Set Theory

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Contents

1 Basic Concepts **1**

1.1 Functions 1

1.2 Properties 2

2 Cardinality **5**

2.1 Relations of Cardinality 5

2.2 Finite, Infinite, and Countably Infinite 6

2.3 Arithmetic Rules of Cardinality 6

2.4 Examples 7

Chapter 1

Basic Concepts

1.1 Functions

Definition (Injective). *Let A and B be two sets. Let f be a function from A to B. We say that f is* ***injective*** *if $\forall y \in B$, $\exists$ at most one $x \in A$ such that $f(x) = y$. Equivalently, if*

$$\forall x_1, x_2 \in A, \quad f(x_1) = f(x_2) \implies x_1 = x_2.$$

Definition (Surjective). *Let A and B be two sets. Let f be a function from A to B. We say that f is* ***surjective*** *if $\forall y \in B$, $\exists$ at least one $x \in A$ such that $f(x) = y$. Equivalently, if $\text{range}(f) = B$.*

Definition (Bijective). *Let A and B be two sets. Let f be a function from A to B. We say that f is* ***bijective*** *if $\forall y \in B$, $\exists$ exactly one $x \in A$ such that $f(x) = y$. Equivalently, if f is both injective and surjective.*

Definition (Left Inverse). *Let A and B be two sets. Let f be a function from A to B. Let g be a function from B to A. We say that g is a* ***left inverse*** *of f if*

$$\forall x \in A, \quad (g \circ f)(x) = x.$$

Equivalently, if $g \circ f = I$.

Definition (Right Inverse). *Let A and B be two sets. Let f be a function from A to B. Let g be a function from B to A. We say that g is a* ***right inverse*** *of f if*

$$\forall x \in B, \quad (f \circ g)(x) = x.$$

Equivalently, if $f \circ g = I$.

1.2 Properties

Proposition 1.2.1. *Let A, B, and C be three sets. Let f be a function from A to B. Let g be a function from B to C. Then*

- *if both f and g are injective, $g \circ f$ is also injective;*
- *if both f and g are surjective, $g \circ f$ is also surjective;*
- *if both f and g are bijective, $g \circ f$ is also bijective.*

Proof of (1). Let z be an arbitrary element in C. If z has no inverse image under $g \circ f$, then we are done. Else, let x_1 and x_2 be inverse images of z, under $g \circ f$. That is, $g(f(x_1)) = g(f(x_2)) = z$. Since g is injective from B to C and $g(f(x_1)) = g(f(x_2)) = z$, we get $f(x_1) = f(x_2)$. Let y denote $f(x_1)$ and $f(x_2)$. Since f is injective from A to B and $f(x_1) = f(x_2) = y$, we get $x_1 = x_2$. So the inverse image of z, under $g \circ f$, is unique. Since $\forall z \in C$, either z has no inverse iamge, under $g \circ f$, or the inverse iamge is unique, by definition, we get $g \circ f$ is injective. ∎

Proof of (2). Let z be an arbitrary element in C. Since g is surjective from B to C, $\exists y \in B$ such that $g(y) = z$. Since f is surjective from A to B, $\exists x \in A$ such that $f(x) = y$. Since $g(y) = z$ and $f(x) = y$, we get $g(f(x)) = z$. That is, $(g \circ f)(x) = z$. Since $\forall z \in C$, $\exists x \in A$, $(g \circ f)(x) = z$, by definition, we get $g \circ f$ is surjective. ∎

Proof of (3). Since both f and g are bijective, they are both injective and surjective. By (1) and (2), $g \circ f$ is both injective and surjective. So $g \circ f$ is bijective. ∎

Proposition 1.2.2. *Let A and B be two sets. Let f be a function from A to B. Then*

- *f is injective if and only if f has a left inverse;*
- *f is surjective if and only if f has a right inverse;*
- *f is bijective if and only if f has a left inverse and a right inverse. In this case, the left inverse and the right inverse are the same.*

Proof of (1). For one direction, assume that f is injective. We are to prove that f has a left inverse. I would assume that A is non-empty. Let a be a fixed

element in A. Define a function g from B to A as follows. Let y be an arbitrary element in B. If $y \in \text{range}(f)$, then $\exists x \in A$ such that $f(x) = y$. Since f is injective, x is unique. Define $g(y) := x$. Else, $y \notin \text{range}(f)$. Define $g(y) := a$. Since $\forall x \in A$, $(g \circ f)(x) = g(f(x)) = x$, by definition, g is a left inverse of f.
For the reverse direction, assume that f has a left inverse. We are to prove that f is injective. Let g denote the left inverse of f. Let y be an arbitrary element in B. If y has no inverse image under f, then we are done. Else, let x_1 and x_2 be inverse images of y, under f. That is, $f(x_1) = f(x_2) = y$. So $g(f(x_1)) = g(f(x_2)) = g(y)$. Since g is a left inverse of f, we get $g(f(x_1)) = x_1$ and $g(f(x_2)) = x_2$. So $x_1 = x_2 = g(y)$. So the inverse image of y, under f, is unique. Since $\forall y \in B$, either y has no inverse image, under f, or the inverse image is unique, by definition, f is injective.

■

Proposition 1.2.3. *Let A and B be two sets. Then there exists an injective map from A to B if and only if there exists a surjective map from B to A.*

Proof. Let f be a function from A to B. If f is injective, then f has a left inverse. Say g is a left inverse of f. Then f is a right inverse of g. Since g has a right inverse, g is surjective. The reverse direction can be proved similarly. ■

Chapter 2

Cardinality

2.1 Relations of Cardinality

Definition (Relations of Cardinality)**.** *Let A and B be two sets. We say that*

- $|A| = |B|$ *if there exists a bijective map between A and B.*
- $|A| \leq |B|$ *if there exists an injective map from A to B. Equivalently, if there exists a surjective map from B to A.*
- $|A| \geq |B|$ *if there exists a surjective map from A to B. Equivalently, if there exists an injective from B to A.*
- $|A| < |B|$ *if* $|A| \leq |B|$ *and* $|A| \neq |B|$.
- $|A| > |B|$ *if* $|A| \geq |B|$ *and* $|A| \neq |B|$.

Proposition 2.1.1 (Equality of Cardinality is an Equivalence Relation)**.**

- *(Reflexivity)*
  For any set A, $|A| = |A|$.
- *(Symmetry)*
  For any sets A and B, if $|A| = |B|$, then $|B| = |A|$.
- *(Transitivity)*
  For any sets A, B, and C, if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proposition 2.1.2 (Transitivity of $\leq$)**.** *For any sets A, B, and C, if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.*

2.2 Finite, Infinite, and Countably Infinite

Definition (Finite)**.** *Let S be a set. For $n \in \mathbb{N}$, let S_n denote the set $\{0, ..., n-1\}$. We say that S is* ***finite*** *if there exists a bijection between S and S_n for some $n \in \mathbb{N}$.*

Definition (Infinite)**.** *Let S be a set. For $n \in \mathbb{N}$, let S_n denote the set $\{0, ..., n-1\}$. We say that S is* ***infinite*** *if S is not finite. Equivalently, if $\forall n \in \mathbb{N}$, there is no bijection between S and S_n.*

Definition (Countably Infinite)**.** *Let S be a set. We say that S is* ***countably infinite*** *if there exists a bijection between S and $\mathbb{N}$.*

Proposition 2.2.1 (Comparison to $\mathbb{N}$)**.** *Let S be a set. Then*

- *S is finite if and only if $|S| < |\mathbb{N}|$.*
- *S is countably infinite if and only if $|S| = |\mathbb{N}|$.*
- *S is infinite if and only if $|S| \geq |\mathbb{N}|$.*

Proposition 2.2.2. *Let A and B be two sets. Then we have the following implications:*

- *If $|A| \leq |B|$ and B is finite, then so is A.*
- *If $|A| \leq |B|$ and A is infinite, then so is B.*

Proposition 2.2.3 (Products)**.** *Let S_1 and S_2 be countably infinite sets. Then the set $S_1 \times S_2$ is also countably infinite.*

Proposition 2.2.4 (Finite Unions)**.** *Let S_1 and S_2 be countably infinite sets. Then the set $S_1 \cup S_2$ is also countably infinite.*

Proposition 2.2.5 (Countable Unions)**.** *Let $\{S_i\}_{i \in \mathbb{N}}$ be a sequence of countably infinite sets. Then the set $\bigcup_{i \in \mathbb{N}} S_i$ is also countably infinite.*

Theorem 1 (Schröder–Bernstein Theorem)**.** *Let A and B be two sets. If $|A| \leq |B|$ and $|A| \geq |B|$, then $|A| = |B|$.*

2.3 Arithmetic Rules of Cardinality

Proposition 2.3.1. *Let S_1 and S_2 be finite sets. Then we have the following rules:*

- *If* S_1 *and* S_2 *are disjoint, then* $|S_1 \cup S_2| = |S_1| + |S_2|$.
- $|S_1 \times S_2| = |S_1| \cdot |S_2|$.
- $|S_1^{S_2}| = |S_1|^{|S_2|}$.

Proposition 2.3.2 (Substitution Rules)**.** *Let* A*,* B*,* C*, and* D *be sets. Suppose that* $|A| = |C|$ *and that* $|B| = |D|$*. Then we have the following rules:*

- *If* A *and* B *are disjoint and* C *and* D *are disjoint, then* $|A \cup B| = |C \cup D|$.
- $|A \times B| = |C \times D|$.
- $|A^B| = |C^D|$.

2.4 Examples

Example 2.4.1. *The cardinality of the set of all open sets in* $\mathbb{R}$ *is* $2^{\aleph_0}$.