Algebraic Topology

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Chapter 1

Homotopy

1.1 Definitions

DEFINITION 1.1 (Homotopy, Homotopic)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces. Let $f, g \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$. Let $H \in \mathcal{C}(\mathfrak{X} \times [0,1], \mathfrak{Y})$. We say that H is a **homotopy** between f and g if and only if $\forall x \in \mathfrak{X}$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. We say that f and g are **homotopic**, denoted by $f \equiv g$, if and only if there is some homotopy between them.

PROPOSITION 1.2. Being homotopic is an equivalence relation on $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$.

1.2 Properties

PROPOSITION 1.3. Let $f_1, g_1, f_2, g_2 \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$. Suppose that $f_1 \equiv g_1$ and $f_2 \equiv g_2$. Then $f_2 \circ f_1 \equiv g_2 \circ g_1$.

1.3 Homotopy Equivalent

DEFINITION 1.4 (Homotopy Equivalence, Homotopy Equivalent)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces. Let $f \in \mathcal{C}(\mathfrak{X})$. Let $g \in \mathcal{C}(\mathfrak{Y})$. We say that the pair (f, g) is a **homotopy equivalence** between $\mathfrak{X}$ and $\mathfrak{Y}$ if and only if $g \circ f \equiv \mathrm{Id}_{\mathfrak{X}}$ and $f \circ g \equiv \mathrm{Id}_{\mathfrak{Y}}$.

We say that $\mathfrak{X}$ and $\mathfrak{Y}$ are **homotopy equivalent** if and only if there is some homotopy equivalence between $\mathfrak{X}$ and $\mathfrak{Y}$.

PROPOSITION 1.5. If $\mathfrak{X}$ and $\mathfrak{Y}$ are homeomorphic topological spaces, then they are homotopy equivalent.

1.4 Invariance Properties

PROPOSITION 1.6. If $\mathfrak{X}$ and $\mathfrak{Y}$ are homotopy equivalent topological spaces, then $\mathfrak{X}$ is path-connected if and only if $\mathfrak{Y}$ is.