Convex Analysis

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Contents

Chapter 1

Affine Sets

1.1 Definitions

Definition (Affine Combination)**.** *Let S be a set in $\mathbb{E}$. We define an* ***affine combination*** *of S to be a point x in the space of the form*

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where (1) $n \in \mathbb{N}$, (2) $v_i \in S$ for all i, (3) $\lambda_i \in \mathbb{R}$ for all i, and (4) $\sum_{i=1}^{n} \lambda_i = 1$.

Definition (Affine Span)**.** *Let S be a set in $\mathbb{E}$. We define the* ***affine span*** *of S, denoted by $\operatorname{affspan}(S)$, to be the set of all affine combinations of S.*

Definition (Affine Set)**.** *Let S be a set in $\mathbb{E}$. We say that S is an* ***affine set*** *if $S = \operatorname{aff}(S)$.*

Definition (Affine Hull)**.** *Let S be a set in $\mathbb{E}$. We define the* ***affine hull*** *of S, denoted by $\operatorname{affhull}(S)$, to be the smallest affine set containing S.*

Chapter 2

Relative Topology

2.1 Definitions

Definition (Relative Interior)**.** *Let $\mathbb{E}$ be some Euclidean space. Let S be a set in the space. We define the **relative interior** of S, denoted by $\mathrm{ri}(S)$, to be the interior of S for the topology relative to the affine hull $\mathrm{aff}(S)$. i.e., the set given by*

$$\mathrm{ri}(S) := \{x \in \mathrm{aff}(S) : \exists r > 0, ball(x, r) \cap \mathrm{aff}(S) \subseteq S\}.$$

A quick result. For a singleton set S, $\mathrm{ri}(S) = S = \mathrm{cl}(S)$.

2.2 Basic Properties

Proposition 2.2.1. *For any set S, we have $\mathrm{ri}(S) \subseteq S$.*

Remark. *The relative interior operator is not monotonic.*

Example 2.2.1. *Consider $\mathbb{R}$ with the usual topology and sets $\{0\}$ and $[0,1]$. Then $\mathrm{ri}(\{0\}) = \{0\}$ and $\mathrm{ri}([0,1]) = (0,1)$.*

Proposition 2.2.2. *Let S be a set in some Euclidean space $\mathbb{E}$. Then if $int(S) \neq \emptyset$, $\mathrm{ri}(S) = int(S)$.*

Proof.

It suffices to show that $\mathrm{aff}(S) = \mathbb{R}^n$.

Since $int(S) \neq \emptyset$, $\exists x \in int(S)$.

Since $x \in int(S)$, $\exists r > 0, ball(x, r) \subseteq S$.

$\mathbb{E} = \mathrm{aff}(ball(x, r)) \subseteq \mathrm{aff}(S) \subseteq \mathbb{E}$.

This shows $\mathrm{aff}(S) = \mathbb{E}$.

■

2.3 Arithmetic Properties

Proposition 2.3.1. *Let C_1 and C_2 be convex subsets of $\mathbb{E}$. Let λ_1 and λ_2 be scalars in $\mathbb{R}$. Then*

$$\operatorname{ri}(\lambda_1 C_1 + \lambda_2 C_2) = \lambda_1 \operatorname{ri}(C_1) + \lambda_2 \operatorname{ri}(C_2).$$

Proposition 2.3.2. *Let C_1 be a convex set in $\mathbb{E}_1$. Let C_2 be a convex set in $\mathbb{E}_2$. Then*

$$\operatorname{ri}(C_1 \oplus C_2) = \operatorname{ri}(C_1) \oplus \operatorname{ri}(C_2).$$

Chapter 3

Convex Sets

3.1 Definitions

Definition (Convex Combination)**.** *Let S be a set in $\mathbb{E}$. We define a* ***convex combination*** *of S to be a point x in the space of the form*

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where (1) $n \in \mathbb{N}$, (2) $v_i \in S$ for all i, (3) $\lambda_i \in \mathbb{R}_+$ for all i, and (4) $\sum_{i=1}^{n} \lambda_i = 1$.

Definition (Convex Span)**.** *Let S be a set in $\mathbb{E}$. We define a* ***convex span*** *of S, denoted by* $\operatorname{convspan}(S)$*, to be the set of all convex combinations of S.*

Definition (Convex Sets)**.** *Let S be a set in $\mathbb{E}$. We say that S is* **convex** *if $S = convspan(S)$, or equivalently, $\alpha x + \beta y \in S$ for any $x, y \in S$ and any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$.*

Definition (Convex Hull)**.** *Let S be a set in $\mathbb{E}$. We define the* ***convex hull*** *of S, denoted by* $\operatorname{convhull}(S)$*, to be the smallest convex set containing S.*

Definition (Closed Convex Hull)**.** *Let S be a set in some Euclidean space. We define the* ***closed convex hull*** *of S to be the intersection of all closed convex supersets of S.*

Proposition 3.1.1. *For any set S,* $\operatorname{convspan}(S) = \operatorname{convhull}(S)$*. They will both be denoted by* $\operatorname{conv}(S)$ *from now on.*

Proof.

For one direction, let x be an arbitrary point in $convspan(S)$.

We are to prove that $x \in convhull(S)$.

Let C be an arbitrary convex set containing S.

Since x is a convex combination of S, x is also a convex combination of C.

Since x is a convex combination of C and C is convex, $x \in C$.
Since x is in any convex set containing S, $x \in convhull(S)$.
Since $x \in convhull(S)$ for any $x \in convspan(S)$, $convspan(S) \subseteq convhull(S)$.
For the reverse direction,

proof incomplete.

■

Proposition 3.1.2. *The closed convex hull is the closure of the convex hull.*

3.2 Basic Properties

Proposition 3.2.1 (The conv Operator)**.**

(1) For any set S in a Euclidean space, we have

$$S \subseteq \text{conv}(S).$$

(2) (Monotonic) For any sets S_1 and S_2 in $\mathbb{E}$, if $S_1 \subseteq S_2$, then

$$\text{conv}(S_1) \subseteq \text{conv}(S_2).$$

(3) (Idempotent) For any set S in $\mathbb{E}$, we have

$$\text{conv}(\text{conv}(S)) = \text{conv}(S).$$

Theorem 1 (Carathéodory)**.** *Let $\mathbb{E}$ be some Euclidean space. Let S be some set in the space. Let x be some point in $\text{conv}(S)$. Then x can be represented as a convex combination of at most $d+1$ points in S. i.e., x lies in some r-simplex with vertices in S, where $r \leq d$.*

3.3 Arithmetic Properties

Proposition 3.3.1. *Let C be a convex set. Let λ_1 and λ_2 be in $\mathbb{R}_+$. Then*

$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C.$$

Proof.

The case where any of λ_1 and λ_2 is 0 is trivial. I will assume that $\lambda_1, \lambda_2 > 0$.
For one direction, let x be an arbitrary point in $(\lambda_1 + \lambda_2)C$.
Since $x \in (\lambda_1 + \lambda_2)C$, $\exists c \in C, x = (\lambda_1 + \lambda_2)c$.
Since $\begin{cases} (\lambda_1 + \lambda_2)c = \lambda_1 c + \lambda_2 c \\ x = (\lambda_1 + \lambda_2)c \end{cases}$, we get $x = \lambda_1 c + \lambda_2 c$.

Since $\begin{cases} x = \lambda_1 c + \lambda_2 c \\ \lambda_1 c \in \lambda_1 C \\ \lambda_2 c \in \lambda_2 C \end{cases}$, we get $x \in \lambda_1 C + \lambda_2 C$.

Since $x \in \lambda_1 C + \lambda_2 C$ for any $x \in (\lambda_1 + \lambda_2)C$, $\lambda_1 C + \lambda_2 C \subseteq (\lambda_1 + \lambda_2)C$.

For the reverse direction, let x be an arbitrary point in $\lambda_1 C + \lambda_2 C$.

Since $x \in \lambda_1 C + \lambda_2 C$, $\exists c_1, c_2 \in C, x = \lambda_1 c_1 + \lambda_2 c_2$.

Define scalars $\mu_1 := \frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\mu_2 := \frac{\lambda_2}{\lambda_1+\lambda_2}$.

Then $x = (\lambda_1 + \lambda_2)c$.

Since $\lambda_1, \lambda_2 > 0$, $\mu_1, \mu_2 \in [0, 1]$.

Define point $c := \mu_1 c_1 + \mu_2 c_2$.

Since $\begin{cases} c = \mu_1 c_1 + \mu_2 c_2 \\ c_1, c_2 \in C \\ \mu_1, \mu_2 \in [0,1] \\ \mu_1 + \mu_2 = 1 \\ C \text{ is convex} \end{cases}$, we get $c \in C$.

Since $x = (\lambda_1 + \lambda_2)c$ and $c \in C$, $x \in \lambda_1 + \lambda_2)C$.

Since $x \in \lambda_1 + \lambda_2)C$ for any $x \in \lambda_1 C + \lambda_2 C$, $(\lambda_1 + \lambda_2)C \subseteq \lambda_1 C + \lambda_2 C$.

Since $\lambda_1 C + \lambda_2 C \subseteq (\lambda_1 + \lambda_2)C$ and $(\lambda_1 + \lambda_2)C \subseteq \lambda_1 C + \lambda_2 C$, $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$

■

3.4 Stability of Convexity

Proposition 3.4.1. *Convexity is stable under intersection. i.e., the intersection of any collection of convex sets is convex.*

Proof. Let $\{C_i\}_{i\in I}$ be an arbitrary collection of convex sets where I is an index set and C_i is convex for any $i \in I$. Let C denote their intersection. If $C = \emptyset$, then we are done. Else, let x and y be two arbitrary points in C. Let λ be an arbitrary number in $(0, 1)$. Define a point $z := \lambda x + (1 - \lambda)y$. Since $x \in C$ and $C = \bigcap_{i\in I} C_i$, we get $x \in C_i$ for any $i \in I$. Since $y \in C$ and $C = \bigcap_{i\in I} C_i$, we get $y \in C_i$ for any $i \in I$. Let i be an arbitrary index in I. Since $x \in C_i$ and $y \in C_i$ and $\lambda \in (0, 1)$ and C_i is convex and $z = \lambda x + (1 - \lambda)y$, we get $z \in C_i$. Since $z \in C_i$ for any $i \in I$ and $C = \bigcap_{i\in I} C_i$, we get $z \in C$. Since

$$\forall x, y \in C, \forall \lambda \in (0, 1), \quad \lambda x + (1 - \lambda)y \in C,$$

by definition of convex sets, we get C is convex. ■

Proposition 3.4.2. *Convexity is stable under affine mapping. i.e., the affine image of a convex set is convex.*

Proposition 3.4.3 (Linear Combinations). *Convexity is stable under linear combination. i.e., if C_1 and C_2 are convex sets and λ_1 and λ_2 are real numbers, then the set $C := \lambda_1 C_1 + \lambda_2 C_2$ is convex.*

Proof.

If $C_1 = \emptyset$ or $C_2 = \emptyset$, then $\lambda_1 C_1 + \lambda_2 C_2 = \emptyset$ and we are done.
Now assume $C_1, C_2 \neq \emptyset$.
Let x and y be arbitrary points in C.
Since $x \in C$,

$$\exists x_1 \in C_1, x_2 \in C_2 \text{ such that } x = \lambda_1 x_1 + \lambda_2 x_2.$$

Since $y \in C$,

$$\exists y_1 \in C_1, y_2 \in C_2 \text{ such that } y = \lambda_1 y_1 + \lambda_2 y_2.$$

Let $\lambda \in [0, 1]$ be arbitrary.
Define $z := \lambda x + (1 - \lambda) y$.

$$\text{Since } \begin{cases} z = \lambda x + (1-\lambda) y \\ x = \lambda_1 x_1 + \lambda_2 x_2 \\ y = \lambda_1 y_1 + \lambda_2 y_2 \end{cases} \text{, substitute } x \text{ and } y \text{ into } z \text{, we get}$$

$$\begin{aligned} z &= \lambda\big(\lambda_1 x_1 + \lambda_2 x_2\big) + (1-\lambda)\big(\lambda_1 y_1 + \lambda_2 y_2\big) \\ &= \lambda_1\big(\lambda x_1 + (1-\lambda) y_1\big) + \lambda_2\big(\lambda x_2 + (1-\lambda) y_2\big). \end{aligned}$$

Since (1) $x_1, y_1 \in C_1$ and (2) $\lambda \in [0, 1]$ and (3) C_1 is convex, we get

$$\lambda x_1 + (1-\lambda) y_1 \in C_1.$$

Since (1) $x_2, y_2 \in C_2$ and (2) $\lambda \in [0, 1]$ and (3) C_2 is convex, we get

$$\lambda x_2 + (1-\lambda) y_2 \in C_2.$$

$$\text{Since } \begin{cases} z = \lambda_1\big(\lambda x_1 + (1-\lambda) y_1\big) + \lambda_2\big(\lambda x_2 + (1-\lambda) y_2\big) \\ \lambda x_1 + (1-\lambda) y_1 \in C_1 \\ \lambda x_2 + (1-\lambda) y_2 \in C_2 \end{cases} \text{, we get}$$

$$z \in \lambda_1 C_1 + \lambda_2 C_2.$$

Since $\forall x \in C, \forall y \in C, \forall \lambda \in [0, 1], \lambda x + (1-\lambda) y \in C$, we get C is convex.

∎

Corollary. *The Minkowski sum of two convex sets is convex.*

Lemma 1. *Let C be a convex set in $\mathbb{E}$. Let $x \in int(C)$. Let $y \in \mathrm{cl}(C)$. Then*

$$\forall \lambda \in (0, 1], \quad \lambda x + (1-\lambda) y \in C.$$

Proof.

Since $x \in int(S)$, there exists some radius r_x such that $\text{ball}(x, r_x) \subseteq S$.

Define $r_z := \lambda r_x$.

Let z' be an arbitrary point in $\text{ball}(z, r_z)$.

Define $x' := \frac{1}{\lambda}(z' - (1-\lambda)y)$.

Notice

$$\begin{aligned}
&\|x - x'\| \\
&= \frac{1}{|\lambda|}\|\lambda x - \lambda x'\| \\
&= \frac{1}{|\lambda|}\big\|(z - (1-\lambda)y) - (z' - (1-\lambda)y)\big\| \\
&= \frac{1}{|\lambda|}\|z - z'\| \\
&\leq \frac{1}{|\lambda|}r_z, \text{ since } z' \in \text{ball}(z, r_z) \\
&= \frac{1}{|\lambda|}\lambda r_x \\
&= r_x.
\end{aligned}$$

That is,

$$\|x - x'\| \leq r_x.$$

So $x' \in \text{ball}(x, r_x)$.

Since $x' \in ball(x, r_x)$ and $ball(x, r_x) \subseteq S$, we get $x' \in S$.

$$\text{Since } \begin{cases} z' = \lambda x' + (1-\lambda)y \\ x', y \in S \\ \lambda \in (0, 1] \\ S \text{ is convex} \end{cases}, \text{ we get } z' \in S.$$

Since $z' \in S$ for any $z' \in ball(z, r_z)$, $ball(z, r_z) \subseteq S$.

Since there exists some radius r_z such that $ball(z, r_z) \subseteq S$, $z \in int(S)$.

Alternative Expressing:

Define $B := ball(0, 1)$.

$$\begin{aligned}
&(1-\lambda)x + \lambda y + \varepsilon B \\
&\subseteq (1-\lambda)x + \lambda(C + \varepsilon B) + \varepsilon B \\
&= (1-\lambda)x + \lambda C + \lambda\varepsilon B + \varepsilon B \\
&= (1-\lambda)x + (1+\lambda)\varepsilon B + \lambda C \\
&= (1-\lambda)(x + \frac{1+\lambda}{1-\lambda})\varepsilon B) + \lambda C
\end{aligned}$$

$$\subseteq (1-\lambda)C + \lambda C$$
$$= C.$$

■

Lemma 2. *Let C be a convex set in $\mathbb{E}$. Let $x \in \mathrm{ri}(C)$. Let $y \in \mathrm{cl}(C)$. Then*

$$\forall \lambda \in (0,1], \quad \lambda x + (1-\lambda)y \in C.$$

Proof.

Case 1. $int(C) \neq \emptyset$.

Then $int(C) = \mathrm{ri}(C)$.

Since $x \in int(C)$ and $y \in \mathrm{cl}(C)$, $\forall t \in (0,1], z := tx + (1-t)y \in C$.

Case 2. $int(C) = \emptyset$.

Now $\dim(C) < d$.

Say $\dim(C) = l$.

Apply case 1 in $\mathbb{R}^l$.

■

Proposition 3.4.4. *Convexity is stable under interior. i.e., the interior of a convex set is convex.*

Proof. Let S be a convex set in $\mathbb{E}$. If $int(S) = \emptyset$, then we are done. Else: let x and y be two arbitrary points in $int(S)$. Let λ be an arbitrary number in $(0,1)$. Define a point z by $z := \lambda x + (1-\lambda)y$. Since $x, y \in int(S)$ and $\lambda \in (0,1)$, by the lemma, we get $z \in int(S)$. Since

$$\forall x, y \in int(S), \forall \lambda \in (0,1), \quad \lambda x + (1-\lambda)y \in int(S),$$

we get $int(S)$ is convex. ■

Proposition 3.4.5. *Convexity is stable under relative interior. i.e., the relative interior of a convex set is convex.*

Proof. Let S be a convex set in $\mathbb{E}$. If $\mathrm{ri}(S) = \emptyset$, then we are done. Else: let x and y be two arbitrary points in $\mathrm{ri}(S)$. Let λ be an arbitrary number in $(0,1)$. Define a point z by $z := \lambda x + (1-\lambda)y$. Since $x, y \in \mathrm{ri}(S)$ and $\lambda \in (0,1)$, by the lemma, we get $z \in \mathrm{ri}(S)$. Since

$$\forall x, y \in \mathrm{ri}(S), \forall \lambda \in (0,1), \quad \lambda x + (1-\lambda)y \in \mathrm{ri}(S),$$

we get $\mathrm{ri}(S)$ is convex. ■

Proposition 3.4.6. *Convexity is stable under closure. i.e., the closure of a convex set is convex.*

Proof Approach 1.

Let $x, y \in \text{cl}(C)$.

Let $t \in [0,1]$.

Since $x \in \text{cl}(C)$, $\exists \{x_i\}_{i\in\mathbb{N}} \subseteq C, \lim_{i\to\infty} x_i = x$.

Since $y \in \text{cl}(C)$, $\exists \{y_i\}_{i\in\mathbb{N}} \subseteq C, \lim_{i\to\infty} y_i = y$.

Since $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} y_i = y$, $\lim_{i\to\infty} \big(tx_i + (1-t)y_i\big) = tx + (1-t)y$.

Since $x_i, y_i \in C$ and C is convex, $tx_i + (1-t)y_i \in C$.

Since $tx_i + (1-t)y_i \in C$ $\lim_{i\to\infty} \big(tx_i + (1-t)y_i\big) = tx + (1-t)y$, $tx + (1-t)y \in \text{cl}(C)$.

Since $\forall x, y \in \text{cl}(C), \forall t \in [0,1], tx + (1-t)y \in \text{cl}(C)$, we get $\text{cl}(C)$ is convex.

■

Proof Approach 2.

$\text{cl}(C) = \bigcap_{\varepsilon>0} \big[C + \varepsilon\,\text{ball}(0,1)\big]$. This is an intersection of linear combinations of convex sets and hence convex.

■

Proposition 3.4.7. *Convexity is stable under conical hull. i.e., if C is convex, then* $\text{cone}(C)$ *is convex.*

Proof.

Let x and y be arbitrary points in $\text{cone}(C)$.

Let λ be an arbitrary number in $(0,1)$.

Define point z as $z := \lambda x + (1-\lambda)y$.

Since $x \in \text{cone}(C)$, $\exists x' \in C$ and $\exists \alpha > 0$ such that $x = \alpha x'$.

Since $y \in \text{cone}(C)$, $\exists y' \in C$ and $\exists \beta > 0$ such that $y = \beta y'$.

Define point z' as $z' := \frac{\lambda\alpha}{\lambda\alpha+(1-\lambda)\beta}x' + \frac{(1-\lambda)\beta}{\lambda\alpha+(1-\lambda)\beta}y'$.

Since $x', y' \in C$ and $\frac{\lambda\alpha}{\lambda\alpha+(1-\lambda)\beta} \in (0,1)$ and $\frac{\lambda\alpha}{\lambda\alpha+(1-\lambda)\beta} + \frac{(1-\lambda)\beta}{\lambda\alpha+(1-\lambda)\beta} = 1$ and C is convex and $z' := \frac{\lambda\alpha}{\lambda\alpha+(1-\lambda)\beta}x' + \frac{(1-\lambda)\beta}{\lambda\alpha+(1-\lambda)\beta}y'$, we get $z' \in C$.

Since $z' \in C$ and $z = (\lambda\alpha + (1-\lambda)\beta)z'$, $z \in \text{cone}(C)$.

That is, $\lambda x + (1-\lambda)y \in \text{cone}(C)$.

Since $\forall x, y \in \text{cone}(C)$, $\forall \lambda \in (0,1)$, $\lambda x + (1-\lambda)y \in \text{cone}(C)$, we get $\text{cone}(C)$ is convex.

■

3.5 Topological Properties

Proposition 3.5.1. *A closed convex hull does not distinguish a set from its closure. i.e., for any set S, we have $\overline{\text{conv}}(S) = \overline{\text{conv}}(cl(S))$.*

Proposition 3.5.2. *The convex hull of a bounded set is bounded.*

Proposition 3.5.3. *The convex hull of a compact set is compact.*

Proposition 3.5.4. *If S is bounded, then the closure operation and the convex hull operation commute. i.e.,* $\operatorname{conv}(\operatorname{cl}(S)) = \operatorname{cl}(\operatorname{conv}(S))$.

Remark. *The closure operation and the convex hull operation do not commute in general.*

Theorem 2. *Let C be a convex set such that $int(C) \neq \emptyset$. Then*

(1) $int(C) = int(\operatorname{cl}(C))$, and

(2) $\operatorname{cl}(C) = \operatorname{cl}(int(C))$.

Proof of (1). $int(C) \subseteq int(\operatorname{cl}(C))$ is clear. For $int(\operatorname{cl}(C)) \subseteq int(C)$, let x be an arbitrary point in $int(\operatorname{cl}(C))$.

Since $x \in int(\operatorname{cl}(C))$,

$$\exists r > 0 \text{ such that } \operatorname{ball}(x, r) \subseteq \operatorname{cl}(C).$$

Since $int(C) \neq \emptyset$, pick $y \in int(C)$.

Define a scalar λ by

$$\lambda := \frac{r}{2\|x - y\|}.$$

Define a point z by

$$z := x + \lambda(x - y).$$

Since $\lambda = \frac{r}{2\|x-y\|}$ and $z = x + \lambda(x - y)$,

$$\begin{aligned}
&\|z - x\| \\
&= \big\|x + \lambda(x - y) - x\big\| \\
&= \big\|\lambda(x - y)\big\| \\
&= \lambda\|x - y\| \\
&= \frac{r}{2\|x - y\|}\|x - y\| \\
&= \frac{r}{2} \\
&< r.
\end{aligned}$$

That is,

$$\|z - x\| < r.$$

So $z \in \operatorname{ball}(x, r)$. It follows that $z \in \operatorname{cl}(C)$.

Since $z = x + \lambda(x - y)$, rearranging this yields

$$x = \frac{1}{1 + \lambda} z + \frac{\lambda}{1 + \lambda} y.$$

Since $\begin{cases} x = \frac{1}{1+\lambda} z + \frac{\lambda}{1+\lambda} y \\ z \in \mathrm{cl}(C) \\ y \in int(C) \\ \frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda} \in (0,1) \\ \frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} = 1 \end{cases}$, by the lemma, we get

$$x \in int(C).$$

Since $\forall x \in int(\mathrm{cl}(C)), x \in int(C)$, we get $int(\mathrm{cl}(C)) \subseteq int(C)$.

■

Proof of (2). $\mathrm{cl}(int(C)) \subseteq \mathrm{cl}(C)$ is clear. For $\mathrm{cl}(C) \subseteq \mathrm{cl}(int(C))$, let x be an arbitrary point in $\mathrm{cl}(C)$.

Since $int(C) \neq \emptyset$, pick $y \in int(C)$.

Let $\lambda \in [0,1)$.

Define a point z by

$$z(\lambda) := \lambda x + (1-\lambda) y.$$

Since $\begin{cases} z(\lambda) := \lambda x + (1-\lambda) y \\ x \in \mathrm{cl}(C) \\ y \in int(C) \\ \lambda \in [0,1) \end{cases}$, by the lemma, we get

$$z(\lambda) \in int(C).$$

Since $\begin{cases} z(\lambda) \in int(C) \\ \lim_{\lambda \to 1} z(\lambda) = x \end{cases}$, we get

$$x \in \mathrm{cl}(int(C)).$$

Since $\forall x \in \mathrm{cl}(C), x \in \mathrm{cl}(int(C))$, we get $\mathrm{cl}(C) \subseteq \mathrm{cl}(int(C))$.

■

Proposition 3.5.5. *If C is convex, then*

(1) $\mathrm{aff}(\mathrm{ri}(C)) = \mathrm{aff}(C) = \mathrm{aff}(\mathrm{cl}(C))$,

(2) $\mathrm{ri}(\mathrm{ri}(C)) = \mathrm{ri}(C) = \mathrm{ri}(\mathrm{cl}(C))$, *and*

(3) $\mathrm{cl}(\mathrm{ri}(C)) = \mathrm{cl}(C) = \mathrm{cl}(\mathrm{cl}(C))$.

Proposition 3.5.6. *Let C be a convex set. Then*

$$C \neq \emptyset \Longleftrightarrow \mathrm{ri}(C) \neq \emptyset.$$

Proof.

For one direction, assume that $C \neq \emptyset$. We are to prove that $\mathrm{ri}(C) \neq \emptyset$. Since $C \neq \emptyset$, $\mathrm{aff}(C) \neq \emptyset$. Since C is convex, $\mathrm{aff}(C) = \mathrm{aff}(\mathrm{ri}(C))$. Since $\begin{cases} \mathrm{aff}(C) \neq \emptyset \\ \mathrm{aff}(C) = \mathrm{aff}(\mathrm{ri}(C)) \end{cases}$, we get

$$\mathrm{aff}(\mathrm{ri}(C)) \neq \emptyset.$$

Since $\mathrm{aff}(\mathrm{ri}(C)) \neq \emptyset$, we get $\mathrm{ri}(C) \neq \emptyset$.

For the reverse direction, assume that $\mathrm{ri}(C) \neq \emptyset$. We are to prove that $C \neq \emptyset$. Since $\mathrm{ri}(C) \neq \emptyset$ and $\mathrm{ri}(C) \subseteq C$, we get $C \neq \emptyset$.

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3.6 Examples of Convex Sets

Example 3.6.1. *Let I be an index set. Let b_i for $i \in I$ be vectors in $\mathbb{E}$. Let β_i for $i \in I$ be reals. Then the set C given by*

$$C := \{x \in \mathbb{E} : \forall i \in I, \langle x, b_i \rangle \leq \beta_i\}$$

is convex.

Proof.

Each of $C_i := \{x \in \mathbb{E} : \langle x, b_i \rangle \leq \beta_i\}$ is convex and $C = \bigcap_{i \in I} C_i$.

$$\begin{aligned} \langle z, b_i \rangle &= \langle \lambda x + (1-\lambda) y, b_i \rangle \\ &= \lambda \langle x, b_i \rangle + (1-\lambda) \langle y, b_i \rangle \\ &\leq \lambda \beta_i + (1-\lambda) \beta_i \\ &= \beta_i. \end{aligned}$$

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Chapter 4

Cones (in Analysis)

4.1 Definitions

Definition (Conical Combination)**.** *Let S be a set in $\mathbb{E}$. We define a **conical combination** of S to be a point x in the space of the form*

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where (1) $n \in \mathbb{N}$, (2) $v_i \in S$ for all i, and (3) $\lambda_i \in \mathbb{R}_{++}$ for all i.

Definition (Cone)**.** *Let S be a set in $\mathbb{E}$. We say that S is a **cone** if $S = \mathbb{R}_{++}S$.*

Definition (Conical Hull)**.** *Let S be a set in $\mathbb{E}$. We define the **conical hull** of S, denoted by $\operatorname{cone}(S)$, to be the intersection of all cones containing C.*

Proposition 4.1.1. *Let S be a set in $\mathbb{E}$. Then $\operatorname{cone}(S) = \mathbb{R}_{++}S$.*

Proof. For one direction, we are to prove $\operatorname{cone}(S) \subseteq \mathbb{R}_{++}S$. Since $\mathbb{R}_{++}\mathbb{R}_{++}S = \mathbb{R}_{++}S$, $\mathbb{R}_{++}S$ is a cone. Since $1 \in \mathbb{R}_{++}$, $S \subseteq \mathbb{R}_{++}$. Since $\mathbb{R}_{++}S$ is a cone containing S and $\operatorname{cone}(S)$ is the smallest cone containing S, we get

$$\operatorname{cone}(S) \subseteq \mathbb{R}_{++}S.$$

For the reverse direction, we are to prove $\mathbb{R}_{++}S \subseteq \operatorname{cone}(S)$. Let C be an arbitrary cone containing S. Since $S \subseteq C$, $\mathbb{R}_{++}S \subseteq \mathbb{R}_{++}C$. Since C is a cone, $\mathbb{R}_{++}C = C$. So $\mathbb{R}_{++}S \subseteq C$. Since $\mathbb{R}_{++}S \subseteq C$ for any cone C containing S, we get

$$\mathbb{R}_{++}S \subseteq \operatorname{cone}(S).$$

■

Definition (Closed Conical Hull)**.** *Let S be a set in $\mathbb{E}$. We define the **closed conical hull** of S, denoted by* $\operatorname{clcone}(S)$*, to be the intersection of all closed cones containing C.*

Proposition 4.1.2. *For any set S in $\mathbb{E}$, we have*

$$\operatorname{clcone}(S) = \operatorname{cl}(\operatorname{cone}(S)).$$

Proof.

For $\operatorname{clcone}(S) \subseteq \operatorname{cl}(\operatorname{cone}(S))$.

Since $\operatorname{cl}(\operatorname{cone}(S))$ is a closed cone containing S and $\operatorname{clcone}(S)$ is the smallest closed cone containing S, $\operatorname{clcone}(S) \subseteq \operatorname{cl}(\operatorname{cone}(S))$.

For $\operatorname{cl}(\operatorname{cone}(S)) \subseteq \operatorname{clcone}(S)$.

Since $S \subseteq \operatorname{clcone}(S)$, by the monotonicity of the cone operator, $\operatorname{cone}(S) \subseteq \operatorname{cone}(\operatorname{clcone}(S))$.

Since $\operatorname{cone}(S) \subseteq \operatorname{cone}(\operatorname{clcone}(S))$, by the monotonicity of the closure operator, $\operatorname{cl}(\operatorname{cone}(S)) \subseteq \operatorname{cl}(\operatorname{cone}(\operatorname{clcone}(S)))$.

Since $\operatorname{clcone}(S)$ is a cone, $\operatorname{cone}(\operatorname{clcone}(S)) = \operatorname{clcone}(S)$.

Since $\operatorname{clcone}(S)$ is closed, $\operatorname{cl}(\operatorname{clcone}(S)) = \operatorname{clcone}(S)$.

Since $\operatorname{cone}(\operatorname{clcone}(S)) = \operatorname{clcone}(S)$ and $\operatorname{cl}(\operatorname{clcone}(S)) = \operatorname{clcone}(S)$, we get $\operatorname{cl}(\operatorname{cone}(\operatorname{clcone}(S))) = \operatorname{clcone}(S)$.

Since $\operatorname{cl}(\operatorname{cone}(S)) \subseteq \operatorname{cl}(\operatorname{cone}(\operatorname{clcone}(S)))$ and $\operatorname{cl}(\operatorname{cone}(\operatorname{clcone}(S))) = \operatorname{clcone}(S)$, we get $\operatorname{cl}(\operatorname{cone}(S)) \subseteq \operatorname{clcone}(S)$.

Since $\operatorname{clcone}(S) \subseteq \operatorname{cl}(\operatorname{cone}(S))$ and $\operatorname{cl}(\operatorname{cone}(S)) \subseteq \operatorname{clcone}(S)$, we get $\operatorname{clcone}(S) = \operatorname{cl}(\operatorname{cone}(S))$.

■

4.2 The cone Operator

Proposition 4.2.1 (The cone Operator)**.** *The* cone *operator has the following properties.*

(1) $\forall S \subseteq \mathbb{E}$,

$$S \subseteq \operatorname{cone}(S).$$

(2) $\forall S_1, S_2 \subseteq \mathbb{E}$,

$$S_1 \subseteq S_2 \implies \operatorname{cone}(S_1) \subseteq \operatorname{cone}(S_2).$$

(3) $\forall S \subseteq \mathbb{E}$,

$$\operatorname{cone}(\operatorname{cone}(S)) = \operatorname{cone}(S).$$

Proposition 4.2.2. *The* conv *operator and the* cone *operator commute. Let S be a set in $\mathbb{E}$. Then*

$$\operatorname{conv}(\operatorname{cone}(S)) = \operatorname{cone}(\operatorname{conv}(S)).$$

Proof.

For $\text{cone}(\text{conv}(S)) \subseteq \text{conv}(\text{cone}(S))$, let x be an arbitrary point in $\text{cone}(\text{conv}(S))$.

Since $x \in \text{cone}(\text{conv}(S))$, we get $\exists \lambda \in \mathbb{R}_+$, $\exists n \in \mathbb{N}$, $\exists v_1, ..., v_n \in S$, $\exists \mu_1, ..., \mu_n \in [0,1], \sum_{i=1}^n \mu_i = 1$ such that $x = \lambda \sum_{i=1}^n \mu_i v_i$.

Since $x = \lambda \sum_{i=1}^n \mu_i v_i$, $x = \sum_{i=1}^n \mu_i (\lambda v_i)$.

Since $\lambda \in \mathbb{R}_+$ and $v_i \in S$, $\lambda v_i \in \text{cone}(S)$.

Since $\lambda v_i \in \text{cone}(S)$ and $\mu_i \in [0,1]$, $\sum_{i=1}^n \mu_i = 1$, $\sum_{i=1}^n \mu_i (\lambda v_i) \in \text{conv}(\text{cone}(S))$.

Since $\forall x \in \text{cone}(\text{conv}(S)), x \in \text{conv}(\text{cone}(S))$, $\text{cone}(\text{conv}(S)) \subseteq \text{conv}(\text{cone}(S))$.

For $\text{conv}(\text{cone}(S)) \subseteq \text{cone}(\text{conv}(S))$, let x be an arbitrary point in $\text{conv}(\text{cone}(S))$.

Since $x \in \text{conv}(\text{cone}(S))$, $\exists n \in \mathbb{N}$, $\exists \lambda_i \in [0,1], \sum_{i=1}^n \lambda_i = 1$, $\exists \mu_i \in \mathbb{R}_+$, $\exists v_i \in S$ such that $x = \sum_{i=1}^n \lambda_i \mu_i v_i$.

Define $\alpha := \sum_{i=1}^n \lambda_i \mu_i$.

Define $\beta_i := \lambda_i \mu_i / \alpha$.

Then $\alpha \in \mathbb{R}_+$ and $\beta_i \in [0,1]$ and $\sum_{i=1}^n \beta_i = 1$ and $x = \alpha \sum_{i=1}^n \beta_i v_i$.

Since $\beta_i \in [0,1]$ and $\sum_{i=1}^n \beta_i = 1$ and $v_i \in S$, we get $\sum_{i=1}^n \beta_i v_i \in \text{conv}(S)$.

Since $\alpha \in \mathbb{R}_+$ and $\sum_{i=1}^n \beta_i v_i \in \text{conv}(S)$ and $x = \alpha \sum_{i=1}^n \beta_i v_i$, we get $x \in \text{cone}(\text{conv}(S))$.

Since $\forall x \in \text{conv}(\text{cone}(S)), x \in \text{cone}(\text{conv}(S))$, we get $\text{conv}(\text{cone}(S)) \subseteq \text{cone}(\text{conv}(S))$.

Since $\text{cone}(\text{conv}(S)) \subseteq \text{conv}(\text{cone}(S))$ and $\text{conv}(\text{cone}(S)) \subseteq \text{cone}(\text{conv}(S))$, we get $\text{conv}(\text{cone}(S)) = \text{cone}(\text{conv}(S))$.

■

4.3 Other Properties

Proposition 4.3.1. *Let C be a convex set in $\mathbb{E}$. Assume $int(C) \neq \emptyset$ and $0 \in C$. Then $int(\text{cone}(C)) = \text{cone}(int(C))$.*

Proof.

For one direction, let x be an arbitrary point in $int(\text{cone}(C))$. We are to prove that $x \in \text{cone}(int(C))$.

Since $x \in int(\text{cone}(C))$, $\exists r$ such that $\text{ball}(x, r) \subseteq \text{cone}(C)$.

Since $x \in int(\text{cone}(C))$, $x \in \text{cone}(C)$.

Since $x \in \text{cone}(C)$, $\exists n \in \mathbb{N}$, $\exists \lambda_1, ..., \lambda_n > 0$, $\exists v_1, ..., v_n \in C$ such that $x = \sum_{i=1}^n \lambda_i v_i$.

Assume for the sake of contradiction that $\exists k \in \{1, ..., n\}$ such that $\forall r_k > 0$, $\text{ball}(v_k, r_k) \cap \mathbb{E} \setminus C \neq \emptyset$.

not finished

For the reverse direction, let x be an arbitrary point in $\text{cone}(int(C))$. We are to prove that $x \in int(\text{cone}(C))$.

Since $x \in \text{cone}(int(C))$, $\exists n \in \mathbb{N}$, $\exists \lambda_1, ..., \lambda_n > 0$, $\exists v_1, ..., v_n \in int(C)$ such that $x = \sum_{i=1}^n \lambda_i v_i$.

Since $v_i \in int(C)$ for each $i \in \{1, ..., n\}$, $\exists r_i$ such that $\text{ball}(v_i, r_i) \subseteq C$.

Define $R := \min\{\lambda_i r_i\}_{i=1}^n$.

Say $R = \lambda_k r_k$ for some $k \in \{1, ..., n\}$.

Let y be an arbitrary point in $\text{ball}(x, R)$.

Since $y \in \text{ball}(x, R)$, $\exists w$ such that $\|w\| < R$ and $y = x + w$.

$$\begin{aligned} y &= \sum_{i=1}^n \lambda_i v_i + w \\ &= \sum_{i \neq k} \lambda_i v_i + \lambda_k v_k + w \\ &= \sum_{i \neq k} \lambda_i v_i + \lambda_k (v_k + w/\lambda_k). \end{aligned}$$

Since $\|w\| < R$, $\|w/\lambda_k\| < R/\lambda_k = r_k$.

Since $\|w/\lambda_k\| < r_k$, $v_k + w/\lambda_k \in \text{ball}(v_k, r_k)$.

So $v_k + w/\lambda_k \in C$.

So $y \in \text{cone}(C)$.

Since $\forall y \in \text{ball}(x, R)$, $y \in \text{cone}(C)$, $\text{ball}(x, R) \subseteq \text{cone}(C)$.

Since $\exists r$ such that $\text{ball}(x, r) \subseteq \text{cone}(C)$, $x \in int(\text{cone}(C))$.

This proves $\text{cone}(int(C)) \subseteq int(\text{cone}(C))$.

■

Proposition 4.3.2. *Let C be a convex set in $\mathbb{E}$. Assume $int(C) \neq \emptyset$ and $0 \in C$. Then*

$$0 \in int(C) \iff \text{cone}(C) = \mathbb{E}.$$

Proof. For one direction, assume that $0 \in int(C)$. We are to prove that $\text{cone}(C) = \mathbb{E}$. Clearly

$$\text{cone}(C) \subseteq \mathbb{E}.$$

Since $0 \in int(C)$, $\exists r > 0$ such that $\text{ball}(0, r) \subseteq C$. Since $\text{ball}(0, r) \subseteq C$, $\text{cone}(\text{ball}(0, r)) \subseteq \text{cone}(C)$. Since $\text{cone}(\text{ball}(0, r)) = \mathbb{E}$ and $\text{cone}(\text{ball}(0, r)) \subseteq \text{cone}(C)$, we get

$$\mathbb{E} \subseteq \text{cone}(C).$$

For the reverse direction, assume that $\text{cone}(C) = \mathbb{E}$. We are to prove that $0 \in int(C)$.

$$\mathbb{E} = int(\mathbb{E}) = int(\text{cone}(C)) = \text{cone}(int(C)).$$

If $0 \notin int(C)$, then $0 \notin \text{cone}(int(C))$. So $0 \in int(C)$.

■

Chapter 5

Tangent Cones and Normal Cones

5.1 Definitions

Definition (Tangent Cones)**.** *Let C be a non-empty convex set in $\mathbb{E}$. Let x be a point in $\mathbb{E}$. We define the **tangent cone** to C at point x, denoted by $T_C(x)$, to be a set given by*

$$T_C(x) := \begin{cases} \operatorname{clcone}(C - x), & \text{if } x \in C, \text{ or} \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Definition (Normal Cones)**.** *Let C be a non-empty convex set in $\mathbb{E}$. Let x be a point in $\mathbb{E}$. We define the **normal cone** to C at point x, denoted by $N_C(x)$, to be a set given by*

$$N_C(x) := \begin{cases} \{v \in \mathbb{E} : \forall y \in C - x, \langle y, v \rangle \leq 0\}, & \text{if } x \in C, \text{ or} \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

5.2 Basic Properties

Proposition 5.2.1. *Let C be a closed convex set in $\mathbb{E}$. Let x be a point in $\mathbb{E}$. Then $T_C(x)$ and $N_C(x)$ are closed convex cones.*

Proof.

If $C = \emptyset$, then $T_C(x) = N_C(x) = \emptyset$.

If $C \neq \emptyset$ and $x \notin C$, then $T_C(x) = N_C(x) = \emptyset$.

So now I assume that $C \neq \emptyset$ and $x \in C$.

Tangent Cone is Closed:

By definition, $T_C(x) = \text{clcone}(C - x)$. So $T_C(x)$ is a closed.

Tangent Cone is Convex:

$$
\begin{array}{cl}
C \text{ is convex} & \\
\Downarrow & \text{since affine mapping preserves convexity} \\
C - x \text{ is convex} & \\
\Downarrow & \text{since the cone operator preserves convexity} \\
\text{cone}(C - x) \text{ is convex} & \\
\Downarrow & \text{since the cl operator preserves convexity} \\
\text{cl}(\text{cone}(C - x)) \text{ is convex} & \\
\Downarrow & \text{since } \text{cl} \circ \text{cone} = \text{clcone} \\
\text{clcone}(C - x) \text{ is convex} &
\end{array}
$$

That is, $T_C(x)$ is convex.

Tangent Cone is a Cone

By definition, $T_C(x) = \text{clcone}(C - x)$. So $T_C(x)$ is a cone.

Normal Cone is Closed:

Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in $N_C(x)$ that converges to some point in $\mathbb{E}$.

Say $x_i \to x_\infty$.

Let y be an arbitrary point in $C - x$.

Since $x_i \in N_C(x)$ and $y \in C - x$, by definition of $N_C(x)$, we get $\langle x_i, y \rangle \leq 0$.

Since $\langle x_i, y \rangle \leq 0$ for any $i \in \mathbb{N}$ and $x_i \to x_\infty$, we get $\langle x_\infty, y \rangle \leq 0$.

Since $\forall y \in C - x$, $\langle x_\infty, y \rangle \leq 0$, by definition of $N_C(x)$, we get $x_\infty \in N_C(x)$.

Since any convergent sequence whose terms are in $N_C(x)$ has its limit also in $N_C(x)$, $N_C(x)$ is closed.

Normal Cone is Convex:

Let u and v be arbitrary points in $N_C(x)$.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda u + (1 - \lambda)v$.

Let y be an arbitrary point in $C - x$.

Since $u \in N_C(x)$, $\langle u, y \rangle \leq 0$.

Since $v \in N_C(x)$, $\langle v, y \rangle \leq 0$.

$$
\begin{aligned}
& \langle z, y \rangle \\
= & \langle \lambda u + (1 - \lambda)v, y \rangle \\
= & \lambda \langle u, y \rangle + (1 - \lambda) \langle v, y \rangle \\
\leq & \lambda 0 + (1 - \lambda) 0
\end{aligned}
$$

$$= 0.$$

That is, $\langle z, y \rangle \leq 0$.
Since $\forall y \in C - x$, $\langle z, y \rangle \leq 0$, we get $z \in N_C(x)$.
That is, $\lambda u + (1-\lambda)v \in N_C(x)$.
Since $\forall u, v \in N_C(x)$, $\forall \lambda \in (0,1)$, $\lambda u + (1-\lambda)v \in N_C(x)$, we get $N_C(x)$ is convex.
Normal Cone is a Cone:
Let v be an arbitrary point in $N_C(x)$.
Let λ be an arbitrary number such that $\lambda > 0$.
Let y be an arbitrary point in $C - x$.
Since $v \in N_C(x)$, $\langle v, y \rangle \leq 0$.
Since $\langle v, y \rangle \leq 0$ and $\lambda > 0$, $\langle \lambda v, y \rangle \leq 0$.
Since $\forall y \in C - x$, $\langle \lambda v, y \rangle \leq 0$, we get $\lambda v \in N_C(x)$.
Since $\forall v \in N_C(x)$, $\forall \lambda > 0$, $\lambda v \in N_C(x)$, we get $N_C(x)$ is a cone.

■

Proposition 5.2.2. *Let C be a non-empty closed convex set in $\mathbb{E}$. Let x be a point in C. Let n be a point in $\mathbb{E}$. Then*

$$n \in N_C(x) \iff \forall t \in T_C(x), \langle n, t \rangle \leq 0.$$

Proof.
For one direction, assume that $n \in N_C(x)$.
We are to prove that

$$\forall t \in T_C(x), \quad \langle n, t \rangle \leq 0.$$

Let t be an arbitrary point in $T_C(x)$.
Since $t \in T_C(x) = \mathrm{cl}(\mathrm{cone}(C - x))$,

$$\exists \{t_i\}_{i \in \mathbb{N}} \subseteq \mathrm{cone}(C - x), \text{ such that } t_i \to t. \tag{1}$$

Since $t_i \in \mathrm{cone}(C - x)$,

$$\forall i \in \mathbb{N}, \exists \lambda_i \in \mathbb{R}_{++}, \exists c_i \in C \text{ such that } t_i = \lambda_i(c_i - x). \tag{2}$$

Since $n \in N_C(x)$ and $c_i \in C$,

$$\langle n, c_i - x \rangle \leq 0. \tag{3}$$

Now using (2) and (3), we have

$$\begin{aligned} & \langle n, t_i \rangle \\ =\ & \langle n, \lambda_i(c_i - x) \rangle, && \text{since } t_i = \lambda_i(c_i - x)s \\ =\ & \lambda_i \langle n, c_i - x \rangle \end{aligned}$$

$$\begin{aligned} &\leq \lambda_i \cdot 0, && \text{since } \langle n, c_i - x \rangle \leq 0 \\ &= 0. \end{aligned}$$

That is,

$$\forall i \in \mathbb{N}, \quad \langle n, t_i \rangle \leq 0.$$

Since $\langle n, t_i \rangle \leq 0$ for each $i \in \mathbb{N}$ and $t_i \to t$, we get

$$\langle n, t \rangle \leq 0.$$

For the reverse direction, assume that n is a vector such that

$$\forall t \in T_C(x), \quad \langle n, t \rangle \leq 0.$$

We are to prove that $n \in N_C(x)$.
Let y be an arbitrary point in $C - x$.
Since $C - x \subseteq \text{clcone}(C - x) = T_C(x)$ and $y \in C - x$, we get $y \in T_C(x)$.
Since $y \in T_C(x)$ and $\forall t \in T_C(x), \langle n, t \rangle \leq 0$, we get $\langle n, y \rangle \leq 0$.
Since $\forall y \in C - x, \langle n, y \rangle \leq 0$, we get $n \in N_C(x)$.

∎

Theorem 3. *Let C be a closed convex set in $\mathbb{E}$ such that $int(C) \neq \emptyset$. Let x be a point in $\mathbb{E}$. Then*

$$x \in int(C) \iff T_C(x) = \mathbb{E} \iff N_C(x) = \{0\}.$$

Proof.

Part 1.
$x \in int(C)$ if and only if $0 \in int(C - x)$, if and only if $\text{clcone}(C - x) = \mathbb{E}$.
Part 2.
For one direction, assume that $T_C(x) = \mathbb{E}$.
We are to prove that $N_C(x) = \{0\}$.
Consider $n = 0$.
Since

$$\forall t \in T_C(x), \quad \langle 0, t \rangle = 0 \leq 0,$$

we get $0 \in N_C(x)$.
Let n be an arbitrary vector in $N_C(x)$.
By another proposition, we have

$$\begin{aligned} &n \in N_C(x) \\ \iff &\forall t \in T_C(x) = \mathbb{E}, \langle n, t \rangle \leq 0 \\ \implies &\text{for } t = n, \langle n, t \rangle = \langle n, n \rangle \leq 0 \end{aligned}$$

$$\implies n = 0.$$

That is, $n \in N_C(x) \implies n = 0$.
So $N_C(x) = \{0\}$.
For the reverse direction, assume that $N_C(x) = \{0\}$.
We are to prove that $T_C(x) = \mathbb{E}$.
Clearly $T_C(x) \subseteq \mathbb{E}$.
For $\mathbb{E} \subseteq T_C(x)$, let x be an arbitrary point in $\mathbb{E}$.
Define $p := \mathrm{proj}_{T_C(x)}(x)$.
Since $p = \mathrm{proj}_{T_C(x)}(x)$,

$$\forall y \in T_C(x), \quad \langle x - p, y - p \rangle \leq 0. \tag{1}$$

Since $p = \mathrm{proj}_{T_C(x)}(x)$, $p \in T_C(x)$.
Since $p \in T_C(x)$ and $T_C(x)$ is a cone,

$$2p \in T_C(x). \tag{2}$$

Apply (1) to $y = 2p$, we get

$$\langle x - p, 2p - p \rangle = \langle x - p, p \rangle \leq 0. \tag{3}$$

Since $T_C(x)$ is a closed cone,

$$0 \in T_C(x). \tag{4}$$

Apply (1) to $y = 0$, we get

$$\langle x - p, 0 - p \rangle = \langle x - p, -p \rangle \leq 0. \tag{5}$$

From (3) and (5), we get

$$\langle x - p, p \rangle = 0.$$

So (1) becomes

$$\forall y \in T_C(x), \quad \langle x - p, y \rangle \leq 0.$$

So $x - p \in N_C(x)$.
So $x - p = 0$.
So $x = p$.
So $x \in T_C(x)$.
Since $\forall x \in \mathbb{E}, x \in T_C(x)$, we get

$$\mathbb{E} \subseteq T_C(x).$$

■

5.3 Arithmetic Properties

Proposition 5.3.1. *Let C and D be convex subsets of $\mathbb{E}$. Let x be a point in $\mathbb{E}$. Then*

$$N_C(x) + N_D(x) \subseteq N_{C\cap D}(x).$$

Proof.

If C or D is empty, then $N_C(x) + N_D(x) = N_{C\cap D}(x) = \emptyset$.

So now I assume that $C, D \neq \emptyset$.

If $x \notin C \cap D$, then $N_C(x) + N_D(x) = N_{C\cap D}(x) = \emptyset$.

So now I assume that $x \in C \cap D$.

Let v be an arbitrary point in $N_C(x) + N_D(x)$.

Since $v \in N_C(x) + N_D(x)$, $\exists u \in N_C(x)$, $\exists w \in N_D(x)$ such that $v = u + w$.

Since $u \in N_C(x)$, $\forall y \in C - x$, $\langle u, y\rangle \leq 0$.

Since $w \in N_D(x)$, $\forall y \in D - x$, $\langle w, y\rangle \leq 0$.

Let y be an arbitrary point in $C \cap D - x$.

Since $y \in C \cap D - x$, we get $y \in C - x$ and $y \in D - x$.

$$\begin{aligned}
&\langle v, y\rangle \\
&= \langle u + w, y\rangle \\
&= \langle u, y\rangle + \langle w, y\rangle \\
&\leq 0 + 0 = 0.
\end{aligned}$$

This is true for any $y \in C \cap D - x$.

So $v \in N_{C\cap D}(x)$.

This is true for any $v \in N_C(x) + N_D(x)$.

So $N_C(x) + N_D(x) \subseteq N_{C\cap D}(x)$.

■

Theorem 4. *Let C and D be convex sets in $\mathbb{E}$. Assume that $\mathrm{ri}(C) \cap \mathrm{ri}(D) \neq \emptyset$. Let x be a point in $C \cap D$. Then*

$$N_{C\cap D}(x) = N_C(x) + N_C(x).$$

5.4 Other Properties

Proposition 5.4.1. *Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x be a point in $\mathrm{dom}(f)$. Let u be a point in $\mathbb{E}$. Then $u \in \partial f(x)$ if and only if $(u, -1) \in N_{\mathrm{epi}(f)}(x, f(x))$.*

Proof.

$$
\begin{aligned}
& u \in \partial f(x) \\
\iff\ & \forall y \in \mathbb{E}, f(y) \geq f(x) + \langle u, y - x \rangle \\
\iff\ & \forall y \in \operatorname{dom}(f), f(y) \geq f(x) + \langle u, y - x \rangle \\
\iff\ & \forall (y, \beta) \in \operatorname{epi}(f), f(x) + \langle u, y - x \rangle \leq \beta \\
\iff\ & \forall (y, \beta) \in \operatorname{epi}(f), \big\langle (u, -1), (y - x, \beta - f(x)) \big\rangle \leq 0 \\
\iff\ & \forall (y, \beta) \in \operatorname{epi}(f), \big\langle (u, -1), (y, \beta) - (x, f(x)) \big\rangle \leq 0 \\
\iff\ & (u, -1) \in N_{\operatorname{epi}(f)}(x, f(x)).
\end{aligned}
$$

■

Chapter 6

Dual Cones and Polar Cones

6.1 Definitions

Definition (Dual Cone)**.** *Let K be a cone in $\mathbb{E}$. We define the* ***dual cone*** *of K, denoted by K^*, to be the set given by*

$$K^* := \{x \in \mathbb{E} : \forall k \in K, x \cdot k \geq 0\}.$$

Definition (Polar Cone)**.** *Let $\mathbb{E}$ be some Euclidean space. Let S be some set in the space. We define the* ***polar cone*** *of S, denoted by C°, to be the set given by*

$$C^\circ := \{y \in \mathbb{E} : \forall x \in C, \langle y, x \rangle \leq 0\}.$$

6.2 Properties

Proposition 6.2.1. *If S is a linear subspace of some Euclidean space $\mathbb{E}$, then $S^\circ = S^\perp$.*

Chapter 7

Extreme Points

7.1 Definitions

Definition (Extreme Points)**.** *Let $\mathbb{E}$ be some Euclidean space. Let C be a nonempty convex set in the space. Let x be some point in C. We say that x is an **extreme point** of C if it does not lie between any two distinct points in C.*

Definition (Extreme Points)**.** *Let $\mathbb{E}$ be some Euclidean space. Let C be a nonempty convex set in the space. Let x be some point in C. We say that x is an **extreme point** of C if $C \setminus \{x\}$ is convex.*

Definition (Extreme Points)**.** *Let $\mathbb{E}$ be some Euclidean space. Let C be a nonempty convex set in the space. Let x be some point in C. We say that x is an **extreme point** of C if $\{x\}$ is a face of C.*

Proposition 7.1.1. *The three definitions of extreme point are equivalent.*

Def 1 $\iff$ Def 2.

For one direction, assume that x does not lie between any two distinct points in C. We are to prove that $C \setminus \{x\}$ is convex. Let x_1 and x_2 be two arbitrary distinct points in $C \setminus \{x\}$. Let λ be an arbitrary number in $(0,1)$. Define a point y as $y := \lambda x_1 + (1-\lambda)x_2$. Since C is convex, $x_1, x_2 \in C$, and $\lambda \in (0,1)$, we get $y \in C$. Since x does not lie between any two distinct points in C, $y \neq x$. So $y \in C \setminus \{x\}$. That is, I have proved that

$$\forall x_1, x_2 \in C \setminus \{x\}, \forall \lambda \in (0,1), \quad y = \lambda x_1 + (1-\lambda)x_2 \in C \setminus \{x\}.$$

By definition, $C \setminus \{x\}$ is convex.

For the reverse direction, assume that $C \setminus \{x\}$ is convex. We are to prove that x does not lie between any two distinct points in C. Assume for the sake of contradiction that

x does lie between two distinct points in C. Say $x = \lambda x_1 + (1-\lambda)x_2$ where $x_1, x_2 \in C$, $x_1 \neq x_2$, and $\lambda \in (0,1)$. Clearly $x \neq x_1$ and $x \neq x_2$. So $x_1, x_2 \in C \setminus \{x\}$. Since $C \setminus \{x\}$ is convex, $x_1, x_2 \in C \setminus \{x\}$, and $\lambda \in (0,1)$, we get $x = \lambda x_1 + (1-\lambda)x_2 \in C \setminus \{x\}$. This leads to a contradiction. So the assumption that x lies between two distinct points in C does not hold. i.e. x does not lie between two distinct points in C.

■

7.2 Properties

Proposition 7.2.1. *If C is nonempty, convex, and compact, then* $\operatorname{extr}(C) \neq \emptyset$.

Theorem 5 (Minkowski)**.** *A compact convex set is the convex hull of its extreme points.*

Chapter 8

Projection

8.1 Definitions

Definition (Projection)**.** *Let $\mathcal{H}$ be a Hilbert space. Let S be a non-empty set in the space. Let x be a point in the space. We define the* ***projection*** *of x onto S, denoted by* $\mathrm{proj}_S(x)$*, to be a point given by*

$$\mathrm{proj}_S(x) := \mathrm{argmin}_{p \in S} \|p - x\|.$$

i.e., $\mathrm{proj}_S(x)$ *is the closest point in S to x.*

Proposition 8.1.1 (Existence)**.** *If S is non-empty and closed, then the projection* $\mathrm{proj}_S(x)$ *exists.*

Proof. Define for an $n \in \mathbb{N}$ a point c_m to be a point in S that satisfies

$$\lim_{i \in \mathbb{N}} \|c_i - x\| = d_S(x) \text{ where } d_S(x) = \inf_{p \in S} \|p - x\|.$$

Since $\mathcal{H}$ is a Hilbert space, the norm $\|\cdot\|$ on $\mathcal{H}$ satisfies the Parallelogram Law. So

$$\begin{aligned}
\|c_m - c_n\|^2 &= 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - \|c_m + c_n - 2x\|^2 \\
&= 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - 4\left\|\frac{c_m + c_n}{2} - x\right\|^2 \\
&\leq 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - 4d_S(x) \\
&\to 2d_S(x) + 2d_S(x) - 4d_S(x) = 0.
\end{aligned}$$

So the sequence $(c_i)_{i \in \mathbb{N}}$ is Cauchy. Since $\mathcal{H}$ is a Hilbert space, it is complete. So $(c_i)_{i \in \mathbb{N}}$ converges. Since S is closed, and $(c_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in S, $p := \lim_{i \in \mathbb{N}} c_i \in S$. So $\|p - x\| = \|\lim_{i \in \mathbb{N}} c_i - x\| = \lim_{i \in \mathbb{N}} \|c_i - x\| = d_S(x)$. So p is the minimizer of the distance to the point x over S. So $p = \mathrm{proj}_S(x)$. ∎

Proposition 8.1.2 (Uniqueness)**.** *If S is non-empty, closed, and convex, then the projection* $\mathrm{proj}_S(x)$ *is unique.*

Proof. Let p denote $\mathrm{proj}_S(x)$. Then $\|p - x\| = d_S(x)$. Let q be a point in S such that $\|q - x\| = d_S(x)$. Then by the Parallelogram Law,

$$\begin{aligned} 0 \le \|p-q\|^2 &= 2\|x-p\|^2 + 2\|q-x\| - 4\left\|x - \frac{1}{2}(p+q)\right\|^2 \\ &\le 2d_S^2(x) + 2d_S^2(x) - 4d_S^2(x) \\ &= 0. \end{aligned}$$

This shows $\|p - q\| = 0$ and hence $p = q$. Thus the projection is unique. ∎

8.2 Properties of the Projection Operator

Proposition 8.2.1 (Idempotent)**.** *The projection operator is idempotent. i.e., if C is a nonempty closed convex set in $\mathbb{E}$, then* $\mathrm{proj}_C = \mathrm{proj}_C \mathrm{proj}_C$.

Proof. Let x be an arbitrary point in $\mathbb{E}$. By definition, $\mathrm{proj}_C(x) \in C$. Since $\mathrm{proj}_C(x) \in C$, the closest point in C to $\mathrm{proj}_C(x)$ is $\mathrm{proj}_C(x)$. So $\mathrm{proj}_C \mathrm{proj}_C(x) = \mathrm{proj}_C(x)$. This is true for any $x \in \mathbb{E}$. So $\mathrm{proj}_C = \mathrm{proj}_C \mathrm{proj}_C$. ∎

Proposition 8.2.2. *Let C be a nonempty closed convex set in $\mathbb{E}$. Then the set of fixed points of the operator* proj_C *is C.*

Proof. For one direction, let x be an arbitrary fixed point of proj_C. We are to prove that $x \in C$. Since x is a fixed point of proj_C, $x = \mathrm{proj}_C(x)$. By definition of projection, $\mathrm{proj}_C(x) \in C$. So $x = \mathrm{proj}_C(x) \in C$.

For the reverse direction, let x be an arbitrary point in C. We are to prove that x is a fixed point of C. Since $x \in C$, the closest point in C to x is x. So $x = \mathrm{proj}_C(x)$. So x is a fixed point of proj_C. ∎

Proposition 8.2.3 (Linearity)**.** *Let C be a nonempty closed convex set in $\mathbb{E}$. Then the operator* proj_C *is linear if and only if C is a linear subspace.*

Proposition 8.2.4 (Non-expansive)**.** *The projection operator is non-expansive. i.e., if C is a nonempty closed convex set in $\mathbb{E}$, then $\|\mathrm{proj}\, C(x)\| \le \|x\|$ for any $x \in \mathbb{E}$.*

this is not true. I guess it will be true when C is a linear subspace.

Proposition 8.2.5. *Let C be a nonempty closed convex set in $\mathbb{E}$. Then* proj_C *is Lipschitz with constant 1.*

Proof. Let x and y be two arbitrary points in $\mathbb{E}$. If $\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\| = 0$, then $\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\| \leq \|x - y\|$. Otherwise,

$$\begin{aligned}
&\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|^2 \\
&= \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), \operatorname{proj}_C(x) - \operatorname{proj}_C(y)\rangle \\
&= \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), \operatorname{proj}_C(x) - x\rangle \\
&+ \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&+ \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), y - \operatorname{proj}_C(y)\rangle \\
&= \langle x - \operatorname{proj}_C(x), \operatorname{proj}_C(y) - \operatorname{proj}_C(x)\rangle \\
&+ \langle y - \operatorname{proj}_C(y), \operatorname{proj}_C(x) - \operatorname{proj}_C(y)\rangle \\
&+ \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&\leq 0 + 0 + \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&= \langle \operatorname{proj}_C(x) - \operatorname{proj}_C(y), x - y\rangle \\
&\leq \|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|\|x - y\|.
\end{aligned}$$

That is,

$$\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|^2 \leq \|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|\|x - y\|.$$

Dividing both sides by $\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\|$ gives

$$\|\operatorname{proj}_C(x) - \operatorname{proj}_C(y)\| \leq \|x - y\|.$$

So proj_C is Lipschitz with constant 1. ■

Proposition 8.2.6 (Firmly Non-expansive)**.** *Let C be a nonempty closed convex set in $\mathbb{E}$. Then proj_C is firmly non-expansive.*

Proof. This is to prove.

$$\forall x, y \in \mathbb{E}, \quad \left\|\operatorname{proj}_C(y) - \operatorname{proj}_C(x)\right\|^2 \leq \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), y - x\right\rangle.$$

$$\begin{aligned}
&\left\|\operatorname{proj}_C(y) - \operatorname{proj}_C(x)\right\|^2 \\
&= \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), \operatorname{proj}_C(y) - \operatorname{proj}_C(x)\right\rangle \\
&= \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), \operatorname{proj}_C(y) - y\right\rangle \\
&+ \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), y - x\right\rangle \\
&+ \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), x - \operatorname{proj}_C(x)\right\rangle \\
&\leq 0 + \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), y - x\right\rangle + 0 \\
&= \left\langle \operatorname{proj}_C(y) - \operatorname{proj}_C(x), y - x\right\rangle.
\end{aligned}$$

■

8.3 Characterization

Theorem 6 (Projection Theorem)**.** *Let C be a nonempty closed convex set in $\mathbb{E}$. Let x and p be points in $\mathbb{E}$. Then $p = \mathrm{proj}_C(x)$ if and only if*

$$\forall y \in C, \quad \langle y - p, x - p \rangle \leq 0.$$

Proof. Let y be an arbitrary point in C. Let α be an arbitrary number in $[0, 1]$. Define $y_\alpha := \alpha y + (1 - \alpha)p$. Now

$$\begin{aligned}
& p = \mathrm{proj}_C(x) \\
\Longleftrightarrow\ & \forall y \in C, \forall \alpha \in [0, 1], \|x - p\|^2 \leq \|x - y_\alpha\|^2 \\
\Longleftrightarrow\ & \forall y \in C, \forall \alpha \in [0, 1], \|x - p\|^2 \leq \|x - p - \alpha(y - p)\|^2 \\
\Longleftrightarrow\ & \forall y \in C, \langle x - p, y - p \rangle \leq 0.
\end{aligned}$$

∎

Chapter 9

Separation

9.1 Definitions

Definition (Separated)**.** *Let S_1 and S_2 be two sets in $\mathbb{E}$. We say that S_1 and S_2 are **separated** if $\exists b \in \mathbb{E} \setminus \{\vec{0}\}$ such that*

$$\sup_{s_1 \in S_1} \langle s_1, b \rangle \leq \inf_{s_2 \in S_2} \langle s_2, b \rangle.$$

*We say that they are **strongly separated** if the inequality holds strictly.*

Definition (Strongly Separated)**.** *Let S_1 and S_2 be two sets in $\mathbb{E}$. We say that they are **strongly separated** if the inequality holds strictly.*

Definition (Properly Separated)**.** *Let S_1 and S_2 be two sets in $\mathbb{E}$. We say that S_1 and S_2 are **properly separated** if $\exists b \in \mathbb{E}$ such that*

$$\sup_{x \in S_1} \langle x, b \rangle \leq \inf_{y \in S_2} \langle y, b \rangle, \text{ and}$$
$$\inf_{x \in S_1} \langle x, b \rangle > \sup_{y \in S_2} \langle y, b \rangle.$$

Definition (Hyperplane)**.** *A **hyperplane** is a set in some Euclidean space $\mathbb{E}$ of the form $\{x \in \mathbb{E} : a^T x = b\}$ for some $a, b \in \mathbb{E}$ and $a \neq \vec{0}$.*

Definition (Polyhedrons)**.** *A **polyhedron** is a set that is the solution set of finitely many linear inequalities, or equivalently, the intersection of finitely many half spaces.*

9.2 Main Results

Proposition 9.2.1. *Let C be a nonempty closed convex set in $\mathbb{E}$. Let x be a point in $\mathbb{E}$ such that $x \notin C$. Then x and C are strongly separated.*

Proof.

Define a point p by

$$p := \mathrm{proj}_C(x).$$

Define a point a by

$$a := x - p.$$

To prove that x is strongly separated from C, it suffices to prove that

$$\forall y \in C, \quad \langle y, a\rangle < \langle x, a\rangle.$$

Since $x \notin C$ and C is closed,

$$a \neq 0. \tag{1}$$

Let y be an arbitrary point in C. Since $p = \mathrm{proj}_C(x)$ and $y \in C$,

$$\langle y - p, x - p\rangle \leq 0. \tag{2}$$

$$\begin{aligned}
&\langle y, a\rangle \\
&< \langle y, a\rangle + \langle a, a\rangle, \text{ since } a \neq 0 \\
&= \langle y + a, a\rangle \\
&= \langle y + x - p, x - p\rangle, \text{ substitute } a = x - p \\
&= \langle y - p, x - p\rangle + \langle x, x - p\rangle \\
&\leq 0 + \langle x, x - p\rangle, \text{ since } \langle y - p, x - p\rangle \leq 0 \\
&= \langle x, x - p\rangle \\
&= \langle x, a\rangle.
\end{aligned}$$

That is,

$$\forall y \in C, \quad \langle y, a\rangle < \langle x, a\rangle.$$

So x is strongly separated from C.

■

Proposition 9.2.2. *Let C_1 be a non-empty closed convex set in $\mathbb{E}$. Let C_2 be a non-empty compact convex set in $\mathbb{E}$. Assume that C_1 and C_2 are disjoint. Then C_1 and C_2 are strongly separated.*

Proof. Since C_1 is non-empty closed and convex and C_2 is non-empty compact and convex, we get $C_1 - C_2$ is non-empty closed and convex. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C_1 - C_2$. Since $C_1 - C_2$ is non-empty closed and convex and $0 \in C_1 - C_2$, 0 and $C_1 - C_2$ are strongly separated. Since 0 is strongly separated from $C_1 - C_2$,

$$\exists a \neq 0 \text{ such that } \forall c_1 \in C_1, c_2 \in C_2, \quad \langle c_1 - c_2, a\rangle < \langle 0, a\rangle.$$

That is,

$$\langle c_1, a\rangle < \langle c_2, a\rangle.$$

So C_1 and C_2 are strongly separated. ■

Theorem 7. *Let C_1 and C_2 be non-empty closed convex sets in $\mathbb{E}$. Assume that C_1 and C_2 are disjoint. Then C_1 and C_2 are separated.*

Proof. For $n \in \mathbb{N}$, define

$$D_n := C_2 \cap \text{ball}(0, n).$$

Then D_n is compact for any $n \in \mathbb{N}$. Since $\begin{cases} C_1 \text{ is non-empty closed and convex } D_n \text{ is non-empty compact and convex}\end{cases}$ we get C_1 and D_n are strongly separated for any $n \in \mathbb{N}$. So

$$\forall n \in \mathbb{N}, \exists a_n \in \mathbb{E}, \|a_n\| = 1 \text{ such that } \forall c_1 \in C_1, \forall d_2 \in D_n, \quad \langle c_1, a_n\rangle < \langle d_2, a_n\rangle.$$

Since $\|a_n\| = 1$ for any $n \in \mathbb{N}$, there exists a subsequence $\{a_n\}_{n\in I}$ where I is some infinite subset of $\mathbb{N}$ such that $\{a_n\}_{n\in I}$ converges to some point $a \in \mathbb{E}$. Let x be an arbitrary point in C_1. Let y be an arbitrary point in C_2. For large enough n, $y \in D_n$. Since $\begin{cases} \langle x, a_n\rangle < \langle y, a_n\rangle \text{ for large enough } n \\ \lim_{n\in I, n\to\infty} \langle x, a_n\rangle = \langle x, a\rangle \\ \lim_{n\in I, n\to\infty} \langle y, a_n\rangle = \langle y, a\rangle \end{cases}$, we get

$$\langle x, a\rangle \leq \langle y, a\rangle.$$

Since

$$\exists a \neq 0 \text{ such that } \forall x \in C_1, \forall y \in C_2, \quad \langle x, a\rangle \leq \langle y, a\rangle,$$

by definition of separated, C_1 and C_2 are separated. ■

Proposition 9.2.3. *Let C_1 and C_2 be non-empty convex subsets of $\mathbb{E}$. Then C_1 and C_2 are properly separated if and only if*

$$\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset.$$

Proposition 9.2.4. *Polyhedrons are convex.*

Chapter 10

Convex Functions

10.1 Preliminaries

Definition (Epigraph)**.** *Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **epigraph** of f, denoted by* $\operatorname{epi}(f)$*, to be the set given by*

$$\operatorname{epi}(f) := \big\{(x, \alpha) \in \mathbb{E} \times \mathbb{R} : f(x) \leq \alpha\big\}.$$

Definition (Domain)**.** *Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the **domain** of f, denoted by* $\operatorname{dom}(f)$*, to be a set given by*

$$\operatorname{dom}(f) := \big\{x \in \mathbb{E} : f(x) < +\infty\big\}.$$

Definition (Proper)**.** *Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **proper** if*

$$\begin{aligned} &\exists x \in \mathbb{E}, \quad f(x) \neq +\infty, \ \textit{and} \\ &\forall x \in \mathbb{E}, \quad f(x) \neq -\infty \end{aligned}$$

10.2 The Indicator Function

Definition (The Indicator Function)**.** *Let S be a subset of $\mathbb{E}$. We define the **indicator function** of S, denoted by δ_S, to be a function from $\mathbb{E}$ to $\mathbb{R}^*$ given by*

$$\delta_S(x) = \begin{cases} 0, & x \in S \\ +\infty, & x \notin S. \end{cases}$$

Proposition 10.2.1. *Let S be a subset of $\mathbb{E}$. Then*

(1) S is non-empty if and only if δ_S is proper.

(2) S is convex if and only if δ_S is convex.

(3) S is closed if and only if δ_S is lower semi-continuous.

Proof of (1).

For one direction, assume that S is not empty.

We are to prove that δ_S is proper.

Since $S \neq \emptyset$, pick $p \in S$.

Since $p \in S$, $\delta_S(p) = 0$.

Since $\delta_S(p) = 0$, $\exists x_0 \in \mathbb{E}$ such that $\delta_S(x_0) \neq +\infty$.

By definition of the indicator function, it never takes $-\infty$.

Since $\exists x_0 \in \mathbb{E}$ such that $\delta_S(x_0) \neq +\infty$ and $\forall x \in \mathbb{E}$, $\delta_S(x) \neq -\infty$, we get δ_S is proper.

For the reverse direction, assume that δ_S is proper.

We are to prove that S is non-empty.

Assume for the sake of contradiction that S is empty.

Let x be an arbitrary point in $\mathbb{E}$.

Since $S = \emptyset$, $x \notin S$.

Since $x \notin S$, $\delta_S(x) = +\infty$.

Since $\forall x \in \mathbb{E}$, $\delta_S(x) = +\infty$, by definition of proper function, δ_S is not proper.

This contradicts to the assumption that δ_S is proper.

So the assumption that $S = \emptyset$ is false.

i.e., S is non-empty.

■

Proof of (2).

For one direction, assume that S is convex.

We are to prove that δ_S is convex.

Let x and y be arbitrary points in $\mathrm{dom}(\delta_S)$.

By definition of indicator functions, $\mathrm{dom}(\delta_S) = S$.

So $x, y \in S$.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda x + (1 - \lambda)y$.

Since $x, y \in S$ and $\lambda \in (0, 1)$ and S is convex and $z = \lambda x + (1 - \lambda)y$, we get $z \in S$.

Since $z \in S$, $\delta_S(z) = 0$.

Since $\lambda \in (0, 1)$ and $\mathrm{range}(\delta_S) = \{0, +\infty\}$, we get $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) \geq 0$.

Since $\delta_S(z) = 0$ and $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) \geq 0$, we get $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$.

That is, $\delta_S(\lambda x + (1 - \lambda)y) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$.

Since $\forall x, y \in \mathrm{dom}(\delta_S)$, $\forall \lambda \in (0, 1)$, $\delta_S(\lambda x + (1 - \lambda)y) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$, we get δ_S is convex.

For the reverse direction, assume that δ_S is convex.

We are to prove that S is convex.

The case where S is empty is trivial.

So now I assume $S \neq \emptyset$.

Let x and y be arbitrary points in S.

Let λ be an arbitrary number in $(0, 1)$.

Define point z as $z := \lambda x + (1 - \lambda)y$.

Since $x \in S$, $\delta_S(x) = 0$.

Since $y \in S$, $\delta_S(y) = 0$.

Since $\delta_S(x) = \delta_S(y) = 0$, we get $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) = 0$.

Since $\lambda \in (0, 1)$ and δ_S is convex, $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$.

Since $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$ and $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) = 0$, we get $\delta_S(z) \leq 0$.

By definition of the indicator function, $\delta_S(z) \geq 0$.

Since $\delta_S(z) \leq 0$ and $\delta_S(z) \geq 0$, we get $\delta_S(z) = 0$.

Since $\delta_S(z) = 0$, $z \in S$.

That is, $\lambda x + (1 - \lambda)y \in S$.

Since $\forall x, y \in S$, $\forall \lambda \in (0, 1)$, $\lambda x + (1 - \lambda)y \in S$, we get S is convex.

■

Proof of (3).

For one direction, assume that S is closed.

We are to prove that δ_S is lower semi-continuous.

Let $\{(x_i, \alpha_i)\}_{i \in \mathbb{N}}$ be an arbitrary sequence in $\mathrm{epi}(\delta_S)$ that converges.

Say its limit is $(x_\infty, \alpha_\infty)$.

Since $(x_i, \alpha_i) \to (x_\infty, \alpha_\infty)$, $x_i \to x_\infty$.

Since $(x_i, \alpha_i) \in \mathrm{epi}(\delta_S)$, $\delta_S(x_i) \leq \alpha_i$.

Since $\delta_S(x_i) \leq \alpha_i$ and $\alpha_i \in \mathbb{R}$, we get $\delta_S(x_i) \neq +\infty$.

Since $\delta_S(x_i) \neq +\infty$, $x_i \in S$.

Since $x_i \in S$ and $x_i \to x_\infty$ and S is closed, $x_\infty \in S$.

Since $x_\infty \in S$, $\delta_S(x_\infty) = 0$.

Since $x_i \in S$, $\delta_S(x_i) = 0$.

Since $\delta_S(x_i) = 0$ and $\delta_S(x_i) \leq \alpha_i$, $\alpha_i \geq 0$.

Since $(x_i, \alpha_i) \to (x_\infty, \alpha_\infty)$, $\alpha_i \to \alpha_\infty$.

Since $\alpha_i \geq 0$ and $\alpha \to \alpha_\infty$, $\alpha_\infty \geq 0$.

Since $\delta_S(x_\infty) = 0$ and $\alpha_\infty \geq 0$, $\delta_S(x_\infty) \leq \alpha_\infty$.

Since $\delta_S(x_\infty) \leq \alpha_\infty$, $(x_\infty, \alpha_\infty) \in \mathrm{epi}(\delta_S)$.

Since for any convergent sequence in $\mathrm{epi}(\delta_S)$, its limit is also in $\mathrm{epi}(\delta_S)$, we get $\mathrm{epi}(\delta_S)$ is closed.

For the reverse direction, assume that δ_S is lower semi-continuous.

We are to prove that S is closed.

Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence in S that converges.

Say its limit is x_∞.

Since $x_i \in S$, $\delta_S(x_i) = 0$.

Since $\delta_S(x_i) = 0$, $(x_i, 0) \in \mathrm{epi}(\delta_S)$.

Since $x_i \to x_\infty$, $(x_i, 0) \to (x_\infty, 0)$.

Since $(x_i, 0) \in \mathrm{epi}(\delta_S)$ and $(x_i, 0) \to (x_\infty, 0)$, $(x_\infty, 0) \in \mathrm{epi}(\delta_S)$.

Since $(x_\infty, 0) \in \mathrm{epi}(\delta_S)$, $\delta_S(x_\infty) \leq 0$.

By definition of the indicator function, $\delta_S(x_\infty) \geq 0$.

Since $\delta_S(x_\infty) \leq 0$ and $\delta_S(x_\infty) \geq 0$, we get $\delta_S(x_\infty) = 0$.

Since $\delta_S(x_\infty) = 0$, $x_\infty \in S$.

Since for any convergent sequence in S, its limit is also in S, we get S is closed.

■

10.3 Definitions

Definition (Convex Function)**.** *Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **convex** if*

$$\forall x, y \in \mathrm{dom}(f), \forall \lambda \in [0,1], \quad f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y).$$

Definition (Convex Function)**.** *Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is **convex** if the epigraph of f is convex.*

Proposition 10.3.1. *The two definitions of convexity of functions are equivalent.*

Proof.

The case where $\mathrm{dom}(f), \mathrm{epi}(f) = \emptyset$ is trivial.

So now I assume that $\mathrm{dom}(f), \mathrm{epi}(f) \neq \emptyset$.

For one direction, assume that $\forall x, y \in \mathrm{dom}(f)$, $\forall \lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$.

We are to prove that the epigraph of f is convex.

Let (x, α) and (y, β) be two arbitrary points in $\mathrm{epi}(f)$.

Since $(x, \alpha), (y, \beta) \in \mathrm{epi}(f)$, $x, y \in \mathrm{dom}(f)$.

Let λ be an arbitrary number in $[0,1]$.

Define a point $(z, \gamma) := \lambda(x, \alpha) + (1-\lambda)(y, \beta)$.

Then $z = \lambda x + (1-\lambda)y$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$.

Since $x, y \in \mathrm{dom}(f)$, $\lambda \in [0,1]$, we get $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$.

Since $(x, \alpha) \in \mathrm{epi}(f)$, $f(x) \leq \alpha$.

Since $(y, \beta) \in \mathrm{epi}(f)$, $f(y) \leq \beta$.

Since $f(x) \leq \alpha$ and $f(y) \leq \beta$ and $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, we get $f(\lambda x + (1-\lambda)y) \leq \lambda\alpha + (1-\lambda)\beta$.

Since $z = \lambda x + (1-\lambda)y$ and $\gamma = \lambda\alpha + (1-\lambda)\beta$ and $f(\lambda x + (1-\lambda)y) \leq \lambda\alpha + (1-\lambda)\beta$, we get $f(z) \leq \gamma$.

Since $f(z) \leq \gamma$, $(z, \gamma) \in \text{epi}(f)$.

For the reverse direction, assume that $\text{epi}(f)$ is convex.

We are to prove that $\forall x, y \in \text{dom}(f)$, $\forall \lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

Let x and y be two arbitrary points in $\text{dom}(f)$.

Let λ be an arbitrary number in $[0,1]$.

Define $z := \lambda x + (1-\lambda)y$.

Define $\gamma := \lambda f(x) + (1-\lambda)f(y)$.

Since $(x, f(x)) \in \text{epi}(f)$ and $(y, f(y)) \in \text{epi}(f)$ and $\lambda \in [0,1]$ and $\text{epi}(f)$ is convex, we get $\lambda(x, f(x)) + (1-\lambda)(y, f(y)) \in \text{epi}(f)$.

Since $z = \lambda x + (1-\lambda)y$ and $\gamma = \lambda f(x) + (1-\lambda)f(y)$ and $\lambda(x, f(x)) + (1-\lambda)(y, f(y)) \in \text{epi}(f)$, we get $(z, \gamma) \in \text{epi}(f)$.

Since $(z, \gamma) \in \text{epi}(f)$, $f(z) \leq \gamma$.

That is, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

■

10.4 Basic Properties

Proposition 10.4.1 (Necessary Condition)**.** *The domain of a convex function is convex.*

Proof. Follows from the fact that convexity is stable under affine transformations. Define $A((x,\alpha)) := x$. Then $\text{dom}(f) = A(\text{epi}(f))$. ■

Proposition 10.4.2. *The level sets of a convex function are convex.*

Proposition 10.4.3 (Restriction to a Line)**.** *A function $f : \mathbb{E} \to \mathbb{R}$ is convex if and only if $\forall x \in \text{dom}(f), \forall v \in \mathbb{E}$, the function $g_{x,v} : \mathbb{R} \to \mathbb{R}$ given by*

$$g_{x,v}(t) = f(x + tv)$$

is convex.

10.5 Differentiable Convex Functions

Proposition 10.5.1. *Let f be a proper convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Let $x \in \text{dom}(f)$. If f is differentiable at point x, then $\nabla(f)(x)$ is the unique subgradient of f at point x. i.e., $\partial(f)(x) = \{\nabla(f)(x)\}$. Conversely, if the subgradient $\partial(f)(x)$ of f at point x is a singleton set $\{v\}$, then f is differentiable at point x and $\nabla(f)(x) = v$.*

Proof. ∎

Proposition 10.5.2 (First-Order Condition)**.** *Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Assume that* $\mathrm{dom}(f)$ *is convex and open and that f is differentiable on* $\mathrm{dom}(f)$*. Then f is convex if and only if*

$$\forall x, y \in \mathrm{dom}(f), \quad f(y) - f(x) \geq \big\langle \nabla(f)(x), y - x \big\rangle.$$

i.e., the first-order approximation of f is a global under-estimator.

Proof.

Part 1.

For one direction, assume that f is convex. We are to prove that

$$\forall x, y \in \mathrm{dom}(f), \quad f(y) - f(x) \geq \big\langle \nabla(f)(x), y - x \big\rangle.$$

Let x and y be arbitrary points in $\mathrm{dom}(f)$. Since f is convex and differentiable at point x, $\nabla(f)(x) = \partial(f)(x)$. So $\nabla(f)(x)$ satisfies the subgradient inequality. That is,

$$f(y) - f(x) \geq \big\langle \nabla(f)(x), y - x \big\rangle.$$

Part 2.

For the reverse direction, assume that

$$\forall x, y \in \mathrm{dom}(f), \quad f(y) - f(x) \geq \big\langle \nabla(f)(x), y - x \big\rangle.$$

We are to prove that f is convex.

Not Finished.

∎

Proposition 10.5.3. *Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Assume that* $\mathrm{dom}(f)$ *is convex and open and that f is differentiable on* $\mathrm{dom}(f)$*. Then f is convex if and only if*

$$\forall x, y \in \mathrm{dom}(f), \quad \big\langle \nabla(f)(x) - \nabla(f)(y), x - y \big\rangle \geq 0.$$

Proof.

Part 1.

For one direction, assume that f is convex. We are to prove that

$$\forall x, y \in \mathrm{dom}(f), \quad \big\langle \nabla(f)(x) - \nabla(f)(y), x - y \big\rangle \geq 0.$$

Let x and y be arbitrary points in $\mathrm{dom}(f)$. Since f is convex and differentiable at point x, $\nabla(f)(x) = \partial(f)(x)$. So $\nabla(f)(x)$ satisfies the subgradient inequality. That is,

$$f(y) - f(x) \geq \big\langle \nabla(f)(x), y - x \big\rangle. \tag{1}$$

Since f is convex and differentiable at point y, $\nabla(f)(y) = \partial(f)(y)$. So $\nabla(f)(y)$ satisfies the subgradient inequality. That is,

$$f(x) - f(y) \geq \big\langle \nabla(f)(y), x - y \big\rangle. \tag{2}$$

Take the sum of inequalities (1) and (2), we get

$$\begin{aligned}
& (f(y) - f(x)) + (f(x) - f(y)) \geq \big\langle \nabla(f)(x), y - x \big\rangle + \big\langle \nabla(f)(y), x - y \big\rangle \\
\Longrightarrow \quad & 0 \geq -\big\langle \nabla(f)(x), x - y \big\rangle + \big\langle \nabla(f)(y), x - y \big\rangle \\
\Longrightarrow \quad & \big\langle \nabla(f)(x), x - y \big\rangle - \big\langle \nabla(f)(y), x - y \big\rangle \geq 0 \\
\Longrightarrow \quad & \big\langle \nabla(f)(x) - \nabla(f)(x), x - y \big\rangle \geq 0.
\end{aligned}$$

Part 2.

For the reverse direction, assume that

$$\forall x, y \in \operatorname{dom}(f), \quad \big\langle \nabla(f)(x) - \nabla(f)(y), x - y \big\rangle \geq 0.$$

We are to prove that f is convex. Let x and y be arbitrary points in $\operatorname{dom}(f)$. Define a function φ on $(0, 1)$ by

$$\varphi(\lambda) := f\big(\lambda x + (1 - \lambda) y\big).$$

Notice φ is differentiable and

$$\varphi'(\lambda) = \big\langle \nabla(f)\big(\lambda x + (1 - \lambda) y\big), x - y \big\rangle.$$

Let α and β be arbitrary numbers in $(0, 1)$. Assume that $\alpha < \beta$. Define two points z_α and z_β by $z_\alpha := \alpha x + (1 - \alpha) y$ and $z_\beta := \beta x + (1 - \beta) y$. Then

$$\begin{aligned}
& \varphi'(\beta) - \varphi'(\alpha) \\
= & \big\langle \nabla(f)(\beta x + (1 - \beta) y), x - y \big\rangle - \big\langle \nabla(f)(\alpha x + (1 - \alpha) y), x - y \big\rangle \\
= & \big\langle \nabla(f)(z_\beta), x - y \big\rangle - \big\langle \nabla(f)(z_\alpha), x - y \big\rangle \\
= & \big\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), x - y \big\rangle \\
= & \left\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), \frac{z_\beta - z_\alpha}{\beta - \alpha} \right\rangle \\
= & \frac{1}{\beta - \alpha} \big\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), z_\beta - z_\alpha \big\rangle \\
\geq & \frac{1}{\beta - \alpha} \cdot 0, \text{ by assumption} \\
= & 0.
\end{aligned}$$

That is,

$$\forall \alpha, \beta \in (0, 1), \quad \beta > \alpha \implies \varphi'(\beta) - \varphi'(\alpha) \geq 0.$$

So φ' is increasing. So φ is convex. So

$$\varphi(\lambda) \leq \lambda\varphi(1) + (1-\lambda)\varphi(0).$$

That is,

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y).$$

By definition, f is convex.

■

Proposition 10.5.4 (Second-Order Condition)**.** *A twice continuously differentiable real-valued function f defined on a convex set is convex if and only if*

$$\forall x \in \mathrm{dom}(f), \quad \nabla^2 f(x) \geq 0$$

where $\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_n \partial x_n} \end{bmatrix}$ *denotes the Hessian matrix of f at x_0.*

Proposition 10.5.5. *Let f be a twice continuously differentiable function from $\mathbb{E}$ to $\mathbb{R}$. Then f is convex if and only if $\forall x \in \mathbb{E}$, $\nabla^2 f(x)$ is positive semi-definite.*

10.6 Convexity and Lipschitz-ness

Theorem 8. *Let f be a differentiable convex function from $\mathbb{E}$ to $\mathbb{R}$. Then the following statements are equivalent.*

(1) ∇f *is Lipschitz with constant L.*

(2) $\forall x, y \in \mathbb{E}$, *we have*

$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2.$$

(3) $\forall x, y \in \mathbb{E}$, *we have*

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle + \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|^2.$$

(4) $\forall x, y \in \mathbb{E}$, *we have*

$$L\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \|\nabla f(y) - \nabla f(x)\|^2.$$

(1) $\implies$ *(2).*

Assume that ∇f is Lipschitz with constant 1.

Let x and y be two arbitrary points in $\mathbb{E}$.

$$
\begin{aligned}
& f(y) - f(x) \\
= & \int_0^1 \langle \nabla f(x + t(y-x)), y - x \rangle dt \\
= & \langle \nabla f(x), y - x \rangle + \int_0^1 \langle \nabla f(x + t(y-x)) - \nabla f(x), y - x \rangle dt \\
\leq & \langle \nabla f(x), y - x \rangle + \int_0^1 \| \langle \nabla f(x + t(y-x)) - \nabla f(x) \rangle \| \|y - x\| dt \\
\leq & \langle \nabla f(x), y - x \rangle + \int_0^1 L \|x + t(y-x) - x\| \|y - x\| dt \\
= & \langle \nabla f(x), y - x \rangle + L \|y - x\|^2 \int_0^1 t dt \\
= & \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2
\end{aligned}
$$

That is,

$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2.$$

■

Theorem 9. *Let f be a twice continuously differentiable function from $\mathbb{E}$ to $\mathbb{R}$. Let L be some non-negative number. Then the following statements are equivalent.*

(1) ∇f is L-Lipschitz.

(2) $\forall x \in \mathbb{E}$, $\left\| \nabla^2 f(x) \right\| \leq L$.

10.7 Stability of Convexity

Proposition 10.7.1 (Non-Negative Linear Combination)**.** *A non-negative linear combination of proper convex functions is again convex.*

Proof. It suffices to prove that non-negative scalar multiples of convex functions are convex and sums of two convex functions are convex.
<u>Part 1</u>.

Let f be a proper convex function. Let $\alpha \geq 0$ be an arbitrary scalar. We are to prove that αf is convex. Notice $\text{dom}(f) = \text{dom}(\alpha f)$. Since f is proper, $\text{dom}(f) \neq \emptyset$. So $\text{dom}(\alpha f) \neq \emptyset$. Let x and y be two arbitrary points in $\text{dom}(\alpha f)$. Let λ be an arbitrary number in $(0, 1)$. Define a point z as $z := \lambda x + (1 - \lambda) y$. Then

$$(\alpha f)(\lambda x + (1 - \lambda) y) = \alpha f(\lambda x + (1 - \lambda) y)$$

$$
\begin{aligned}
&\leq \alpha(\lambda f(x) + (1-\lambda)f(y)) \\
&= \lambda\alpha f(x) + (1-\lambda)\alpha f(y) \\
&= \lambda(\alpha f)(x) + (1-\lambda)(\alpha f)(y).
\end{aligned}
$$

That is,

$$\forall x, y \in \operatorname{dom}(\alpha f), \forall \lambda \in (0,1), \quad (\alpha f)(\lambda x + (1-\lambda)y) \leq \lambda(\alpha f)(x) + (1-\lambda)(\alpha f)(y).$$

So by definition, αf is convex.

Part 2.

Let f and g be proper convex functions. We are to prove that $f+g$ is convex. Notice $\operatorname{dom}(f+g) = \operatorname{dom}(f) \cap \operatorname{dom}(g)$. Since f is proper, $\operatorname{dom}(f) \neq \emptyset$. Since g is proper, $\operatorname{dom}(g) \neq \emptyset$. So $\operatorname{dom}(f+g) \neq \emptyset$. Let x and y be two arbitrary points in $\operatorname{dom}(f+g)$. Let λ be an arbitrary number in $(0,1)$. Define a point z as $z := \lambda x + (1-\lambda)y$. Then

$$
\begin{aligned}
(f+g)(\lambda x + (1-\lambda)y) &= f(\lambda x + (1-\lambda)y) + g(\lambda x + (1-\lambda)y) \\
&\leq \lambda f(x) + (1-\lambda)f(y) + \lambda g(x) + (1-\lambda)g(y) \\
&= \lambda(f(x) + g(x)) + (1-\lambda)(f(y) + g(y)) \\
&= \lambda(f+g)(x) + (1-\lambda)(f+g)(y).
\end{aligned}
$$

That is,

$$\forall x, y \in \operatorname{dom}(f+g), \forall \lambda \in (0,1), \quad (f+g)(\lambda x + (1-\lambda)y) = \lambda(f+g)(x) + (1-\lambda)(f+g)(y).$$

So by definition, $f+g$ is convex.

■

Proposition 10.7.2 (Direct Sum)**.** *Direct sums of convex functions are convex.*

Proof. Let z and w be two arbitrary points in $\operatorname{dom}(f \oplus g)$. Let $\lambda \in (0,1)$ be arbitrary. Say $z = x \oplus y$ and $w = u \oplus v$ where $x, u \in \mathbb{R}^m$ and $y, v \in \mathbb{R}^p$. Since $z \in \operatorname{dom}(f \oplus g)$, $(f \oplus g)(z) \neq +\infty$. That is, $f(x) + g(y) \neq +\infty$. So neither $f(x)$ nor $g(y)$ is $+\infty$. So both $x \in \operatorname{dom}(f)$ and $y \in \operatorname{dom}(g)$. Similarly, we have $u \in \operatorname{dom}(f)$ and $v \in \operatorname{dom}(g)$. Consider the point

$$
\begin{aligned}
&\lambda z + (1-\lambda)w \\
&= \lambda x \oplus y + (1-\lambda)u \oplus v \\
&= \big(\lambda x + (1-\lambda)u\big) \oplus \big(\lambda y + (1-\lambda)v\big).
\end{aligned}
$$

Apply $f \oplus g$ to both sides, we get

$$(f \oplus g)(\lambda z + (1-\lambda)w)$$

$$\begin{aligned}&= (f \oplus g)\left[\big(\lambda x + (1-\lambda)u\big) \oplus \big(\lambda y + (1-\lambda)v\big)\right]\\ &= f(\lambda x + (1-\lambda)u) + g(\lambda y + (1-\lambda)v).\end{aligned}$$

Since f and g are convex, we get

$$\begin{aligned}&f(\lambda x + (1-\lambda)u) \leq \lambda f(x) + (1-\lambda) f(u), \text{ and}\\ &g(\lambda y + (1-\lambda)v) \leq \lambda g(y) + (1-\lambda) g(v).\end{aligned}$$

So

$$\begin{aligned}&(f \oplus g)(\lambda z + (1-\lambda)w)\\ \leq\ & \lambda f(x) + (1-\lambda) f(u) + \lambda g(y) + (1-\lambda) g(v)\\ =\ & \lambda(f(x) + g(y)) + (1-\lambda)(f(u) + g(v))\\ =\ & \lambda(f \oplus g)(x \oplus y) + (1-\lambda)(f \oplus g)(u \oplus v)\\ =\ & \lambda(f \oplus g)(z) + (1-\lambda)(f \oplus g)(w).\end{aligned}$$

That is,

$$(f \oplus g)(\lambda z + (1-\lambda)w) \leq \lambda(f \oplus g)(z) + (1-\lambda)(f \oplus g)(w).$$

This holds for any $z, w \in \operatorname{dom}(f \oplus g)$ and any $\lambda \in (0,1)$. So $(f \oplus g)$ is convex.

■

Proposition 10.7.3 (Composition)**.** *The composition of a convex function with an affine function is convex. i.e., if f is convex, then $f(Ax+b)$ is convex.*

Proof. Let x an y be arbitrary points in $\mathbb{E}$. Let λ be an arbitrary number in $(0,1)$. Define a point z by $z := \lambda x + (1-\lambda)y$.

$$\begin{aligned}&g\big(\lambda x + (1-\lambda y\big)\\ =\ & f\big(A(\lambda x + (1-\lambda)y) + b\big)\\ =\ & f\big(\lambda A x + (1-\lambda)Ay + b\big), && \text{by linearity of } A\\ =\ & f\big(\lambda A x + (1-\lambda)Ay + \lambda b + (1-\lambda)b\big), && \text{decomposite } b\\ =\ & f\big(\lambda(Ax+b) + (1-\lambda)(Ay+b)\big)\\ \leq\ & \lambda f(Ax+b) + (1-\lambda) f(Ay+b), && \text{by convexity of } f\\ =\ & \lambda g(x) + (1-\lambda) g(y).\end{aligned}$$

That is,

$$\forall x, y \in \mathbb{E}, \forall \lambda \in (0,1), \quad g\big(\lambda x + (1-\lambda)y\big) \leq \lambda g(x) + (1-\lambda) g(y).$$

So g is convex. ■

Proposition 10.7.4 (Supremum)**.** *The supremum of a collection of convex functions is again convex. i.e., Let $\{f_i\}_{i\in I}$ be a collection of convex functions where I is some index set. Then the function F given by $F := \sup_{i\in I} f_i$ is convex.*

Proof.

$$
\begin{aligned}
&(x,\alpha) \in \operatorname{epi}(F) \\
\iff &\sup_{i\in I} f_i(x) \le \alpha \\
\iff &\forall i \in I, f_i(x) \le \alpha \\
\iff &\forall i \in I, (x,\alpha) \in \operatorname{epi}(f_i \\
\iff &(x,\alpha) \in \bigcap_{i\in I} \operatorname{epi}(f_i).
\end{aligned}
$$

So $\operatorname{epi}(F) = \bigcap_{i\in I}\operatorname{epi}(f_i)$. Since f_i are convex, $\operatorname{epi}(f_i)$ are convex. Since $\operatorname{epi}(f_i)$ are convex, $\bigcap_{i\in I}\operatorname{epi}(f_i)$ is convex. That is, $\operatorname{epi}(F)$ is convex. Since $\operatorname{epi}(F)$ is convex, F is convex.

■

Proposition 10.7.5 (Pointwise Supremum)**.** *If $f(x,y)$ is convex in x for each y in some set $\mathcal{A}$, then the function g given by*

$$g(x) = \sup_{y\in\mathcal{A}} f(x,y)$$

is convex.

10.8 Examples

Example 10.8.1. *Affine functions are convex.*

Example 10.8.2. *Norms are convex.*

Proof.

$$
\begin{aligned}
&\|\alpha x + \beta y\| \\
\le &\|\alpha x\| + \|\beta y\| \\
= &|\alpha|\|x\| + |\beta|\|y\| \\
= &\alpha\|x\| + \beta\|y\|.
\end{aligned}
$$

■

Example 10.8.3. *Square norms are convex.*

Proof Approach 1. Notice $\|\cdot\|^2$ is the direct sum of m squares and squares are convex. So by CO 463 Assignment 2 Problem 3, $\|\cdot\|^2$ is convex. ∎

Proof Approach 2. The domain is $\mathbb{E}$. Let x and y be two points in $\mathbb{E}$. Let λ be an arbitrary number in $(0,1)$. Define a point z as $z := \lambda x + (1-\lambda)y$.

$$\begin{aligned}
&\|\lambda x + (1-\lambda)y\|^2 \\
&= \|\lambda x\|^2 + \|(1-\lambda)y\|^2 + 2\langle \lambda x, (1-\lambda)y\rangle \\
&= \lambda^2\|x\|^2 + (1-\lambda)^2\|y\|^2 + 2\lambda(1-\lambda)\langle x, y\rangle \\
&\leq \lambda^2\|x\|^2 + (1-\lambda)^2\|y\|^2 + 2\lambda(1-\lambda)\|x\|\|y\| \\
&\leq \lambda(\lambda-1)\|x\|^2 + \lambda(\lambda-1)\|y\|^2 + 2\lambda(1-\lambda)\|x\|\|y\| \\
&\quad + \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
&= \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
&\quad + \lambda(\lambda-1)\left[\|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\right] \\
&\leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2
\end{aligned}$$

That is,

$$\forall x, y \in \mathbb{E}, \forall \lambda \in (0,1), \quad \|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2.$$

So by definition, $\|\cdot\|^2$ is convex. ∎

Example 10.8.4. *The distance function to a convex set is convex.*

Example 10.8.5. *The perspective of a convex function is convex. i.e., if* $f : \mathbb{E} \to \mathbb{R}$

Chapter 11

More Convex Functions

11.1 Strictly Convex

Definition (Strictly Convex)**.** *Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is* ***strictly convex*** *if $\forall x, y \in \operatorname{dom}(f)$, $\forall \lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$, except when $\lambda x + (1-\lambda)y = x$ or y.*

Proposition 11.1.1. *Strictly convex functions are convex.*

11.2 Strongly Convex

Definition (Strongly Convex)**.** *Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. We say that f is* ***strongly convex*** *with constant β if $\forall x, y \in \operatorname{dom}(f)$, $\forall \lambda \in [0,1]$, we have*

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) - \frac{\beta}{2}\lambda(1-\lambda)\|x-y\|^2,$$

for some positive constant β.

Proposition 11.2.1. *Strongly convex functions are strictly convex.*

Proposition 11.2.2. *Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. Then f is β-strongly convex if and only if $f - \frac{\beta}{2}\|\cdot\|^2$ is convex.*

Proof. Let β be some positive constnat. Let g denote $f - \frac{\beta}{2}\|\cdot\|^2$. Let $x, y \in \mathbb{E}$ be arbitrary. Let $\lambda \in (0,1)$ be arbitrary.

$$\begin{aligned} & f \text{ is } \beta\text{-strongly convex} \\ \iff & f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) \\ & \qquad\qquad\qquad - \frac{\beta}{2}\lambda(1-\lambda)\|x-y\|^2 \end{aligned}$$

$$
\begin{aligned}
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \\
&\qquad\qquad\qquad - \frac{\beta}{2}\lambda(1-\lambda)\big(\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle\big) \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \\
&\qquad\qquad\qquad - \lambda\frac{\beta}{2}\|x\|^2 + \frac{\beta}{2}\lambda^2\|x\|^2 \\
&\qquad\qquad\qquad - (1-\lambda)\frac{\beta}{2}\|y\|^2 + \frac{\beta}{2}(1-\lambda)^2\|y\|^2 \\
&\qquad\qquad\qquad + \beta\lambda(1-\lambda)\langle x, y\rangle \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \\
&\qquad\qquad\qquad - \lambda\frac{\beta}{2}\|x\|^2 - (1-\lambda)\frac{\beta}{2}\|y\|^2 \\
&\qquad\qquad\qquad + \frac{\beta}{2}\|\lambda x\|^2 + \frac{\beta}{2}\|(1-\lambda)y\|^2 + \beta\langle \lambda x, (1-\lambda)y\rangle \\
\iff\ & f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \\
&\qquad\qquad\qquad - \lambda\frac{\beta}{2}\|x\|^2 - (1-\lambda)\frac{\beta}{2}\|y\|^2 \\
&\qquad\qquad\qquad + \frac{\beta}{2}\|\lambda x + (1-\lambda)y\|^2 \\
\iff\ & g\big(\lambda x + (1-\lambda)y\big) \le \lambda g(x) + (1-\lambda) g(y) \\
\iff\ & f - \frac{\beta}{2}\|\cdot\|^2 \text{ is } \beta \text{ convex.}
\end{aligned}
$$

■

Question: Can we allow f to take on $-\infty$? Do we need f to be proper?

Proposition 11.2.3. *Let f and g be functions from $\mathbb{E}$ to $\mathbb{R}^*$. Assume that f is β-strongly convex for some positive constant β and that g is convex. Then $f+g$ is β-strongly convex.*

Question: Can we allow f or g to take on $-\infty$? Do we need f and g to be proper?

Proof.

Since f is β-strongly convex, $f - \frac{\beta}{2}\|\cdot\|^2$ is convex.
Since $f - \frac{\beta}{2}\|\cdot\|^2$ and g are convex, $f + g - \frac{\beta}{2}\|\cdot\|^2$ is convex.
Since $f + g - \frac{\beta}{2}\|\cdot\|^2$ is convex, $f+g$ is β-strongly convex.

■

11.3 Quasiconvex

Definition (Quasiconvex)**.** *Let $f : \mathbb{E} \to \mathbb{R}$ be a function with convex domain. We say that f is* ***quasiconvex*** *if any level set of f is convex.*

Proposition 11.3.1 (Jensen's Inequality for Quasiconvex Functions)**.** *Let f be a quasiconvex function. Then $\forall x, y \in \mathrm{dom}(f)$, $\forall \alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$,*

$$f(\alpha x + \beta y) \leq \max\{f(x), f(y)\}.$$

Proposition 11.3.2. *A differentiable real-valued function f with convex domain is convex if and only if $\forall x, y \in \mathrm{dom}(f)$,*

$$f(y) \leq f(x) \implies \nabla f(x) \cdot (y - x) \leq 0. \quad ???$$

Not sure where did this come from but I don't think this is correct.

Chapter 12

Support

12.1 Definitions

Definition (Support Function)**.** *Let S be a subset of $\mathbb{E}$. We define the* ***support function*** *of S, denoted by σ_S, to be a function from $\mathbb{E}$ to $\mathbb{R}^*$ given by*

$$\sigma_S(x) = \sup_{s \in S} \langle x, s \rangle.$$

Definition (Supporting Hyperplane)**.** *Let S be a set in $\mathbb{E}$ with nonempty boundary. Let x_0 be a point in the boundary of S. We define a* ***supporting hyperplane*** *H to set S at point x_0 to be a set of the form*

$$H = \{x \in \mathbb{E} : a^T x = a^T x_0\},$$

such that $a \in \mathbb{E}$ and $a \neq \vec{0}$ and $\forall x \in S, a^T x \leq a^T x_0$.

12.2 Properties

Proposition 12.2.1. *The support function of a non-empty set S is proper, convex, and lower semi-continuous.*

Proof.

Part 1. Proper.

Define f_s to be a function from $\mathbb{E}$ to $\mathbb{R}$ by $f_s(x) = \langle s, x \rangle$.

These functions are linear and hence proper, convex, and lower semi-continuous.

Notice $\sigma_S = \sup_{s \in S} f_s$.

So σ_S is convex and lower semi-continuous.

Since $\sigma_S(0) = \sup_{s \in S} \langle 0, s \rangle = 0$, $\exists x_0 \in \mathbb{E}, \sigma_S(x) \neq +\infty$.

Since $\sigma_S(x) = \sup_{s \in S} \langle x, s \rangle \geq \langle x, s \rangle \neq -\infty$, $\forall x \in \mathbb{E}, \sigma_S(x) \neq -\infty$.

Since $\exists x_0 \in \mathbb{E}, \sigma_S(x) \neq +\infty$ and $\forall x \in \mathbb{E}, \sigma_S(x) \neq -\infty$, by definition, σ_S is proper.

∎

Proposition 12.2.2. *The support function of a non-empty and bounded set is continuous.*

Proof.

Let x_0 be an arbitrary point in $\mathbb{E}$. Let ε be an arbitrary positive number. Define $M := \sup_{y \in C} \|y\| + 1$. Since C is bounded, M is finite. Define $\delta := \varepsilon / M$. Let x be an arbitrary point such that $\|x - x_0\| < \delta$. Let y be an arbitrary point in $\mathbb{E}$. Then by the Cauchy Schwarz inequality, we have

$$\langle x - x_0, y \rangle \leq \|x - x_0\| \|y\|.$$

That is,

$$\langle x, y \rangle \leq \|x - x_0\| \|y\| + \langle x_0, y \rangle.$$

It follows that

$$\begin{aligned} \sup_{y \in C} \langle x, y \rangle &\leq \sup_{y \in C} \left(\|x - x_0\| \|y\| + \langle x_0, y \rangle \right) \\ &\leq \|x - x_0\| \sup_{y \in C} \|y\| + \sup_{y \in C} \langle x_0, y \rangle. \end{aligned}$$

That is,

$$\sigma_C(x) \leq \sigma_C(x_0) + \|x - x_0\| \sup_{y \in C} \|y\|.$$

By definition of δ and M,

$$\sigma_C(x) < \sigma_C(x_0) + \varepsilon. \tag{1}$$

Similarly, reversing the role of x and x_0, we can prove that

$$\sigma_C(x_0) < \sigma_C(x) + \varepsilon. \tag{2}$$

From (1) and (2) we get

$$|\sigma_C(x) - \sigma_C(x_0)| < \varepsilon.$$

Since $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|\sigma_C(x) - \sigma_C(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \delta$, by definition, δ_C is continuous.

∎

Proposition 12.2.3. *Let S be a subset of $\mathbb{E}$. Then $\sigma_S = \sigma_{\mathrm{conv}(S)} = \sigma_{\mathrm{clconv}(S)}$.*

Proof.

Let x be an arbitrary point in $\mathbb{E}$.

$$\sigma_S(x) = \sup \left\{ \langle x, s \rangle : s \in S \right\}$$

$$\sigma_{\mathrm{conv}(S)}(x) = \sup\left\{\langle x, s\rangle : s \in \mathrm{conv}(S)\right\}$$
$$\sigma_{\mathrm{clconv}(S)}(x) = \sup\left\{\langle x, s\rangle : s \in \mathrm{clconv}(S)\right\}.$$

It is easy to see that by the linearity of inner products,

$$\mathrm{conv}\left\{\langle x, s\rangle : s \in S\right\} = \left\{\langle x, s\rangle : s \in \mathrm{conv}(S)\right\}.$$

It is easy to see that by the linearity and the continuity of inner products,

$$\mathrm{clconv}\left\{\langle x, s\rangle : s \in S\right\} = \left\{\langle x, s\rangle : s \in \mathrm{clconv}(S)\right\}.$$

It is also easy to see that for any subset A of the reals,

$$\sup(A) = \sup(\mathrm{conv}(A)),$$

and

$$\sup(A) = \sup(\mathrm{cl}(A)).$$

So

$$\begin{aligned}
&\sigma_S(x)\\
=&\ \sup\left\{\langle x, s\rangle : s \in S\right\}\\
=&\ \sup\mathrm{conv}\left\{\langle x, s\rangle : s \in S\right\}\\
=&\ \sup\left\{\langle x, s\rangle : s \in \mathrm{conv}(S)\right\}\\
=&\ \sigma_{\mathrm{conv}(S)}(x).
\end{aligned}$$

That is, $\sigma_S(x) = \sigma_{\mathrm{conv}(S)}(x)$.

$$\begin{aligned}
&\sigma_S(x)\\
=&\ \sup\left\{\langle x, s\rangle : s \in S\right\}\\
=&\ \sup\mathrm{conv}\left\{\langle x, s\rangle : s \in S\right\}\\
=&\ \sup\left\{\langle x, s\rangle : s \in \mathrm{conv}(S)\right\}\\
=&\ \sup\mathrm{cl}\left\{\langle x, s\rangle : s \in \mathrm{conv}(S)\right\}\\
=&\ \sup\left\{\langle x, s\rangle : s \in \mathrm{cl}(\mathrm{conv}(S))\right\}\\
=&\ \sup\left\{\langle x, s\rangle : s \in \mathrm{clconv}(S)\right\}\\
=&\ \sigma_{\mathrm{clconv}(S)}(x).
\end{aligned}$$

That is, $\sigma_S(x) = \sigma_{\mathrm{clconv}(S)}(x)$.

■

12.3 Supporting Hyperplane

Theorem 10 (Supporting Hyperplane Theorem)**.** *For any boundary point x_0 of a convex set C, there exists a supporting hyperplane to C at x_0.*

Chapter 13

Conjugacy

13.1 Definition and Examples

Definition (Convex Conjugate (Legendre–Fenchel Transformation))**.** *Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the* ***convex conjugate*** *of f, denoted by f^*, to be a function also from $\mathbb{E}$ to $\mathbb{R}^*$ given by*

$$f^*(x) := \sup_{y \in \mathbb{E}} \big\{ \langle y, x \rangle - f(y) \big\}.$$

Example 13.1.1. *Let S be a subset of $\mathbb{E}$. Then $\delta_S^* = \sigma_S$.*

Proof. Recall that

$$\delta_S(x) = \begin{cases} 0, & x \in S, \\ +\infty, & x \notin S, \end{cases}$$
$$\sigma_S(x) = \sup_{y \in S} \langle x, y \rangle.$$

Now for any $x \in \mathbb{E}$,

$$\begin{aligned} & \delta_S^*(x) \\ = & \sup_{y \in S} \big(\langle x, y \rangle - \delta_S(y) \big) \\ = & \sup_{y \in S} \big(\langle x, y \rangle - 0 \big) \\ = & \sup_{y \in S} \langle x, y \rangle \\ = & \sigma_S(x). \end{aligned}$$

So $\delta_S^* = \sigma_S$. ∎

13.2 Basic Properties

Proposition 13.2.1. *The convex conjugate function is convex.*

Proof. If $\mathrm{dom}(f) = \emptyset$, then one can see that $f^* \equiv -\infty$. It is a pointwise supremum of affine functions. ∎

Proposition 13.2.2. *The convex conjugate function is lower semi-continuous.*

13.3 Double Conjugate

Proposition 13.3.1. *Let f be any function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $f^{**} \leq f$.*

Proof. Let x be an arbitrary point in $\mathbb{E}$.

$$\begin{aligned}
& f^{**}(x) \\
= \; & \sup_{y\in\mathbb{E}} \left\{ \langle y, x\rangle - f^*(y) \right\} \\
= \; & \sup_{y\in\mathbb{E}} \left\{ \langle y, x\rangle - \sup_{z\in\mathbb{E}} \left\{ \langle z, y\rangle - f(z) \right\} \right\} \\
\leq \; & \sup_{y\in\mathbb{E}} \left\{ \langle y, x\rangle - \left\{ \langle x, y\rangle - f(x) \right\} \right\} \\
= \; & \sup_{y\in\mathbb{E}} \left\{ f(x) \right\} \\
= \; & f(x).
\end{aligned}$$

That is, $f^{**}(x) \leq f(x)$. Since $\forall x \in \mathbb{E}$, $f^{**}(x) \leq f(x)$, we get $f^{**} \leq f$. ∎

Proposition 13.3.2. *Let f be a proper function. Then f is convex and lower semi-continuous if and only if*

$$f^{**} = f.$$

Proposition 13.3.3. *Let f and g be functions from $\mathbb{E}$ to $\mathbb{R}^*$. Then $f \leq g$ implies $f^* \geq g^*$ and $f^{**} \leq g^{**}$.*

Proof. Let x be an arbitrary point in $\mathbb{E}$.

$$\begin{aligned}
& f^*(x) \\
= \; & \sup_{y\in\mathbb{E}} \left\{ \langle y, x\rangle - f(y) \right\} \\
\geq \; & \sup_{y\in\mathbb{E}} \left\{ \langle y, x\rangle - g(y) \right\} \\
= \; & g^*(x).
\end{aligned}$$

That is, $f^*(x) \geq g^*(x)$. Since $\forall x \in \mathbb{E}$, $f^*(x) \geq g^*(x)$, we get $f^* \geq g^*$.

Let x be an arbitrary point in $\mathbb{E}$.

$$\begin{aligned}
& f^{**}(x) \\
= \; & \sup_{y \in \mathbb{E}} \big\{ \langle y, x \rangle - f^*(y) \big\} \\
= \; & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \sup_{z \in \mathbb{E}} \big\{ \langle z, y \rangle - f(z) \big\} \right\} \\
\leq \; & \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \sup_{z \in \mathbb{E}} \big\{ \langle z, y \rangle - g(z) \big\} \right\} \\
= \; & \sup_{y \in \mathbb{E}} \big\{ \langle y, x \rangle - g^*(y) \big\} \\
= \; & g^{**}(x).
\end{aligned}$$

That is, $f^{**}(x) \leq g^{**}(x)$. Since $\forall x \in \mathbb{E}$, $f^{**}(x) \leq g^{**}(x)$, we get $f^{**} \leq g^{**}$. ■

Proposition 13.3.4.

$$\operatorname{epi}(f^{**}) = \operatorname{conv}(\operatorname{epi}(f)).$$

13.4 Conjugates and Sub-Differentials

Theorem 11 (Fenchel-Young)**.** *Let f be a proper function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x, y \in \mathbb{E}$, we have*

$$f(x) + f^*(y) \geq \langle x, y \rangle.$$

Proposition 13.4.1. *Let f be a proper closed convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x, y \in \mathbb{E}$,*

$$y \in \partial f(x) \iff x \in \partial f^*(y) \iff f(x) + f^*(y) = \langle x, y \rangle.$$

Proof of $y \in \partial f(x) \iff x \in \partial f^*(y)$. For one direction, assume that $y \in \partial f(x)$. We are to prove that $x \in \partial f^*(y)$. Consider an arbitrary point $z \in \mathbb{E}$. Since $y \in \partial f(x)$, we get

$$\forall u \in \mathbb{E}, \quad \langle y, u - x \rangle \leq f(u) - f(x).$$

Rearranging yields

$$\forall u \in \mathbb{E}, \quad \langle y, u \rangle - f(u) \leq \langle y, x \rangle - f(x).$$

It follows that

$$\sup_{u \in \mathbb{E}} \big(\langle y, u \rangle - f(u) \big) \leq \langle y, x \rangle - f(x). \tag{1}$$

By definition of supremum, we have

$$\sup_{u \in \mathbb{E}} \big(\langle y, u \rangle - f(u) \big) \geq \langle y, x \rangle - f(x). \tag{2}$$

From (1) and (2), we get

$$\sup_{u\in\mathbb{E}}\big(\langle y,u\rangle - f(u)\big) = \langle y,x\rangle - f(x).$$

That is,

$$f^*(y) = \langle y,x\rangle - f(x).$$

Then

$$\begin{aligned}
& f^*(z) - f^*(y) \\
= \; & \sup_{u\in\mathbb{E}}\big(\langle z,u\rangle - f(u)\big) - \sup_{u\in\mathbb{E}}\big(\langle y,u\rangle\rangle - f(u)\big) \\
= \; & \sup_{u\in\mathbb{E}}\big(\langle z,u\rangle - f(u)\big) - \langle y,x\rangle + f(x) \\
\geq \; & \langle z,x\rangle - f(x) - \langle y,x\rangle + f(x) \\
= \; & \langle z-y,x\rangle.
\end{aligned}$$

That is,

$$\langle x, z-y\rangle \leq f^*(z) - f^*(y).$$

So $x \in \partial f^*(y)$. This proves

$$y \in \partial f(x) \implies x \in \partial f^*(y).$$

Since $f^{**} = f$, similarly, we can prove that

$$x \in \partial f^*(y) \implies y \in \partial f(x).$$

■

Proposition 13.4.2. *Let f be a proper convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x be a point in $\mathbb{E}$. Assume that $\partial f(x) \neq \emptyset$. Then $f^{**}(x) = f(x)$.*

Chapter 14

Proximal

14.1 Definition

Definition (Proximal Operator). *Let f be a function from $\mathbb{E}$ to $\mathbb{R}^*$. We define the* ***proximal operator*** *of f, denoted by* prox_f*, to be a function from $\mathbb{E}$ to $\mathcal{P}(\mathbb{E})$ given by*

$$\text{prox}_f(x) := \underset{y \in \mathbb{E}}{\text{argmin}}\{f(y) + \frac{1}{2}\|y - x\|^2\}.$$

14.2 Basic Properties

Proposition 14.2.1. *Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Then $\forall x \in \mathbb{E}$, $\text{prox}_f(x)$ is a singleton set.*

Proposition 14.2.2. *Let C be a nonempty closed convex set in $\mathbb{E}$. Then $\text{prox}_{\delta_C} = \text{proj}_C$.*

Proposition 14.2.3 (Firmly Non-Expansive). *Let f be a proper closed convex function. Then prox_f is firmly non-expansive.*

14.3 Prox Calculus Rules

Proposition 14.3.1 (Scaling and Translation).

Theorem 12 (Norm Composition).

14.4 The Second Prox Theorem

Proposition 14.4.1. *Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x and p be points in $\mathbb{E}$. Then $p = \operatorname{prox}_f(x)$ if and only if*

$$x - p \in \partial f(p).$$

Proposition 14.4.2. *Let f be a proper, convex, and lower semi-continuous function from $\mathbb{E}$ to $\mathbb{R}^*$. Let x and p be points in $\mathbb{E}$. Then $p = \operatorname{prox}_f(x)$ if and only if*

$$\forall y \in \mathbb{E}, \quad \langle y - p, x - p \rangle \leq f(y) - f(p).$$

14.5 Moreau Decomposition

Theorem 13 (Moreau Decomposition). *Let f be a proper closed convex function from $\mathbb{E}$ to $\mathbb{R}^*$. Then*

$$\operatorname{prox}_f + \operatorname{prox}_{f^*} = \operatorname{id}.$$

Proof. Let x be an arbitrary point in $\mathbb{E}$. We are to prove that

$$\operatorname{prox}_f(x) + \operatorname{prox}_{f^*}(x) = x.$$

Let p denote $\operatorname{prox}_f(x)$. Since f is proper convex and lower semi-continuous and $p = \operatorname{prox}_f(x)$, we get

$$x - p \in \partial f(p).$$

Since $x - p \in \partial f(p)$, we get $p \in \partial f^*(x - p)$. It follows that $x - p = \operatorname{prox}_{f^*}(x)$. Substitute $p = \operatorname{prox}_f(x)$ and rearrange the equation, we get

$$\operatorname{prox}_f(x) + \operatorname{prox}_{f^*}(x) = x.$$

■

Chapter 15

Ellipsoids

Definition (Ellipsoid)**.** *Let v be a point in some Euclidean space $\mathbb{E}$. We define an **ellipsoid**, centered at point v, to be a set of the form*

$$\{x \in \mathbb{E} : (x - v)^T A(x - v) = 1\}$$

where A is some d by d positive definite matrix.

15.1 Properties

Proposition 15.1.1. *The eigenvectors of A define the principal axes of the ellipsoid.*