Chapter 1

Experimental Design

Contents

1.1 Completely Random Design

DEFINITION 1.1 (Completely Random Design)**.** Let k denote the number of treatments. Let n_i denote the number of units that receive the i-th treatment. We model the population as

$$y_{ij} = \mu_i + e_{ij}, \text{ for } i \in \{1, ..., k\} \text{ and } j \in \{1, ..., n_i\}$$

where $e_{ij} \sim \mathcal{N}(0, \sigma^2)$ are assumed to be independent. The total number of parameters in this model is $k + 1$

1.1.1 Estimation of Mean

PROPOSITION 1.2. Let y_{ij} for $i \in \{1, ..., k\}$ and $j \in \{1, ..., n_i\}$ be given. Consider the following optimization problem:

$$\begin{array}{llll} \text{(P)} & \min & f(\mu) := \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 \\ & \text{subject to:} & \mu \in \mathbb{R}^k. \end{array}$$

Then the minimizer $\hat{\mu} \in \mathbb{R}^k$ of (P) is given by

$$\hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}, \text{ for } i \in \{1, ..., k\}.$$

Proof. Let $p \in \{1, ..., k\}$ be arbitrary. Then

$$\frac{\partial}{\partial \mu_p} f(\mu) = \frac{\partial}{\partial \mu_p} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 = \sum_{j=1}^{n_p} \frac{\partial}{\partial \mu_p} (y_{pj} - \mu_p)^2 = -2 \sum_{j=1}^{n_p} (y_{pj} - \mu_p), \text{ and}$$

$$\frac{\partial^2}{\partial \mu_p^2} f(\mu) = \frac{\partial}{\partial \mu_p} \left[-2 \sum_{j=1}^{n_p} (y_{pj} - \mu_p) \right] = 2n_p > 0.$$

Suppose $\hat{\mu} \in \mathbb{R}^k$ is a minimizer of f. Then we have $\nabla f(\hat{\mu}) = \mathbb{0} \in \mathbb{R}^k$. So

$$\frac{\partial}{\partial \mu_i} f(\hat{\mu}) = 0 \implies \hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}.$$

Testing the Hessian of f at point $\hat{\mu} \in \mathbb{R}^k$ confirms that it is indeed a minimizer of f. $\square$

PROPOSITION 1.3 (Mean of the Mean Estimator)**.** We have

$$\mathbb{E}(\hat{\mu}) = \mu \in \mathbb{R}^k.$$

i.e., $\hat{\mu}$ is an unbiased estimator for μ.

Proof. Recall that $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, we have $y_{ij} \sim \mathcal{N}(\mu_i, \sigma^2)$. So $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, $\mathbb{E}(y_{ij}) = \mu_i$. Now we can compute

$$\mathbb{E}(\hat{\mu}_i) = \mathbb{E}(\frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}) = \frac{1}{n_i}\sum_{j=1}^{n_i}\mathbb{E}(y_{ij}) = \frac{1}{n_i}\sum_{j=1}^{n_i}\mu_i = \mu_i.$$

That is, $\mathbb{E}(\hat{\mu}) = \mu$, as desired. □

PROPOSITION 1.4 (Variance of the Mean Estimator)**.** We have

$$\forall i \in \{1, ..., k\}, \quad \mathbb{V}(\hat{\mu}_i) = \frac{\sigma^2}{n_i}.$$

Proof. Recall that $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, we have $y_{ij} \sim \mathcal{N}(\mu_i, \sigma^2)$. So $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, $\mathbb{V}(y_{ij}) = \sigma^2$. Now we can compute

$$\mathbb{V}(\hat{\mu}_i) = \mathbb{V}(\frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}) = \sum_{j=1}^{n_i}\frac{1}{n_i^2}\mathbb{V}(y_{ij}) = \sum_{j=1}^{n_i}\frac{1}{n_i^2}\sigma^2 = \frac{\sigma^2}{n_i}.$$

□

1.1.2 Estimation of Variance

In this subsection, we assume that $\forall i \in \{1, ..., k\}$, $n_i = n$ for some $n \in \mathbb{Z}_{++}$.

DEFINITION 1.5 (Sum of Squares)**.** We define the following terms:

$$\mathrm{SS}_{\mathrm{trt}} := n\sum_{i=1}^{k}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2,$$

$$\mathrm{SS}_{\mathrm{err}} := \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i\cdot})^2,$$

$$\mathrm{SS}_{\mathrm{tot}} := \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2.$$

PROPOSITION 1.6 (Decomposition of $\mathrm{SS}_{\mathrm{tot}}$)**.** We have

$$\mathrm{SS}_{\mathrm{tot}} = \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{err}}.$$

Proof.

$$\begin{aligned}
\mathrm{SS}_{\mathrm{tot}} &= \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.} + \bar{y}_{i.} - \bar{y}_{..})^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n}\left[(y_{ij} - \bar{y}_{i.})^2 + (\bar{y}_{i.} - \bar{y}_{..})^2 + 2(y_{ij} - \bar{y}_{i.})(\bar{y}_{i.} - \bar{y}_{..})\right] \\
&= \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{err}} + \sum_{i=1}^{k}\sum_{j=1}^{n} 2(y_{ij} - \bar{y}_{i.})(\bar{y}_{i.} - \bar{y}_{..}) \\
&= \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{err}} + 2\sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}\bar{y}_{i.} - 2\sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}\bar{y}_{..} - 2\sum_{i=1}^{k}\sum_{j=1}^{n}\bar{y}_{i.}^2 + 2\sum_{i=1}^{k}\sum_{j=1}^{n}\bar{y}_{i.}\bar{y}_{..} \\
&= \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{err}} + 2n\sum_{i=1}^{k}\bar{y}_{i.}^2 - 2n\bar{y}_{..}\sum_{i=1}^{k}\bar{y}_{i.} - 2n\sum_{i=1}^{k}\bar{y}_{i.}^2 + 2n\bar{y}_{..}\sum_{i=1}^{k}\bar{y}_{i.} \\
&= \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{err}} + 0 = \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{err}}.
\end{aligned}$$

□

DEFINITION 1.7 (Mean Squares)**.** We define the following estimators for the variance σ^2.

$$\begin{aligned}
\mathrm{MS}_{\mathrm{trt}} &:= \mathrm{SS}_{\mathrm{trt}}/(k-1), \\
\mathrm{MS}_{\mathrm{err}} &:= \mathrm{SS}_{\mathrm{err}}/(k(n-1)).
\end{aligned}$$

PROPOSITION 1.8 (Mean of $\mathrm{MS}_{\mathrm{err}}$)**.** We have

$$\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \sigma^2.$$

i.e., $\mathrm{MS}_{\mathrm{err}}$ is an unbiased estimator for σ^2.

Proof. Recall that $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n\}$, we have $y_{ij} \sim \mathcal{N}(\mu_i, \sigma^2)$. So

$$\bar{y}_{i\cdot} = \frac{1}{n} \sum_{j=1}^{n} y_{ij} \sim \mathcal{N}(\mu_i, \frac{\sigma^2}{n}), \quad \forall i \in \{1, ..., k\}.$$

So

$$\begin{aligned} \mathbb{E}(y_{ij}^2) &= \mathbb{V}(y_{ij}) + \mathbb{E}^2(y_{ij}) = \sigma^2 + \mu_i^2, \quad \forall i, j, \text{ and} \\ \mathbb{E}(\bar{y}_{i\cdot}^2) &= \mathbb{V}(\bar{y}_{i\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot}) = \frac{\sigma^2}{n} + \mu_i^2, \quad \forall i. \end{aligned}$$

Now we can compute

$$\begin{aligned} \mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \mathbb{E}(\mathrm{SS}_{\mathrm{err}}/(k(n-1))) &= \mathbb{E}(\frac{1}{k(n-1)} \sum_{i=1}^{k} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{i\cdot})^2) \\ &= \mathbb{E}(\frac{1}{k(n-1)} \sum_{i=1}^{k} \sum_{j=1}^{n} (y_{ij}^2 + \bar{y}_{i\cdot}^2 - 2 y_{ij} \bar{y}_{i\cdot})) \\ &= \mathbb{E}(\frac{1}{k(n-1)} \left[\sum_{i=1}^{k} \sum_{j=1}^{n} y_{ij}^2 + n \sum_{i=1}^{k} \bar{y}_{i\cdot}^2 - 2 \sum_{i=1}^{k} \bar{y}_{i\cdot} \sum_{j=1}^{n} y_{ij} \right]) \\ &= \mathbb{E}(\frac{1}{k(n-1)} \left[\sum_{i=1}^{k} \sum_{j=1}^{n} y_{ij}^2 + n \sum_{i=1}^{k} \bar{y}_{i\cdot}^2 - 2n \sum_{i=1}^{k} \bar{y}_{i\cdot}^2 \right]) \\ &= \mathbb{E}(\frac{1}{k(n-1)} \left[\sum_{i=1}^{k} \sum_{j=1}^{n} y_{ij}^2 - n \sum_{i=1}^{k} \bar{y}_{i\cdot}^2 \right]) \\ &= \frac{1}{k(n-1)} \left[\sum_{i=1}^{k} \sum_{j=1}^{n} \mathbb{E}(y_{ij}^2) - n \sum_{i=1}^{k} \mathbb{E}(\bar{y}_{i\cdot}^2) \right], \text{ by linearity} \\ &= \frac{1}{k(n-1)} \left[\sum_{i=1}^{k} \sum_{j=1}^{n} (\sigma^2 + \mu_i^2) - n \sum_{i=1}^{k} (\frac{\sigma^2}{n} + \mu_i^2) \right], \text{ by above} \\ &= \frac{1}{k(n-1)} \left[(kn - k)\sigma^2 + n \sum_{i=1}^{k} (\mu_i^2 - \mu_i^2) \right] = \sigma^2. \end{aligned}$$

That is, $\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \sigma^2$, as desired. ☐

PROPOSITION 1.9 (Mean of $\mathrm{MS}_{\mathrm{trt}}$)**.** Under the assumption that $\mu = \mathbb{1}\mu_0 \in \mathbb{R}^k$ for some $\mu_0 \in \mathbb{R}$, we have

$$\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) = \sigma^2.$$

i.e., $\mathrm{MS}_{\mathrm{trt}}$ is an unbiased estimator for σ^2 given that $\mu = \mathbb{1}\mu_0$ for some μ_0.

Proof. Assume that $\mu = \mathbb{1}\mu_0 \in \mathbb{R}^k$ for some $\mu_0 \in \mathbb{R}$. Then $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n\}$, we have $y_{ij} \sim \mathcal{N}(\mu_0, \sigma^2)$. So

$$\begin{aligned} \bar{y}_{i\cdot} &= \frac{1}{n}\sum_{j=1}^{n} y_{ij} \sim \mathcal{N}(\mu_0, \frac{\sigma^2}{n}), \quad \forall i \in \{1, ..., k\}, \text{ and} \\ \bar{y}_{\cdot\cdot} &= \frac{1}{Kn}\sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij} \sim \mathcal{N}(\mu_0, \frac{\sigma^2}{Kn}). \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}(\bar{y}_{i\cdot}^2) &= \mathbb{V}(\bar{y}_{i\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot}) = \frac{\sigma^2}{n} + \mu_0^2, \quad \forall i, \text{ and} \\ \mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \mathbb{V}(\bar{y}_{\cdot\cdot}) + \mathbb{E}^2(\bar{y}_{\cdot\cdot}) = \frac{\sigma^2}{Kn} + \mu_0^2. \end{aligned}$$

Now we can compute

$$\begin{aligned} \mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) = \mathbb{E}(\mathrm{SS}_{\mathrm{trt}}/(k-1)) &= \mathbb{E}(\frac{n}{k-1}\sum_{i=1}^{k}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2) \\ &= \mathbb{E}(\frac{n}{k-1}\sum_{i=1}^{k}(\bar{y}_{i\cdot}^2 + \bar{y}_{\cdot\cdot}^2 - 2\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot})) \\ &= \mathbb{E}(\frac{n}{k-1}\left[\sum_{i=1}^{k}\bar{y}_{i\cdot}^2 + k\bar{y}_{\cdot\cdot}^2 - 2\bar{y}_{\cdot\cdot}\sum_{i=1}^{k}\bar{y}_{i\cdot}\right]) \\ &= \mathbb{E}(\frac{n}{k-1}\left[\sum_{i=1}^{k}\bar{y}_{i\cdot}^2 + k\bar{y}_{\cdot\cdot}^2 - 2K\bar{y}_{\cdot\cdot}^2\right]) \\ &= \mathbb{E}(\frac{n}{k-1}\left[\sum_{i=1}^{k}\bar{y}_{i\cdot}^2 - k\bar{y}_{\cdot\cdot}^2\right]) \\ &= \frac{n}{k-1}\left[\sum_{i=1}^{k}\mathbb{E}(\bar{y}_{i\cdot}^2) - k\mathbb{E}(\bar{y}_{\cdot\cdot}^2)\right], \text{ by linearity} \\ &= \frac{n}{k-1}\left[\sum_{i=1}^{k}(\frac{\sigma^2}{n} + \mu_0^2) - k(\frac{\sigma^2}{Kn} + \mu_0^2)\right], \text{ by above} \\ &= \frac{n}{k-1}\left[(\frac{k}{n} - \frac{1}{n})\sigma^2 + (k-k)\mu_0^2\right] = \sigma^2. \end{aligned}$$

That is, $\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) = \sigma^2$, as desired. $\square$

1.1.3 Hypothesis Testing

DEFINITION 1.10 (ANOVA Table)**.**

Table 1.1: ANOVA Table for Completely Randomized Design

	Sum of Squares	Degree of Freedom	Mean Squares	F_0
Treatment	SS_{trt}	$k-1$	MS_{trt}	$\text{MS}_{\text{trt}}/\text{MS}_{\text{err}}$
Error	SS_{err}	$k(n-1)$	MS_{err}	
Total	SS_{tot}	$kn-1$		

1.2 Randomized Block Design

DEFINITION 1.11 (Randomized Block Design)**.** Let $a \in \mathbb{Z}_{++}$ denote the number of treatments. Let $b \in \mathbb{Z}_{++}$ denote the number of blocks. We model the population as

$$y_{ij} = \mu + \alpha_i + \beta_j + e_{ij}, \text{ for } i \in \{1, ..., a\} \text{ and } j \in \{1, ..., b\}$$

with constraints $\mathbb{1}^\top \alpha = 0$ and $\mathbb{1}^\top \beta = 0$, and $e_{ij} \sim \mathcal{N}(0, \sigma^2)$ are assumed to be independent. The total number of parameters in this model is $2 + a + b$.

1.2.1 Estimation of Mean

PROPOSITION 1.12. Let y_{ij} for $i \in \{1, ..., a\}$ and $j \in \{1, ..., b\}$ be given. Consider the following optimization problem:

$$\begin{aligned} \text{(P)} \qquad \min \qquad & f(\mu, \alpha, \beta) := \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \mu - \alpha_i - \beta_j)^2 \\ \text{subject to:} \quad & \mu \in \mathbb{R}, \alpha \in \mathbb{R}^a, \beta \in \mathbb{R}^b, \\ & \mathbb{1}^\top \alpha = 0, \mathbb{1}^\top \beta = 0. \end{aligned}$$

Then the minimizer $(\hat{\mu}, \hat{\alpha}, \hat{\beta}) \in \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b$ of (P) is given by

$$\begin{aligned} \hat{\mu} &= \bar{y}_{\cdot\cdot} = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}, \\ \hat{\alpha}_i &= \bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot} = \frac{1}{b} \sum_{j=1}^{b} y_{ij} - \bar{y}_{\cdot\cdot}, \text{ for } i \in \{1, ..., a\}, \\ \hat{\beta}_j &= \bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot} = \frac{1}{a} \sum_{i=1}^{a} y_{ij} - \bar{y}_{\cdot\cdot}, \text{ for } j \in \{1, ..., b\}. \end{aligned}$$

Proof. Form the Lagrangian function $\mathcal{L} : \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b \oplus \mathbb{R} \oplus \mathbb{R}$

$$\mathcal{L}(\mu, \alpha, \beta, \xi, \eta) := f(\mu, \alpha, \beta) - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta.$$

Compute the derivatives:

$$\begin{aligned} \frac{\partial}{\partial \mu} \mathcal{L}(\mu, \alpha, \beta, \xi, \eta) &= \frac{\partial}{\partial \mu} \left[f(\mu, \alpha, \beta) - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta \right] \\ &= \frac{\partial}{\partial \mu} \left[\sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \mu - \alpha_i - \beta_j)^2 - \xi \mathbb{1}^\top \alpha - \eta \mathbb{1}^\top \beta \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\partial}{\partial\mu}(y_{ij}-\mu-\alpha_i-\beta_j)^2 \\
&= -2\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\mu-\alpha_i-\beta_j), \\
\frac{\partial}{\partial\alpha_p}\mathcal{L}(\mu,\alpha,\beta,\xi,\eta) &= \frac{\partial}{\partial\alpha_p}\left[f(\mu,\alpha,\beta)-\xi\mathbb{1}^\top\alpha-\eta\mathbb{1}^\top\beta\right] \\
&= \frac{\partial}{\partial\alpha_p}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\mu-\alpha_i-\beta_j)^2-\xi\mathbb{1}^\top\alpha-\eta\mathbb{1}^\top\beta\right] \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\partial}{\partial\alpha_p}(y_{ij}-\mu-\alpha_i-\beta_j)^2-\xi\frac{\partial}{\partial\alpha_p}\mathbb{1}^\top\alpha \\
&= -2\sum_{j=1}^{b}(y_{pj}-\mu-\alpha_p-\beta_j)-\xi, \\
\frac{\partial}{\partial\beta_q}\mathcal{L}(\mu,\alpha,\beta,\xi,\eta) &= \frac{\partial}{\partial\beta_q}\left[f(\mu,\alpha,\beta)-\xi\mathbb{1}^\top\alpha-\eta\mathbb{1}^\top\beta\right] \\
&= \frac{\partial}{\partial\beta_q}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\mu-\alpha_i-\beta_j)^2-\xi\mathbb{1}^\top\alpha-\eta\mathbb{1}^\top\beta\right] \\
&= \sum_{i=1}^{a}\frac{\partial}{\partial\beta_q}(y_{ij}-\mu-\alpha_i-\beta_j)^2-\eta\frac{\partial}{\partial\beta_q}\mathbb{1}^\top\beta \\
&= -2\sum_{i=1}^{a}(y_{iq}-\mu-\alpha_i-\beta_q)-\eta, \\
\frac{\partial}{\partial\xi}\mathcal{L}(\mu,\alpha,\beta,\xi,\eta) &= \frac{\partial}{\partial\xi}\left[f(\mu,\alpha,\beta)-\xi\mathbb{1}^\top\alpha-\eta\mathbb{1}^\top\beta\right] = -\mathbb{1}^\top\alpha, \\
\frac{\partial}{\partial\eta}\mathcal{L}(\mu,\alpha,\beta,\xi,\eta) &= \frac{\partial}{\partial\eta}\left[f(\mu,\alpha,\beta)-\xi\mathbb{1}^\top\alpha-\eta\mathbb{1}^\top\beta\right] = -\mathbb{1}^\top\beta.
\end{aligned}
$$

Let $(\hat\mu,\hat\alpha,\hat\beta,\hat\xi,\hat\eta)\in\mathbb{R}\oplus\mathbb{R}^a\oplus\mathbb{R}^b\oplus\mathbb{R}\oplus\mathbb{R}$ be such that $\nabla\mathcal{L}(\hat\mu,\hat\alpha,\hat\beta,\hat\xi,\hat\eta)=\mathbb{0}\in\mathbb{R}^{a+b+3}$. Then we get the following system of equations:

$$
\begin{cases}
-2\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\hat\mu-\hat\alpha_i-\hat\beta_j)=0 \\
-2\sum_{j=1}^{b}(y_{pj}-\hat\mu-\hat\alpha_p-\hat\beta_j)-\hat\xi=0, \forall p \\
-2\sum_{i=1}^{a}(y_{iq}-\hat\mu-\hat\alpha_i-\hat\beta_q)-\hat\eta=0, \forall q \\
-\mathbb{1}^\top\hat\alpha=0 \\
-\mathbb{1}^\top\hat\beta=0
\end{cases}
\implies
\begin{cases}
\hat\mu=\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij} \\
\hat\alpha_i=\bar y_{i\cdot}-\hat\mu, \forall i \\
\hat\beta_j=\bar y_{\cdot j}-\hat\mu, \forall j \\
\hat\xi=0 \\
\hat\eta=0.
\end{cases}
$$

Testing the Hessian of $\mathcal{L}$ at point $(\hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\xi}, \hat{\eta}) \in \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b \oplus \mathbb{R} \oplus \mathbb{R}$ confirms that it is indeed a minimizer of $\mathcal{L}$. $\square$

PROPOSITION 1.13 (Mean of the Mean Estimator)**.** We have

$$\mathbb{E}(\hat{\mu}) = \mu \in \mathbb{R}, \ \mathbb{E}(\hat{\alpha}) = \alpha \in \mathbb{R}^a, \ \text{and } \mathbb{E}(\hat{\beta}) = \beta \in \mathbb{R}^b.$$

i.e., $\hat{\mu}$ is an unbiased estimator for μ, $\hat{\alpha} \in \mathbb{R}^a$ is an unbiased estimator for $\alpha \in \mathbb{R}^a$, and $\hat{\beta} \in \mathbb{R}^b$ is an unbiased estimator for $\beta \in \mathbb{R}^b$.

Proof. Recall that $\forall i \in \{1, ..., a\}$, $\forall j \in \{1, ..., b\}$, we have $y_{ij} \sim \mathcal{N}(\mu + \alpha_i + \beta_j, \sigma^2)$. So

$$\begin{aligned}
\bar{y}_{i\cdot} &= \frac{1}{b}\sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu + \alpha_i, \frac{\sigma^2}{b}), \quad \forall i \in \{1, ..., a\}, \\
\bar{y}_{\cdot j} &= \frac{1}{a}\sum_{i=1}^{a} y_{ij} \sim \mathcal{N}(\mu + \beta_j, \frac{\sigma^2}{a}), \quad \forall j \in \{1, ..., b\}, \\
\bar{y}_{\cdot\cdot} &= \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu, \frac{\sigma^2}{ab}).
\end{aligned}$$

Now we can compute

$$\begin{aligned}
\mathbb{E}(\hat{\mu}) &= \mathbb{E}(\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}) = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} \mathbb{E}(y_{ij}) = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}(\mu + \alpha_i + \beta_j) \\
&= \mu + \frac{1}{a}\sum_{i=1}^{a}\alpha_i + \frac{1}{b}\sum_{j=1}^{b}\beta_j = \mu, \\
\mathbb{E}(\hat{\alpha}_i) &= \mathbb{E}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) = \mathbb{E}(\bar{y}_{i\cdot}) - \mathbb{E}(\bar{y}_{\cdot\cdot}) = \mu + \alpha_i - \mu = \alpha_i, \quad \forall i, \\
\mathbb{E}(\hat{\beta}_j) &= \mathbb{E}(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) = \mathbb{E}(\bar{y}_{\cdot j}) - \mathbb{E}(\bar{y}_{\cdot\cdot}) = \mu + \beta_j - \mu = \beta_j, \quad \forall j.
\end{aligned}$$

That is, $\mathbb{E}(\hat{\mu}) = \mu$, $\mathbb{E}(\hat{\alpha}) = \alpha$, and $\mathbb{E}(\hat{\beta}) = \beta$, as desired. $\square$

PROPOSITION 1.14 (Variance of the Mean Estimator)**.**

1.2.2 Estimation of Variance

DEFINITION 1.15 (Sum of Squares)**.** We define the following terms:

$$\begin{aligned}
\mathrm{SS}_{\mathrm{trt}} &:= b\sum_{i=1}^{a}(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2,\\
\mathrm{SS}_{\mathrm{blk}} &:= a\sum_{j=1}^{b}(\bar{y}_{\cdot j}-\bar{y}_{\cdot\cdot})^2,\\
\mathrm{SS}_{\mathrm{err}} &:= \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i\cdot}-\bar{y}_{\cdot j}+\bar{y}_{\cdot\cdot})^2,\\
\mathrm{SS}_{\mathrm{tot}} &:= \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{\cdot\cdot})^2.
\end{aligned}$$

PROPOSITION 1.16 (Decomposition of $\mathrm{SS}_{\mathrm{tot}}$)**.** We have

$$\mathrm{SS}_{\mathrm{tot}} = \mathrm{SS}_{\mathrm{trt}} + \mathrm{SS}_{\mathrm{blk}} + \mathrm{SS}_{\mathrm{err}}.$$

DEFINITION 1.17 (Mean Squares)**.** We define the following estimators for the variance σ^2.

$$\begin{aligned}
\mathrm{MS}_{\mathrm{trt}} &:= \mathrm{SS}_{\mathrm{trt}}/(a-1),\\
\mathrm{MS}_{\mathrm{err}} &:= \mathrm{SS}_{\mathrm{err}}/((a-1)(b-1)).
\end{aligned}$$

PROPOSITION 1.18. We have

$$\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \sigma^2.$$

i.e., $\mathrm{MS}_{\mathrm{err}}$ is an unbiased estimator for σ^2.

Proof. Recall that $\forall i \in \{1, ..., a\}$, $\forall j \in \{1, ..., b\}$, we have $y_{ij} \sim \mathcal{N}(\mu + \alpha_i + \beta_j, \sigma^2)$. So

$$\begin{aligned}
\bar{y}_{i\cdot} &= \frac{1}{b}\sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu+\alpha_i, \frac{\sigma^2}{b}), \quad \forall i \in \{1, ..., a\},\\
\bar{y}_{\cdot j} &= \frac{1}{a}\sum_{i=1}^{a} y_{ij} \sim \mathcal{N}(\mu+\beta_j, \frac{\sigma^2}{a}), \quad \forall j \in \{1, ..., b\}, \text{ and}
\end{aligned}$$

$$\bar{y}_{\cdot\cdot} = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu, \frac{\sigma^2}{ab}).$$

So

$$\begin{aligned}
\mathbb{E}(y_{ij}^2) &= \mathbb{V}(y_{ij}) + \mathbb{E}^2(y_{ij}) = \sigma^2 + (\mu+\alpha_i+\beta_j)^2, \quad \forall i,j,\\
\mathbb{E}(\bar{y}_{i\cdot}^2) &= \mathbb{V}(\bar{y}_{i\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot}) = \frac{\sigma^2}{b} + (\mu+\alpha_i)^2, \quad \forall i,\\
\mathbb{E}(\bar{y}_{\cdot j}^2) &= \mathbb{V}(\bar{y}_{\cdot j}) + \mathbb{E}^2(\bar{y}_{\cdot j}) = \frac{\sigma^2}{a} + (\mu+\beta_j)^2, \quad \forall j, \text{ and}\\
\mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \mathbb{V}(\bar{y}_{\cdot\cdot}) + \mathbb{E}^2(\bar{y}_{\cdot\cdot}) = \frac{\sigma^2}{ab} + \mu^2.
\end{aligned}$$

Now we can compute

$$\begin{aligned}
\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \mathbb{E}(\mathrm{SS}_{\mathrm{err}}/((a-1)(b-1))) &= \mathbb{E}(\frac{1}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i\cdot}-\bar{y}_{\cdot j}+\bar{y}_{\cdot\cdot})^2)\\
&= \frac{1}{(a-1)(b-1)}\mathbb{E}\Bigg(\Bigg[\sum_{i=1}^{a}\sum_{j=1}^{b}\begin{matrix}
+y_{ij}^2 & -y_{ij}\bar{y}_{i\cdot} & -y_{ij}\bar{y}_{\cdot j} & +y_{ij}\bar{y}_{\cdot\cdot}\\
-\bar{y}_{i\cdot}y_{ij} & +\bar{y}_{i\cdot}^2 & +\bar{y}_{i\cdot}\bar{y}_{\cdot j} & -\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot}\\
-\bar{y}_{\cdot j}y_{ij} & +\bar{y}_{\cdot j}\bar{y}_{i\cdot} & +\bar{y}_{\cdot j}^2 & -\bar{y}_{\cdot j}\bar{y}_{\cdot\cdot}\\
+\bar{y}_{\cdot\cdot}y_{ij} & -\bar{y}_{\cdot\cdot}\bar{y}_{i\cdot} & -\bar{y}_{\cdot\cdot}\bar{y}_{\cdot j} & +\bar{y}_{\cdot\cdot}^2
\end{matrix}\Bigg]\Bigg)\\
&= \frac{1}{(a-1)(b-1)}\mathbb{E}\Bigg(\begin{matrix}
+\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}^2 & -\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}\bar{y}_{i\cdot} & -\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}\bar{y}_{\cdot j} & +\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}\bar{y}_{\cdot\cdot}\\
-\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{i\cdot}y_{ij} & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{i\cdot}^2 & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{i\cdot}\bar{y}_{\cdot j} & -\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot}\\
-\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot j}y_{ij} & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot j}\bar{y}_{i\cdot} & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 & -\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot j}\bar{y}_{\cdot\cdot}\\
+\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot\cdot}y_{ij} & -\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot\cdot}\bar{y}_{i\cdot} & -\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot\cdot}\bar{y}_{\cdot j} & +\sum_{i=1}^{a}\sum_{j=1}^{b}\bar{y}_{\cdot\cdot}^2
\end{matrix}\Bigg)\\
&= \frac{1}{(a-1)(b-1)}\mathbb{E}\Bigg(\begin{matrix}
+\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}^2 & -b\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 & -a\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 & +ab\bar{y}_{\cdot\cdot}^2\\
-b\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 & +b\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 & +ab\bar{y}_{\cdot\cdot}^2 & -ab\bar{y}_{\cdot\cdot}^2\\
-a\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 & +ab\bar{y}_{\cdot\cdot}^2 & +a\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 & -ab\bar{y}_{\cdot\cdot}^2\\
+ab\bar{y}_{\cdot\cdot}^2 & -ab\bar{y}_{\cdot\cdot}^2 & -ab\bar{y}_{\cdot\cdot}^2 & +ab\bar{y}_{\cdot\cdot}^2
\end{matrix}\Bigg)\\
&= \frac{1}{(a-1)(b-1)}\mathbb{E}\Bigg(\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}^2 - a\sum_{j=1}^{b}\bar{y}_{\cdot j}^2 - b\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 + ab\bar{y}_{\cdot\cdot}^2\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(a-1)(b-1)}\Big[\sum_{i=1}^{a}\sum_{j=1}^{b}\mathbb{E}(y_{ij}^2) - a\sum_{j=1}^{b}\mathbb{E}(\bar{y}_{\cdot j}^2) - b\sum_{i=1}^{a}\mathbb{E}(\bar{y}_{i\cdot}^2) + ab\mathbb{E}(\bar{y}_{\cdot\cdot}^2)\Big] \\
&= \frac{1}{(a-1)(b-1)}\Big[\sum_{i=1}^{a}\sum_{j=1}^{b}(\sigma^2 + (\mu+\alpha_i+\beta_j)^2) + ab(\frac{\sigma^2}{ab} + \mu^2) \\
&\qquad - a\sum_{j=1}^{b}(\frac{\sigma^2}{a} + (\mu+\beta_j)^2) - b\sum_{i=1}^{a}(\frac{\sigma^2}{b} + (\mu+\alpha_i)^2)\Big] \\
&= \frac{1}{(a-1)(b-1)}\Big[(ab+1-a-b)\sigma^2 + (ab+ab-ab-ab)\mu^2 \\
&\qquad + 0\mu + (b-b)\sum_{i=1}^{a}\alpha_i^2 + (a-a)\sum_{j=1}^{b}\beta_j^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i\beta_j\Big] \\
&= \frac{1}{(a-1)(b-1)}(ab+1-a-b)\sigma^2 = \sigma^2.
\end{aligned}
$$

That is, $\mathbb{E}(\mathrm{MS}_{\mathrm{err}}) = \sigma^2$, as desired. □

PROPOSITION 1.19. Under the assumption that $\alpha = \mathbb{0} \in \mathbb{R}^a$, we have

$$\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) = \sigma^2.$$

i.e., $\mathrm{MS}_{\mathrm{trt}}$ is an unbiased estimator for σ^2 given that $\alpha = \mathbb{0}$.

Proof. Under the assumption that $\alpha = \mathbb{0} \in \mathbb{R}^a$, we have $\forall i \in \{1, ..., a\}$, $\forall j \in \{1, ..., b\}$, $y_{ij} \sim \mathcal{N}(\mu + \beta_j, \sigma^2)$. So

$$
\begin{aligned}
\bar{y}_{i\cdot} &= \frac{1}{b}\sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu, \frac{\sigma^2}{b}), \quad \forall i \in \{1, ..., a\}, \text{ and} \\
\bar{y}_{\cdot\cdot} &= \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij} \sim \mathcal{N}(\mu, \frac{\sigma^2}{ab}).
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbb{E}(\bar{y}_{i\cdot}^2) &= \mathbb{V}(\bar{y}_{i\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot}) = \frac{\sigma^2}{b} + \mu^2, \quad \forall i, \text{ and} \\
\mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \mathbb{V}(\bar{y}_{\cdot\cdot}) + \mathbb{E}^2(\bar{y}_{\cdot\cdot}) = \frac{\sigma^2}{ab} + \mu^2.
\end{aligned}
$$

Now we can compute

$$\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) = \mathbb{E}(\mathrm{SS}_{\mathrm{trt}}/(a-1)) = \mathbb{E}(\frac{b}{a-1}\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2)$$

$$= \frac{b}{a-1}\mathbb{E}(\sum_{i=1}^{a}(\bar{y}_{i\cdot}^2 + \bar{y}_{\cdot\cdot}^2 - 2\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot})) = \frac{b}{a-1}\mathbb{E}(\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 + a\bar{y}_{\cdot\cdot}^2 - 2a\bar{y}_{\cdot\cdot}^2)$$
$$= \frac{b}{a-1}\mathbb{E}(\sum_{i=1}^{a}\bar{y}_{i\cdot}^2 - a\bar{y}_{\cdot\cdot}^2) = \frac{b}{a-1}\left[\sum_{i=1}^{a}\mathbb{E}(\bar{y}_{i\cdot}^2) - a\mathbb{E}(\bar{y}_{\cdot\cdot}^2)\right]$$
$$= \frac{b}{a-1}\left[\sum_{i=1}^{a}(\frac{\sigma^2}{b} + \mu^2) - a(\frac{\sigma^2}{ab} + \mu^2)\right]$$
$$= \frac{b}{a-1}\left[(\frac{a}{b} - \frac{1}{b})\sigma^2 + (a-a)\mu^2\right] = \sigma^2.$$

That is, $\mathbb{E}(\mathrm{MS}_{\mathrm{trt}}) = \sigma^2$, as desired. □

1.2.3 Hypothesis Testing

We are interested in testing the following hypothesis:

- $H_0 : \alpha = \mathbb{0} \in \mathbb{R}^a$ vs $H_1 : \alpha \neq \mathbb{0} \in \mathbb{R}^a$.

DEFINITION 1.20. We define the F-statistic as

$$F_0 := \frac{\mathrm{MS}_{\mathrm{trt}}}{\mathrm{MS}_{\mathrm{err}}} \sim \mathcal{F}(a-1, (a-1)(b-1)).$$

DEFINITION 1.21 (ANOVA Table)**.**

Table 1.2: ANOVA Table for Randomized Block Design

	Sum of Squares	Degrees of Freedom	Mean Squares	F_0
Treatment	$\mathrm{SS}_{\mathrm{trt}}$	$a-1$	$\mathrm{MS}_{\mathrm{trt}}$	$\mathrm{MS}_{\mathrm{trt}}/\mathrm{MS}_{\mathrm{err}}$
Block	$\mathrm{SS}_{\mathrm{blk}}$	$b-1$	$\mathrm{MS}_{\mathrm{blk}}$	
Error	$\mathrm{SS}_{\mathrm{err}}$	$(a-1)(b-1)$	$\mathrm{MS}_{\mathrm{err}}$	
Total	$\mathrm{SS}_{\mathrm{tot}}$	$ab-1$		

1.3 Two-Way Factorial Design

DEFINITION 1.22. Let $a \in \mathbb{Z}_{++}$ denote the number of treatments of factor A. Let $b \in \mathbb{Z}_{++}$ denote the number of treatments of factor B. Let $n \in \mathbb{Z}_{++}$ denote the number of repetitions for each combination of treatments. Let $\mu \in \mathbb{R}$ denote the overall mean. We model the population as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}, \text{ for } i \in \{1, ..., a\}, j \in \{1, ..., b\}, k \in \{1, ..., n\}$$

with constraints $\mathbb{1}^\top \alpha = 0$, $\mathbb{1}^\top \beta = 0$, $\gamma^\top \mathbb{1} = \mathbb{0}$, and $\gamma \mathbb{1} = \mathbb{0}$, and $e_{ijk} \sim \mathcal{N}(0, \sigma^2)$ are assumed to be independent. The total number of parameters in this model is $2 + a + b + ab$.

1.3.1 Estimation of Mean

1.3.2 Estimation of Variance

DEFINITION 1.23 (Sum of Squared Errors)**.** We define the following terms:

$$\begin{aligned}
\mathrm{SS_A} &:= bn \sum_{i=1}^{a} (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2 \\
\mathrm{SS_B} &:= an \sum_{j=1}^{b} (\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})^2 \\
\mathrm{SS_{AB}} &:= n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})^2 \\
\mathrm{SS_{err}} &:= \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (\bar{y}_{ijk} - \bar{y}_{ij\cdot})^2 \\
\mathrm{SS_{tot}} &:= \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (\bar{y}_{ijk} - \bar{y}_{\cdot\cdot\cdot})^2.
\end{aligned}$$

PROPOSITION 1.24 (Decomposition of $\mathrm{SS_{tot}}$)**.** We have

$$\mathrm{SS_{tot}} = \mathrm{SS_A} + \mathrm{SS_B} + \mathrm{SS_{AB}} + \mathrm{SS_{err}}.$$

DEFINITION 1.25 (Variance Estimator)**.** We define the following estimators for the variance σ^2.

$$\begin{aligned} \mathrm{MS_A} &:= \mathrm{SS_A}/(a-1), \\ \mathrm{MS_B} &:= \mathrm{SS_B}/(b-1), \\ \mathrm{MS_{AB}} &:= \mathrm{SS_{AB}}/((a-1)(b-1)), \\ \mathrm{MS_{err}} &:= \mathrm{SS_{err}}/(ab(n-1)). \end{aligned}$$

PROPOSITION 1.26. We have

$$\mathbb{E}(\mathrm{MS_{err}}) = \sigma^2.$$

i.e., $\mathrm{MS_{err}}$ is an unbiased estimator for σ^2.

PROPOSITION 1.27. Under the assumption that $\alpha = \mathbb{0} \in \mathbb{R}^a$, we have

$$\mathbb{E}(\mathrm{MS_A}) = \sigma^2.$$

i.e., $\mathrm{MS_A}$ is an unbiased estimator for σ^2 given that $\alpha = \mathbb{0}$.

PROPOSITION 1.28. Under the assumption that $\beta = \mathbb{0} \in \mathbb{R}^b$, we have

$$\mathbb{E}(\mathrm{MS_B}) = \sigma^2.$$

i.e., $\mathrm{MS_B}$ is an unbiased estimator for σ^2 given that $\beta = \mathbb{0}$.

PROPOSITION 1.29. Under the assumption that $\gamma = \mathbb{0} \in \mathbb{R}^{a\times b}$, we have

$$\mathbb{E}(\mathrm{MS_{AB}}) = \sigma^2.$$

i.e., $\mathrm{MS_{AB}}$ is an unbiased estimator for σ^2 given that $\gamma = \mathbb{0}$.

1.3.3 Hypothesis Testing

We are interested in testing the following hypothesis

- $H_0 : \alpha = \mathbb{0} \in \mathbb{R}^a$ vs $H_1 : \alpha \neq \mathbb{0}$.
- $H_0 : \beta = \mathbb{0} \in \mathbb{R}^b$ vs $H_1 : \beta \neq \mathbb{0}$.
- $H_0 : \gamma = \mathbb{0} \in \mathbb{R}^{a\times b}$ vs $H_1 : \gamma \neq \mathbb{0}$.

DEFINITION 1.30 (ANOVA Table)**.**

Table 1.3: ANOVA Table for Two-Way Factorial Design

	Sum of Squares	Degrees of Freedom	Mean Squares	F_0
A	$\mathrm{SS_A}$	$a-1$	$\mathrm{MS_A}$	$\mathrm{MS_A}/\mathrm{MS_{err}}$
B	$\mathrm{SS_B}$	$b-1$	$\mathrm{MS_B}$	$\mathrm{MS_B}/\mathrm{MS_{err}}$
AB	$\mathrm{SS_{AB}}$	$(a-1)(b-1)$	$\mathrm{MS_{AB}}$	$\mathrm{MS_{AB}}/\mathrm{MS_{err}}$
Error	$\mathrm{SS_{err}}$	$ab(n-1)$	$\mathrm{MS_{err}}$	
Total	$\mathrm{SS_{tot}}$	$abn-1$		

1.4 Two-Level Factorial Design