

# Convex Analysis

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# Chapter 1

## Affine Sets

### 1.1 Definitions

**DEFINITION 1.1** (Affine Combination). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define an **affine combination** of  $S$  to be a point  $x$  in the space of the form

$$x = \sum_{i=1}^n \lambda_i v_i$$

where  $n \in \mathbb{N}$ ,  $v_i \in S$ ,  $\forall i \in [n]$ ,  $\lambda_i \in \mathbb{R}$ ,  $\forall i \in [n]$ , and  $\sum_{i=1}^n \lambda_i = 1$ .

**DEFINITION 1.2** (Affine Span). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define the **affine span** of  $S$ , denoted by  $\text{affspan}(S)$ , to be the set of all affine combinations of  $S$ .

**DEFINITION 1.3** (Affine Set). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We say that  $S$  is an **affine set** if and only if  $S = \text{affspan}(S)$ .

**DEFINITION 1.4** (Affine Hull). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset

of  $\mathcal{V}$ . We define the **affine hull** of  $S$ , denoted by  $\text{affhull}(S)$ , to be the smallest affine set containing  $S$ .

**THEOREM 1.5.** Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . Then

$$\text{affspan}(S) = \text{affhull}(S).$$



## Chapter 2

# Generalized Interiors

### 2.1 Definitions

**DEFINITION 2.1** (Relative Interior - 1). Let  $\mathcal{V}$  be a normed linear space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define the **relative interior** of  $S$ , denoted by  $\text{ri}(S)$ , to be the interior of  $S$  for the topology relative to the affine hull  $\text{aff}(S)$ . i.e., the set given by

$$\text{ri}(S) := \left\{ x \in \text{aff}(S) : \exists r > 0, \text{ball}(x, r) \cap \text{aff}(S) \subseteq S \right\}.$$

**DEFINITION 2.2** (Relative Interior - 2). Let  $\mathcal{V}$  be a normed linear space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define the **relative interior** of  $S$ , denoted by  $\text{ri}(S)$ , to be a subset of  $S$  given by

$$\text{ri}(S) := \left\{ x \in S : \text{cone}(S - x) = \text{span}(S - x) \right\}.$$

**DEFINITION 2.3** (Strong Relative Interior). Let  $\mathcal{V}$  be a normed linear space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define the **strong relative interior** of  $S$ , denoted by  $\text{sri}(S)$ , to be a subset of  $S$  given by

$$\text{sri}(S) := \left\{ x \in S : \text{cone}(S - x) = \overline{\text{span}}(S - x) \right\}.$$

**DEFINITION 2.4** (Quasi-Relative Interior). Let  $\mathcal{V}$  be a normed linear space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define the **quasi-relative interior** of  $S$ , denoted by  $\text{qri}(S)$ , to be a subset of  $S$  given by

$$\text{qri}(S) := \left\{ x \in S : \overline{\text{cone}}(S - x) = \overline{\text{span}}(S - x) \right\}.$$

## 2.2 Basic Properties

**PROPOSITION 2.5.** For a singleton set  $S$ ,  $\text{ri}(S) = S = \text{cl}(S)$ .

**PROPOSITION 2.6.** For any set  $S$ , we have  $\text{ri}(S) \subseteq S$ .

**REMARK 2.7.** The relative interior operator is not monotonic. Consider  $\mathbb{R}$  with the usual topology and sets  $\{0\}$  and  $[0, 1]$ . Then  $\text{ri}(\{0\}) = \{0\}$  and  $\text{ri}([0, 1]) = (0, 1)$ .

**PROPOSITION 2.8.** Let  $\mathcal{V}$  be a normed linear space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . Then if  $\text{int}(S) \neq \emptyset$  we have  $\text{ri}(S) = \text{int}(S)$ .

*Proof.* It suffices to show that  $\text{aff}(S) = \mathcal{V}$ . Since  $\text{int}(S) \neq \emptyset$ , we can take  $x \in \text{int}(S)$ . Then  $\exists r > 0$ ,  $\text{ball}(x, r) \subseteq S$ . Then

$$\mathcal{V} = \text{aff}(\text{ball}(x, r)) \subseteq \text{aff}(S) \subseteq \mathcal{V}.$$

So  $\text{aff}(S) = \mathcal{V}$ . □

## 2.3 Arithmetic Properties

**PROPOSITION 2.9** (Linearity). Let  $\mathcal{V}$  be a normed linear space over  $\mathbb{R}$ . Let  $C_1$

and  $C_2$  be convex subsets of  $\mathbb{R}$ . Let  $\lambda_1, \lambda_2 \in \mathbb{R}$ . Then

$$\text{ri}(\lambda_1 C_1 + \lambda_2 C_2) = \lambda_1 \text{ri}(C_1) + \lambda_2 \text{ri}(C_2).$$

**PROPOSITION 2.10.** Let  $C_1$  be a convex set in  $\mathbb{E}_1$ . Let  $C_2$  be a convex set in  $\mathbb{E}_2$ . Then

$$\text{ri}(C_1 \oplus C_2) = \text{ri}(C_1) \oplus \text{ri}(C_2).$$



## Chapter 3

# Convex Sets

### 3.1 Definitions (bug)

**DEFINITION 3.1** (Convex Combination). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define a **convex combination** of  $S$  to be a point  $x$  in  $\mathcal{V}$  of the form

$$x = \sum_{i=1}^n \lambda_i v_i$$

where (1)  $n \in \mathbb{N}$ , (2)  $v_1, \dots, v_n \in S$ , (3)  $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ , and (4)  $\sum_{i=1}^n \lambda_i = 1$ .

**DEFINITION 3.2** (Convex Span). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define a **convex span** of  $S$ , denoted by  $\text{convspan}(S)$ , to be the set of all possible convex combinations of  $S$ .

**DEFINITION 3.3** (Convex). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathbb{E}$ . We say that  $S$  is **convex** if  $S = \text{convspan}(S)$ , or equivalently, if

$$\forall x, y \in S, \forall \alpha, \beta \in [0, 1] : \alpha + \beta = 1, \quad \alpha x + \beta y \in S.$$

**DEFINITION 3.4** (Convex Hull). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define the **convex hull** of  $S$ , denoted by  $\text{convhull}(S)$ , to be the smallest convex set containing  $S$ .

**PROPOSITION 3.5.** Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . For any subset  $S$  of  $\mathcal{V}$ , we have  $\text{convspan}(S) = \text{convhull}(S)$ . They will both be denoted by  $\text{conv}(S)$  from now on.

*Proof. Forward Inclusion:* Let  $x$  be an arbitrary element of  $\text{convspan}(S)$ . I will show that  $x \in \text{convhull}(S)$ . Let  $C$  be an arbitrary convex set containing  $S$ . Since  $x$  is a convex combination of elements in  $S$ ,  $x$  is also a convex combination of elements in  $C$ . So  $x \in C$ . This holds for any convex set in  $\mathcal{V}$  containing  $S$ . So  $x \in \text{convhull}(S)$ . So  $\text{convspan}(S) \subseteq \text{convhull}(S)$ .

**Backward Inclusion:** I will show that  $\text{convhull}(S) \subseteq \text{convspan}(S)$ . □

not finished

**DEFINITION 3.6** (Pointed). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We say that  $S$  is **pointed** if and only if  $S$  contains no line.

## 3.2 Arithmetic Properties of Convex Sets

**PROPOSITION 3.7.** Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $C$  be a convex subset of  $\mathcal{V}$ . Let  $\lambda_1, \lambda_2 \in \mathbb{R}_+$ . Then

$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C.$$

*Proof.* The case where  $\lambda_1 = 0$  or  $\lambda_2 = 0$  is trivial. Now suppose that  $\lambda_1 \neq 0$  and  $\lambda_2 \neq 0$ .

**Forward Inclusion:** Let  $x$  be an arbitrary element of  $(\lambda_1 + \lambda_2)C$ . I will show that  $x \in \lambda_1 C + \lambda_2 C$ . Since  $x \in (\lambda_1 + \lambda_2)C$ ,  $x = (\lambda_1 + \lambda_2)c$  for some  $c \in C$ . Since  $\mathcal{V}$  is a vector space over  $\mathbb{R}$ ,  $c \in C \subseteq \mathcal{V}$ , and  $\lambda_1, \lambda_2 \in \mathbb{R}_+ \subseteq \mathbb{R}$ , we get  $x = (\lambda_1 + \lambda_2)c = \lambda_1 c + \lambda_2 c$ . Notice  $\lambda_1 c \in \lambda_1 C$ , and  $\lambda_2 c \in \lambda_2 C$ . So  $x \in \lambda_1 C + \lambda_2 C$ . So  $(\lambda_1 + \lambda_2)C \subseteq \lambda_1 C + \lambda_2 C$ .

**Backward Inclusion:** Let  $x$  be an arbitrary element of  $\lambda_1 C + \lambda_2 C$ . I will show that

$x \in (\lambda_1 + \lambda_2)C$ . Since  $x \in \lambda_1 C + \lambda_2 C$ ,  $x = \lambda_1 c_1 + \lambda_2 c_2$  for some  $c_1, c_2 \in C$ . Notice

$$x = \lambda_1 c_1 + \lambda_2 c_2 = (\lambda_1 + \lambda_2) \left( \underbrace{\frac{\lambda_1}{\lambda_1 + \lambda_2}}_{\in [0,1]} \underbrace{c_1}_{\in C} + \underbrace{\frac{\lambda_2}{\lambda_1 + \lambda_2}}_{\in [0,1]} \underbrace{c_2}_{\in C} \right).$$

Notice the second term is a convex combination of two points in  $C$  and hence is in  $C$ . So  $x \in (\lambda_1 + \lambda_2)C$ . So  $\lambda_1 C + \lambda_2 C \subseteq (\lambda_1 + \lambda_2)C$ .  $\square$

### 3.3 The Convex Hull Operator

**PROPOSITION 3.8** (The Convex Hull Operator). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ .

1. Expansive

$$\forall S \subseteq \mathcal{V}, \quad S \subseteq \text{conv}(S).$$

2. Monotonic Increasing

$$\forall S_1, S_2 \subseteq \mathcal{V} : S_1 \subseteq S_2, \quad \text{conv}(S_1) \subseteq \text{conv}(S_2).$$

3. Idempotent

$$\forall S \subseteq \mathcal{V}, \quad \text{conv}(\text{conv}(S)) = \text{conv}(S).$$

**PROPOSITION 3.9** (Bounded). The convex hull of a bounded set is bounded.

*Proof.* Let  $\mathcal{V}$  be a normed linear space over  $\mathbb{R}$ . Let  $C$  be a bounded subset of  $\mathcal{V}$ . Then  $\exists R > 0$  such that  $\forall c \in C, \|c\| < R$ . Let  $x$  be an arbitrary element of  $\text{conv}(C)$ . Then  $\exists n \in \mathbb{Z}_{++}, \exists \lambda_1, \dots, \lambda_n \in [0, 1], \exists c_1, \dots, c_n \in C$  such that  $\sum_{i \in [n]} \lambda_i = 1$  and  $x = \sum_{i \in [n]} \lambda_i c_i$ . Then by the triangle inequality of  $\|\cdot\|$ , we get

$$\|x\| = \left\| \sum_{i \in [n]} \lambda_i c_i \right\| \leq \sum_{i \in [n]} \lambda_i \|c_i\| < \sum_{i \in [n]} \lambda_i R = 1 \cdot R = R.$$

So  $\forall x \in \text{conv}(C), \|x\| < R$ . So  $\text{conv}(C)$  is bounded.  $\square$

**PROPOSITION 3.10** (Open). The convex hull of an open set is open.

*Proof Approach (1).* Let  $\mathcal{V}$  be a topological vector space. Let  $G$  be an open subset of  $\mathcal{V}$ . I will show that  $\text{conv}(G)$  is open. Let  $x \in \text{conv}(G)$  be arbitrary. Then  $x = \sum_{i \in [n]} \lambda_i g_i$  for some  $n \in \mathbb{Z}_{++}$ ,  $\lambda_i \in [0, 1]$ ,  $\forall i \in [n]$ , and  $g_i \in G$ ,  $\forall i \in [n]$ . Since  $G$  is an open set and  $g_i \in G$ ,  $\forall i \in [n]$ , there exist neighborhoods  $\mathcal{N}_1, \dots, \mathcal{N}_n$  of  $g_1, \dots, g_n$ , respectively, such that  $g_i \in \mathcal{N}_i \subseteq G$ ,  $\forall i \in [n]$ . Define  $\mathcal{N} := \sum_{i \in [n]} \lambda_i \mathcal{N}_i$ . Then  $\mathcal{N}$  is a neighborhood of  $g$ , and  $\mathcal{N} \subseteq \text{conv}(G)$ . So  $g \in \text{int}(\text{conv}(G))$ . So  $\text{conv}(G) \subseteq \text{int}(\text{conv}(G))$ . So  $\text{conv}(G) = \text{int}(\text{conv}(G))$ . So  $\text{conv}(G)$  is open.  $\square$

*Proof Approach (2).* Let  $\mathcal{V}$  be a topological vector space. Let  $G$  be an open subset of  $\mathcal{V}$ . I will show that  $\text{conv}(G)$  is open. Let  $x \in \text{conv}(G)$  be arbitrary. Then  $x = \sum_{i \in [n]} \lambda_i g_i$  for some  $n \in \mathbb{Z}_{++}$ ,  $\lambda_i \in [0, 1]$ ,  $\forall i \in [n]$ , and  $g_i \in G$ ,  $\forall i \in [n]$ . Let  $i_0 \in [n]$  be such that  $\lambda_{i_0} \neq 0$ . Then

$$x = \sum_{i=1}^n \lambda_i g_i = \left( \sum_{i \neq i_0} \lambda_i g_i \right) + \lambda_{i_0} g_{i_0} \in \left( \sum_{i \neq i_0} \lambda_i g_i \right) + i_0 G \subseteq \text{conv}(G).$$

So

$$\text{conv}(G) = \bigcup_{x \in \text{conv}(G)} \left\{ \sum_{i \neq i_0} \lambda_i g_i + i_0 G \right\}.$$

Note that for each  $x \in \text{conv}(G)$ , the function  $f_x : \mathcal{V} \rightarrow \mathcal{V}$  given by  $f_x(v) := \sum_{i \neq i_0} \lambda_i g_i + i_0 v$  is a homeomorphism. So

$$\text{conv}(G) = \bigcup_{x \in \text{conv}(G)} f_x(G)$$

is the union of a collection of open sets and hence is open.  $\square$

**REMARK 3.11** (Closed). The convex hull of a closed set need not be closed.

- Example in  $\mathbb{R}^2$ : The set  $S := \{(x, y) \in \mathbb{R}^2 : y \geq \frac{1}{1+x^2}\}$  is closed. However,  $\text{conv}(S) = \{(x, y) \in \mathbb{R}^2 : y > 0\}$  is open.
- Example in  $\ell^\infty$ : Define for each  $n \in \mathbb{Z}_{++}$  a sequence  $x_n$  by  $x_n^{(n)} := \frac{1}{n}$  and  $x_n^{(i)} := 0$ ,  $\forall i \neq n$ . Consider the set  $S := \{x_n\}_{n \in \mathbb{N}} \cup \{0\}$ . Then  $S$  is a compact subset of  $\ell^\infty$ . However,  $\text{conv}(S)$  contains the elements  $\sum_{n=1}^k 2^{-n} x_n$  for  $k \in \mathbb{Z}_{++}$ . Notice they converge to the sequence  $\sum_{n=1}^{\infty} 2^{-n} x_n$ , which is not in  $\text{conv}(K)$  (it has infinitely many non-zero entries).



**PROPOSITION 3.12** (Compact in  $\mathbb{R}^n$ ). Let  $K$  be a compact subset of  $\mathbb{R}^n$ . Then  $\text{conv}(K)$  is also compact.

**PROPOSITION 3.13.** Let  $\mathcal{V}$  be a normed linear space. Let  $K$  be a compact subset of  $\mathcal{V}$ . Then  $\overline{\text{conv}}(K)$  is pre-compact (totally bounded). Moreover, if  $\mathcal{V}$  is complete, then  $\overline{\text{conv}}(K)$  is also compact.

### 3.4 The Closed Convex Hull Operator

**DEFINITION 3.14** (Closed Convex Hull). Let  $S$  be a set in some Euclidean space. We define the **closed convex hull** of  $S$ , denoted by  $\overline{\text{conv}}(S)$ , to be the smallest closed convex containing  $S$ .

**PROPOSITION 3.15.** The closed convex hull is the closure of the convex hull.

**PROPOSITION 3.16.** A closed convex hull does not distinguish a set from its closure. i.e., for any set  $S$ , we have  $\overline{\text{conv}}(S) = \overline{\text{conv}}(\text{cl}(S))$ .

**PROPOSITION 3.17.** If  $S$  is bounded, then the closure operation and the convex hull operation commute. i.e.,  $\text{conv}(\text{cl}(S)) = \text{cl}(\text{conv}(S))$ .

**REMARK 3.18.** The closure operation and the convex hull operation do not commute in general.

### 3.5 Stability of Convexity

**PROPOSITION 3.19** (Intersection). Convexity is stable under intersection. i.e., the intersection of any collection of convex sets is convex.

*Proof.* Let  $\{C_i\}_{i \in I}$  be an arbitrary collection of convex sets where  $I$  is an index set and  $C_i$  is convex for any  $i \in I$ . Let  $C$  denote their intersection. If  $C = \emptyset$ , then we are done. Else, let  $x$  and  $y$  be two arbitrary points in  $C$ . Let  $\lambda$  be an arbitrary number in  $(0, 1)$ . Define a point  $z := \lambda x + (1 - \lambda)y$ . Since  $x \in C$  and  $C = \bigcap_{i \in I} C_i$ , we get  $x \in C_i$  for any  $i \in I$ . Since  $y \in C$  and  $C = \bigcap_{i \in I} C_i$ , we get  $y \in C_i$  for any  $i \in I$ . Let  $i$  be an arbitrary index in  $I$ . Since  $x \in C_i$  and  $y \in C_i$  and  $\lambda \in (0, 1)$  and  $C_i$  is convex and  $z = \lambda x + (1 - \lambda)y$ , we get  $z \in C_i$ . Since  $z \in C_i$  for any  $i \in I$  and  $C = \bigcap_{i \in I} C_i$ , we get  $z \in C$ . Since

$$\forall x, y \in C, \forall \lambda \in (0, 1), \quad \lambda x + (1 - \lambda)y \in C,$$

by definition of convex sets, we get  $C$  is convex.  $\square$

**PROPOSITION 3.20** (Affine Map). Convexity is stable under affine mapping. i.e., the affine image of a convex set is convex.

**PROPOSITION 3.21** (Linear Combinations). Convexity is stable under linear combinations. i.e., if  $C_1$  and  $C_2$  are convex sets and  $\lambda_1$  and  $\lambda_2$  are real numbers, then the set  $C$  defined as

$$C := \lambda_1 C_1 + \lambda_2 C_2$$

is convex.

*Proof.* If  $C_1 = \emptyset$  or  $C_2 = \emptyset$ , then  $\lambda_1 C_1 + \lambda_2 C_2 = \emptyset$  and we are done. Now assume that  $C_1, C_2 \neq \emptyset$ . Then  $C = \lambda_1 C_1 + \lambda_2 C_2 \neq \emptyset$ . Let  $x$  and  $y$  be arbitrary points in  $C$ .

Since  $x \in C$ ,  $\exists x_1 \in C_1, x_2 \in C_2$  such that  $x = \lambda_1 x_1 + \lambda_2 x_2$ .

Since  $y \in C$ ,  $\exists y_1 \in C_1, y_2 \in C_2$  such that  $y = \lambda_1 y_1 + \lambda_2 y_2$ .

Let  $\lambda \in [0, 1]$  be arbitrary. Define a point  $z$  as  $z := \lambda x + (1 - \lambda)y$ . Then

$$\begin{aligned} z &= \lambda x + (1 - \lambda)y \\ &= \lambda(\lambda_1 x_1 + \lambda_2 x_2) + (1 - \lambda)(\lambda_1 y_1 + \lambda_2 y_2) \\ &= \lambda_1(\lambda x_1 + (1 - \lambda)y_1) + \lambda_2(\lambda x_2 + (1 - \lambda)y_2). \end{aligned}$$

Since  $x_1, y_1 \in C_1$ ,  $\lambda \in [0, 1]$  and  $C_1$  is convex, we get  $\lambda x_1 + (1 - \lambda)y_1 \in C_1$ .

Since  $x_2, y_2 \in C_2$ ,  $\lambda \in [0, 1]$  and  $C_2$  is convex, we get  $\lambda x_2 + (1 - \lambda)y_2 \in C_2$ .

So  $z = \lambda_1(\lambda x_1 + (1 - \lambda)y_1) + \lambda_2(\lambda x_2 + (1 - \lambda)y_2) \in \lambda_1 C_1 + \lambda_2 C_2$ .

That is,  $\forall x \in C, \forall y \in C, \forall \lambda \in [0, 1]$ , we have  $\lambda x + (1 - \lambda)y \in C$ .

So by definition,  $C$  is convex.  $\square$

**COROLLARY 3.22.** The Minkowski sum of two convex sets is convex.

**LEMMA 3.23.** Let  $\mathcal{V}$  be a normed linear space. Let  $C$  be a convex subset of  $\mathcal{V}$ . Let  $x \in \text{int}(C)$ . Let  $y \in \text{cl}(C)$ . Then

$$\forall t \in (0, 1], \quad tx + (1 - t)y \in C.$$

*Proof Approach (1).* Let  $t \in (0, 1]$  be arbitrary. Define a point  $z \in \mathcal{V}$  by  $z := tx + (1 - t)y$ . If  $t = 1$ , then  $z = x \in \text{int}(C) \subseteq C$  and we are done. Otherwise,  $t \in (0, 1)$ . Since  $x \in \text{int}(C)$ ,  $\exists r_x > 0$  such that  $\text{ball}(x, r_x) \subseteq C$ . Define  $r_y := \frac{t}{1-t}r_x$ . Since  $y \in \text{cl}(C)$ ,  $\exists y' \in \text{ball}(y, r_y) \cap C$ . Define a point  $z' \in \mathcal{V}$  by  $z' := tx + (1 - t)y'$ . Since  $x, y' \in C$ ,  $t \in (0, 1)$ , and  $C$  is convex, we get  $z' \in C$ . Define a point  $x' \in \mathcal{V}$  by  $x' := \frac{1}{t}(z - (1 - t)y')$  so that  $z = tx' + (1 - t)y'$ . Notice  $x = \frac{1}{t}(z' - (1 - t)y')$ . So

$$\begin{aligned} \|x - x'\| &= \left\| x - \frac{1}{t}(z - (1 - t)y') \right\| = \left\| \frac{1}{t}(z' - (1 - t)y') - \frac{1}{t}(z - (1 - t)y') \right\| \\ &= \frac{1}{t} \|z - z'\| = \frac{1}{t} \|tx + (1 - t)y - tx - (1 - t)y'\| \\ &= \frac{1-t}{t} \|y - y'\| \leq \frac{1-t}{t} r_y = \frac{1-t}{t} \frac{t}{1-t} r_x = r_x. \end{aligned}$$

That is,  $\|x - x'\| \leq r_x$ . So  $x' \in \text{ball}(x, r_x) \subseteq C$ . Since  $x', y' \in C$ ,  $t \in (0, 1)$ , and  $C$  is convex, we get  $z \in C$ .  $\square$

*Proof Approach (2).* Let  $t \in (0, 1]$  be arbitrary. If  $t = 1$ , then  $tx + (1 - t)y = x \in \text{int}(C) \subseteq C$  and we are done. Otherwise,  $t \in (0, 1)$ . Define  $B := \text{ball}(0, 1)$ . Then for some small enough  $\varepsilon > 0$ , we have

$$\begin{aligned} tx + (1 - t)y + \varepsilon B &\subseteq tx + (1 - t)(C + \varepsilon B) + \varepsilon B, \text{ since } y \in \text{cl}(C) \\ &= tx + (1 - t)C + (1 - t)\varepsilon B + \varepsilon B \\ &= tx + (2 - t)\varepsilon B + (1 - t)C \end{aligned}$$

$$\begin{aligned}
&= t\left(x + \frac{2-t}{t}\varepsilon B\right) + (1-t)C \\
&\subseteq tC + (1-t)C, \text{ since } x \in \text{int}(C) \\
&= C.
\end{aligned}$$

□

**LEMMA 3.24.** Let  $C$  be a convex set in  $\mathbb{E}$ . Let  $x \in \text{ri}(C)$ . Let  $y \in \text{cl}(C)$ . Then

$$\forall \lambda \in (0, 1], \quad \lambda x + (1 - \lambda)y \in C.$$

*Proof.*

**Case 1.**  $\text{int}(C) \neq \emptyset$ .

Then  $\text{int}(C) = \text{ri}(C)$ .

Since  $x \in \text{int}(C)$  and  $y \in \text{cl}(C)$ ,  $\forall t \in (0, 1]$ ,  $z := tx + (1 - t)y \in C$ .

**Case 2.**  $\text{int}(C) = \emptyset$ .

Now  $\dim(C) < d$ .

Say  $\dim(C) = l$ .

Apply case 1 in  $\mathbb{R}^l$ .

□

**PROPOSITION 3.25** (Interior). Convexity is stable under interior. i.e., the interior of a convex set is convex.

*Proof.* Let  $\mathcal{V}$  be a normed linear space. Let  $S$  be a convex subset of  $\mathcal{V}$ . If  $\text{int}(S) = \emptyset$ , then we are done. Else: let  $x$  and  $y$  be two arbitrary points in  $\text{int}(S)$ . Let  $\lambda$  be an arbitrary number in  $(0, 1)$ . Define a point  $z$  by  $z := \lambda x + (1 - \lambda)y$ . Since  $x, y \in \text{int}(S)$  and  $\lambda \in (0, 1)$ , by the lemma, we get  $z \in \text{int}(S)$ . Since

$$\forall x, y \in \text{int}(S), \forall \lambda \in (0, 1), \quad \lambda x + (1 - \lambda)y \in \text{int}(S),$$

we get  $\text{int}(S)$  is convex.

□

**PROPOSITION 3.26** (Relative Interior). Convexity is stable under relative interior. i.e., the relative interior of a convex set is convex.

*Proof.* Let  $\mathcal{V}$  be a normed linear space. Let  $S$  be a convex subset of  $\mathcal{V}$ . If  $\text{ri}(S) = \emptyset$ , then we are done. Otherwise, let  $x$  and  $y$  be two arbitrary points in  $\text{ri}(S)$ . Let  $\lambda$  be an arbitrary number in  $(0, 1)$ . Define a point  $z$  by  $z := \lambda x + (1 - \lambda)y$ . Since  $x, y \in \text{ri}(S)$  and  $\lambda \in (0, 1)$ , by the lemma, we get  $z \in \text{ri}(S)$ . Since

$$\forall x, y \in \text{ri}(S), \forall \lambda \in (0, 1), \quad \lambda x + (1 - \lambda)y \in \text{ri}(S),$$

we get  $\text{ri}(S)$  is convex.  $\square$

**PROPOSITION 3.27** (Closure). Convexity is stable under closure. i.e., the closure of a convex set is convex.

*Proof Approach 1.*

Let  $x, y \in \text{cl}(C)$ .

Let  $t \in [0, 1]$ .

Since  $x \in \text{cl}(C)$ ,  $\exists \{x_i\}_{i \in \mathbb{N}} \subseteq C$ ,  $\lim_{i \rightarrow \infty} x_i = x$ .

Since  $y \in \text{cl}(C)$ ,  $\exists \{y_i\}_{i \in \mathbb{N}} \subseteq C$ ,  $\lim_{i \rightarrow \infty} y_i = y$ .

Since  $\lim_{i \rightarrow \infty} x_i = x$  and  $\lim_{i \rightarrow \infty} y_i = y$ ,  $\lim_{i \rightarrow \infty} (tx_i + (1 - t)y_i) = tx + (1 - t)y$ .

Since  $x_i, y_i \in C$  and  $C$  is convex,  $tx_i + (1 - t)y_i \in C$ .

Since  $tx_i + (1 - t)y_i \in C$ ,  $\lim_{i \rightarrow \infty} (tx_i + (1 - t)y_i) = tx + (1 - t)y$ ,  $tx + (1 - t)y \in \text{cl}(C)$ .

Since  $\forall x, y \in \text{cl}(C), \forall t \in [0, 1], tx + (1 - t)y \in \text{cl}(C)$ , we get  $\text{cl}(C)$  is convex.  $\square$

*Proof Approach 2.*

$\text{cl}(C) = \bigcap_{\varepsilon > 0} [C + \varepsilon \text{ball}(0, 1)]$ . This is an intersection of linear combinations of convex sets and hence convex.  $\square$

**PROPOSITION 3.28** (Conical Hull). Convexity is stable under conical hull. i.e., if  $C$  is convex, then  $\text{cone}(C)$  is convex.

*Proof.*

Let  $x$  and  $y$  be arbitrary points in  $\text{cone}(C)$ .

Let  $\lambda$  be an arbitrary number in  $(0, 1)$ .

Define point  $z$  as  $z := \lambda x + (1 - \lambda)y$ .

Since  $x \in \text{cone}(C)$ ,  $\exists x' \in C$  and  $\exists \alpha > 0$  such that  $x = \alpha x'$ .

Since  $y \in \text{cone}(C)$ ,  $\exists y' \in C$  and  $\exists \beta > 0$  such that  $y = \beta y'$ .

Define point  $z'$  as  $z' := \frac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta}x' + \frac{(1-\lambda)\beta}{\lambda\alpha + (1-\lambda)\beta}y'$ .

Since  $x', y' \in C$  and  $\frac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta} \in (0, 1)$  and  $\frac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta} + \frac{(1-\lambda)\beta}{\lambda\alpha + (1-\lambda)\beta} = 1$  and  $C$  is convex and  $z' := \frac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta}x' + \frac{(1-\lambda)\beta}{\lambda\alpha + (1-\lambda)\beta}y'$ , we get  $z' \in C$ .

Since  $z' \in C$  and  $z = (\lambda\alpha + (1-\lambda)\beta)z'$ ,  $z \in \text{cone}(C)$ .

That is,  $\lambda x + (1-\lambda)y \in \text{cone}(C)$ .

Since  $\forall x, y \in \text{cone}(C)$ ,  $\forall \lambda \in (0, 1)$ ,  $\lambda x + (1-\lambda)y \in \text{cone}(C)$ , we get  $\text{cone}(C)$  is convex.  $\square$

### 3.6 Topological Properties of Convex Sets

**THEOREM 3.29.** Let  $\mathcal{V}$  be a topological vector space. Let  $C$  be a convex subset of  $\mathcal{V}$  with  $\text{int}(C) \neq \emptyset$ . Then

1.  $\text{int}(C) = \text{int}(\text{cl}(C))$ , and
2.  $\text{cl}(C) = \text{cl}(\text{int}(C))$ .

*Proof of 1.* Since  $C \subseteq \text{cl}(C)$ , we have  $\text{int}(C) \subseteq \text{int}(\text{cl}(C))$ . So there remains only to show that  $\text{int}(\text{cl}(C)) \subseteq \text{int}(C)$ . Let  $x$  be an arbitrary element of  $\text{int}(\text{cl}(C))$ . Then  $\exists \varepsilon > 0$  such that  $\text{ball}(x, \varepsilon) \subseteq \text{cl}(C)$ .  $\square$

*Proof of (1).*  $\text{int}(C) \subseteq \text{int}(\text{cl}(C))$  is clear. For  $\text{int}(\text{cl}(C)) \subseteq \text{int}(C)$ , let  $x$  be an arbitrary point in  $\text{int}(\text{cl}(C))$ .

Since  $x \in \text{int}(\text{cl}(C))$ ,

$$\exists r > 0 \text{ such that } \text{ball}(x, r) \subseteq \text{cl}(C).$$

Since  $\text{int}(C) \neq \emptyset$ , pick  $y \in \text{int}(C)$ .

Define a scalar  $\lambda$  by

$$\lambda := \frac{r}{2\|x - y\|}.$$

Define a point  $z$  by

$$z := x + \lambda(x - y).$$

Since  $\lambda = \frac{r}{2\|x - y\|}$  and  $z = x + \lambda(x - y)$ ,

$$\begin{aligned} & \|z - x\| \\ &= \|x + \lambda(x - y) - x\| \end{aligned}$$

$$\begin{aligned}
&= \|\lambda(x - y)\| \\
&= \lambda\|x - y\| \\
&= \frac{r}{2\|x - y\|}\|x - y\| \\
&= \frac{r}{2} \\
&< r.
\end{aligned}$$

That is,

$$\|z - x\| < r.$$

So  $z \in \text{ball}(x, r)$ . It follows that  $z \in \text{cl}(C)$ .

Since  $z = x + \lambda(x - y)$ , rearranging this yields

$$x = \frac{1}{1 + \lambda}z + \frac{\lambda}{1 + \lambda}y.$$

$$\text{Since } \begin{cases} x = \frac{1}{1 + \lambda}z + \frac{\lambda}{1 + \lambda}y \\ z \in \text{cl}(C) \\ y \in \text{int}(C) \\ \frac{1}{1 + \lambda}, \frac{\lambda}{1 + \lambda} \in (0, 1) \\ \frac{1}{1 + \lambda} + \frac{\lambda}{1 + \lambda} = 1 \end{cases}, \text{ by the lemma, we get}$$

$$x \in \text{int}(C).$$

Since  $\forall x \in \text{int}(\text{cl}(C)), x \in \text{int}(C)$ , we get  $\text{int}(\text{cl}(C)) \subseteq \text{int}(C)$ .

□

*Proof of (2).*  $\text{cl}(\text{int}(C)) \subseteq \text{cl}(C)$  is clear. For  $\text{cl}(C) \subseteq \text{cl}(\text{int}(C))$ , let  $x$  be an arbitrary point in  $\text{cl}(C)$ .

Since  $\text{int}(C) \neq \emptyset$ , pick  $y \in \text{int}(C)$ .

Let  $\lambda \in [0, 1)$ .

Define a point  $z$  by

$$z(\lambda) := \lambda x + (1 - \lambda)y.$$

$$\text{Since } \begin{cases} z(\lambda) := \lambda x + (1 - \lambda)y \\ x \in \text{cl}(C) \\ y \in \text{int}(C) \\ \lambda \in [0, 1) \end{cases}, \text{ by the lemma, we get}$$

$$z(\lambda) \in \text{int}(C).$$

Since  $\begin{cases} z(\lambda) \in \text{int}(C) \\ \lim_{\lambda \rightarrow 1} z(\lambda) = x \end{cases}$ , we get

$$x \in \text{cl}(\text{int}(C)).$$

Since  $\forall x \in \text{cl}(C), x \in \text{cl}(\text{int}(C))$ , we get  $\text{cl}(C) \subseteq \text{cl}(\text{int}(C))$ .

□

**PROPOSITION 3.30.** Let  $C$  be a convex set. Then

1.  $\text{aff}(\text{ri}(C)) = \text{aff}(C) = \text{aff}(\text{cl}(C))$ ,
2.  $\text{ri}(\text{ri}(C)) = \text{ri}(C) = \text{ri}(\text{cl}(C))$ , and
3.  $\text{cl}(\text{ri}(C)) = \text{cl}(C) = \text{cl}(\text{cl}(C))$ .

**PROPOSITION 3.31.** Let  $C$  be a convex set. Then

$$C \neq \emptyset \iff \text{ri}(C) \neq \emptyset.$$

*Proof. Forward Direction:* Assume that  $C \neq \emptyset$ . I will show that  $\text{ri}(C) \neq \emptyset$ . Since  $C \neq \emptyset$ ,  $\text{aff}(C) \neq \emptyset$ . Since  $C$  is convex,  $\text{aff}(C) = \text{aff}(\text{ri}(C))$ . Since  $\begin{cases} \text{aff}(C) \neq \emptyset \\ \text{aff}(C) = \text{aff}(\text{ri}(C)) \end{cases}$ , we get

$$\text{aff}(\text{ri}(C)) \neq \emptyset.$$

Since  $\text{aff}(\text{ri}(C)) \neq \emptyset$ , we get  $\text{ri}(C) \neq \emptyset$ .

**Backward Direction:** Assume that  $\text{ri}(C) \neq \emptyset$ . I will show that  $C \neq \emptyset$ . Since  $\text{ri}(C) \neq \emptyset$  and  $\text{ri}(C) \subseteq C$ , we get  $C \neq \emptyset$ . □

### 3.7 Examples of Convex Sets

**EXAMPLE 3.32.** Let  $I$  be an index set. Let  $b_i$  for  $i \in I$  be vectors in  $\mathbb{E}$ . Let  $\beta_i$  for  $i \in I$  be reals. Then the set  $C$  given by

$$C := \{x \in \mathbb{E} : \forall i \in I, \langle x, b_i \rangle \leq \beta_i\}$$

is convex.



*Proof.*

Each of  $C_i := \{x \in \mathbb{E} : \langle x, b_i \rangle \leq \beta_i\}$  is convex and  $C = \bigcap_{i \in I} C_i$ .

$$\begin{aligned} \langle z, b_i \rangle &= \langle \lambda x + (1 - \lambda)y, b_i \rangle \\ &= \lambda \langle x, b_i \rangle + (1 - \lambda) \langle y, b_i \rangle \\ &\leq \lambda \beta_i + (1 - \lambda) \beta_i \\ &= \beta_i. \end{aligned}$$

□

### 3.8 The Carathéodory Theorem

**THEOREM 3.33** (Carathéodory). Let  $S$  be a subset of  $\mathbb{R}^n$ . Let  $x$  be some point in  $\text{conv}(S)$ . Then  $x$  can be represented as a convex combination of at most  $n + 1$  points in  $S$ . i.e.,  $x$  lies in some  $r$ -simplex with vertices in  $S$ , where  $r \leq n$ .



## Chapter 4

# Geometric Objects

### 4.1 Definitions

**DEFINITION 4.1** (Hyperplane). Let  $\mathbb{E}$  be a Euclidean space over  $\mathbb{R}$ . Let  $H$  be a subset of  $\mathbb{E}$ . We say that  $H$  is a **hyperplane** if and only if  $H$  can be expressed as

$$H = \{x \in \mathbb{E} : a^\top x = b\}$$

for some  $a \in \mathbb{E} \setminus \{0\}$  and  $b \in \mathbb{R}$ .

**DEFINITION 4.2** (Closed Half-Space). Let  $\mathbb{E}$  be a Euclidean space over  $\mathbb{R}$ . Let  $P$  be a subset of  $\mathbb{E}$ . We say that  $P$  is a **closed half-space** if and only if  $P$  can be expressed as

$$P = \{x \in \mathbb{E} : a^\top x \leq b\}$$

for some  $a \in \mathbb{E} \setminus \{0\}$  and  $b \in \mathbb{R}$ .

**DEFINITION 4.3** (Polyhedron). Let  $\mathbb{E}$  be a Euclidean space over  $\mathbb{R}$ . Let  $P$  be a subset of  $\mathbb{E}$ . We say that  $P$  is a **polyhedron** if and only if  $P$  can be expressed as the intersection of finitely many closed half-spaces in  $\mathbb{E}$ .

### 4.2 Properties

**PROPOSITION 4.4.** Polyhedrons are convex.

# Chapter 5

## Cones

### 5.1 Definitions

**DEFINITION 5.1** (Conical Combination). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define a **conical combination** of  $S$  to be a point  $x$  in the space of the form

$$x = \sum_{i=1}^n \lambda_i v_i$$

where (1)  $n \in \mathbb{N}$ , (2)  $v_i \in S$  for all  $i$ , and (3)  $\lambda_i \in \mathbb{R}_{++}$  for all  $i$ .

**DEFINITION 5.2** (Cone). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We say that  $S$  is a **cone** if and only if  $S = \mathbb{R}_{++}S$ .

**DEFINITION 5.3** (Conical Hull). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . We define the **conical hull** of  $S$ , denoted by  $\text{cone}(S)$ , to be the intersection of all cones containing  $S$ .

**PROPOSITION 5.4.** Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{V}$ . Then  $\text{cone}(S) = \mathbb{R}_{++}S$ .

*Proof. Forward Direction:* I will show that  $\text{cone}(S) \subseteq \mathbb{R}_{++}S$ . Since  $\mathbb{R}_{++}\mathbb{R}_{++}S = \mathbb{R}_{++}S$ ,  $\mathbb{R}_{++}S$  is a cone. Since  $1 \in \mathbb{R}_{++}$ ,  $S \subseteq \mathbb{R}_{++}S$ . Since  $\mathbb{R}_{++}S$  is a cone containing  $S$  and  $\text{cone}(S)$  is the smallest cone containing  $S$ , we get

$$\text{cone}(S) \subseteq \mathbb{R}_{++}S.$$

**Backward Direction:** I will show that  $\mathbb{R}_{++}S \subseteq \text{cone}(S)$ . Let  $C$  be an arbitrary cone containing  $S$ . Since  $S \subseteq C$ ,  $\mathbb{R}_{++}S \subseteq \mathbb{R}_{++}C$ . Since  $C$  is a cone,  $\mathbb{R}_{++}C = C$ . So  $\mathbb{R}_{++}S \subseteq C$ . Since  $\mathbb{R}_{++}S \subseteq C$  for any cone  $C$  containing  $S$ , we get

$$\mathbb{R}_{++}S \subseteq \text{cone}(S).$$

□

## 5.2 Stability of the Cone Structure

**PROPOSITION 5.5.** The closure of a cone is a cone.

*Proof.* Let  $\mathcal{V}$  be a normed linear space. Let  $C$  be a cone in  $\mathcal{V}$ . I will show that  $\text{cl}(C)$  is also a cone. Let  $x$  be an arbitrary element of  $\text{cl}(C)$ . Then  $\exists (x_n)_{n \in \mathbb{N}} \subseteq C$  such that  $\lim_{n \in \mathbb{N}} x_n = x$ . Let  $\lambda \in \mathbb{R}_{++}$  be arbitrary. Since  $C$  is a cone and  $x_n \in C$ ,  $\forall n \in \mathbb{N}$ , we have  $\lambda x_n \in C$ . So  $(\lambda x_n)_{n \in \mathbb{N}} \subseteq C$ . Moreover, notice  $\lim_{n \in \mathbb{N}} \lambda x_n = \lambda \lim_{n \in \mathbb{N}} x_n = \lambda x$ . So  $\lambda x \in \text{cl}(C)$ . This holds for any  $\lambda \in \mathbb{R}_{++}$ . So  $\mathbb{R}_{++}\text{cl}(C) \subseteq \text{cl}(C)$ . It is clear that  $\text{cl}(C) \subseteq \mathbb{R}_{++}\text{cl}(C)$ . So we have  $\text{cl}(C) = \mathbb{R}_{++}\text{cl}(C)$ . So  $\text{cl}(C)$  is a cone. □

## 5.3 Other Properties

**PROPOSITION 5.6.** Let  $C$  be a convex set in  $\mathbb{E}$ . Assume  $\text{int}(C) \neq \emptyset$  and  $0 \in C$ . Then  $\text{int}(\text{cone}(C)) = \text{cone}(\text{int}(C))$ .

*Proof.*

For one direction, let  $x$  be an arbitrary point in  $\text{int}(\text{cone}(C))$ . We are to prove that  $x \in \text{cone}(\text{int}(C))$ .

Since  $x \in \text{int}(\text{cone}(C))$ ,  $\exists r$  such that  $\text{ball}(x, r) \subseteq \text{cone}(C)$ .

Since  $x \in \text{int}(\text{cone}(C))$ ,  $x \in \text{cone}(C)$ .

Since  $x \in \text{cone}(C)$ ,  $\exists n \in \mathbb{N}$ ,  $\exists \lambda_1, \dots, \lambda_n > 0$ ,  $\exists v_1, \dots, v_n \in C$  such that  $x = \sum_{i=1}^n \lambda_i v_i$ .

Assume for the sake of contradiction that  $\exists k \in \{1, \dots, n\}$  such that  $\forall r_k > 0$ ,  $\text{ball}(v_k, r_k) \cap \mathbb{E} \setminus C \neq \emptyset$ .

**### not finished**

For the reverse direction, let  $x$  be an arbitrary point in  $\text{cone}(\text{int}(C))$ . We are to prove that  $x \in \text{int}(\text{cone}(C))$ .

Since  $x \in \text{cone}(\text{int}(C))$ ,  $\exists n \in \mathbb{N}$ ,  $\exists \lambda_1, \dots, \lambda_n > 0$ ,  $\exists v_1, \dots, v_n \in \text{int}(C)$  such that  $x = \sum_{i=1}^n \lambda_i v_i$ .

Since  $v_i \in \text{int}(C)$  for each  $i \in \{1, \dots, n\}$ ,  $\exists r_i$  such that  $\text{ball}(v_i, r_i) \subseteq C$ .

Define  $R := \min\{\lambda_i r_i\}_{i=1}^n$ .

Say  $R = \lambda_k r_k$  for some  $k \in \{1, \dots, n\}$ .

Let  $y$  be an arbitrary point in  $\text{ball}(x, R)$ .

Since  $y \in \text{ball}(x, R)$ ,  $\exists w$  such that  $\|w\| < R$  and  $y = x + w$ .

$$\begin{aligned} y &= \sum_{i=1}^n \lambda_i v_i + w \\ &= \sum_{i \neq k} \lambda_i v_i + \lambda_k v_k + w \\ &= \sum_{i \neq k} \lambda_i v_i + \lambda_k (v_k + w/\lambda_k). \end{aligned}$$

Since  $\|w\| < R$ ,  $\|w/\lambda_k\| < R/\lambda_k = r_k$ .

Since  $\|w/\lambda_k\| < r_k$ ,  $v_k + w/\lambda_k \in \text{ball}(v_k, r_k)$ .

So  $v_k + w/\lambda_k \in C$ .

So  $y \in \text{cone}(C)$ .

Since  $\forall y \in \text{ball}(x, R)$ ,  $y \in \text{cone}(C)$ ,  $\text{ball}(x, R) \subseteq \text{cone}(C)$ .

Since  $\exists r$  such that  $\text{ball}(x, r) \subseteq \text{cone}(C)$ ,  $x \in \text{int}(\text{cone}(C))$ .

This proves  $\text{cone}(\text{int}(C)) \subseteq \text{int}(\text{cone}(C))$ .

□

**PROPOSITION 5.7.** Let  $C$  be a convex set in  $\mathbb{E}$ . Assume  $\text{int}(C) \neq \emptyset$  and  $0 \in C$ . Then

$$0 \in \text{int}(C) \iff \text{cone}(C) = \mathbb{E}.$$

*Proof.* For one direction, assume that  $0 \in \text{int}(C)$ . We are to prove that  $\text{cone}(C) = \mathbb{E}$ . Clearly

$$\text{cone}(C) \subseteq \mathbb{E}.$$

Since  $0 \in \text{int}(C)$ ,  $\exists r > 0$  such that  $\text{ball}(0, r) \subseteq C$ . Since  $\text{ball}(0, r) \subseteq C$ ,  $\text{cone}(\text{ball}(0, r)) \subseteq \text{cone}(C)$ . Since  $\text{cone}(\text{ball}(0, r)) = \mathbb{E}$  and  $\text{cone}(\text{ball}(0, r)) \subseteq \text{cone}(C)$ , we get

$$\mathbb{E} \subseteq \text{cone}(C).$$

For the reverse direction, assume that  $\text{cone}(C) = \mathbb{E}$ . We are to prove that  $0 \in \text{int}(C)$ .

$$\mathbb{E} = \text{int}(\mathbb{E}) = \text{int}(\text{cone}(C)) = \text{cone}(\text{int}(C)).$$

If  $0 \notin \text{int}(C)$ , then  $0 \notin \text{cone}(\text{int}(C))$ . So  $0 \in \text{int}(C)$ .

□

## 5.4 Closed Conical Hull

**DEFINITION 5.8** (Closed Conical Hull). Let  $\mathcal{V}$  be a normed linear space. Let  $S$  be a subset of  $\mathcal{V}$ . We define the **closed conical hull** of  $S$ , denoted by  $\overline{\text{cone}}(S)$ , to be the intersection of all closed cones containing  $C$ .

**PROPOSITION 5.9.** Let  $\mathcal{V}$  be a normed linear space. Let  $S$  be a subset of  $\mathcal{V}$ . Then

$$\overline{\text{cone}}(S) = \text{cl}(\text{cone}(S)).$$

*Proof. Forward Direction:* I will show that  $\overline{\text{cone}}(S) \subseteq \text{cl}(\text{cone}(S))$ . By Proposition 5.5,  $\text{cl}(\text{cone}(S))$  is a closed cone containing  $S$ . Since  $\overline{\text{cone}}(S)$  is the smallest closed cone containing  $S$ , we have  $\overline{\text{cone}}(S) \subseteq \text{cl}(\text{cone}(S))$ .

**Backward Direction:** I will show that  $\text{cl}(\text{cone}(S)) \subseteq \overline{\text{cone}}(S)$ .

$$\begin{aligned} S &\subseteq \overline{\text{cone}}(S) \\ \implies \text{cone}(S) &\subseteq \text{cone}(\overline{\text{cone}}(S)), \text{ since cone is monotonic increasing} \\ \implies \text{cl}(\text{cone}(S)) &\subseteq \text{cl}(\text{cone}(\overline{\text{cone}}(S))), \text{ since cl is monotonic increasing} \\ \iff \text{cl}(\text{cone}(S)) &\subseteq \overline{\text{cone}}(S), \text{ since } \overline{\text{cone}}(S) \text{ is a closed cone.} \end{aligned}$$

This completes the proof.

□

## 5.5 The cone and $\overline{\text{cone}}$ Operators



**PROPOSITION 5.10** (The cone Operator). The cone operator has the following properties.

1. Expansive:  $\forall S \subseteq \mathbb{E}$ ,

$$S \subseteq \text{cone}(S).$$

2. Monotone:  $\forall S_1, S_2 \subseteq \mathbb{E}$ ,

$$S_1 \subseteq S_2 \implies \text{cone}(S_1) \subseteq \text{cone}(S_2).$$

3. Idempotence:  $\forall S \subseteq \mathbb{E}$ ,

$$\text{cone}(\text{cone}(S)) = \text{cone}(S).$$

**PROPOSITION 5.11** (Bauschke-Combettes, 2017 Book). Let  $\mathcal{V}$  be a normed linear space. Let  $S$  be a subset of  $\mathcal{V}$ . Then

$$\text{cone}(\text{conv}(S)) = \text{conv}(\text{cone}(S)).$$

This is the smallest convex cone containing  $S$ .

*Proof.*

For  $\text{cone}(\text{conv}(S)) \subseteq \text{conv}(\text{cone}(S))$ , let  $x$  be an arbitrary point in  $\text{cone}(\text{conv}(S))$ .

Since  $x \in \text{cone}(\text{conv}(S))$ , we get  $\exists \lambda \in \mathbb{R}_+$ ,  $\exists n \in \mathbb{N}$ ,  $\exists v_1, \dots, v_n \in S$ ,  $\exists \mu_1, \dots, \mu_n \in [0, 1]$ ,  $\sum_{i=1}^n \mu_i = 1$  such that  $x = \lambda \sum_{i=1}^n \mu_i v_i$ .

Since  $x = \lambda \sum_{i=1}^n \mu_i v_i$ ,  $x = \sum_{i=1}^n \mu_i (\lambda v_i)$ .

Since  $\lambda \in \mathbb{R}_+$  and  $v_i \in S$ ,  $\lambda v_i \in \text{cone}(S)$ .

Since  $\lambda v_i \in \text{cone}(S)$  and  $\mu_i \in [0, 1]$ ,  $\sum_{i=1}^n \mu_i = 1$ ,  $\sum_{i=1}^n \mu_i (\lambda v_i) \in \text{conv}(\text{cone}(S))$ .

Since  $\forall x \in \text{cone}(\text{conv}(S))$ ,  $x \in \text{conv}(\text{cone}(S))$ ,  $\text{cone}(\text{conv}(S)) \subseteq \text{conv}(\text{cone}(S))$ .

For  $\text{conv}(\text{cone}(S)) \subseteq \text{cone}(\text{conv}(S))$ , let  $x$  be an arbitrary point in  $\text{conv}(\text{cone}(S))$ .

Since  $x \in \text{conv}(\text{cone}(S))$ ,  $\exists n \in \mathbb{N}$ ,  $\exists \lambda_i \in [0, 1]$ ,  $\sum_{i=1}^n \lambda_i = 1$ ,  $\exists \mu_i \in \mathbb{R}_+$ ,  $\exists v_i \in S$  such that

$$x = \sum_{i=1}^n \lambda_i \mu_i v_i.$$

Define  $\alpha := \sum_{i=1}^n \lambda_i \mu_i$ .

Define  $\beta_i := \lambda_i \mu_i / \alpha$ .

Then  $\alpha \in \mathbb{R}_+$  and  $\beta_i \in [0, 1]$  and  $\sum_{i=1}^n \beta_i = 1$  and  $x = \alpha \sum_{i=1}^n \beta_i v_i$ .

Since  $\beta_i \in [0, 1]$  and  $\sum_{i=1}^n \beta_i = 1$  and  $v_i \in S$ , we get  $\sum_{i=1}^n \beta_i v_i \in \text{conv}(S)$ .

Since  $\alpha \in \mathbb{R}_+$  and  $\sum_{i=1}^n \beta_i v_i \in \text{conv}(S)$  and  $x = \alpha \sum_{i=1}^n \beta_i v_i$ , we get  $x \in \text{cone}(\text{conv}(S))$ .

Since  $\forall x \in \text{conv}(\text{cone}(S)), x \in \text{cone}(\text{conv}(S))$ , we get  $\text{conv}(\text{cone}(S)) \subseteq \text{cone}(\text{conv}(S))$ .

Since  $\text{cone}(\text{conv}(S)) \subseteq \text{conv}(\text{cone}(S))$  and  $\text{conv}(\text{cone}(S)) \subseteq \text{cone}(\text{conv}(S))$ , we get  $\text{conv}(\text{cone}(S)) = \text{cone}(\text{conv}(S))$ .

□

**PROPOSITION 5.12** (Bauschke-Combettes, 2017 Book). Let  $\mathcal{V}$  be a normed linear space. Let  $S$  be a subset of  $\mathcal{V}$ . Then

$$\overline{\text{cone}}(\text{conv}(S)) = \overline{\text{conv}}(\text{cone}(S)).$$

This is the smallest closed convex cone containing  $S$ .

## 5.6 Dual Cone

**DEFINITION 5.13** (Dual of a Convex Cone). Let  $\mathfrak{X}$  be vector space over  $\mathbb{R}$ . Let  $C$  be a subset of  $\mathfrak{X}$ . We define the **dual cone** of  $C$ , denoted by  $C^*$ , to be the subset of  $\mathfrak{X}$  given by

$$C^* := \{x \in \mathfrak{X} : \forall y \in C, \langle x, y \rangle \geq 0\}.$$

**PROPOSITION 5.14.** The dual of a convex cone is always a closed convex cone.

**PROPOSITION 5.15.** Let  $\mathbb{E}$  be a Euclidean space. Let  $K$  be a convex cone in  $\mathbb{E}$ . Then  $K^{**} = \text{cl}(K)$ .

**PROPOSITION 5.16.** Let  $\mathbb{E}$  be a Euclidean space. Let  $K$  be a pointed, closed convex cone with nonempty interior. Then so is  $K^*$ .

**PROPOSITION 5.17.** Let  $\mathbb{E}$  be a Euclidean space. Let  $K_1$  and  $K_2$  be nonempty convex cones. Then

1.  $(K_1 + K_2)^* = K_1^* \cap K_2^*$ .
2.  $(\text{cl}(K_1) \cap \text{cl}(K_2))^* = \text{cl}(K_1^* + K_2^*)$ .
3. If  $K_1$  and  $K_2$  are closed and  $\text{ri}(K_1) \cap \text{ri}(K_2) \neq \emptyset$ , then  $(K_1 \cap K_2)^* = K_1^* + K_2^*$ .

*Proof of (1). Forward Direction:* Let  $x$  be an arbitrary element of  $(K_1 + K_2)^*$ . I will show that  $x \in K_1^* \cap K_2^*$ . Since  $x \in (K_1 + K_2)^*$ ,  $\forall k \in K_1 + K_2$ , we have  $\langle x, k \rangle \geq 0$ . Let  $k_1$  be an arbitrary element of  $K_1$ . Let  $k_2$  be an arbitrary element of  $K_2$ . Then

$$\begin{aligned} \langle x, k_1 \rangle &= \left\langle x, \lim_{n \rightarrow \infty} \left( k_1 + \frac{1}{n} k_2 \right) \right\rangle \\ &= \lim_{n \rightarrow \infty} \left\langle x, k_1 + \frac{1}{n} k_2 \right\rangle, \text{ since } \langle x, \cdot \rangle \text{ is continuous} \\ &\geq \lim_{n \rightarrow \infty} 0, \text{ since } k_1 + \frac{1}{n} k_2 \in K_1 + K_2 \\ &= 0. \end{aligned}$$

That is,  $\langle x, k_1 \rangle \geq 0$ . A similar argument can show that  $\langle x, k_2 \rangle \geq 0$ . So  $x \in K_1^*$  and  $x \in K_2^*$ . So  $x \in K_1^* \cap K_2^*$ .

**Backward Direction:** Let  $x$  be an arbitrary element of  $K_1^* \cap K_2^*$ . I will show that  $x \in (K_1 + K_2)^*$ . Let  $k$  be an arbitrary element of  $K_1 + K_2$ . Then  $k$  can be written as  $k = k_1 + k_2$  where  $k_1 \in K_1$  and  $k_2 \in K_2$ . Since  $x \in K_1^* \cap K_2^*$ ,  $x \in K_1^*$ . Since  $x \in K_1^*$  and  $k_1 \in K_1$ , we get  $\langle x, k_1 \rangle \geq 0$ . A similar argument can show that  $\langle x, k_2 \rangle \geq 0$ . So

$$\langle x, k \rangle = \langle x, k_1 + k_2 \rangle = \langle x, k_1 \rangle + \langle x, k_2 \rangle \geq 0 + 0 = 0.$$

That is,  $\langle x, k \rangle \geq 0$ . So  $x \in (K_1 + K_2)^*$ . □

## 5.7 Polar Cone

**DEFINITION 5.18** (Polar Cone). Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{R}$ . Let  $S$  be a subset of  $\mathcal{H}$ . We define the **polar cone** of  $S$ , denoted by  $S^\circ$ , to be a subset of  $\mathcal{H}$  given

by

$$S^\circ := \{x \in \mathcal{H} : \forall y \in S, \langle x, y \rangle \leq 0\}.$$

**PROPOSITION 5.19** (Bauschke-Combettes, 2017 Book). The polar cone of any set is a closed convex cone.

**PROPOSITION 5.20** (Bauschke-Combettes, 2017 Book). Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{R}$ . Let  $A, B \subseteq \mathcal{H}$ . Then  $A \subseteq B \implies B^\circ \subseteq A^\circ$ .

**PROPOSITION 5.21.** Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{R}$ . Let  $C$  be a subset of  $\mathcal{H}$ . Then  $C^\circ = -C^*$ .

**PROPOSITION 5.22.** If  $S$  is a linear subspace of some Euclidean space  $\mathbb{E}$ , then  $S^\circ = S^\perp$ .

## 5.8 Extreme Rays

**DEFINITION 5.23** (Rays). Let  $\mathcal{V}$  be a vector space. Let  $R$  be a subset of  $\mathcal{V}$ . We say that  $R$  is a **ray** if and only if  $R$  can be expressed as

$$R = \{\alpha v : \alpha \in \mathbb{R}_+\}$$

for some  $v \in \mathbb{E} \setminus \{0\}$ .

**DEFINITION 5.24** (Extreme Rays). Let  $\mathcal{V}$  be a vector space. Let  $K$  be a convex cone in  $\mathcal{V}$ . Let  $R$  be a ray in  $K$ . We say that  $R$  is an **extreme ray** in  $K$  if and only if for any pair of rays  $R_1$  and  $R_2$  in  $K$  such that  $R_1 + R_2 \supseteq R$ , we have either  $R_1 = R$  or  $R_2 = R$  (or both).

## Chapter 6

# Tangent Cones and Normal Cones

### 6.1 Definitions

**DEFINITION 6.1** (Tangent Cones). Let  $C$  be a non-empty convex set in  $\mathbb{R}^n$ . Let  $x$  be a point in  $\mathbb{R}^n$ . We define the **tangent cone** to  $C$  at point  $x$ , denoted by  $T_C(x)$ , to be the subset of  $\mathbb{R}^n$  given by

$$T_C(x) := \begin{cases} \overline{\text{cone}}(C - x), & \text{if } x \in C \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

**DEFINITION 6.2** (Normal Cones). Let  $C$  be a non-empty convex set in  $\mathbb{R}^n$ . Let  $x$  be a point in  $\mathbb{R}^n$ . We define the **normal cone** to  $C$  at point  $x$ , denoted by  $N_C(x)$ , to be the subset of  $\mathbb{R}^n$  given by

$$N_C(x) := \begin{cases} \{v \in \mathbb{R}^n : \forall y \in C - x, \langle y, v \rangle \leq 0\}, & \text{if } x \in C \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

### 6.2 Basic Properties

**PROPOSITION 6.3.** Let  $C$  be a closed convex set in  $\mathbb{E}$ . Let  $x$  be a point in  $\mathbb{E}$ . Then  $T_C(x)$  and  $N_C(x)$  are closed convex cones.

*Proof.*

If  $C = \emptyset$ , then  $T_C(x) = N_C(x) = \emptyset$ .

If  $C \neq \emptyset$  and  $x \notin C$ , then  $T_C(x) = N_C(x) = \emptyset$ .

So now I assume that  $C \neq \emptyset$  and  $x \in C$ .

**Tangent Cone is Closed:**

By definition,  $T_C(x) = \overline{\text{cone}}(C - x)$ . So  $T_C(x)$  is a closed.

**Tangent Cone is Convex:**

$C$ is convex	
$\Downarrow$	since affine mapping preserves convexity
$C - x$ is convex	
$\Downarrow$	since the cone operator preserves convexity
$\text{cone}(C - x)$ is convex	
$\Downarrow$	since the cl operator preserves convexity
$\text{cl}(\text{cone}(C - x))$ is convex	
$\Downarrow$	since $\text{cl} \circ \text{cone} = \overline{\text{cone}}$
$\overline{\text{cone}}(C - x)$ is convex	

That is,  $T_C(x)$  is convex.

**Tangent Cone is a Cone**

By definition,  $T_C(x) = \overline{\text{cone}}(C - x)$ . So  $T_C(x)$  is a cone.

**Normal Cone is Closed:**

Let  $\{x_i\}_{i \in \mathbb{N}}$  be an arbitrary sequence in  $N_C(x)$  that converges to some point in  $\mathbb{E}$ .

Say  $x_i \rightarrow x_\infty$ .

Let  $y$  be an arbitrary point in  $C - x$ .

Since  $x_i \in N_C(x)$  and  $y \in C - x$ , by definition of  $N_C(x)$ , we get  $\langle x_i, y \rangle \leq 0$ .

Since  $\langle x_i, y \rangle \leq 0$  for any  $i \in \mathbb{N}$  and  $x_i \rightarrow x_\infty$ , we get  $\langle x_\infty, y \rangle \leq 0$ .

Since  $\forall y \in C - x$ ,  $\langle x_\infty, y \rangle \leq 0$ , by definition of  $N_C(x)$ , we get  $x_\infty \in N_C(x)$ .

Since any convergent sequence whose terms are in  $N_C(x)$  has its limit also in  $N_C(x)$ ,  $N_C(x)$  is closed.

**Normal Cone is Convex:**

Let  $u$  and  $v$  be arbitrary points in  $N_C(x)$ .

Let  $\lambda$  be an arbitrary number in  $(0, 1)$ .

Define point  $z$  as  $z := \lambda u + (1 - \lambda)v$ .

Let  $y$  be an arbitrary point in  $C - x$ .

Since  $u \in N_C(x)$ ,  $\langle u, y \rangle \leq 0$ .

Since  $v \in N_C(x)$ ,  $\langle v, y \rangle \leq 0$ .

$$\begin{aligned}
 & \langle z, y \rangle \\
 &= \langle \lambda u + (1 - \lambda)v, y \rangle \\
 &= \lambda \langle u, y \rangle + (1 - \lambda) \langle v, y \rangle \\
 &\leq \lambda 0 + (1 - \lambda) 0 \\
 &= 0.
 \end{aligned}$$

That is,  $\langle z, y \rangle \leq 0$ .

Since  $\forall y \in C - x$ ,  $\langle z, y \rangle \leq 0$ , we get  $z \in N_C(x)$ .

That is,  $\lambda u + (1 - \lambda)v \in N_C(x)$ .

Since  $\forall u, v \in N_C(x)$ ,  $\forall \lambda \in (0, 1)$ ,  $\lambda u + (1 - \lambda)v \in N_C(x)$ , we get  $N_C(x)$  is convex.

**Normal Cone is a Cone:**

Let  $v$  be an arbitrary point in  $N_C(x)$ .

Let  $\lambda$  be an arbitrary number such that  $\lambda > 0$ .

Let  $y$  be an arbitrary point in  $C - x$ .

Since  $v \in N_C(x)$ ,  $\langle v, y \rangle \leq 0$ .

Since  $\langle v, y \rangle \leq 0$  and  $\lambda > 0$ ,  $\langle \lambda v, y \rangle \leq 0$ .

Since  $\forall y \in C - x$ ,  $\langle \lambda v, y \rangle \leq 0$ , we get  $\lambda v \in N_C(x)$ .

Since  $\forall v \in N_C(x)$ ,  $\forall \lambda > 0$ ,  $\lambda v \in N_C(x)$ , we get  $N_C(x)$  is a cone.

□

**PROPOSITION 6.4.** Let  $C$  be a non-empty closed convex set in  $\mathbb{E}$ . Let  $x$  be a point in  $C$ . Let  $n$  be a point in  $\mathbb{E}$ . Then

$$n \in N_C(x) \iff \forall t \in T_C(x), \langle n, t \rangle \leq 0.$$

*Proof.*

**For one direction,** assume that  $n \in N_C(x)$ .

We are to prove that

$$\forall t \in T_C(x), \quad \langle n, t \rangle \leq 0.$$

Let  $t$  be an arbitrary point in  $T_C(x)$ .

Since  $t \in T_C(x) = \text{cl}(\text{cone}(C - x))$ ,

$$\exists \{t_i\}_{i \in \mathbb{N}} \subseteq \text{cone}(C - x), \text{ such that } t_i \rightarrow t. \quad (1)$$

Since  $t_i \in \text{cone}(C - x)$ ,

$$\forall i \in \mathbb{N}, \exists \lambda_i \in \mathbb{R}_{++}, \exists c_i \in C \text{ such that } t_i = \lambda_i(c_i - x). \quad (2)$$

Since  $n \in N_C(x)$  and  $c_i \in C$ ,

$$\langle n, c_i - x \rangle \leq 0. \quad (3)$$

Now using (2) and (3), we have

$$\begin{aligned} & \langle n, t_i \rangle \\ &= \langle n, \lambda_i(c_i - x) \rangle, & \text{since } t_i = \lambda_i(c_i - x)s \\ &= \lambda_i \langle n, c_i - x \rangle \\ &\leq \lambda_i \cdot 0, & \text{since } \langle n, c_i - x \rangle \leq 0 \\ &= 0. \end{aligned}$$

That is,

$$\forall i \in \mathbb{N}, \quad \langle n, t_i \rangle \leq 0.$$

Since  $\langle n, t_i \rangle \leq 0$  for each  $i \in \mathbb{N}$  and  $t_i \rightarrow t$ , we get

$$\langle n, t \rangle \leq 0.$$

**For the reverse direction,** assume that  $n$  is a vector such that

$$\forall t \in T_C(x), \quad \langle n, t \rangle \leq 0.$$

We are to prove that  $n \in N_C(x)$ .

Let  $y$  be an arbitrary point in  $C - x$ .

Since  $C - x \subseteq \overline{\text{cone}}(C - x) = T_C(x)$  and  $y \in C - x$ , we get  $y \in T_C(x)$ .

Since  $y \in T_C(x)$  and  $\forall t \in T_C(x), \langle n, t \rangle \leq 0$ , we get  $\langle n, y \rangle \leq 0$ .

Since  $\forall y \in C - x, \langle n, y \rangle \leq 0$ , we get  $n \in N_C(x)$ .

□

**THEOREM 6.5.** Let  $C$  be a closed convex set in  $\mathbb{E}$  such that  $\text{int}(C) \neq \emptyset$ . Let  $x$  be a point in  $\mathbb{E}$ . Then

$$x \in \text{int}(C) \iff T_C(x) = \mathbb{E} \iff N_C(x) = \{0\}.$$

*Proof.*

**Part 1.**

$x \in \text{int}(C)$  if and only if  $0 \in \text{int}(C - x)$ , if and only if  $\overline{\text{cone}}(C - x) = \mathbb{E}$ .



**Part 2.**

**For one direction,** assume that  $T_C(x) = \mathbb{E}$ .

We are to prove that  $N_C(x) = \{0\}$ .

Consider  $n = 0$ .

Since

$$\forall t \in T_C(x), \quad \langle 0, t \rangle = 0 \leq 0,$$

we get  $0 \in N_C(x)$ .

Let  $n$  be an arbitrary vector in  $N_C(x)$ .

By another proposition, we have

$$\begin{aligned} n &\in N_C(x) \\ \iff \forall t \in T_C(x) = \mathbb{E}, \langle n, t \rangle &\leq 0 \\ \implies \text{for } t = n, \langle n, t \rangle = \langle n, n \rangle &\leq 0 \\ \implies n &= 0. \end{aligned}$$

That is,  $n \in N_C(x) \implies n = 0$ .

So  $N_C(x) = \{0\}$ .

**For the reverse direction,** assume that  $N_C(x) = \{0\}$ .

We are to prove that  $T_C(x) = \mathbb{E}$ .

Clearly  $T_C(x) \subseteq \mathbb{E}$ .

For  $\mathbb{E} \subseteq T_C(x)$ , let  $x$  be an arbitrary point in  $\mathbb{E}$ .

Define  $p := \text{proj}_{T_C(x)}(x)$ .

Since  $p = \text{proj}_{T_C(x)}(x)$ ,

$$\forall y \in T_C(x), \quad \langle x - p, y - p \rangle \leq 0. \quad (1)$$

Since  $p = \text{proj}_{T_C(x)}(x)$ ,  $p \in T_C(x)$ .

Since  $p \in T_C(x)$  and  $T_C(x)$  is a cone,

$$2p \in T_C(x). \quad (2)$$

Apply (1) to  $y = 2p$ , we get

$$\langle x - p, 2p - p \rangle = \langle x - p, p \rangle \leq 0. \quad (3)$$

Since  $T_C(x)$  is a closed cone,

$$0 \in T_C(x). \quad (4)$$

Apply (1) to  $y = 0$ , we get

$$\langle x - p, 0 - p \rangle = \langle x - p, -p \rangle \leq 0. \quad (5)$$

From (3) and (5), we get

$$\langle x - p, p \rangle = 0.$$

So (1) becomes

$$\forall y \in T_C(x), \quad \langle x - p, y \rangle \leq 0.$$

So  $x - p \in N_C(x)$ .

So  $x - p = 0$ .

So  $x = p$ .

So  $x \in T_C(x)$ .

Since  $\forall x \in \mathbb{E}, x \in T_C(x)$ , we get

$$\mathbb{E} \subseteq T_C(x).$$

□

### 6.3 Arithmetic Properties

**PROPOSITION 6.6.** Let  $C$  and  $D$  be convex subsets of  $\mathbb{E}$ . Let  $x$  be a point in  $\mathbb{E}$ . Then

$$N_C(x) + N_D(x) \subseteq N_{C \cap D}(x).$$

*Proof.*

If  $C$  or  $D$  is empty, then  $N_C(x) + N_D(x) = N_{C \cap D}(x) = \emptyset$ .

So now I assume that  $C, D \neq \emptyset$ .

If  $x \notin C \cap D$ , then  $N_C(x) + N_D(x) = N_{C \cap D}(x) = \emptyset$ .

So now I assume that  $x \in C \cap D$ .

Let  $v$  be an arbitrary point in  $N_C(x) + N_D(x)$ .

Since  $v \in N_C(x) + N_D(x)$ ,  $\exists u \in N_C(x)$ ,  $\exists w \in N_D(x)$  such that  $v = u + w$ .

Since  $u \in N_C(x)$ ,  $\forall y \in C - x$ ,  $\langle u, y \rangle \leq 0$ .

Since  $w \in N_D(x)$ ,  $\forall y \in D - x$ ,  $\langle w, y \rangle \leq 0$ .

Let  $y$  be an arbitrary point in  $C \cap D - x$ .

Since  $y \in C \cap D - x$ , we get  $y \in C - x$  and  $y \in D - x$ .

$$\begin{aligned} & \langle v, y \rangle \\ &= \langle u + w, y \rangle \\ &= \langle u, y \rangle + \langle w, y \rangle \\ &\leq 0 + 0 = 0. \end{aligned}$$

This is true for any  $y \in C \cap D - x$ .

So  $v \in N_{C \cap D}(x)$ .

This is true for any  $v \in N_C(x) + N_D(x)$ .

So  $N_C(x) + N_D(x) \subseteq N_{C \cap D}(x)$ .

□

**THEOREM 6.7.** Let  $C$  and  $D$  be convex sets in  $\mathbb{E}$ . Assume that  $\text{ri}(C) \cap \text{ri}(D) \neq \emptyset$ . Let  $x$  be a point in  $C \cap D$ . Then

$$N_{C \cap D}(x) = N_C(x) + N_D(x).$$

## 6.4 Other Properties

**PROPOSITION 6.8.** Let  $f$  be a proper, convex, and lower semi-continuous function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Let  $x$  be a point in  $\text{dom}(f)$ . Let  $u$  be a point in  $\mathbb{E}$ . Then  $u \in \partial f(x)$  if and only if  $(u, -1) \in N_{\text{epi}(f)}(x, f(x))$ .

*Proof.*

$$\begin{aligned} u &\in \partial f(x) \\ \iff \forall y \in \mathbb{E}, f(y) &\geq f(x) + \langle u, y - x \rangle \\ \iff \forall y \in \text{dom}(f), f(y) &\geq f(x) + \langle u, y - x \rangle \\ \iff \forall (y, \beta) \in \text{epi}(f), f(x) + \langle u, y - x \rangle &\leq \beta \\ \iff \forall (y, \beta) \in \text{epi}(f), \langle (u, -1), (y - x, \beta - f(x)) \rangle &\leq 0 \\ \iff \forall (y, \beta) \in \text{epi}(f), \langle (u, -1), (y, \beta) - (x, f(x)) \rangle &\leq 0 \\ \iff (u, -1) \in N_{\text{epi}(f)}(x, f(x)). \end{aligned}$$

□



## Chapter 7

# Extreme Points and Faces

### 7.1 Extreme Points

**DEFINITION 7.1** (Extreme Points - 1). Let  $\mathcal{V}$  be a vector space. Let  $C$  be a nonempty convex subset of  $\mathcal{V}$ . Let  $z$  be some point in  $C$ . We say that  $z$  is an **extreme point** of  $C$  if and only if it does not lie between any two distinct points in  $C$ . i.e.,

$$\forall x, y \in C, \forall t \in (0, 1), \quad tx + (1 - t)y = z \implies x = y = z.$$

**DEFINITION 7.2** (Extreme Points - 2). Let  $\mathcal{V}$  be a vector space. Let  $C$  be a nonempty convex subset of  $\mathcal{V}$ . Let  $x$  be some point in  $C$ . We say that  $x$  is an **extreme point** of  $C$  if and only if  $C \setminus \{x\}$  is still convex.

**PROPOSITION 7.3.** The two definitions of extreme point are equivalent.

*Proof. Forward Direction:* Assume that  $x$  does not lie between any two distinct points in  $C$ . I will show that  $C \setminus \{x\}$  is convex. Let  $x_1$  and  $x_2$  be two arbitrary distinct points in  $C \setminus \{x\}$ . Let  $\lambda$  be an arbitrary number in  $(0, 1)$ . Define a point  $y$  as  $y := \lambda x_1 + (1 - \lambda)x_2$ . Since  $C$  is convex,  $x_1, x_2 \in C$ , and  $\lambda \in (0, 1)$ , we get  $y \in C$ . Since  $x$  does not lie between any two distinct points in  $C$ ,  $y \neq x$ . So  $y \in C \setminus \{x\}$ . That is, I have proved that

$$\forall x_1, x_2 \in C \setminus \{x\}, \forall \lambda \in (0, 1), \quad y = \lambda x_1 + (1 - \lambda)x_2 \in C \setminus \{x\}.$$

By definition,  $C \setminus \{x\}$  is convex.

**Backward Direction:** Assume that  $C \setminus \{x\}$  is convex. I will show that  $x$  does not lie between any two distinct points in  $C$ . Assume for the sake of contradiction that  $x$  does lie between two distinct points in  $C$ . Say  $x = \lambda x_1 + (1 - \lambda)x_2$  where  $x_1, x_2 \in C$ ,  $x_1 \neq x_2$ , and  $\lambda \in (0, 1)$ . Clearly  $x \neq x_1$  and  $x \neq x_2$ . So  $x_1, x_2 \in C \setminus \{x\}$ . Since  $C \setminus \{x\}$  is convex,  $x_1, x_2 \in C \setminus \{x\}$ , and  $\lambda \in (0, 1)$ , we get  $x = \lambda x_1 + (1 - \lambda)x_2 \in C \setminus \{x\}$ . This leads to a contradiction. So the assumption that  $x$  lies between two distinct points in  $C$  does not hold. i.e.  $x$  does not lie between any two distinct points in  $C$ .  $\square$

**PROPOSITION 7.4.** If  $C$  is nonempty, convex, and compact, then  $\text{Ext}(C) \neq \emptyset$ .

**PROPOSITION 7.5.** Let  $\mathcal{V}$  be a locally convex space. Let  $K$  be a nonempty, compact, and convex set in  $\mathcal{V}$ . Then  $\text{Ext}(K) \neq \emptyset$ .

## 7.2 Faces

**DEFINITION 7.6** (Faces - 1). Let  $\mathcal{V}$  be a vector space. Let  $C$  be a nonempty convex subset of  $\mathcal{V}$ . Let  $F \subseteq C$ . We say that  $F$  is a **face** of  $C$ , denoted by  $F \trianglelefteq C$ , if and only if  $F$  is a nonempty convex subset of  $C$  such that

$$\forall x, y \in C, \forall t \in (0, 1), \quad tx + (1 - t)y \in F \implies x, y \in F.$$

**DEFINITION 7.7** (Faces - 2). Let  $\mathcal{V}$  be a vector space. Let  $C$  be a nonempty convex subset of  $\mathcal{V}$ . Let  $F \subseteq C$ . We say that  $F$  is a **face** of  $C$ , denoted by  $F \trianglelefteq C$ , if and only if  $F$  is a nonempty convex subset of  $C$  such that

$$\forall n \in \mathbb{N}, \forall x \in C^n, \forall t \in (0, 1)^n : \sum_{i=1}^n t_i = 1, \quad \sum_{i=1}^n t_i x_i \in F \implies x \in F^n.$$

**PROPOSITION 7.8.** The two definitions of faces above are equivalent.

*Proof.* **Forward Direction:** Suppose that  $\forall x, y \in C, \forall t \in (0, 1)$  such that  $tx + (1 - t)y \in F$ , we have  $x, y \in F$ . Let  $n \in \mathbb{N}$ ,  $x \in C^n$ ,  $t \in (0, 1)^n$  be arbitrary such that  $\sum_{i=1}^n t_i = 1$ . Define

a point  $z$  by  $z := \sum_{i=1}^n t_i x_i$ . Suppose that  $z \in F$ . I will show that  $x \in F^n$ . Note that  $\forall i \in \{1, \dots, n\}$ , we have

$$z = \sum_{j=1}^n t_j x_j = t_i x_i + (1 - t_i) \sum_{j \neq i} \frac{t_j}{1 - t_i} x_j.$$

Consider the point  $z_i := \sum_{j \neq i} \frac{t_j}{1 - t_i} x_j$ . Note that  $\forall j \neq i, \frac{t_j}{1 - t_i} \in (0, 1)$  and that  $\sum_{j \neq i} \frac{t_j}{1 - t_i} = 1$ . So since  $\forall j \neq i, x_j \in C$  and  $C$  is convex, we get  $z_i \in C$ . By assumption, we get  $x_i, z_i \in F$ . In particular,  $x_i \in F$ . So  $x \in F^n$ .

**Backward Direction:** Suppose that  $\forall n \in \mathbb{N}, \forall x \in C^n, \forall t \in (0, 1)^n$  such that  $\sum_{i=1}^n t_i = 1$  and that  $\sum_{i=1}^n t_i x_i \in F$ , we have  $x \in F^n$ . Take  $n := 2$ , then  $\forall x, y \in C, \forall t \in (0, 1)$  such that  $tx + (1 - t)y \in F$ , we have  $x, y \in F$ .  $\square$

Faces are generalizations of extreme points.

**PROPOSITION 7.9** (Transitivity). Let  $\mathcal{V}$  be a vector space. Let  $A, B$ , and  $C$  be nonempty convex subsets of  $\mathcal{V}$ . Suppose that  $A \trianglelefteq B$  and  $B \trianglelefteq C$ . Then  $A \trianglelefteq C$ .

*Proof.* Let  $x$  and  $y$  be two arbitrary elements of  $C$ . Let  $t$  be an arbitrary element of  $(0, 1)$ . Define a point  $z$  by  $z := tx + (1 - t)y$ . Suppose that  $z \in A$ . I will show that  $x, y \in A$ . Note that since  $A \trianglelefteq B$ , we have  $A \subseteq B$ . So  $z \in A \subseteq B$ . Since  $x, y \in C, t \in (0, 1), z \in B$ , and  $B \trianglelefteq C$ , we get  $x, y \in B$ . Since  $x, y \in B, t \in (0, 1), z \in A$ , and  $A \trianglelefteq B$ , we get  $x, y \in A$ . So  $A \trianglelefteq C$ .  $\square$

**PROPOSITION 7.10** (Intersection). Let  $\mathcal{V}$  be a vector space. Let  $C$  be a nonempty convex subset of  $\mathcal{V}$ . Let  $A, B \trianglelefteq C$ . Then  $(A \cap B) \trianglelefteq C$ .

*Proof.* Let  $x$  and  $y$  be two arbitrary elements of  $C$ . Let  $t$  be an arbitrary element of  $(0, 1)$ . Define a point  $z$  by  $z := tx + (1 - t)y$ . Suppose that  $z \in A \cap B$ . I will show that  $x, y \in A \cap B$ . Since  $A \trianglelefteq C, x, y \in C, t \in (0, 1)$ , and  $z \in A \cap B \subseteq A$ , we get  $x, y \in A$ . Similarly, we get  $x, y \in B$ . So  $x, y \in A \cap B$ . So  $(A \cap B) \trianglelefteq C$ .  $\square$

### 7.2.1 Exposed Faces

**DEFINITION 7.11** (Exposed Face of a Convex Cone). Let  $\mathcal{V}$  be an inner product space with inner product  $\langle \cdot, \cdot \rangle$ . Let  $C$  be a nonempty convex conic subset of  $\mathcal{V}$ . Let  $F \trianglelefteq C$ . We say that  $F$  is **exposed** if and only if  $\exists a \in \mathcal{V} \setminus \{0\}$  such that

$$F = \{x \in C : \langle a, x \rangle = 0\} \text{ and } C \subseteq \{x \in \mathcal{V} : \langle a, x \rangle \leq 0\}.$$

**PROPOSITION 7.12.** Let  $\mathcal{V}$  be a vector space. Let  $C$  be nonempty convex subset of  $\mathcal{V}$ . Then every face of  $C$  is contained in some exposed face of  $C$ .

### 7.2.2 Relation Between Extreme Points and Faces

**DEFINITION 7.13** (Extreme Points - 3). Let  $\mathcal{V}$  be a vector space. Let  $C$  be a nonempty convex subset of  $\mathcal{V}$ . Let  $x$  be some point in  $C$ . We say that  $x$  is an **extreme point** of  $C$  if and only if  $\{x\}$  is a face of  $C$ .

**PROPOSITION 7.14.** This definition of extreme points is equivalent to the previous two.

**PROPOSITION 7.15.** If  $F$  is a face of  $C$ , then  $\text{Ext}(F) \subseteq \text{Ext}(C)$ .

## 7.3 The Krein-Milman Theorem

**LEMMA 7.16.** Let  $\mathcal{V}$  be a locally convex space. Let  $K$  be a nonempty compact convex subset of  $\mathcal{V}$ . Let  $\rho \in \mathcal{V}^*$ . Define  $r := \sup\{\Re\rho(x) : x \in K\}$ . Define  $F := \{x \in K : \Re\rho(x) = r\}$ . Then  $F$  is a nonempty compact face of  $K$ .

*Proof. Nonempty:* Since  $\Re\rho$  is continuous and  $K$  is compact,  $\{\Re\rho(x) : x \in K\}$  is a compact set in  $\mathbb{R}$ . So  $r = \sup\{\Re\rho(x) : x \in K\}$  is attained. So  $F \neq \emptyset$ .

**Compact:** Notice  $F = (\Re\rho)^{-1}(\{r\})$ . Since  $\Re\rho$  is continuous and  $\{r\} \subseteq \mathbb{R}$  is closed,  $F$  is closed. Since  $F$  is a closed subset of  $K$  and  $K$  is compact,  $F$  is compact.



**Convex:** Let  $x$  and  $y$  be arbitrary elements of  $F$ . Let  $t \in (0, 1)$ . Since  $x, y \in F$ , we have  $\Re \rho(x) = \Re \rho(y) = r$ . So

$$\Re \rho(tx + (1-t)y) = t\Re \rho(x) + (1-t)\Re \rho(y) = tr + (1-t)r = r.$$

So  $tx + (1-t)y \in F$ . So  $F$  is convex.

**Face:** Let  $x$  and  $y$  be arbitrary elements of  $K$ . Let  $t \in (0, 1)$ . Suppose that  $tx + (1-t)y \in F$ . Since  $x, y \in K$ , we have  $\Re \rho(x) \leq r$  and  $\Re \rho(y) \leq r$ . Since  $tx + (1-t)y \in F$ , we have

$$t\Re \rho(x) + (1-t)\Re \rho(y) = \Re \rho(tx + (1-t)y) = r.$$

So we must have  $\Re \rho(x) = \Re \rho(y) = r$ . So  $x, y \in F$ . So  $F$  is a face of  $K$ .

□

**THEOREM 7.17** (Krein-Milman Theorem). A compact convex set in a locally convex space is the closed convex hull of its extreme points.

*Proof.* Let  $\mathcal{V}$  be a locally convex space. Let  $K$  be a nonempty, compact, and convex set in  $\mathcal{V}$ .

**Forward Direction:** Show that  $K \subseteq \overline{\text{conv}}(\text{Ext}(K))$ . Let  $m$  be an arbitrary element of  $K$ . Assume for the sake of contradiction that  $m \notin \overline{\text{conv}}(\text{Ext}(K))$ . By the Hahn-Banach Theorem, there is some  $\tau \in \mathcal{V}^*$  and  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha > \beta$  and

$$\forall b \in \overline{\text{conv}}(\text{Ext}(K)), \quad \Re \tau(m) \geq \alpha > \beta \geq \Re \tau(b).$$

Define  $s := \sup\{\Re \tau(w) : w \in K\}$ . Define  $L := \{z \in K : \Re \tau(z) = s\}$ . Then  $L$  is a nonempty compact face of  $K$ . So  $\text{Ext}(L) \neq \emptyset$ . Let  $e$  be an element of  $\text{Ext}(L)$ . Then  $e \in \text{Ext}(L) \subseteq L$ . So  $\Re \tau(e) = s$ . So

$$\forall b \in \overline{\text{conv}}(\text{Ext}(K)), \quad \Re \tau(e) = s \geq \Re \tau(m) \geq \alpha > \beta \geq \Re \tau(b).$$

That is,  $\Re \tau(e) > \Re \tau(b)$ . Since  $L$  is a face of  $K$ ,  $\text{Ext}(L) \subseteq \text{Ext}(K)$ . Notice  $e \in \text{Ext}(L) \subseteq \text{Ext}(K) \subseteq \overline{\text{conv}}(\text{Ext}(K))$ . So in particular,  $\Re \tau(e) > \Re \tau(e)$ . This is a contradiction. So  $m \in \overline{\text{conv}}(\text{Ext}(K))$ . So  $K \subseteq \overline{\text{conv}}(\text{Ext}(K))$ .

**Backward Direction:** Show that  $\overline{\text{conv}}(\text{Ext}(K)) \subseteq K$ . Note that  $\text{Ext}(K) \subseteq K$ . Since  $K$  is closed and convex and  $\text{Ext}(K) \subseteq K$ , we get  $\overline{\text{conv}}(\text{Ext}(K)) \subseteq K$ .

□

**PROPOSITION 7.18.** Let  $\mathcal{V}$  be a vector space. Let  $K$  be a nonempty compact convex subset of  $\mathcal{V}$ . Let  $\mathcal{F}(K)$  denote the set of faces of  $K$ , partially ordered by

inclusion. Then the minimal proper faces in  $\mathcal{F}(K)$  are the extreme points of  $K$ .

**PROPOSITION 7.19.** Let  $\mathcal{V}$  be a vector space. Let  $K$  be a nonempty compact convex subset of  $\mathcal{V}$ . Let  $\mathcal{F}(K)$  denote the set of faces of  $K$ , partially ordered by inclusion. Then the maximal proper faces in  $\mathcal{F}(K)$  are exposed.

## Chapter 8

# Projection Operators

### 8.1 Definitions

**DEFINITION 8.1** (Projection). Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{R}$ . Let  $S$  be a non-empty set in the space. Let  $x$  be a point in the space. We define the **projection** of  $x$  onto  $S$ , denoted by  $\text{proj}_S(x)$ , to be a point given by

$$\text{proj}_S(x) := \operatorname{argmin}_{p \in S} \|p - x\|.$$

i.e.,  $\text{proj}_S(x)$  is the closest point in  $S$  to  $x$ .

**PROPOSITION 8.2** (Existence). If  $S$  is non-empty and closed, then the projection  $\text{proj}_S(x)$  exists.

*Proof.* Define for an  $n \in \mathbb{N}$  a point  $c_n$  to be a point in  $S$  that satisfies

$$\lim_{i \in \mathbb{N}} \|c_i - x\| = d_S(x) \text{ where } d_S(x) = \inf_{p \in S} \|p - x\|.$$

Since  $\mathcal{H}$  is a Hilbert space, the norm  $\|\cdot\|$  on  $\mathcal{H}$  satisfies the Parallelogram Law. So

$$\begin{aligned} \|c_m - c_n\|^2 &= 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - \|c_m + c_n - 2x\|^2 \\ &= 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - 4\left\|\frac{c_m + c_n}{2} - x\right\|^2 \\ &\leq 2\|c_m - x\|^2 + 2\|c_n - x\|^2 - 4d_S(x)^2 \\ &\rightarrow 2d_S(x)^2 + 2d_S(x)^2 - 4d_S(x)^2 = 0. \end{aligned}$$

So the sequence  $(c_i)_{i \in \mathbb{N}}$  is Cauchy. Since  $\mathcal{H}$  is a Hilbert space, it is complete. So  $(c_i)_{i \in \mathbb{N}}$  converges. Since  $S$  is closed, and  $(c_i)_{i \in \mathbb{N}}$  is a Cauchy sequence in  $S$ ,  $p := \lim_{i \in \mathbb{N}} c_i \in S$ . So  $\|p - x\| = \|\lim_{i \in \mathbb{N}} c_i - x\| = \lim_{i \in \mathbb{N}} \|c_i - x\| = d_S(x)$ . So  $p$  is the minimizer of the distance to the point  $x$  over  $S$ . So  $p = \text{proj}_S(x)$ .  $\square$

**PROPOSITION 8.3** (Uniqueness). If  $S$  is non-empty, closed, and convex, then the projection  $\text{proj}_S(x)$  is unique.

*Proof.* Let  $p$  denote  $\text{proj}_S(x)$ . Then  $\|p - x\| = d_S(x)$ . Let  $q$  be a point in  $S$  such that  $\|q - x\| = d_S(x)$ . Then by the Parallelogram Law,

$$\begin{aligned} 0 &\leq \|p - q\|^2 = 2\|x - p\|^2 + 2\|q - x\|^2 - 4\left\|x - \frac{1}{2}(p + q)\right\|^2 \\ &\leq 2d_S^2(x) + 2d_S^2(x) - 4d_S^2(x) \\ &= 0. \end{aligned}$$

This shows  $\|p - q\| = 0$  and hence  $p = q$ . Thus the projection is unique.  $\square$

## 8.2 Properties

**PROPOSITION 8.4** (Idempotent). The projection operator is idempotent. i.e., if  $C$  is a nonempty closed convex set in  $\mathbb{E}$ , then  $\text{proj}_C = \text{proj}_C \text{proj}_C$ .

*Proof.* Let  $x$  be an arbitrary point in  $\mathbb{E}$ . By definition,  $\text{proj}_C(x) \in C$ . Since  $\text{proj}_C(x) \in C$ , the closest point in  $C$  to  $\text{proj}_C(x)$  is  $\text{proj}_C(x)$ . So  $\text{proj}_C \text{proj}_C(x) = \text{proj}_C(x)$ . This is true for any  $x \in \mathbb{E}$ . So  $\text{proj}_C = \text{proj}_C \text{proj}_C$ .  $\square$

**PROPOSITION 8.5.** Let  $C$  be a nonempty closed convex set in  $\mathbb{E}$ . Then the set of fixed points of the operator  $\text{proj}_C$  is  $C$ .

*Proof.* For one direction, let  $x$  be an arbitrary fixed point of  $\text{proj}_C$ . We are to prove that  $x \in C$ . Since  $x$  is a fixed point of  $\text{proj}_C$ ,  $x = \text{proj}_C(x)$ . By definition of projection,  $\text{proj}_C(x) \in C$ . So  $x = \text{proj}_C(x) \in C$ .

For the reverse direction, let  $x$  be an arbitrary point in  $C$ . We are to prove that  $x$  is a fixed point of  $\text{proj}_C$ . Since  $x \in C$ , the closest point in  $C$  to  $x$  is  $x$ . So  $x = \text{proj}_C(x)$ . So  $x$  is a fixed point of  $\text{proj}_C$ .  $\square$

**PROPOSITION 8.6** (Linearity). Let  $C$  be a nonempty closed convex set in  $\mathbb{E}$ . Then the operator  $\text{proj}_C$  is linear if and only if  $C$  is a linear subspace.

**PROPOSITION 8.7** (Non-expansive). The projection operator is non-expansive. i.e., if  $C$  is a nonempty closed convex set in  $\mathbb{E}$ , then  $\|\text{proj}_C(x) - \text{proj}_C(y)\| \leq \|x - y\|$  for any  $x, y \in \mathbb{E}$ .

this is not true. I guess it will be true when  $C$  is a linear subspace.

**PROPOSITION 8.8.** Let  $C$  be a nonempty closed convex set in  $\mathbb{E}$ . Then  $\text{proj}_C$  is Lipschitz with constant 1.

*Proof.* Let  $x$  and  $y$  be two arbitrary points in  $\mathbb{E}$ . If  $\|\text{proj}_C(x) - \text{proj}_C(y)\| = 0$ , then  $\|\text{proj}_C(x) - \text{proj}_C(y)\| \leq \|x - y\|$ . Otherwise,

$$\begin{aligned}
 & \|\text{proj}_C(x) - \text{proj}_C(y)\|^2 \\
 &= \langle \text{proj}_C(x) - \text{proj}_C(y), \text{proj}_C(x) - \text{proj}_C(y) \rangle \\
 &= \langle \text{proj}_C(x) - \text{proj}_C(y), \text{proj}_C(x) - x \rangle \\
 &+ \langle \text{proj}_C(x) - \text{proj}_C(y), x - y \rangle \\
 &+ \langle \text{proj}_C(x) - \text{proj}_C(y), y - \text{proj}_C(y) \rangle \\
 &= \langle x - \text{proj}_C(x), \text{proj}_C(y) - \text{proj}_C(x) \rangle \\
 &+ \langle y - \text{proj}_C(y), \text{proj}_C(x) - \text{proj}_C(y) \rangle \\
 &+ \langle \text{proj}_C(x) - \text{proj}_C(y), x - y \rangle \\
 &\leq 0 + 0 + \langle \text{proj}_C(x) - \text{proj}_C(y), x - y \rangle \\
 &= \langle \text{proj}_C(x) - \text{proj}_C(y), x - y \rangle \\
 &\leq \|\text{proj}_C(x) - \text{proj}_C(y)\| \|x - y\|.
 \end{aligned}$$

That is,

$$\|\text{proj}_C(x) - \text{proj}_C(y)\|^2 \leq \|\text{proj}_C(x) - \text{proj}_C(y)\| \|x - y\|.$$

Dividing both sides by  $\|\text{proj}_C(x) - \text{proj}_C(y)\|$  gives

$$\|\text{proj}_C(x) - \text{proj}_C(y)\| \leq \|x - y\|.$$

So  $\text{proj}_C$  is Lipschitz with constant 1. □

**PROPOSITION 8.9** (Firmly Non-expansive). Let  $C$  be a nonempty closed convex set in  $\mathbb{E}$ . Then  $\text{proj}_C$  is firmly non-expansive.

*Proof.* This is to prove.

$$\forall x, y \in \mathbb{E}, \quad \|\text{proj}_C(y) - \text{proj}_C(x)\|^2 \leq \langle \text{proj}_C(y) - \text{proj}_C(x), y - x \rangle.$$

$$\begin{aligned} & \|\text{proj}_C(y) - \text{proj}_C(x)\|^2 \\ &= \langle \text{proj}_C(y) - \text{proj}_C(x), \text{proj}_C(y) - \text{proj}_C(x) \rangle \\ &= \langle \text{proj}_C(y) - \text{proj}_C(x), \text{proj}_C(y) - y \rangle \\ &+ \langle \text{proj}_C(y) - \text{proj}_C(x), y - x \rangle \\ &+ \langle \text{proj}_C(y) - \text{proj}_C(x), x - \text{proj}_C(x) \rangle \\ &\leq 0 + \langle \text{proj}_C(y) - \text{proj}_C(x), y - x \rangle + 0 \\ &= \langle \text{proj}_C(y) - \text{proj}_C(x), y - x \rangle. \end{aligned}$$

□

### 8.3 Examples

**EXAMPLE 8.10** (Projection onto Half Spaces, Bauschke-Combettes, 2017 Book). Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{R}$ . Let  $C \subseteq \mathcal{H}$  be given by

$$C := \{x \in \mathcal{H} : \langle x, u \rangle \leq \eta\}$$

where  $u \in \mathcal{H}$  and  $\eta \in \mathbb{R}$  are constants. Then if  $u = 0$  and  $\eta \geq 0$ ,  $C = \mathcal{H}$  and  $\text{proj}_C = \text{Id}$ ; if  $u = 0$  and  $\eta < 0$ ,  $C = \emptyset$ ; if  $u \neq 0$ , we have

$$\text{proj}_C(x) = \begin{cases} x, & \text{if } \langle x, u \rangle \leq \eta \\ x + \frac{\eta - \langle x, u \rangle}{\|u\|^2} u, & \text{if } \langle x, u \rangle > \eta. \end{cases}$$

### 8.4 Characterizations

**THEOREM 8.11** (Projection Theorem). Let  $C$  be a nonempty closed convex set in

$\mathbb{E}$ . Let  $x$  and  $p$  be points in  $\mathbb{E}$ . Then  $p = \text{proj}_C(x)$  if and only if

$$\forall y \in C, \quad \langle y - p, x - p \rangle \leq 0.$$

*Proof.* Let  $y$  be an arbitrary point in  $C$ . Let  $\alpha$  be an arbitrary number in  $[0, 1]$ . Define  $y_\alpha := \alpha y + (1 - \alpha)p$ . Now

$$\begin{aligned} p &= \text{proj}_C(x) \\ \iff \forall y \in C, \forall \alpha \in [0, 1], \|x - p\|^2 &\leq \|x - y_\alpha\|^2 \\ \iff \forall y \in C, \forall \alpha \in [0, 1], \|x - p\|^2 &\leq \|x - p - \alpha(y - p)\|^2 \\ \iff \forall y \in C, \langle x - p, y - p \rangle &\leq 0. \end{aligned}$$

□

**THEOREM 8.12** (Bauschke-Combettes, 2017 Book). Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{R}$ . Let  $K$  be a nonempty closed convex cone in  $\mathcal{H}$ . Let  $x, p \in \mathcal{H}$ . Then  $p = \text{proj}_K(x)$  if and only if  $p \in K$ ,  $\langle x - p, p \rangle = 0$ , and  $x - p \in K^\circ$ .





## Chapter 9

# Separation

### 9.1 Definitions

**DEFINITION 9.1** (Separated). Let  $S_1$  and  $S_2$  be two sets in  $\mathbb{E}$ . We say that  $S_1$  and  $S_2$  are **separated** if  $\exists b \in \mathbb{E} \setminus \{\vec{0}\}$  such that

$$\sup_{s_1 \in S_1} \langle s_1, b \rangle \leq \inf_{s_2 \in S_2} \langle s_2, b \rangle.$$

**DEFINITION 9.2** (Strongly Separated). Let  $S_1$  and  $S_2$  be two sets in  $\mathbb{E}$ . We say that they are **strongly separated** if the inequality holds strictly.

**DEFINITION 9.3** (Properly Separated). Let  $S_1$  and  $S_2$  be two sets in  $\mathbb{E}$ . We say that  $S_1$  and  $S_2$  are **properly separated** if  $\exists b \in \mathbb{E}$  such that

$$\begin{aligned} \sup_{x \in S_1} \langle x, b \rangle &\leq \inf_{y \in S_2} \langle y, b \rangle, \text{ and} \\ \inf_{x \in S_1} \langle x, b \rangle &> \sup_{y \in S_2} \langle y, b \rangle. \end{aligned}$$

### 9.2 Main Results

**PROPOSITION 9.4.** Let  $C$  be a nonempty closed convex set in  $\mathbb{E}$ . Let  $x$  be a point in  $\mathbb{E}$  such that  $x \notin C$ . Then  $x$  and  $C$  are strongly separated.

*Proof.* Define a point  $p$  by

$$p := \text{proj}_C(x).$$

Define a point  $a$  by

$$a := x - p.$$

To prove that  $x$  is strongly separated from  $C$ , it suffices to prove that

$$\forall y \in C, \quad \langle y, a \rangle < \langle x, a \rangle.$$

Since  $x \notin C$  and  $C$  is closed,

$$a \neq 0. \tag{1}$$

Let  $y$  be an arbitrary point in  $C$ . Since  $p = \text{proj}_C(x)$  and  $y \in C$ ,

$$\langle y - p, x - p \rangle \leq 0. \tag{2}$$

$$\begin{aligned} & \langle y, a \rangle \\ & < \langle y, a \rangle + \langle a, a \rangle, \text{ since } a \neq 0 \\ & = \langle y + a, a \rangle \\ & = \langle y + x - p, x - p \rangle, \text{ substitute } a = x - p \\ & = \langle y - p, x - p \rangle + \langle x, x - p \rangle \\ & \leq 0 + \langle x, x - p \rangle, \text{ since } \langle y - p, x - p \rangle \leq 0 \\ & = \langle x, x - p \rangle \\ & = \langle x, a \rangle. \end{aligned}$$

That is,

$$\forall y \in C, \quad \langle y, a \rangle < \langle x, a \rangle.$$

So  $x$  is strongly separated from  $C$ . □

**PROPOSITION 9.5.** Let  $C_1$  be a non-empty closed convex set in  $\mathbb{E}$ . Let  $C_2$  be a non-empty compact convex set in  $\mathbb{E}$ . Assume that  $C_1$  and  $C_2$  are disjoint. Then  $C_1$  and  $C_2$  are strongly separated.

*Proof.* Since  $C_1$  is non-empty closed and convex and  $C_2$  is non-empty compact and convex, we get  $C_1 - C_2$  is non-empty closed and convex. Since  $C_1 \cap C_2 = \emptyset$ ,  $0 \notin C_1 - C_2$ . Since  $C_1 - C_2$  is non-empty closed and convex and  $0 \in C_1 - C_2$ ,  $0$  and  $C_1 - C_2$  are strongly separated. Since  $0$  is strongly separated from  $C_1 - C_2$ ,

$$\exists a \neq 0 \text{ such that } \forall c_1 \in C_1, c_2 \in C_2, \quad \langle c_1 - c_2, a \rangle < \langle 0, a \rangle.$$

That is,

$$\langle c_1, a \rangle < \langle c_2, a \rangle.$$

So  $C_1$  and  $C_2$  are strongly separated.  $\square$

**THEOREM 9.6.** Let  $C_1$  and  $C_2$  be non-empty closed convex sets in  $\mathbb{E}$ . Assume that  $C_1$  and  $C_2$  are disjoint. Then  $C_1$  and  $C_2$  are separated.

*Proof.* For  $n \in \mathbb{N}$ , define

$$D_n := C_2 \cap \text{ball}(0, n).$$

Then  $D_n$  is compact for any  $n \in \mathbb{N}$ . Since  $\left\{ C_1 \text{ is non-empty closed and convex } D_n \text{ is non-empty compact and convex} \right\}$  we get  $C_1$  and  $D_n$  are strongly separated for any  $n \in \mathbb{N}$ . So

$$\forall n \in \mathbb{N}, \exists a_n \in \mathbb{E}, \|a_n\| = 1 \text{ such that } \forall c_1 \in C_1, \forall d_2 \in D_n, \quad \langle c_1, a_n \rangle < \langle d_2, a_n \rangle.$$

Since  $\|a_n\| = 1$  for any  $n \in \mathbb{N}$ , there exists a subsequence  $\{a_n\}_{n \in I}$  where  $I$  is some infinite subset of  $\mathbb{N}$  such that  $\{a_n\}_{n \in I}$  converges to some point  $a \in \mathbb{E}$ . Let  $x$  be an arbitrary point in  $C_1$ . Let  $y$  be an arbitrary point in  $C_2$ . For large enough  $n$ ,  $y \in D_n$ . Since

$$\begin{cases} \langle x, a_n \rangle < \langle y, a_n \rangle \text{ for large enough } n \\ \lim_{n \in I, n \rightarrow \infty} \langle x, a_n \rangle = \langle x, a \rangle \\ \lim_{n \in I, n \rightarrow \infty} \langle y, a_n \rangle = \langle y, a \rangle \end{cases}, \text{ we get}$$

$$\langle x, a \rangle \leq \langle y, a \rangle.$$

Since

$$\exists a \neq 0 \text{ such that } \forall x \in C_1, \forall y \in C_2, \quad \langle x, a \rangle \leq \langle y, a \rangle,$$

by definition of separated,  $C_1$  and  $C_2$  are separated.  $\square$

**PROPOSITION 9.7.** Let  $C_1$  and  $C_2$  be non-empty convex subsets of  $\mathbb{E}$ . Then  $C_1$

and  $C_2$  are properly separated if and only if

$$\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset.$$

## Chapter 10

# Convex Functions

### 10.1 Preliminaries

**DEFINITION 10.1** (Epigraph). Let  $f$  be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . We define the **epigraph** of  $f$ , denoted by  $\text{epi}(f)$ , to be the set given by

$$\text{epi}(f) := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} : f(x) \leq \alpha\}.$$

**DEFINITION 10.2** (Domain). Let  $f$  be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . We define the **domain** of  $f$ , denoted by  $\text{dom}(f)$ , to be a set given by

$$\text{dom}(f) := \{x \in \mathbb{E} : f(x) < +\infty\}.$$

**DEFINITION 10.3** (Proper). Let  $f$  be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . We say that  $f$  is **proper** if

$$\begin{aligned} \exists x \in \mathbb{E}, \quad f(x) &\neq +\infty, \text{ and} \\ \forall x \in \mathbb{E}, \quad f(x) &\neq -\infty \end{aligned}$$

### 10.2 The Indicator Function

**DEFINITION 10.4** (The Indicator Function). Let  $S$  be a subset of  $\mathbb{E}$ . We define the **indicator function** of  $S$ , denoted by  $\delta_S$ , to be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$  given by

$$\delta_S(x) = \begin{cases} 0, & \text{if } x \in S \\ +\infty, & \text{if } x \notin S. \end{cases}$$

**PROPOSITION 10.5.** Let  $S$  be a subset of  $\mathbb{E}$ . Then

1.  $S$  is non-empty if and only if  $\delta_S$  is proper.
2.  $S$  is convex if and only if  $\delta_S$  is convex.
3.  $S$  is closed if and only if  $\delta_S$  is lower semi-continuous.

*Proof of (1).*

For one direction, assume that  $S$  is not empty.

We are to prove that  $\delta_S$  is proper.

Since  $S \neq \emptyset$ , pick  $p \in S$ .

Since  $p \in S$ ,  $\delta_S(p) = 0$ .

Since  $\delta_S(p) = 0$ ,  $\exists x_0 \in \mathbb{E}$  such that  $\delta_S(x_0) \neq +\infty$ .

By definition of the indicator function, it never takes  $-\infty$ .

Since  $\exists x_0 \in \mathbb{E}$  such that  $\delta_S(x_0) \neq +\infty$  and  $\forall x \in \mathbb{E}$ ,  $\delta_S(x) \neq -\infty$ , we get  $\delta_S$  is proper.

For the reverse direction, assume that  $\delta_S$  is proper.

We are to prove that  $S$  is non-empty.

Assume for the sake of contradiction that  $S$  is empty.

Let  $x$  be an arbitrary point in  $\mathbb{E}$ .

Since  $S = \emptyset$ ,  $x \notin S$ .

Since  $x \notin S$ ,  $\delta_S(x) = +\infty$ .

Since  $\forall x \in \mathbb{E}$ ,  $\delta_S(x) = +\infty$ , by definition of proper function,  $\delta_S$  is not proper.

This contradicts to the assumption that  $\delta_S$  is proper.

So the assumption that  $S = \emptyset$  is false.

i.e.,  $S$  is non-empty.

□

*Proof of (2).*

For one direction, assume that  $S$  is convex.

We are to prove that  $\delta_S$  is convex.

Let  $x$  and  $y$  be arbitrary points in  $\text{dom}(\delta_S)$ .

By definition of indicator functions,  $\text{dom}(\delta_S) = S$ .

So  $x, y \in S$ .

Let  $\lambda$  be an arbitrary number in  $(0, 1)$ .

Define point  $z$  as  $z := \lambda x + (1 - \lambda)y$ .

Since  $x, y \in S$  and  $\lambda \in (0, 1)$  and  $S$  is convex and  $z = \lambda x + (1 - \lambda)y$ , we get  $z \in S$ .

Since  $z \in S$ ,  $\delta_S(z) = 0$ .

Since  $\lambda \in (0, 1)$  and  $\text{range}(\delta_S) = \{0, +\infty\}$ , we get  $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) \geq 0$ .

Since  $\delta_S(z) = 0$  and  $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) \geq 0$ , we get  $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$ .

That is,  $\delta_S(\lambda x + (1 - \lambda)y) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$ .

Since  $\forall x, y \in \text{dom}(\delta_S)$ ,  $\forall \lambda \in (0, 1)$ ,  $\delta_S(\lambda x + (1 - \lambda)y) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$ , we get  $\delta_S$  is convex.

For the reverse direction, assume that  $\delta_S$  is convex.

We are to prove that  $S$  is convex.

The case where  $S$  is empty is trivial.

So now I assume  $S \neq \emptyset$ .

Let  $x$  and  $y$  be arbitrary points in  $S$ .

Let  $\lambda$  be an arbitrary number in  $(0, 1)$ .

Define point  $z$  as  $z := \lambda x + (1 - \lambda)y$ .

Since  $x \in S$ ,  $\delta_S(x) = 0$ .

Since  $y \in S$ ,  $\delta_S(y) = 0$ .

Since  $\delta_S(x) = \delta_S(y) = 0$ , we get  $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) = 0$ .

Since  $\lambda \in (0, 1)$  and  $\delta_S$  is convex,  $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$ .

Since  $\delta_S(z) \leq \lambda\delta_S(x) + (1 - \lambda)\delta_S(y)$  and  $\lambda\delta_S(x) + (1 - \lambda)\delta_S(y) = 0$ , we get  $\delta_S(z) \leq 0$ .

By definition of the indicator function,  $\delta_S(z) \geq 0$ .

Since  $\delta_S(z) \leq 0$  and  $\delta_S(z) \geq 0$ , we get  $\delta_S(z) = 0$ .

Since  $\delta_S(z) = 0$ ,  $z \in S$ .

That is,  $\lambda x + (1 - \lambda)y \in S$ .

Since  $\forall x, y \in S$ ,  $\forall \lambda \in (0, 1)$ ,  $\lambda x + (1 - \lambda)y \in S$ , we get  $S$  is convex.

□

*Proof of (3).*

For one direction, assume that  $S$  is closed.

We are to prove that  $\delta_S$  is lower semi-continuous.

Let  $\{(x_i, \alpha_i)\}_{i \in \mathbb{N}}$  be an arbitrary sequence in  $\text{epi}(\delta_S)$  that converges.

Say its limit is  $(x_\infty, \alpha_\infty)$ .

Since  $(x_i, \alpha_i) \rightarrow (x_\infty, \alpha_\infty)$ ,  $x_i \rightarrow x_\infty$ .

Since  $(x_i, \alpha_i) \in \text{epi}(\delta_S)$ ,  $\delta_S(x_i) \leq \alpha_i$ .

Since  $\delta_S(x_i) \leq \alpha_i$  and  $\alpha_i \in \mathbb{R}$ , we get  $\delta_S(x_i) \neq +\infty$ .

Since  $\delta_S(x_i) \neq +\infty$ ,  $x_i \in S$ .

Since  $x_i \in S$  and  $x_i \rightarrow x_\infty$  and  $S$  is closed,  $x_\infty \in S$ .

Since  $x_\infty \in S$ ,  $\delta_S(x_\infty) = 0$ .

Since  $x_i \in S$ ,  $\delta_S(x_i) = 0$ .

Since  $\delta_S(x_i) = 0$  and  $\delta_S(x_i) \leq \alpha_i$ ,  $\alpha_i \geq 0$ .

Since  $(x_i, \alpha_i) \rightarrow (x_\infty, \alpha_\infty)$ ,  $\alpha_i \rightarrow \alpha_\infty$ .

Since  $\alpha_i \geq 0$  and  $\alpha \rightarrow \alpha_\infty$ ,  $\alpha_\infty \geq 0$ .

Since  $\delta_S(x_\infty) = 0$  and  $\alpha_\infty \geq 0$ ,  $\delta_S(x_\infty) \leq \alpha_\infty$ .

Since  $\delta_S(x_\infty) \leq \alpha_\infty$ ,  $(x_\infty, \alpha_\infty) \in \text{epi}(\delta_S)$ .

Since for any convergent sequence in  $\text{epi}(\delta_S)$ , its limit is also in  $\text{epi}(\delta_S)$ , we get  $\text{epi}(\delta_S)$  is closed.

For the reverse direction, assume that  $\delta_S$  is lower semi-continuous.

We are to prove that  $S$  is closed.

Let  $\{x_i\}_{i \in \mathbb{N}}$  be an arbitrary sequence in  $S$  that converges.

Say its limit is  $x_\infty$ .

Since  $x_i \in S$ ,  $\delta_S(x_i) = 0$ .

Since  $\delta_S(x_i) = 0$ ,  $(x_i, 0) \in \text{epi}(\delta_S)$ .

Since  $x_i \rightarrow x_\infty$ ,  $(x_i, 0) \rightarrow (x_\infty, 0)$ .

Since  $(x_i, 0) \in \text{epi}(\delta_S)$  and  $(x_i, 0) \rightarrow (x_\infty, 0)$ ,  $(x_\infty, 0) \in \text{epi}(\delta_S)$ .

Since  $(x_\infty, 0) \in \text{epi}(\delta_S)$ ,  $\delta_S(x_\infty) \leq 0$ .

By definition of the indicator function,  $\delta_S(x_\infty) \geq 0$ .

Since  $\delta_S(x_\infty) \leq 0$  and  $\delta_S(x_\infty) \geq 0$ , we get  $\delta_S(x_\infty) = 0$ .

Since  $\delta_S(x_\infty) = 0$ ,  $x_\infty \in S$ .

Since for any convergent sequence in  $S$ , its limit is also in  $S$ , we get  $S$  is closed. □

### 10.3 Definitions

**DEFINITION 10.6** (Convex Function). Let  $f$  be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . We say that  $f$  is **convex** if and only if

$$\forall x, y \in \text{dom}(f), \forall \lambda \in [0, 1], \quad f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

**DEFINITION 10.7** (Convex Function). Let  $f$  be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . We say that  $f$  is **convex** if and only if the epigraph of  $f$  is convex.



**PROPOSITION 10.8.** The two definitions of convexity of functions are equivalent.

*Proof.*

The case where  $\text{dom}(f), \text{epi}(f) = \emptyset$  is trivial.

So now I assume that  $\text{dom}(f), \text{epi}(f) \neq \emptyset$ .

For one direction, assume that  $\forall x, y \in \text{dom}(f), \forall \lambda \in [0, 1]$ , we have  $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ .

We are to prove that the epigraph of  $f$  is convex.

Let  $(x, \alpha)$  and  $(y, \beta)$  be two arbitrary points in  $\text{epi}(f)$ .

Since  $(x, \alpha), (y, \beta) \in \text{epi}(f)$ ,  $x, y \in \text{dom}(f)$ .

Let  $\lambda$  be an arbitrary number in  $[0, 1]$ .

Define a point  $(z, \gamma) := \lambda(x, \alpha) + (1 - \lambda)(y, \beta)$ .

Then  $z = \lambda x + (1 - \lambda)y$  and  $\gamma = \lambda\alpha + (1 - \lambda)\beta$ .

Since  $x, y \in \text{dom}(f)$ ,  $\lambda \in [0, 1]$ , we get  $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ .

Since  $(x, \alpha) \in \text{epi}(f)$ ,  $f(x) \leq \alpha$ .

Since  $(y, \beta) \in \text{epi}(f)$ ,  $f(y) \leq \beta$ .

Since  $f(x) \leq \alpha$  and  $f(y) \leq \beta$  and  $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ , we get  $f(\lambda x + (1 - \lambda)y) \leq \lambda\alpha + (1 - \lambda)\beta$ .

Since  $z = \lambda x + (1 - \lambda)y$  and  $\gamma = \lambda\alpha + (1 - \lambda)\beta$  and  $f(\lambda x + (1 - \lambda)y) \leq \lambda\alpha + (1 - \lambda)\beta$ , we get  $f(z) \leq \gamma$ .

Since  $f(z) \leq \gamma$ ,  $(z, \gamma) \in \text{epi}(f)$ .

For the reverse direction, assume that  $\text{epi}(f)$  is convex.

We are to prove that  $\forall x, y \in \text{dom}(f), \forall \lambda \in [0, 1]$ , we have  $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ .

Let  $x$  and  $y$  be two arbitrary points in  $\text{dom}(f)$ .

Let  $\lambda$  be an arbitrary number in  $[0, 1]$ .

Define  $z := \lambda x + (1 - \lambda)y$ .

Define  $\gamma := \lambda f(x) + (1 - \lambda)f(y)$ .

Since  $(x, f(x)) \in \text{epi}(f)$  and  $(y, f(y)) \in \text{epi}(f)$  and  $\lambda \in [0, 1]$  and  $\text{epi}(f)$  is convex, we get  $\lambda(x, f(x)) + (1 - \lambda)(y, f(y)) \in \text{epi}(f)$ .

Since  $z = \lambda x + (1 - \lambda)y$  and  $\gamma = \lambda f(x) + (1 - \lambda)f(y)$  and  $\lambda(x, f(x)) + (1 - \lambda)(y, f(y)) \in \text{epi}(f)$ , we get  $(z, \gamma) \in \text{epi}(f)$ .

Since  $(z, \gamma) \in \text{epi}(f)$ ,  $f(z) \leq \gamma$ .

That is,  $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ .

□

## 10.4 Basic Properties

**PROPOSITION 10.9** (Necessary Condition). The domain of a convex function is convex.

*Proof.* Follows from the fact that convexity is stable under affine transformations. Define  $A((x, \alpha)) := x$ . Then  $\text{dom}(f) = A(\text{epi}(f))$ .  $\square$

**PROPOSITION 10.10.** The level sets of a convex function are convex.

**PROPOSITION 10.11** (Restriction to a Line). A function  $f : \mathbb{E} \rightarrow \mathbb{R}$  is convex if and only if  $\forall x \in \text{dom}(f), \forall v \in \mathbb{E}$ , the function  $g_{x,v} : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$g_{x,v}(t) = f(x + tv)$$

is convex.

## 10.5 Differentiable Convex Functions

**PROPOSITION 10.12.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable and convex. Then  $f \in \mathbb{C}^1$ . i.e.,  $f$  is continuously differentiable.

**PROPOSITION 10.13.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable. Then  $f$  is convex if and only if  $f'$  is non-decreasing.

**PROPOSITION 10.14.** Let  $f$  be a proper convex function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Let  $x \in \text{dom}(f)$ . If  $f$  is differentiable at point  $x$ , then  $\nabla(f)(x)$  is the unique subgradient of  $f$  at point  $x$ . i.e.,  $\partial(f)(x) = \{\nabla(f)(x)\}$ . Conversely, if the subgradient  $\partial(f)(x)$  of  $f$  at point  $x$  is a singleton set  $\{v\}$ , then  $f$  is differentiable at point  $x$  and  $\nabla(f)(x) = v$ .

**PROPOSITION 10.15** (First-Order Condition). Let  $f$  be a proper function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Assume that  $\text{dom}(f)$  is convex and open and that  $f$  is differentiable on  $\text{dom}(f)$ . Then  $f$  is convex if and only if

$$\forall x, y \in \text{dom}(f), \quad f(y) - f(x) \geq \langle \nabla(f)(x), y - x \rangle.$$

i.e., the first-order approximation of  $f$  is a global under-estimator.

*Proof.*

Part 1.

For one direction, assume that  $f$  is convex. We are to prove that

$$\forall x, y \in \text{dom}(f), \quad f(y) - f(x) \geq \langle \nabla(f)(x), y - x \rangle.$$

Let  $x$  and  $y$  be arbitrary points in  $\text{dom}(f)$ . Since  $f$  is convex and differentiable at point  $x$ ,  $\nabla(f)(x) = \partial(f)(x)$ . So  $\nabla(f)(x)$  satisfies the subgradient inequality. That is,

$$f(y) - f(x) \geq \langle \nabla(f)(x), y - x \rangle.$$

Part 2.

For the reverse direction, assume that

$$\forall x, y \in \text{dom}(f), \quad f(y) - f(x) \geq \langle \nabla(f)(x), y - x \rangle.$$

We are to prove that  $f$  is convex.

**Not Finished.**

□

**PROPOSITION 10.16.** Let  $f$  be a proper function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Assume that  $\text{dom}(f)$  is convex and open and that  $f$  is differentiable on  $\text{dom}(f)$ . Then  $f$  is convex if and only if

$$\forall x, y \in \text{dom}(f), \quad \langle \nabla(f)(x) - \nabla(f)(y), x - y \rangle \geq 0.$$

*Proof.*

Part 1.

For one direction, assume that  $f$  is convex. We are to prove that

$$\forall x, y \in \text{dom}(f), \quad \langle \nabla(f)(x) - \nabla(f)(y), x - y \rangle \geq 0.$$

Let  $x$  and  $y$  be arbitrary points in  $\text{dom}(f)$ . Since  $f$  is convex and differentiable at point  $x$ ,  $\nabla(f)(x) = \partial(f)(x)$ . So  $\nabla(f)(x)$  satisfies the subgradient inequality. That is,

$$f(y) - f(x) \geq \langle \nabla(f)(x), y - x \rangle. \quad (1)$$

Since  $f$  is convex and differentiable at point  $y$ ,  $\nabla(f)(y) = \partial(f)(y)$ . So  $\nabla(f)(y)$  satisfies the subgradient inequality. That is,

$$f(x) - f(y) \geq \langle \nabla(f)(y), x - y \rangle. \quad (2)$$

Take the sum of inequalities (1) and (2), we get

$$\begin{aligned} & (f(y) - f(x)) + (f(x) - f(y)) \geq \langle \nabla(f)(x), y - x \rangle + \langle \nabla(f)(y), x - y \rangle \\ \implies & 0 \geq -\langle \nabla(f)(x), x - y \rangle + \langle \nabla(f)(y), x - y \rangle \\ \implies & \langle \nabla(f)(x), x - y \rangle - \langle \nabla(f)(y), x - y \rangle \geq 0 \\ \implies & \langle \nabla(f)(x) - \nabla(f)(y), x - y \rangle \geq 0. \end{aligned}$$

Part 2.

For the reverse direction, assume that

$$\forall x, y \in \text{dom}(f), \quad \langle \nabla(f)(x) - \nabla(f)(y), x - y \rangle \geq 0.$$

We are to prove that  $f$  is convex. Let  $x$  and  $y$  be arbitrary points in  $\text{dom}(f)$ . Define a function  $\varphi$  on  $(0, 1)$  by

$$\varphi(\lambda) := f(\lambda x + (1 - \lambda)y).$$

Notice  $\varphi$  is differentiable and

$$\varphi'(\lambda) = \langle \nabla(f)(\lambda x + (1 - \lambda)y), x - y \rangle.$$

Let  $\alpha$  and  $\beta$  be arbitrary numbers in  $(0, 1)$ . Assume that  $\alpha < \beta$ . Define two points  $z_\alpha$  and  $z_\beta$  by  $z_\alpha := \alpha x + (1 - \alpha)y$  and  $z_\beta := \beta x + (1 - \beta)y$ . Then

$$\begin{aligned} & \varphi'(\beta) - \varphi'(\alpha) \\ &= \langle \nabla(f)(\beta x + (1 - \beta)y), x - y \rangle - \langle \nabla(f)(\alpha x + (1 - \alpha)y), x - y \rangle \\ &= \langle \nabla(f)(z_\beta), x - y \rangle - \langle \nabla(f)(z_\alpha), x - y \rangle \\ &= \langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), x - y \rangle \\ &= \left\langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), \frac{z_\beta - z_\alpha}{\beta - \alpha} \right\rangle \\ &= \frac{1}{\beta - \alpha} \langle \nabla(f)(z_\beta) - \nabla(f)(z_\alpha), z_\beta - z_\alpha \rangle \\ &\geq \frac{1}{\beta - \alpha} \cdot 0, \text{ by assumption} \end{aligned}$$

$$= 0.$$

That is,

$$\forall \alpha, \beta \in (0, 1), \quad \beta > \alpha \implies \varphi'(\beta) - \varphi'(\alpha) \geq 0.$$

So  $\varphi'$  is increasing. So  $\varphi$  is convex. So

$$\varphi(\lambda) \leq \lambda\varphi(1) + (1 - \lambda)\varphi(0).$$

That is,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

By definition,  $f$  is convex. □

**PROPOSITION 10.17** (Second-Order Condition). A twice continuously differentiable real-valued function  $f$  defined on a convex set is convex if and only if

$$\forall x \in \text{dom}(f), \quad \nabla^2 f(x) \geq 0$$

where  $\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_n \partial x_n} \end{bmatrix}$  denotes the Hessian matrix of  $f$  at  $x_0$ .

**PROPOSITION 10.18.** Let  $f$  be a twice continuously differentiable function from  $\mathbb{E}$  to  $\mathbb{R}$ . Then  $f$  is convex if and only if  $\forall x \in \mathbb{E}$ ,  $\nabla^2 f(x)$  is positive semi-definite.

## 10.6 Convexity and Lipschitz-ness

**THEOREM 10.19.** Let  $f$  be a differentiable convex function from  $\mathbb{E}$  to  $\mathbb{R}$ . Then the following statements are equivalent.

1.  $\nabla f$  is Lipschitz with constant  $L$ .
2.  $\forall x, y \in \mathbb{E}$ , we have

$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2.$$

3.  $\forall x, y \in \mathbb{E}$ , we have

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2.$$

4.  $\forall x, y \in \mathbb{E}$ , we have

$$L \langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \|\nabla f(y) - \nabla f(x)\|^2.$$

(1)  $\implies$  (2).

Assume that  $\nabla f$  is Lipschitz with constant 1.

Let  $x$  and  $y$  be two arbitrary points in  $\mathbb{E}$ .

$$\begin{aligned} & f(y) - f(x) \\ &= \int_0^1 \langle \nabla f(x + t(y - x)), y - x \rangle dt \\ &= \langle \nabla f(x), y - x \rangle + \int_0^1 \langle \nabla f(x + t(y - x)) - \nabla f(x), y - x \rangle dt \\ &\leq \langle \nabla f(x), y - x \rangle + \int_0^1 \|\nabla f(x + t(y - x)) - \nabla f(x)\| \|y - x\| dt \\ &\leq \langle \nabla f(x), y - x \rangle + \int_0^1 L \|x + t(y - x) - x\| \|y - x\| dt \\ &= \langle \nabla f(x), y - x \rangle + L \|y - x\|^2 \int_0^1 t dt \\ &= \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2 \end{aligned}$$

That is,

$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2.$$

□

**THEOREM 10.20.** Let  $f$  be a twice continuously differentiable function from  $\mathbb{E}$  to  $\mathbb{R}$ . Let  $L$  be some non-negative number. Then the following statements are equivalent.

1.  $\nabla f$  is  $L$ -Lipschitz.
2.  $\forall x \in \mathbb{E}$ ,  $\|\nabla^2 f(x)\| \leq L$ .

## 10.7 Stability of Convexity

**PROPOSITION 10.21** (Non-Negative Linear Combination). A non-negative linear combination of proper convex functions is again convex.

*Proof.* It suffices to prove that non-negative scalar multiples of convex functions are convex and sums of two convex functions are convex.

Part 1.

Let  $f$  be a proper convex function. Let  $\alpha \geq 0$  be an arbitrary scalar. We are to prove that  $\alpha f$  is convex. Notice  $\text{dom}(f) = \text{dom}(\alpha f)$ . Since  $f$  is proper,  $\text{dom}(f) \neq \emptyset$ . So  $\text{dom}(\alpha f) \neq \emptyset$ . Let  $x$  and  $y$  be two arbitrary points in  $\text{dom}(\alpha f)$ . Let  $\lambda$  be an arbitrary number in  $(0, 1)$ . Define a point  $z$  as  $z := \lambda x + (1 - \lambda)y$ . Then

$$\begin{aligned} (\alpha f)(\lambda x + (1 - \lambda)y) &= \alpha f(\lambda x + (1 - \lambda)y) \\ &\leq \alpha(\lambda f(x) + (1 - \lambda)f(y)) \\ &= \lambda \alpha f(x) + (1 - \lambda) \alpha f(y) \\ &= \lambda(\alpha f)(x) + (1 - \lambda)(\alpha f)(y). \end{aligned}$$

That is,

$$\forall x, y \in \text{dom}(\alpha f), \forall \lambda \in (0, 1), \quad (\alpha f)(\lambda x + (1 - \lambda)y) \leq \lambda(\alpha f)(x) + (1 - \lambda)(\alpha f)(y).$$

So by definition,  $\alpha f$  is convex.

Part 2.

Let  $f$  and  $g$  be proper convex functions. We are to prove that  $f + g$  is convex. Notice  $\text{dom}(f + g) = \text{dom}(f) \cap \text{dom}(g)$ . Since  $f$  is proper,  $\text{dom}(f) \neq \emptyset$ . Since  $g$  is proper,  $\text{dom}(g) \neq \emptyset$ . So  $\text{dom}(f + g) \neq \emptyset$ . Let  $x$  and  $y$  be two arbitrary points in  $\text{dom}(f + g)$ . Let  $\lambda$  be an arbitrary number in  $(0, 1)$ . Define a point  $z$  as  $z := \lambda x + (1 - \lambda)y$ . Then

$$\begin{aligned} (f + g)(\lambda x + (1 - \lambda)y) &= f(\lambda x + (1 - \lambda)y) + g(\lambda x + (1 - \lambda)y) \\ &\leq \lambda f(x) + (1 - \lambda)f(y) + \lambda g(x) + (1 - \lambda)g(y) \\ &= \lambda(f(x) + g(x)) + (1 - \lambda)(f(y) + g(y)) \\ &= \lambda(f + g)(x) + (1 - \lambda)(f + g)(y). \end{aligned}$$

That is,

$$\forall x, y \in \text{dom}(f + g), \forall \lambda \in (0, 1), \quad (f + g)(\lambda x + (1 - \lambda)y) = \lambda(f + g)(x) + (1 - \lambda)(f + g)(y).$$

So by definition,  $f + g$  is convex.

□

**PROPOSITION 10.22** (Direct Sum). Direct sums of convex functions are convex.

*Proof.* Let  $z$  and  $w$  be two arbitrary points in  $\text{dom}(f \oplus g)$ . Let  $\lambda \in (0, 1)$  be arbitrary. Say  $z = x \oplus y$  and  $w = u \oplus v$  where  $x, u \in \mathbb{R}^m$  and  $y, v \in \mathbb{R}^p$ . Since  $z \in \text{dom}(f \oplus g)$ ,  $(f \oplus g)(z) \neq +\infty$ . That is,  $f(x) + g(y) \neq +\infty$ . So neither  $f(x)$  nor  $g(y)$  is  $+\infty$ . So both  $x \in \text{dom}(f)$  and  $y \in \text{dom}(g)$ . Similarly, we have  $u \in \text{dom}(f)$  and  $v \in \text{dom}(g)$ . Consider the point

$$\begin{aligned} & \lambda z + (1 - \lambda)w \\ &= \lambda x \oplus y + (1 - \lambda)u \oplus v \\ &= (\lambda x + (1 - \lambda)u) \oplus (\lambda y + (1 - \lambda)v). \end{aligned}$$

Apply  $f \oplus g$  to both sides, we get

$$\begin{aligned} & (f \oplus g)(\lambda z + (1 - \lambda)w) \\ &= (f \oplus g)[(\lambda x + (1 - \lambda)u) \oplus (\lambda y + (1 - \lambda)v)] \\ &= f(\lambda x + (1 - \lambda)u) + g(\lambda y + (1 - \lambda)v). \end{aligned}$$

Since  $f$  and  $g$  are convex, we get

$$\begin{aligned} f(\lambda x + (1 - \lambda)u) &\leq \lambda f(x) + (1 - \lambda)f(u), \text{ and} \\ g(\lambda y + (1 - \lambda)v) &\leq \lambda g(y) + (1 - \lambda)g(v). \end{aligned}$$

So

$$\begin{aligned} & (f \oplus g)(\lambda z + (1 - \lambda)w) \\ &\leq \lambda f(x) + (1 - \lambda)f(u) + \lambda g(y) + (1 - \lambda)g(v) \\ &= \lambda(f(x) + g(y)) + (1 - \lambda)(f(u) + g(v)) \\ &= \lambda(f \oplus g)(x \oplus y) + (1 - \lambda)(f \oplus g)(u \oplus v) \\ &= \lambda(f \oplus g)(z) + (1 - \lambda)(f \oplus g)(w). \end{aligned}$$

That is,

$$(f \oplus g)(\lambda z + (1 - \lambda)w) \leq \lambda(f \oplus g)(z) + (1 - \lambda)(f \oplus g)(w).$$

This holds for any  $z, w \in \text{dom}(f \oplus g)$  and any  $\lambda \in (0, 1)$ . So  $(f \oplus g)$  is convex. □



**PROPOSITION 10.23** (Composition). The composition of a convex function with an affine function is convex. i.e., if  $f$  is convex, then  $f(Ax + b)$  is convex.

*Proof.* Let  $x$  and  $y$  be arbitrary points in  $\mathbb{E}$ . Let  $\lambda$  be an arbitrary number in  $(0, 1)$ . Define a point  $z$  by  $z := \lambda x + (1 - \lambda)y$ .

$$\begin{aligned}
 & g(\lambda x + (1 - \lambda)y) \\
 &= f(A(\lambda x + (1 - \lambda)y) + b) \\
 &= f(\lambda Ax + (1 - \lambda)Ay + b), && \text{by linearity of } A \\
 &= f(\lambda Ax + (1 - \lambda)Ay + \lambda b + (1 - \lambda)b), && \text{decompose } b \\
 &= f(\lambda(Ax + b) + (1 - \lambda)(Ay + b)) \\
 &\leq \lambda f(Ax + b) + (1 - \lambda)f(Ay + b), && \text{by convexity of } f \\
 &= \lambda g(x) + (1 - \lambda)g(y).
 \end{aligned}$$

That is,

$$\forall x, y \in \mathbb{E}, \forall \lambda \in (0, 1), \quad g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y).$$

So  $g$  is convex. □

**PROPOSITION 10.24** (Supremum). The supremum of a collection of convex functions is again convex. i.e., Let  $\{f_i\}_{i \in I}$  be a collection of convex functions where  $I$  is some index set. Then the function  $F$  given by  $F := \sup_{i \in I} f_i$  is convex.

*Proof.*

$$\begin{aligned}
 & (x, \alpha) \in \text{epi}(F) \\
 & \iff \sup_{i \in I} f_i(x) \leq \alpha \\
 & \iff \forall i \in I, f_i(x) \leq \alpha \\
 & \iff \forall i \in I, (x, \alpha) \in \text{epi}(f_i) \\
 & \iff (x, \alpha) \in \bigcap_{i \in I} \text{epi}(f_i).
 \end{aligned}$$

So  $\text{epi}(F) = \bigcap_{i \in I} \text{epi}(f_i)$ . Since  $f_i$  are convex,  $\text{epi}(f_i)$  are convex. Since  $\text{epi}(f_i)$  are convex,

$\bigcap_{i \in I} \text{epi}(f_i)$  is convex. That is,  $\text{epi}(F)$  is convex. Since  $\text{epi}(F)$  is convex,  $F$  is convex. □

**PROPOSITION 10.25** (Pointwise Supremum). If  $f(x, y)$  is convex in  $x$  for each  $y$  in some set  $\mathcal{A}$ , then the function  $g$  given by

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex.

## 10.8 Examples

**EXAMPLE 10.26.** Affine functions are convex.

**EXAMPLE 10.27.** Norms are convex.

*Proof.*

$$\begin{aligned} & \|\alpha x + \beta y\| \\ & \leq \|\alpha x\| + \|\beta y\| \\ & = |\alpha| \|x\| + |\beta| \|y\| \\ & = \alpha \|x\| + \beta \|y\|. \end{aligned}$$

□

**EXAMPLE 10.28.** Square norms are convex.

*Proof Approach 1.* Notice  $\|\cdot\|^2$  is the direct sum of  $m$  squares and squares are convex. So by CO 463 Assignment 2 Problem 3,  $\|\cdot\|^2$  is convex. □

*Proof Approach 2.* The domain is  $\mathbb{E}$ . Let  $x$  and  $y$  be two points in  $\mathbb{E}$ . Let  $\lambda$  be an arbitrary number in  $(0, 1)$ . Define a point  $z$  as  $z := \lambda x + (1 - \lambda)y$ .

$$\begin{aligned} & \|\lambda x + (1 - \lambda)y\|^2 \\ & = \|\lambda x\|^2 + \|(1 - \lambda)y\|^2 + 2\langle \lambda x, (1 - \lambda)y \rangle \\ & = \lambda^2 \|x\|^2 + (1 - \lambda)^2 \|y\|^2 + 2\lambda(1 - \lambda)\langle x, y \rangle \end{aligned}$$

$$\begin{aligned}
&\leq \lambda^2 \|x\|^2 + (1-\lambda)^2 \|y\|^2 + 2\lambda(1-\lambda)\|x\|\|y\| \\
&\leq \lambda(\lambda-1)\|x\|^2 + \lambda(\lambda-1)\|y\|^2 + 2\lambda(1-\lambda)\|x\|\|y\| \\
&\quad + \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
&= \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
&\quad + \lambda(\lambda-1)[\|x\|^2 + \|y\|^2 - 2\|x\|\|y\|] \\
&\leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2
\end{aligned}$$

That is,

$$\forall x, y \in \mathbb{E}, \forall \lambda \in (0, 1), \quad \|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2.$$

So by definition,  $\|\cdot\|^2$  is convex. □

**EXAMPLE 10.29.** The distance function to a convex set is convex.

**EXAMPLE 10.30.** The perspective of a convex function is convex. i.e., if  $f : \mathbb{E} \rightarrow \mathbb{R}$



# Chapter 11

## More Convex Functions

### 11.1 Strictly Convex

**DEFINITION 11.1** (Strictly Convex). Let  $\mathcal{V}$  be a vector space over  $\mathbb{R}$ . Let  $f$  be a proper function from  $\mathcal{V}$  to  $\mathbb{R}^*$ . We say that  $f$  is **strictly convex** if and only if  $\forall x, y \in \text{dom}(f)$ ,  $\forall \lambda \in (0, 1)$ , we have

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

**PROPOSITION 11.2.** Strictly convex functions are convex.

### 11.2 Strongly Convex

**DEFINITION 11.3** (Strongly Convex). Let  $f$  be a proper function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Let  $\beta$  be a positive constant. We say that  $f$  is  $\beta$  **-strongly convex** if  $\forall x, y \in \text{dom}(f)$ ,  $\forall \lambda \in [0, 1]$ , we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \frac{\beta}{2}\lambda(1 - \lambda)\|x - y\|^2.$$

**PROPOSITION 11.4.** Strongly convex functions are strictly convex.

**PROPOSITION 11.5.** Let  $f$  be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Then  $f$  is  $\beta$ -strongly convex if and only if  $f - \frac{\beta}{2}\|\cdot\|^2$  is convex.

*Proof.* Let  $\beta$  be a positive constant. Let  $g$  denote  $f - \frac{\beta}{2}\|\cdot\|^2$ . Let  $x$  and  $y$  be arbitrary elements of  $\mathbb{E}$ . Let  $\lambda \in (0, 1)$  be arbitrary.

$$\begin{aligned}
& f \text{ is } \beta\text{-strongly convex} \\
\iff & f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \\
& \quad - \frac{\beta}{2}\lambda(1 - \lambda)\|x - y\|^2 \\
\iff & f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \\
& \quad - \frac{\beta}{2}\lambda(1 - \lambda)(\|x\|^2 + \|y\|^2 - 2\langle x, y \rangle) \\
\iff & f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \\
& \quad - \lambda\frac{\beta}{2}\|x\|^2 + \frac{\beta}{2}\lambda^2\|x\|^2 \\
& \quad - (1 - \lambda)\frac{\beta}{2}\|y\|^2 + \frac{\beta}{2}(1 - \lambda)^2\|y\|^2 \\
& \quad + \beta\lambda(1 - \lambda)\langle x, y \rangle \\
\iff & f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \\
& \quad - \lambda\frac{\beta}{2}\|x\|^2 - (1 - \lambda)\frac{\beta}{2}\|y\|^2 \\
& \quad + \frac{\beta}{2}\|\lambda x\|^2 + \frac{\beta}{2}\|(1 - \lambda)y\|^2 + \beta\langle \lambda x, (1 - \lambda)y \rangle \\
\iff & f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \\
& \quad - \lambda\frac{\beta}{2}\|x\|^2 - (1 - \lambda)\frac{\beta}{2}\|y\|^2 \\
& \quad + \frac{\beta}{2}\|\lambda x + (1 - \lambda)y\|^2 \\
\iff & g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y) \\
\iff & f - \frac{\beta}{2}\|\cdot\|^2 \text{ is } \beta \text{ convex.}
\end{aligned}$$

□

Question: Can we allow  $f$  to take on  $-\infty$ ? Do we need  $f$  to be proper?

**PROPOSITION 11.6.** Let  $f$  and  $g$  be functions from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Suppose  $f$  is  $\beta$ -

strongly convex for some positive constant  $\beta$  and  $g$  is convex. Then  $f + g$  is also  $\beta$ -strongly convex.

Question: Can we allow  $f$  or  $g$  to take on  $-\infty$ ? Do we need  $f$  and  $g$  to be proper?

*Proof.*

$$\begin{aligned} & f \text{ is } \beta\text{-strongly convex} \\ \implies & f - \frac{\beta}{2} \|\cdot\|^2 \text{ is convex} \\ \implies & f + g - \frac{\beta}{2} \|\cdot\|^2 \text{ is convex} \\ \implies & f + g \text{ is } \beta\text{-strongly convex.} \end{aligned}$$

□

## 11.3 Quasiconvex

**DEFINITION 11.7** (Quasiconvex). Let  $f : \mathbb{E} \rightarrow \mathbb{R}$  be a function with convex domain. We say that  $f$  is **quasiconvex** if any level set of  $f$  is convex.

**PROPOSITION 11.8** (Jensen's Inequality for Quasiconvex Functions). Let  $f$  be a quasiconvex function. Then  $\forall x, y \in \text{dom}(f)$ ,  $\forall \alpha, \beta \in [0, 1]$  such that  $\alpha + \beta = 1$ ,

$$f(\alpha x + \beta y) \leq \max\{f(x), f(y)\}.$$

**PROPOSITION 11.9.** A differentiable real-valued function  $f$  with convex domain is convex if and only if  $\forall x, y \in \text{dom}(f)$ ,

$$f(y) \leq f(x) \implies \nabla f(x) \cdot (y - x) \leq 0. \quad ???$$

Not sure where did this come from but I don't think this is correct.





# Chapter 12

## Support

### 12.1 Definitions

**DEFINITION 12.1** (Support Function). Let  $S$  be a subset of  $\mathbb{E}$ . We define the **support function** of  $S$ , denoted by  $\sigma_S$ , to be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$  given by

$$\sigma_S(x) = \sup_{s \in S} \langle x, s \rangle.$$

**DEFINITION 12.2** (Supporting Hyperplane). Let  $S$  be a set in  $\mathbb{E}$  with nonempty boundary. Let  $x_0$  be a point in the boundary of  $S$ . We define a **supporting hyperplane**  $H$  to set  $S$  at point  $x_0$  to be a set of the form

$$H = \{x \in \mathbb{E} : a^T x = a^T x_0\},$$

such that  $a \in \mathbb{E}$  and  $a \neq \vec{0}$  and  $\forall x \in S, a^T x \leq a^T x_0$ .

### 12.2 Properties

**PROPOSITION 12.3.** The support function of a non-empty set  $S$  is proper, convex, and lower semi-continuous.

*Proof.*

**Part 1.** Proper.

Define  $f_s$  to be a function from  $\mathbb{E}$  to  $\mathbb{R}$  by  $f_s(x) = \langle s, x \rangle$ .

These functions are linear and hence proper, convex, and lower semi-continuous.

Notice  $\sigma_S = \sup_{s \in S} f_s$ .

So  $\sigma_S$  is convex and lower semi-continuous.

Since  $\sigma_S(0) = \sup_{s \in S} \langle 0, s \rangle = 0$ ,  $\exists x_0 \in \mathbb{E}$ ,  $\sigma_S(x) \neq +\infty$ .

Since  $\sigma_S(x) = \sup_{s \in S} \langle x, s \rangle \geq \langle x, s \rangle \neq -\infty$ ,  $\forall x \in \mathbb{E}$ ,  $\sigma_S(x) \neq -\infty$ .

Since  $\exists x_0 \in \mathbb{E}$ ,  $\sigma_S(x) \neq +\infty$  and  $\forall x \in \mathbb{E}$ ,  $\sigma_S(x) \neq -\infty$ , by definition,  $\sigma_S$  is proper.

□

**PROPOSITION 12.4.** The support function of a non-empty and bounded set is continuous.

*Proof.*

Let  $x_0$  be an arbitrary point in  $\mathbb{E}$ . Let  $\varepsilon$  be an arbitrary positive number. Define  $M := \sup_{y \in C} \|y\| + 1$ . Since  $C$  is bounded,  $M$  is finite. Define  $\delta := \varepsilon/M$ . Let  $x$  be an arbitrary point such that  $\|x - x_0\| < \delta$ . Let  $y$  be an arbitrary point in  $\mathbb{E}$ . Then by the Cauchy Schwarz inequality, we have

$$\langle x - x_0, y \rangle \leq \|x - x_0\| \|y\|.$$

That is,

$$\langle x, y \rangle \leq \|x - x_0\| \|y\| + \langle x_0, y \rangle.$$

It follows that

$$\begin{aligned} \sup_{y \in C} \langle x, y \rangle &\leq \sup_{y \in C} (\|x - x_0\| \|y\| + \langle x_0, y \rangle) \\ &\leq \|x - x_0\| \sup_{y \in C} \|y\| + \sup_{y \in C} \langle x_0, y \rangle. \end{aligned}$$

That is,

$$\sigma_C(x) \leq \sigma_C(x_0) + \|x - x_0\| \sup_{y \in C} \|y\|.$$

By definition of  $\delta$  and  $M$ ,

$$\sigma_C(x) < \sigma_C(x_0) + \varepsilon. \tag{1}$$

Similarly, reversing the role of  $x$  and  $x_0$ , we can prove that

$$\sigma_C(x_0) < \sigma_C(x) + \varepsilon. \tag{2}$$

From (1) and (2) we get

$$|\sigma_C(x) - \sigma_C(x_0)| < \varepsilon.$$

Since  $\forall \varepsilon > 0, \exists \delta > 0$  such that  $|\sigma_C(x) - \sigma_C(x_0)| < \varepsilon$  whenever  $\|x - x_0\| < \delta$ , by definition,  $\delta_C$  is continuous.

□

**PROPOSITION 12.5.** Let  $S$  be a subset of  $\mathbb{E}$ . Then  $\sigma_S = \sigma_{\text{conv}(S)} = \sigma_{\overline{\text{conv}}(S)}$ .

*Proof.*

Let  $x$  be an arbitrary point in  $\mathbb{E}$ .

$$\begin{aligned}\sigma_S(x) &= \sup \{ \langle x, s \rangle : s \in S \} \\ \sigma_{\text{conv}(S)}(x) &= \sup \{ \langle x, s \rangle : s \in \text{conv}(S) \} \\ \sigma_{\overline{\text{conv}}(S)}(x) &= \sup \{ \langle x, s \rangle : s \in \overline{\text{conv}}(S) \}.\end{aligned}$$

It is easy to see that by the linearity of inner products,

$$\text{conv} \{ \langle x, s \rangle : s \in S \} = \{ \langle x, s \rangle : s \in \text{conv}(S) \}.$$

It is easy to see that by the linearity and the continuity of inner products,

$$\overline{\text{conv}} \{ \langle x, s \rangle : s \in S \} = \{ \langle x, s \rangle : s \in \overline{\text{conv}}(S) \}.$$

It is also easy to see that for any subset  $A$  of the reals,

$$\sup(A) = \sup(\text{conv}(A)),$$

and

$$\sup(A) = \sup(\text{cl}(A)).$$

So

$$\begin{aligned}\sigma_S(x) &= \sup \{ \langle x, s \rangle : s \in S \} \\ &= \sup \text{conv} \{ \langle x, s \rangle : s \in S \} \\ &= \sup \{ \langle x, s \rangle : s \in \text{conv}(S) \} \\ &= \sigma_{\text{conv}(S)}(x).\end{aligned}$$

That is,  $\sigma_S(x) = \sigma_{\text{conv}(S)}(x)$ .

$$\begin{aligned}\sigma_S(x) &= \sup \{ \langle x, s \rangle : s \in S \}\end{aligned}$$

$$\begin{aligned}
&= \sup \operatorname{conv} \{ \langle x, s \rangle : s \in S \} \\
&= \sup \{ \langle x, s \rangle : s \in \operatorname{conv}(S) \} \\
&= \sup \operatorname{cl} \{ \langle x, s \rangle : s \in \operatorname{conv}(S) \} \\
&= \sup \{ \langle x, s \rangle : s \in \operatorname{cl}(\operatorname{conv}(S)) \} \\
&= \sup \{ \langle x, s \rangle : s \in \overline{\operatorname{conv}}(S) \} \\
&= \sigma_{\overline{\operatorname{conv}}(S)}(x).
\end{aligned}$$

That is,  $\sigma_S(x) = \sigma_{\overline{\operatorname{conv}}(S)}(x)$ .

□

### 12.3 Supporting Hyperplane

**THEOREM 12.6** (Supporting Hyperplane Theorem). For any boundary point  $x_0$  of a convex set  $C$ , there exists a supporting hyperplane to  $C$  at  $x_0$ .

## Chapter 13

# Conjugacy

### 13.1 Definition and Examples

**DEFINITION 13.1** (Convex Conjugate (Legendre–Fenchel Transformation)). Let  $f$  be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . We define the **convex conjugate** of  $f$ , denoted by  $f^*$ , to be a function also from  $\mathbb{E}$  to  $\mathbb{R}^*$  given by

$$f^*(x) := \sup_{y \in \mathbb{E}} \{ \langle y, x \rangle - f(y) \}.$$

**EXAMPLE 13.2.** Let  $S$  be a subset of  $\mathbb{E}$ . Then  $\delta_S^* = \sigma_S$ .

*Proof.* Recall that

$$\delta_S(x) = \begin{cases} 0, & x \in S, \\ +\infty, & x \notin S, \end{cases}$$
$$\sigma_S(x) = \sup_{y \in S} \langle x, y \rangle.$$

Now for any  $x \in \mathbb{E}$ ,

$$\begin{aligned} \delta_S^*(x) &= \sup_{y \in S} (\langle x, y \rangle - \delta_S(y)) \\ &= \sup_{y \in S} (\langle x, y \rangle - 0) \end{aligned}$$

$$\begin{aligned}
&= \sup_{y \in S} \langle x, y \rangle \\
&= \sigma_S(x).
\end{aligned}$$

So  $\delta_S^* = \sigma_S$ . □

## 13.2 Basic Properties

**PROPOSITION 13.3.** The convex conjugate function is convex.

*Proof.* If  $\text{dom}(f) = \emptyset$ , then one can see that  $f^* \equiv -\infty$ . It is a pointwise supremum of affine functions. □

**PROPOSITION 13.4.** The convex conjugate function is lower semi-continuous.

## 13.3 Double Conjugate

**PROPOSITION 13.5.** Let  $f$  be any function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Then  $f^{**} \leq f$ .

*Proof.* Let  $x$  be an arbitrary point in  $\mathbb{E}$ .

$$\begin{aligned}
&f^{**}(x) \\
&= \sup_{y \in \mathbb{E}} \{ \langle y, x \rangle - f^*(y) \} \\
&= \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \sup_{z \in \mathbb{E}} \{ \langle z, y \rangle - f(z) \} \right\} \\
&\leq \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \{ \langle x, y \rangle - f(x) \} \right\} \\
&= \sup_{y \in \mathbb{E}} \{ f(x) \} \\
&= f(x).
\end{aligned}$$

That is,  $f^{**}(x) \leq f(x)$ . Since  $\forall x \in \mathbb{E}$ ,  $f^{**}(x) \leq f(x)$ , we get  $f^{**} \leq f$ . □

**PROPOSITION 13.6.** Let  $f$  be a proper function. Then  $f$  is convex and lower semi-continuous if and only if

$$f^{**} = f.$$

**PROPOSITION 13.7.** Let  $f$  and  $g$  be functions from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Then  $f \leq g$  implies  $f^* \geq g^*$  and  $f^{**} \leq g^{**}$ .

*Proof.* Let  $x$  be an arbitrary point in  $\mathbb{E}$ .

$$\begin{aligned} f^*(x) &= \sup_{y \in \mathbb{E}} \{ \langle y, x \rangle - f(y) \} \\ &\geq \sup_{y \in \mathbb{E}} \{ \langle y, x \rangle - g(y) \} \\ &= g^*(x). \end{aligned}$$

That is,  $f^*(x) \geq g^*(x)$ . Since  $\forall x \in \mathbb{E}$ ,  $f^*(x) \geq g^*(x)$ , we get  $f^* \geq g^*$ .

Let  $x$  be an arbitrary point in  $\mathbb{E}$ .

$$\begin{aligned} f^{**}(x) &= \sup_{y \in \mathbb{E}} \{ \langle y, x \rangle - f^*(y) \} \\ &= \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \sup_{z \in \mathbb{E}} \{ \langle z, y \rangle - f(z) \} \right\} \\ &\leq \sup_{y \in \mathbb{E}} \left\{ \langle y, x \rangle - \sup_{z \in \mathbb{E}} \{ \langle z, y \rangle - g(z) \} \right\} \\ &= \sup_{y \in \mathbb{E}} \{ \langle y, x \rangle - g^*(y) \} \\ &= g^{**}(x). \end{aligned}$$

That is,  $f^{**}(x) \leq g^{**}(x)$ . Since  $\forall x \in \mathbb{E}$ ,  $f^{**}(x) \leq g^{**}(x)$ , we get  $f^{**} \leq g^{**}$ . □

**PROPOSITION 13.8.**

$$\text{epi}(f^{**}) = \text{conv}(\text{epi}(f)).$$

## 13.4 Conjugates and Sub-Differentials

**THEOREM 13.9** (Fenchel-Young). Let  $f$  be a proper function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Then  $\forall x, y \in \mathbb{E}$ , we have

$$f(x) + f^*(y) \geq \langle x, y \rangle.$$

**PROPOSITION 13.10.** Let  $f$  be a proper closed convex function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Then  $\forall x, y \in \mathbb{E}$ ,

$$y \in \partial f(x) \iff x \in \partial f^*(y) \iff f(x) + f^*(y) = \langle x, y \rangle.$$

*Proof of  $y \in \partial f(x) \iff x \in \partial f^*(y)$ .* For one direction, assume that  $y \in \partial f(x)$ . We are to prove that  $x \in \partial f^*(y)$ . Consider an arbitrary point  $z \in \mathbb{E}$ . Since  $y \in \partial f(x)$ , we get

$$\forall u \in \mathbb{E}, \quad \langle y, u - x \rangle \leq f(u) - f(x).$$

Rearranging yields

$$\forall u \in \mathbb{E}, \quad \langle y, u \rangle - f(u) \leq \langle y, x \rangle - f(x).$$

It follows that

$$\sup_{u \in \mathbb{E}} (\langle y, u \rangle - f(u)) \leq \langle y, x \rangle - f(x). \quad (1)$$

By definition of supremum, we have

$$\sup_{u \in \mathbb{E}} (\langle y, u \rangle - f(u)) \geq \langle y, x \rangle - f(x). \quad (2)$$

From (1) and (2), we get

$$\sup_{u \in \mathbb{E}} (\langle y, u \rangle - f(u)) = \langle y, x \rangle - f(x).$$

That is,

$$f^*(y) = \langle y, x \rangle - f(x).$$

Then

$$\begin{aligned} & f^*(z) - f^*(y) \\ &= \sup_{u \in \mathbb{E}} (\langle z, u \rangle - f(u)) - \sup_{u \in \mathbb{E}} (\langle y, u \rangle - f(u)) \\ &= \sup_{u \in \mathbb{E}} (\langle z, u \rangle - f(u)) - \langle y, x \rangle + f(x) \\ &\geq \langle z, x \rangle - f(x) - \langle y, x \rangle + f(x) \\ &= \langle z - y, x \rangle. \end{aligned}$$



That is,

$$\langle x, z - y \rangle \leq f^*(z) - f^*(y).$$

So  $x \in \partial f^*(y)$ . This proves

$$y \in \partial f(x) \implies x \in \partial f^*(y).$$

Since  $f^{**} = f$ , similarly, we can prove that

$$x \in \partial f^*(y) \implies y \in \partial f(x).$$

□

**PROPOSITION 13.11.** Let  $f$  be a proper convex function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Let  $x$  be a point in  $\mathbb{E}$ . Assume that  $\partial f(x) \neq \emptyset$ . Then  $f^{**}(x) = f(x)$ .



## Chapter 14

# Proximal Operators

### 14.1 Definitions

**DEFINITION 14.1** (Proximal Operator). Let  $f$  be a function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . We define the **proximal operator** of  $f$ , denoted by  $\text{prox}_f$ , to be a function from  $\mathbb{E}$  to  $\mathcal{P}(\mathbb{E})$  given by

$$\text{prox}_f(x) := \underset{y \in \mathbb{E}}{\operatorname{argmin}} \left\{ f(y) + \frac{1}{2} \|y - x\|^2 \right\}.$$

### 14.2 Examples

**EXAMPLE 14.2** (Soft Threshold). Let  $\lambda \geq 0$ . Let  $f$  be a function from  $\mathbb{R}$  to  $\mathbb{R}$  given by  $f(x) := \lambda|x|$ . Then

$$\text{prox}_f(x) = \begin{cases} x + \lambda, & \text{if } x < -\lambda \\ 0, & \text{if } -\lambda \leq x \leq \lambda \\ x - \lambda, & \text{if } x > \lambda. \end{cases}$$

### 14.3 Basic Properties

(bug)

**PROPOSITION 14.3.** Let  $f$  be a proper, convex, and lower semi-continuous function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Then  $\forall x \in \mathbb{E}$ ,  $\text{prox}_f(x)$  is a singleton set.

*Proof.* Let  $x$  be an arbitrary element of  $\mathbb{E}$ . Define a function  $h : \mathbb{E} \rightarrow \mathbb{R}^*$  by  $h(y) := \frac{1}{2}\|y-x\|^2$ . Define a function  $g : \mathbb{E} \rightarrow \mathbb{R}^*$  by  $g(y) := f(y) + h(y)$ . Then  $\text{prox}_f(x) = \underset{y \in \mathbb{E}}{\operatorname{argmin}} g(y)$ . Note that  $h$  is proper, lower semi-continuous, and  $\beta$ -strongly convex for any  $\beta \in (0, 1)$ . Since  $f$  and  $h$  are proper,  $g$  is proper (why?). Since  $f$  and  $h$  are lower semi-continuous,  $g$  is lower semi-continuous. Since  $f$  is convex and  $h$  is  $\beta$ -strongly convex,  $g$  is  $\beta$ -strongly convex. Since  $g$  is proper, lower semi-continuous, and strongly convex,  $g$  has a unique minimizer (why?). So  $\text{prox}_f(x)$  is a singleton set.  $\square$

not fully understood

**PROPOSITION 14.4.** Let  $C$  be a nonempty closed convex subset of  $\mathbb{E}$ . Then  $\text{prox}_{\delta_C}$  and  $\text{proj}_C$  are both singleton and  $\text{prox}_{\delta_C} = \text{proj}_C$ .

*Proof.* Since  $C$  is nonempty, convex, and closed,  $\delta_C$  is proper, convex, and lower semi-continuous and hence  $\text{prox}_{\delta_C}$  is singleton. Since  $C$  is nonempty, convex, and closed,  $\text{proj}_C$  is singleton. Let  $x$  and  $p$  be arbitrary elements of  $\mathbb{E}$ . Then

$$\begin{aligned}
 & p \in \text{prox}_{\delta_C}(x) \\
 \iff & p \in \underset{y \in \mathbb{E}}{\operatorname{argmin}} \{ \delta_C(y) + \frac{1}{2}\|y-x\|^2 \} \\
 \iff & \forall y \in \mathbb{E}, \delta_C(y) + \frac{1}{2}\|y-x\|^2 \geq \delta_C(p) + \frac{1}{2}\|p-x\|^2 \\
 \iff & p \in C \text{ and } \forall y \in C, \frac{1}{2}\|y-x\|^2 \geq \frac{1}{2}\|p-x\|^2 \\
 \iff & p \in C \text{ and } \forall y \in C, \|y-x\| \geq \|p-x\| \\
 \iff & p \in \underset{y \in C}{\operatorname{argmin}} \|y-x\| \\
 \iff & p \in \text{proj}_C(x).
 \end{aligned}$$

$\square$

**PROPOSITION 14.5** (Firmly Non-Expansive). Let  $f$  be a proper closed convex function. Then  $\text{prox}_f$  is firmly non-expansive.

## 14.4 Prox Calculus Rules

**PROPOSITION 14.6** (Scaling and Translation).

**THEOREM 14.7** (Norm Composition).

**PROPOSITION 14.8.** Let  $f_1, \dots, f_m$  be proper, convex, and lower semi-continuous functions from  $\mathbb{R}$  to  $\mathbb{R}^*$ . Define a function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^*$  by  $f((x_i)_{i=1}^m) := \sum_{i=1}^m f_i(x_i)$ . Then

$$\text{prox}_f((x_i)_{i=1}^m) = (\text{prox}_{f_i}(x_i))_{i=1}^m.$$

*Proof.* Since each  $f_i$  is proper, convex, and lower semi-continuous,  $f$  is proper, convex, and lower semi-continuous. Let  $(x_i)_{i=1}^m$  and  $(p_i)_{i=1}^m$  be arbitrary elements of  $\mathbb{R}^m$ . Then

$$(p_i)_{i=1}^m = \text{prox}_f((x_i)_{i=1}^m)$$

$$\iff$$

□

## 14.5 The Second Prox Theorem

**PROPOSITION 14.9.** Let  $f$  be a proper, convex, and lower semi-continuous function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Let  $x$  and  $p$  be points in  $\mathbb{E}$ . Then  $p = \text{prox}_f(x)$  if and only if

$$x - p \in \partial f(p).$$

**PROPOSITION 14.10.** Let  $f$  be a proper, convex, and lower semi-continuous function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Let  $x$  and  $p$  be elements of  $\mathbb{E}$ . Then  $p = \text{prox}_f(x)$  if and only if

$$\forall y \in \mathbb{E}, \quad \langle y - p, x - p \rangle \leq f(y) - f(p).$$

*Proof. Forward Direction:*

Assume that  $p = \text{prox}_f(x)$ . I will show that  $\forall y \in \mathbb{E}$ ,  $\langle y - p, x - p \rangle \leq f(y) - f(p)$ . Let  $y$  be an arbitrary element of  $\mathbb{E}$ . Define for each  $\lambda \in (0, 1)$  a point  $p_\lambda$  by  $p_\lambda := \lambda y + (1 - \lambda)p$ . Then

$$\begin{aligned}
& p = \text{prox}_f(x) \\
& \implies f(p) + \frac{1}{2}\|x - p\|^2 \leq f(p_\lambda) + \frac{1}{2}\|x - p_\lambda\|^2 \\
& \iff f(p) \leq f(p_\lambda) + \frac{1}{2}\|x - p_\lambda\|^2 - \frac{1}{2}\|x - p\|^2 \\
& \iff f(p) \leq f(p_\lambda) + \frac{1}{2} \left\langle \left[ (x - p_\lambda) + (x - p) \right], \left[ (x - p_\lambda) - (x - p) \right] \right\rangle \\
& \iff f(p) \leq f(p_\lambda) + \frac{1}{2} \left\langle \left[ 2x - \lambda y - (1 - \lambda)p - p \right], \left[ p - \lambda y - (1 - \lambda)p \right] \right\rangle \\
& \iff f(p) \leq f(p_\lambda) + \frac{1}{2} \left\langle \left[ 2(x - p) + \lambda(p - y) \right], \left[ \lambda(p - y) \right] \right\rangle \\
& \iff f(p) \leq f(p_\lambda) + \lambda \langle x - p, p - y \rangle + \frac{1}{2}\lambda^2\|p - y\|^2 \\
& \iff f(p) \leq f(\lambda y + (1 - \lambda)p) + \lambda \langle x - p, p - y \rangle + \frac{1}{2}\lambda^2\|p - y\|^2 \\
& \implies f(p) \leq \lambda f(y) + (1 - \lambda)f(p) + \lambda \langle x - p, p - y \rangle + \frac{1}{2}\lambda^2\|p - y\|^2 \\
& \iff \lambda \langle y - p, x - p \rangle \leq \lambda f(y) - \lambda f(p) + \frac{1}{2}\lambda^2\|p - y\|^2 \\
& \iff \langle y - p, x - p \rangle \leq f(y) - f(p) + \frac{1}{2}\lambda\|p - y\|^2 \\
& \iff \langle y - p, x - p \rangle \leq f(y) - f(p).
\end{aligned}$$

**Backward Direction:**

Assume that  $\forall y \in \mathbb{E}$ ,  $\langle y - p, x - p \rangle \leq f(y) - f(p)$ . I will show that  $p = \text{prox}_f(x)$ . Let  $y$  be an arbitrary element of  $\mathbb{E}$ . Then

$$\begin{aligned}
& \langle y - p, x - p \rangle \leq f(y) - f(p) \\
& \iff f(p) + \frac{1}{2}\|x - p\|^2 \leq f(y) + \langle x - p, p - y \rangle + \frac{1}{2}\|x - p\|^2 \\
& \implies f(p) + \frac{1}{2}\|x - p\|^2 \leq f(y) + \langle x - p, p - y \rangle + \frac{1}{2}\|x - p\|^2 + \frac{1}{2}\|p - y\|^2 \\
& \iff f(p) + \frac{1}{2}\|x - p\|^2 \leq f(y) + \frac{1}{2}\|(x - p) + (p - y)\|^2 \\
& \iff f(p) + \frac{1}{2}\|x - p\|^2 \leq f(y) + \frac{1}{2}\|x - y\|^2 \\
& \implies p = \text{prox}_f(x).
\end{aligned}$$

This completes the proof. □

## 14.6 Moreau Decomposition

**THEOREM 14.11** (Moreau Decomposition). Let  $f$  be a proper closed convex function from  $\mathbb{E}$  to  $\mathbb{R}^*$ . Then

$$\text{prox}_f + \text{prox}_{f^*} = \text{Id}.$$

*Proof.* Let  $x$  be an arbitrary point in  $\mathbb{E}$ . We are to prove that

$$\text{prox}_f(x) + \text{prox}_{f^*}(x) = x.$$

Let  $p$  denote  $\text{prox}_f(x)$ . Since  $f$  is proper convex and lower semi-continuous and  $p = \text{prox}_f(x)$ , we get

$$x - p \in \partial f(p).$$

Since  $x - p \in \partial f(p)$ , we get  $p \in \partial f^*(x - p)$ . It follows that  $x - p = \text{prox}_{f^*}(x)$ . Substitute  $p = \text{prox}_f(x)$  and rearrange the equation, we get

$$\text{prox}_f(x) + \text{prox}_{f^*}(x) = x.$$

□





## Chapter 15

# Ellipsoids

**DEFINITION 15.1** (Ellipsoid). Let  $v$  be a point in some Euclidean space  $\mathbb{E}$ . We define an **ellipsoid**, centered at point  $v$ , to be a set of the form

$$\{x \in \mathbb{E} : (x - v)^T A (x - v) = 1\}$$

where  $A$  is some  $d$  by  $d$  positive definite matrix.

### 15.1 Properties

**PROPOSITION 15.2.** The eigenvectors of  $A$  define the principal axes of the ellipsoid.