General Topology

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Topological Spaces

1.1 Topology

Definition (Topology)**.** *Let X be a set. We define a* ***topology*** *on X, denoted by τ_X, to be a collection of subsets of X that satisfies all of the following conditions.*

(1) $\emptyset, X \in \tau_X$.

(2) τ_X *is closed under union.*

(3) τ_X *is closed under finite intersection.*

Definition (Finer, Coarser)**.** *Let τ_1 and τ_2 be two topologies on X.*

- *We say that τ_1 is* ***finer*** *than τ_2 if $\tau_1 \supseteq \tau_2$.*
- *We say that τ_1 is* ***coarser*** *than τ_2 if $\tau_1 \subseteq \tau_2$.*

Proposition 1.1.1. *Let X be a set. Let $\{\tau_\lambda\}_{\lambda\in\Lambda}$ be a collection of topologies on X. Then their intersection $\bigcap_{\lambda\in\Lambda} \tau_\lambda$ is also a topology on X.*

1.2 Examples of Topology

Example 1.2.1 (Trivial Topology)**.** *Let X be a set. We define the* ***trivial topology*** *on X to be $\tau := \{\emptyset, X\}$.*

Example 1.2.2 (Discrete Topology)**.** *Let X be a set. We define the* ***discrete topology*** *on X to be $\tau := \mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the power set of X.*

Example 1.2.3 (Metric Topology)**.** *Let (X, d) be a metric space. We define the* ***metric topology*** *on X, induced by the metric d, to be*

$$\tau := \{G \subseteq X : \forall x \in G, \exists r > 0, \mathrm{ball}(x, r) \subseteq G\}.$$

Example 1.2.4 (Cofinite Topology). *Let X be a set. We define the* ***cofinite topology*** *on X to be*

$$\tau := \{\emptyset\} \cup \{Y \subseteq X : X \setminus Y \text{ is finite }\}.$$

1.3 Open Sets and Closed Sets

Definition (Open Sets). *Let (X, τ) be a topological space. Let S be a set in the space. We say that S is* ***open*** *if $S \in \tau$.*

Definition (Closed Sets). *Let (X, τ) be a topological space. Let S be a set in the space. We say that S is* ***closed*** *if $\bar{S} \in \tau$.*

1.4 Neighborhood

Definition (Neighborhood of a Point). *Let (X, τ) be a topological space. Let x be a point in X. Let $\mathcal{N}$ be a subset of X. We say that $\mathcal{N}$ is a* ***neighborhood*** *of point x if*

$$\exists G \in \tau, \quad x \in G \text{ and } G \subseteq \mathcal{N}.$$

Definition (Neighborhood of a Set). *Let (X, τ) be a topological space. Let S be a subset of X. Let $\mathcal{N}$ be a subset of X. We say that $\mathcal{N}$ is a* ***neighborhood*** *of set S if*

$$\exists G \in \tau, \quad S \subseteq G \text{ and } G \subseteq \mathcal{N}.$$

Definition (Neighborhood System). *Let (X, τ) be a topological space. Let x be a point in X. We define a* ***neighborhood system*** *at point x, denoted by $\mathcal{U}_x$, to be the set of all neighborhoods of x.*

Proposition 1.4.1 (Properties of Neighborhood System). *Let (X, τ) be a topological space. Let x be a point in the space. Then*

(1) Neighborhood systems are closed under finite intersections. i.e.,

$$\forall U, V \in \mathcal{U}_x, \quad U \cap V \in \mathcal{U}_x.$$

(2) There always exists smaller neighborhoods. i.e.,

$$\forall U \in \mathcal{U}_x, \quad \exists V \in \mathcal{U}_x, \quad \forall y \in V, \quad U \in \mathcal{U}_y.$$

Note that here V is not only a subset of U, but U is a neighborhood of V.

(3) Any superset of a neighborhood is also a neighborhood. i.e.,

$$\forall U \in \mathcal{U}_x, \forall V \subseteq X, V \supseteq U \implies V \in \mathcal{U}_x.$$

Proposition 1.4.2. *Let (X, τ) be a topological space. Let G be a set in the space. Then*

$$G \in \tau \iff \forall x \in G, \exists \mathcal{N} \subseteq G, \mathcal{N} \in \mathcal{U}_x.$$

1.5 Base for Topologies

Definition (Base). *Let (X,τ) be a topological space. We define a* ***base*** *for τ, denoted by $\mathcal{B}$, to be a subset of τ such that any set in τ can be written as a union of elements of $\mathcal{B}$. i.e.,*

$$\forall G \in \tau, \quad \exists \mathcal{C} \subseteq \mathcal{B}, \quad G = \bigcup_{B \in \mathcal{C}} B.$$

Definition (Subbase). *Let (X,τ) be a topological space. We define a* ***subbase*** *for τ, denoted by $\mathcal{S}$, to be a subset of τ such that the collection of all finite intersections of elements of $\mathcal{S}$ forms a base for τ.*

Definition (Neighborhood Base). *Let (X,τ) be a topological space. Let x be a point in X. We define a* ***neighborhood base*** *at point x, denoted by $\mathcal{B}_x$, to be a sub-collection of the neighborhood system $\mathcal{U}_x$ at x such that*

$$\forall U \in \mathcal{U}_x, \quad \exists B \in \mathcal{B}_x, \quad B \subseteq U.$$

That is, a neighborhood base at a point is a sub-collection of all the neighborhoods that are "small".

Proposition 1.5.1 (Base and Neighborhood Base - 1). *Let (X,τ) be a topological space. Let x be an arbitrary point in the space. Define a set $\mathcal{B}_x$ to be the neighborhood base at point x consisting of only open sets. Define a set $\mathcal{B}$ as $\mathcal{B} := \bigcup_{x \in X} \mathcal{B}_x$. Then $\mathcal{B}$ is a base for the space.*

Proof. Clearly $\mathcal{B} \subseteq \tau$. Let G be an arbitrary set in τ. We are to prove that G can be written as a union of elements of $\mathcal{B}$. If $G = \emptyset$, then we are done. Otherwise, let x be an arbitrary point in G. Since $G \in \tau$ and $x \in G$, $G \in \mathcal{U}_x$. Since $G \in \mathcal{U}_x$ and $\mathcal{B}_x$ is a neighborhood base at point x, $\exists B(x) \in \mathcal{B}_x$ such that $B(x) \subseteq G$. Since $\forall x \in G$, $x \in B(x)$, we get $\bigcup_{x \in G} B(x) \supseteq G$. Since $\forall x \in G$, $B(x) \subseteq G$, we get $\bigcup_{x \in G} B(x) \subseteq G$. Since $\bigcup_{x \in G} B(x) \supseteq G$ and $\bigcup_{x \in G} B(x) \subseteq G$, we get $\bigcup_{x \in G} B(x) = G$. So G can be written as a union of elements of $\mathcal{B}$. Since any open set can be written as a union of elements of $\mathcal{B}$, $\mathcal{B}$ is a base for the space. ■

Proposition 1.5.2 (Base and Neighborhood Base - 2). *Let (X,τ) be a topological space. Let $\mathcal{B}$ be a subset of τ. Let x be an arbitrary point in the space. Define a set $\mathcal{B}_x$ by $\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\}$. Then $\mathcal{B}_x$ is a neighborhood base at x.*

Proof. Clearly, $\mathcal{B}_x \subseteq \mathcal{U}_x$. Let N_x be an arbitrary neighborhood of x. Since N_x is a neighborhood of x, $\exists G \in \tau$ such that $x \in G \subseteq N_x$. Since $G \in \tau$ and $\mathcal{B}$ is a base for τ, $\exists \mathcal{B}' \subseteq \mathcal{B}$ such that $G = \bigcup_{B \in \mathcal{B}'} B$. Since $x \in G$ and $G = \bigcup_{B \in \mathcal{B}'} B$, $\exists B_0 \in \mathcal{B}'$ such that $x \in B_0$. So $x \in B_0 \subseteq \bigcup_{B \in \mathcal{B}'} B = G \subseteq N_x$. Notice $B_0 \in \mathcal{B}_x$. That is,

$$\forall N_x \in \mathcal{U}_x, \quad \exists B_0 \in \mathcal{B}_x, \quad B_0 \subseteq N_x.$$

So $\forall x \in X$, $\mathcal{B}_x$ is a neighborhood base at x. ■

Remark. *Note that the above two propositions are converges of each other.*

1.6 Examples of Base

Example 1.6.1. *Consider the set of real numbers with the usual topology. The collection*

$$\mathcal{B} := \{(a, b) : a, b \in \mathbb{R}, a < b\}$$

is a base for the space.

Example 1.6.2. *Consider the set of real numbers with the usual topology. The collection*

$$\mathcal{S} := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$$

is a subbase for the the space.

1.7 Generating Topology

Proposition 1.7.1. *Let X be a set. Let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a base for some topology on X if and only if*

- $X = \bigcup_{B \in \mathcal{B}} B$*, and*
- $\forall B_1, B_2 \in \mathcal{B}$*,* $\forall p \in B_1 \cap B_2$*,* $\exists B_3 \in \mathcal{B}$ *such that* $p \in B_3$ *and* $B_3 \subseteq B_1 \cap B_2$*.*

Proof. For one direction, assume that $\mathcal{B}$ is a base for some topology X. We are to prove that the two conditions hold. Let τ denote the topology. Since $\mathcal{B}$ is a base for τ and $X \in \tau$, $\exists \mathcal{B}' \subseteq \mathcal{B}$ such that $X = \bigcup_{B \in \mathcal{B}'} B$. So $X = \bigcup_{B \in \mathcal{B}} B$. Let B_1 and B_2 be arbitrary sets in $\mathcal{B}$. If $B_1 \cap B_2 = \emptyset$, then we are done. Otherwise, let p be an arbitrary point in $B_1 \cap B_2$. Since $\mathcal{B}$ is a base, $\mathcal{B} \subseteq \tau$. Since $B_1, B_2 \in \mathcal{B}$ and $\mathcal{B} \subseteq \tau$, $B_1, B_2 \in \tau$. Since $B_1, B2 \in \tau$ and τ is a topology, we get $B_1 \cap B_2 \in \tau$. Since $B_1 \cap B_2 \in \tau$ and $\mathcal{B}$ is a base, $\exists \mathcal{B}' \subseteq \mathcal{B}$ such that $B_1 \cap B_2 = \bigcup_{B \in \mathcal{B}'} B$. Since $p \in B_1 \cap B_2 = \bigcup_{B \in \mathcal{B}'} B$, $\exists B_3 \in \mathcal{B}' \subseteq \mathcal{B}$ such that $p \in B_3$. Notice $B_3 \subseteq \bigcup_{B \in \mathcal{B}'} B = B_1 \cap B_2$. That is, $B_3 \subseteq B_1 \cap B_2$.
For the reverse direction, assume that the two conditions hold. We are to prove that $\mathcal{B}$ is a base for some topology on X. Define a topology τ on X by

$$\tau := \{ \bigcup_{B \in \mathcal{B}'} : \mathcal{B}' \subseteq \mathcal{B}\}.$$

Now I will verify that τ is indeed a topology on X.

- Since $\emptyset \subseteq \mathcal{B}$, $\emptyset \in \tau$. Since $\mathcal{B} \subseteq \mathcal{B}$ and $X = \bigcup_{B \in \mathcal{B}} B$, $X \in \tau$.

- Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a subset of τ for some index set Λ. Then $\forall\lambda \in \Lambda$, $\exists\mathcal{B}_\lambda \subseteq \mathcal{B}$ such that $G_\lambda = \bigcup_{B\in\mathcal{B}_\lambda} B$. So $\bigcup_{\lambda\in\Lambda} G_\lambda = \bigcup_{\lambda\in\Lambda}\bigcup_{B\in\mathcal{B}_\lambda} B = \bigcup_{B\in\mathcal{C}} B$ where $\mathcal{C} = \bigcup_{\lambda\in\Lambda}\mathcal{B}_\lambda$. Notice $\mathcal{C} \subseteq \mathcal{B}$. So $\bigcup_{\lambda\in\Lambda} G_\lambda \in \tau$.

- Let H_1 and H_2 be subsets of τ. Then $\exists\mathcal{B}_1 \subseteq \mathcal{B}$ such that $H_1 = \bigcup_{B\in\mathcal{B}_1} B$ and $\exists\mathcal{B}_2 \subseteq \mathcal{B}$ such that $H_2 = \bigcup_{B\in\mathcal{B}_2} B$. Let p be an arbitrary point in $H_1 \cap H_2$. Since $p \in H_1 \cap H_2$, $\exists B_1 \in \mathcal{B}_1 \subseteq \mathcal{B}$ and $\exists B_2 \in \mathcal{B}_2 \subseteq \mathcal{B}$ such that $p \in B_1$ and $p \in B_2$. Since $B_1, B_2 \subseteq \mathcal{B}$ and $p \in B_1 \cap B_2$, by assumption, $\exists B_3 \in \mathcal{B}$ such that $p \in B_3$ and $B_3 \subseteq B_1 \cap B_2$. So $p \in B_3$ and $B_3 \subseteq H_1 \cap H_2$. Since $\forall p \in H_1 \cap H_2$, $\exists B_3 \in \mathcal{B}$ such that $p \in B_3 \subseteq H_1 \cap H_2$, we get $H_1 \cap H_2 = \bigcup_{p\in H_1\cap H_2} B_3(p)$. So $H_1 \cap H_2 \in \tau$.

So τ is a topology on X. Since τ is defined to be $\tau = \{\bigcup_{B\in\mathcal{B}'} : \mathcal{B}' \subseteq \mathcal{B}\}$, $\mathcal{B}$ is clearly a base for τ. This completes the proof. ■

1.8 Weak Topology

Definition (Weak Topology)**.** *Let X be a non-empty set. Let $\{(Y_\gamma, \tau_\gamma)\}_{\gamma\in\Gamma}$ be a collection of topological spaces where Γ is an index set. Let $\mathcal{F} = \{f_\gamma\}_{\gamma\in\Gamma}$ be a collection of functions where f_γ is a function from X to Y_γ for each γ. Define a set $\mathcal{S}$ as*

$$\mathcal{S} := \{f_\gamma^{-1}(G_\gamma) : G_\gamma \in \tau_\gamma, \gamma \in \Gamma\}.$$

*We define the **weak topology** on X induced by $\mathcal{F}$, denoted by $\sigma(X, \mathcal{F})$, to be a topology which has $\mathcal{S}$ as a subbase.*

Proposition 1.8.1. *Let (X, τ) be a topological space. Let $\{(Y_\gamma, \tau_\gamma)\}_{\gamma\in\Gamma}$ be a collection of topological spaces where Γ is an index set. Let $\mathcal{F} = \{f_\gamma\}_{\gamma\in\Gamma}$ be a collection of functions where f_γ is a function from X to Y_γ for each $\gamma \in \Gamma$. Suppose f_γ is continuous for all $\gamma \in \Gamma$. Then $\sigma(X, \mathcal{F}) \subseteq \tau$. i.e., $\sigma(X, \mathcal{F})$ is the weakest topology on X under which all f_γ's are continuous.*

Proposition 1.8.2. *Let X be a non-empty set. Let $\{(Y_\gamma, \tau_\gamma)\}_{\gamma\in\Gamma}$ be a collection of topological spaces where Γ is an index set. Let $\mathcal{F} = \{f_\gamma\}_{\gamma\in\Gamma}$ be a collection of functions where f_γ is a function from X to Y_γ for each γ. Let (Z, τ_Z) be a topological space. Let g be a function from (Z, τ_Z) to $(X, \sigma(X, \mathcal{F}))$. Then g is continuous if and only if $f_\gamma \circ g$ is continuous for all $\gamma \in \Gamma$.*

1.9 Product Topology

Definition (Projection Map)**.** *Let $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ be a collection of topological spaces where Λ is an index set. Let β be an index in Λ. We define the β^{th} **projection map**, denoted by*

π_β, *to be a function from* $\prod_{\lambda\in\Lambda} X_\lambda$ *to* X_β *given by*

$$\pi_\beta(x) := x_\beta.$$

Definition (Product Topology)**.** *Let* $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ *be a collection of topological spaces where* Λ *is an index set. We define a* ***product topology*** *on* $\prod_{\lambda\in\Lambda} X_\lambda$ *to be the weak topology on* $\prod_{\lambda\in\Lambda} X_\lambda$ *induced by* $\{\pi_\lambda\}_{\lambda\in\Lambda}$.

2

Metric Spaces

2.1 Metrics

Definition (Metric)**.** *Let X be a non-empty set. Let d be a function from $X \times X$ to $\mathbb{R}_+$. We say that d is a* ***metric*** *(or distance function) on X if it satisfies all of the following conditions.*

(1) Non-negativity

$$\forall x, y \in X, \quad d(x, y) \geq 0.$$

(2) Identity of Indiscernible

$$\forall x, y \in X, \quad d(x, y) = 0 \iff x = y.$$

(3) Symmetry

$$\forall x, y \in X, \quad d(x, y) = d(y, x).$$

(4) Sub-Additivity (Triangle Inequality)

$$\forall x, y, z \in X, \quad d(x, y) \leq d(x, z) + d(z, y).$$

Definition (Metric Space)**.** *Let X be a non-empty set. Let d be a metric on X. We call the pair (X, d) a* ***metric space****.*

2.2 Continuity of Metrics

Proposition 2.2.1. *Let (X, d) be a metric space. Let x_0 be a point in X. Let d_{x_0} be a function from X to $\mathbb{R}$ given by $d_{x_0}(x) := d(x, x_0)$. Then d_{x_0} is continuous.*

Proposition 2.2.2. *Let (X, d) be a metric space. Let S be a subset of X. Let d_S be a function from X to $\mathbb{R}$ given by $d_S(x) := d(x, S)$. Then d_S is continuous.*

Proof.
Let x_0 be an arbitrary point in X.
Let ε be an arbitrary positive number.
Let $\delta(\varepsilon)$ be a positive number given by $\delta(\varepsilon)\varepsilon$.
Let x be an arbitrary point in X such that $d(x, x_0) < \delta(\varepsilon)$.
Let x' be an arbitrary point in S.
Since $x' \in S$, by definition of distance from a point to a set, $d(x_0, S) \leq d(x_0, x')$.
Since d is a metric on X and x_0 and x' are points in X, by the triangle inequality, $d(x_0, x') \leq d(x_0, x) + d(x, x')$.
Since $d(x_0, S) \leq d(x_0, x')$ and $d(x_0, x') \leq d(x_0, x) + d(x, x')$, $d(x_0, S) - d(x_0, x) \leq d(x, x')$.
Since $d(x_0, S) - d(x_0, x) \leq d(x, x')$ for any $x' \in S$, $d(x_0, S) - d(x_0, x)$ is a lower bound for the set $\{d(x, x') : x' \in S\}$.
Since $d(x_0, S) - d(x_0, x)$ is a lower bound for the set $\{d(x, x') : x' \in S\}$, by definition infimum, $d(x_0, S) - d(x_0, x) \leq \inf\{d(x, x') : x' \in S\} = d(x, S)$.
Since $x' \in S$, by definition of distance from a point to a set, $d(x, S) \leq d(x, x')$.
Since d is a metric on X and x and x' are points in X, by the triangle inequality, $d(x, x') \leq d(x, x_0) + d(x_0, x')$.
Since $d(x, S) \leq d(x, x')$ and $d(x, x') \leq d(x, x_0) + d(x_0, x')$, $d(x, S) - d(x, x_0) \leq d(x_0, x')$.
Since $d(x, S) - d(x, x_0) \leq d(x_0, x')$ for any $x' \in S$, $d(x, S) - d(x, x_0)$ is a lower bound for the set $\{d(x_0, x') : x' \in S\}$.
Since $d(x, S) - d(x, x_0)$ is a lower bound for the set $\{d(x_0, x') : x' \in S\}$, by definition of infimum, $d(x, S) - d(x, x_0) \leq \inf\{d(x_0, x') : x' \in S\} = d(x_0, S)$.
Since $d(x_0, S) - d(x_0, x) \leq d(x, S)$ and $d(x, S) - d(x, x_0) \leq d(x_0, S)$, $|d(x, S) - d(x_0, S)| \leq d(x, x_0)$.
Since $|d(x, S) - d(x_0, S)| \leq d(x, x_0)$ and $d(x, x_0) < \delta(\varepsilon) = \varepsilon$, $|d(x, S) - d(x_0, S)| < \varepsilon$.
Since for any positive number ε, there exists a positive number $\delta(\varepsilon)$ such that if for any point x in X, if $d(x, x_0) < \delta(\varepsilon)$, then $|d(x, S) - d(x_0, S)| < \varepsilon$, by definition of continuity, the function $x \mapsto d(x, S)$ is continuous.

■

2.3 Open Sets and Closed Sets in Metric Spaces

Definition (Openness). *Let (X, d) be a metric space. Let S be a subset of X. We say that S is **open** in (X, d) if for any sequence $\{x_k\}_{k=1}^{\infty}$ in X that converges to some point in S, there exists an integer N such that for any index k with $k > N$, we have $x_k \in S$.*

Definition (Openness)**.** *Let (X, d) be a metric space. Let S be a subset of X. We say that S is **open** in (X, d) if*

$$\forall x \in S, \exists r_0 \in \mathbb{R}_{++}, \mathrm{ball}(x, r_0) \in S.$$

Proposition 2.3.1. *Two definitions of open-ness are equivalent.*

Proof.
For one direction, assume that for any sequence $\{x_k\}_{k=1}^{\infty}$ in X that converges to some point in S, there exists an integer N such that for any index k with $k > N$, we have $x_k \in S$.
We are to prove that

$$\forall x \in S, \exists r_0 \in \mathbb{R}_{++}, \mathrm{ball}(x, r_0) \subseteq S.$$

Let x be an arbitrary point in S.
Let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary sequence in X that converges to x.
By assumption, there exists an integer N such that $x_N \in S$.
Consider the open ball $B_0(x, d(x_N, x))$.
By definition, there exists another integer N' such that for any index k with $k > N'$, we have $x_k \in B_0(x, d(x_N, x))$.
Assume that for any point x in G, there exists an open ball $B(x, r)$ centered at x, of some radius r, such that $B(x, r) \subseteq G$.
Let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary sequence in X that converges to some point x_0 in G. We are to show that $\{x_k\}$ is eventually in G.
By assumption, there exists an open ball $B_0(x_0, r_0)$ centered at x_0, of some radius r_0, such that $B_0(x_0, r_0) \subseteq G$.

$$d(x_k, x_0) < r_0$$

By definition of open balls, x_k is in $B_0(x_0, r_0)$. Thus x_k is also in G.
i.e., any sequence in X that converges to some point in G is eventually in G.

■

Proof. For the reverse direction, assume that any sequence in X that converges to some point in G is eventually in G.
Let x_0 be an arbitrary point in G.
We are to prove that there exists an open ball $B_0(x_0, r_0)$ centered at x_0, of some radius r_0, such that $B_0(x_0, r_0) \subseteq G$.

$$d(x_k, x_0) < 1/k$$

Then $\{x_k\}$ converges to x_0.
By assumption, there exists an integer N such that $x_N \in G$.
Take $r_0 = 1/N$. Then the open ball $B_0(x_0, r_0)$ is contained in G.

i.e., for any point x in G, there exists an open ball $B(x, r)$ centered at x, of some radius r, such that $B(x, r) \subseteq G$.

■

Definition (Closedness). *Let (X, d) be a metric space. Let S be a subset of X. We say that S is **closed** in (X, d) if for any sequence in S that converges to some point x_0 in S, x_0 is also in S.*

Proposition 2.3.2 (Stability under Set Operations).

(1) The complement of an open set is closed.

(2) The complement of a closed set is open.

(3) The union of an arbitrary collection of open sets is open.

(4) The union of a finite collection of closed sets is closed.

(5) The intersection of a finite collection of open sets is open.

(6) The intersection of an arbitrary collection of closed sets is closed.

(7) If S_1 is open and S_2 is closed, then $S_1 \setminus S_2$ is open.

(8) If S_1 is closed and S_2 is open, then $S_1 \setminus S_2$ is closed.

Proof of (1). Let $\{U_\alpha\}$ be a set of open sets.
By Lemma 2.1, each of the open set can be written as a union of open balls.
i.e. for all α, there exists a set of open balls $\{B_{\alpha\beta}\}$ such that

$$U_\alpha = \bigcup_\beta B_{\alpha\beta}.$$

Then the union of the open sets $\{U_\alpha\}$ is

$$\bigcup_\alpha U_\alpha = \bigcup_\alpha \bigcup_\beta B_{\alpha\beta}.$$

By Lemma 2.1 again, the union is open.

■

Proof of (2). Let $\{U_k\}_{k=1}^{k=n}$ be a finite sequence of open sets.
Let x be an arbitrary element in their intersection.
Then x is in each of U_k.
By definition of open sets, for each U_k, there exists an open ball $B_k(r_k, x)$ such that $B_k \subseteq U_k$
Define $r = \min\{r_k\}$.
Consider the open ball $B(r, x)$.

Then B is a subset of each U_k and hence a subset of their intersection.
By definition of open sets, the intersection is open.

■

Proof of (3). Let $\{F_\alpha\}$ be a set of closed subsets of X.
Then each of $(F_\alpha)^c$ is open.
By Proposition 2.1, we get $\bigcup_\alpha (F_\alpha)^c$ is open.
By the De Morgan's Laws, we get $(\bigcap_\alpha F_\alpha)^c$ is open.
It follows that $\bigcap_\alpha F_\alpha$ is closed.

■

Proof of (4). Let $\{F_\alpha\}$ be a set of closed subsets of X.
Then each of $(F_\alpha)^c$ is open.
By Proposition 2.2, we get $\bigcap_\alpha (F_\alpha)^c$ is open.
By the De Morgan's Laws, we get $(\bigcup_\alpha F_\alpha)^c$ is open.
It follows that $\bigcup_\alpha F_\alpha$ is closed.

■

2.4 The Discrete Metric

Definition (Discrete Metric)**.** *Let X be a set. We define a* ***discrete metric*** *on X to be a function from X to $\mathbb{R}_+$ given by*

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

Proposition 2.4.1. *In a discrete metric space, any set is both open and closed.*

Proposition 2.4.2. *In a discrete metric space, the unit closed ball is the whole space, and the unit open ball is the singleton set consisting of only the center.*

Proposition 2.4.3. *Discrete metric spaces are always bounded.*

Proof. Let (X,d) be a discrete metric space and S be a subset of X.
Case 1. S is empty.
Since S is empty, by definition of boundedness, S is bounded.
Case 2. S is not empty.
Since S is not empty, pick a point x_0 in S.
Since (S,d) is discrete, $d(x,x_0) \leq 1$ for any $x \in S$.
Since $d(x,x_0) \leq 1$ for any $x \in S$, by definition of boundedness, S is bounded.
Summary.

Since in all cases, S is bounded, S is bounded.

■

Proposition 2.4.4. *A discrete metric space is totally bounded only if it is empty or it is a singleton set.* ***(unconfirmed)***

Proposition 2.4.5. *Discrete metric spaces are always complete.*

Proof. Let (X, d) be a discrete metric space and $\{x_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in X.
We are to prove that $\{x_i\}_{i\in\mathbb{N}}$ is convergent.
Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy, there exists an integer $N(1)$ such that for any indices m and n, if $m, n > N(1)$, then $d(x_m, x_n) < 1$.
Since $d(x_m, x_n) < 1$ and d is discrete, $d(x_m, x_n) = 0$.
Since $d(x_m, x_n) = 0$ and d is a metric, $x_m = x_n$.
Since $m, n > N(1)$ implies $x_m = x_n$ for any indices m and n, in particular, $i > N(1)$ implies $x_i = x_{N(1)+1}$ for any index i.
Since $i > N(1)$ implies $x_i = x_{N(1)+1}$ for any index i, for any positive number ε and any index i, if $i > N(1)$, then $d(x_i, x_{N(1)+1}) < \varepsilon$.
Since for any positive number ε and any index i, if $i > N(1)$, then $d(x_i, x_{N(1)+1}) < \varepsilon$, by definition of convergence, $\{x_i\}_{i\in\mathbb{N}}$ converges to $x_{N(1)+1}$.
Since any Cauchy sequence in (X, d) converges in (X, d), by definition of completeness, (X, d) is complete.

■

Proposition 2.4.6. *Discrete metric spaces are totally disconnected.*

Proposition 2.4.7. *Any function defined on a discrete metric space is uniformly continuous.*

not sure what the codomain could be.

2.5 The Hausdorff Metric

Proposition 2.5.1. *Let (X, d) be a metric space. Let $(\mathcal{H}(X), d_H)$ be the induced Hausdorff space. Then if (X, d) is complete, $(\mathcal{H}(X), d)$ is complete.*

Proposition 2.5.2. *Let (X, d) be a metric space. Let $(\mathcal{H}(X), d_H)$ be the induced Hausdorff space. Then if (X, d) is totally bounded, $(\mathcal{H}(X), d_H)$ is totally bounded.*

Proof. Let r be an arbitrary radius.
Since (X, d) is totally bounded, there exists a finite collection $\S = \{x_i\}_{i\in I}$ of points in X such that $\{\text{ball}(x_i, r)\}_{i\in I}$ covers X.

Let S be an arbitrary set in $\mathcal{H}(X)$.
Let $\S'(S) = \{x_i\}_{i \in I'(S)}$ be the subcollection of $\S$ of points x_i such that $\text{ball}(x_i, r) \cap S \neq \emptyset$.
Since S is a nonempty subset of X and $\{\text{ball}(x_i, r)\}_{i \in I}$ covers X, $\S'(S)$ is nonempty.
Since $\S'(S)$ is finite, $\S'(S)$ is closed and bounded in (X, d).
Since $\S'(S)$ is nonempty, closed, and bounded in (X, d), $\S'(S) \in \mathcal{H}(X)$.
Since $\text{ball}(x_i, r) \cap S \neq \emptyset$ for each $i \in I'(S)$, $(S)_r \supseteq \S'(S)$.
Since $\{\text{ball}(x_i, r)\}_{i \in I}$ covers X and S is a subset of X, $\{\text{ball}(x_i, r)\}_{i \in I}$ covers S.
Since $\{\text{ball}(x_i, r)\}_{i \in I}$ covers S and $\text{ball}(x_i, r) \cap S = \emptyset$ for any $i \in I \backslash I'(S)$, $\{\text{ball}(x_i, r)\}_{i \in I'(S)}$ covers S.
Since $\{\text{ball}(x_i, r)\}_{i \in I'(S)}$ covers S, $(\S'(S))_r \supseteq S$.
Since $(S)_r \supseteq \S'(S)$ and $(\S'(S))_r \supseteq S$, by definition of Hausdorff metric, $d_H(S, \S'(S)) < r$.
Since $d_H(S, \S'(S)) < r$, $S \in \text{ball}(\S'(S), r)$.
Since for any S in $\mathcal{H}(X)$, there exists another set $\S'(S)$ in $\mathcal{H}(X)$ such that $S \in \text{ball}(\S'(S), r)$, $\{\text{ball}(\S', r)\}_{\S' \in \mathcal{P}(\S)}$ covers $\mathcal{H}(X)$.
Since for any radius r, there exists a collection $\mathcal{C}$ of sets in $\mathcal{H}(X)$ such that $\{\text{ball}(\S', r)\}_{\S' \in \mathcal{C}}$ covers $\mathcal{H}(X)$, by definition of total boundedness, $\mathcal{H}(X)$ is totally bounded.

■

3

Interior, Closure, and Boundary

3.1 Definitions

3.1.1 Interior

Definition (Interior Point)**.** *Let (X, τ) be a topological space. Let S be a set in the space. Let x be a point in the space. We say that x is an **interior point** of S if*

$$\exists \text{ neighborhood } \mathcal{N} \text{ of } x, \quad \mathcal{N} \subseteq S.$$

Equivalently,

$$\mathcal{N} \cap X \setminus S = \emptyset.$$

Definition (Interior)**.** *Let (X, τ) be a topological space. Let S be a set in (X, τ). We define the **interior** of S, denoted by $\int(S)$, to be the set of all interior points of S.*

Definition (Interior)**.** *Let (X, τ) be a topological space. Let S be a set in (X, τ). We define the **interior** of S, denoted by $\int(S)$, to be the union of all open subsets of S. Equivalently, the largest open subset of S.*

Proposition 3.1.1. *The two definitions of interior are equivalent.*

3.1.2 Boundary

Definition (Boundary Point)**.** *Let (X, τ) be a topological space. Let S be a set in the space. Let x be a point in the space. We say that x is a **boundary point** of S if for any neighborhood $\mathcal{N}(x)$ of x, we have $\mathcal{N}(x) \cap S \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S) \neq \emptyset$.*

Definition (Boundary)**.** *Let (X, τ) be a topological space. Let S be a set in (X, τ). We define the **boundary** of S, denoted by $bd(S)$, to be the set of all boundary points of S.*

Definition (Boundary)**.** *Let (X, τ) be a topological space. Let S be a set in (X, τ). We define the **boundary** of S, denoted by $\mathrm{bd}(S)$, to be the set given by $C_X(\int(S) \cup \int(C_X(S))$.*

Definition (Boundary)**.** *Let (X, τ) be a topological space. Let S be a set in (X, τ). We define the **boundary** of S, denoted by $\mathrm{bd}(S)$, to be the set given by $\mathrm{cl}(S) \cap \mathrm{cl}(C_X(S))$.*

Proposition 3.1.2. *The three definitions of boundary are equivalent.*

3.1.3 Closure

Definition (Adherent Point)**.** *Let (X, τ) be a topological space. Let S be a set in the space. Let x be a point in the space. We say that x is an **adherent point** of S if for any neighborhood $\mathcal{N}(x)$ of x, we have $\mathcal{N}(x) \cap S \neq \emptyset$.*

Definition (Closure)**.** *Let (X, τ) be a topological space. Let S be a set in (X, τ). We define the **closure** of S, denoted by $\mathrm{cl}(S)$, to be the set of all adherent points of S.*

Definition (Closure)**.** *Let (X, τ) be a topological space. Let S be a set in (X, τ). We define the **closure** of S, denoted by $\mathrm{cl}(S)$, to be the intersection of all closed superset of S, or equivalently, the smallest closed superset of S.*

Proposition 3.1.3. *The two definitions of closure are equivalent.*

Proof.
For one direction, let x be an arbitrary point in the intersection of all closed supersets of S.
We are to prove that x is an adherent point of S.
Assume for the sake of contradiction that x is not an adherent point of S.
Since x is not an adherent point of S, there exists some open set U containing x such that $U \cap S = \emptyset$.
Since U is open in (X, τ), $C_X(U)$ is closed in (X, τ).
Since $U \cap S = \emptyset$, $C_X(U) \supseteq S$.
Since $x \in U$, $x \notin C_X(U)$.
Since $C_X(U)$ is a closed superset of S and $x \notin C_X(U)$, x is not in the intersection of all closed supersets of S.
This contradicts to the assumption that x is in the intersection of all closed supersets of S.
So the assumption that x is not an adherent point of S is false.
i.e., x is an adherent point of S.
For the other direction, let x be an adherent point of S.
We are to prove that x is in the intersection of all closed supersets of S.
Assume for the sake of contradiction that x is not in the intersection of all closed supersets of S.

Since x is not in the intersection of all closed supersets of S, there exists some closed superset E of S such that $x \notin E$.
Since E is closed in (X, τ), $C_X(E)$ is open in (X, τ).
Since $E \supseteq S$, $C_X(E) \cap S = \emptyset$.
Since $x \notin E$, $x \in C_X(E)$.
Since there exists some open set U such that $x \in U$ and $U \cap S = \emptyset$, x is not an adherent point of S.
This contradicts to the assumption that x is an adherent point of S.
So the assumption that x is not in the intersection of all closed supersets of S is false.
i.e., x is in the intersection of all closed supersets of S.

■

3.2 Basic Properties

Proposition 3.2.1.

(1) Interiors are open.

(2) Closures are closed.

(3) Boundaries are closed.

Proposition 3.2.2. *For any set in any topological space, the closure is the disjoint union of the interior and the boundary. i.e., an adherent point is exactly one of an interior point or a boundary point.*

Proposition 3.2.3. *Let (X, τ) be a topological space. Let S be a set in the space. Then*

(1) $\int(X \setminus S) = X \setminus \mathrm{cl}(S)$.

(2) $\mathrm{cl}(X \setminus S) = X \setminus \int(S)$.

(3) $\mathrm{bd}(X \setminus S) = \mathrm{bd}(S)$.

Proof of (1). For one direction, let x be an arbitrary point in $\int(C_X(S))$.
We are to prove that $x \notin \mathrm{cl}(S)$.
Since $x \in \int(C_X(S))$, by definition of interior, there exists an open set U such that $x \in U$ and $U \subseteq C_X(S)$.
Since U is an open subset of S, $C_X(U)$ is a closed superset of S.
Since $x \in U$, $x \notin C_X(U)$.
Since $C_X(U)$ a closed superset of S and $x \notin C_X(U)$, x is not in the intersection of all closed superset of S.
Since x is not in the intersection of all closed superset of S, by definition of closure, $x \notin \mathrm{cl}(S)$.

Since $x \notin \text{cl}(S)$, we get $x \in C_X(\text{cl}(S))$.
For the reverse direction, let x be an arbitrary point in $\text{cl}(S)$.
We are to prove that $x \notin \int(C_X(S))$.
Since $x \in \text{cl}(S)$, by definition of closure, x is in any closed superset F of S.
Since F is a closed superset of S, $C_X(F)$ is an open subset of $C_X(S)$.
Since $x \in F$, $x \notin C_X(F)$.
Since $C_X(F)$ is an arbitrary open subset of $C_X(S)$ and $x \notin C_X(F)$, x is not in the union of all open subsets of $C_X(S)$.
Since x is not in the union of all open subsets of $C_X(S)$, by definition of interior, $x \notin \int(C_X(S))$.

■

Proof of (2).

$$\text{cl}(S^c) \subseteq (\int(S))^c \# (*) \backslash n (\int(S))^c \subseteq \text{cl}(S^c) \# (**)$$

Let x be an arbitrary point in $\text{cl}(S^c)$.
By definition of closure, x is in every closed superset of S^c.
Note that x is not in any open subset of S.
By definition, we conclude that $x \notin \int(S)$ and hence $x \in (\int(S))^c$.

$$\text{cl}(S^c) \subseteq (\int(S))^c \# (*)$$

Let x be an arbitrary point in $(\int(S))^c$.
By definition of interior, x is not in any open subset of S.
Note that the complement of open subsets of S are closed supersets of S^c.
Thus x is in every closed superset of S^c.
By definition, we conclude that $x \in \text{cl}(S^c)$.

$$(\int(S))^c \subseteq \text{cl}(S^c) \# (**)$$

■

Proposition 3.2.4. *Let (X, τ) be a topological space. Let S be a set in the space. Then*

(1) S is open if and only if $S = \int(S)$.

(2) S is closed if and only if $S = \text{cl}(S)$.

Proof of (1). For one direction, assume that S is open.
We are to prove that $S = \int(S)$.
Let x be an arbitrary point in S.
Since x is in S and S is open and S is a subset of S, x is in some open subset of S.
Since x is in some open subset of S, x is in the union of all open subsets of S.

Since x is in the union of all open subsets of S, by definition, $x \in \int(S)$.
Since for any point $x \in S$, we have $x \in \int(S)$, we get $S \subseteq \int(S)$.
Let x be an arbitrary point in $\int(S)$.
Since $x \in \int(S)$, by definition, x is in the union of all subsets of S.
Since x is in the union of all subsets of S, x is in some open subset S' of S.
Since $x \in S'$ and $S' \subseteq S$, $x \in S$.
Since for any point $x \in \int(S)$, we have $x \in S$, we get $\int(S) \subseteq S$.
Since $S \subseteq \int(S)$ and $\int(S) \subseteq S$, $S = \int(S)$.
For the reverse direction, assume that $S = \int(S)$.
We are to prove that S is open.
Since $S = \int(S)$ and $\int(S)$ is open, S is open.

■

Proposition 3.2.5.

(1) A set S is open if and only if S and $\mathrm{bd}(S)$ *are disjoint.*

(2) A set S is closed if and only if S contains $\mathrm{bd}(S)$.

Proposition 3.2.6. *The boundary of some open set or of some closed set has no interior.*

Proof for Open Sets. Let (X, τ) be a topological space and S be an open set in (X, τ).
Let x be an arbitrary point in $bd(S)$.
Let $\mathcal{N}(x)$ be an arbitrary open neighborhood of x in (X, τ).
Since x is a boundary point of S and $\mathcal{N}(x)$ is some open neighborhood of x in (X, τ), by definition of boundary, $\mathcal{N}(x) \cap S \neq \emptyset$.
Since S is open, $S \cap bd(S) = \emptyset$.
Since $\mathcal{N}(x) \cap S \neq \emptyset$ and $S \cap bd(S) = \emptyset$, $\mathcal{N}(x) \nsubseteq bd(S)$.
Since $\mathcal{N}(x) \nsubseteq bd(S)$ for any open neighborhood of x in (X, τ), by definition of interior, $x \notin \int(bd(S))$.
Since $x \notin \int(bd(S))$ for any $x \in bd(S)$, $bd(S)$ has no interior.

■

Proof for Closed Sets. Let (X, τ) be a topological space and S be a closed set in (X, τ).
Let x be an arbitrary point in $bd(S)$.
Let $\mathcal{N}(x)$ be an arbitrary open neighborhood of x in (X, τ).
Since x is a boundary point of S and $\mathcal{N}(x)$ is some open neighborhood of x in (X, τ), by definition of boundary, $\mathcal{N}(x) \cap C_X(S) \neq \emptyset$.
Since S is closed, $C_X(S) \cap bd(S) = \emptyset$.
Since $\mathcal{N}(x) \cap C_X(S) \neq \emptyset$ and $C_X(S) \cap bd(S) = \emptyset$, $\mathcal{N}(x) \nsubseteq bd(S)$.
Since $\mathcal{N}(x) \nsubseteq bd(S)$ for any open neighborhood of x in (X, τ), by definition of interior, $x \notin \int(bd(S))$.

Since $x \notin \int(bd(S))$ for any $x \in bd(S)$, $bd(S)$ has no interior.

■

Proposition 3.2.7. *The boundary of a set is empty if and only if the set is both open and closed.*

Proof.
Let (X, d) be a metric space and S be a subset of X.
For one direction, assume that $bd(S) = \emptyset$. We are to prove that S is both open and closed.
Let x be an arbitrary point in S. Then $x \in \int(S)$. Thus $S = \int(S)$.
Note that $\mathrm{cl}(S) = \int(S) \cup bd(S) = \int(S)$. Thus $S = \mathrm{cl}(S)$.
By definition, we conclude that S is both open and closed.
For the reverse direction, assume that S is both open and closed. We are to prove that $bd(S) = \emptyset$.
Since S is open, $S = \int(S)$.
Since S is closed, $S = \mathrm{cl}(S)$.
Since $S = \int(S)$ and $S = \mathrm{cl}(S)$, $\int(S) = \mathrm{cl}(S)$.
Since $bd(S) = \mathrm{cl}(S) - \int(S)$, $bd(S) = \emptyset$.

■

3.3 As Operators

Proposition 3.3.1 (The Interior Operator)**.** *In any topological space,*

(1)

$$\forall S \subseteq X, \quad \int(S) \subseteq S.$$

(2) Monotonic:

$$\forall S_1, S_2 \subseteq X, \quad S_1 \subseteq S_2 \implies \int(S_1) \subseteq \int(S_2).$$

(3) Idempotent:

$$\forall S \subseteq X, \quad \int(S) = \int(\int(S)).$$

Proof of (2). Let x be an arbitrary point in $\int(S_1)$.
Since $x \in \int(S_1)$, by definition of interior, x is in some open subset S' of S_1.
Since S' is an open subset of S_1 and $S_1 \subseteq S_2$, S' is an open subset of S_2.
Since S' is an open subset of S_2 and $x \in S'$, by definition of interior, $x \in \int(S_2)$.
Since for any point in $\int(S_1)$ is also in $\int(S_2)$, $\int(S_1) \subseteq \int(S_2)$.

■

Proposition 3.3.2 (The Closure Operator)**.** *In any topological space,*

(1)

$$\forall S \subseteq X, \quad S \subseteq \mathrm{cl}(S).$$

(2) Monotonic:

$$\forall S_1, S_2 \subseteq X, \quad S_1 \subseteq S_2 \implies \mathrm{cl}(S_1) \subseteq \mathrm{cl}(S_2).$$

(3) Idempotent:

$$\forall S \subseteq X, \quad \mathrm{cl}(S) = \mathrm{cl}(\mathrm{cl}(S)).$$

Proof of (2). Let x be an arbitrary point in $\mathrm{cl}(S_1)$.
Let S' be an arbitrary closed superset of S_2.
Since S' is a closed superset of S_2 and $S_1 \subseteq S_2$, S' is a closed superset of S_1.
Since S' is a closed superset of S_1 and $x \in \mathrm{cl}(S_1)$, by definition of closure, $x \in S'$.
Since x is in any closed superset of S_2, by definition, $x \in \mathrm{cl}(S_2)$.
Since any point in $\mathrm{cl}(S_1)$ is in $\mathrm{cl}(S_2)$, $\mathrm{cl}(S_1) \subseteq \mathrm{cl}(S_2)$.

■

Proposition 3.3.3 (The Exterior Operator)**.** *In any topological space,*

(1) (Monotonic) For any sets S and T, if $S \subseteq T$, then ext$(S) \supseteq T$.

Remark. *The exterior operator is not idempotent.*

Proposition 3.3.4. *Let (X, τ) be a topological space and S be a subset of X. Then*

(1) $\int(S) \subseteq \int(\mathrm{cl}(S))$.

(2) $\mathrm{cl}(S) \supseteq \mathrm{cl}(\int(S))$.

Proof. By the properties of the closure operator, $S \subseteq \mathrm{cl}(S)$. Since $S \subseteq \mathrm{cl}(S)$ and the interior operator is monotonic increasing, $\int(S) \subseteq \int(\mathrm{cl}(S))$. By the properties of the interior operator, $\int(S) \subseteq S$. Since $\int(S) \subseteq S$ and the closure operator is monotonic increasing, $\mathrm{cl}(\int(S)) \subseteq \mathrm{cl}(S)$. ■

Remark. *The point of this proposition is to remind the readers that the equalities might not hold. See section 3.4 for counter examples.*

Proposition 3.3.5. *Let S be a set in some topological space. Then*

(1) $\mathrm{bd}(\int(S)) \subseteq \mathrm{bd}(S)$.

(2) $\mathrm{bd}(\mathrm{cl}(S)) \subseteq \mathrm{bd}(S)$.

(3) $\mathrm{bd}(\mathrm{bd}(S)) \subseteq \mathrm{bd}(S)$

Proof of (1). Let x be an arbitrary element in $bd(\int(S))$.
We are to prove that $x \in bd(S)$.
Since $x \in bd(\int(S))$, by definition, for any neighborhood $N(x)$ of x, there exist a point x_1 in $N(x) \cap \int(S)$ and a point x_2 in $N(x) \cap (\int(S))^c$.
Since $x_1 \in \int(S)$ and $\int(S) \subseteq S$, $x_1 \in S$.
Since $x_1 \in S$ and $x_1 \in N(x)$, $x_1 \in N(x) \cap S$.
To prove that $N(x) \cap S^c \neq \emptyset$, assume for the sake of contradiction that $N(x) \subseteq S$.
Since $x_2 \in N(x)$ and $N(x) \subseteq S$, by definition, $x_2 \in \int(S)$.
This contradicts to the fact that $x_2 \in (\int(S))^c$.
Thus the assumption that $N(x) \subseteq S$ is false.
i.e., there exists a point x_2' in $N(x) \cap S^c$.
In short, I have proved that for any neighborhood $N(x)$ of x, there exist a point x_1 in $N(x) \cap S$ and a point x_2' in $N(x) \cap S^c$.
By definition, I conclude that $x \in bd(S)$.

■

Proof of (2). Let x be an arbitrary element in $bd(\mathrm{cl}(S))$.
We are to prove that $x \in bd(S)$.
Since $x \in bd(\mathrm{cl}(S))$, by definition, for any neighborhood $N(x)$ of x, there exist a point x_1 in $N(x) \cap \mathrm{cl}(S)$ and a point x_2 in $N(x) \in (\mathrm{cl}(S))^c$.
To prove that $N(x) \cap S \neq \emptyset$, assume for the sake of contradiction that $N(x) \subseteq S^c$.
Since $x_1 \in N(x)$ and $N(x) \subseteq S^c$, $x_1 \in \mathrm{ext}(S)$.
This contradicts to the fact that $x_1 \in \mathrm{cl}(S)$.
Thus the assumption that $N(x) \subseteq S^c$ is false.
i.e., there exists a point x_1' in $N(x) \cap S$.
Since $x_2 \in (\mathrm{cl}(S))^c$ and $(\mathrm{cl}(S))^c \subseteq S^c$, $x_2 \in S^c$.
Since $x_2 \in S^c$ and $x_2 \in N(x)$, $x_2 \in N(x) \cap S^c$.
In short, I have proved that for any neighborhood $N(x)$ of x, there exist a point x_1 in $N(x) \cap S$ and a point $x_2 \in N(x) \cap S^c$.
By definition, I conclude that $x \in bd(S)$.

■

Proposition 3.3.6 (Set Operations)**.** *Let S_1 and S_2 be sets in some topological space. Then*

(1) $\int(S_1 \cup S_2) \supseteq \int(S_1) \cup \int(S_2)$.

(2) $\int(S_1 \cap S_2) = \int(S_1) \cap \int(S_2)$.

(3) $\mathrm{cl}(S_1 \cup S_2) = \mathrm{cl}(S_1) \cup \mathrm{cl}(S_2)$.

(4) $\mathrm{cl}(S_1 \cap S_2) \subseteq \mathrm{cl}(S_1) \cap \mathrm{cl}(S_2)$.

(5) $\mathrm{bd}(S_1 \cup S_2) \subseteq \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2)$.

(6) $\mathrm{bd}(S_1 \cap S_2) \subseteq \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2)$.

(7) $\mathrm{bd}(S_1 \setminus S_2) \subseteq \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2)$.

Remark (Remark on (1).). *"$\int(S_1 \cup S_2) = \int(S_1) \cup \int(S_2)$" may not be true. For example, consider $X = \mathbb{R}$, $S_1 = \mathbb{Q}$, and $S_2 = X \setminus \mathbb{Q}$. Then $LHS = \int(S_1 \cup S_2) = \mathbb{R}$ and $RHS = \int(S_1) \cup \int(S_2) = \emptyset$.*

Proof of (2).
It is clear that $\int(S_1 \cap S_2) \subseteq \int(S_1) \cap \int(S_2)$.
So it suffices to prove that $\int(S_1 \cap S_2) \supseteq \int(S_1) \cap \int(S_2)$.
Let x be an arbitrary point in $\int(S_1) \cap \int(S_2)$.
Then $x \in \int(S_1)$ and $x \in \int(S_2)$.
Since $x \in \int(S_1)$, there exists some open set G_1 such that $x \in G_1 \subseteq S_1$.
Since $x \in \int(S_2)$, there exists some open set G_2 such that $x \in G_2 \subseteq S_2$.
Since $x \in G_1$ and $x \in G_2$, $x \in G_1 \cap G_2$.
Since G_1 and G_2 are both open, $G_1 \cap G_2$ is open.
Since $G_1 \subseteq S_1$ and $G_2 \subseteq S_2$, $G_1 \cap G_2 \subseteq S_1 \cap S_2$.
Since $G_1 \cap G_2$ is open and $x \in G_1 \cap G_2 \subseteq S_1 \cap S_2$, by definition of interior points, $x \in \int(S_1 \cap S_2)$.
Since

$$\forall x \in \int(S_1) \cap \int(S_2), \quad x \in \int(S_1 \cap S_2),$$

we get $\int(S_1 \cap S_2) \supseteq \int(S_1) \cap \int(S_2)$.

■

Proof of (3).

$$\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B).$$

Let x be an arbitrary point in $\mathrm{cl}(A \cup B)$.
By definition of closure, x is in every closed superset of $(A \cup B)$.
Note that the union of an arbitrary closed superset of A and an arbitrary closed superset of B is a closed superset of $(A \cup B)$.
Thus x is in the union of the intersection of all closed supersets of A and the intersection of all closed supersets of B.
By definition again, we conclude that $x \in \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
Since x is an arbitrary point in $\mathrm{cl}(A \cup B)$, $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

■

Remark (Remark on (4).). *"*$\mathrm{cl}(S_1 \cap S_2) \subseteq \mathrm{cl}(S_1) \cap \mathrm{cl}(S_2)$*" may not be true. For example, consider* $X = \mathbb{R}$*,* $S_1 = \mathbb{Q}$*, and* $S_2 = X \setminus \mathbb{Q}$*. Then* $LHS = \mathrm{cl}(S_1 \cap S_2) = \emptyset$ *and* $RHS = \mathrm{cl}(S_1) \cap \mathrm{cl}(S_2) = \mathbb{R}$.

Proof of (5).
Let x be an arbitrary point in $\mathrm{bd}(S_1 \cup S_2)$.
Since $x \in \mathrm{bd}(S_1 \cup S_2)$, by definition of boundary, for any neighborhood $\mathcal{N}(x)$ around x, $\mathcal{N}(x) \cap (S_1 \cup S_2) \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_1 \cup S_2) \neq \emptyset$.
Since $\mathcal{N}(x) \cap (S_1 \cup S_2) \neq \emptyset$, either $\mathcal{N}(x) \cap S_1 \neq \emptyset$ or $\mathcal{N}(x) \cap S_2 \neq \emptyset$.
Since $\mathcal{N}(x) \cap C_X(S_1 \cup S_2) \neq \emptyset$, $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$.
Case 1. $\mathcal{N}(x) \cap S_1 \neq \emptyset$.
Since $\mathcal{N}(x) \cap S_1 \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$ for any neighborhood $\mathcal{N}(x)$ around x, by definition of boundary, $x \in \mathrm{bd}(S_1)$.
Since $x \in \mathrm{bd}(S_1)$, $x \in \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2)$.
Case 2. $\mathcal{N}(x) \cap S_2 = \emptyset$.
Since $\mathcal{N}(x) \cap S_2 = \emptyset$ and $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$ for any neighborhood $\mathcal{N}(x)$ around x, by definition of boundary, $x \in \mathrm{bd}(S_2)$.
Since $x \in \mathrm{bd}(S_2)$, $x \in \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2)$.
Summary.
Since "$x \in \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2)$" holds in all cases, we conclude that it is true.
Since

$$\forall x \in \mathrm{bd}(S_1 \cup S_2), \quad x \in \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2),$$

we get $\mathrm{bd}(S_1 \cup S_2) \subseteq \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2)$.

■

Proof of (6).
Let x be an arbitrary point in $\mathrm{bd}(S_1 \cap S_2)$.
Since $x \in \mathrm{bd}(S_1 \cap S_2)$, by definition of boundary, for any neighborhood $\mathcal{N}(x)$ around x, $\mathcal{N}(x) \cap (S_1 \cap S_2) \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_1 \cap S_2) \neq \emptyset$.
Since $\mathcal{N}(x) \cap (S_1 \cap S_2) \neq \emptyset$, $\mathcal{N}(x) \cap S_1 \neq \emptyset$ and $\mathcal{N}(x) \cap S_1 \neq \emptyset$ and $\mathcal{N}(x) \cap S_2 \neq \emptyset$.
Since $\mathcal{N}(x) \cap C_X(S_1 \cap S_2) \neq \emptyset$, either $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$ or $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$.
Case 1. $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$.
Since $\mathcal{N}(x) \cap S_1 \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_1) \neq \emptyset$ for any neighborhood $\mathcal{N}(x)$ around x, by definition of boundary, $x \in \mathrm{bd}(S_1)$.
Since $x \in \mathrm{bd}(S_1)$, $x \in \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2)$.
Case 2. $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$.
Since $\mathcal{N}(x) \cap S_2 \neq \emptyset$ and $\mathcal{N}(x) \cap C_X(S_2) \neq \emptyset$ for any neighborhood $\mathcal{N}(x)$ around x, by definition of boundary, $x \in \mathrm{bd}(S_2)$.
Since $x \in \mathrm{bd}(S_2)$, $x \in \mathrm{bd}(S_1) \cup \mathrm{bd}(S_2)$.

Summary.
Since "$x \in \operatorname{bd}(S_1) \cup \operatorname{bd}(S_2)$" holds in all cases, we conclude that it is true.
Since

$$\forall x \in \operatorname{bd}(S_1 \cap S_2), \quad x \in \operatorname{bd}(S_1) \cup \operatorname{bd}(S_2),$$

we get $\operatorname{bd}(S_1 \cap S_2) \subseteq \operatorname{bd}(S_1) \cup \operatorname{bd}(S_2)$.

■

3.4 Examples

Example 3.4.1. *It is **not** true that if S is open, then $\int(\operatorname{cl}(S)) = \int(S)$. Consider the set $\mathbb{R} \setminus \{0\}$.*

Example 3.4.2. *It is **not** true that if S is closed, then $\operatorname{cl}(\int(S)) = \operatorname{cl}(S)$. Consider the set $\{0\}$.*

4

Dense and Nowhere Dense

4.1 Dense

4.1.1 Definitions

Definition (Dense-1)**.** *Let (X, τ) be a topological space. Let S be a set in the space. We say that S is **dense** in the space if any point in the space is an adherent point of S, or equivalently, the closure of S equals the whole space.*

Definition (Dense-2)**.** *Let (X, τ) be a topological space. Let S be a set in the space. We say that S is **dense** in the space if S has nonempty intersection with any nonempty open set in the space.*

Proposition 4.1.1. *The two definitions of dense sets are equivalent.*

Proof.
For one direction, assume that $cl(S) = X$.
We are to prove that $\forall$ nonempty open $\mathcal{O}$, $S \cap \mathcal{O} \neq \emptyset$.
Let $\mathcal{O}$ be an arbitrary nonempty open set in the space.
Assume for the sake of contradiction that $\mathcal{O} \cap S = \emptyset$.
Define $\mathcal{C} := X \setminus \mathcal{O}$.
Since $\mathcal{O} \cap S = \emptyset$, $\mathcal{C} \supseteq S$.
Since $\mathcal{O}$ is open, $\mathcal{C}$ is closed.
Since $\mathcal{C}$ is a closed superset of S, $cl(S) \subseteq \mathcal{C}$.
Since $cl(S) \subseteq \mathcal{C}$ and $cl(S) = X$, $\mathcal{C} \supseteq X$.
Since $\mathcal{C} = X \setminus \mathcal{O}$, $\mathcal{C} \subseteq X$.
Since $\mathcal{C} \subseteq X$ and $\mathcal{C} \supseteq X$, $\mathcal{C} = X$.
Since $\mathcal{C} = X$, $\mathcal{O} = \emptyset$.

This contradicts to the assumption that $\mathcal{O}$ is nonempty.
So the assumption that $\mathcal{O} \cap S = \emptyset$ is false.
i.e., $\mathcal{O} \cap S \neq \emptyset$.
For the reverse direction, assume that $\forall$ nonempty open $\mathcal{O}$, $S \cap \mathcal{O} \neq \emptyset$.
We are to prove that $cl(S) = X$.
Let $\mathcal{C}$ be an arbitrary closed superset of S.
Define $\mathcal{O} := X \setminus \mathcal{C}$.
Since $\mathcal{C}$ is closed, $\mathcal{O}$ is open.
Since $\mathcal{C} \supseteq S$, $\mathcal{O} \cap S = \emptyset$.
Assume for the sake of contradiction that $\mathcal{C} \neq X$.
Since $\mathcal{C} \neq X$, $\mathcal{O} \neq \emptyset$.
Since $\forall$ nonempty open $\mathcal{O}$, $S \cap \mathcal{O} \neq \emptyset$, in particular, $\mathcal{O} \cap S \neq \emptyset$.
This contradicts to the fact that $\mathcal{O} \cap S = \emptyset$.
So the assumption that $\mathcal{C} \neq X$ is false.
i.e., $\mathcal{C} = X$.
Since $\mathcal{C} = X$ for any closed superset $\mathcal{C}$ of S, $cl(S) = X$.

■

4.1.2 Properties

Proposition 4.1.2 (Transitivity). *Denseness is transitive. i.e.: Let (X, τ) be a topological space. Let S_1 and S_2 and S_3 be subsets of X. Suppose S_1 is dense in S_2 and S_2 is dense in S_3, then S_1 is dense in S_3.*

Proposition 4.1.3. *A superset of a dense set is dense.*

Proposition 4.1.4 (Images). *A continuous image of a dense set is dense in the range.*

Proof.
Let (X, τ_X) and (Y, τ_Y) be topological spaces.
Let f be a surjective function from X to Y.
Let S be a dense set in (X, τ_X).
Let T denote $f(S)$.
We are to prove that T is dense in (Y, τ_Y).
Let y be an arbitrary point in Y.
Since $y \in Y$, there exists some point x in X such that $f(x) = y$.
Since $x \in X$ and S is dense in (X, τ_X), there exists a sequence § in S that converges to x in (X, τ_X).
Let † denote $f(\S)$.
Since $\S \subseteq S$, $\dagger \subseteq T$.

Since § converges to x in (X, τ_X) and f is continuous, † converges to y in (Y, τ_Y).
Since for any point y in Y, there exists a sequence † in T that converges to y in (Y, τ_Y), by definition of dense sets, T is dense in (Y, τ_Y).

∎

4.2 Nowhere Dense

4.2.1 Definitions

Definition (Nowhere Dense)**.** *Let (X, τ) be a topological space. Let S be a set in the space. We say that S is **nowhere dense** in (X, τ) if there is no nonempty open set in (X, τ) in which S is dense.*

Definition (Nowhere Dense)**.** *Let (X, τ) be a topological space. Let S be a set in the space. We say that S is **nowhere dense** in (X, τ) if $int(\mathrm{cl}(S)) = \emptyset$.*

Proposition 4.2.1. *The two definitions of nowhere dense sets are equivalent.*

4.2.2 Properties

Proposition 4.2.2 (Subspaces)**.** *Let (X, τ) be a topological space. Let (Y, τ) be a subspace of (X, τ). Then*

(1) Let S be a nowhere dense set in (Y, τ). Then S is also nowhere dense in (X, τ).

(2) Let S be a nowhere dense set in (X, τ). Then if Y is open in (X, τ), S is also nowhere dense in (Y, τ).

Proposition 4.2.3. *The nowhere dense sets in a space form an ideal of sets.*

Proposition 4.2.4 (Closure)**.** *A set is nowhere dense if and only if its closure is nowhere dense.*

Proposition 4.2.5 (Exterior)**.** *A set is nowhere dense if and only if its exterior is dense.*

Proof. Sketch
Let (X, τ) be a topological space.
Let S be a set in the space.
S is nowhere dense if and only if $int(cl(S)) = \emptyset$.
$ext(S)$ is dense if and only if $cl(ext(S)) = X$.
So it suffices to prove that $int(cl(S)) = \emptyset$ if and only if $cl(ext(S)) = X$.
It suffices to prove that $C_X(int(cl(S))) = cl(int(C_X(S)))$.

$$int(C_X(S)) = C_X(cl(S))$$
$$\Downarrow$$
$$cl(int(C_X(S))) = cl(C_X(cl(S)))$$
$$\Downarrow$$
$$cl(int(C_X(S))) = C_X(int(cl(S)))$$

This completes the proof.

■

Proposition 4.2.6. *A set is open and dense if and only if its complement is closed and nowhere dense.*

Proposition 4.2.7. *The boundary of an open set or of a closed set is nowhere dense.*

Proposition 4.2.8 (not sure...)**.** *A singleton set is nowhere dense if and only if the point in it is not an isolated point.*

5

Meager and Residual

5.1 Definitions

Definition (Meager, or First Category)**.** *Let (X, τ) be a topological space. Let S be a set in the space. We say S is **meager** if S is the union of some countable collection of nowhere dense sets in the space.*

Definition (Residual)**.** *Let (X, τ) be a topological space. Let S be a subset of X. We say S is **residual** if S is the intersection of some countable collection of sets with dense interior.*

5.2 Sufficient Conditions

Proposition 5.2.1. *A set is meager if and only if its complement is residual.*

Proposition 5.2.2 (Set Operations)**.**

(1) A subset of a meager set is meager.

(2) A countable union of meager sets is meager.

Proof.
Proof of (1).
Let (X, τ) be a topological space and S_1 be a meager set in (X, τ).
Let S_2 be a subset of S_1.
Since S_1 is meager in (X, τ), by definition of meager, there exist a countable collection $\{A_i\}_{i \in \mathbb{N}}$ of nowhere dense sets in (X, τ) such that $S_1 = \bigcup_{i \in \mathbb{N}} A_i$.
Define $B_i A_i \cap S_2$.
Since $S_2 \subseteq S_1$ and $S_1 = \bigcup_{i \in \mathbb{N}} A_i$ and $B_i = A_i \cap S_2$, $S_2 = \bigcup_{i \in \mathbb{N}} B_i$.
Since $B_i \subseteq A_i$ and A_i is nowhere dense in (X, τ), B_i is nowhere dense in (X, τ).

Since B_i is nowhere dense in (X,τ) for any $i\in\mathbb{N}$ and $S_2=\bigcup_{i\in\mathbb{N}}B_i$, by definition of meager, S_2 is meager in (X,τ).

$\blacksquare$

Proposition 5.2.3 (Set Operations)**.** *A countable intersection of residual sets is residual.*

6

Sequences

6.1 Definitions

Definition (Sequence)**.** *Let (X, τ) be a topological space. We define a* ***sequence*** *in X, denoted by $\{x_i\}_{i\in I}$ where I is a subset of the natural numbers, to be a* <u>*function*</u> *from I to X.*

6.2 Convergence of Sequences

Definition (Convergence)**.** *Let (X, τ) be a topological space. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in X. Let x_0 be a point in X. We say that $\{x_i\}_{i\in\mathbb{N}}$* ***converges*** *to x_0 if for any neighborhood $\mathcal{N}_X(x_0)$ of x_0 in X, there exists an integer $N(\mathcal{N}_X(x_0))$ such that for any index i greater than $N(\mathcal{N}_X(x_0))$, we have $x_i \in \mathcal{N}_X(x_0)$.*

Proposition 6.2.1. *Let (X, τ) be a topological space. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in X. Let x_0 be a point in X. Then if $\{x_i\}_{i\in\mathbb{N}}$ converges to x_0, any subsequence of $\{x_i\}_{i\in\mathbb{N}}$ also converges to x_0.*

Proof. Let $\{x_i\}_{i\in I}$ be an arbitrary subsequence of $\{x_i\}_{i\in\mathbb{N}}$. Let $\mathcal{N}(x_0)$ be an arbitrary neighborhood of x_0. Since $\{x_i\}_{i\in\mathbb{N}}$ converges to x_0, there exists some cutoff N such that $x_i \in \mathcal{N}(x_0)$ whenever $i > N$. Since $x_i \in \mathcal{N}(x_0)$ whenever $i > N$, in particular, $x_i \in \mathcal{N}(x_0)$ whenever $i \in I$ and $i > N$. Since for any positive number ε, there exists some neighborhood $\mathcal{N}(x_0)$ such that $x_i \in \mathcal{N}(x_0)$ whenever $i \in I$ and $i > N$, $\{x_i\}_{i\in I}$ converges to x_0. ■

Proposition 6.2.2. *Let (X, τ) be a topological space. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in X. Let x_0 be a point in X. Then if any subsequence of $\{x_i\}_{i\in\mathbb{N}}$ has a subsequence that converges to x_0, $\{x_i\}_{i\in\mathbb{N}}$ also converges to x_0.*

6.3 Cauchyness of Sequences

Definition (Cauchy Sequence)**.** *Let (X, d) be a metric space. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in X. We say that $\{x_i\}_{i\in\mathbb{N}}$ is **Cauchy** if for any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices m and n with $m, n > N(\varepsilon)$, we have $d(x_m, x_n) < \varepsilon$.*

Proposition 6.3.1. *Cauchy sequences are bounded.*

Proof.
Let (X, d) be a metric space.
Let $\{x_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in X.
Let x_0 be some fixed point in the space.
Let ε be some fixed positive number.
Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy, there exists some cutoff N such that $d(x_m, x_n) < \varepsilon$ whenever $m, n > N$.
Let j be some fixed index such that $j > N$.
Since $d(x_m, x_n) < \varepsilon$ for any $m, n > N$, in particular, $d(x_i, x_j) < \varepsilon$ for any $i > N$.
Define $M_j := d(x_j, x_0) + \varepsilon$.
Define $M_i := d(x_i, x_0)$ for $i \in \{1, ..., N\}$.
Define $M := \max\{M_i\}$, $i \in \{1, ..., N, j\}$.
Since $M = \max\{M_i\}$, $i \in \{1, ..., N, j\}$, $M_i \leq M$ for $i \in \{1, ..., N, j\}$.
Let i be an arbitrary index.
If $i \in \{1, ..., N\}$, then $d(x_i, x_0) = M_i \leq M$.
If $i > N$, then $d(x_i, x_0) \leq d(x_i, x_j) + d(x_j, x_0) < \varepsilon + d(x_j, x_0) = M_j \leq M$.
Since $d(x_i, x_0) \leq M$ for any $i \in \mathbb{N}$, $\{x_i\}_{i\in\mathbb{N}}$ is bounded.

∎

Proposition 6.3.2. *Let (X, d) be a metric space. Let $\{x_k\}$ be a Cauchy sequence in X. Then if there exists a convergent subsequence, $\{x_k\}$ also converges to the same limit.*

Proof.
Let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_k\}$ that converges to point x in X.
By definition of convergence, for any positive number ε, there exists an integer $N_1(\varepsilon)$ such that for any index k with $k > N_1$, we have $d(x_{n_k}, x) < \varepsilon/2$. (**)
By definition of Cauchy, there exists an integer $N_2(\varepsilon)$ such that for any indices $m, n > N_2$, we have $d(x_m, x_n) < \varepsilon/2$. (*)
Take $N = \max\{N_1, N_2\} + 1$. Then we have $N > N_1$, $N > N_2$ and $n_N > N_2$.
Apply statement (*) with $k = N > N_1$, we get $d(x_{n_N}, x) < \varepsilon/2$.
Apply statement (**) with $m > N > N_2$ and $n = n_N > N_2$, we get $d(x_m, x_{n_N}) < \varepsilon/2$.
Combining the preceding two inequalities, we get $d(x_k, x) < \varepsilon$ for any $k > N$.

In short, for any positive number ε, there exists an integer $N(\varepsilon)$ such that for any index k with $k > N$, we have $d(x_k, x) < \varepsilon$.
By definition, we conclude that $\{x_k\}$ converges to x.

■

Proposition 6.3.3. *Convergent sequences are Cauchy.*

Proof. Let (X, d) be a metric space and $\{x_k\}$ be an arbitrary convergent sequence in X.
By definition of convergence, for all $\varepsilon > 0$, there exists an integer N such that for all $k > N$, we have $d(x_k, x) < \varepsilon/2$. (*)
Let $m, n > N$ be arbitrary.
Apply statement (*) to m and n, we get

$$d(x_m, x) < \varepsilon/2\#(1)$$

$$d(x_n, x) < \varepsilon/2\#(2)$$

By the triangle inequality, we get

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n)\#(3)$$

From inequations (1) ˜ (3), we get

$$d(x_m, x_n) < \varepsilon$$

In short, we have proved that for all $\varepsilon > 0$, there exists an integer N such that for all $m, n > N$, we have $d(x_m, x_n) < \varepsilon$.
By definition, $\{x_k\}$ is Cauchy.

■

7

Nets

7.1 Definitions

Definition (Net)**.** *Let X be a set. Let Λ be a directed set. We define a* ***net****, denoted by $\{x_\lambda\}_{\lambda\in\Lambda}$, to be a function from Λ to X.*

Definition (Subnet)**.** *Let X be a set. Let Λ be a directed set. Let $P : \Lambda \to X$ be a net. Let M be another directed set. Let φ be an increasing and cofinal function from M to Λ. We define a* ***subnet*** *of P, denoted by $\{x_{\lambda_\mu}\}_{\mu\in M}$. to be a composition of the functions φ and P.*

7.2 Convergence of Nets

Definition (Convergence)**.** *Let (X, τ) be a topological space. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in X. Let x be a point in X. We say that the net $\{x_\lambda\}_{\lambda\in\Lambda}$* ***converge****s to the point x, denoted by $\lim_{\lambda\in\Lambda} x_\lambda = x$, if*

$$\forall U \in \mathcal{U}_x, \quad \exists\lambda_0 \in \Lambda, \quad \forall\lambda \geq \lambda_0, \quad x_\lambda \in U.$$

Proposition 7.2.1. *Let (X, τ) be a topological space. Then the space is Hausdorff if and only if the limits of nets in the space are unique.*

Proof. For one direction, assume that the space is Hausdorff. We are to prove that limits of nets in the space are unique. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net where Λ is a directed set. Let x_1 and x_2 be points in the space. Suppose $\lim_{\lambda\in\Lambda} x_\lambda = x_1$ and $\lim_{\lambda\in\Lambda} x_\lambda = x_2$. Let $\mathcal{N}_1$ be an arbitrary neighborhood of x_1. Let $\mathcal{N}_2$ be an arbitrary neighborhood of x_2. Then $\exists\lambda_1 \in \Lambda$ such that $\forall\lambda \geq \lambda_1$, $x_\lambda \in \mathcal{N}_1$; and $\exists\lambda_2 \in \Lambda$ such that $\forall\lambda \geq \lambda_2$, $x_\lambda \in \mathcal{N}_2$. Let λ_3 be an index such that $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$. Then $x_{\lambda_3} \in \mathcal{N}_1$ and $x_{\lambda_3} \in \mathcal{N}_2$. So $\mathcal{N}_1$ and $\mathcal{N}_2$ are not disjoint. So x_1 and x_2 are not separated. Since the space is Hausdorff and x_1 and x_2 are not separated, x_1 and x_2 are not distinct. So the limit of $\{x_\lambda\}_{\lambda\in\Lambda}$ is unique.

For the reverse direction, assume that the limits of nets in the space are unique. We are to prove that the space is Hausdorff. Assume for the sake of contradiction that the space is not Hausdorff. Then $\exists x, y \in X : x \neq y$ such that $\forall$ neighborhood $\mathcal{N}_x$ of x and $\mathcal{N}_y$ of y, $\mathcal{N}_x \cap \mathcal{N}_y = \emptyset$. Define a directed set Λ by

$$\Lambda := \{(\mathcal{N}_x, \mathcal{N}_y)\}$$

with partial order

$$(\mathcal{N}_x, \mathcal{N}_y) \leq (\mathcal{M}_x, \mathcal{M}_y) \iff \mathcal{M}_x \subseteq \mathcal{N}_x \text{ and } \mathcal{M}_y \subseteq \mathcal{N}_y.$$

Define a net $\{x_\lambda\}_{\lambda \in \Lambda}$ by

$$x_{(\mathcal{N}_x, \mathcal{N}_y)} \in \mathcal{N}_x \cap \mathcal{N}_y.$$

Then

$$\forall \mathcal{N}_x \in \mathcal{U}_x, \quad \forall (\mathcal{M}_x, \mathcal{M}_y) \geq (\mathcal{N}_x, X), \quad x_{(\mathcal{M}_x, \mathcal{M}_y)} \in \mathcal{M}_x \cap \mathcal{M}_y \subseteq \mathcal{M}_x \subseteq \mathcal{N}_x.$$

So $\lim_{\lambda \in \Lambda} x_\lambda = x$. Similarly, $\lim_{\lambda \in \Lambda} x_\lambda = y$. So limits of nets in the space are not unique. This completes the proof. ■

7.3 Cauchyness of Nets

Proposition 7.3.1. *Convergent nets are Cauchy.*

Proof. Let (X, τ) be a topological space. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a convergent net in (X, τ) where Λ is some directed index set. Define a point x in X as $x := \lim_{\lambda \in \Lambda} x_\lambda$. Let U be an arbitrary neighborhood of x. Since $\{x_\lambda\}_{\lambda \in \Lambda}$ converges to x, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda \in \Lambda, \quad \lambda \geq \lambda_0 \implies x_\lambda \in U$. ■

7.4 Examples of Nets

Example 7.4.1. *Let $\mathcal{P}$ denote the set of all finite partitions of $[0, 1]$, partially ordered by inclusion. Let f be a continuous function on $[0, 1]$. Define a mapping x from $\mathcal{P}$ to $\mathbb{R}$ by*

$$x_P := L(P, f) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}).$$

Then $\{x_P\}_{P \in \mathcal{P}}$ is a net and

$$\lim_{P \in \mathcal{P}} x(P) = \int_0^1 f(x)dx.$$

Example 7.4.2. *Let* (X, τ) *be a topological space. Let* x *be a point in the space. Let* $\mathcal{U}_x$ *be the neighborhood system at point* x. *Define a relation* $\leq$ *on* $\mathcal{U}_x$ *by* $U < V$ *if* $V \subseteq U$. *Then* $(\mathcal{U}_x, \leq)$ *forms a directed set. Define a mapping* x *from* $\mathcal{U}_x$ *to* X *by* x_U *is a point in* U. *Then* $\{x_U\}_{U \in \mathcal{U}_x}$ *is a net and*

$$\lim_{U \in \mathcal{U}_x} x_U = x.$$

8

Continuous Functions

8.1 Continuity in General

Definition (Continuity)**.** *Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y. We say that f is **continuous** if*

$$\forall G \in \tau_Y, f^{-1}(G) \in \tau_X.$$

Definition (Continuity)**.** *Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y. We say that f is **continuous** if f preserves convergence of nets.*

Proposition 8.1.1. *The two definitions of continuity are equivalent.*

Proposition 8.1.2 (Restriction)**.** *Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a continuous function from X to Y. Let S be a subset of X. Then the restriction f_S of f to (S, τ_X) is continuous.*

Proposition 8.1.3 (Composition)**.** *Let (X, τ_X), (Y, τ_Y), and (Z, τ_Z) be topological spaces. Let f be a continuous map from X to Y. Let g be a continuous map from Y to Z. Then the composition $g \circ f$ is a continuous map from X to Z.*

Proposition 8.1.4. *The limit, with respect to the infinity norm, of a convergent sequence of continuous functions is continuous.*

8.2 Sequential Continuity

Definition (Sequential Continuity)**.** *Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y. We say that f is **sequentially continuous** at a point x_0 in X if f preserves convergence of sequences.*

Proposition 8.2.1. *Continuity implies sequential continuity.*

Proof.
Let U be an arbitrary neighborhood of $f(x_0)$ in Y.
Since U is a neighborhood of $f(x_0)$, by definition of neighborhood, U is open.
Since f is continuous and U is open, by definition of continuity, $f^{-1}(U)$ is open in X.
Since U is a neighborhood of $f(x_0)$, by definition of neighborhood, $f(x_0) \in U$.
Since $f(x_0) \in U$, $x_0 \in f^{-1}(U)$.
Since $x_0 \in f^{-1}(U)$ and $f^{-1}(U)$ is open in X, $f^{-1}(U)$ is a neighborhood of x_0 in X.
Since $f^{-1}(U)$ is a neighborhood of x_0 in X and $\{x_i\}_{i\in\mathbb{N}}$ converges to x_0, by definition of convergence, there exists an integer $N(U)$ such that for any index i with $i > N$, we have $x_i \in f^{-1}(U)$.
Since for any index i with $i > N$, $x_i \in f^{-1}(U)$, for any index i with $i > N$, we have $f(x_i) \in U$.
In short, we have proved that for any neighborhood U of $f(x_0)$ in Y, there exists an integer $N(U)$ such that for any index i with $i > N$, we have $f(x_i) \in U$.
By definition, we conclude that $\{f(x_i)\}_{i\in\mathbb{N}}$ converges to $f(x_0)$.

■

Proposition 8.2.2. *Sequential continuity can imply continuity if the space is first-countable.*

Proof. Let (X, τ) be a topological space. Suppose the space is first-countable.

■

8.3 Cauchy Continuity

Definition (Cauchy Continuous)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y. We say f is **Cauchy continuous** if it preserves Cauchyness of sequences.*

Proposition 8.3.1. *Cauchy continuous functions are continuous.*

Proposition 8.3.2. *Continuous functions are Cauchy continuous if the domain is complete.*

8.4 Uniform Continuity

Definition (Uniform Continuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y. We say that f is **uniformly continuous** on X if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all elements x and y in X such that $d(x, y) < \delta$, we have $\rho(f(x), f(y)) < \varepsilon$.*

Proposition 8.4.1. *Uniformly continuous functions are Cauchy continuous.*

Proof.
Let (X, d_X) and (Y, d_Y) be metric spaces.
Let f be a uniformly continuous function from X to Y.
Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in (X, d_X).
Let ε be an arbitrary positive number.
Since f is uniformly continuous, there exists a positive number $\delta(\varepsilon)$ such that for any points x_1 and x_2 in X, if $d_X(x_1, x_2) < \delta(\varepsilon)$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$.
Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in (X, d_X), there exists an integer $N(\varepsilon)$ such that for any indices m and n, if $m, n > N$, then $d_X(x_m, x_n) < \delta(\varepsilon)$.
Since $d_X(x_m, x_n) < \delta(\varepsilon)$, $d_Y(f(x_m), f(x_n)) < \varepsilon$.
Since for any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices m and n such that $m, n > N$, $d_Y(f(x_m), f(x_n)) < \varepsilon$, $\{f(x_i)\}_{i\in\mathbb{N}}$ is Cauchy in (Y, d_Y).
■

Proposition 8.4.2. *Cauchy continuous functions are uniformly continuous if the domain is totally bounded.*

8.5 Lipschitz Continuity

Definition (Lipschitz)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y. We say that f is **Lipschitz** on X if there exists a non-negative constant c such that for any points x_1 and x_2 in X, we have*

$$d_Y(f(x_1), f(x_2)) \leq c d_X(x_1, x_2).$$

Definition (Bi-Lipschitz)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y. We say that f is **bi-Lipschitz** on X if there exist positive constants c_1 and c_2 such that for any points x_1 and x_2 in X, we have*

$$c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq c_2 d_X(x_1, x_2).$$

Proposition 8.5.1. *Lipschitz continuous functions are uniformly continuous.*

Proof.
Let (X, d_X) and (Y, d_Y) be metric spaces and f be a Lipschitz continuous function from X to Y.
Since f is Lipschitz continuous, by definition, there exists a non-negative constant c such that for any points x_1 and x_2 in X, $d_Y(f(x_1), f(x_2)) \leq c d_X(x_1, x_2)$.
Since for any points x_1 and x_2 in X, $d_Y(f(x_1), f(x_2)) \leq c d_X(x_1, x_2)$, for any positive number ε and any points x_1 and x_2 in X, if $d_X(x_1, x_2) < \varepsilon/c$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$.

Since for any positive number ε and any points x_1 and x_2 in X, if $d_X(x_1, x_2) < \varepsilon/c$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$, for any positive number ε, there exists a positive number $\delta(\varepsilon)$ such that for any points x_1 and x_2 in X, if $d_X(x_1, x_2) < \delta(\varepsilon)$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$.
Since for any positive number ε, there exists a positive number $\delta(\varepsilon)$ such that for any points x_1 and x_2 in X, if $d_X(x_1, x_2) < \delta(\varepsilon)$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$, by definition of uniform continuity, f is uniformly continuous.

■

Proposition 8.5.2. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of Lipschitz continuous functions from X to Y with the sequence $\{K_k\}_{k=1}^{\infty}$ of Lipschitz constants bounded by some non-negative number K. Then if $\{f_k\}$ converges to some function f, f is also Lipschitz continuous with Lipschitz constant also bounded by K.*

Proposition 8.5.3. *Bi-Lipschitz functions are injective.*

Proof.
Let (X, d_X) and (Y, d_Y) be metric spaces and f be a bi-Lipschitz function from X to Y with Lipschitz constants c_1 and c_2.
Let x_1 and x_2 be distinct points in X.
Since f is bi-Lipschitz, by definition, we have $c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq c_2 d_X(x_1, x_2)$.
Assume for the sake of contradiction that $f(x_1) = f(x_2)$.
Since $f(x_1) = f(x_2)$ and d_Y is a metric, we get $d_Y(f(x_1), f(x_2)) = 0$.
Since $d_Y(f(x_1), f(x_2)) = 0$ and $c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2))$, we get $c_1 d_X(x_1, x_2) = 0$.
Since $c_1 d_X(x_1, x_2) = 0$ and $c_1 > 0$, we get $d_X(x_1, x_2) = 0$.
Since $d_X(x_1, x_2) = 0$ and d_X is a metric, by definition, $x_1 = x_2$.
This contradicts to the assumption that $x_1 \neq x_2$.

■

8.6 Other Forms of Continuity

Definition (Hölder continuous)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y. We say that f is **Hölder continuous** with exponent α on X if there exists a number K such that for any points x and y in X, we have $d_Y(f(x), f(y)) \leq K(d_X(x, y))^{\alpha}$.*

8.7 Oscillation in Metric Spaces

Definition (Oscillation)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y. Let x_0 be a point in X. We define the **oscillation** of f at point x_0, denoted* *by $\omega(f, x_0$, to be the number given by*

$$\omega(f, x_0) \inf\{\omega(f, x_0, \delta) : \delta > 0\}$$

where $\omega(f, x_0, \delta)$ is given by

$$\omega(f, x_0, \delta) \sup\{d_Y(f(x_1), f(x_2)) : d_X(x_1, x_0) < \delta, d_X(x_2, x_0) < \delta\}.$$

Proposition 8.7.1. *A function is continuous at a point if and only if the oscillation at the point is 0.*

Proof.
Let (X, d_X) and (Y, d_Y) be metric spaces.
Let f be a function from X to Y.
Let x_0 be a point in X.
For one direction, assume that f is continuous at point x_0.
We are to prove that $\omega(f, x_0) = 0$.
Let ε be an arbitrary positive number.
Since f is continuous at point x_0, by definition of continuity, there exists some $\delta(\varepsilon)$ such that $d_Y(f(x), f(x_0)) < \varepsilon/4$ whenever $d_X(x, x_0) < \delta(\varepsilon)$.
Let x_1 and x_2 be arbitrary points such that $d_X(x_1, x_0) < \delta(\varepsilon)$ and $d_X(x_2, x_0) < \delta(\varepsilon)$.
Since $d_Y(f(x_1), f(x_0)) < \varepsilon/4$ and $d_Y(f(x_2), f(x_0)) < \varepsilon/4$, by the triangle inequality, $d_Y(f(x_1), f(x_2)) < \varepsilon/2$.
Since $d_Y(f(x_1), f(x_2)) < \varepsilon/2$ for any x_1 and x_2 such that $d_X(x_1, x_0) < \delta(\varepsilon)$ and $d_X(x_2, x_0) < \delta(\varepsilon)$, by definition of supremum, $\omega(f, x_0, \delta(\varepsilon)) \leq \varepsilon/2$.
Since for any positive number ε, there exists a $\delta(\varepsilon)$ such that $\omega(f, x_0, \delta(\varepsilon)) < \varepsilon$, by definition of infimum, $\omega(f, x_0) = 0$.
For the reverse direction, assume that $\omega(f, x_0) = 0$.
We are to prove that f is continuous at point x_0.
Let ε be an arbitrary positive number.
Since $\omega(f, x_0) = 0$, by definition of infimum, there exists some $\delta(\varepsilon)$ such that $\omega(f, x_0, \delta(\varepsilon)) < \varepsilon$.
Since $\omega(f, x_0, \delta(\varepsilon)) < \varepsilon$, by definition of supremum, $d_Y(f(x_1), f(x_2)) < \varepsilon$ for any x_1 and x_2 such that $d_X(x_1, x_0) < \delta(\varepsilon)$ and $d_X(x_2, x_0) < \delta(\varepsilon)$.
Since $d_Y(f(x_1), f(x_2)) < \varepsilon$ for any x_1 and x_2 such that $d_X(x_1, x_0) < \delta(\varepsilon)$ and $d_X(x_2, x_0) < \delta(\varepsilon)$, in particular, $d_Y(f(x), f(x_0)) < \varepsilon$ for any point x such that $d_X(x, x_0) < \delta(\varepsilon)$.
Since for any positive number ε, there exists some $\delta(\varepsilon)$ such that for any point x such that $d_X(x, x_0) < \delta(\varepsilon)$, $d_Y(f(x), f(x_0)) < \varepsilon$, by definition of continuity, f is continuous at point x_0.

■

8.8 Isomorphisms

Definition (Isometry)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let f be a function from X to Y. We say that f is **isometric** if for any points x_1 and x_2 in X, we have*

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Proposition 8.8.1. *Isometries are injective.*

8.9 Homeomorphisms

Definition (Homeomorphism)**.** *Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a function from X to Y. We say that f is a **homeomorphism** if f is a bijection and both f and f^{-1} are continuous.*

Proposition 8.9.1. *Homeomorphisms preserve interior. i.e., Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a homeomorphism from X to Y. Let S be a subset of X. Then $f(int(S)) = int(f(S))$.*

Proof.
If S is empty, then both sides evaluate to the empty set and the equality holds.
Now assume $S \neq \varnothing$.
For one direction, let y be an arbitrary point in $f(int(S))$.
We are to prove that $y \in int(f(S))$.
Since $y \in f(int(S))$, there exists some point x in $int(S)$ such that $f(x) = y$.
Since $x \in int(S)$, by definition of interior, there exists some open set U in (X, τ_X) such that $x \in U \subseteq S$.
Let V denote $f(U)$.
Since U is open in (X, τ_X) and f is a homeomorphism, V is open in (Y, τ_Y).
Since $x \in U \subseteq S$, $y \in V \subseteq f(S)$.
Since there exists some open set V in (Y, τ_Y) such that $y \in V \subseteq f(S)$, by definition of interior, $y \in int(f(S))$.
Since $y \in int(f(S))$ for any $y \in f(int(S))$, $f(int(S)) \subseteq int(f(S))$.
For the reverse direction, let y be an arbitrary point in $int(f(S))$.
We are to prove that $y \in f(int(S))$.
Let x denote $f^{-1}(y)$.
Since $y \in int(f(S))$, by definition of interior, there exists some open subset V of $f(S)$ such that $y \in V \subseteq f(S)$.
Let U denote $f^{-1}(V)$.
Since V is open in (Y, τ_Y) and f is a homeomorphism, U is open in (X, τ_X).
Since f is a homeomorphism, $f^{-1}(f(S)) = S$.

Since $y \in V \subseteq f(S)$, $x \in U \subseteq S$.

Since there exists some open set U such that $x \in U \subseteq S$, by definition of interior, $x \in int(S)$.

Since $x \in int(S)$, $y \in f(int(S))$.

Since $y \in f(int(S))$ for any $y \in int(f(S))$, $int(f(S)) \subseteq f(int(S))$.

■

Proposition 8.9.2. *Homeomorphisms preserve closure. i.e., Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let f be a homeomorphism from X to Y. Let S be a subset of X. Then $f(cl(S)) = cl(f(S))$.*

Proof.

If S is empty, then both sides evaluate to the empty set and the equality holds.

Now assume $S \neq \varnothing$.

For one direction, let y be an arbitrary point in $f(cl(S))$.

We are to prove that $y \in cl(f(S))$.

Let F be an arbitrary closed superset of $f(S)$.

Let E denote $f^{-1}(F)$.

Since $y \in f(cl(S))$, there exists some point x in $cl(S)$ such that $f(x) = y$.

Since F is closed in (Y, τ_Y) and f is a homeomorphism, E is closed in (X, τ_X).

Since f is a homeomorphism, $f^{-1}(f(S)) = S$.

Since $f(S) \subseteq F$ and $f^{-1}(f(S)) = S$ and $f^{-1}(F) = E$, we get $S \subseteq E$.

Since $x \in cl(S)$ and E is a closed superset of S, we get $x \in E$.

Since $x \in E$ and $f(x) = y$ and $f(E) = F$, $y \in F$.

Since $y \in F$ for any closed superset of $f(S)$, by definition of closure, $y \in cl(f(S))$.

For the reverse direction, let y be an arbitrary point in $cl(f(S))$.

We are to prove that $y \in f(cl(S))$.

Let x denote $f^{-1}(y)$.

Let E be an arbitrary closed superset of S.

Let F denote $f(E)$.

Since E is closed in (X, τ_X) and f is a homeomorphism, F is closed in (Y, τ_Y).

Since $S \subseteq E$ and $f(E) = F$, $f(S) \subseteq F$.

Since F is a closed superset of $f(S)$ and $y \in cl(f(S))$, by definition of closure, $y \in F$.

Since $y \in F$ and $f^{-1}(y) = x$ and $f^{-1}(F) = E$, $x \in E$.

Since $x \in E$ for any closed superset E of S, by definition of closure, $x \in cl(S)$.

Since $x \in cl(S)$ and $y = f(x)$, $y \in f(cl(S))$.

■

Proposition 8.9.3. *Homeomorphisms preserve nowhere denseness.*

Proof.
Let (X, τ_X) and (Y, τ_Y) be topological spaces.
Let f be a homeomorphism from X to Y.
Let S be a nowhere dense set in (X, τ_X).
We are to prove that $f(S)$ is nowhere dense in (Y, τ_Y).
Assume for the sake of contradiction that $f(S)$ is not nowhere dense in (Y, τ_Y).
Since $f(S)$ is not nowhere dense in (Y, τ_Y), by definition of nowhere dense, there exists some nonempty open set V in (Y, τ_Y) such that $f(S)$ is dense in V.
Let E denote $f^{-1}(V)$.
Since $f(S)$ is dense in V, $V \subseteq cl(f(S))$.
Since V is open in (Y, τ_Y) and f is a homeomorphism, E is open in (X, τ_X).
Since $V \subseteq cl(f(S))$ and $f^{-1}(V) = E$, $E \subseteq f^{-1}(cl(f(S)))$.
Since f is a homeomorphism, f^{-1} is also a homeomorphism.
Since f^{-1} is a homeomorphism, $f^{-1}(cl(f(S))) = cl(f^{-1}(f(S)))$.
Since f is a homeomorphism, $f^{-1}(f(S)) = S$.
Since $E \subseteq f^{-1}(cl(f(S)))$ and $f^{-1}(cl(f(S))) = cl(f^{-1}(f(S)))$ and $f^{-1}(f(S)) = S$, $E \subseteq cl(S)$.
Since $E \subseteq cl(S)$, by definition of denseness, S is dense in E.
Since S is dense in E and E is open in (X, τ_X), by definition of nowhere dense, S is not nowhere dense in (X, τ_X).
This contradicts to the assumption that S is nowhere dense in (X, τ_X).
So the assumption that $f(S)$ is not nowhere dense in (Y, τ_Y) is false.
i.e., $f(S)$ is nowhere dense in (Y, τ_Y).

■

Proposition 8.9.4. *Homeomorphisms preserve meager sets.*

9

Separation Axioms

9.1 Definitions

Definition (Topologically Distinguishable Points)**.** *Let (X, τ) be a topological space. Let x and y be two points in the space. We say that the points x and y are **topologically distinguishable** if*

$$(\exists G \in \tau, x \in G \text{ and } y \notin G) \text{ or } (\exists G \in \tau, y \in G \text{ and } x \notin G).$$

Definition (Separated Sets)**.** *Let (X, τ) be a topological space. Let A and B be two sets in the space. We say that A and B are **separated** by τ if*

$$\exists U, V \in \tau : U \cap V = \emptyset, \quad A \subseteq U \text{ and } B \subseteq V.$$

Definition (T_0 Space)**.** *Let (X, τ) be a topological space. We say that the space is **T_0**, or **Kolmogorov**, if any two distinct points in the space are topologically distinguishable. i.e.,*

$$\forall x, y \in X : x \neq y, \quad (\exists U \in \tau, x \in U, y \notin U) \text{ or } (\exists V \in \tau, y \in V, x \notin V).$$

Definition (T_1 Space)**.** *Let (X, τ) be a topological space. We say that the space is **T_1**, or **Fréchet**, if*

$$\forall x, y \in X : x \neq y, \quad (\exists U \in \tau, x \in U, y \notin U) \text{ and } (\exists V \in \tau, y \in V, x \notin V).$$

Definition (T_2 Space)**.** *Let (X, τ) be a topological space. We say that the space is **T_2**, or **Hausdorff**, if any two distinct points in the space are separated by τ. i.e.,*

$$\forall x, y \in X : x \neq y, \quad (\exists U \in \tau, x \in U) \text{ and } (\exists V \in \tau, y \in V) \text{ and } (U \cap V = \emptyset).$$

Definition (Regular Space)**.** *Let (X, τ) be a topological space. We say that the space is **regular** if any closed set F and a point x such that $x \notin F$ are separated by τ. i.e.,*

$$\forall F \subseteq X, x \in X : F \text{ is closed and } x \notin F, \quad (\exists U \in \tau, F \subseteq U) \text{ and } (\exists V \in \tau, x \in V) \text{ and } (U \cap V = \emptyset).$$

Definition (Normal Space)**.** *Let (X,τ) be a topological space. We say that the space is* ***normal*** *if any two disjoint closed sets in the space are separated by τ.*

$$\forall E,F\subseteq X : E \text{ and } F \text{ are closed and } E\cap F=\emptyset,\quad (\exists U\in\tau, E\subseteq U) \text{ and } (\exists V\in\tau, F\subseteq V) \text{ and } (U\cap V=\emptyset).$$

Definition (T_3 Space)**.** *Let (X,τ) be a topological space. We say that the space is* $\boldsymbol{T_3}$ *if it is T_1 and regular.*

Definition (T_4 Space)**.** *Let (X,τ) be a topological space. We say that the space is* $\boldsymbol{T_4}$ *if it is T_1 and normal.*

9.2 Sufficient Conditions

Proposition 9.2.1. *Compact T_2 spaces are normal.*

Proof. Let (X,τ) be a topological space. Suppose that the space is compact and T_2. We are to prove that the space is normal. Let E and F be two arbitrary disjoint closed sets in the space. Let x be an arbitrary point in E. Let y be an arbitrary point in F. Since $E\cap F=\emptyset$ and $x\in E$ and $y\in F$, $x\neq y$. Since (X,τ) is T_2 and $x\neq y$, x and y are separated. So $\exists U,V\in\tau : U\cap V=\emptyset$ such that $x\in U, y\notin U, y\in V, x\notin V$. The sets $\{V(x,y)\}_{y\in F}$ form an open cover of F. Since the space is compact and F is closed, F has a finite subcover. Say F' is a finite subset of F such that $\{V(x,y)\}_{y\in F'}$ is a finite open cover of F. Define sets $\mathcal{U}(x)$ and $\mathcal{V}(x)$ as

$$\mathcal{U}(x) := \bigcap_{y\in F'} U(x,y) \text{ and } \mathcal{V}(x) := \bigcup_{y\in F'} V(x,y).$$

Then $\mathcal{U}(x)$ and $\mathcal{V}(x)$ form a separation of x and F. The sets $\{\mathcal{U}(x)\}_{x\in U}$ form an open cover of E. Since the space is compact and E is closed, E has a finite subcover. Say E' is a finite subset of E such that $\{\mathcal{U}(x)\}_{x\in E'}$ is a finite open cover of E. Define sets $\mathfrak{U}$ and $\mathfrak{V}$ as

$$\mathfrak{U} := \bigcup_{x\in E'} \mathcal{U}(x) \text{ and } \mathfrak{V} := \bigcap_{x\in E'} \mathcal{V}(x).$$

Then $\mathfrak{U}$ and $\mathfrak{V}$ form a separation of E and F. Since any two disjoint closed sets can be separated by τ, by definition, the space is normal. ■

9.3 Examples

Example 9.3.1. *Metric spaces are T_4.*

10

Countability

10.1 Definitions

Definition (First Countability). *We say that a topological space is first-countable if every point has a countable local base.*

Definition (Second Countability). *We say that a topological space is second-countable if it has a countable base.*

10.2 Properties

Proposition 10.2.1. *Second-countable spaces are separable.*

Proof. Let (X, τ) be a non-empty second-countable topological space.
We are to prove that (X, τ) is separable.
Let G be an arbitrary non-empty open set in (X, τ).
Since (X, τ) is second-countable, by definition of second-countability, there exists a countable basis $\mathcal{B} = \{B_i\}_{i \in I_{\mathcal{B}}} \cup \{\emptyset\}$ for (X, τ) where $I_{\mathcal{B}}$ is a non-empty subset of $\mathbb{N}$ and each B_i is not empty.
Let x_i be some point in B_i.
Let $D\{x_i\}_{i \in I_{\mathcal{B}}}$.
Since $\mathbb{N}$ is countable, D is countable.
Since G is a non-empty open set in (X, τ) and $\mathcal{B}$ is a basis for (X, τ), G can be written as $G = \bigcup_{i \in I'_{\mathcal{B}}(G)} B_i$ where $I'_{\mathcal{B}}(G)$ is a non-empty subset of $I_{\mathcal{B}}$.
Since $x_i \in B_i$ for each $i \in I'_{\mathcal{B}}(G)$ and $G = \bigcup_{i \in I'_{\mathcal{B}}(G)} B_i$, $x_i \in G$ for each $i \in I'_{\mathcal{B}}(G)$.
Since $x_i \in G, D$ for each $i \in I'_{\mathcal{B}}(G)$, $G \cap D$ is not empty.
Since D intersects any non-empty open set in (X, τ) non-trivially, by definition of density, D is dense in (X, τ).

Since D is countable and dense in (X, τ), by definition of separability, (X, τ) is separable.

■

Proposition 10.2.2. *Second-countable spaces are Lindelöf.*

Proof. Let (X, τ) be a second-countable space.
We are to prove that (X, τ) is Lindelöf.
Since (X, τ) is second-countable, by definition of second-countability, there exists a countable basis $\mathcal{B} = \{B_i\}_{i \in I_{\mathcal{B}}}$ for (X, τ) where $I_{\mathcal{B}}$ is a non-empty subset of $\mathbb{N}$.
Let $\mathcal{U}$ =$\{U_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary open cover of (X, τ).
Since each U_λ is open in (X, τ) and $\mathcal{B}$ is a basis for (X, τ), each U_λ can be written as $U_\lambda = \bigcup_{i \in I'_{\mathcal{B}}(\lambda)} B_i$ where $I'_{\mathcal{B}}(\lambda)$ is a non-empty subset of $I_{\mathcal{B}}$.
Let $I \bigcup_{\lambda \in \Lambda} I'_{\mathcal{B}}(\lambda)$.
Since $I'_{\mathcal{B}}(\lambda) \subseteq I_{\mathcal{B}}$ for each $\lambda \in \Lambda$, $I \subseteq I_{\mathcal{B}}$.
Since I is countable, $\{B_i\}_{i \in I}$ is countable.
Since $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ and $U_\lambda = \bigcup_{i \in I'_{\mathcal{B}}(\lambda)} B_i$ for each $\lambda \in \Lambda$, $X = \bigcup_{i \in I} B_i$.
Since any open cover of X has a countable subcover, by definition of Lindelöf, (X, τ) is Lindelöf.

■

10.3 Sufficient Conditions

Proposition 10.3.1. *Subspaces of a second-countable space are second-countable.*

Proposition 10.3.2 (Product)**.** *The product of a countable collection of second-countable spaces is second-countable.*

11

Separability

11.1 Definitions

Definition (Separable). *Let (X, d) be a metric space. Let S be a subset of X. We say that S is **separable** if S has a countable dense subset.*

11.2 Properties

Proposition 11.2.1. *Separable metric spaces are Lindelöf.*

Proof. Let (X, d) be a metric space and S be a separable subset of X.
Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of S.
Since S is separable, S has a countable dense subset S'.
$\{U_\alpha\}$ is an open cover of S'.
Find for each element in S' a U_α. Construct $\{U_{\alpha_k}\}_{k=1}^{\infty}$. Then $\{U_{\alpha_k}\}$ is an open cover of S'.
Since S' is dense, for any $x \in S$ and any neighborhood $N(x)$, there exists $x_0 \in S' \cap N(x)$.
■

Proposition 11.2.2. *Separable metric spaces are second-countable.*

Proof. Let (X, d) be a non-empty separable metric space.
We are to prove that (X, d) is second-countable.
Since (X, d) is separable, there exists a countable dense subset $D = \{d_i\}_{i \in \mathbb{N}}$ of X.
Let $\mathcal{B}\{\text{ball}(d_i, 1/n) : d_i \in D, n \in \mathbb{N}\}$.
Since D and $\mathbb{N}$ are countable, $\mathcal{B}$ is countable.
Let G be an arbitrary non-empty open set in (X, d).
Let x be an arbitrary point in G.

Since $x \in G$ and G is open in (X, d), there exists some radius $r(x)$ of the form $r(x) = 1/n$ where $n \in \mathbb{N}$ such that $\text{ball}(x, 2r(x)) \subseteq G$.
Since $\text{ball}(x, r(x))$ is open in (X, d) and D is dense in (X, d), by definition of density, there exists some $d_{i_0(x)}$ in D such that $d_{i_0(x)} \in \text{ball}(x, r(x))$.
Since $d_{i_0(x)} \in \text{ball}(x, r(x))$, $x \in \text{ball}(d_{i_0(x)}, r(x))$.
Since $d_{i_0(x)} \in \text{ball}(x, r(x))$, $\text{ball}(d_{i_0(x)}, r(x)) \subseteq \text{ball}(x, 2r(x))$.
Since $\text{ball}(d_{i_0(x)}, r(x)) \subseteq \text{ball}(x, 2r(x))$ and $\text{ball}(x, 2r(x)) \subseteq G$, $\text{ball}(d_{i_0(x)}, r(x)) \subseteq G$.
Since for any $x \in G$, there exists some open ball $B(x)$ in $\mathcal{B}$ such that $x \in B(x) \subseteq G$, $G = \bigcup_{x \in G} B(x)$.
Since any non-empty open set in (X, d) can be written as a union of open balls in $\mathcal{B}$, $\mathcal{B} \cup \{\emptyset\}$ is a basis.
Since $\mathcal{B} \cup \{\emptyset\}$ is a countable set and $\mathcal{B} \cup \{\emptyset\}$ is a basis for (X, d), by definition of second-countability, (X, d) is second-countable.

■

11.3 Stability of Separability

Proposition 11.3.1. *An open subspace of a separable space is separable.*

Proposition 11.3.2. *Subsets of a separable metric space are also separable.*

Proposition 11.3.3 (Images)**.** *A continuous image of a separable space is also separable.*

Proof. Let (X, τ_X) and (Y, τ_Y) be topological spaces.
Let f be a continuous function from (X, τ_X) to (Y, τ_Y).
Let (S, τ_X) be a separable subspace of (X, τ_X).
Let T denote $f(S)$.
We are to prove that (T, τ_Y) is separable.
Since (S, τ_X) is separable, by definition, there exists a countable dense set S' in (S, τ_X).
Let T' denote $f(S')$.
Since S' is countable, T' is countable.
Since S' is dense in (S, τ_X), T' is dense in (T, τ_Y).
Since T' is a countable dense set in (T, τ_Y), by definition, (T, τ_Y) is separable.

■

Proposition 11.3.4 (Set Operations)**.**

(1) The union of a countable collection of separable spaces is separable.

11.4 Separation Properties

Proposition 11.4.1. *Let (X, d) be a metric space and $\mathcal{B}$ be a set of open subsets of X. Then $\mathcal{B}$ is a basis for (X, d) if and only if for all $x \in X$ and all open neighborhood U of x, there exists a set $B_0 \in \mathcal{B}$ such that $x \in B_0 \subseteq U$.*

Proposition 11.4.2. *Let (X, d) be a metric space. Let D be a subset of X. Then D is dense if and only if for any element x in X and any radius r, $B(x, r) \cap D \neq \emptyset$.*

12

Totally Bounded Space

12.1 Diameter

Definition (Diameter)**.** *Let (X, d) be a metric space. Let S be a set in the space. We define the* ***diameter*** *of S, denoted by $\operatorname{diam}(S)$, to be a number given by*

$$\operatorname{diam}(S) := \sup\{d(x, y) : x, y \in S\}.$$

Proposition 12.1.1. *If the diameter of a set is zero, then the set is a singleton set.*

Proof. Let (X, d) be a metric space and S be a subset of X with $\operatorname{diam}(S) = 0$.
Assume for the sake of contradiction that there exist distinct points x_1 and x_2 in S.
Since $x_1 \neq x_2$, by definition of metric, $d(x_1, x_2) \neq 0$.
Since $d(x_1, x_2) \in \{d(x, y) : x, y \in S\}$, by definition of supremum, $\sup\{d(x, y) : x, y \in S\} \geq d(x, y)$.
Since $\sup\{d(x, y) : x, y \in S\} \geq d(x, y)$, by definition of diameter, $\operatorname{diam}(S) \geq d(x, y)$.
This contradicts to the assumption that $\operatorname{diam}(S) = 0$.
Thus S is a singleton set.

■

Proposition 12.1.2 (Monotonicity)**.** *The diameter function is increasing.*

Proposition 12.1.3. *The diameter of the closure of a set is equal to the diameter of the set itself.*

Proof. Let (X, d) be a metric space and S be a subset of X.
Let ε be an arbitrary positive number.
Let x and y be arbitrary points in $\operatorname{cl}(S)$.
Since $x \in \operatorname{cl}(S)$, there exists a point x' in S such that $d(x, x') < \varepsilon/2$.

Since $y \in \mathrm{cl}(S)$, there exists a point y' in S such that $d(y, y') < \varepsilon/2$.
Since $x', y' \in S$, by definition of diameter, $d(x', y') \leq \mathrm{diam}(S)$.
Since d is a metric on X, by the triangle inequality, $d(x, y) \leq d(x, x') + d(x', y') + d(y', y)$.
Since $d(x, y) \leq d(x, x') + d(x', y') + d(y', y)$ and $d(x, x') < \varepsilon/2$ and $d(y, y') < \varepsilon/2$ and $d(x', y') \leq \mathrm{diam}(S)$, $d(x, y) < \mathrm{diam}(S) + \varepsilon$.
Since for any positive number ε, $d(x, y) < \mathrm{diam}(S) + \varepsilon$, $d(x, y) \leq \mathrm{diam}(S)$.
Let x_0 and y_0 be points in S such that $d(x_0, y_0) > \mathrm{diam}(S) - \varepsilon$.
Since $x_0 \in S$ and $S \subseteq \mathrm{cl}(S)$, $x_0 \in \mathrm{cl}(S)$.
Since $y_0 \in S$ and $S \subseteq \mathrm{cl}(S)$, $y_0 \in \mathrm{cl}(S)$.
Since for any points x and y in $\mathrm{cl}(S)$, $d(x, y) \leq \mathrm{diam}(S)$ and for any positive number ε, there exists $x_0, y_0 \in S$ such that $d(x_0, y_0) > \mathrm{diam}(S) - \varepsilon$, by definition of supremum, $\sup\{d(x, y) : x, y \in \mathrm{cl}(S)\} = \mathrm{diam}(S)$.
Since $\sup\{d(x, y) : x, y \in \mathrm{cl}(S)\} = \mathrm{diam}(S)$, by definition of diameter, $\mathrm{diam}(\mathrm{cl}(S)) = \mathrm{diam}(S)$.

■

12.2 Boundedness

Definition (Boundedness)**.** *Let (X, d) be a metric space. Let S be a subset of X. We say S is **bounded** if its diameter is finite.*

Proposition 12.2.1. *The closure of a bounded set is bounded.*

12.3 Total Boundedness

Definition (Total Boundedness)**.** *Let (X, d) be a metric space. We say that the space is **totally bounded** if*

$$\forall r > 0, \exists n \in \mathbb{N}, \exists \{x_i\}_{i=1}^{n} \subseteq X, \quad \bigcup_{i=1}^{n} \mathrm{ball}(x_i, r) = X.$$

Proposition 12.3.1. *A space is totally bounded if and only if any sequence in the space has a Cauchy subsequence.*

Proof. For one direction, assume that for any radius r, there exists a finite collection $\mathfrak{p}(r) = \{p_i\}_{i=1}^{i=k}$ of points in X such that $\{\mathrm{ball}(p_i, r)\}_{i=1}^{i=k}$ covers S.
We are to prove that any sequence in (S, d) has a Cauchy subsequence in (S, d).
Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in (S, d).
Let r be an arbitrary radius.
Since $\{x_i\}_{i \in \mathbb{N}}$ is a sequence in (S, d) and $\{\mathrm{ball}(p_i, r)\}_{i=1}^{i=k}$ covers S, there exists some p_{i_0} among $\mathfrak{p}(r)$ such that $\mathrm{ball}(p_{i_0}, r)$ contains a subsequence $\{x_i\}_{i \in I}$ of $\{x_i\}_{i \in \mathbb{N}}$.

Since $\{x_i\}_{i\in I} \subseteq \text{ball}(p_{i_0}, r)$, for any indices m and n in I, $d(x_m, x_n) < r$.
Since $d(x_m, x_n) < r$ for any radius r, by definition of Cauchy-ness, $\{x_i\}_{i\in I}$ is Cauchy.
Since $\{x_i\}_{i\in I}$ is Cauchy, any sequence in (S, d) has a Cauchy subsequence in (S, d).
For the reverse direction, assume that any sequence in (S, d) has a Cauchy sequence in (S, d).
We are to prove that for any radius r, there exists a finite collection $\mathfrak{p}(r) = \{p_i\}_{i=1}^{i=k}$ of points in X such that $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$ covers S.
Case 1. S is empty.
Since S is empty, for any radius r, there exists a finite collection $\mathfrak{p}(r) = \{p_i\}_{i=1}^{i=k}$ of points in X such that $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$ covers S.
Case 2. S is not empty.
Assume for the sake of contradiction that there exists some radius r_0 such that for any finite collection $\S(r_0) = \{x_i\}_{i=1}^{i=k}$ of points in X, $\{\text{ball}(x_i, r_0)\}_{i=1}^{i=k}$ does not cover S.
Since S is not empty, pick x_1 from S.
Since $\{\text{ball}(x_i, r_0)\}_{i=1}^{i=1}$ does not cover S, pick x_2 from $C_S(\bigcup_{i=1}^{i=1} \text{ball}(x_i, r_0))$.
In general, since $\{\text{ball}(x_i, r_0)\}_{i=1}^{i=k}$ does not cover S, pick x_{k+1} from $C_S(\bigcup_{i=1}^{i=k} \text{ball}(x_i, r_0))$.
Let m and n be arbitrary indices such that $m > n$.
Since $x_m \in C_S(\bigcup_{i=1}^{i=m-1} \text{ball}(x_i, r_0))$, $x_m \notin \text{ball}(x_n, r_0)$.
Since $x_m \notin \text{ball}(x_n, r_0)$, by definition of open ball, $d(x_m, x_n) \geq r_0$.
Since for any indices m and n, $d(x_m, x_n) \geq r_0$, by definition of Cauchy-ness, $\{x_i\}_{i\in\mathbb{N}}$ does not have a Cauchy subsequence in (S, d).
This contradicts to the fact that any sequence in (S, d) has a Cauchy sequence in (S, d).
Thus the assumption that there exists some radius r_0 such that for any finite collection $\S(r_0) = \{x_i\}_{i=1}^{i=k}$ of points in X, $\{\text{ball}(x_i, r_0)\}_{i=1}^{i=k}$ does not cover S is false.
i.e., for any radius r, there exists a finite collection $\mathfrak{p}(r) = \{p_i\}_{i=1}^{i=k}$ of points in X such that $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$ covers S.
Summary.
Since in either cases, for any radius r, there exists a finite collection $\mathfrak{p}(r) = \{p_i\}_{i=1}^{i=k}$ of points in X such that $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$ covers S, I conclude that for any radius r, there exists a finite collection $\mathfrak{p}(r) = \{p_i\}_{i=1}^{i=k}$ of points in X such that $\{\text{ball}(p_i, r)\}_{i=1}^{i=k}$ covers S.

■

12.4 Properties of Totally Bounded Spaces

Proposition 12.4.1. *Totally bounded spaces are bounded.*

Proposition 12.4.2. *Totally bounded metric spaces are separable.*

Proof. Let (X, d) be a metric space. Suppose that the space is totally bounded. We are to prove that the space has a countable dense subset. Let n be an arbitrary natural number.

Since the space is totally bounded, there exists a finite collection of points $\mathfrak{p}_n$ such that $\bigcup_{p\in\mathfrak{p}_n} \mathrm{ball}(p, 1/n) = X$. Define a set $\mathfrak{p}$ by $\mathfrak{p} := \bigcup_{n\in\mathbb{N}} \mathfrak{p}_n$. Then $\mathfrak{p}$ is a countable dense subset of X. ■

12.5 Stability of Total Boundedness

Proposition 12.5.1. *The closure of a totally bounded set is totally bounded.*

Proof. Let (X, d) be a metric space and S be a totally bounded subset of X.
We are to prove that $\mathrm{cl}(S)$ is also totally bounded.
Let r be an arbitrary radius.
Since S is totally bounded, by definition of total boundedness, there exist a finite collection $\S(r) = \{x_i\}_{i=1}^{i=k}$ of points in X such that $\{\mathrm{ball}(x_i, r/2)\}_{i=1}^{i=k}$ covers S.
Let x be an arbitrary point in $\mathrm{cl}(S)$.
Since $x \in \mathrm{cl}(S)$, by definition of closure, there exists a point $x'(x)$ in S such that $d(x, x'(x)) < r/2$.
Since $x'(x) \in S$ and $\{\mathrm{ball}(x_i, r/2)\}_{i=1}^{i=k}$ covers S, there exists some $x_{i_0(x)}$ among $\{x_i\}_{i=1}^{i=k}$ such that $x'(x) \in \mathrm{ball}(x_{i_0(x)}, r/2)$.
Since $x'(x) \in \mathrm{ball}(x_{i_0(x)}, r/2)$, by definition of open ball, $d(x'(x), x_{i_0(x)}) < r/2$.
Since $d(x, x'(x)) < r/2$ and $d(x'(x), x_{i_0(x)}) < r/2$, by the triangle inequality, $d(x, x_{i_0(x)}) < r$.
Since $d(x, x_{i_0(x)}) < r$, by definition of open ball, $x \in \mathrm{ball}(x_{i_0(x)}, r)$.
Since for any $x \in \mathrm{cl}(S)$, there exists a point $x_{i_0(x)}$ among $\{x_i\}_{i=1}^{i=k}$ such that $x \in \mathrm{ball}(x_{i_0(x)}, r)$, $\{\mathrm{ball}(x_i, r)\}_{i=1}^{i=k}$ covers $\mathrm{cl}(S)$.
Since for any radius r, there exists a finite collection $\S(r) = \{x_i\}_{i=1}^{i=k}$ of points in X such that $\{\mathrm{ball}(x_i, r)\}_{i=1}^{i=k}$ covers $\mathrm{cl}(S)$, by definition, $\mathrm{cl}(S)$ is totally bounded.
■

Proposition 12.5.2 (Set Operations)**.** *(1) A subset of a totally bounded set is totally bounded.*

(2) The union of a finite collection of totally bounded sets is totally bounded.

Proposition 12.5.3 (Images)**.** *A uniformly continuous image of a totally bounded set is totally bounded.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces.
Let S be a totally bounded set in (X, d_X).
Let f be a uniformly continuous function from X to Y.
We are to prove that $f(S)$ is totally bounded.
Let r be an arbitrary radius.

Since f is uniformly continuous on S, there exists a $\delta(r)$ such that $d_Y(f(x_1), f(x_2)) < r$ whenever $d_X(x_1, x_2) < \delta(r)$.
Since S is totally bounded in (X, d_X), there exists a finite collection § of points in X such that $\{(x, \delta(r))\}_{x \in \S}$ covers S.
Let † denote $f(\S)$.
Let y be an arbitrary point in $f(S)$.
Since $y \in f(S)$, there exists a point x in S such that $f(x) = y$.
Since $x \in S$ and $\{(x, \delta(r))\}_{x \in \S}$ covers S, there exists a point $x_0 \in \S$ such that $x \in (x_0, \delta(r))$.
Let y_0 denote $f(x_0)$.
Since $x \in (x_0, \delta(r))$, by definition of $\delta(r)$, $y \in (y_0, r)$.
Since $y \in (y_0, r)$ and $y_0 \in \dagger$, $\{(y, r)\}_{y \in \dagger}$ covers y.
Since $\{(y, r)\}_{y \in \dagger}$ covers y for any $y \in f(S)$, $\{(y, r)\}_{y \in \dagger}$ covers $f(S)$.
Since for any radius r, there exists a collection † of points in $f(S)$ such that $\{(y, r)\}_{y \in \dagger}$ covers $f(S)$, by definition of total boundedness, $f(S)$ is totally bounded in (Y, d_Y).

$\blacksquare$

13

Lindelöf Space

13.1 Definition

Definition (Lindelöf Space)**.** *Let (X, τ) be a topological space. We say that the space is **Lindelöf** if any open cover has a countable subcover.*

14

Compact Space

14.1 Definitions

Definition (Open Cover)**.** *Let (X, τ) be a topological space. Let S be a set in the space. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a collection of sets in the space. We say that $\{U_\lambda\}_{\lambda \in \Lambda}$ is an* ***open cover*** *of S if U_λ is open for each $\lambda \in \Lambda$ and*

$$\bigcup_{\lambda \in \Lambda} U_\lambda \supseteq S.$$

Definition (Compactness)**.** *Let (X, τ) be a topological space. We say that the space is* ***compact*** *if any open cover has a finite subcover.*

Definition (Finite Intersection Property)**.** *Let (X, τ) be a topological space. Let $\{S_i\}_{i \in I}$ be a collection of sets in the space. We say that $\{S_i\}_{i \in I}$ has the* ***finite intersection property*** *if for any finite subcollection $\{S_i\}_{i \in I'}$, where I' is a finite subset of I,*

$$\bigcap_{i \in I'} S_i \neq \emptyset.$$

Definition (Compactness)**.** *Let (X, τ) be a topological space. Let S be a set in the space. We say that S is* ***compact*** *if for any collection $\{S_i\}_{i \in I}$ of relatively closed subsets of S with the finite intersection property,*

$$\bigcap_{i \in I} S_i \neq \emptyset.$$

Definition (Compactness)**.** *Let (X, τ) be a topological space. Let S be a set in the space. We say that S is* ***compact*** *if any net on S has a convergent subnet.*

Definition (Compactness)**.** *Let (X, τ) be a topological space. Let S be a set in the space. We say that S is* ***compact*** *if any filter on S has a convergent refinement.*

Proposition 14.1.1. *Definitions 1 and 2 of compactness are equivalent.*

Proof. [***Definition 1***] $\implies$ [***Definition 2***].
For one direction, assume that any open cover of the space has a finite subcover.
We are to prove that any collection of closed sets in the space with the finite intersection property has nonempty intersection.
Let $\mathcal{F}$ be an arbitrary collection of closed sets in the space with the finite intersection property.
Say $\mathcal{F} = \{F_\lambda\}_{\lambda \in \Lambda}$ where Λ is an index set and F_λ is a closed set in the space for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda'} F_\lambda \neq \emptyset$ for any finite subset Λ' of Λ.
Define $U_\lambda := X \setminus F_\lambda$.
Define $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$.
Since F_λ is closed for each $\lambda \in \Lambda$, U_λ is open for each $\lambda \in \Lambda$.
Assume for the sake of contradiction that $\mathcal{F}$ has empty intersection.
Since $\bigcap_{\lambda \in \Lambda} F_\lambda = \emptyset$, by the De Morgan's Law, $\bigcup_{\lambda \in \Lambda} U_\lambda = X$.
Since $\mathcal{U}$ is an open cover of the space and any open cover of the space has a finite subcover, in particular, $\mathcal{U}$ has a finite subcover $\mathcal{U}'$.
Say $\mathcal{U}' = \{U_\lambda\}_{\lambda \in \Lambda'}$ where Λ' is a finite subset of Λ and $\bigcup_{\lambda \in \Lambda'} U_\lambda = X$.
Since $\bigcup_{\lambda \in \Lambda'} U_\lambda = X$, by the De Morgan's Law, $\bigcap_{\lambda \in \Lambda'} F_\lambda = \emptyset$.
This contradicts to the assumption that $\mathcal{F}$ has the finite intersection property.
So the assumption that $\mathcal{F}$ has empty intersection is false.
i.e., $\mathcal{F}$ has nonempty intersection.

■

Proof. [***Definition 2***] $\implies$ [***Definition 1***].
For the reverse direction, assume that any collection of closed sets in the space with the finite intersection property has nonempty intersection.
We are to prove that any open cover has a finite subcover.
Let $\mathcal{U}$ be an arbitrary open cover of the space.
Say $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ where Λ is an index set and U_λ is an open set in the space for any $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} U_\lambda = X$.
Define $F_\lambda := X \setminus U_\lambda$ for each $\lambda \in \Lambda$.
Define $\mathcal{F} := \{F_\lambda\}_{\lambda \in \Lambda}$.
Since U_λ is open for each $\lambda \in \Lambda$, F_λ is closed for each $\lambda \in \Lambda$.
Assume for the sake of contradiction that $\mathcal{U}$ does not have a finite subcover.
Let Λ' be an arbitrary finite subset of Λ.
Define $\mathcal{U}' := \{U_\lambda\}_{\lambda \in \Lambda'}$.
Define $\mathcal{F}' := \{F_\lambda\}_{\lambda \in \Lambda'}$.
Since $\mathcal{U}'$ is a finite subcollection of $\mathcal{U}$ and $\mathcal{U}$ does not have a finite subcover, in particular, $\mathcal{U}'$ cannot cover the whole space.

Since $\bigcup_{\lambda\in\Lambda'} U_\lambda \neq X$, by the De Morgan's Law, $\bigcap_{\lambda\in\Lambda'} F_\lambda \neq \emptyset$.
Since $\bigcap_{\lambda\in\Lambda'} F_\lambda \neq \emptyset$ for any finite subcollection Λ' of Λ, $\mathcal{F}$ has the finite intersection property.
Since $\mathcal{F}$ has the finite intersection property and any collection of closed sets with the finite intersection property has nonempty intersection, in particular, $\bigcap_{\lambda\in\Lambda} F_\lambda \neq \emptyset$.
Since $\bigcap_{\lambda\in\Lambda} F_\lambda \neq \emptyset$, by the De Morgan's Law, $\bigcup_{\lambda\in\Lambda} U_\lambda \neq X$.
This contradicts to the assumption that $\mathcal{U}$ is an open cover of the space.
So the assumption that $\mathcal{U}$ does not have a finite subcover is false.
i.e., $\mathcal{U}$ has a finite subcover.

■

14.2 Properties

Proposition 14.2.1. *A compact subspace of a Hausdorff space is closed.*

Proposition 14.2.2. *Compact metric spaces are totally bounded.*

Proof.
Assume for the sake of contradiction that the space is not totally bounded.
Since the space is not totally bounded, by definition of total boundedness, there exists some radius r_0 such that the space cannot be covered by finitely many open balls of radius r_0.
Let x_1 be an arbitrary point in the space.
Since $\{ball(x_i, r_0)\}_{i\leq n}$ cannot cover the space for any $n \in \mathbb{N}$, pick x_{n+1} from $X \backslash \bigcup_{i\leq n} ball(x_i, r_0)$.
Define $\mathfrak{x} := \{x_n\}_{n\in\mathbb{N}}$.
Let m and n be arbitrary indices in $\mathbb{N}$.
Assume without loss of generality that $m > n$.
Since $x_m \in X \setminus \bigcup_{i\leq m} ball(x_i, r_0)$, $x_m \notin ball(x_n, r_0)$.
Since $x_m \notin ball(x_n, r_0)$, $d(x_m, x_n) \geq r_0$.
Since $d(x_m, x_n) \geq r_0$ for any $m, n \in \mathbb{N}$, $\mathfrak{x}$ has no convergent subsequence.
Since $\mathfrak{x}$ is a sequence in the space and $\mathfrak{x}$ has no convergent subsequence, by definition of compactness, the space is not compact.
This contradicts to the assumption that the space is compact.
Thus the assumption that the space is not totally bounded is false.
i.e., the space is totally bounded.

■

Proposition 14.2.3. *Compact metric spaces are complete.*

Proof. Let $\mathfrak{x}$ be an arbitrary Cauchy sequence in the space.
Since the space is compact, any sequence has a convergent subsequence.
Since $\mathfrak{x}$ is a sequence in the space and any sequence has a convergent subsequence, $\mathfrak{x}$ has a convergent subsequence.

Since $\mathfrak{x}$ is Cauchy and has a convergent subsequence, $\mathfrak{x}$ converges.
Since any Cauchy sequence in the space converges, by definition of completeness, the space is complete.

■

Proposition 14.2.4. *Compact metric spaces are separable.*

Theorem 1 (Extension)**.** *Let (X, d_X) be a compact metric space and (Y, d_Y) be an arbitrary metric spaces. Let f be a continuous function from X to Y. Then f is uniformly continuous on X.*

Proof.
Since f is continuous on X, for all $\varepsilon > 0$ and all $x \in X$, there exists a $\delta(x) > 0$ such that for all $x' \in X$ with $d(x, x') < \delta(x)$, we have $\rho(f(x), f(x')) < \varepsilon/2$.
Consider the set of open balls $\mathcal{B} = \{B(\frac{1}{2}\delta(x), x)\}$.
Then $\mathcal{B}$ is a cover of X.
By definition of compactness, there exists a finite set of open balls $\mathcal{B}_n = \{B_k\}_{k=1}^{k=n}$.
Define $\delta > 0$ by $\delta = \min\{\frac{1}{2}\delta(x_k)\}$.
Let x and y be arbitrary elements in X with $d(x, y) < \delta$.
Since $x \in X$ and $\mathcal{B}_n$ is a cover of X, there exists an open ball $B_0(\frac{1}{2}\delta(x_0), x_0)$ such that $x \in B_0$.
By our choice of x, y and δ, we have

$$d(y, x) < \delta \leq \tfrac{1}{2}\delta(x_0)\#(1)$$

Since $x \in B_0$, we get

$$d(x, x_0) < \tfrac{1}{2}\delta(x_0)\#(2)$$

By the triangle inequality, we get

$$d(y, x_0) \leq d(y, x) + d(x, x_0)\#(3)$$

From inequations (1) ˜ (3), we get

$$d(y, x_0) < \delta(x_0)\#(4)$$

Since f is continuous at point x_0 and $d(x, x_0) < \delta(x_0)$, we get

$$\rho(f(x), f(x_0)) < \tfrac{\varepsilon}{2}\#(5)$$

Since f is continuous at pint x_0 and $d(y, x_0) < \delta(x_0)$, we get

$$\rho(f(y), f(x_0)) < \frac{\varepsilon}{2}\#(6)$$

By the triangle inequality again, we get

$$\rho(f(x), f(y)) < \rho(f(x), f(x_0)) + \rho(f(y), f(x_0))\#(7)$$

From inequations (5) ˜ (7), we get

$$\rho(f(x), f(y)) < \varepsilon$$

In short, we have proved that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all elements x and y in X with $d(x, y) < \delta$, we have $\rho(f(x), f(y)) < \varepsilon$.
By definition of uniform continuity, f is uniformly continuous on X.

■

14.3 Sufficient Conditions

Proposition 14.3.1. *A closed subspace of a compact space is compact.*

Proof.
Let (X, τ) be a compact topological space.
Let (S, τ) be a closed subspace of (X, τ).
We are to prove that (S, τ) is compact.
Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary open cover of S.
Since S is closed, $X \setminus S$ is open.
Since $\{U_\lambda\}_{\lambda\in\Lambda}$ covers S, $\{U_\lambda\}_{\lambda\in\Lambda} \cup \{X \setminus S\}$ covers X.
Since each set in $\{U_\lambda\}_{\lambda\in\Lambda} \cup \{X \setminus S\}$ is open and $\{U_\lambda\}_{\lambda\in\Lambda} \cup \{X \setminus S\}$ covers X, by definition of open cover, $\{U_\lambda\}_{\lambda\in\Lambda} \cup \{X \setminus S\}$ is an open cover of X.
Since (X, τ) is compact, by definition of compactness, any open cover of (X, τ) has a finite subcover.
Since $\mathcal{U} \cup \{X \setminus S\}$ is an open cover of (X, τ) and any open cover has a finite subcover, in particular, $\mathcal{U}$ has a finite subcover $\mathcal{U}'$.
Say $\mathcal{U}' = \{U_\lambda\}_{\lambda\in\Lambda'}$ where Λ' is a finite subset of Λ and $\bigcup_{\lambda\in\Lambda'} U_\lambda = X$
Since $\{U_\lambda\}_{\lambda\in\Lambda'} \cup \{X \setminus S\}$ covers X, $\{U_\lambda\}_{\lambda\in\Lambda'}$ covers S.
Since any open cover of S has a finite subcover, by definition of compactness, (S, τ) is compact.

■

Proposition 14.3.2 (Set Operations)**.**

(1) *The intersection of any collection of compact spaces is compact.* ***### got some problem with this! Need the space to be Hausdorff.***

(2) *The union of any finite collection of compact spaces is compact.*

Proof.
Proof of (1).
Let X be a topological space and $\{S_i\}_{i\in I}$ be a collection of compact sets in X.
Let
Proof of (1).
Let $\{S_\alpha\}_{\alpha\in A}$ be an arbitrary collection of compact sets and let S be their intersection.
Then there exists a compact set S_{α_0} such that $S \subseteq S_{\alpha_0}$.
Since for any $\alpha \in A$, S_α is closed, S is closed.
Since S is closed and S_{α_0} is compact, S also compact.
Proof of (2).
Let (X,τ) be a topological space and $\{S_\lambda\}_{\lambda\in\Lambda}$ be a finite collection of compact subsets of X.
Let $\{U_\lambda\}$
Let $\{S_k\}_{k=1}^N$ be an arbitrary finite collection of compact sets and let S be their union.
Let $\mathcal{U}$ be an arbitrary open cover of S.
Since $\mathcal{U}$ is an open cover of S and for each k, $S_k \subseteq S$, for each k, $\mathcal{U}$ is an open cover of S_k.
Since $\mathcal{U}$ is an open cover of S_k and S_k is compact, by definition, there exists a finite subcover $\mathcal{U}_k$ of S_k.
Let $\mathcal{U}'$ be the union of $\{\mathcal{U}_k\}_{k=1}^N$. Then $\mathcal{U}'$ is finite.
In short, we have proved that any open cover of S has a finite subcover.
By definition, we conclude that S is compact.

■

Proposition 14.3.3 (Product, Tychonoff)**.** *The product of any collection of compact sets is compact.*

Proposition 14.3.4 (Image)**.** *The continuous image of a compact set is compact.*

Proof.
Let (X,τ_X) and (Y,τ_Y) be topological space.
Let f be a continuous function from X to Y.
Let S be a compact set in (X,τ_X).
Define $T := f(S)$.
We are to prove that T is a compact set in (Y,τ_Y).
Let $\mathcal{V}$ be an arbitrary open cover of T.

Say $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ where Λ is an index set and V_λ is an open set in (Y, τ_Y) for each $\lambda \in \Lambda$ and $T \subseteq \bigcup_{\lambda\in\Lambda} V_\lambda$.
Define $U_\lambda := f^{-1}(V_\lambda)$ for each $\lambda \in \Lambda$.
Define $\mathcal{U} := \{U_\lambda\}_{\lambda\in\Lambda}$.
Since $V_\lambda \neq \emptyset$ for each $\lambda \in \Lambda$, $U_\lambda \neq \emptyset$ for each $\lambda \in \Lambda$.
Since V_λ is open in (Y, τ_Y) for each $\lambda \in \Lambda$ and f is a continuous function from (X, τ_X) to (Y, τ_Y), by definition of continuity, U_λ is open in (X, τ_X) for each $\lambda \in \Lambda$.
Since $T \subseteq \bigcup_{\lambda\in\Lambda} V_\lambda$, $S \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$.
Since U_λ is a nonempty open set in (X, τ_X) for each $\lambda \in \Lambda$ and $S \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$, $\mathcal{U}$ is an open cover of S in (X, τ_X).
Since S is a compact set in (X, τ_X), by definition of compactness, any open cover of S in (X, τ_X) has a finite subcover.
Since $\mathcal{U}$ is an open cover of S in (X, τ_X) and any open cover of S in (X, τ_X) has a finite subcover, in particular, $\mathcal{U}$ has a finite subcover $\mathcal{U}'$.
Say $\mathcal{U}' = \{U_\lambda\}_{\lambda\in\Lambda'}$ where Λ' is a finite subset of Λ and $S \subseteq \bigcup_{\lambda\in\Lambda'} U_\lambda$.
Define $\mathcal{V}' := \{V_\lambda\}_{\lambda\in\Lambda'}$.
Since $S \subseteq \bigcup_{\lambda\in\Lambda'} U_\lambda$, $T \subseteq \bigcup_{\lambda\in\Lambda'} V_\lambda$.
Since V_λ is a nonempty open set in (Y, τ_Y) for each $\lambda \in \Lambda'$ and $T \subseteq \bigcup_{\lambda\in\Lambda'} V_\lambda$ and Λ' is finite, $\mathcal{V}'$ is a finite open cover of T in (Y, τ_Y).
Since any open cover of T in (Y, τ_Y) has a finite subcover, by definition of compactness, T is a compact set in (Y, τ_Y).

■

Proposition 14.3.5. *Finite spaces are compact.*

Proof.
Let (X, τ) be a finite topological space.
Say $X = \{x_i\}_{i\in I}$ where I is a finite index set.
Let $\mathcal{U}$ be an arbitrary open cover of the space.
Say $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ where Λ is an index set and U_λ is an open set in (X, τ_X) for each $\lambda \in \Lambda$ and $\bigcup_{\lambda\in\Lambda} U_\lambda = X$.
Since $\mathcal{U}$ covers X and $x_i \in X$ for each $i \in I$, $\mathcal{U}$ covers x_i for each $i \in I$.
Since $x_i \in \mathcal{U}$ for each $i \in I$, there exists some λ_i for each $i \in I$ such that $x_i \in U_{\lambda_i}$.
Define $\mathcal{U}' := \{U_{\lambda_i}\}_{i\in I}$.
Since $x_i \in U_{\lambda_i}$ for each $i \in I$, $X \subseteq \bigcup_{i\in I} U_{\lambda_i}$.
Since each U_{λ_i} is open in (X, τ) and $X \subseteq \bigcup_{i\in I} U_{\lambda_i}$ and I is finite, $\mathcal{U}'$ is a finite opsn cover of the space.
Since any open cover of the space has a finite subcover, by definition, of compactness, (X, τ) is compact.

■

Proposition 14.3.6. *A complete and totally bounded metric space is (sequentially) compact.*

Proof. Every sequence in a totally bounded metric space admits Cauchy subsequence. Every Cauchy sequence in a complete metric space converges. So every sequence has a convergent subsequence.

■

14.4 Countably Compact

14.4.1 Definitions

Definition (Countably Compact)**.** *Let (X, τ) be a topological space. We say that (X, τ) is **countably compact** if every countable open cover has a finite subcover.*

Definition (Countably Compact)**.** *Let (X, τ) be a topological space. We say that (X, τ) is **countably compact** if every infinite set in the space has a ω-limit point.*

Definition (Weakly Countably Compact)**.** *Let (X, τ) be a topological space. We say that (X, τ) is **weakly countably compact** if every infinite set in the space has a limit point.*

Proposition 14.4.1. *The two definitions of countably compact are equivalent.*

14.4.2 Sufficient Conditions

Proposition 14.4.2 (Image)**.** *A continuous image of a countably compact set is countably compact.*

Remark. *Continuous images of weakly countably compact sets may not be weakly countably compact.*

14.4.3 Relation to Other Forms of Compactness

Proposition 14.4.3. *Compactness implies countable compactness.*

Proposition 14.4.4. *Countable compactness can imply compactness if the space is Lindelöf.*

Proposition 14.4.5. *Countable compactness implies weak countable compactness.*

Proposition 14.4.6. *Weak countable compactness can imply countable compactness if the space is metrizable.*

14.5 Sequentially Compact

14.5.1 Definitions

Definition (Sequentially Compact)**.** *Let (X, τ) be a topological space. We say that (X, τ) is **sequentially compact** if every sequence in the space has a convergent subsequence.*

14.5.2 Relation to Other Forms of Compactness

Proposition 14.5.1. *Compactness can imply sequential compactness if the space is first countable.*

Proof.
Assume that any open cover has a finite subcover.
We are to prove that any sequence has a convergent subsequence.
Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in (X, d).
Define $F_n := cl(\{x_i\}_{i \geq n})$ for each $n \in \mathbb{N}$.
Define $U_n := X \setminus F_n$ for each $n \in \mathbb{N}$.
Define $\mathcal{U} := \{U_n\}_{n \in \mathbb{N}}$.
Assume for the sake of contradiction that $\bigcup_{n \in \mathbb{N}} U_n = X$.
Since $\{U_n\}_{n \in \mathbb{N}}$ is an open cover of X and any open cover of X has a finite subcover, there exists a finite index set I such that $\{U_n\}_{n \in I}$ covers X.
Since $\{U_n\}_{n \in I}$ covers X, $\bigcap_{n \in I} F_n = \emptyset$.
Define $i_0 := max(I)$.
Since $\{F_n\}_{n \in I}$ is decreasing, $\bigcap_{n \in I} F_n = F_{i_0}$.
Since $\bigcap_{n \in I} F_n = \emptyset$ and $\bigcap_{n \in I} F_n = F_{i_0}$, $F_{i_0} = \emptyset$.
This contradicts to the assumption that $F_n = cl(\{x_i\}_{i \geq n})$.
So the assumption that $\bigcup_{n \in \mathbb{N}} U_n = X$ is false.
i.e., $\bigcup_{n \in \mathbb{N}} U_n \neq X$.
Since $U_n \subseteq X$ for any $n \in \mathbb{N}$, $\bigcup_{n \in \mathbb{N}} U_n \subseteq X$.
Since $\bigcup_{n \in \mathbb{N}} U_n \subseteq X$ and $\bigcup_{n \in \mathbb{N}} U_n \neq X$, there exists a point x_0 in $X \setminus \bigcup_{n \in \mathbb{N}} U_n$.
Since $x_0 \in X \setminus \bigcup_{n \in \mathbb{N}} U_n$ and $X \setminus \bigcup_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} F_n$, $x_0 \in \bigcap_{n \in \mathbb{N}} F_n$.

■

Remark. *Compactness does not imply sequential compactness in general.*

Proposition 14.5.2. *Sequential compactness can imply compactness if the space is metrizable.*

Proof.
Let (X, d) be a metrizable topological space.
Assume that any sequence has a convergent subsequence.

We are to prove that any open cover has a finite subcover.
Let $\mathcal{U}$ be an arbitrary open cover of the space.
Say $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ where Λ is an index set and U_λ is open for each $\lambda \in \Lambda$ and $\bigcup_{\lambda\in\Lambda} U_\lambda = X$.
Assume for the sake of contradiction that $\mathcal{U}$ does not have a finite subcover.
Let x_1 be some point in the space.
Since $\{x_1\}$ is closed, $X \setminus \{x_1\}$ is open.
Since

∎

Remark. *I do not know if the statement is still true if the condition "the space is metrizable" is replaced with a weaker one. But this is at least true for metric spaces.*

Proposition 14.5.3. *Sequential compactness implies countable compactness.*

Proof.
Assume that every sequence has a convergent subsequence.
We are to prove that every countable open cover has a finite subcover.
Let $\mathcal{U}$ be an arbitrary countable open cover of the space.
Say $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ where each U_i is an open set in the space and $\bigcup_{\lambda\in\mathbb{N}} U_\lambda = X$.
Assume for the sake of contradiction that $\mathcal{U}$ has no finite subcover.
Since $\{U_i\}_{i\leq n}$ is a finite subcollection of $\mathcal{U}$ and $\mathcal{U}$ has no finite subcover, in particular, $\{U_i\}_{i\leq n}$ does not cover the whole space.
Since $\{U_i\}_{i\leq n}$ does not cover the whole space, take $x_n \in X \setminus \bigcup_{i\leq n} U_i$.
Define $\mathfrak{x} := \{x_n\}_{n\in\mathbb{N}}$.
Since $\mathfrak{x}$ is a sequence and any sequence has a convergent subsequence, in particular, $\mathfrak{x}$ has a convergent subsequence $\mathfrak{x}'$.
Say $\mathfrak{x}$ converges to a point x_∞.
Since $\mathcal{U}$ covers the whole space, there exists an index $i_\infty \in \mathbb{N}$ such that $x_\infty \in U_{i_\infty}$.
Since $\mathfrak{x}$ converges to x_∞ and $x_\infty \in U_{i_\infty}$, U_{i_∞} contains a tail of $\mathfrak{x}$.
Since $x_n \in X \setminus \bigcup_{i\leq n} U_i$ for all n, $x_n \notin U_{i_\infty}$ for all $n > i_\infty$.
This contradicts to the fact that U_{i_∞} contains a tail of $\mathfrak{x}$.
So the assumption that $\mathcal{U}$ has no finite subcover is false.
i.e., $\mathcal{U}$ has a finite subcover.
Since any countable open cover has a finite subcover, by definition of countable compactness, the space is countably compact.

∎

Proposition 14.5.4. *Countable compactness can imply sequential compactness if the space is metrizable.*

Proof.

Assume that any countable open cover has a finite subcover.

We are to prove that any sequence has a convergent subsequence.

Let $\mathfrak{x}$ be an arbitrary sequence in the space.

Say $\mathfrak{x} = \{x_i\}_{i \in \mathbb{N}}$ where x_i is a point in the space for each $i \in \mathbb{N}$.

Define $U_i := X \setminus \{x_i\}$.

Define $\mathcal{U} := \{U_i\}_{i \in \mathbb{N}}$.

Since X is open and $\{x_i\}$ is closed, $X \setminus \{x_I\}$ is open.

Since $U_i = X \setminus \{x_i\}$ for each $i \in \mathbb{N}$, $X = \bigcup_{i \in \mathbb{N}} U_i$.

Since U_i is open and $X = \bigcup_{i \in \mathbb{N}} U_i$ and $\mathbb{N}$ is countable, $\mathcal{U}$ is a countable open cover of the space.

Since $\mathcal{U}$ is a countable open cover of the space and any countable open cover of the space has a finite subcover, in particular, $\mathcal{U}$ has a finite subcover $\mathcal{U}'$.

Say $\mathcal{U}' = \{U_i\}_{i \in I}$ where I is a finite subset of $\mathbb{N}$.

■

15

Connectedness

15.1 Definitions

Definition (Separation)**.** *Let (X, τ) be a topological space. Let S be a subset of X. Let S_1 and S_2 be two sets. We say the pair (S_1, S_2) is a* ***separation*** *of S if all of the following conditions hold.*

(1) S_1 and S_2 are both subsets of S.

(2) S_1 and S_2 are both non-empty.

(3) S_1 and S_2 are both open.

(4) $S_1 \cap S_2 = \emptyset$.

(5) $S_1 \cup S_2 = S$.

Definition (Disconnected)**.** *Let (X, τ) be a topological space. We say X is* ***disconnected*** *if there exists a separation of X.*

Definition (Connected)**.** *Let (X, τ) be a topological space. We say X is* ***connected*** *if it is not disconnected.*

15.2 Properties

Proposition 15.2.1. *A space X is connected if and only if the only subsets of X that are both open and closed are $\emptyset$ and X.*

Proof. For one direction, assume that X is connected. We are to prove that the only subsets of X that are both open and closed are $\emptyset$ and X.

Assume for the sake of contradiction that there exists a non-empty proper subset S_0 of X that is both open and closed.
Since S_0 is a non-empty proper subset of X, both S_0 and $C_X(S_0)$ are non-empty.
Since S_0 is closed, $C_X(S_0)$ is open.
Since S_0 and $C_X(S_0)$ are both non-empty and open, $S_0 \cap C_X(S_0) = \emptyset$, and $S_0 \cup C_X(S_0) = X$, we get the pair $(S_0, C_X(S_0))$ is a separation of X.
By definition, we conclude that X is disconnected.
This contradicts to the fact that X is connected.
Thus the only subsets of X that are both open and closed are $\emptyset$ and X.
For the reverse direction, assume that the only subsets of X that are both open and closed are $\emptyset$ and X. We are to prove that X is connected.
Assume for the sake of contradiction that there exists a pair (S_1, S_2) of non-empty proper open subsets of X such that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = X$.
Since S_2 is open, $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2 = X$, we get S_1 is closed.
This contradicts to the fact that the only subsets of X that are both open and closed are $\emptyset$ and X.
Thus X is connected.

■

Proposition 15.2.2. *Let (X, d) be a connected metric space. The only subsets of X that has empty boundaries are X and $\emptyset$.*

Proof. Assume for the sake of contradiction that there exists a non-empty proper-subset S_0 of X such that the boundary of S_0 is empty.
Since $\partial(S_0) = \emptyset$, S_0 is both open and closed.
This contradicts to the fact that the only subsets of X that are both open and closed are X and $\emptyset$.
It follows that the only subsets of X that has empty boundaries are X and $\emptyset$.

■

Proposition 15.2.3. *Let X be a disconnected topological space. Let S_1 and S_2 be a separation of X. Let Y be a connected subset of X. Then either $Y \subseteq S_1$ or $Y \subseteq S_2$.*

Proof. Assume for the sake of contradiction that $Y \nsubseteq S_1$ and $Y \nsubseteq S_2$.
Since $Y \nsubseteq S_1$, there exists a point x_1 such that $x_1 \in Y$ and $x_1 \notin S_1$.
Since $x_1 \notin S_1$, $x_1 \in X$, and $S_1 \cup S_2 = X$, we have $x_1 \in S_2$.
Since $x_1 \in Y$ and $x_1 \in S_2$, we have $x_1 \in Y \cap S_2$.
Since $x_1 \in Y \cap S_2$, $Y \cap S_2$ is not empty.
Since $Y \nsubseteq S_2$, there exists a point x_2 such that $x_2 \in Y$ and $x_2 \notin S_2$.
Since $x_2 \notin S_2$, $x_2 \in X$, and $S_1 \cup S_2 = X$, we have $x_2 \in S_1$.

Since $x_2 \in Y$ and $x_2 \in S_1$, we have $x_2 \in Y \cap S_1$.
Since $x_2 \in Y \cap S_1$, $Y \cap S_1$ is not empty.
Since S_1 is open in X and $Y \subseteq X$, we have $Y \cap S_1$ is open in Y.
Since S_2 is open in X and $Y \subseteq X$, we have $Y \cap S_2$ is open in Y.
Since $S_1 \cap S_2 = \emptyset$, $(Y \cap S_1) \cap (Y \cap S_2) = \emptyset$.
Since $S_1 \cup S_2 = X$ and $Y \subseteq X$, we have $(Y \cap S_1) \cup (Y \cap S_2) = Y$.
Since $Y \cap S_1$ and $Y \cap S_2$ are both not empty, open in Y, $(Y \cap S_1) \cap (Y \cap S_2) = \emptyset$, and $(Y \cap S_1) \cup (Y \cap S_2) = Y$, we have $Y \cap S_1$ and $Y \cap S_2$ is a separation of Y.
Since $Y \cap S_1$ and $Y \cap S_2$ is a separation of Y, by definition, Y is disconnected.
This contradicts to the assumption that Y is connected.
Thus the assumption that $Y \nsubseteq S_1$ and $Y \nsubseteq S_2$ is false.
i.e., $Y \subseteq S_1$ or $Y \subseteq S_2$.

■

15.3 Sufficient Conditions

Proposition 15.3.1. *Let S be a connected topological space. Let T be a set such that $S \subseteq T \subseteq \mathrm{cl}(S)$. Then T is also connected.*

Proof. Assume for the sake of contradiction that there exists a separation (T_1, T_2) of T.
Since (T_1, T_2) is a separation of T and $S \subseteq T$ and S is connected, either $S \subseteq T_1$ or $S \subseteq T_2$.
Assume without loss of generality that $S \subseteq T_1$.
Since $S \subseteq T_1$, by properties of the closure operator, $\mathrm{cl}(S) \subseteq \mathrm{cl}(T_1)$.
Since T_1 and T_2 are both open and $T_1 \cap T_2 = \emptyset$, $\mathrm{cl}(T_1) \cap T_2 = \emptyset$.
Since $\mathrm{cl}(S) \subseteq \mathrm{cl}(T_1)$ and $\mathrm{cl}(T_1) \cap T_2 = \emptyset$, $\mathrm{cl}(S) \cap T_2 = \emptyset$.
Since $T \subseteq \mathrm{cl}(S)$ and $\mathrm{cl}(S) \cap T_2 = \emptyset$, $T \cap T_2 = \emptyset$.
This contradicts to the fact that T_2 is non-empty.
Thus T is connected.

■

Corollary. *The closure of a connected space is also connected.*

Proposition 15.3.2 (Set Operations)**.** *Let X be a topological space. Let $\{S_i\}_{i \in \mathbb{N}}$ be a sequence of connected subspaces of X with $S_i \cap S_{i+1} \neq \emptyset$ for each i. Then the set $\bigcup_{i \in \mathbb{N}} S_i$ is also connected.*

Proof. Let U denote the set $\bigcup_{i \in \mathbb{N}} S_i$. We are to prove that U is connected.
Assume for the sake of contradiction that there exists a separation (U_1, U_2) of U.
Since each S_i is a subset of U and U is separated by (U_1, U_2), either $S_i \subseteq U_1$ or $S_i \subseteq U_2$.
Assume for the sake of contradiction that either for any index i in $\mathbb{N}$, both S_i and S_{i+1} lie in U_1, or for any $i \in \mathbb{N}$, both S_i and S_{i+1} lie in U_2.

Assume without loss of generality that for any index i in $\mathbb{N}$, both S_i and S_{i+1} lie in U_1.
Since for any index i in $\mathbb{N}$, both S_i and S_{i+1} lie in U_1, all of $\{S_i\}_{i\in\mathbb{N}}$ lie in U_1.
Since all of $\{S_i\}_{i\in\mathbb{N}}$ lie in U_1, $U \subseteq U_1$.
Since $U \subseteq U_1$ and $U_1 \subseteq U$, $U_1 = U$.
Since $U_1 = U$ and $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = U$, $U_2 = \emptyset$.
This contradicts to the fact that U_1 is non-empty.
Thus there exists an index i_0 in $\mathbb{N}$ such that either $S_{i_0} \subseteq U_1$ and $S_{i_0+1} \subseteq U_2$ or $S_{i_0} \subseteq U_2$ and $S_{i_0+1} \subseteq U_1$.
Assume without loss of generality that $S_{i_0} \subseteq U_1$ and $S_{i_0+1} \subseteq U_2$.
Since $S_{i_0} \subseteq U_1$ and $S_{i_0+1} \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$, $S_{i_0} \cap S_{i_0+1} = \emptyset$.
This contradicts to the fact that for any index i in $\mathbb{N}$, $S_i \cap S_{i+1} = \emptyset$.
Thus U is connected.

■

Proposition 15.3.3 (Set Operations)**.** *Let X be a topological space. Let $\{S_i\}_{i\in I}$ where I is an index set be a collection of connected subspaces of X such that there exists a point x_0 in X such that $x_0 \in S_i$ for each i. Then the set $\bigcup_{i\in I} S_i$ is also connected.*

Proof. Let U denote the set $\bigcup_{i\in I} S_i$. We are to prove that U is also connected.
Assume for the sake of contradiction that there exists a separation (U_1, U_2) of U.
Since $x_0 \in U$ and $U = U_1 \cup U_2$, either $x_0 \in U_1$ or $x_0 \in U_2$.
Assume without loss of generality that $x_0 \in U_1$.
Since $x_0 \in U_1$ and $U_1 \cap U_2 = \emptyset$, $x_0 \notin U_2$.
Since each S_i is a subset of U and U is separated by (U_1, U_2), either $S_i \subseteq U_1$ or $S_i \subseteq U_2$.
Since U_2 is non-empty, there exists a set S_0 in $\{S_i\}_{i\in I}$ such that $S_0 \subseteq U_2$.
Since $x_0 \notin U_2$ and $S_0 \subseteq U_2$, $x_0 \notin S_0$.
This contradicts to the assumption that $x_0 \in S_0$.
Thus U is connected.

■

Proposition 15.3.4 (Set Operations)**.** *Let X be a topological space. Let $\{S_i\}_{i\in I}$ where I is an index set be a collection of connected sets such that there exists a set S such that $S \cap S_i \neq \emptyset$ for any i. Then the set $S \cup (\bigcup_{i\in I} S_I)$ is also connected.*

Proposition 15.3.5 (Continuous Maps)**.** *The continuous image of a connected set is also connected.*

Comment.

Continuity only guarantees open-ness. The other parts follows from basic set theory and are trivial.

Proof. Let X and Y be topological spaces and f be a function from X to Y.
Let S be a connected subset of X. We are to prove that $f(S)$ is also connected.
Assume for the sake of contradiction that $f(S)$ is disconnected.
Since $f(S)$ is disconnected, there exists a separation (T_1, T_2) of $f(S)$.
Since T_1 is non-empty, $f^{-1}(T_1)$ is non-empty.
Since T_2 is non-empty, $f^{-1}(T_2)$ is non-empty.
Since T_1 is open and f is continuous, $f^{-1}(T_1)$ is open.
Since T_2 is open and f is continuous, $f^{-1}(T_2)$ is open.
Since $T_1 \cap T_2 \neq \emptyset$, from set theory we know that $f^{-1}(T_1) \cap f^{-1}(T_2) \neq \emptyset$.
Since $f(S) \subseteq T_1 \cup T_2$, from set theory we know that $S \subseteq f^{-1}(T_1) \cup f^{-1}(T_2)$.
Since $f^{-1}(T_1)$ and $f^{-1}(T_2)$ are both non-empty and open, $f^{-1}(T_1) \cap f^{-1}(T_2) = \emptyset$, and $S \subseteq f^{-1}(T_1) \cup f^{-1}(T_2)$, we get $(f^{-1}(T_1), f^{-1}(T_2))$ is a separation of S.
Since $(f^{-1}(T_1), f^{-1}(T_2))$ is a separation of S, by definition, S is disconnected.
This contradicts to the fact that S is connected.
Thus $f(S)$ is connected.

■

Proposition 15.3.6 (Cartesian Product)**.** *The cartesian product of a finite collection of connected sets is also connected.*

15.4 Connected Components

Definition (Connected Component)**.** *Let (X, τ) be a topological space. We define the **connected components** of (X, τ) to be the maximal connected subsets of (X, τ).*

Proposition 15.4.1. *The connected components of a space form a partition of it.*

Proposition 15.4.2. *Connected components are closed.*

Proposition 15.4.3. *If the number of connected components of a space is finite, then each component is open.*

15.5 Connectedness on the Real Line

Proposition 15.5.1. *Intervals are connected.*

Theorem 2 (Intermediate Value Theorem)**.** *Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.*

15.6 Pathwise Connectedness

Definition (Pathwise Connectedness)**.** *Let (X, d) be a metric space. We say that X is **pathwise connected** if for any points x and y in X, there exists a continuous function f from $[0, 1]$ to X such that $f(0) = x$ and $f(1) = y$.*

Proposition 15.6.1. *Pathwise connected spaces are connected.*

Claim 1. *The combination of two paths is also a path. i.e., if there exists a path p_1 from x to y and another path p_2 from y to z, then there exists a path p from x to z.*

Proof. By definition of path, we have $p_1(0) = x$, $p_1(1) = y$, $p_2(0) = y$, and $p_2(1) = z$. Define p by $p(x) = p_1(2x)$ if $0 \leq x \leq 1/2$ and $p(x) = p_2(2x - 1)$ if $1/2 \leq x \leq 1$. Notice that at point $x = 1/2$, $p_1(2x) = p_1(1) = y = p_2(2x - 1) = p_2(0) = y$. Since φ_1 given by $\varphi_1(x) = 2x$ is continuous, φ_2 given by $\varphi_2(x) = 2x - 1$ is continuous, and p_1 and p_2 are continuous, p is continuous on both $[0, 1/2]$ and $[1/2, 1]$. Since the value of p given by the two pieces agree at the overlapping point, p is continuous.

■

Claim 2. *If there exists a path from x to y, then there exists a path from y to x.*

Proof. Say p is the path from x to y. i.e., we have p is continuous and $p(0) = x$ and $p(1) = y$. Define p' by $p'(x) = p(1 - x)$. Since the function φ given by $\varphi(x) = 1 - x$ is continuous and p is continuous, $p'(x)$ is continuous. Further, $p'(0) = p(1) = y$ and $p'(1) = p(0) = x$. So p' is a path from y to x.

■

15.7 Totally Disconnectedness

Definition (Totally Disconnected)**.** *Let (X, τ) be a topological space. We say that (X, τ) is **totally disconnected** if all connected components of (X, τ) are singleton sets.*

Proposition 15.7.1. *Subspaces of a totally disconnected space are totally disconnected.*

Proposition 15.7.2. *Cartesian products of totally disconnected spaces are totally disconnected.*

Proposition 15.7.3. *Countable metric spaces are totally disconnected.*

16

Cauchy Completeness

16.1 Definitions

Definition (Cauchy Completeness)**.** *Let (X, τ) be a topological space. We say that (X, τ) is **cauchy complete** if every Cauchy net in (X, τ) converges to some point in X.*

17

Completeness in Metric Spaces

17.1 Definitions

Definition (Completeness)**.** *Let (X, d) be a metric space. Let S be a subset of X. We say that S is* ***complete*** *if any Cauchy sequence in S converges in S.*

Definition (Completeness)**.** *Let (X, d) be a metric space. Let S be a subset of X. We say that S is* ***complete*** *if any infinite totally bounded subset of S has an accumulation point in S.*

Proposition 17.1.1. *The two definitions of completeness are equivalent.*

17.2 Properties

Proposition 17.2.1. *Complete metric spaces are closed.*

Proof. Let (X, d) be a metric space and S be a complete subspace of X. We are to prove that S is closed.
Let $\{x_k\}$ be a convergent sequence in S. Then $\{x_k\}$ is Cauchy.
By definition of completeness, $\{x_k\}$ converges to some point in S.
By definition, we conclude that S is closed.

■

17.3 Sufficient Conditions

Proposition 17.3.1. *Closed subspaces of a complete space are also complete.*

Proof. Let (X, d) be a complete metric space and S be a closed subset of X.

Let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary Cauchy sequence in S. We are to prove that $\{x_k\}$ converges to some point in S.

Since $\{x_k\}$ is Cauchy in X and X is complete, by definition, $\{x_k\}$ converges to some point in X.

Since $\{x_k\}$ converges in X and S is closed, by definition, $\{x_k\}$ converges to some point in S.

In short, we have proved that any Cauchy sequence in S converges to some point in S.

By definition, we conclude that S is complete.

■

Proposition 17.3.2 (Cartesian Product)**.** *Let (X_1, d_1) and (X_2, d_2) be metric spaces. Then the product space $(X_1 \times X_2, D)$ is complete if and only if both (X_1, d_1) and (X_2, d_2) are complete.*

Proof. For one direction, assume $(X_1 \times X_2, D)$ is complete.

We are to prove that (X_1, d_1) and (X_2, d_2) are complete.

Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in (X_1, d_1) and c be an arbitrary point in X_2.

Let ε be an arbitrary positive number.

Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in (X_1, d_1), there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N(\varepsilon)$, we have $d_1(x_m, x_n) < \varepsilon$.

Since $d_1(x_m, x_n) < \varepsilon$ and $d_2(c, c) = 0$, by definition of D, $D((x_m, c), (x_n, c)) < \varepsilon$.

Since for any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N(\varepsilon)$, we have $D((x_m, c), (x_n, c)) < \varepsilon$, $\{(x_i, c)\}_{i\in\mathbb{N}}$ is Cauchy in $(X_1 \times X_2, D)$.

Since $\{(x_i, c)\}_{i\in\mathbb{N}}$ is Cauchy in $(X_1 \times X_2, D)$ and $(X_1 \times X_2, D)$ is complete, $\{(x_i, c)\}_{i\in\mathbb{N}}$ converges in $(X_1 \times X_2, D)$.

Since $\{(x_i, c)\}_{i\in\mathbb{N}}$ converges in $(X_1 \times X_2, D)$, $\{x_i\}_{i\in\mathbb{N}}$ converges in (X_1, d_1).

Since any Cauchy sequence in (X_1, d_1) converges in (X_1, d_1), (X_1, d_1) is complete.

Similarly, (X_2, d_2) is also complete.

For the reverse direction, assume that both (X_1, d_1) and (X_2, d_2) are complete.

We are to prove that $(X_1 \times X_2, D)$ is complete.

Let $\{(x_1^{(i)}, x_2^{(i)})\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $(X_1 \times X_2, D)$.

Since $\{(x_1^{(i)}, x_2^{(i)})\}_{i\in\mathbb{N}}$ is Cauchy in $(X_1 \times X_2, D)$, $\{x_1^{(i)}\}_{i\in\mathbb{N}}$ is Cauchy in (X_1, d_1) and $\{x_2^{(i)}\}_{i\in\mathbb{N}}$ is Cauchy in (X_2, d_2).

Since $\{x_1^{(i)}\}_{i\in\mathbb{N}}$ is Cauchy in (X_1, d_1) and (X_1, d_1) is complete, $\{x_1^{(i)}\}_{i\in\mathbb{N}}$ converges in (X_1, d_1).

Since $\{x_2^{(i)}\}_{i\in\mathbb{N}}$ is Cauchy in (X_2, d_2) and (X_2, d_2) is complete, $\{x_2^{(i)}\}_{i\in\mathbb{N}}$ converges in (X_2, d_2).

Since $\{x_1^{(i)}\}_{i\in\mathbb{N}}$ converges in (X_1, d_1) and $\{x_2^{(i)}\}_{i\in\mathbb{N}}$ converges in (X_2, d_2), $\{(x_1^{(i)}, x_2^{(i)})\}_{i\in\mathbb{N}}$ converges in $(X_1 \times X_2, D)$.

Since any Cauchy sequence in $(X_1 \times X_2, D)$ converges in $(X_1 \times X_2, D)$, $(X_1 \times X_2, D)$ is complete.

■

Theorem 3 (Cantor's Intersection Theorem)**.** *Let (X, d) be a metric space. Then (X, d) is complete if and only if for any decreasing sequence $\{S_i\}_{i\in\mathbb{N}}$ of non-empty closed sets with* $\mathrm{diam}(S_i) = 0$, $\bigcap_{i\in\mathbb{N}} S_i$ *is a singleton set.*

###question. How does this theorem relate to compactness?

Proof. For one direction, assume that (X, d) is complete.
Since S_i is non-empty for all $i \in \mathbb{N}$, there exists a point x_i in S_i for each $i \in \mathbb{N}$.
Let ε be an arbitrary positive number.
Since $\mathrm{diam}(S_i) = 0$, there exists an integer $N(\varepsilon)$ such that $\mathrm{diam}(S_{N(\varepsilon)}) < \varepsilon$.
Since $\{S_i\}_{i\in\mathbb{N}}$ is decreasing, for any indices m and n, if $m, n > N(\varepsilon)$, we have $x_m, x_n \in S_{N(\varepsilon)}$.
Since $\mathrm{diam}(S_{N(\varepsilon)}) < \varepsilon$, by definition of diameter, $d(x_m, x_n) < \varepsilon$.
Since for any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices m and n, if $m, n > N(\varepsilon)$, then $d(x_m, x_n) < \varepsilon$, by definition of Cauchy-ness, $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in (X, d).
Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in (X, d) and (X, d) is complete, by definition of completeness, $\{x_i\}_{i\in\mathbb{N}}$ is convergent in (X, d).
Let x_0 be the limit of $\{x_i\}_{i\in\mathbb{N}}$.
Let S_{i_0} be an arbitrary set in $\{S_i\}_{i\in\mathbb{N}}$.
Since $\{S_i\}_{i\in\mathbb{N}}$ is decreasing, by construction of $\{x_i\}_{i\in\mathbb{N}}$, $\{x_i\}_{i\geq i_0}$ is a sequence in S_{i_0}.
Since $\{x_i\}_{i\in\mathbb{N}}$ converges to x_0 in (X, d), $\{x_i\}_{i\geq i_0}$ also converges to x_0 in (X, d).
Since $\{x_i\}_{i\geq i_0}$ is a sequence in S_{i_0} and converges to x_0 in (X, d) and S_{i_0} is closed, $x_0 \in S_{i_0}$.
Since x_0 is in any set in $\{S_i\}_{i\in\mathbb{N}}$, $x_0 \in \bigcap_{i\in\mathbb{N}} S_i$.
Since $\mathrm{diam}(S_i) = 0$, $\mathrm{diam}(\bigcap_{i\in\mathbb{N}} S_i) = 0$.
Since $\mathrm{diam}(\bigcap_{i\in\mathbb{N}} S_i) = 0$ and $x_0 \in \bigcap_{i\in\mathbb{N}} S_i$, $\bigcap_{i\in\mathbb{N}} S_i$ is a singleton set consisting of only x_0.
For the reverse direction, assume that for any decreasing sequence $\{S_i\}_{i\in\mathbb{N}}$ of non-empty closed sets with $\mathrm{diam}(S_i) = 0$, $\bigcap_{i\in\mathbb{N}} S_i$ is a singleton set.
We are to prove that (X, d) is complete.
Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in (X, d).
Let $S_k = \mathrm{cl}(\{x_i\}_{i\geq k})$.
Since $S_k = \mathrm{cl}(\{x_i\}_{i\geq k})$, $\{S_k\}_{k\in\mathbb{N}}$ is a decreasing sequence of non-empty closed sets in (X, d).
Let ε be an arbitrary positive number.
Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in (X, d), there exists an integer $N(\varepsilon)$ such that for any indices m and n, if $m, n > N(\varepsilon)$, then $d(x_m, x_n) < \varepsilon/2$.
Since for any indices m and n, if $m, n > N$, then $d(x_m, x_n) < \varepsilon/2$, $\mathrm{diam}(\{x_i\}_{i\geq N(\varepsilon)}) < \varepsilon$.
Since $\mathrm{diam}(\{x_i\}_{i\geq N(\varepsilon)}) < \varepsilon$, and $\mathrm{diam}(\{x_i\}_{i\geq N(\varepsilon)}) = \mathrm{diam}(S_{N(\varepsilon)})$, $\mathrm{diam}(S_{N(\varepsilon)}) < \varepsilon$.

Since for any index i, if $i > N(\varepsilon)$, then $\operatorname{diam}(S_i) \leq \operatorname{diam}(S_{N(\varepsilon)})$, and $\operatorname{diam}(S_{N(\varepsilon)}) < \varepsilon$, for any index i, if $i > N(\varepsilon)$, then $\operatorname{diam}(S_i) < \varepsilon$.
Since for any positive number ε, there exists an integer $N(\varepsilon)$ such that for any index i, if $i > N(\varepsilon)$, then $\operatorname{diam}(S_i) < \varepsilon$, by definition of limits, $\operatorname{diam}(S_i) = 0$.
Since $\{S_i\}_{i\in\mathbb{N}}$ is a decreasing sequence of non-empty closed sets with $\operatorname{diam}(S_i) = 0$, by assumption, $\bigcap_{i\in\mathbb{N}} S_i$ is a singleton set.
Let $x_0 \in \bigcap_{i\in\mathbb{N}} S_i$.
Let ε be an arbitrary positive number.
Since $\operatorname{diam}(S_i) = 0$, there exists an integer $N(\varepsilon)$ such that $\operatorname{diam}(S_{N(\varepsilon)}) < \varepsilon$.
Since for any index i, if $i > N(\varepsilon)$, then $x_i \in S_{N(\varepsilon)}$, and $x_0 \in S_{N(\varepsilon)}$, by definition of diameter, for any index i, if $i > N(\varepsilon)$, $d(x_i, x_0) < \operatorname{diam}(S_{N(\varepsilon)})$.
Since $d(x_i, x_0) < \operatorname{diam}(S_{N(\varepsilon)})$ and $\operatorname{diam}(S_{N(\varepsilon)}) < \varepsilon$, $d(x_i, x_0) < \varepsilon$.
Since for any positive number ε, there exists an integer $N(\varepsilon)$ such that for any index i, if $i > N(\varepsilon)$, then $d(x_i, x_0) < \varepsilon$, by definition of limits, $\{x_i\}_{i\in\mathbb{N}}$ converges to x_0.
Since any Cauchy sequence in (X, d) converges, (X, d) is complete.

■

17.4 Metric Completion

$$Cd_1(x, y) \leq d_2(x, y) \leq Dd_1(x, y)$$

Proposition 17.4.1. *Let (X, d_1) and (X, d_2) be equivalent metric spaces. Then they have the same class of convergent sequences and Cauchy sequences.*

18

Baire Space

18.1 Definitions

Definition (Baire Space)**.** *Let (X, τ) be a topological space. We say that (X, τ) is a **Baire Space** if any nonempty open set in the space is not meager.*

Definition (Baire Space)**.** *Let (X, τ) be a topological space. We say that (X, τ) is a **Baire Space** if any comeager set in the space is dense.*

Definition (Baire Space)**.** *Let (X, τ) be a topological space. We say that (X, τ) is a **Baire Space** if the intersection of any countable collection of dense open sets is again dense.*

Definition (Baire Space)**.** *Let (X, τ) be a topological space. We say that (X, τ) is a **Baire Space** if the union of any countable collection of closed nowhere dense sets has empty interior.*

Proposition 18.1.1. *The four definitions of Baire space are equivalent.*

18.2 Properties

Proposition 18.2.1. *An open subspace of a Baire space is again Baire.*

Remark. *A closed subspace of a Baire space is not necessarily Baire.*

19

Uniform Spaces

19.1 Entourages

Definition (Uniform Structure, Entourages)**.** *Let X be a set. Let Φ be a collection of subsets of $X \times X$. We say that Φ is a **uniform structure** and the elements of Φ are **entourages** if Φ satisfies all of the following conditions.*

- *If $U \in \Phi$, then $\Delta \subseteq U$ where $\Delta := \{(x, x) : x \in X\}$ is the diagonal on $X \times X$.*
- *If $U \in \Phi$ and V is a set such that $U \subseteq V \subseteq X \times X$, then $V \in \Phi$.*
- $\forall U, V \in \Phi, \quad U \cap V \in \Phi.$
- $\forall U \in \Phi, \quad \exists V \in \Phi, \quad V \circ V \subseteq U$ *where $\circ$ is a function given by*

$$S \circ T := \{(x, z) : \exists y \in X \text{ such that } (x, z) \in T \text{ and } (z, y) \in S.$$

- $\forall U \in \Phi, \quad U^{-1} \in \Phi$ *where $U^{-1} := \{(y, x) : (x, y) \in U\}$.*

20

Fixed Point Theory

20.1 Definitions

Definition (Contraction Maps)**.** *Let (X, d_X) be a metric space. Let f be a function from X to X. We say that f is a* ***contraction map*** *if $\exists c \in [0,1)$ such that*

$$\forall x_1, x_2 \in X, \quad d(f(x_1), f(x_2)) \leq c \cdot d(x_1, x_2).$$

Definition (Non-Expansive Operator)**.** *Let (X, d) be a metric space. Let f be a function from X to X. We say that f is* ***non-expansive*** *if*

$$\forall x_1, x_2 \in X, \quad d(f(x_1, x_2)) \leq d(x_1, x_2).$$

Or equivalently, $\exists c \in [0,1]$ such that

$$\forall x_1, x_2 \in X, \quad d(f(x_1), f(x_2)) \leq c \cdot d(x_1, x_2).$$

Definition (Firmly Non-Expansive Operator)**.** *Let $\mathcal{H}$ be a Hilbert space. Let f be a function from $\mathcal{H}$ to $\mathcal{H}$. We say that f is* ***firmly non-expansive*** *if*

$$\forall x, y \in \mathcal{H}, \quad \langle f(y) - f(x), f(y) - f(x) \rangle \leq \langle y - x, f(y) - f(x) \rangle.$$

Or equivalently,

$$\forall x, y \in \mathcal{H}, \quad \|f(y) - f(x)\|^2 \leq \langle y - x, f(y) - f(x) \rangle.$$

Definition (Firmly Non-Expansive Operator)**.** *Let $\mathcal{H}$ be a Hilbert space. Let f be a function from $\mathcal{H}$ to $\mathcal{H}$. We say that f is* ***firmly non-expansive*** *if*

$$\forall x, y \in \mathcal{H}, \quad \|f(y) - f(x)\|^2 + \|(I - f)(y) - (I - f)(x)\|^2 \leq \|y - x\|^2.$$

Definition (Firmly Non-Expansive Operator)**.** *Let $\mathcal{H}$ be a Hilbert space. Let f be a function from $\mathcal{H}$ to $\mathcal{H}$. We say that f is **firmly non-expansive** if*

$$\forall x, y \in \mathcal{H}, \quad \langle f(y) - f(x), (I - f)(y) - (I - f)(x) \rangle \geq 0.$$

Definition (Averaged Non-Expansive Operator)**.** *Let $\mathcal{H}$ be a Hilbert space. Let f be a function from $\mathcal{H}$ to $\mathcal{H}$. We say that f is a α**-averaged non-expansive operator** if $\exists \alpha \in (0,1)$ and $\exists$ non-expansive operator N from $\mathcal{H}$ to $\mathcal{H}$ such that*

$$f = \alpha N + (1 - \alpha) I$$

where I denoted the identity operator from $\mathcal{H}$ to $\mathcal{H}$.

20.2 Properties

Proposition 20.2.1. *Firmly non-expansive operators are special cases of averaged non-expansive operators when $\alpha = \frac{1}{2}$.*

Proposition 20.2.2. *Averaged non-expansive operators are non-expansive.*

Proof. Let T be an α-average of a non-expansive operator N. i.e. $T = (1-\alpha)\operatorname{id} + \alpha N$. Let x and y be arbitrary points in $\mathcal{H}$. Then

$$\begin{aligned}
&\|Tx - Ty\| \\
&= \big\| \big[(1-\alpha)\operatorname{id} + \alpha N\big]x - \big[(1-\alpha)\operatorname{id} + \alpha N\big]y \big\| \\
&= \big\| \big[(1-\alpha)x + \alpha Nx\big] - \big[(1-\alpha)y + \alpha Ny\big] \big\| \\
&= \big\| (1-\alpha)(x-y) + \alpha(Nx - Ny) \big\| \\
&\leq \big\| (1-\alpha)(x-y) \big\| + \big\| \alpha(Nx - Ny) \big\| \\
&= (1-\alpha)\big\| x - y \big\| + \alpha \big\| Nx - Ny \big\| \\
&\leq (1-\alpha)\big\| x - y \big\| + \alpha \big\| x - y \big\| \\
&\leq \big\| x - y \big\|.
\end{aligned}$$

That is,

$$\forall x, y \in \mathcal{H}, \quad \|Tx - Ty\| \leq \|x - y\|.$$

So by definition, T is non-expansive. ■

Proposition 20.2.3. *If f is linear, then f being firmly non-expansive is equivalent to the followings.*

- $\forall x \in \mathbb{R}^d$, $\|fx\|^2 \leq \langle x, fx \rangle$.
- $\forall x \in \mathbb{R}^d$, $\langle fx, x - fx \rangle \geq 0$.

20.3 Algebra & Stability of Non-Expansiveness

Proposition 20.3.1. *f is firmly non-expansive if and only if $I - f$ is firmly non-expansive.*

Proposition 20.3.2. *Let M be a linear operator from $\mathbb{R}^d$ to $\mathbb{R}^d$. Then M is non-expansive if and only if M^T is non-expansive.*

Proof. ∎

Proposition 20.3.3 (Convex Combination)**.** *The class of firmly non-expansive maps is closed under convex combinations.*

Proposition 20.3.4 (Composition)**.** *Let f_1 be an α_1-averaged non-expansive operator. Let f_2 be an α_2-averaged non-expansive operator. Define an operator f by $f := f_1 f_2$. Then f is also averaged non-expansive with "rate"*

$$\alpha_f = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2}.$$

20.4 Fixed Points

Proposition 20.4.1. *Contraction maps has at most one fixed point.*

Proposition 20.4.2. *Contraction maps on a complete space has a unique fixed point.*

Proof.
Part 1: construction
Let (X, d) be a complete metric space and f be a contraction map on X.
Let x_0 be a point in X. Construct a sequence $\{x_k\}_{k=1}^{\infty}$ by $x_k = f(x_{k-1})$. We are to prove that $\{x_k\}$ converges to some point x^* in X.

$$d(x_m, x_n) \leq \sum_{k=n+1}^{m} d(x_k, x_{k-1}) \leq \sum_{k=n+1}^{m} K^{k-1} d(x_1, x_0) = \frac{K^n - K^m}{1 - K} d(x_1, x_0)$$

Since $K < 1$, for any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices m and n with $m, n > N$, we have $K^n - K^m < \varepsilon(1 - K)/d(x_1, x_0)$.
Since $d(x_m, x_n) < (K^n - K^m)d(x_1, x_0)/(1 - K)$ and $K^n - K^m < \varepsilon(1 - K)/d(x_1, x_0)$, we get $d(x_m, x_n) < \varepsilon$.
In short, we have proved that for any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices m and n with $m, n > N$, we have $d(x_m, x_n) < \varepsilon$.
By definition, $\{x_k\}$ is Cauchy in X.
Since $\{x_k\}$ is Cauchy in X and X is complete, by definition of completeness, $\{x_k\}$ converges to some point x^* in X.
Part 2: existence.

Since f is a contraction map, f is continuous.
Since f is continuous and $x_k \to x^*$, $f(x^*) = f(\ x_k) = \ f(x_k)$.
Since $x_k \to x^*$ and $x_k = f(x_{k-1})$, $f(x_k) = \ x_k = x^*$.
Since $f(x^*) = \ f(x_k)$ and $f(x_k) = x^*$, $f(x^*) = x^*$.
Part 3: uniqueness.
Assume that there exists a point $x^{'}$ in X such that $f(x^{'}) = x^{'}$.
Compute $d(x^*, x^{'}) = d(f(x^*), f(x^{'})) \leq Kd(x^*, x^{'})$.
That is, $(1 - K)d(x^*, x^{'}) \leq 0$.
Since $K < 1$ and $(1 - K)d(x^*, x^{'}) \leq 0$, $d(x^*, x^{'}) = 0$.
It follows that $x^* = x^{'}$.
Thus x^* is a unique fixed point.

■

20.5 Fejér Monotonic Sequences

Definition (Fejér Monotonic Sequences)**.** *Let $\mathcal{H}$ be a real Hilbert space. Let S be a non-empty subset of $\mathcal{H}$. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathcal{H}$. We say that $\{x_i\}_{i\in\mathbb{N}}$ is **Fejér monotonic** with respect to S if $\forall p \in S$, the sequence $\{\|x_i - p\|\}_{i\in\mathbb{N}}$ is decreasing. i.e.*

$$\forall p \in S, \forall i \in \mathbb{N}, \quad \|x_{i+1} - p\| \leq \|x_i - p\|.$$

Proposition 20.5.1. *Fejér monotonic sequences are bounded.*

Proof.
Let $\mathcal{H}$ be a real Hilbert space.
Let S be a non-empty subset of $\mathcal{H}$.
Let $\mathfrak{x}$ be a Fejér monotonic sequence in $\mathcal{H}$ with respect to S.
Let p be an arbitrary point in S.

$$\begin{aligned} \|x_i\| &\leq \|p\| + \|x_i - p\| \\ &\leq \|p\| + \|x_{i-1} - p\| \\ &\leq \cdots \\ &\leq \|p\| + \|x_1 - p\|. \end{aligned}$$

Since $\forall i \in \mathbb{N}$, $\|x_i\| \leq \|p\| + \|x_1 - p\|$, we get $\{x_i\}_{i\in\mathbb{N}}$ is bounded.

■

Proposition 20.5.2. *Let $\mathcal{H}$ be a real Hilbert space. Let S be a non-empty subset of $\mathcal{H}$. Let $\{x_i\}_{i\in\mathbb{N}}$ be a Fejér monotonic sequence in $\mathcal{H}$ with respect to S. Then $\forall p \in S$, the sequence $\{x_i - p\}_{i\in\mathbb{N}}$ converges.*

Proof. $\|x_i - p\|_{i\in\mathbb{N}}$ is a decreasing sequence bounded below by 0. ■