Convex Analysis

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Contents

Chapter 1

Affine Sets

1.1 Definitions

DEFINITION (Affine Combination)**.** Let S be a set in $\mathbb{E}$. We define an **affine combination** of S to be a point x in the space of the form

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where (1) $n \in \mathbb{N}$, (2) $v_i \in S$ for all i, (3) $\lambda_i \in \mathbb{R}$ for all i, and (4) $\sum_{i=1}^{n} \lambda_i = 1$.

DEFINITION (Affine Span)**.** Let S be a set in $\mathbb{E}$. We define the **affine span** of S, denoted by $\operatorname{affspan}(S)$, to be the set of all affine combinations of S.

DEFINITION (Affine Set)**.** Let S be a set in $\mathbb{E}$. We say that S is an **affine set** if $S = \operatorname{aff}(S)$.

DEFINITION (Affine Hull)**.** Let S be a set in $\mathbb{E}$. We define the **affine hull** of S, denoted by $\operatorname{affhull}(S)$, to be the smallest affine set containing S.

Chapter 2

Relative Topology

2.1 Definitions

DEFINITION (Relative Interior)**.** Let $\mathbb{E}$ be some Euclidean space. Let S be a set in the space. We define the **relative interior** of S, denoted by $\mathrm{ri}(S)$, or $\mathrm{relint}(S)$, to be the interior of S for the topology relative to the affine hull $\mathrm{aff}(S)$. i.e., the set given by

$$\mathrm{ri}(S) := \{x \in \mathrm{aff}(S) : \exists r > 0, \mathrm{ball}(x, r) \cap \mathrm{aff}(S) \subseteq S\}.$$

A quick result. For a singleton set S, $\mathrm{ri}(S) = S = \mathrm{cl}(S)$.

2.2 Basic Properties

PROPOSITION 2.2.1. For any set S, we have $\mathrm{ri}(S) \subseteq S$.

REMARK. The relative interior operator is not monotonic.

EXAMPLE 2.2.1. Consider $\mathbb{R}$ with the usual topology and sets $\{0\}$ and $[0, 1]$. Then $\mathrm{ri}(\{0\}) = \{0\}$ and $\mathrm{ri}([0, 1]) = (0, 1)$.

PROPOSITION 2.2.2. Let S be a set in some Euclidean space $\mathbb{E}$. Then if $int(S) \neq \emptyset$, $\mathrm{ri}(S) = int(S)$.

Proof.

It suffices to show that $\mathrm{aff}(S) = \mathbb{R}^n$.

Since $int(S) \neq \emptyset$, $\exists x \in int(S)$.

Since $x \in int(S)$, $\exists r > 0, ball(x, r) \subseteq S$.

$\mathbb{E} = \mathrm{aff}(ball(x, r)) \subseteq \mathrm{aff}(S) \subseteq \mathbb{E}$.

This shows $\mathrm{aff}(S) = \mathbb{E}$.

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2.3 Arithmetic Properties

PROPOSITION 2.3.1. Let C_1 and C_2 be convex subsets of $\mathbb{E}$. Let λ_1 and λ_2 be scalars in $\mathbb{R}$. Then

$$\mathrm{ri}(\lambda_1 C_1 + \lambda_2 C_2) = \lambda_1 \, \mathrm{ri}(C_1) + \lambda_2 \, \mathrm{ri}(C_2).$$

PROPOSITION 2.3.2. Let C_1 be a convex set in $\mathbb{E}_1$. Let C_2 be a convex set in $\mathbb{E}_2$. Then

$$\mathrm{ri}(C_1 \oplus C_2) = \mathrm{ri}(C_1) \oplus \mathrm{ri}(C_2).$$

Chapter 3

Convex Sets

3.1 Definitions

DEFINITION (Convex Combination)**.** Let $\mathcal{V}$ be a vector space over field $\mathbb{F}$. Let S be a subset of $\mathcal{V}$. We define a **convex combination** of S to be a point x in $\mathcal{V}$ of the form

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where (1) $n \in \mathbb{N}$, (2) $v_1, ..., v_n \in S$, (3) $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$, and (4) $\sum_{i=1}^{n} \lambda_i = 1$.

DEFINITION (Convex Span)**.** Let $\mathbb{E}$ be a Euclidean space. Let S be a subset of $\mathbb{E}$. We define a **convex span** of S, denoted by $\operatorname{convspan}(S)$, to be the set of all convex combinations of S.

DEFINITION (Convex)**.** Let $\mathbb{E}$ be a Euclidean space. Let S be a subset of $\mathbb{E}$. We say that S is **convex** if $S = \operatorname{convspan}(S)$, or equivalently, if

$$\forall x, y \in S, \forall \alpha, \beta \in [0, 1] : \alpha + \beta = 1, \quad \alpha x + \beta y \in S.$$

DEFINITION (Pointed)**.** Let $\mathbb{E}$ be a Euclidean space. Let S be a subset of $\mathbb{E}$. We say that S is **pointed** if S contains no line.

3.2 Arithmetic Properties of Convex Sets

PROPOSITION 3.2.1. Let C be a convex set. Let λ_1 and λ_2 be in $\mathbb{R}_+$. Then

$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C.$$

Proof.

The case where any of λ_1 and λ_2 is 0 is trivial. I will assume that $\lambda_1, \lambda_2 > 0$.

For one direction, let x be an arbitrary point in $(\lambda_1 + \lambda_2)C$.

Since $x \in (\lambda_1 + \lambda_2)C$, $\exists c \in C, x = (\lambda_1 + \lambda_2)c$.

Since $\begin{cases} (\lambda_1 + \lambda_2)c = \lambda_1 c + \lambda_2 c \\ x = (\lambda_1 + \lambda_2)c \end{cases}$, we get $x = \lambda_1 c + \lambda_2 c$.

Since $\begin{cases} x = \lambda_1 c + \lambda_2 c \\ \lambda_1 c \in \lambda_1 C \\ \lambda_2 c \in \lambda_2 C \end{cases}$, we get $x \in \lambda_1 C + \lambda_2 C$.

Since $x \in \lambda_1 C + \lambda_2 C$ for any $x \in (\lambda_1 + \lambda_2)C$, $\lambda_1 C + \lambda_2 C \subseteq (\lambda_1 + \lambda_2)C$.

For the reverse direction, let x be an arbitrary point in $\lambda_1 C + \lambda_2 C$.

Since $x \in \lambda_1 C + \lambda_2 C$, $\exists c_1, c_2 \in C, x = \lambda_1 c_1 + \lambda_2 c_2$.

Define scalars $\mu_1 := \frac{\lambda_1}{\lambda_1 + \lambda_2}$ and $\mu_2 := \frac{\lambda_2}{\lambda_1 + \lambda_2}$.

Then $x = (\lambda_1 + \lambda_2)c$.

Since $\lambda_1, \lambda_2 > 0$, $\mu_1, \mu_2 \in [0, 1]$.

Define point $c := \mu_1 c_1 + \mu_2 c_2$.

Since $\begin{cases} c = \mu_1 c_1 + \mu_2 c_2 \\ c_1, c_2 \in C \\ \mu_1, \mu_2 \in [0, 1] \\ \mu_1 + \mu_2 = 1 \\ C \text{ is convex} \end{cases}$, we get $c \in C$.

Since $x = (\lambda_1 + \lambda_2)c$ and $c \in C$, $x \in \lambda_1 + \lambda_2)C$.

Since $x \in \lambda_1 + \lambda_2)C$ for any $x \in \lambda_1 C + \lambda_2 C$, $(\lambda_1 + \lambda_2)C \subseteq \lambda_1 C + \lambda_2 C$.

Since $\lambda_1 C + \lambda_2 C \subseteq (\lambda_1 + \lambda_2)C$ and $(\lambda_1 + \lambda_2)C \subseteq \lambda_1 C + \lambda_2 C$, $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$

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3.3 Topological Properties of Convex Sets

THEOREM 3.1. Let C be a convex set such that $\mathrm{int}(C) \neq \emptyset$. Then

(1) $\mathrm{int}(C) = \mathrm{int}(\mathrm{cl}(C))$, and

(2) $\mathrm{cl}(C) = \mathrm{cl}(\mathrm{int}(C))$.

Proof of (1). $int(C) \subseteq int(\mathrm{cl}(C))$ is clear. For $int(\mathrm{cl}(C)) \subseteq int(C)$, let x be an arbitrary point in $int(\mathrm{cl}(C))$.

Since $x \in int(\mathrm{cl}(C))$,

$$\exists r > 0 \text{ such that } \mathrm{ball}(x, r) \subseteq \mathrm{cl}(C).$$

Since $int(C) \neq \emptyset$, pick $y \in int(C)$.

Define a scalar λ by

$$\lambda := \frac{r}{2\|x - y\|}.$$

Define a point z by

$$z := x + \lambda(x - y).$$

Since $\lambda = \frac{r}{2\|x-y\|}$ and $z = x + \lambda(x - y)$,

$$\begin{aligned}
&\|z - x\| \\
&= \big\|x + \lambda(x - y) - x\big\| \\
&= \big\|\lambda(x - y)\big\| \\
&= \lambda\|x - y\| \\
&= \frac{r}{2\|x - y\|}\|x - y\| \\
&= \frac{r}{2} \\
&< r.
\end{aligned}$$

That is,

$$\|z - x\| < r.$$

So $z \in \mathrm{ball}(x, r)$. It follows that $z \in \mathrm{cl}(C)$.

Since $z = x + \lambda(x - y)$, rearranging this yields

$$x = \frac{1}{1 + \lambda} z + \frac{\lambda}{1 + \lambda} y.$$

Since $\begin{cases} x = \frac{1}{1+\lambda}z + \frac{\lambda}{1+\lambda}y \\ z \in \text{cl}(C) \\ y \in int(C) \\ \frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda} \in (0,1) \\ \frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} = 1 \end{cases}$, by the lemma, we get

$$x \in int(C).$$

Since $\forall x \in int(\text{cl}(C)), x \in int(C)$, we get $int(\text{cl}(C)) \subseteq int(C)$.

∎

Proof of (2). $\text{cl}(int(C)) \subseteq \text{cl}(C)$ is clear. For $\text{cl}(C) \subseteq \text{cl}(int(C))$, let x be an arbitrary point in $\text{cl}(C)$.

Since $int(C) \neq \emptyset$, pick $y \in int(C)$.

Let $\lambda \in [0,1)$.

Define a point z by

$$z(\lambda) := \lambda x + (1-\lambda)y.$$

Since $\begin{cases} z(\lambda) := \lambda x + (1-\lambda)y \\ x \in \text{cl}(C) \\ y \in int(C) \\ \lambda \in [0,1) \end{cases}$, by the lemma, we get

$$z(\lambda) \in int(C).$$

Since $\begin{cases} z(\lambda) \in int(C) \\ \lim_{\lambda \to 1} z(\lambda) = x \end{cases}$, we get

$$x \in \text{cl}(int(C)).$$

Since $\forall x \in \text{cl}(C), x \in \text{cl}(int(C))$, we get $\text{cl}(C) \subseteq \text{cl}(int(C))$.

∎

PROPOSITION 3.3.1. Let C be a convex set. Then

(1) $\text{aff}(\text{ri}(C)) = \text{aff}(C) = \text{aff}(\text{cl}(C))$,

(2) $\text{ri}(\text{ri}(C)) = \text{ri}(C) = \text{ri}(\text{cl}(C))$, and

(3) $\text{cl}(\text{ri}(C)) = \text{cl}(C) = \text{cl}(\text{cl}(C))$.

PROPOSITION 3.3.2. Let C be a convex set. Then

$$C \neq \emptyset \iff \mathrm{ri}(C) \neq \emptyset.$$

Proof. **Forward Direction**: Assume that $C \neq \emptyset$. I will show that $\mathrm{ri}(C) \neq \emptyset$. Since $C \neq \emptyset$, $\mathrm{aff}(C) \neq \emptyset$. Since C is convex, $\mathrm{aff}(C) = \mathrm{aff}(\mathrm{ri}(C))$. Since $\begin{cases} \mathrm{aff}(C) \neq \emptyset \\ \mathrm{aff}(C) = \mathrm{aff}(\mathrm{ri}(C)) \end{cases}$, we get

$$\mathrm{aff}(\mathrm{ri}(C)) \neq \emptyset.$$

Since $\mathrm{aff}(\mathrm{ri}(C)) \neq \emptyset$, we get $\mathrm{ri}(C) \neq \emptyset$.

Backward Direction: Assume that $\mathrm{ri}(C) \neq \emptyset$. I will show that $C \neq \emptyset$. Since $\mathrm{ri}(C) \neq \emptyset$ and $\mathrm{ri}(C) \subseteq C$, we get $C \neq \emptyset$. ■

3.4 The Convex Hull Operator

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DEFINITION (Convex Hull)**.** Let S be a set in $\mathbb{E}$. We define the **convex hull** of S, denoted by $\mathrm{convhull}(S)$, to be the smallest convex set containing S.

PROPOSITION 3.4.1. For any set S, $\mathrm{convspan}(S) = \mathrm{convhull}(S)$. They will both be denoted by $\mathrm{conv}(S)$ from now on.

Proof. **Forward Direction**: I will show that $\mathrm{convspan}(S) \subseteq \mathrm{convhull}(S)$. Let x be an arbitrary element of $\mathrm{convspan}(S)$. We are to prove that $x \in convhull(S)$. Let C be an arbitrary convex set containing S. Since x is a convex combination of S, x is also a convex combination of C. Since x is a convex combination of C and C is convex, $x \in C$. Since x is in any convex set containing S, $x \in convhull(S)$. Since $x \in convhull(S)$ for any $x \in convspan(S)$, $convspan(S) \subseteq convhull(S)$.

Backward Direction: I will show that $\mathrm{convhull}(S) \subseteq \mathrm{convspan}(S)$.

proof incomplete. ■

not finished

PROPOSITION 3.4.2 (The Convex Hull Operator)**.** Let $\mathbb{E}$ be a Euclidean space.

(1) Expansive
$$\forall S \subseteq \mathbb{E}, \quad S \subseteq \operatorname{conv}(S).$$

(2) Monotonic Increasing
$$\forall S_1, S_2 \subseteq \mathbb{E} : S_1 \subseteq S_2, \quad \operatorname{conv}(S_1) \subseteq \operatorname{conv}(S_2).$$

(3) Idempotent
$$\forall S \subseteq \mathbb{E}, \quad \operatorname{conv}(\operatorname{conv}(S)) = \operatorname{conv}(S).$$

PROPOSITION 3.4.3 (Bounded)**.** The convex hull of a bounded set is bounded.

PROPOSITION 3.4.4 (Open)**.** The convex hull of an open set is open.

Proof. Let $\mathcal{V}$ be a topological vector space. Let G be an open subset of $\mathcal{V}$. I will show that $\operatorname{conv}(G)$ is open. Let $\sum_{i=1}^{n} \lambda_i x_i$ be an arbitrary convex combination of elements of G. Let $i_0 \in \{1..n\}$ be such that $\lambda_{i_0} \neq 0$. Then
$$\sum_{i=1}^{n} \lambda_i x_i \in \sum_{i \neq i_0} \lambda_i x_i + i_0 G \subseteq \operatorname{conv}(G).$$
So
$$\operatorname{conv}(G) = \bigcup \left\{ \sum_{i \neq i_0} \lambda_i x_i + i_0 G \right\}.$$
Note that the function $f(x) := \sum_{i \neq i_0} \lambda_i x_i + i_0 x$ is a homeomorphism. So $\operatorname{conv}(G)$ is a union of open sets and hence open. ■

REMARK (Closed)**.** The convex hull of a closed set need not be closed.

EXAMPLE 3.4.1. The set $S := \{(x, y) \in \mathbb{R}^2 : y \geq \frac{1}{1+x^2}\}$ is closed. However, $\operatorname{conv}(S) = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is open.

PROPOSITION 3.4.5 (Compact)**.** The convex hull of a compact set is compact.

3.5 The Closed Convex Hull Operator

DEFINITION (Closed Convex Hull)**.** Let S be a set in some Euclidean space. We define the **closed convex hull** of S, denoted by $\overline{\text{conv}}(S)$, to be the smallest <u>closed</u> convex containing S.

PROPOSITION 3.5.1. The closed convex hull is the closure of the convex hull.

PROPOSITION 3.5.2. A closed convex hull does not distinguish a set from its closure. i.e., for any set S, we have $\overline{\text{conv}}(S) = \overline{\text{conv}}(\text{cl}(S))$.

PROPOSITION 3.5.3. If S is bounded, then the closure operation and the convex hull operation commute. i.e., $\text{conv}(\text{cl}(S)) = \text{cl}(\text{conv}(S))$.

REMARK. The closure operation and the convex hull operation do not commute in general.

3.6 Stability of Convexity

PROPOSITION 3.6.1 (Intersection)**.** Convexity is stable under intersection. i.e., the intersection of any collection of convex sets is convex.

Proof. Let $\{C_i\}_{i\in I}$ be an arbitrary collection of convex sets where I is an index set and C_i is convex for any $i \in I$. Let C denote their intersection. If $C = \emptyset$, then we are done. Else, let x and y be two arbitrary points in C. Let λ be an arbitrary number in $(0,1)$. Define a point $z := \lambda x + (1-\lambda)y$. Since $x \in C$ and $C = \bigcap_{i\in I} C_i$, we get $x \in C_i$ for any $i \in I$. Since $y \in C$ and $C = \bigcap_{i\in I} C_i$, we get $y \in C_i$ for any $i \in I$. Let i be an arbitrary index in I. Since $x \in C_i$ and $y \in C_i$ and $\lambda \in (0,1)$ and C_i is convex and $z = \lambda x + (1-\lambda)y$, we get $z \in C_i$. Since $z \in C_i$ for any $i \in I$ and $C = \bigcap_{i\in I} C_i$, we get $z \in C$. Since

$$\forall x, y \in C, \forall \lambda \in (0,1), \quad \lambda x + (1-\lambda)y \in C,$$

by definition of convex sets, we get C is convex. ∎

PROPOSITION 3.6.2 (Affine Map)**.** Convexity is stable under affine mapping. i.e., the affine image of a convex set is convex.

PROPOSITION 3.6.3 (Linear Combinations)**.** Convexity is stable under linear combinations. i.e., if C_1 and C_2 are convex sets and λ_1 and λ_2 are real numbers, then the set C defined as

$$C := \lambda_1 C_1 + \lambda_2 C_2$$

is convex.

Proof. If $C_1 = \emptyset$ or $C_2 = \emptyset$, then $\lambda_1 C_1 + \lambda_2 C_2 = \emptyset$ and we are done. Now assume that $C_1, C_2 \neq \emptyset$. Then $C = \lambda_1 C_1 + \lambda_2 C_2 \neq \emptyset$. Let x and y be arbitrary points in C.
Since $x \in C$, $\exists x_1 \in C_1, x_2 \in C_2$ such that $x = \lambda_1 x_1 + \lambda_2 x_2$.
Since $y \in C$, $\exists y_1 \in C_1, y_2 \in C_2$ such that $y = \lambda_1 y_1 + \lambda_2 y_2$.
Let $\lambda \in [0,1]$ be arbitrary. Define a point z as $z := \lambda x + (1-\lambda)y$. Then

$$\begin{aligned} z &= \lambda x + (1-\lambda)y \\ &= \lambda\big(\lambda_1 x_1 + \lambda_2 x_2\big) + (1-\lambda)\big(\lambda_1 y_1 + \lambda_2 y_2\big) \\ &= \lambda_1\big(\lambda x_1 + (1-\lambda)y_1\big) + \lambda_2\big(\lambda x_2 + (1-\lambda)y_2\big). \end{aligned}$$

Since $x_1, y_1 \in C_1$, $\lambda \in [0,1]$ and C_1 is convex, we get $\lambda x_1 + (1-\lambda)y_1 \in C_1$.
Since $x_2, y_2 \in C_2$, $\lambda \in [0,1]$ and C_2 is convex, we get $\lambda x_2 + (1-\lambda)y_2 \in C_2$.
So $z = \lambda_1(\lambda x_1 + (1-\lambda)y_1) + \lambda_2(\lambda x_2 + (1-\lambda)y_2) \in \lambda_1 C_1 + \lambda_2 C_2$.
That is, $\forall x \in C$, $\forall y \in C$, $\forall \lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in C$.
So by definition, C is convex. ∎

COROLLARY 3.1. The Minkowski sum of two convex sets is convex.

LEMMA 3.1. Let C be a convex set in $\mathbb{E}$. Let $x \in \operatorname{int}(C)$. Let $y \in \operatorname{cl}(C)$. Then

$$\forall \lambda \in (0,1], \quad \lambda x + (1-\lambda) y \in C.$$

Proof.

Since $x \in int(S)$, there exists some radius r_x such that $\operatorname{ball}(x, r_x) \subseteq S$.

Define $r_z := \lambda r_x$.

Let z' be an arbitrary point in $\operatorname{ball}(z, r_z)$.

Define $x' := \frac{1}{\lambda}(z' - (1-\lambda)y)$.

Notice

$$\begin{aligned}
&\|x - x'\| \\
&= \frac{1}{|\lambda|}\|\lambda x - \lambda x'\| \\
&= \frac{1}{|\lambda|}\big\|(z - (1-\lambda)y) - (z' - (1-\lambda)y)\big\| \\
&= \frac{1}{|\lambda|}\|z - z'\| \\
&\leq \frac{1}{|\lambda|} r_z, \text{ since } z' \in \operatorname{ball}(z, r_z) \\
&= \frac{1}{|\lambda|}\lambda r_x \\
&= r_x.
\end{aligned}$$

That is,

$$\|x - x'\| \leq r_x.$$

So $x' \in \operatorname{ball}(x, r_x)$.

Since $x' \in ball(x, r_x)$ and $ball(x, r_x) \subseteq S$, we get $x' \in S$.

Since $\begin{cases} z' = \lambda x' + (1-\lambda)y \\ x', y \in S \\ \lambda \in (0,1] \\ S \text{ is convex} \end{cases}$, we get $z' \in S$.

Since $z' \in S$ for any $z' \in ball(z, r_z)$, $ball(z, r_z) \subseteq S$.

Since there exists some radius r_z such that $ball(z, r_z) \subseteq S$, $z \in int(S)$.

Alternative Expressing:

Define $B := ball(0,1)$.

$$\begin{aligned}
& (1-\lambda)x + \lambda y + \varepsilon B \\
\subseteq\ & (1-\lambda)x + \lambda(C + \varepsilon B) + \varepsilon B \\
=\ & (1-\lambda)x + \lambda C + \lambda\varepsilon B + \varepsilon B \\
=\ & (1-\lambda)x + (1+\lambda)\varepsilon B + \lambda C \\
=\ & (1-\lambda)(x + \frac{1+\lambda}{1-\lambda})\varepsilon B) + \lambda C \\
\subseteq\ & (1-\lambda)C + \lambda C \\
=\ & C.
\end{aligned}$$

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LEMMA 3.2. Let C be a convex set in $\mathbb{E}$. Let $x \in \mathrm{ri}(C)$. Let $y \in \mathrm{cl}(C)$. Then

$$\forall \lambda \in (0,1], \quad \lambda x + (1-\lambda)y \in C.$$

Proof.

Case 1. $int(C) \neq \emptyset$.

Then $int(C) = \mathrm{ri}(C)$.

Since $x \in int(C)$ and $y \in \mathrm{cl}(C)$, $\forall t \in (0,1], z := tx + (1-t)y \in C$.

Case 2. $int(C) = \emptyset$.

Now $\dim(C) < d$.

Say $\dim(C) = l$.

Apply case 1 in $\mathbb{R}^l$.

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PROPOSITION 3.6.4 (Interior)**.** Convexity is stable under interior. i.e., the interior of a convex set is convex.

Proof. Let S be a convex set in $\mathbb{E}$. If $int(S) = \emptyset$, then we are done. Else: let x and y be two arbitrary points in $int(S)$. Let λ be an arbitrary number in $(0,1)$. Define a point z by $z := \lambda x + (1-\lambda)y$. Since $x, y \in int(S)$ and $\lambda \in (0,1)$, by the lemma, we get $z \in int(S)$. Since

$$\forall x, y \in int(S), \forall \lambda \in (0,1), \quad \lambda x + (1-\lambda)y \in int(S),$$

we get $int(S)$ is convex. ■

PROPOSITION 3.6.5 (Relative Interior)**.** Convexity is stable under relative interior. i.e., the relative interior of a convex set is convex.

Proof. Let S be a convex set in $\mathbb{E}$. If $\mathrm{ri}(S) = \emptyset$, then we are done. Else: let x and y be two arbitrary points in $\mathrm{ri}(S)$. Let λ be an arbitrary number in $(0,1)$. Define a point z by $z := \lambda x + (1-\lambda)y$. Since $x, y \in \mathrm{ri}(S)$ and $\lambda \in (0,1)$, by the lemma, we get $z \in \mathrm{ri}(S)$. Since

$$\forall x, y \in \mathrm{ri}(S), \forall \lambda \in (0,1), \quad \lambda x + (1-\lambda)y \in \mathrm{ri}(S),$$

we get $\mathrm{ri}(S)$ is convex. ■

PROPOSITION 3.6.6 (Closure)**.** Convexity is stable under closure. i.e., the closure of a convex set is convex.

Proof Approach 1.

Let $x, y \in \mathrm{cl}(C)$.

Let $t \in [0,1]$.

Since $x \in \mathrm{cl}(C)$, $\exists \{x_i\}_{i\in\mathbb{N}} \subseteq C, \lim_{i\to\infty} x_i = x$.

Since $y \in \mathrm{cl}(C)$, $\exists \{y_i\}_{i\in\mathbb{N}} \subseteq C, \lim_{i\to\infty} y_i = y$.

Since $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} y_i = y$, $\lim_{i\to\infty} \big(tx_i + (1-t)y_i\big) = tx + (1-t)y$.

Since $x_i, y_i \in C$ and C is convex, $tx_i + (1-t)y_i \in C$.

Since $tx_i + (1-t)y_i \in C$ $\lim_{i\to\infty} \big(tx_i + (1-t)y_i\big) = tx + (1-t)y$, $tx + (1-t)y \in \mathrm{cl}(C)$.

Since $\forall x, y \in \mathrm{cl}(C), \forall t \in [0,1], tx + (1-t)y \in \mathrm{cl}(C)$, we get $\mathrm{cl}(C)$ is convex.

■

Proof Approach 2.

$\mathrm{cl}(C) = \bigcap_{\varepsilon>0} \big[C + \varepsilon\, \mathrm{ball}(0,1)\big]$. This is an intersection of linear combinations of convex sets and hence convex.

■

PROPOSITION 3.6.7 (Conical Hull)**.** Convexity is stable under conical hull. i.e., if C is convex, then $\mathrm{cone}(C)$ is convex.

Proof.

Let x and y be arbitrary points in $\mathrm{cone}(C)$.

Let λ be an arbitrary number in $(0,1)$.

Define point z as $z := \lambda x + (1-\lambda)y$.

Since $x \in \text{cone}(C)$, $\exists x' \in C$ and $\exists \alpha > 0$ such that $x = \alpha x'$.

Since $y \in \text{cone}(C)$, $\exists y' \in C$ and $\exists \beta > 0$ such that $y = \beta y'$.

Define point z' as $z' := \frac{\lambda\alpha}{\lambda\alpha+(1-\lambda)\beta}x' + \frac{(1-\lambda)\beta}{\lambda\alpha+(1-\lambda)\beta}y'$.

Since $x', y' \in C$ and $\frac{\lambda\alpha}{\lambda\alpha+(1-\lambda)\beta} \in (0,1)$ and $\frac{\lambda\alpha}{\lambda\alpha+(1-\lambda)\beta} + \frac{(1-\lambda)\beta}{\lambda\alpha+(1-\lambda)\beta} = 1$ and C is convex and $z' := \frac{\lambda\alpha}{\lambda\alpha+(1-\lambda)\beta}x' + \frac{(1-\lambda)\beta}{\lambda\alpha+(1-\lambda)\beta}y'$, we get $z' \in C$.

Since $z' \in C$ and $z = (\lambda\alpha + (1-\lambda)\beta)z'$, $z \in \text{cone}(C)$.

That is, $\lambda x + (1-\lambda)y \in \text{cone}(C)$.

Since $\forall x, y \in \text{cone}(C)$, $\forall \lambda \in (0,1)$, $\lambda x + (1-\lambda)y \in \text{cone}(C)$, we get $\text{cone}(C)$ is convex.

■

3.7 Examples of Convex Sets

EXAMPLE 3.7.1. Let I be an index set. Let b_i for $i \in I$ be vectors in $\mathbb{E}$. Let β_i for $i \in I$ be reals. Then the set C given by

$$C := \{x \in \mathbb{E} : \forall i \in I, \langle x, b_i \rangle \leq \beta_i\}$$

is convex.

Proof.

Each of $C_i := \{x \in \mathbb{E} : \langle x, b_i \rangle \leq \beta_i\}$ is convex and $C = \bigcap_{i \in I} C_i$.

$$\begin{aligned}
\langle z, b_i \rangle &= \langle \lambda x + (1-\lambda)y, b_i \rangle \\
&= \lambda\langle x, b_i \rangle + (1-\lambda)\langle y, b_i \rangle \\
&\leq \lambda\beta_i + (1-\lambda)\beta_i \\
&= \beta_i.
\end{aligned}$$

■

3.8 The Carathéodory Theorem

THEOREM 3.2 (Carathéodory)**.** Let $\mathbb{E}$ be some Euclidean space. Let S be some set in the space. Let x be some point in $\text{conv}(S)$. Then x can be represented as a convex combination of at most $d+1$ points in S. i.e., x lies in some r-simplex with vertices in S, where $r \leq d$.

Chapter 4

Geometric Objects

4.1 Definitions

DEFINITION (Hyperplane)**.** Let $\mathbb{E}$ be a Euclidean space over $\mathbb{R}$. Let H be a subset of $\mathbb{E}$. We say that H is a **hyperplane** if and only if H can be expressed as

$$H = \{x \in \mathbb{E} : a^\top x = b\}$$

for some $a \in \mathbb{E} \setminus \{0\}$ and $b \in \mathbb{R}$.

DEFINITION (Closed Half-Space)**.** Let $\mathbb{E}$ be a Euclidean space over $\mathbb{R}$. Let P be a subset of $\mathbb{E}$. We say that P is a **closed half-space** if and only if P can be expressed as

$$P = \{x \in \mathbb{E} : a^\top x \leq b\}$$

for some $a \in \mathbb{E} \setminus \{0\}$ and $b \in \mathbb{R}$.

DEFINITION (Polyhedron)**.** Let $\mathbb{E}$ be a Euclidean space over $\mathbb{R}$. Let P be a subset of $\mathbb{E}$. We say that P is a **polyhedron** if and only if P can be expressed as the intersection of finitely many closed half-spaces in $\mathbb{E}$.

4.2 Properties

PROPOSITION 4.2.1. Polyhedrons are convex.

Chapter 5

Cones

5.1 Definitions

DEFINITION (Conical Combination)**.** Let S be a set in $\mathbb{E}$. We define a **conical combination** of S to be a point x in the space of the form

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where (1) $n \in \mathbb{N}$, (2) $v_i \in S$ for all i, and (3) $\lambda_i \in \mathbb{R}_{++}$ for all i.

DEFINITION (Cone)**.** Let S be a set in $\mathbb{E}$. We say that S is a **cone** if and only if $S = \mathbb{R}_{++}S$.

DEFINITION (Conical Hull)**.** Let S be a set in $\mathbb{E}$. We define the **conical hull** of S, denoted by $\operatorname{cone}(S)$, to be the intersection of all cones containing C.

PROPOSITION 5.1.1. Let S be a set in $\mathbb{E}$. Then $\operatorname{cone}(S) = \mathbb{R}_{++}S$.

Proof. **Forward Direction**: I will show that $\operatorname{cone}(S) \subseteq \mathbb{R}_{++}S$. Since $\mathbb{R}_{++}\mathbb{R}_{++}S = \mathbb{R}_{++}S$, $\mathbb{R}_{++}S$ is a cone. Since $1 \in \mathbb{R}_{++}$, $S \subseteq \mathbb{R}_{++}$. Since $\mathbb{R}_{++}S$ is a cone containing S and $\operatorname{cone}(S)$

is the smallest cone containing S, we get

$$\text{cone}(S) \subseteq \mathbb{R}_{++}S.$$

Backward Direction: I will show that $\mathbb{R}_{++}S \subseteq \text{cone}(S)$. Let C be an arbitrary cone containing S. Since $S \subseteq C$, $\mathbb{R}_{++}S \subseteq \mathbb{R}_{++}C$. Since C is a cone, $\mathbb{R}_{++}C = C$. So $\mathbb{R}_{++}S \subseteq C$. Since $\mathbb{R}_{++}S \subseteq C$ for any cone C containing S, we get

$$\mathbb{R}_{++}S \subseteq \text{cone}(S).$$

■

DEFINITION (Closed Conical Hull)**.** Let S be a set in $\mathbb{E}$. We define the **closed conical hull** of S, denoted by $\text{clcone}(S)$, to be the intersection of all closed cones containing C.

PROPOSITION 5.1.2. For any set S in $\mathbb{E}$, we have

$$\text{clcone}(S) = \text{cl}(\text{cone}(S)).$$

Proof. **Forward Direction**: show that $\text{clcone}(S) \subseteq \text{cl}(\text{cone}(S))$. Since $\text{cl}(\text{cone}(S))$ is a closed cone containing S and $\text{clcone}(S)$ is the smallest closed cone containing S, $\text{clcone}(S) \subseteq \text{cl}(\text{cone}(S))$.

Backward Direction: show that $\text{cl}(\text{cone}(S)) \subseteq \text{clcone}(S)$. Since $S \subseteq \text{clcone}(S)$, $\text{cone}(S) \subseteq \text{cone}(\text{clcone}(S))$. So $\text{cl}(\text{cone}(S)) \subseteq \text{cl}(\text{cone}(\text{clcone}(S)))$. Since $\text{clcone}(S)$ is a cone, $\text{cone}(\text{clcone}(S)) = \text{clcone}(S)$. Since $\text{clcone}(S)$ is closed, $\text{cl}(\text{clcone}(S)) = \text{clcone}(S)$. So $\text{cl}(\text{cone}(\text{clcone}(S))) = \text{clcone}(S)$. So $\text{cl}(\text{cone}(S)) \subseteq \text{clcone}(S)$. This completes the proof. ■

5.2 The cone Operator

PROPOSITION 5.2.1 (The cone Operator)**.** The cone operator has the following properties.

(1) $\forall S \subseteq \mathbb{E}$,

$$S \subseteq \text{cone}(S).$$

(2) $\forall S_1, S_2 \subseteq \mathbb{E}$,

$$S_1 \subseteq S_2 \implies \text{cone}(S_1) \subseteq \text{cone}(S_2).$$

(3) $\forall S \subseteq \mathbb{E}$,

$$\operatorname{cone}(\operatorname{cone}(S)) = \operatorname{cone}(S).$$

PROPOSITION 5.2.2. The conv operator and the cone operator commute. Let S be a set in $\mathbb{E}$. Then

$$\operatorname{conv}(\operatorname{cone}(S)) = \operatorname{cone}(\operatorname{conv}(S)).$$

Proof.

For $\operatorname{cone}(\operatorname{conv}(S)) \subseteq \operatorname{conv}(\operatorname{cone}(S))$, let x be an arbitrary point in $\operatorname{cone}(\operatorname{conv}(S))$.

Since $x \in \operatorname{cone}(\operatorname{conv}(S))$, we get $\exists \lambda \in \mathbb{R}_+$, $\exists n \in \mathbb{N}$, $\exists v_1, ..., v_n \in S$, $\exists \mu_1, ..., \mu_n \in [0,1], \sum_{i=1}^n \mu_i = 1$ such that $x = \lambda \sum_{i=1}^n \mu_i v_i$.

Since $x = \lambda \sum_{i=1}^n \mu_i v_i$, $x = \sum_{i=1}^n \mu_i(\lambda v_i)$.

Since $\lambda \in \mathbb{R}_+$ and $v_i \in S$, $\lambda v_i \in \operatorname{cone}(S)$.

Since $\lambda v_i \in \operatorname{cone}(S)$ and $\mu_i \in [0,1]$, $\sum_{i=1}^n \mu_i = 1$, $\sum_{i=1}^n \mu_i(\lambda v_i) \in \operatorname{conv}(\operatorname{cone}(S))$.

Since $\forall x \in \operatorname{cone}(\operatorname{conv}(S)), x \in \operatorname{conv}(\operatorname{cone}(S))$, $\operatorname{cone}(\operatorname{conv}(S)) \subseteq \operatorname{conv}(\operatorname{cone}(S))$.

For $\operatorname{conv}(\operatorname{cone}(S)) \subseteq \operatorname{cone}(\operatorname{conv}(S))$, let x be an arbitrary point in $\operatorname{conv}(\operatorname{cone}(S))$.

Since $x \in \operatorname{conv}(\operatorname{cone}(S))$, $\exists n \in \mathbb{N}$, $\exists \lambda_i \in [0,1], \sum_{i=1}^n \lambda_i = 1$, $\exists \mu_i \in \mathbb{R}_+$, $\exists v_i \in S$ such that $x = \sum_{i=1}^n \lambda_i \mu_i v_i$.

Define $\alpha := \sum_{i=1}^n \lambda_i \mu_i$.

Define $\beta_i := \lambda_i \mu_i / \alpha$.

Then $\alpha \in \mathbb{R}_+$ and $\beta_i \in [0,1]$ and $\sum_{i=1}^n \beta_i = 1$ and $x = \alpha \sum_{i=1}^n \beta_i v_i$.

Since $\beta_i \in [0,1]$ and $\sum_{i=1}^n \beta_i = 1$ and $v_i \in S$, we get $\sum_{i=1}^n \beta_i v_i \in \operatorname{conv}(S)$.

Since $\alpha \in \mathbb{R}_+$ and $\sum_{i=1}^n \beta_i v_i \in \operatorname{conv}(S)$ and $x = \alpha \sum_{i=1}^n \beta_i v_i$, we get $x \in \operatorname{cone}(\operatorname{conv}(S))$.

Since $\forall x \in \operatorname{conv}(\operatorname{cone}(S)), x \in \operatorname{cone}(\operatorname{conv}(S))$, we get $\operatorname{conv}(\operatorname{cone}(S)) \subseteq \operatorname{cone}(\operatorname{conv}(S))$.

Since $\operatorname{cone}(\operatorname{conv}(S)) \subseteq \operatorname{conv}(\operatorname{cone}(S))$ and $\operatorname{conv}(\operatorname{cone}(S)) \subseteq \operatorname{cone}(\operatorname{conv}(S))$, we get $\operatorname{conv}(\operatorname{cone}(S)) = \operatorname{cone}(\operatorname{conv}(S))$.

■

5.3 Other Properties

PROPOSITION 5.3.1. Let C be a convex set in $\mathbb{E}$. Assume $\operatorname{int}(C) \neq \emptyset$ and $0 \in C$. Then $\operatorname{int}(\operatorname{cone}(C)) = \operatorname{cone}(\operatorname{int}(C))$.

Proof.

For one direction, let x be an arbitrary point in $int(\text{cone}(C))$. We are to prove that $x \in \text{cone}(int(C))$.

Since $x \in int(\text{cone}(C))$, $\exists r$ such that $\text{ball}(x, r) \subseteq \text{cone}(C)$.

Since $x \in int(\text{cone}(C))$, $x \in \text{cone}(C)$.

Since $x \in \text{cone}(C)$, $\exists n \in \mathbb{N}$, $\exists \lambda_1, ..., \lambda_n > 0$, $\exists v_1, ..., v_n \in C$ such that $x = \sum_{i=1}^{n} \lambda_i v_i$.

Assume for the sake of contradiction that $\exists k \in \{1, ..., n\}$ such that $\forall r_k > 0$, $\text{ball}(v_k, r_k) \cap \mathbb{E} \setminus C \neq \emptyset$.

not finished

For the reverse direction, let x be an arbitrary point in $\text{cone}(int(C))$. We are to prove that $x \in int(\text{cone}(C))$.

Since $x \in \text{cone}(int(C))$, $\exists n \in \mathbb{N}$, $\exists \lambda_1, ..., \lambda_n > 0$, $\exists v_1, ..., v_n \in int(C)$ such that $x = \sum_{i=1}^{n} \lambda_i v_i$.

Since $v_i \in int(C)$ for each $i \in \{1, ..., n\}$, $\exists r_i$ such that $\text{ball}(v_i, r_i) \subseteq C$.

Define $R := \min\{\lambda_i r_i\}_{i=1}^{n}$.

Say $R = \lambda_k r_k$ for some $k \in \{1, ..., n\}$.

Let y be an arbitrary point in $\text{ball}(x, R)$.

Since $y \in \text{ball}(x, R)$, $\exists w$ such that $\|w\| < R$ and $y = x + w$.

$$\begin{aligned} y &= \sum_{i=1}^{n} \lambda_i v_i + w \\ &= \sum_{i \neq k} \lambda_i v_i + \lambda_k v_k + w \\ &= \sum_{i \neq k} \lambda_i v_i + \lambda_k (v_k + w/\lambda_k). \end{aligned}$$

Since $\|w\| < R$, $\|w/\lambda_k\| < R/\lambda_k = r_k$.

Since $\|w/\lambda_k\| < r_k$, $v_k + w/\lambda_k \in \text{ball}(v_k, r_k)$.

So $v_k + w/\lambda_k \in C$.

So $y \in \text{cone}(C)$.

Since $\forall y \in \text{ball}(x, R)$, $y \in \text{cone}(C)$, $\text{ball}(x, R) \subseteq \text{cone}(C)$.

Since $\exists r$ such that $\text{ball}(x, r) \subseteq \text{cone}(C)$, $x \in int(\text{cone}(C))$.

This proves $\text{cone}(int(C)) \subseteq int(\text{cone}(C))$.

■

PROPOSITION 5.3.2. Let C be a convex set in $\mathbb{E}$. Assume $\text{int}(C) \neq \emptyset$ and $0 \in C$.

Then

$$0 \in \text{int}(C) \iff \text{cone}(C) = \mathbb{E}.$$

Proof. For one direction, assume that $0 \in int(C)$. We are to prove that $\text{cone}(C) = \mathbb{E}$. Clearly

$$\text{cone}(C) \subseteq \mathbb{E}.$$

Since $0 \in int(C)$, $\exists r > 0$ such that $\text{ball}(0, r) \subseteq C$. Since $\text{ball}(0, r) \subseteq C$, $\text{cone}(\text{ball}(0, r)) \subseteq \text{cone}(C)$. Since $\text{cone}(\text{ball}(0, r)) = \mathbb{E}$ and $\text{cone}(\text{ball}(0, r)) \subseteq \text{cone}(C)$, we get

$$\mathbb{E} \subseteq \text{cone}(C).$$

For the reverse direction, assume that $\text{cone}(C) = \mathbb{E}$. We are to prove that $0 \in int(C)$.

$$\mathbb{E} = int(\mathbb{E}) = int(\text{cone}(C)) = \text{cone}(int(C)).$$

If $0 \notin int(C)$, then $0 \notin \text{cone}(int(C))$. So $0 \in int(C)$.

■

5.4 Dual Cone

DEFINITION (Dual of a Convex Cone)**.** Let $\mathfrak{X}$ be vector space over $\mathbb{R}$. Let C be a subset of $\mathfrak{X}$. We define the **dual cone** of C, denoted by C^*, to be the subset of $\mathfrak{X}$ given by

$$C^* := \{x \in \mathfrak{X} : \forall y \in C, \langle x, y \rangle \geq 0\}.$$

PROPOSITION 5.4.1. The dual of a convex cone is always a closed convex cone.

PROPOSITION 5.4.2. Let $\mathbb{E}$ be a Euclidean space. Let K be a convex cone in $\mathbb{E}$. Then $K^{**} = \text{cl}(K)$.

PROPOSITION 5.4.3. Let $\mathbb{E}$ be a Euclidean space. Let K be a pointed, closed convex cone with nonempty interior. Then so is K^*.

PROPOSITION 5.4.4. Let $\mathbb{E}$ be a Euclidean space. Let K_1 and K_2 be nonempty convex cones. Then

(1) $(K_1 + K_2)^* = K_1^* \cap K_2^*$.

(2) $(\mathrm{cl}(K_1) \cap \mathrm{cl}(K_2))^* = \mathrm{cl}(K_1^* + K_2^*)$.

(3) If K_1 and K_2 are closed and $\mathrm{relint}(K_1) \cap \mathrm{relint}(K_2) \neq \emptyset$, then $(K_1 \cap K_2)^* = K_1^* + K_2^*$.

Proof of (1). **Forward Direction:** Let x be an arbitrary element of $(K_1 + K_2)^*$. I will show that $x \in K_1^* \cap K_2^*$. Since $x \in (K_1 + K_2)^*$, $\forall k \in K_1 + K_2$, we have $\langle x, k \rangle \geq 0$. Let k_1 be an arbitrary element of K_1. Let k_2 be an arbitrary element of K_2. Then

$$\begin{aligned}
\langle x, k_1 \rangle &= \left\langle x, \lim_{n \to \infty} (k_1 + \frac{1}{n} k_2) \right\rangle \\
&= \lim_{n \to \infty} \left\langle x, k_1 + \frac{1}{n} k_2 \right\rangle, \text{ since } \langle x, \cdot \rangle \text{ is continuous} \\
&\geq \lim_{n \to \infty} 0, \text{ since } k_1 + \frac{1}{n} k_2 \in K_1 + K_2 \\
&= 0.
\end{aligned}$$

That is, $\langle x, k_1 \rangle \geq 0$. A similar argument can show that $\langle x, k_2 \rangle \geq 0$. So $x \in K_1^*$ and $x \in K_2^*$. So $x \in K_1^* \cap K_2^*$.

Backward Direction: Let x be an arbitrary element of $K_1^* \cap K_2^*$. I will show that $x \in (K_1 + K_2)^*$. Let k be an arbitrary element of $K_1 + K_2$. Then k can be written as $k = k_1 + k_2$ where $k_1 \in K_1$ and $k_2 \in K_2$. Since $x \in K_1^* \cap K_2^*$, $x \in K_1^*$. Since $x \in K_1^*$ and $k_1 \in K_1$, we get $\langle x, k_1 \rangle \geq 0$. A similar argument can show that $\langle x, k_2 \rangle \geq 0$. So

$$\langle x, k \rangle = \langle x, k_1 + k_2 \rangle = \langle x, k_1 \rangle + \langle x, k_2 \rangle \geq 0 + 0 = 0.$$

That is, $\langle x, k \rangle \geq 0$. So $x \in (K_1 + K_2)^*$. ■

5.5 Polar Cone

DEFINITION (Polar Cone)**.** Let $\mathfrak{X}$ be vector space over $\mathbb{R}$. Let C be a subset of $\mathfrak{X}$. We define the **polar cone** of C, denoted by C°, to be the subset of $\mathfrak{X}$ given by

$$C^\circ := \{x \in \mathfrak{X} : \forall y \in C, \langle x, y \rangle \leq 0\}.$$

PROPOSITION 5.5.1. Let $\mathfrak{X}$ be vector space over $\mathbb{R}$. Let C be a subset of $\mathfrak{X}$. Then $C^\circ = -C^*$.

PROPOSITION 5.5.2. If S is a linear subspace of some Euclidean space $\mathbb{E}$, then $S^\circ = S^\perp$.

5.6 Extreme Rays

DEFINITION (Rays)**.** Let $\mathcal{V}$ be a vector space. Let R be a subset of $\mathcal{V}$. We say that R is a **ray** if and only if R can be expressed as

$$R = \{\alpha v : \alpha \in \mathbb{R}_+\}$$

for some $v \in \mathbb{E} \setminus \{0\}$.

DEFINITION (Extreme Rays)**.** Let $\mathcal{V}$ be a vector space. Let K be a convex cone in $\mathcal{V}$. Let R be a ray in K. We say that R is an **extreme ray** in K if and only if for any pair of rays R_1 and R_2 in K such that $R_1 + R_2 \supseteq R$, we have either $R_1 = R$ or $R_2 = R$ (or both).

Chapter 6

Tangent Cones and Normal Cones

6.1 Definitions

DEFINITION (Tangent Cones)**.** Let C be a non-empty convex set in $\mathbb{R}^n$. Let x be a point in $\mathbb{R}^n$. We define the **tangent cone** to C at point x, denoted by $T_C(x)$, to be the subset of $\mathbb{R}^n$ given by

$$T_C(x) := \begin{cases} \operatorname{clcone}(C - x), & \text{if } x \in C \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

DEFINITION (Normal Cones)**.** Let C be a non-empty convex set in $\mathbb{R}^n$. Let x be a point in $\mathbb{R}^n$. We define the **normal cone** to C at point x, denoted by $N_C(x)$, to be the subset of $\mathbb{R}^n$ given by

$$N_C(x) := \begin{cases} \{v \in \mathbb{R}^n : \forall y \in C - x, \langle y, v \rangle \leq 0\}, & \text{if } x \in C \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

6.2 Basic Properties

PROPOSITION 6.2.1. Let C be a closed convex set in $\mathbb{E}$. Let x be a point in $\mathbb{E}$. Then $T_C(x)$ and $N_C(x)$ are closed convex cones.

Proof.

If $C = \emptyset$, then $T_C(x) = N_C(x) = \emptyset$.

If $C \neq \emptyset$ and $x \notin C$, then $T_C(x) = N_C(x) = \emptyset$.

So now I assume that $C \neq \emptyset$ and $x \in C$.

Tangent Cone is Closed:

By definition, $T_C(x) = \text{clcone}(C - x)$. So $T_C(x)$ is a closed.

Tangent Cone is Convex:

$$\begin{array}{cl}
C \text{ is convex} & \\
\Downarrow & \text{since affine mapping preserves convexity} \\
C - x \text{ is convex} & \\
\Downarrow & \text{since the cone operator preserves convexity} \\
\text{cone}(C - x) \text{ is convex} & \\
\Downarrow & \text{since the cl operator preserves convexity} \\
\text{cl}(\text{cone}(C - x)) \text{ is convex} & \\
\Downarrow & \text{since } \text{cl} \circ \text{cone} = \text{clcone} \\
\text{clcone}(C - x) \text{ is convex} &
\end{array}$$

That is, $T_C(x)$ is convex.

Tangent Cone is a Cone

By definition, $T_C(x) = \text{clcone}(C - x)$. So $T_C(x)$ is a cone.

Normal Cone is Closed:

Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in $N_C(x)$ that converges to some point in $\mathbb{E}$.

Say $x_i \to x_\infty$.

Let y be an arbitrary point in $C - x$.

Since $x_i \in N_C(x)$ and $y \in C - x$, by definition of $N_C(x)$, we get $\langle x_i, y \rangle \leq 0$.

Since $\langle x_i, y \rangle \leq 0$ for any $i \in \mathbb{N}$ and $x_i \to x_\infty$, we get $\langle x_\infty, y \rangle \leq 0$.

Since $\forall y \in C - x$, $\langle x_\infty, y \rangle \leq 0$, by definition of $N_C(x)$, we get $x_\infty \in N_C(x)$.

Since any convergent sequence whose terms are in $N_C(x)$ has its limit also in $N_C(x)$, $N_C(x)$ is closed.

Normal Cone is Convex:

Let u and v be arbitrary points in $N_C(x)$.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda u + (1 - \lambda)v$.

Let y be an arbitrary point in $C - x$.
Since $u \in N_C(x)$, $\langle u, y \rangle \leq 0$.
Since $v \in N_C(x)$, $\langle v, y \rangle \leq 0$.

$$\begin{aligned}
&\langle z, y \rangle \\
&= \langle \lambda u + (1-\lambda)v, y \rangle \\
&= \lambda \langle u, y \rangle + (1-\lambda)\langle v, y \rangle \\
&\leq \lambda 0 + (1-\lambda)0 \\
&= 0.
\end{aligned}$$

That is, $\langle z, y \rangle \leq 0$.
Since $\forall y \in C - x$, $\langle z, y \rangle \leq 0$, we get $z \in N_C(x)$.
That is, $\lambda u + (1-\lambda)v \in N_C(x)$.
Since $\forall u, v \in N_C(x)$, $\forall \lambda \in (0,1)$, $\lambda u + (1-\lambda)v \in N_C(x)$, we get $N_C(x)$ is convex.
Normal Cone is a Cone:
Let v be an arbitrary point in $N_C(x)$.
Let λ be an arbitrary number such that $\lambda > 0$.
Let y be an arbitrary point in $C - x$.
Since $v \in N_C(x)$, $\langle v, y \rangle \leq 0$.
Since $\langle v, y \rangle \leq 0$ and $\lambda > 0$, $\langle \lambda v, y \rangle \leq 0$.
Since $\forall y \in C - x$, $\langle \lambda v, y \rangle \leq 0$, we get $\lambda v \in N_C(x)$.
Since $\forall v \in N_C(x)$, $\forall \lambda > 0$, $\lambda v \in N_C(x)$, we get $N_C(x)$ is a cone.

■

PROPOSITION 6.2.2. Let C be a non-empty closed convex set in $\mathbb{E}$. Let x be a point in C. Let n be a point in $\mathbb{E}$. Then

$$n \in N_C(x) \iff \forall t \in T_C(x), \langle n, t \rangle \leq 0.$$

Proof.
For one direction, assume that $n \in N_C(x)$.
We are to prove that

$$\forall t \in T_C(x), \quad \langle n, t \rangle \leq 0.$$

Let t be an arbitrary point in $T_C(x)$.
Since $t \in T_C(x) = \mathrm{cl}(\mathrm{cone}(C - x))$,

$$\exists \{t_i\}_{i \in \mathbb{N}} \subseteq \mathrm{cone}(C - x), \text{ such that } t_i \to t. \tag{1}$$

Since $t_i \in \text{cone}(C - x)$,

$$\forall i \in \mathbb{N}, \exists \lambda_i \in \mathbb{R}_{++}, \exists c_i \in C \text{ such that } t_i = \lambda_i(c_i - x). \tag{2}$$

Since $n \in N_C(x)$ and $c_i \in C$,

$$\langle n, c_i - x \rangle \leq 0. \tag{3}$$

Now using (2) and (3), we have

$$\begin{aligned}
&\langle n, t_i \rangle \\
&= \langle n, \lambda_i(c_i - x) \rangle, && \text{since } t_i = \lambda_i(c_i - x)s \\
&= \lambda_i \langle n, c_i - x \rangle \\
&\leq \lambda_i \cdot 0, && \text{since } \langle n, c_i - x \rangle \leq 0 \\
&= 0.
\end{aligned}$$

That is,

$$\forall i \in \mathbb{N}, \quad \langle n, t_i \rangle \leq 0.$$

Since $\langle n, t_i \rangle \leq 0$ for each $i \in \mathbb{N}$ and $t_i \to t$, we get

$$\langle n, t \rangle \leq 0.$$

For the reverse direction, assume that n is a vector such that

$$\forall t \in T_C(x), \quad \langle n, t \rangle \leq 0.$$

We are to prove that $n \in N_C(x)$.
Let y be an arbitrary point in $C - x$.
Since $C - x \subseteq \text{clcone}(C - x) = T_C(x)$ and $y \in C - x$, we get $y \in T_C(x)$.
Since $y \in T_C(x)$ and $\forall t \in T_C(x), \langle n, t \rangle \leq 0$, we get $\langle n, y \rangle \leq 0$.
Since $\forall y \in C - x, \langle n, y \rangle \leq 0$, we get $n \in N_C(x)$.

■

THEOREM 6.1. Let C be a closed convex set in $\mathbb{E}$ such that $\text{int}(C) \neq \emptyset$. Let x be a point in $\mathbb{E}$. Then

$$x \in \text{int}(C) \iff T_C(x) = \mathbb{E} \iff N_C(x) = \{0\}.$$

Proof.

Part 1.
$x \in \text{int}(C)$ if and only if $0 \in \text{int}(C - x)$, if and only if $\text{clcone}(C - x) = \mathbb{E}$.

Part 2.

For one direction, assume that $T_C(x) = \mathbb{E}$.

We are to prove that $N_C(x) = \{0\}$.

Consider $n = 0$.

Since

$$\forall t \in T_C(x), \quad \langle 0, t\rangle = 0 \leq 0,$$

we get $0 \in N_C(x)$.

Let n be an arbitrary vector in $N_C(x)$.

By another proposition, we have

$$\begin{aligned}
& n \in N_C(x) \\
\iff & \forall t \in T_C(x) = \mathbb{E}, \langle n, t\rangle \leq 0 \\
\implies & \text{for } t = n, \langle n, t\rangle = \langle n, n\rangle \leq 0 \\
\implies & n = 0.
\end{aligned}$$

That is, $n \in N_C(x) \implies n = 0$.

So $N_C(x) = \{0\}$.

For the reverse direction, assume that $N_C(x) = \{0\}$.

We are to prove that $T_C(x) = \mathbb{E}$.

Clearly $T_C(x) \subseteq \mathbb{E}$.

For $\mathbb{E} \subseteq T_C(x)$, let x be an arbitrary point in $\mathbb{E}$.

Define $p := \mathrm{proj}_{T_C(x)}(x)$.

Since $p = \mathrm{proj}_{T_C(x)}(x)$,

$$\forall y \in T_C(x), \quad \langle x - p, y - p\rangle \leq 0. \tag{1}$$

Since $p = \mathrm{proj}_{T_C(x)}(x)$, $p \in T_C(x)$.

Since $p \in T_C(x)$ and $T_C(x)$ is a cone,

$$2p \in T_C(x). \tag{2}$$

Apply (1) to $y = 2p$, we get

$$\langle x - p, 2p - p\rangle = \langle x - p, p\rangle \leq 0. \tag{3}$$

Since $T_C(x)$ is a closed cone,

$$0 \in T_C(x). \tag{4}$$

Apply (1) to $y = 0$, we get

$$\langle x - p, 0 - p\rangle = \langle x - p, -p\rangle \leq 0. \tag{5}$$

From (3) and (5), we get

$$\langle x - p, p \rangle = 0.$$

So (1) becomes

$$\forall y \in T_C(x), \quad \langle x - p, y \rangle \leq 0.$$

So $x - p \in N_C(x)$.
So $x - p = 0$.
So $x = p$.
So $x \in T_C(x)$.
Since $\forall x \in \mathbb{E}, x \in T_C(x)$, we get

$$\mathbb{E} \subseteq T_C(x).$$

■

6.3 Arithmetic Properties

PROPOSITION 6.3.1. Let C and D be convex subsets of $\mathbb{E}$. Let x be a point in $\mathbb{E}$. Then

$$N_C(x) + N_D(x) \subseteq N_{C \cap D}(x).$$

Proof.

If C or D is empty, then $N_C(x) + N_D(x) = N_{C\cap D}(x) = \emptyset$.
So now I assume that $C, D \neq \emptyset$.
If $x \notin C \cap D$, then $N_C(x) + N_D(x) = N_{C\cap D}(x) = \emptyset$.
So now I assume that $x \in C \cap D$.
Let v be an arbitrary point in $N_C(x) + N_D(x)$.
Since $v \in N_C(x) + N_D(x)$, $\exists u \in N_C(x)$, $\exists w \in N_D(x)$ such that $v = u + w$.
Since $u \in N_C(x)$, $\forall y \in C - x$, $\langle u, y \rangle \leq 0$.
Since $w \in N_D(x)$, $\forall y \in D - x$, $\langle w, y \rangle \leq 0$.
Let y be an arbitrary point in $C \cap D - x$.
Since $y \in C \cap D - x$, we get $y \in C - x$ and $y \in D - x$.

$$\begin{aligned}
&\langle v, y \rangle \\
&= \langle u + w, y \rangle \\
&= \langle u, y \rangle + \langle w, y \rangle \\
&\leq 0 + 0 = 0.
\end{aligned}$$

This is true for any $y \in C \cap D - x$.
So $v \in N_{C\cap D}(x)$.
This is true for any $v \in N_C(x) + N_D(x)$.
So $N_C(x) + N_D(x) \subseteq N_{C\cap D}(x)$.

■

THEOREM 6.2. Let C and D be convex sets in $\mathbb{E}$. Assume that $\mathrm{ri}(C) \cap \mathrm{ri}(D) \neq \emptyset$. Let x be a point in $C \cap D$. Then

$$N_{C\cap D}(x) = N_C(x) + N_C(x).$$

6.4 Other Properties

PROPOSITION 6.4.1. Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x be a point in $\mathrm{dom}(f)$. Let u be a point in $\mathbb{E}$. Then $u \in \partial f(x)$ if and only if $(u, -1) \in N_{\mathrm{epi}(f)}(x, f(x))$.

Proof.

$$\begin{aligned}
& u \in \partial f(x) \\
\iff & \forall y \in \mathbb{E}, f(y) \geq f(x) + \langle u, y - x\rangle \\
\iff & \forall y \in \mathrm{dom}(f), f(y) \geq f(x) + \langle u, y - x\rangle \\
\iff & \forall (y, \beta) \in \mathrm{epi}(f), f(x) + \langle u, y - x\rangle \leq \beta \\
\iff & \forall (y, \beta) \in \mathrm{epi}(f), \big\langle (u, -1), (y - x, \beta - f(x))\big\rangle \leq 0 \\
\iff & \forall (y, \beta) \in \mathrm{epi}(f), \big\langle (u, -1), (y, \beta) - (x, f(x))\big\rangle \leq 0 \\
\iff & (u, -1) \in N_{\mathrm{epi}(f)}(x, f(x)).
\end{aligned}$$

■

Chapter 7

Extreme Points and Faces

7.1 Extreme Points

DEFINITION (Extreme Points - 1)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex set in $\mathcal{V}$. Let x be some point in C. We say that x is an **extreme point** of C if it does not lie strictly between any two distinct points in C.

DEFINITION (Extreme Points - 2)**.** Let $\mathbb{E}$ be some Euclidean space. Let C be a nonempty convex set in the space. Let x be some point in C. We say that x is an **extreme point** of C if $C \setminus \{x\}$ is convex.

PROPOSITION 7.1.1. The two definitions of extreme point are equivalent.

Proof. **Definition 1** $\iff$ **Definition 2**:

Forward Direction: Assume that x does not lie between any two distinct points in C. I will show that $C \setminus \{x\}$ is convex. Let x_1 and x_2 be two arbitrary distinct points in $C \setminus \{x\}$. Let λ be an arbitrary number in $(0, 1)$. Define a point y as $y := \lambda x_1 + (1 - \lambda)x_2$. Since C is convex, $x_1, x_2 \in C$, and $\lambda \in (0, 1)$, we get $y \in C$. Since x does not lie between any two distinct points in C, $y \neq x$. So $y \in C \setminus \{x\}$. That is, I have proved that

$$\forall x_1, x_2 \in C \setminus \{x\}, \forall \lambda \in (0, 1), \quad y = \lambda x_1 + (1 - \lambda)x_2 \in C \setminus \{x\}.$$

By definition, $C \setminus \{x\}$ is convex.

Backward Direction: Assume that $C \setminus \{x\}$ is convex. I will show that x does not lie between any two distinct points in C. Assume for the sake of contradiction that x does lie between two distinct points in C. Say $x = \lambda x_1 + (1-\lambda)x_2$ where $x_1, x_2 \in C$, $x_1 \neq x_2$, and $\lambda \in (0,1)$. Clearly $x \neq x_1$ and $x \neq x_2$. So $x_1, x_2 \in C \setminus \{x\}$. Since $C \setminus \{x\}$ is convex, $x_1, x_2 \in C \setminus \{x\}$, and $\lambda \in (0,1)$, we get $x = \lambda x_1 + (1-\lambda)x_2 \in C \setminus \{x\}$. This leads to a contradiction. So the assumption that x lies between two distinct points in C does not hold. i.e. x does not lie between two distinct points in C.

■

PROPOSITION 7.1.2. If C is nonempty, convex, and compact, then $\mathrm{Ext}(C) \neq \emptyset$.

PROPOSITION 7.1.3. Let $\mathcal{V}$ be a locally convex space. Let K be a nonempty, compact, and convex set in $\mathcal{V}$. Then $\mathrm{Ext}(K) \neq \emptyset$.

7.2 Faces

DEFINITION (Faces - 1)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex subset of $\mathcal{V}$. Let $F \subseteq \mathcal{V}$. We say that F is a **face** of C, denoted by $F \trianglelefteq C$, if and only if F is a nonempty convex subset of C such that

$$\forall x, y \in C, \forall t \in (0,1), \quad tx + (1-t)y \in F \implies x, y \in F.$$

DEFINITION (Faces - 2)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex subset of $\mathcal{V}$. Let $F \subseteq \mathcal{V}$. We say that F is a **face** of C, denoted by $F \trianglelefteq C$, if and only if F is a nonempty convex subset of C such that

$$\forall n \in \mathbb{N}, \forall x \in C^n, \forall t \in (0,1)^n : \sum_{i=1}^{n} t_i = 1, \quad \sum_{i=1}^{n} t_i x_i \in F \implies x \in F^n.$$

PROPOSITION 7.2.1. The two definitions of faces above are equivalent.

Proof. **Forward Direction**: Suppose that $\forall x, y \in C$, $\forall t \in (0,1)$ such that $tx + (1-t)y \in F$, we have $x, y \in F$. Let $n \in \mathbb{N}$, $x \in C^n$, $t \in (0,1)^n$ be arbitrary such that $\sum_{i=1}^n t_i = 1$. Define a point z by $z := \sum_{i=1}^n t_i x_i$. Suppose that $z \in F$. I will show that $x \in F^n$. Note that $\forall i \in \{1, ..., n\}$, we have

$$z = \sum_{j=1}^n t_j x_j = t_i x_i + (1 - t_i) \sum_{j \neq i} \frac{t_j}{1 - t_i} x_j.$$

Consider the point $z_i := \sum_{j \neq i} \frac{t_j}{1-t_i} x_j$. Note that $\forall j \neq i$, $\frac{t_j}{1-t_i} \in (0,1)$ and that $\sum_{j \neq i} \frac{t_j}{1-t_i} = 1$. So since $\forall j \neq i$, $x_j \in C$ and C is convex, we get $z_i \in C$. By assumption, we get $x_i, z_i \in F$. In particular, $x_i \in F$. So $x \in F^n$.

Backward Direction: Suppose that $\forall n \in \mathbb{N}$, $\forall x \in C^n$, $\forall t \in (0,1)^n$ such that $\sum_{i=1}^n t_i = 1$ and that $\sum_{i=1}^n t_i x_i \in F$, we have $x \in F^n$. Take $n := 2$, then $\forall x, y \in C$, $\forall t \in (0,1)$ such that $tx + (1-t)y \in F$, we have $x, y \in F$. ■

Faces are generalizations of extreme points.

PROPOSITION 7.2.2 (Transitivity)**.** Let $\mathcal{V}$ be a vector space. Let A, B, and C be nonempty convex subsets of $\mathcal{V}$. Suppose that $A \trianglelefteq B$ and $B \trianglelefteq C$. Then $A \trianglelefteq C$.

Proof. Let x and y be two arbitrary elements of C. Let t be an arbitrary element of $(0,1)$. Define a point z by $z := tx + (1-t)y$. Suppose that $z \in A$. I will show that $x, y \in A$. Note that since $A \trianglelefteq B$, we have $A \subseteq B$. So $z \in A \subseteq B$. Since $x, y \in C$, $t \in (0,1)$, $z \in B$, and $B \trianglelefteq C$, we get $x, y \in B$. Since $x, y \in B$, $t \in (0,1)$, $z \in A$, and $A \trianglelefteq B$, we get $x, y \in A$. So $A \trianglelefteq C$. ■

PROPOSITION 7.2.3 (Intersection)**.** Let $\mathcal{V}$ be a vector space. Let C be a nonempty convex subset of $\mathcal{V}$. Let $A, B \trianglelefteq C$. Then $(A \cap B) \trianglelefteq C$.

Proof. Let x and y be two arbitrary elements of C. Let t be an arbitrary element of $(0,1)$. Define a point z by $z := tx + (1-t)y$. Suppose that $z \in A \cap B$. I will show that $x, y \in A \cap B$. Since $A \trianglelefteq C$, $x, y \in C$, $t \in (0,1)$, and $z \in A \cap B \subseteq A$, we get $x, y \in A$. Similarly, we get $x, y \in B$. So $x, y \in A \cap B$. So $(A \cap B) \trianglelefteq C$. ■

7.2.1 Exposed Faces

DEFINITION (Exposed Face of a Convex Cone)**.** Let $\mathcal{V}$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Let C be a nonempty convex conic subset of $\mathcal{V}$. Let $F \trianglelefteq C$.

We say that F is **exposed** if and only if $\exists a \in \mathcal{V} \setminus \{0\}$ such that

$$F = \{x \in C : \langle a, x\rangle = 0\} \text{ and } C \subseteq \{x \in \mathcal{V} : \langle a, x\rangle \leq 0\}.$$

PROPOSITION 7.2.4. Let $\mathcal{V}$ be a vector space. Let C be nonempty convex subset of $\mathcal{V}$. Then every face of C is contained in some exposed face of C.

7.3 Relation Between Extreme Points and Faces

DEFINITION (Extreme Points - 3)**.** Let $\mathbb{E}$ be some Euclidean space. Let C be a nonempty convex set in the space. Let x be some point in C. We say that x is an **extreme point** of C if $\{x\}$ is a face of C.

PROPOSITION 7.3.1. This definition of extreme points is equivalent to the previous two.

PROPOSITION 7.3.2. If F is a face of C, then $\operatorname{Ext}(F) \subseteq \operatorname{Ext}(C)$.

7.4 The Krein-Milman Theorem

LEMMA 7.1. Let $\mathcal{V}$ be a locally convex space. Let K be a nonempty compact convex subset of $\mathcal{V}$. Let $\rho \in \mathcal{V}^*$. Define $r := \sup\{\Re\rho(x) : x \in K\}$. Define $F := \{x \in K : \Re\rho(x) = r\}$. Then F is a nonempty compact face of K.

Proof. **Nonempty**: Since $\Re\rho$ is continuous and K is compact, $\{\Re\rho(x) : x \in K\}$ is a compact set in $\mathbb{R}$. So $r = \sup\{\Re\rho(x) : x \in K\}$ is attained. So $F \neq \emptyset$.

Compact: Notice $F = (\Re\rho)^{-1}(\{r\})$. Since $\Re\rho$ is continuous and $\{r\} \subseteq \mathbb{R}$ is closed, F is closed. Since F is a closed subset of K and K is compact, F is compact.

Convex: Let x and y be arbitrary elements of F. Let $t \in (0, 1)$. Since $x, y \in F$, we have $\Re\rho(x) = \Re\rho(y) = r$. So

$$\Re\rho(tx + (1-t)y) = t\Re\rho(x) + (1-t)\Re\rho(y) = tr + (1-t)r = r.$$

So $tx + (1-t)y \in F$. So F is convex.

Face: Let x and y be arbitrary elements of K. Let $t \in (0,1)$. Suppose that $tx+(1-t)y \in F$. Since $x, y \in K$, we have $\Re\rho(x) \leq r$ and $\Re\rho(y) \leq r$. Since $tx + (1-t)y \in F$, we have

$$t\Re\rho(x) + (1-t)\Re\rho(y) = \Re\rho(tx + (1-t)y) = r.$$

So we must have $\Re\rho(x) = \Re\rho(y) = r$. So $x, y \in F$. So F is a face of K.

■

THEOREM 7.1 (Krein-Milman Theorem)**.** A compact convex set in a locally convex space is the closed convex hull of its extreme points.

Proof. Let $\mathcal{V}$ be a locally convex space. Let K be a nonempty, compact, and convex set in $\mathcal{V}$.

Forward Direction: Show that $K \subseteq \mathrm{clconv}(\mathrm{Ext}(K))$. Let m be an arbitrary element of K. Assume for the sake of contradiction that $m \notin \mathrm{clconv}(\mathrm{Ext}(K))$. By the Hahn-Banach Theorem, there is some $\tau \in \mathcal{V}^*$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha > \beta$ and

$$\forall b \in \mathrm{clconv}(\mathrm{Ext}(K)), \quad \Re\tau(m) \geq \alpha > \beta \geq \Re\tau(b).$$

Define $s := \sup\{\Re\tau(w) : w \in K\}$. Define $L := \{z \in K : \Re\tau(z) = s\}$. Then L is a nonempty compact face of K. So $\mathrm{Ext}(L) \neq \emptyset$. Let e be an element of $\mathrm{Ext}(L)$. Then $e \in \mathrm{Ext}(L) \subseteq L$. So $\Re\tau(e) = s$. So

$$\forall b \in \mathrm{clconv}(\mathrm{Ext}(K)), \quad \Re\tau(e) = s \geq \Re\tau(m) \geq \alpha > \beta \geq \Re\tau(b).$$

That is, $\Re\tau(e) > \Re\tau(b)$. Since L is a face of K, $\mathrm{Ext}(L) \subseteq \mathrm{Ext}(K)$. Notice $e \in \mathrm{Ext}(L) \subseteq \mathrm{Ext}(K) \subseteq \mathrm{clconv}(\mathrm{Ext}(K))$. So in particular, $\Re\tau(e) > \Re\tau(e)$. This is a contradiction. So $m \in \mathrm{clconv}(\mathrm{Ext}(K))$. So $K \subseteq \mathrm{clconv}(\mathrm{Ext}(K))$.

Backward Direction: Show that $\mathrm{clconv}(\mathrm{Ext}(K)) \subseteq K$. Note that $\mathrm{Ext}(K) \subseteq K$. Since K is closed and convex and $\mathrm{Ext}(K) \subseteq K$, we get $\mathrm{clconv}(\mathrm{Ext}(K)) \subseteq K$.

■

PROPOSITION 7.4.1. Let $\mathcal{V}$ be a vector space. Let K be a nonempty compact convex subset of $\mathcal{V}$. Let $\mathcal{F}(K)$ denote the set of faces of K, partially ordered by inclusion. Then the minimal proper faces in $\mathcal{F}(K)$ are the extreme points of K.

PROPOSITION 7.4.2. Let $\mathcal{V}$ be a vector space. Let K be a nonempty compact convex subset of $\mathcal{V}$. Let $\mathcal{F}(K)$ denote the set of faces of K, partially ordered by inclusion. Then the maximal proper faces in $\mathcal{F}(K)$ are exposed.

Chapter 8

Projection

8.1 Definitions

DEFINITION (Projection)**.** Let $\mathcal{H}$ be a Hilbert space. Let S be a non-empty set in the space. Let x be a point in the space. We define the **projection** of x onto S, denoted by $\operatorname{proj}_S(x)$, to be a point given by

$$\operatorname{proj}_S(x) := \operatorname{argmin}_{p \in S} \|p - x\|.$$

i.e., $\operatorname{proj}_S(x)$ is the closest point in S to x.

PROPOSITION 8.1.1 (Existence)**.** If S is non-empty and closed, then the projection $\operatorname{proj}_S(x)$ exists.

Proof. Define for an $n \in \mathbb{N}$ a point c_m to be a point in S that satisfies

$$\lim_{i \in \mathbb{N}} \|c_i - x\| = d_S(x) \text{ where } d_S(x) = \inf_{p \in S} \|p - x\|.$$

Since $\mathcal{H}$ is a Hilbert space, the norm $\|\cdot\|$ on $\mathcal{H}$ satisfies the Parallelogram Law. So

$$\begin{aligned}
\|c_m - c_n\|^2 &= 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - \|c_m + c_n - 2x\|^2 \\
&= 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - 4\left\|\frac{c_m + c_n}{2} - x\right\|^2 \\
&\leq 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - 4d_S(x) \\
&\to 2d_S(x) + 2d_S(x) - 4d_S(x) = 0.
\end{aligned}$$

So the sequence $(c_i)_{i\in\mathbb{N}}$ is Cauchy. Since $\mathcal{H}$ is a Hilbert space, it is complete. So $(c_i)_{i\in\mathbb{N}}$ converges. Since S is closed, and $(c_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in S, $p := \lim_{i\in\mathbb{N}} c_i \in S$. So $\|p-x\| = \|\lim_{i\in\mathbb{N}} c_i - x\| = \lim_{i\in\mathbb{N}} \|c_i - x\| = d_S(x)$. So p is the minimizer of the distance to the point x over S. So $p = \mathrm{proj}_S(x)$. ∎

PROPOSITION 8.1.2 (Uniqueness)**.** If S is non-empty, closed, and convex, then the projection $\mathrm{proj}_S(x)$ is unique.

Proof. Let p denote $\mathrm{proj}_S(x)$. Then $\|p-x\| = d_S(x)$. Let q be a point in S such that $\|q-x\| = d_S(x)$. Then by the Parallelogram Law,

$$\begin{aligned} 0 \le \|p-q\|^2 &= 2\|x-p\|^2 + 2\|q-x\| - 4\left\|x - \frac{1}{2}(p+q)\right\|^2 \\ &\le 2d_S^2(x) + 2d_S^2(x) - 4d_S^2(x) \\ &= 0. \end{aligned}$$

This shows $\|p-q\| = 0$ and hence $p = q$. Thus the projection is unique. ∎

8.2 Properties of the Projection Operator

PROPOSITION 8.2.1 (Idempotent)**.** The projection operator is idempotent. i.e., if C is a nonempty closed convex set in $\mathbb{E}$, then $\mathrm{proj}_C = \mathrm{proj}_C \mathrm{proj}_C$.

Proof. Let x be an arbitrary point in $\mathbb{E}$. By definition, $\mathrm{proj}_C(x) \in C$. Since $\mathrm{proj}_C(x) \in C$, the closest point in C to $\mathrm{proj}_C(x)$ is $\mathrm{proj}_C(x)$. So $\mathrm{proj}_C \mathrm{proj}_C(x) = \mathrm{proj}_C(x)$. This is true for any $x \in \mathbb{E}$. So $\mathrm{proj}_C = \mathrm{proj}_C \mathrm{proj}_C$. ∎

PROPOSITION 8.2.2. Let C be a nonempty closed convex set in $\mathbb{E}$. Then the set of fixed points of the operator proj_C is C.

Proof. For one direction, let x be an arbitrary fixed point of proj_C. We are to prove that $x \in C$. Since x is a fixed point of proj_C, $x = \mathrm{proj}_C(x)$. By definition of projection, $\mathrm{proj}_C(x) \in C$. So $x = \mathrm{proj}_C(x) \in C$.

For the reverse direction, let x be an arbitrary point in C. We are to prove that x is a fixed point of C. Since $x \in C$, the closest point in C to x is x. So $x = \mathrm{proj}_C(x)$. So x is a fixed point of proj_C. ∎

PROPOSITION 8.2.3 (Linearity)**.** Let C be a nonempty closed convex set in $\mathbb{E}$. Then the operator proj_C is linear if and only if C is a linear subspace.

PROPOSITION 8.2.4 (Non-expansive)**.** The projection operator is non-expansive. i.e., if C is a nonempty closed convex set in $\mathbb{E}$, then $\|\operatorname{proj} C(x)\| \leq \|x\|$ for any $x \in \mathbb{E}$.

this is not true. I guess it will be true when C is a linear subspace.

PROPOSITION 8.2.5. Let C be a nonempty closed convex set in $\mathbb{E}$. Then proj_C is Lipschitz with constant 1.

Proof. Let x and y be two arbitrary points in $\mathbb{E}$. If $\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\| = 0$, then $\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\| \leq \|x - y\|$. Otherwise,

$$\begin{aligned}
&\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|^2 \\
&= \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), \operatorname{proj}_C(x) - \operatorname{proj}_C(y)\rangle \\
&= \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), \operatorname{proj}_C(x) - x\rangle \\
&+ \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&+ \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), y - \operatorname{proj}_C(y)\rangle \\
&= \langle x - \operatorname{proj}_C(x), \operatorname{proj}_C(y) - \operatorname{proj}_C(x)\rangle \\
&+ \langle y - \operatorname{proj}_C(y), \operatorname{proj}_C(x) - \operatorname{proj}_C(y)\rangle \\
&+ \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&\leq 0 + 0 + \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&= \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&\leq \|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|\|x - y\|.
\end{aligned}$$

That is,

$$\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|^2 \leq \|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|\|x - y\|.$$

Dividing both sides by $\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|$ gives

$$\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\| \leq \|x - y\|.$$

So proj_C is Lipschitz with constant 1. ∎

PROPOSITION 8.2.6 (Firmly Non-expansive)**.** Let C be a nonempty closed convex set in $\mathbb{E}$. Then proj_C is firmly non-expansive.

Proof. This is to prove.

$$\forall x, y \in \mathbb{E}, \quad \left\| \operatorname{proj}_C(y) - \operatorname{proj}_C(x) \right\|^2 \leq \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), y - x \right\rangle.$$

$$\begin{aligned}
&\left\| \operatorname{proj}_C(y) - \operatorname{proj}_C(x) \right\|^2 \\
&= \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), \operatorname{proj}_C(y) - \operatorname{proj}_C(x) \right\rangle \\
&= \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), \operatorname{proj}_C(y) - y \right\rangle \\
&+ \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), y - x \right\rangle \\
&+ \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), x - \operatorname{proj}_C(x) \right\rangle \\
&\leq 0 + \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), y - x \right\rangle + 0 \\
&= \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), y - x \right\rangle.
\end{aligned}$$

■

8.3 Characterization

THEOREM 8.1 (Projection Theorem)**.** Let C be a nonempty closed convex set in $\mathbb{E}$. Let x and p be points in $\mathbb{E}$. Then $p = \operatorname{proj}_C(x)$ if and only if

$$\forall y \in C, \quad \langle y - p, x - p \rangle \leq 0.$$

Proof. Let y be an arbitrary point in C. Let α be an arbitrary number in $[0, 1]$. Define $y_\alpha := \alpha y + (1 - \alpha)p$. Now

$$\begin{aligned}
& p = \operatorname{proj}_C(x) \\
\iff & \forall y \in C, \forall \alpha \in [0, 1], \|x - p\|^2 \leq \|x - y_\alpha\|^2 \\
\iff & \forall y \in C, \forall \alpha \in [0, 1], \|x - p\|^2 \leq \|x - p - \alpha(y - p)\|^2 \\
\iff & \forall y \in C, \langle x - p, y - p \rangle \leq 0.
\end{aligned}$$

■

Chapter 9

Separation

9.1 Definitions

DEFINITION (Separated)**.** Let S_1 and S_2 be two sets in $\mathbb{E}$. We say that S_1 and S_2 are **separated** if $\exists b \in \mathbb{E} \setminus \{\vec{0}\}$ such that

$$\sup_{s_1 \in S_1} \langle s_1, b \rangle \leq \inf_{s_2 \in S_2} \langle s_2, b \rangle.$$

DEFINITION (Strongly Separated)**.** Let S_1 and S_2 be two sets in $\mathbb{E}$. We say that they are **strongly separated** if the inequality holds strictly.

DEFINITION (Properly Separated)**.** Let S_1 and S_2 be two sets in $\mathbb{E}$. We say that S_1 and S_2 are **properly separated** if $\exists b \in \mathbb{E}$ such that

$$\sup_{x \in S_1} \langle x, b \rangle \leq \inf_{y \in S_2} \langle y, b \rangle, \text{ and}$$
$$\inf_{x \in S_1} \langle x, b \rangle > \sup_{y \in S_2} \langle y, b \rangle.$$

9.2 Main Results

PROPOSITION 9.2.1. Let C be a nonempty closed convex set in $\mathbb{E}$. Let x be a point in $\mathbb{E}$ such that $x \notin C$. Then x and C are strongly separated.

Proof. Define a point p by

$$p := \mathrm{proj}_C(x).$$

Define a point a by

$$a := x - p.$$

To prove that x is strongly separated from C, it suffices to prove that

$$\forall y \in C, \quad \langle y, a\rangle < \langle x, a\rangle.$$

Since $x \notin C$ and C is closed,

$$a \neq 0. \tag{1}$$

Let y be an arbitrary point in C. Since $p = \mathrm{proj}_C(x)$ and $y \in C$,

$$\langle y - p, x - p\rangle \leq 0. \tag{2}$$

$$\begin{aligned}
&\langle y, a\rangle \\
&< \langle y, a\rangle + \langle a, a\rangle, \text{ since } a \neq 0 \\
&= \langle y + a, a\rangle \\
&= \langle y + x - p, x - p\rangle, \text{ substitute } a = x - p \\
&= \langle y - p, x - p\rangle + \langle x, x - p\rangle \\
&\leq 0 + \langle x, x - p\rangle, \text{ since } \langle y - p, x - p\rangle \leq 0 \\
&= \langle x, x - p\rangle \\
&= \langle x, a\rangle.
\end{aligned}$$

That is,

$$\forall y \in C, \quad \langle y, a\rangle < \langle x, a\rangle.$$

So x is strongly separated from C. ■

PROPOSITION 9.2.2. Let C_1 be a non-empty closed convex set in $\mathbb{E}$. Let C_2 be a non-empty compact convex set in $\mathbb{E}$. Assume that C_1 and C_2 are disjoint. Then C_1 and C_2 are strongly separated.

Proof. Since C_1 is non-empty closed and convex and C_2 is non-empty compact and convex, we get $C_1 - C_2$ is non-empty closed and convex. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C_1 - C_2$. Since $C_1 - C_2$ is non-empty closed and convex and $0 \in C_1 - C_2$, 0 and $C_1 - C_2$ are strongly separated. Since 0 is strongly separated from $C_1 - C_2$,

$$\exists a \neq 0 \text{ such that } \forall c_1 \in C_1, c_2 \in C_2, \quad \langle c_1 - c_2, a\rangle < \langle 0, a\rangle.$$

That is,

$$\langle c_1, a\rangle < \langle c_2, a\rangle.$$

So C_1 and C_2 are strongly separated. ■

THEOREM 9.1. Let C_1 and C_2 be non-empty closed convex sets in $\mathbb{E}$. Assume that C_1 and C_2 are disjoint. Then C_1 and C_2 are separated.

Proof. For $n \in \mathbb{N}$, define

$$D_n := C_2 \cap \operatorname{ball}(0, n).$$

Then D_n is compact for any $n \in \mathbb{N}$. Since $\begin{cases} C_1 \text{ is non-empty closed and convex } D_n \text{ is non-empty compact and convex} \end{cases}$ we get C_1 and D_n are strongly separated for any $n \in \mathbb{N}$. So

$$\forall n \in \mathbb{N}, \exists a_n \in \mathbb{E}, \|a_n\| = 1 \text{ such that } \forall c_1 \in C_1, \forall d_2 \in D_n, \quad \langle c_1, a_n\rangle < \langle d_2, a_n\rangle.$$

Since $\|a_n\| = 1$ for any $n \in \mathbb{N}$, there exists a subsequence $\{a_n\}_{n\in I}$ where I is some infinite subset of $\mathbb{N}$ such that $\{a_n\}_{n\in I}$ converges to some point $a \in \mathbb{E}$. Let x be an arbitrary point in C_1. Let y be an arbitrary point in C_2. For large enough n, $y \in D_n$. Since $\begin{cases} \langle x, a_n\rangle < \langle y, a_n\rangle \text{ for large enough } n \\ \lim_{n\in I, n\to\infty} \langle x, a_n\rangle = \langle x, a\rangle \\ \lim_{n\in I, n\to\infty} \langle y, a_n\rangle = \langle y, a\rangle \end{cases}$, we get

$$\langle x, a\rangle \le \langle y, a\rangle.$$

Since

$$\exists a \neq 0 \text{ such that } \forall x \in C_1, \forall y \in C_2, \quad \langle x, a\rangle \le \langle y, a\rangle,$$

by definition of separated, C_1 and C_2 are separated. ■

PROPOSITION 9.2.3. Let C_1 and C_2 be non-empty convex subsets of $\mathbb{E}$. Then C_1 and C_2 are properly separated if and only if

$$\operatorname{ri}(C_1) \cap \operatorname{ri}(C_2) = \emptyset.$$

Chapter 10

Convex Functions

10.1 Preliminaries

DEFINITION (Epigraph)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **epigraph** of f, denoted by $\mathrm{epi}(f)$, to be the set given by

$$\mathrm{epi}(f) := \left\{(x, \alpha) \in \mathbb{E} \times \mathbb{R} : f(x) \leq \alpha\right\}.$$

DEFINITION (Domain)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **domain** of f, denoted by $\mathrm{dom}(f)$, to be a set given by

$$\mathrm{dom}(f) := \left\{x \in \mathbb{E} : f(x) < +\infty\right\}.$$

DEFINITION (Proper)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **proper** if

$$\begin{aligned} &\exists x \in \mathbb{E}, \quad f(x) \neq +\infty, \text{ and} \\ &\forall x \in \mathbb{E}, \quad f(x) \neq -\infty \end{aligned}$$

10.2 The Indicator Function

DEFINITION (The Indicator Function)**.** Let S be a subset of $\mathbb{E}$. We define the **indicator function** of S, denoted by δ_S, to be a function from $\mathbb{E}$ to $\mathbb{R}^*$ given by

$$\delta_S(x) = \begin{cases} 0, & \text{if } x \in S \\ +\infty, & \text{if } x \notin S. \end{cases}$$

PROPOSITION 10.2.1. Let S be a subset of $\mathbb{E}$. Then

(1) S is non-empty if and only if δ_S is proper.

(2) S is convex if and only if δ_S is convex.

(3) S is closed if and only if δ_S is lower semi-continuous.

Proof of (1).

For one direction, assume that S is not empty.
We are to prove that δ_S is proper.
Since $S \neq \emptyset$, pick $p \in S$.
Since $p \in S$, $\delta_S(p) = 0$.
Since $\delta_S(p) = 0$, $\exists x_0 \in \mathbb{E}$ such that $\delta_S(x_0) \neq +\infty$.
By definition of the indicator function, it never takes $-\infty$.
Since $\exists x_0 \in \mathbb{E}$ such that $\delta_S(x_0) \neq +\infty$ and $\forall x \in \mathbb{E}$, $\delta_S(x) \neq -\infty$, we get δ_S is proper.
For the reverse direction, assume that δ_S is proper.
We are to prove that S is non-empty.
Assume for the sake of contradiction that S is empty.
Let x be an arbitrary point in $\mathbb{E}$.
Since $S = \emptyset$, $x \notin S$.
Since $x \notin S$, $\delta_S(x) = +\infty$.
Since $\forall x \in \mathbb{E}$, $\delta_S(x) = +\infty$, by definition of proper function, δ_S is not proper.
This contradicts to the assumption that δ_S is proper.
So the assumption that $S = \emptyset$ is false.
i.e., S is non-empty.

∎

Proof of (2).

For one direction, assume that S is convex.
We are to prove that δ_S is convex.

Let x and y be arbitrary points in $\mathrm{dom}(\delta_S)$.

By definition of indicator functions, $\mathrm{dom}(\delta_S) = S$.

So $x, y \in S$.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda x + (1 - \lambda)y$.

Since $x, y \in S$ and $\lambda \in (0, 1)$ and S is convex and $z = \lambda x + (1 - \lambda)y$, we get $z \in S$.

Since $z \in S$, $\delta_S(z) = 0$.

Since $\lambda \in (0, 1)$ and $\mathrm{range}(\delta_S) = \{0, +\infty\}$, we get $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) \geq 0$.

Since $\delta_S(z) = 0$ and $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) \geq 0$, we get $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$.

That is, $\delta_S(\lambda x + (1 - \lambda)y) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$.

Since $\forall x, y \in \mathrm{dom}(\delta_S)$, $\forall \lambda \in (0, 1)$, $\delta_S(\lambda x + (1 - \lambda)y) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$, we get δ_S is convex.

For the reverse direction, assume that δ_S is convex.

We are to prove that S is convex.

The case where S is empty is trivial.

So now I assume $S \neq \emptyset$.

Let x and y be arbitrary points in S.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda x + (1 - \lambda)y$.

Since $x \in S$, $\delta_S(x) = 0$.

Since $y \in S$, $\delta_S(y) = 0$.

Since $\delta_S(x) = \delta_S(y) = 0$, we get $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) = 0$.

Since $\lambda \in (0, 1)$ and δ_S is convex, $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$.

Since $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$ and $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) = 0$, we get $\delta_S(z) \leq 0$.

By definition of the indicator function, $\delta_S(z) \geq 0$.

Since $\delta_S(z) \leq 0$ and $\delta_S(z) \geq 0$, we get $\delta_S(z) = 0$.

Since $\delta_S(z) = 0$, $z \in S$.

That is, $\lambda x + (1 - \lambda)y \in S$.

Since $\forall x, y \in S$, $\forall \lambda \in (0, 1)$, $\lambda x + (1 - \lambda)y \in S$, we get S is convex.

■

Proof of (3).

For one direction, assume that S is closed.

We are to prove that δ_S is lower semi-continuous.

Let $\{(x_i, \alpha_i)\}_{i \in \mathbb{N}}$ be an arbitrary sequence in $\mathrm{epi}(\delta_S)$ that converges.

Say its limit is $(x_\infty, \alpha_\infty)$.

Since $(x_i, \alpha_i) \to (x_\infty, \alpha_\infty)$, $x_i \to x_\infty$.

Since $(x_i, \alpha_i) \in \mathrm{epi}(\delta_S)$, $\delta_S(x_i) \leq \alpha_i$.

Since $\delta_S(x_i) \leq \alpha_i$ and $\alpha_i \in \mathbb{R}$, we get $\delta_S(x_i) \neq +\infty$.

Since $\delta_S(x_i) \neq +\infty$, $x_i \in S$.

Since $x_i \in S$ and $x_i \to x_\infty$ and S is closed, $x_\infty \in S$.

Since $x_\infty \in S$, $\delta_S(x_\infty) = 0$.

Since $x_i \in S$, $\delta_S(x_i) = 0$.

Since $\delta_S(x_i) = 0$ and $\delta_S(x_i) \leq \alpha_i$, $\alpha_i \geq 0$.

Since $(x_i, \alpha_i) \to (x_\infty, \alpha_\infty)$, $\alpha_i \to \alpha_\infty$.

Since $\alpha_i \geq 0$ and $\alpha \to \alpha_\infty$, $\alpha_\infty \geq 0$.

Since $\delta_S(x_\infty) = 0$ and $\alpha_\infty \geq 0$, $\delta_S(x_\infty) \leq \alpha_\infty$.

Since $\delta_S(x_\infty) \leq \alpha_\infty$, $(x_\infty, \alpha_\infty) \in \text{epi}(\delta_S)$.

Since for any convergent sequence in $\text{epi}(\delta_S)$, its limit is also in $\text{epi}(\delta_S)$, we get $\text{epi}(\delta_S)$ is closed.

For the reverse direction, assume that δ_S is lower semi-continuous.

We are to prove that S is closed.

Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in S that converges.

Say its limit is x_∞.

Since $x_i \in S$, $\delta_S(x_i) = 0$.

Since $\delta_S(x_i) = 0$, $(x_i, 0) \in \text{epi}(\delta_S)$.

Since $x_i \to x_\infty$, $(x_i, 0) \to (x_\infty, 0)$.

Since $(x_i, 0) \in \text{epi}(\delta_S)$ and $(x_i, 0) \to (x_\infty, 0)$, $(x_\infty, 0) \in \text{epi}(\delta_S)$.

Since $(x_\infty, 0) \in \text{epi}(\delta_S)$, $\delta_S(x_\infty) \leq 0$.

By definition of the indicator function, $\delta_S(x_\infty) \geq 0$.

Since $\delta_S(x_\infty) \leq 0$ and $\delta_S(x_\infty) \geq 0$, we get $\delta_S(x_\infty) = 0$.

Since $\delta_S(x_\infty) = 0$, $x_\infty \in S$.

Since for any convergent sequence in S, its limit is also in S, we get S is closed.

■

10.3 Definitions

DEFINITION (Convex Function)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **convex** if

$$\forall x, y \in \text{dom}(f), \forall \lambda \in [0, 1], \quad f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y).$$

DEFINITION (Convex Function)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **convex** if the epigraph of f is convex.

PROPOSITION 10.3.1. The two definitions of convexity of functions are equivalent.

Proof.

The case where $\mathrm{dom}(f), \mathrm{epi}(f) = \emptyset$ is trivial.

So now I assume that $\mathrm{dom}(f), \mathrm{epi}(f) \neq \emptyset$.

For one direction, assume that $\forall x, y \in \mathrm{dom}(f)$, $\forall \lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

We are to prove that the epigraph of f is convex.

Let (x, α) and (y, β) be two arbitrary points in $\mathrm{epi}(f)$.

Since $(x, \alpha), (y, \beta) \in \mathrm{epi}(f)$, $x, y \in \mathrm{dom}(f)$.

Let λ be an arbitrary number in $[0,1]$.

Define a point $(z, \gamma) := \lambda(x, \alpha) + (1-\lambda)(y, \beta)$.

Then $z = \lambda x + (1-\lambda)y$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$.

Since $x, y \in \mathrm{dom}(f)$, $\lambda \in [0,1]$, we get $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

Since $(x, \alpha) \in \mathrm{epi}(f)$, $f(x) \leq \alpha$.

Since $(y, \beta) \in \mathrm{epi}(f)$, $f(y) \leq \beta$.

Since $f(x) \leq \alpha$ and $f(y) \leq \beta$ and $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, we get $f(\lambda x + (1-\lambda)y) \leq \lambda\alpha + (1-\lambda)\beta$.

Since $z = \lambda x + (1-\lambda)y$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$ and $f(\lambda x + (1-\lambda)y) \leq \lambda\alpha + (1-\lambda)\beta$, we get $f(z) \leq \gamma$.

Since $f(z) \leq \gamma$, $(z, \gamma) \in \mathrm{epi}(f)$.

For the reverse direction, assume that $\mathrm{epi}(f)$ is convex.

We are to prove that $\forall x, y \in \mathrm{dom}(f)$, $\forall \lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

Let x and y be two arbitrary points in $\mathrm{dom}(f)$.

Let λ be an arbitrary number in $[0,1]$.

Define $z := \lambda x + (1-\lambda)y$.

Define $\gamma := \lambda f(x) + (1-\lambda)f(y)$.

Since $(x, f(x)) \in \mathrm{epi}(f)$ and $(y, f(y)) \in \mathrm{epi}(f)$ and $\lambda \in [0,1]$ and $\mathrm{epi}(f)$ is convex, we get $\lambda(x, f(x)) + (1-\lambda)(y, f(y)) \in \mathrm{epi}(f)$.

Since $z = \lambda x + (1-\lambda)y$ and $\gamma = \lambda f(x) + (1-\lambda)f(y)$ and $\lambda(x, f(x)) + (1-\lambda)(y, f(y)) \in \mathrm{epi}(f)$, we get $(z, \gamma) \in \mathrm{epi}(f)$.

Since $(z, \gamma) \in \mathrm{epi}(f)$, $f(z) \leq \gamma$.

That is, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

$\blacksquare$

10.4 Basic Properties

PROPOSITION 10.4.1 (Necessary Condition)**.** The domain of a convex function is convex.

Proof. Follows from the fact that convexity is stable under affine transformations. Define $A((x, \alpha)) := x$. Then $\mathrm{dom}(f) = A(\mathrm{epi}(f))$. ∎

PROPOSITION 10.4.2. The level sets of a convex function are convex.

PROPOSITION 10.4.3 (Restriction to a Line)**.** A function $f : \mathbb{E} \to \mathbb{R}$ is convex if and only if $\forall x \in \mathrm{dom}(f), \forall v \in \mathbb{E}$, the function $g_{x,v} : \mathbb{R} \to \mathbb{R}$ given by

$$g_{x,v}(t) = f(x + tv)$$

is convex.

10.5 Differentiable Convex Functions

PROPOSITION 10.5.1. Let f be a proper convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Let $x \in \mathrm{dom}(f)$. If f is differentiable at point x, then $\nabla(f)(x)$ is the unique subgradient of f at point x. i.e., $\partial(f)(x) = \{\nabla(f)(x)\}$. Conversely, if the subgradient $\partial(f)(x)$ of f at point x is a singleton set $\{v\}$, then f is differentiable at point x and $\nabla(f)(x) = v$.

Proof. ∎

PROPOSITION 10.5.2 (First-Order Condition)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Assume that $\mathrm{dom}(f)$ is convex and open and that f is differentiable on $\mathrm{dom}(f)$. Then f is convex if and only if

$$\forall x, y \in \mathrm{dom}(f), \quad f(y) - f(x) \geq \big\langle \nabla(f)(x), y - x \big\rangle.$$

i.e., the first-order approximation of f is a global under-estimator.

Proof.

Part 1.

For one direction, assume that f is convex. We are to prove that

$$\forall x, y \in \text{dom}(f), \quad f(y) - f(x) \geq \langle \nabla(f)(x), y - x \rangle.$$

Let x and y be arbitrary points in $\text{dom}(f)$. Since f is convex and differentiable at point x, $\nabla(f)(x) = \partial(f)(x)$. So $\nabla(f)(x)$ satisfies the subgradient inequality. That is,

$$f(y) - f(x) \geq \langle \nabla(f)(x), y - x \rangle.$$

Part 2.

For the reverse direction, assume that

$$\forall x, y \in \text{dom}(f), \quad f(y) - f(x) \geq \langle \nabla(f)(x), y - x \rangle.$$

We are to prove that f is convex.

Not Finished.

■

PROPOSITION 10.5.3. Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Assume that $\text{dom}(f)$ is convex and open and that f is differentiable on $\text{dom}(f)$. Then f is convex if and only if

$$\forall x, y \in \text{dom}(f), \quad \langle \nabla(f)(x) - \nabla(f)(y), x - y \rangle \geq 0.$$

Proof.

Part 1.

For one direction, assume that f is convex. We are to prove that

$$\forall x, y \in \text{dom}(f), \quad \langle \nabla(f)(x) - \nabla(f)(y), x - y \rangle \geq 0.$$

Let x and y be arbitrary points in $\text{dom}(f)$. Since f is convex and differentiable at point x, $\nabla(f)(x) = \partial(f)(x)$. So $\nabla(f)(x)$ satisfies the subgradient inequality. That is,

$$f(y) - f(x) \geq \langle \nabla(f)(x), y - x \rangle. \tag{1}$$

Since f is convex and differentiable at point y, $\nabla(f)(y) = \partial(f)(y)$. So $\nabla(f)(y)$ satisfies the subgradient inequality. That is,

$$f(x) - f(y) \geq \langle \nabla(f)(y), x - y \rangle. \tag{2}$$

Take the sum of inequalities (1) and (2), we get

$$(f(y) - f(x)) + (f(x) - f(y)) \geq \langle \nabla(f)(x), y - x \rangle + \langle \nabla(f)(y), x - y \rangle$$

$$\begin{aligned}
\implies &\quad 0 \geq -\big\langle \nabla(f)(x), x-y \big\rangle + \big\langle \nabla(f)(y), x-y \big\rangle \\
\implies &\quad \big\langle \nabla(f)(x), x-y \big\rangle - \big\langle \nabla(f)(y), x-y \big\rangle \geq 0 \\
\implies &\quad \big\langle \nabla(f)(x) - \nabla(f)(x), x-y \big\rangle \geq 0.
\end{aligned}$$

Part 2.

For the reverse direction, assume that

$$\forall x, y \in \mathrm{dom}(f), \quad \big\langle \nabla(f)(x) - \nabla(f)(y), x-y \big\rangle \geq 0.$$

We are to prove that f is convex. Let x and y be arbitrary points in $\mathrm{dom}(f)$. Define a function φ on $(0,1)$ by

$$\varphi(\lambda) := f\big(\lambda x + (1-\lambda)y\big).$$

Notice φ is differentiable and

$$\varphi'(\lambda) = \big\langle \nabla(f)\big(\lambda x + (1-\lambda)y\big), x-y \big\rangle.$$

Let α and β be arbitrary numbers in $(0,1)$. Assume that $\alpha < \beta$. Define two points z_α and z_β by $z_\alpha := \alpha x + (1-\alpha)y$ and $z_\beta := \beta x + (1-\beta)y$. Then

$$\begin{aligned}
&\varphi'(\beta) - \varphi'(\alpha) \\
&= \big\langle \nabla(f)(\beta x + (1-\beta)y), x-y \big\rangle - \big\langle \nabla(f)(\alpha x + (1-\alpha)y), x-y \big\rangle \\
&= \big\langle \nabla(f)(z_\beta), x-y \big\rangle - \big\langle \nabla(f)(z_\alpha), x-y \big\rangle \\
&= \big\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), x-y \big\rangle \\
&= \left\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), \frac{z_\beta - z_\alpha}{\beta - \alpha} \right\rangle \\
&= \frac{1}{\beta-\alpha} \big\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), z_\beta - z_\alpha \big\rangle \\
&\geq \frac{1}{\beta-\alpha} \cdot 0, \text{ by assumption} \\
&= 0.
\end{aligned}$$

That is,

$$\forall \alpha, \beta \in (0,1), \quad \beta > \alpha \implies \varphi'(\beta) - \varphi'(\alpha) \geq 0.$$

So φ' is increasing. So φ is convex. So

$$\varphi(\lambda) \leq \lambda \varphi(1) + (1-\lambda)\varphi(0).$$

That is,

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y).$$

By definition, f is convex.

■

PROPOSITION 10.5.4 (Second-Order Condition)**.** A twice continuously differentiable real-valued function f defined on a convex set is convex if and only if

$$\forall x \in \mathrm{dom}(f), \quad \nabla^2 f(x) \geq 0$$

where $\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_n \partial x_n} \end{bmatrix}$ denotes the Hessian matrix of f at x_0.

PROPOSITION 10.5.5. Let f be a twice continuously differentiable function from $\mathbb{E}$ to $\mathbb{R}$. Then f is convex if and only if $\forall x \in \mathbb{E}$, $\nabla^2 f(x)$ is positive semi-definite.

10.6 Convexity and Lipschitz-ness

THEOREM 10.1. Let f be a differentiable convex function from $\mathbb{E}$ to $\mathbb{R}$. Then the following statements are equivalent.

(1) ∇f is Lipschitz with constant L.

(2) $\forall x, y \in \mathbb{E}$, we have

$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2.$$

(3) $\forall x, y \in \mathbb{E}$, we have

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle + \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|^2.$$

(4) $\forall x, y \in \mathbb{E}$, we have

$$L\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \|\nabla f(y) - \nabla f(x)\|^2.$$

(1) $\implies$ (2).

Assume that ∇f is Lipschitz with constant 1.

Let x and y be two arbitrary points in $\mathbb{E}$.

$$f(y) - f(x)$$

$$
\begin{aligned}
&= \int_0^1 \langle \nabla f(x+t(y-x)), y-x \rangle dt \\
&= \langle \nabla f(x), y-x \rangle + \int_0^1 \langle \nabla f(x+t(y-x)) - \nabla f(x), y-x \rangle dt \\
&\leq \langle \nabla f(x), y-x \rangle + \int_0^1 \|\langle \nabla f(x+t(y-x)) - \nabla f(x) \rangle\| \|y-x\| dt \\
&\leq \langle \nabla f(x), y-x \rangle + \int_0^1 L\|x+t(y-x)-x\| \|y-x\| dt \\
&= \langle \nabla f(x), y-x \rangle + L\|y-x\|^2 \int_0^1 t dt \\
&= \langle \nabla f(x), y-x \rangle + \frac{L}{2}\|y-x\|^2
\end{aligned}
$$

That is,

$$f(y) - f(x) \leq \langle \nabla f(x), y-x \rangle + \frac{L}{2}\|y-x\|^2.$$

■

THEOREM 10.2. Let f be a twice continuously differentiable function from $\mathbb{E}$ to $\mathbb{R}$. Let L be some non-negative number. Then the following statements are equivalent.

(1) ∇f is L-Lipschitz.

(2) $\forall x \in \mathbb{E}$, $\left\|\nabla^2 f(x)\right\| \leq L$.

10.7 Stability of Convexity

PROPOSITION 10.7.1 (Non-Negative Linear Combination)**.** A non-negative linear combination of proper convex functions is again convex.

Proof. It suffices to prove that non-negative scalar multiples of convex functions are convex and sums of two convex functions are convex.

Part 1.

Let f be a proper convex function. Let $\alpha \geq 0$ be an arbitrary scalar. We are to prove that αf is convex. Notice $\mathrm{dom}(f) = \mathrm{dom}(\alpha f)$. Since f is proper, $\mathrm{dom}(f) \neq \emptyset$. So $\mathrm{dom}(\alpha f) \neq \emptyset$. Let x and y be two arbitrary points in $\mathrm{dom}(\alpha f)$. Let λ be an arbitrary number in $(0,1)$.

Define a point z as $z := \lambda x + (1-\lambda)y$. Then

$$\begin{aligned}(\alpha f)(\lambda x + (1-\lambda)y) &= \alpha f(\lambda x + (1-\lambda)y)\\ &\le \alpha(\lambda f(x) + (1-\lambda)f(y))\\ &= \lambda \alpha f(x) + (1-\lambda)\alpha f(y)\\ &= \lambda(\alpha f)(x) + (1-\lambda)(\alpha f)(y).\end{aligned}$$

That is,

$$\forall x, y \in \mathrm{dom}(\alpha f), \forall \lambda \in (0,1), \quad (\alpha f)(\lambda x + (1-\lambda)y) \le \lambda(\alpha f)(x) + (1-\lambda)(\alpha f)(y).$$

So by definition, αf is convex.
Part 2.

Let f and g be proper convex functions. We are to prove that $f+g$ is convex. Notice $\mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$. Since f is proper, $\mathrm{dom}(f) \neq \emptyset$. Since g is proper, $\mathrm{dom}(g) \neq \emptyset$. So $\mathrm{dom}(f+g) \neq \emptyset$. Let x and y be two arbitrary points in $\mathrm{dom}(f+g)$. Let λ be an arbitrary number in $(0,1)$. Define a point z as $z := \lambda x + (1-\lambda)y$. Then

$$\begin{aligned}(f+g)(\lambda x + (1-\lambda)y) &= f(\lambda x + (1-\lambda)y) + g(\lambda x + (1-\lambda)y)\\ &\le \lambda f(x) + (1-\lambda)f(y) + \lambda g(x) + (1-\lambda)g(y)\\ &= \lambda(f(x) + g(x)) + (1-\lambda)(f(y) + g(y))\\ &= \lambda(f+g)(x) + (1-\lambda)(f+g)(y).\end{aligned}$$

That is,

$$\forall x, y \in \mathrm{dom}(f+g), \forall \lambda \in (0,1), \quad (f+g)(\lambda x + (1-\lambda)y) = \lambda(f+g)(x) + (1-\lambda)(f+g)(y).$$

So by definition, $f+g$ is convex.

■

PROPOSITION 10.7.2 (Direct Sum)**.** Direct sums of convex functions are convex.

Proof. Let z and w be two arbitrary points in $\mathrm{dom}(f \oplus g)$. Let $\lambda \in (0,1)$ be arbitrary. Say $z = x \oplus y$ and $w = u \oplus v$ where $x, u \in \mathbb{R}^m$ and $y, v \in \mathbb{R}^p$. Since $z \in \mathrm{dom}(f \oplus g)$, $(f \oplus g)(z) \neq +\infty$. That is, $f(x) + g(y) \neq +\infty$. So neither $f(x)$ nor $g(y)$ is $+\infty$. So both $x \in \mathrm{dom}(f)$ and $y \in \mathrm{dom}(g)$. Similarly, we have $u \in \mathrm{dom}(f)$ and $v \in \mathrm{dom}(g)$. Consider the point

$$\begin{aligned}&\lambda z + (1-\lambda)w\\ =\ &\lambda x \oplus y + (1-\lambda)u \oplus v\end{aligned}$$

$$= \big(\lambda x + (1-\lambda)u\big) \oplus \big(\lambda y + (1-\lambda)v\big).$$

Apply $f \oplus g$ to both sides, we get

$$\begin{aligned}
&(f \oplus g)(\lambda z + (1-\lambda)w) \\
&= (f \oplus g)\left[\big(\lambda x + (1-\lambda)u\big) \oplus \big(\lambda y + (1-\lambda)v\big)\right] \\
&= f(\lambda x + (1-\lambda)u) + g(\lambda y + (1-\lambda)v).
\end{aligned}$$

Since f and g are convex, we get

$$\begin{aligned}
&f(\lambda x + (1-\lambda)u) \le \lambda f(x) + (1-\lambda)f(u), \text{ and} \\
&g(\lambda y + (1-\lambda)v) \le \lambda g(y) + (1-\lambda)g(v).
\end{aligned}$$

So

$$\begin{aligned}
&(f \oplus g)(\lambda z + (1-\lambda)w) \\
&\le \lambda f(x) + (1-\lambda)f(u) + \lambda g(y) + (1-\lambda)g(v) \\
&= \lambda(f(x) + g(y)) + (1-\lambda)(f(u) + g(v)) \\
&= \lambda(f \oplus g)(x \oplus y) + (1-\lambda)(f \oplus g)(u \oplus v) \\
&= \lambda(f \oplus g)(z) + (1-\lambda)(f \oplus g)(w).
\end{aligned}$$

That is,

$$(f \oplus g)(\lambda z + (1-\lambda)w) \le \lambda(f \oplus g)(z) + (1-\lambda)(f \oplus g)(w).$$

This holds for any $z, w \in \operatorname{dom}(f \oplus g)$ and any $\lambda \in (0,1)$. So $(f \oplus g)$ is convex.

■

PROPOSITION 10.7.3 (Composition)**.** The composition of a convex function with an affine function is convex. i.e., if f is convex, then $f(Ax+b)$ is convex.

Proof. Let x an y be arbitrary points in $\mathbb{E}$. Let λ be an arbitrary number in $(0,1)$. Define a point z by $z := \lambda x + (1-\lambda)y$.

$$\begin{aligned}
&g\big(\lambda x + (1-\lambda y\big) \\
&= f\big(A(\lambda x + (1-\lambda)y) + b\big) \\
&= f\big(\lambda Ax + (1-\lambda)Ay + b\big), && \text{by linearity of } A \\
&= f\big(\lambda Ax + (1-\lambda)Ay + \lambda b + (1-\lambda)b\big), && \text{decomposite } b \\
&= f\big(\lambda(Ax+b) + (1-\lambda)(Ay+b)\big) \\
&\le \lambda f(Ax+b) + (1-\lambda)f(Ay+b), && \text{by convexity of } f
\end{aligned}$$

$$= \lambda g(x) + (1-\lambda) g(y).$$

That is,

$$\forall x, y \in \mathbb{E}, \forall \lambda \in (0,1), \quad g\big(\lambda x + (1-\lambda) y\big) \leq \lambda g(x) + (1-\lambda) g(y).$$

So g is convex. ∎

PROPOSITION 10.7.4 (Supremum)**.** The supremum of a collection of convex functions is again convex. i.e., Let $\{f_i\}_{i \in I}$ be a collection of convex functions where I is some index set. Then the function F given by $F := \sup_{i \in I} f_i$ is convex.

Proof.

$$\begin{aligned}
& (x, \alpha) \in \operatorname{epi}(F) \\
\iff & \sup_{i \in I} f_i(x) \leq \alpha \\
\iff & \forall i \in I, f_i(x) \leq \alpha \\
\iff & \forall i \in I, (x, \alpha) \in \operatorname{epi}(f_i \\
\iff & (x, \alpha) \in \bigcap_{i \in I} \operatorname{epi}(f_i).
\end{aligned}$$

So $\operatorname{epi}(F) = \bigcap_{i \in I} \operatorname{epi}(f_i)$. Since f_i are convex, $\operatorname{epi}(f_i)$ are convex. Since $\operatorname{epi}(f_i)$ are convex, $\bigcap_{i \in I} \operatorname{epi}(f_i)$ is convex. That is, $\operatorname{epi}(F)$ is convex. Since $\operatorname{epi}(F)$ is convex, F is convex.

∎

PROPOSITION 10.7.5 (Pointwise Supremum)**.** If $f(x, y)$ is convex in x for each y in some set $\mathcal{A}$, then the function g given by

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex.

10.8 Examples

EXAMPLE 10.8.1. Affine functions are convex.

EXAMPLE 10.8.2. Norms are convex.

Proof.

$$\begin{aligned}
&\|\alpha x + \beta y\| \\
\leq\ &\|\alpha x\| + \|\beta y\| \\
=\ &|\alpha|\|x\| + |\beta|\|y\| \\
=\ &\alpha\|x\| + \beta\|y\|.
\end{aligned}$$

■

EXAMPLE 10.8.3. Square norms are convex.

Proof Approach 1. Notice $\|\cdot\|^2$ is the direct sum of m squares and squares are convex. So by CO 463 Assignment 2 Problem 3, $\|\cdot\|^2$ is convex. ■

Proof Approach 2. The domain is $\mathbb{E}$. Let x and y be two points in $\mathbb{E}$. Let λ be an arbitrary number in $(0, 1)$. Define a point z as $z := \lambda x + (1-\lambda)y$.

$$\begin{aligned}
&\|\lambda x + (1-\lambda)y\|^2 \\
=\ &\|\lambda x\|^2 + \|(1-\lambda)y\|^2 + 2\langle \lambda x, (1-\lambda)y\rangle \\
=\ &\lambda^2\|x\|^2 + (1-\lambda)^2\|y\|^2 + 2\lambda(1-\lambda)\langle x, y\rangle \\
\leq\ &\lambda^2\|x\|^2 + (1-\lambda)^2\|y\|^2 + 2\lambda(1-\lambda)\|x\|\|y\| \\
\leq\ &\lambda(\lambda-1)\|x\|^2 + \lambda(\lambda-1)\|y\|^2 + 2\lambda(1-\lambda)\|x\|\|y\| \\
&+ \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
=\ &\lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
&+ \lambda(\lambda-1)\big[\|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\big] \\
\leq\ &\lambda\|x\|^2 + (1-\lambda)\|y\|^2
\end{aligned}$$

That is,

$$\forall x, y \in \mathbb{E}, \forall \lambda \in (0,1), \quad \|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2.$$

So by definition, $\|\cdot\|^2$ is convex. ■

EXAMPLE 10.8.4. The distance function to a convex set is convex.

EXAMPLE 10.8.5. The perspective of a convex function is convex. i.e., if $f : \mathbb{E} \to \mathbb{R}$

Chapter 11

More Convex Functions

11.1 Strictly Convex

DEFINITION (Strictly Convex)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **strictly convex** if $\forall x, y \in \text{dom}(f)$, $\forall \lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$, except when $\lambda x + (1-\lambda)y = x$ or y.

PROPOSITION 11.1.1. Strictly convex functions are convex.

11.2 Strongly Convex

DEFINITION (Strongly Convex)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Let β be a positive constant. We say that f is β **-strongly convex** if $\forall x, y \in \text{dom}(f)$, $\forall \lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - \frac{\beta}{2}\lambda(1-\lambda)\|x-y\|^2.$$

PROPOSITION 11.2.1. Strongly convex functions are strictly convex.

PROPOSITION 11.2.2. Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. Then f is β-strongly convex if and only if $f - \frac{\beta}{2}\|\cdot\|^2$ is convex.

Proof. Let β be a positive constant. Let g denote $f - \frac{\beta}{2}\|\cdot\|^2$. Let x and y be arbitrary elements of $\mathbb{E}$. Let $\lambda \in (0,1)$ be arbitrary.

$$\begin{aligned}
& f \text{ is } \beta\text{-strongly convex} \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) \\
& \qquad\qquad - \frac{\beta}{2}\lambda(1-\lambda)\|x-y\|^2 \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) \\
& \qquad\qquad - \frac{\beta}{2}\lambda(1-\lambda)\big(\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle\big) \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) \\
& \qquad\qquad - \lambda\frac{\beta}{2}\|x\|^2 + \frac{\beta}{2}\lambda^2\|x\|^2 \\
& \qquad\qquad - (1-\lambda)\frac{\beta}{2}\|y\|^2 + \frac{\beta}{2}(1-\lambda)^2\|y\|^2 \\
& \qquad\qquad + \beta\lambda(1-\lambda)\langle x, y\rangle \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) \\
& \qquad\qquad - \lambda\frac{\beta}{2}\|x\|^2 - (1-\lambda)\frac{\beta}{2}\|y\|^2 \\
& \qquad\qquad + \frac{\beta}{2}\|\lambda x\|^2 + \frac{\beta}{2}\|(1-\lambda)y\|^2 + \beta\langle \lambda x, (1-\lambda)y\rangle \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) \\
& \qquad\qquad - \lambda\frac{\beta}{2}\|x\|^2 - (1-\lambda)\frac{\beta}{2}\|y\|^2 \\
& \qquad\qquad + \frac{\beta}{2}\|\lambda x + (1-\lambda)y\|^2 \\
\iff\ & g\big(\lambda x + (1-\lambda)y\big) \leq \lambda g(x) + (1-\lambda)g(y) \\
\iff\ & f - \frac{\beta}{2}\|\cdot\|^2 \text{ is } \beta \text{ convex.}
\end{aligned}$$

∎

Question: Can we allow f to take on $-\infty$? Do we need f to be proper?

PROPOSITION 11.2.3. Let f and g be functions from $\mathbb{E}$ to $\mathbb{R}^*$. Suppose f is

β-strongly convex for some positive constant β and g is convex. Then $f + g$ is also β-strongly convex.

Question: Can we allow f or g to take on $-\infty$? Do we need f and g to be proper?

Proof.

$$\begin{aligned} & f \text{ is } \beta\text{-strongly convex} \\ \implies & f - \frac{\beta}{2}\|\cdot\|^2 \text{ is convex} \\ \implies & f + g - \frac{\beta}{2}\|\cdot\|^2 \text{ is convex} \\ \implies & f + g \text{ is } \beta\text{-strongly convex.} \end{aligned}$$

■

11.3 Quasiconvex

DEFINITION (Quasiconvex)**.** Let $f : \mathbb{E} \to \mathbb{R}$ be a function with convex domain. We say that f is **quasiconvex** if any level set of f is convex.

PROPOSITION 11.3.1 (Jensen's Inequality for Quasiconvex Functions)**.** Let f be a quasiconvex function. Then $\forall x, y \in \operatorname{dom}(f)$, $\forall \alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$,

$$f(\alpha x + \beta y) \leq \max\{f(x), f(y)\}.$$

PROPOSITION 11.3.2. A differentiable real-valued function f with convex domain is convex if and only if $\forall x, y \in \operatorname{dom}(f)$,

$$f(y) \leq f(x) \implies \nabla f(x) \cdot (y - x) \leq 0. \quad ???$$

Not sure where did this come from but I don't think this is correct.

Chapter 12

Support

12.1 Definitions

DEFINITION (Support Function)**.** Let S be a subset of $\mathbb{E}$. We define the **support function** of S, denoted by σ_S, to be a function from $\mathbb{E}$ to $\mathbb{R}^*$ given by

$$\sigma_S(x) = \sup_{s \in S} \langle x, s \rangle.$$

DEFINITION (Supporting Hyperplane)**.** Let S be a set in $\mathbb{E}$ with nonempty boundary. Let x_0 be a point in the boundary of S. We define a **supporting hyperplane** H to set S at point x_0 to be a set of the form

$$H = \left\{ x \in \mathbb{E} : a^T x = a^T x_0 \right\},$$

such that $a \in \mathbb{E}$ and $a \neq \vec{0}$ and $\forall x \in S, a^T x \leq a^T x_0$.

12.2 Properties

PROPOSITION 12.2.1. The support function of a non-empty set S is proper, convex, and lower semi-continuous.

Proof.

Part 1. Proper.

Define f_s to be a function from $\mathbb{E}$ to $\mathbb{R}$ by $f_s(x) = \langle s, x\rangle$.

These functions are linear and hence proper, convex, and lower semi-continuous.

Notice $\sigma_S = \sup_{s\in S} f_s$.

So σ_S is convex and lower semi-continuous.

Since $\sigma_S(0) = \sup_{s\in S}\langle 0, s\rangle = 0$, $\exists x_0 \in \mathbb{E}, \sigma_S(x) \neq +\infty$.

Since $\sigma_S(x) = \sup_{s\in S}\langle x, s\rangle \geq \langle x, s\rangle \neq -\infty$, $\forall x \in \mathbb{E}, \sigma_S(x) \neq -\infty$.

Since $\exists x_0 \in \mathbb{E}, \sigma_S(x) \neq +\infty$ and $\forall x \in \mathbb{E}, \sigma_S(x) \neq -\infty$, by definition, σ_S is proper.

■

PROPOSITION 12.2.2. The support function of a non-empty and bounded set is continuous.

Proof.

Let x_0 be an arbitrary point in $\mathbb{E}$. Let ε be an arbitrary positive number. Define $M := \sup_{y\in C}\|y\| + 1$. Since C is bounded, M is finite. Define $\delta := \varepsilon/M$. Let x be an arbitrary point such that $\|x - x_0\| < \delta$. Let y be an arbitrary point in $\mathbb{E}$. Then by the Cauchy Schwarz inequality, we have

$$\langle x - x_0, y\rangle \leq \|x - x_0\|\|y\|.$$

That is,

$$\langle x, y\rangle \leq \|x - x_0\|\|y\| + \langle x_0, y\rangle.$$

It follows that

$$\begin{aligned}\sup_{y\in C}\langle x, y\rangle &\leq \sup_{y\in C}\left(\|x - x_0\|\|y\| + \langle x_0, y\rangle\right)\\ &\leq \|x - x_0\|\sup_{y\in C}\|y\| + \sup_{y\in C}\langle x_0, y\rangle.\end{aligned}$$

That is,

$$\sigma_C(x) \leq \sigma_C(x_0) + \|x - x_0\|\sup_{y\in C}\|y\|.$$

By definition of δ and M,

$$\sigma_C(x) < \sigma_C(x_0) + \varepsilon. \tag{1}$$

Similarly, reversing the role of x and x_0, we can prove that

$$\sigma_C(x_0) < \sigma_C(x) + \varepsilon. \tag{2}$$

From (1) and (2) we get

$$|\sigma_C(x) - \sigma_C(x_0)| < \varepsilon.$$

Since $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $|\sigma_C(x) - \sigma_C(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \delta$, by definition, δ_C is continuous.

■

PROPOSITION 12.2.3. Let S be a subset of $\mathbb{E}$. Then $\sigma_S = \sigma_{\text{conv}(S)} = \sigma_{\text{clconv}(S)}$.

Proof.

Let x be an arbitrary point in $\mathbb{E}$.

$$\begin{aligned}\sigma_S(x) &= \sup\big\{\langle x, s\rangle : s \in S\big\}\\ \sigma_{\text{conv}(S)}(x) &= \sup\big\{\langle x, s\rangle : s \in \text{conv}(S)\big\}\\ \sigma_{\text{clconv}(S)}(x) &= \sup\big\{\langle x, s\rangle : s \in \text{clconv}(S)\big\}.\end{aligned}$$

It is easy to see that by the linearity of inner products,

$$\text{conv}\big\{\langle x, s\rangle : s \in S\big\} = \big\{\langle x, s\rangle : s \in \text{conv}(S)\big\}.$$

It is easy to see that by the linearity and the continuity of inner products,

$$\text{clconv}\big\{\langle x, s\rangle : s \in S\big\} = \big\{\langle x, s\rangle : s \in \text{clconv}(S)\big\}.$$

It is also easy to see that for any subset A of the reals,

$$\sup(A) = \sup(\text{conv}(A)),$$

and

$$\sup(A) = \sup(\text{cl}(A)).$$

So

$$\begin{aligned}&\sigma_S(x)\\ =\;& \sup\big\{\langle x, s\rangle : s \in S\big\}\\ =\;& \sup\text{conv}\big\{\langle x, s\rangle : s \in S\big\}\\ =\;& \sup\big\{\langle x, s\rangle : s \in \text{conv}(S)\big\}\\ =\;& \sigma_{\text{conv}(S)}(x).\end{aligned}$$

That is, $\sigma_S(x) = \sigma_{\text{conv}(S)}(x)$.

$$\begin{aligned}&\sigma_S(x)\\ =\;& \sup\big\{\langle x, s\rangle : s \in S\big\}\end{aligned}$$

$$
\begin{aligned}
&= \sup \operatorname{conv} \left\{ \langle x, s \rangle : s \in S \right\} \\
&= \sup \left\{ \langle x, s \rangle : s \in \operatorname{conv}(S) \right\} \\
&= \sup \operatorname{cl} \left\{ \langle x, s \rangle : s \in \operatorname{conv}(S) \right\} \\
&= \sup \left\{ \langle x, s \rangle : s \in \operatorname{cl}(\operatorname{conv}(S)) \right\} \\
&= \sup \left\{ \langle x, s \rangle : s \in \operatorname{clconv}(S) \right\} \\
&= \sigma_{\operatorname{clconv}(S)}(x).
\end{aligned}
$$

That is, $\sigma_S(x) = \sigma_{\operatorname{clconv}(S)}(x)$.

■

12.3 Supporting Hyperplane

THEOREM 12.1 (Supporting Hyperplane Theorem)**.** For any boundary point x_0 of a convex set C, there exists a supporting hyperplane to C at x_0.

Chapter 13

Conjugacy

13.1 Definition and Examples

DEFINITION (Convex Conjugate (Legendre–Fenchel Transformation))**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **convex conjugate** of f, denoted by f^*, to be a function also from $\mathbb{E}$ to $\mathbb{R}^*$ given by

$$f^*(x) := \sup_{y \in \mathbb{E}} \big\{ \langle y, x \rangle - f(y) \big\}.$$

EXAMPLE 13.1.1. Let S be a subset of $\mathbb{E}$. Then $\delta_S^* = \sigma_S$.

Proof. Recall that

$$\delta_S(x) = \begin{cases} 0, & x \in S, \\ +\infty, & x \notin S, \end{cases}$$
$$\sigma_S(x) = \sup_{y \in S} \langle x, y \rangle.$$

Now for any $x \in \mathbb{E}$,

$$\begin{aligned} & \delta_S^*(x) \\ = \;& \sup_{y \in S} \big(\langle x, y \rangle - \delta_S(y) \big) \\ = \;& \sup_{y \in S} \big(\langle x, y \rangle - 0 \big) \end{aligned}$$

$$
\begin{aligned}
&= \sup_{y\in S}\langle x, y\rangle \\
&= \sigma_S(x).
\end{aligned}
$$

So $\delta_S^* = \sigma_S$. ■

13.2 Basic Properties

PROPOSITION 13.2.1. The convex conjugate function is convex.

Proof. If $\operatorname{dom}(f) = \emptyset$, then one can see that $f^* \equiv -\infty$. It is a pointwise supremum of affine functions. ■

PROPOSITION 13.2.2. The convex conjugate function is lower semi-continuous.

13.3 Double Conjugate

PROPOSITION 13.3.1. Let f be any function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $f^{**} \leq f$.

Proof. Let x be an arbitrary point in $\mathbb{E}$.

$$
\begin{aligned}
& f^{**}(x) \\
={}& \sup_{y\in\mathbb{E}}\left\{\langle y, x\rangle - f^*(y)\right\} \\
={}& \sup_{y\in\mathbb{E}}\left\{\langle y, x\rangle - \sup_{z\in\mathbb{E}}\left\{\langle z, y\rangle - f(z)\right\}\right\} \\
\leq{}& \sup_{y\in\mathbb{E}}\left\{\langle y, x\rangle - \left\{\langle x, y\rangle - f(x)\right\}\right\} \\
={}& \sup_{y\in\mathbb{E}}\left\{f(x)\right\} \\
={}& f(x).
\end{aligned}
$$

That is, $f^{**}(x) \leq f(x)$. Since $\forall x \in \mathbb{E}$, $f^{**}(x) \leq f(x)$, we get $f^{**} \leq f$. ■

PROPOSITION 13.3.2. Let f be a proper function. Then f is convex and lower semi-continuous if and only if

$$f^{**} = f.$$

PROPOSITION 13.3.3. Let f and g be functions from $\mathbb{E}$ to $\mathbb{R}^*$. Then $f \leq g$ implies $f^* \geq g^*$ and $f^{**} \leq g^{**}$.

Proof. Let x be an arbitrary point in $\mathbb{E}$.

$$\begin{aligned}
&f^*(x)\\
=\ &\sup_{y\in\mathbb{E}}\left\{\langle y, x\rangle - f(y)\right\}\\
\geq\ &\sup_{y\in\mathbb{E}}\left\{\langle y, x\rangle - g(y)\right\}\\
=\ &g^*(x).
\end{aligned}$$

That is, $f^*(x) \geq g^*(x)$. Since $\forall x \in \mathbb{E}$, $f^*(x) \geq g^*(x)$, we get $f^* \geq g^*$.

Let x be an arbitrary point in $\mathbb{E}$.

$$\begin{aligned}
&f^{**}(x)\\
=\ &\sup_{y\in\mathbb{E}}\left\{\langle y, x\rangle - f^*(y)\right\}\\
=\ &\sup_{y\in\mathbb{E}}\left\{\langle y, x\rangle - \sup_{z\in\mathbb{E}}\left\{\langle z, y\rangle - f(z)\right\}\right\}\\
\leq\ &\sup_{y\in\mathbb{E}}\left\{\langle y, x\rangle - \sup_{z\in\mathbb{E}}\left\{\langle z, y\rangle - g(z)\right\}\right\}\\
=\ &\sup_{y\in\mathbb{E}}\left\{\langle y, x\rangle - g^*(y)\right\}\\
=\ &g^{**}(x).
\end{aligned}$$

That is, $f^{**}(x) \leq g^{**}(x)$. Since $\forall x \in \mathbb{E}$, $f^{**}(x) \leq g^{**}(x)$, we get $f^{**} \leq g^{**}$. ∎

PROPOSITION 13.3.4.

$$\mathrm{epi}(f^{**}) = \mathrm{conv}(\mathrm{epi}(f)).$$

13.4 Conjugates and Sub-Differentials

THEOREM 13.1 (Fenchel-Young)**.** Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x, y \in \mathbb{E}$, we have

$$f(x) + f^*(y) \geq \langle x, y \rangle.$$

PROPOSITION 13.4.1. Let f be a proper closed convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x, y \in \mathbb{E}$,

$$y \in \partial f(x) \iff x \in \partial f^*(y) \iff f(x) + f^*(y) = \langle x, y \rangle.$$

Proof of $y \in \partial f(x) \iff x \in \partial f^*(y)$. For one direction, assume that $y \in \partial f(x)$. We are to prove that $x \in \partial f^*(y)$. Consider an arbitrary point $z \in \mathbb{E}$. Since $y \in \partial f(x)$, we get

$$\forall u \in \mathbb{E}, \quad \langle y, u - x \rangle \leq f(u) - f(x).$$

Rearranging yields

$$\forall u \in \mathbb{E}, \quad \langle y, u \rangle - f(u) \leq \langle y, x \rangle - f(x).$$

It follows that

$$\sup_{u \in \mathbb{E}} \big(\langle y, u \rangle - f(u) \big) \leq \langle y, x \rangle - f(x). \tag{1}$$

By definition of supremum, we have

$$\sup_{u \in \mathbb{E}} \big(\langle y, u \rangle - f(u) \big) \geq \langle y, x \rangle - f(x). \tag{2}$$

From (1) and (2), we get

$$\sup_{u \in \mathbb{E}} \big(\langle y, u \rangle - f(u) \big) = \langle y, x \rangle - f(x).$$

That is,

$$f^*(y) = \langle y, x \rangle - f(x).$$

Then

$$\begin{aligned}
& f^*(z) - f^*(y) \\
= \; & \sup_{u \in \mathbb{E}} \big(\langle z, u \rangle - f(u) \big) - \sup_{u \in \mathbb{E}} \big(\langle y, u \rangle \rangle - f(u) \big) \\
= \; & \sup_{u \in \mathbb{E}} \big(\langle z, u \rangle - f(u) \big) - \langle y, x \rangle + f(x) \\
\geq \; & \langle z, x \rangle - f(x) - \langle y, x \rangle + f(x)
\end{aligned}$$

$$= \langle z - y, x \rangle.$$

That is,

$$\langle x, z - y \rangle \leq f^*(z) - f^*(y).$$

So $x \in \partial f^*(y)$. This proves

$$y \in \partial f(x) \implies x \in \partial f^*(y).$$

Since $f^{**} = f$, similarly, we can prove that

$$x \in \partial f^*(y) \implies y \in \partial f(x).$$

$\blacksquare$

PROPOSITION 13.4.2. Let f be a proper convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x be a point in $\mathbb{E}$. Assume that $\partial f(x) \neq \emptyset$. Then $f^{**}(x) = f(x)$.

Chapter 14

The Proximal Operator

14.1 Definitions

DEFINITION (Proximal Operator)**.** Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **proximal operator** of f, denoted by prox_f, to be a function from $\mathbb{E}$ to $\mathcal{P}(\mathbb{E})$ given by

$$\text{prox}_f(x) := \operatorname*{argmin}_{y \in \mathbb{E}} \left\{ f(y) + \frac{1}{2}\|y - x\|^2 \right\}.$$

14.2 Examples

EXAMPLE 14.2.1 (Soft Threshold)**.** Let $\lambda \geq 0$. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) := \lambda|x|$. Then

$$\text{prox}_f(x) = \begin{cases} x + \lambda, & \text{if } x < -\lambda \\ 0, & \text{if } -\lambda \leq x \leq \lambda \\ x - \lambda, & \text{if } x > \lambda. \end{cases}$$

14.3 Basic Properties

(bug)

PROPOSITION 14.3.1. Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x \in \mathbb{E}$, $\text{prox}_f(x)$ is a singleton set.

Proof. Let x be an arbitrary element of $\mathbb{E}$. Define a function $h : \mathbb{E} \to \mathbb{R}^*$ by $h(y) := \frac{1}{2}\|y-x\|^2$. Define a function $g : \mathbb{E} \to \mathbb{R}^*$ by $g(y) := f(y) + h(y)$. Then $\text{prox}_f(x) = \underset{y\in\mathbb{E}}{\text{argmin}} g(y)$. Note that h is proper, lower semi-continuous, and β-strongly convex for any $\beta \in (0, 1)$. Since f and h are proper, g is proper (why?). Since f and h are lower semi-continuous, g is lower semi-continuous. Since f is convex and h is β-strongly convex, g is β-strongly convex. Since g is proper, lower semi-continuous, and strongly convex, g has a unique minimizer (why?). So $\text{prox}_f(x)$ is a singleton set. ■

not fully understood

PROPOSITION 14.3.2. Let C be a nonempty closed convex subset of $\mathbb{E}$. Then prox_{δ_C} and proj_C are both singleton and $\text{prox}_{\delta_C} = \text{proj}_C$.

Proof. Since C is nonempty, convex, and closed, δ_C is proper, convex, and lower semi-continuous and hence prox_{δ_C} is singleton. Since C is nonempty, convex, and closed, proj_C is singleton. Let x and p be arbitrary elements of $\mathbb{E}$. Then

$$
\begin{aligned}
&p \in \text{prox}_{\delta_C}(x) \\
\iff &p \in \underset{y\in\mathbb{E}}{\text{argmin}}\{\delta_C(y) + \frac{1}{2}\|y-x\|^2\} \\
\iff &\forall y \in \mathbb{E}, \delta_C(y) + \frac{1}{2}\|y-x\|^2 \geq \delta_C(p) + \frac{1}{2}\|p-x\|^2 \\
\iff &p \in C \text{ and } \forall y \in C, \frac{1}{2}\|y-x\|^2 \geq \frac{1}{2}\|p-x\|^2 \\
\iff &p \in C \text{ and } \forall y \in C, \|y-x\| \geq \|p-x\| \\
\iff &p \in \underset{y\in C}{\text{argmin}}\|y-x\| \\
\iff &p \in \text{proj}_C(x).
\end{aligned}
$$

■

PROPOSITION 14.3.3 (Firmly Non-Expansive)**.** Let f be a proper closed convex function. Then prox_f is firmly non-expansive.

14.4 Prox Calculus Rules

PROPOSITION 14.4.1 (Scaling and Translation)**.**

THEOREM 14.1 (Norm Composition)**.**

PROPOSITION 14.4.2. Let $f_1, ..., f_m$ be proper, convex, and lower semi-continuous functions from $\mathbb{R}$ to $\mathbb{R}^*$. Define a function $f : \mathbb{R}^m \to \mathbb{R}^*$ by $f((x_i)_{i=1}^m) := \sum_{i=1}^m f_i(x_i)$. Then

$$\operatorname{prox}_f((x_i)_{i=1}^m) = (\operatorname{prox}_{f_i}(x_i))_{i=1}^m.$$

Proof. Since each f_i is proper, convex, and lower semi-continuous, f is proper, convex, and lower semi-continuous. Let $(x_i)_{i=1}^m$ and $(p_i)_{i=1}^m$ be arbitrary elements of $\mathbb{R}^m$. Then

$$(p_i)_{i=1}^m = \operatorname{prox}_f((x_i)_{i=1}^m)$$
$$\iff$$

■

14.5 The Second Prox Theorem

PROPOSITION 14.5.1. Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x and p be points in $\mathbb{E}$. Then $p = \operatorname{prox}_f(x)$ if and only if

$$x - p \in \partial f(p).$$

PROPOSITION 14.5.2. Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x and p be elements of $\mathbb{E}$. Then $p = \operatorname{prox}_f(x)$ if and only if

$$\forall y \in \mathbb{E}, \quad \langle y - p, x - p \rangle \leq f(y) - f(p).$$

Proof. **Forward Direction**:

Assume that $p = \text{prox}_f(x)$. I will show that $\forall y \in \mathbb{E}$, $\langle y - p, x - p\rangle \le f(y) - f(p)$. Let y be an arbitrary element of $\mathbb{E}$. Define for each $\lambda \in (0,1)$ a point p_λ by $p_\lambda := \lambda y + (1-\lambda)p$. Then

$$
\begin{aligned}
& p = \text{prox}_f(x) \\
\Longrightarrow\ & f(p) + \frac{1}{2}\|x - p\|^2 \le f(p_\lambda) + \frac{1}{2}\|x - p_\lambda\|^2 \\
\Longleftrightarrow\ & f(p) \le f(p_\lambda) + \frac{1}{2}\|x - p_\lambda\|^2 - \frac{1}{2}\|x - p\|^2 \\
\Longleftrightarrow\ & f(p) \le f(p_\lambda) + \frac{1}{2}\left\langle \Big[(x - p_\lambda) + (x - p)\Big], \Big[(x - p_\lambda) - (x - p)\Big] \right\rangle \\
\Longleftrightarrow\ & f(p) \le f(p_\lambda) + \frac{1}{2}\left\langle \Big[2x - \lambda y - (1-\lambda)p - p\Big], \Big[p - \lambda y - (1-\lambda)p\Big] \right\rangle \\
\Longleftrightarrow\ & f(p) \le f(p_\lambda) + \frac{1}{2}\left\langle \Big[2(x - p) + \lambda(p - y))\Big], \Big[\lambda(p - y)\Big] \right\rangle \\
\Longleftrightarrow\ & f(p) \le f(p_\lambda) + \lambda\langle x - p, p - y\rangle + \frac{1}{2}\lambda^2\|p - y\|^2 \\
\Longleftrightarrow\ & f(p) \le f(\lambda y + (1-\lambda)p) + \lambda\langle x - p, p - y\rangle + \frac{1}{2}\lambda^2\|p - y\|^2 \\
\Longrightarrow\ & f(p) \le \lambda f(y) + (1-\lambda)f(p) + \lambda\langle x - p, p - y\rangle + \frac{1}{2}\lambda^2\|p - y\|^2 \\
\Longleftrightarrow\ & \lambda\langle y - p, x - p\rangle \le \lambda f(y) - \lambda f(p) + \frac{1}{2}\lambda^2\|p - y\|^2 \\
\Longleftrightarrow\ & \langle y - p, x - p\rangle \le f(y) - f(p) + \frac{1}{2}\lambda\|p - y\|^2 \\
\Longleftrightarrow\ & \langle y - p, x - p\rangle \le f(y) - f(p).
\end{aligned}
$$

Backward Direction:

Assume that $\forall y \in \mathbb{E}$, $\langle y - p, x - p\rangle \le f(y) - f(p)$. I will show that $p = \text{prox}_f(x)$. Let y be an arbitrary element of $\mathbb{E}$. Then

$$
\begin{aligned}
& \langle y - p, x - p\rangle \le f(y) - f(p) \\
\Longleftrightarrow\ & f(p) + \frac{1}{2}\|x - p\|^2 \le f(y) + \langle x - p, p - y\rangle + \frac{1}{2}\|x - p\|^2 \\
\Longrightarrow\ & f(p) + \frac{1}{2}\|x - p\|^2 \le f(y) + \langle x - p, p - y\rangle + \frac{1}{2}\|x - p\|^2 + \frac{1}{2}\|p - y\|^2 \\
\Longleftrightarrow\ & f(p) + \frac{1}{2}\|x - p\|^2 \le f(y) + \frac{1}{2}\|(x - p) + (p - y)\|^2 \\
\Longleftrightarrow\ & f(p) + \frac{1}{2}\|x - p\|^2 \le f(y) + \frac{1}{2}\|x - y\|^2 \\
\Longrightarrow\ & p = \text{prox}_f(x).
\end{aligned}
$$

This completes the proof.

■

14.6 Moreau Decomposition

THEOREM 14.2 (Moreau Decomposition)**.** Let f be a proper closed convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Then

$$\operatorname{prox}_f + \operatorname{prox}_{f^*} = \operatorname{Id}.$$

Proof. Let x be an arbitrary point in $\mathbb{E}$. We are to prove that

$$\operatorname{prox}_f(x) + \operatorname{prox}_{f^*}(x) = x.$$

Let p denote $\operatorname{prox}_f(x)$. Since f is proper convex and lower semi-continuous and $p = \operatorname{prox}_f(x)$, we get

$$x - p \in \partial f(p).$$

Since $x - p \in \partial f(p)$, we get $p \in \partial f^*(x-p)$. It follows that $x - p = \operatorname{prox}_{f^*}(x)$. Substitute $p = \operatorname{prox}_f(x)$ and rearrange the equation, we get

$$\operatorname{prox}_f(x) + \operatorname{prox}_{f^*}(x) = x.$$

∎

Chapter 15

Ellipsoids

DEFINITION (Ellipsoid)**.** Let v be a point in some Euclidean space $\mathbb{E}$. We define an **ellipsoid**, centered at point v, to be a set of the form

$$\{x \in \mathbb{E} : (x - v)^T A(x - v) = 1\}$$

where A is some d by d positive definite matrix.

15.1 Properties

PROPOSITION 15.1.1. The eigenvectors of A define the principal axes of the ellipsoid.