mathbb{E}(y_{ijk}^2) = \mathbb{V}(y_{ijk}) + \mathbb{E}^2(y_{ijk}) = \sigma^2 + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2, \quad \forall i, j, k, \\
&\mathbb{E}(\bar{y}_{ij\cdot}^2) = \mathbb{V}(\bar{y}_{ij\cdot}) + \mathbb{E}^2(\bar{y}_{ij\cdot}) = \frac{\sigma^2}{n} + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2, \quad \forall i, j, \\
&\mathbb{E}(\bar{y}_{i\cdot\cdot}^2) = \mathbb{V}(\bar{y}_{i\cdot\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot\cdot}) = \frac{\sigma^2}{bn} + (\mu + \alpha_i)^2, \quad \forall i, \\
&\mathbb{E}(\bar{y}_{\cdot j\cdot}^2) = \mathbb{V}(\bar{y}_{\cdot j\cdot}) + \mathbb{E}^2(\bar{y}_{\cdot j\cdot}) = \frac{\sigma^2}{an} + (\mu + \beta_j)^2, \quad \forall j, \text{ and} \\
&\mathbb{E}(\bar{y}_{\cdots}^2) = \mathbb{V}(\bar{y}_{\cdots}) + \mathbb{E}^2(\bar{y}_{\cdots}) = \frac{\sigma^2}{abn} + \mu^2.
\end{aligned}$$

$\square$

PROPOSITION 1.41. We have

$$\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \sigma^2.$$

i.e., $\mathrm{MS}_{\mathrm{err}}$ is an unbiased estimator for σ^2.

Proof.

$$\begin{aligned}
\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) &= \mathbb{E}(\frac{1}{ab(n-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{ij\cdot})^2) \\
&= \mathbb{E}(\frac{1}{ab(n-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left[y_{ijk}^2 + \bar{y}_{ij\cdot}^2 - 2y_{ijk}\bar{y}_{ij\cdot}\right]), \text{ expand}
\end{aligned}$$

$$= \mathbb{E}(\frac{1}{ab(n-1)}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 + n\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{ij\cdot}^2 - 2\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{ij\cdot}\sum_{k=1}^{n} y_{ijk}\right]), \text{ separate}$$

$$= \mathbb{E}(\frac{1}{ab(n-1)}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 + n\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{ij\cdot}^2 - 2n\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{ij\cdot}^2\right]), \text{ reduce}$$

$$= \mathbb{E}(\frac{1}{ab(n-1)}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - n\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{ij\cdot}^2\right]), \text{ combine}$$

$$= \frac{1}{ab(n-1)}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} \mathbb{E}(y_{ijk}^2) - n\sum_{i=1}^{a}\sum_{j=1}^{b}\mathbb{E}(\bar{y}_{ij\cdot}^2)\right], \text{ by linearity}$$

$$= \frac{1}{ab(n-1)}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(\sigma^2 + (\mu+\alpha_i+\beta_j+\gamma_{ij})^2)\right.$$

$$\left. - n\sum_{i=1}^{a}\sum_{j=1}^{b}(\frac{\sigma^2}{n} + (\mu+\alpha_i+\beta_j+\gamma_{ij})^2)\right], \text{ by Lemma 1.40}$$

$$= \frac{1}{ab(n-1)}\left[abn\sigma^2 - ab\sigma^2\right] = \sigma^2.$$

□

PROPOSITION 1.42 (Mean of $\mathrm{MS_A}$)**.** We have

$$\mathbb{E}(\mathrm{MS_A}) \geq \sigma^2$$

with equality holds if and only if $\alpha = \mathbb{0} \in \mathbb{R}^a$. i.e., $\mathrm{MS_A}$ is an unbiased estimator for σ^2 given that $\alpha = \mathbb{0}$.

Proof.

$$\mathbb{E}(\mathrm{MS_A}) = \mathbb{E}(\frac{1}{a-1}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2)$$

$$= \mathbb{E}(\frac{1}{a-1}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(\bar{y}_{i\cdot\cdot}^2 + \bar{y}_{\cdot\cdot\cdot}^2 - 2\bar{y}_{i\cdot\cdot}\bar{y}_{\cdot\cdot\cdot})), \text{ expand}$$

$$= \mathbb{E}(\frac{1}{a-1}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\bar{y}_{i\cdot\cdot}^2 + abn\bar{y}_{\cdot\cdot\cdot}^2 - 2bn\bar{y}_{\cdot\cdot\cdot}\sum_{i=1}^{a}\bar{y}_{i\cdot\cdot}\right]), \text{ separate}$$

$$= \mathbb{E}(\frac{1}{a-1}\left[bn\sum_{i=1}^{a}\bar{y}_{i\cdot\cdot}^2 + abn\bar{y}_{\cdot\cdot\cdot}^2 - 2abn\bar{y}_{\cdot\cdot\cdot}^2\right]), \text{ reduce}$$

$$= \mathbb{E}(\frac{1}{a-1}\left[bn\sum_{i=1}^{a}\bar{y}_{i\cdot\cdot}^2 - abn\bar{y}_{\cdot\cdot\cdot}^2\right]), \text{ combine}$$

$$= \frac{1}{a-1}\left[bn\sum_{i=1}^{a}\mathbb{E}(\bar{y}_{i\cdot\cdot}^2) - abn\mathbb{E}(\bar{y}_{\cdot\cdot\cdot}^2)\right], \text{ by linearity}$$
$$= \frac{1}{a-1}\left[bn\sum_{i=1}^{a}(\frac{\sigma^2}{bn} + (\mu+\alpha_i)^2) - abn(\frac{\sigma^2}{abn} + \mu^2)\right], \text{ by Lemma 1.40}$$
$$= \frac{1}{a-1}\left[(a-1)\sigma^2 + bn\sum_{i=1}^{a}\alpha_i^2\right] \geq \sigma^2.$$

That is, $\mathbb{E}(\text{MS}_\text{A}) \geq \sigma^2$. From the above derivation we can see that equality holds if and only if $\alpha = \mathbb{0} \in \mathbb{R}^a$. □

PROPOSITION 1.43 (Mean of MS_B)**.** We have

$$\mathbb{E}(\text{MS}_\text{B}) = \sigma^2$$

with equality holds if and only if $\beta = \mathbb{0} \in \mathbb{R}^b$. i.e., MS_B is an unbiased estimator for σ^2 given that $\beta = \mathbb{0}$.

PROPOSITION 1.44. Under the assumption that $\gamma = \mathbb{0} \in \mathbb{R}^{a\times b}$, we have

$$\mathbb{E}(\text{MS}_\text{AB}) = \sigma^2.$$

i.e., MS_AB is an unbiased estimator for σ^2 given that $\gamma = \mathbb{0}$.

1.5.3 Hypothesis Testing for Two-Way Factorial Design

DEFINITION 1.45 (ANOVA Table)**.**

Table 1.3: ANOVA Table for Two-Way Factorial Design

	Sum of Squares	Degrees of Freedom	Mean Squares	F_0
A	SS_A	$a-1$	MS_A	$\text{MS}_\text{A}/\text{MS}_\text{err}$
B	SS_B	$b-1$	MS_B	$\text{MS}_\text{B}/\text{MS}_\text{err}$
AB	SS_AB	$(a-1)(b-1)$	MS_AB	$\text{MS}_\text{AB}/\text{MS}_\text{err}$
Error	SS_err	$ab(n-1)$	MS_err	
Total	SS_tot	$abn-1$		

PROPOSITION 1.46. We are interested in testing the following hypothesis

$$H_0 : \alpha = \mathbb{0} \in \mathbb{R}^a \text{ vs } H_1 : \alpha \neq \mathbb{0}.$$

The F-statistics is

$$F_0 := \mathrm{MS_A}/\mathrm{MS_{err}} \sim \mathcal{F}(a-1, ab(n-1)).$$

We reject the null if $F_0 > \mathcal{F}_\alpha(a-1, ab(n-1))$.

PROPOSITION 1.47. We are interested in testing the following hypothesis

$$H_0 : \beta = \mathbb{0} \in \mathbb{R}^b \text{ vs } H_1 : \beta \neq \mathbb{0}.$$

The F-statistics is

$$F_0 := \mathrm{MS_B}/\mathrm{MS_{err}} \sim \mathcal{F}(b-1, ab(n-1)).$$

We reject the null if $F_0 > \mathcal{F}_\alpha(b-1, ab(n-1))$.

PROPOSITION 1.48. We are interested in testing the following hypothesis

$$H_0 : \gamma = \mathbb{0} \in \mathbb{R}^{a\times b} \text{ vs } H_1 : \gamma \neq \mathbb{0}.$$

The F-statistics is

$$F_0 := \mathrm{MS_{AB}}/\mathrm{MS_{err}} \sim \mathcal{F}((a-1)(b-1), ab(n-1)).$$

We reject the null if $F_0 > \mathcal{F}_\alpha((a-1)(b-1), ab(n-1))$.

1.6 Two-Level Factorial Design

1.7 Multiple Comparison Tests

DEFINITION 1.49.

$$\hat{\theta} := \sum_{i=1}^{k} a_i \hat{\mu}_i = \sum_{i=1}^{k} a_i \bar{y}_{i\cdot}.$$

PROPOSITION 1.50. We have

$$\mathbb{E}(\hat{\theta}) = \theta.$$

1.7.1 Estimation of Variance

DEFINITION 1.51. Let θ be a contrast with coefficient vector $a \in \mathbb{R}^k$. We define

$$\mathrm{SS}_\theta := n\left[\sum_{i=1}^{k} \frac{a_i}{\|a\|_2} \bar{y}_{i\cdot}\right]^2.$$

PROPOSITION 1.52 (Decomposition of $\mathrm{SS}_{\mathrm{trt}}$)**.** If $\theta_1, ..., \theta_{k-1}$ are $k-1$ orthogonal contrasts, then

$$\mathrm{SS}_{\mathrm{trt}} = \sum_{i=1}^{k-1} \mathrm{SS}_{\theta_i}.$$

Proof.

$$\begin{aligned}
\mathrm{SS}_{\mathrm{trt}} &= n\sum_{i=1}^{k}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 = n\sum_{i=1}^{k}\bar{y}_{i\cdot}^2 - nk\bar{y}_{\cdot\cdot}^2 \\
&= n\sum_{j=1}^{k-1}\sum_{p=1}^{k}\sum_{q=1}^{k} \frac{a_p^{(j)}}{\|a^{(j)}\|_2}\frac{a_q^{(j)}}{\|a^{(j)}\|_2}\bar{y}_{p\cdot}\bar{y}_{q\cdot} \\
&= n\sum_{j=1}^{k-1}\left[\sum_{i=1}^{k}\frac{a_i^{(j)}}{\|a^{(j)}\|_2}\bar{y}_{i\cdot}\right]^2.
\end{aligned}$$

□

1.7.2 Hypothesis Testing

PROPOSITION 1.53. We are interested in testing the following hypothesis

$$H_0 : \theta = \sum_{i=1}^{k} a_i \mu_i = 0 \text{ vs } H_1 : \theta = \sum_{i=1}^{k} a_i \mu_i \neq 0.$$

In the case where σ is known, the T-statistics is

$$T_0 := \frac{\hat{\theta}}{\sqrt{\mathbb{V}(\hat{\theta})}} = \frac{\sum_{i=1}^{k} a_i \bar{y}_{i\cdot}}{\sqrt{\frac{\sigma^2}{n} \sum_{i=1}^{k} a_i^2}} \sim \mathcal{N}(0, 1).$$

In the cases where σ is unknown and is estimated by $\hat{\sigma}$, the T-statistics is

$$T_0 := \frac{\hat{\theta}}{\sqrt{v(\hat{\theta})}} = \frac{\sum_{i=1}^{k} a_i \bar{y}_{i\cdot}}{\sqrt{\frac{\hat{\sigma}^2}{n} \sum_{i=1}^{k} a_i^2}} \sim \mathcal{T}(k(n-1)).$$

We reject the null if any of the following conditions hold:

- $|T_0| \geq t_{\alpha/2}(k(n-1))$;
- $\Pr(\mathcal{T}(k(n-1)) \geq |T_0|) \leq \alpha/2$;
- $0 \notin \text{CI} = \left[\hat{\theta} \pm \mathcal{T}_{\alpha/2}(k(n-1))\sqrt{v(\hat{\theta})}\right]$.

PROPOSITION 1.54. The F-statistics is

$$F_0 := \frac{\hat{\theta}^2}{v(\hat{\theta})} \sim \mathcal{F}(1, k(n-1)).$$

We reject the null if any of the following conditions hold:

- $F_0 \geq \mathcal{F}_\alpha(1, k(n-1))$;
- $\Pr(\mathcal{F}(1, k(n-1)) \geq F_0) \leq \alpha$;