Matrix Theory

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Chapter 1

Fundamentals

1.1 Definitions

DEFINITION (Column Space)**.** Let A be an $m \times n$ matrix. We define the **column space** of A, denoted by $\operatorname{col}(A)$, to be the set given by

$$\operatorname{col}(A) := \left\{Av : v \in \mathbb{R}^n\right\}.$$

DEFINITION (Row Space)**.** Let A be an $m \times n$ matrix. We define the **row space** of A, denoted by $\operatorname{row}(A)$, to be the set given by

$$\operatorname{row}(A) := \left\{A^\top v : v \in \mathbb{R}^m\right\}.$$

DEFINITION (Nullspace)**.** Let A be an $m \times n$ matrix. We define the **nullspace** of A, denoted by $\operatorname{null}(A)$, to be the set given by

$$\operatorname{null}(A) := \left\{v \in \mathbb{R}^n : Av = \mathbf{0}\right\}.$$

DEFINITION (Left Nullspace)**.** Let A be an $m \times n$ matrix. We define the **left**

nullspace of A, denoted by $\mathrm{null}(A^\top)$, to be the set given by

$$\mathrm{null}(A^\top) := \{v \in \mathbb{R}^m : A^\top v = \mathbf{0}\}.$$

1.2 Main Results

THEOREM 1.1 (The Fundamental Theorem of Linear Algebra)**.** Let A be an $m \times n$ matrix. Then $\mathrm{col}(A)^\perp = \mathrm{null}(A^\top)$ and $\mathrm{row}(A)^\perp = \mathrm{null}(A)$.

Chapter 2

Rank

2.1 Definitions

DEFINITION (Column Rank)**.** Let A be a matrix. We define the **column rank** of A to be the dimension of the column space of A. i.e.

$$\mathrm{colrank}(A) := \dim(\mathrm{col}(A)).$$

DEFINITION (Row Rank)**.** Let A be a matrix. We define the **row rank** of A to be the dimension of the row space of A. i.e.

$$\mathrm{rowrank}(A) := \dim(\mathrm{row}(A)).$$

DEFINITION (Rank)**.** Let A be a matrix. Then the column rank and the row rank are the same. We define the **rank** of A to be this common number.

DEFINITION (Full Rank)**.** Let A be an $m \times n$ matrix. We say that A has **full rank** if $\mathrm{rank}(A) = \min\{m, n\}$.

2.2 Properties

PROPOSITION 2.2.1. Let A be an $m \times n$ matrix. Then

- A is injective if and only if A has full column rank. i.e. $\operatorname{rank}(A) = n$, and
- A is surjective if and only if A has full row rank. i.e. $\operatorname{rank}(A) = m$.

PROPOSITION 2.2.2. Let A and B be matrices with appropriate dimensions. Then

$$\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

PROPOSITION 2.2.3. Let A, B, and C be matrices with appropriate dimensions. Then

- If B has full row rank, then $\operatorname{rank}(AB) = \operatorname{rank}(A)$, and
- If C has full column rank, then $\operatorname{rank}(CA) = \operatorname{rank}(A)$.

PROPOSITION 2.2.4 (Subadditivity)**.** Let A and B be matrices with appropriate dimensions. Then

$$\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

PROPOSITION 2.2.5. Let A be a matrix over $\mathbb{C}$. Let A^- denote the complex conjugate of A. Let $A^\top$ denote the transpose of A. Let A^* denote the conjugate transpose of A. Then

$$\operatorname{rank}(A) = \operatorname{rank}(A^-) = \operatorname{rank}(A^\top) = \operatorname{rank}(A^*) = \operatorname{rank}(AA^*) = \operatorname{rank}(A^*A).$$

Chapter 3

Matrix Inverse

3.1 Definitions

DEFINITION (Invertible)**.** Let A be an $n \times n$ matrix over $\mathbb{C}$. We say that A is **invertible** if there exists another $n \times n$ matrix B over $\mathbb{C}$ such that $AB = BA = I_n$.

PROPOSITION 3.1.1. Let A be an $n \times n$ invertible matrix over $\mathbb{C}$. Then the $n \times n$ matrix B over $\mathbb{C}$ satisfying $AB = BA = I_n$ is unique.

DEFINITION (Inverse)**.** Let A be an $n \times n$ matrix over $\mathbb{C}$. We define the **inverse** of A, denoted by A^{-1}, to be the unique $n \times n$ matrix over $\mathbb{C}$ satisfying $AA^{-1} = A^{-1}A = I_n$.

DEFINITION (Left/Right Inverse)**.** Let A be an $m \times n$ matrix over $\mathbb{C}$. We define

- the **left inverse** of A, to be an $n \times m$ matrix B over $\mathbb{C}$ such that $BA = I_n$.
- the **right inverse** of A, to be an $n \times m$ matrix B over $\mathbb{C}$ such that $AB = I_n$.

3.2 Characterization

PROPOSITION 3.2.1. Let A be an $n \times n$ matrix over field K. Then the following statements are equivalent.

- A is invertible.
- $\dim(\text{row}(A)) = n$.
- $\dim(\text{col}(A)) = n$.
- $\dim(\text{null}(A)) = 0$.

PROPOSITION 3.2.2. Let A be an $n \times n$ matrix over field K. Then the following statements are equivalent.

- A is invertible.
- A is row-equivalent to I_n.
- A is column-equivalent to I_n.
- A can be written as a finite product of elementary matrices.

PROPOSITION 3.2.3. Let A be an $n \times n$ matrix over field K. Then A is invertible if and only if $\det(A) \neq 0$.

PROPOSITION 3.2.4. Let A be an $n \times n$ matrix over field K. Then A is invertible if and only if 0 is not an eigenvalue of A.

3.3 Arithmetic Properties

PROPOSITION 3.3.1. Let A be an invertible matrix. Then

- $(A^{-1})^{-1} = A$.
- $(kA)^{-1} = k^{-1}A^{-1}$.

- $(AB)^{-1} = B^{-1}A^{-1}$.
- $(A^T)^{-1} = (A^{-1})^T$.

3.4 Pseudo-Inverse

DEFINITION (Moore-Penrose Pseudo-Inverse)**.** Let A be an $n \times d$ matrix. We define the **Moore-Penrose pseudo-inverse** of A, denoted by $A^\dagger$, to be a $d \times n$ matrix G such that

$$AGA = A, \quad GAG = G, \quad (AG)^\top = AG, \quad (GA)^\top = GA.$$

Chapter 4

Determinant

4.1 Definitions

DEFINITION (Cofactor)**.** Let M be an $n \times n$ matrix over field $\mathbb{F}$. We define the $(i,j)^{th}$ **cofactor** of A, denoted by $C_{ij}(A)$, to be a number given by

$$C_{ij}(A) := (-1)^{i+j} \det(M_{ij})$$

where M_{ij} denotes the submatrix obtained from A by removing the i^{th} row and the j^{th} column.

4.2 Properties

PROPOSITION 4.2.1. Let A be a matrix. Then

$$\det(A^T) = \det(A).$$

PROPOSITION 4.2.2. Let A and B be positive semi-definite matrices with appropriate dimensions. Then

$$\det(A+B) \geq \det(A) + \det(B).$$

PROPOSITION 4.2.3. Let A be an $n \times n$ matrix. Let c be some scalar. Then

$$\det(cA) = c^n \det(A).$$

PROPOSITION 4.2.4. Let A be an invertible matrix. Then

$$\det(A^{-1}) = \det(A)^{-1}.$$

PROPOSITION 4.2.5. Let A and B be matrices with appropriate dimensions. Then

$$\det(AB) = \det(A)\det(B).$$

PROPOSITION 4.2.6. The determinant operator is a multi-linear operator on the rows/columns.

4.3 Adjoint of a Matrix

DEFINITION (Adjoint)**.** Let M be an $n \times n$ matrix. We define the **adjoint** of M, denoted by $\text{adj}(M)$, to be an $n \times n$ matrix given by

$$(\text{adj}(M))_{ij} = C_{ji}(M),$$

for $i, j = 1, ..., n$.

PROPOSITION 4.3.1. Let M be an $n \times n$ matrix. Then

$$M \,\text{adj}(M) = \text{adj}(M)M = \det(M)I_n.$$

Chapter 5

Trace

DEFINITION. Let A be a square matrix. We define the trace of A, denoted by $\operatorname{tr}(A)$, to be the sum of the entries on the main diagonal of A.

5.1 Properties

PROPOSITION 5.1.1. Trace is a linear operator.

PROPOSITION 5.1.2. The trace of the transpose of a matrix equals the trace of the matrix itself. i.e. if M is a square matrix, then

$$\operatorname{tr}(M) = \operatorname{tr}(M^\top).$$

PROPOSITION 5.1.3. If $A \in M_{m\times n}$ and $B \in M_{n\times m}$, then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

PROPOSITION 5.1.4. Trace is similarity-invariant. i.e., if A is similar to B, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.

PROPOSITION 5.1.5. The trace of an idempotent matrix is equal to its rank.

PROPOSITION 5.1.6. The trace of a matrix equals the sum of its eigenvalues.

Chapter 6

Matrix Norm

DEFINITION. $\|A\| := \sup_{\|x\|=1} \|Ax\|$

6.1 Properties

PROPOSITION 6.1.1. Let A be an $n \times n$ matrix. Then if A is symmetric, we have

$$\|A\| = \max\{\lambda_i\}_{i=1}^n$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues of A.

Chapter 7

Eigenvalues and Eigenvectors

7.1 Definitions

DEFINITION (Eigenvalue and Eigenvector)**.** Let A be a matrix. Let x be a vector. Let λ be a scalar. We say that x is an **eigenvector** of A and that λ is an **eigenvalue** of A if $x \neq 0$ and

$$Ax = \lambda x.$$

7.2 Properties

PROPOSITION 7.2.1. Let A be an invertible matrix. Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of A. Then the eigenvalues of A^{-1} are $\{\lambda_i^{-1}\}_{i=1}^n$.

Proof.

$$\begin{aligned}
&Av = \lambda v \\
\iff &A^{-1}Av = A^{-1}\lambda v \\
\iff &v = \lambda A^{-1} v \\
\iff &A^{-1}v = \lambda^{-1} v.
\end{aligned}$$

■

PROPOSITION 7.2.2. Let A be an invertible matrix. Let $\{x_i\}_{i=1}^n$ be the eigenvectors of A. Then the eigenvectors of A^{-1} are also $\{x_i\}_{i=1}^n$.

PROPOSITION 7.2.3. Let A be a matrix. Let n be a positive integer. Let (x, λ) be an eigenpair of A. Then

$$A^n x = \lambda^n x.$$

Proof. I will prove by induction on n.

Base Case: $n = 1$.

This is to prove that $Ax = \lambda x$. This holds since (x, λ) is an eigenpair of A.

Inductive Step:

Assume that $A^n x = \lambda^n x$ for some $n \in \mathbb{N}$. We are to prove that $A^{n+1}x = \lambda^{n+1}x$.

$$\begin{aligned} A^{n+1}x &= A^n Ax \\ &= A^n \lambda x \\ &= \lambda A^n x \\ &= \lambda\lambda^n x \text{ by the inductive hypothesis} \\ &= \lambda^{n+1}x. \end{aligned}$$

That is,

$$A^{n+1}x = \lambda^{n+1}x.$$

Summary:

By the principle of mathematical induction,

$$\forall n \in \mathbb{N}, \quad A^n x = \lambda^n x.$$

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PROPOSITION 7.2.4. If a square matrix is idempotent, then its eigenvalues are either 0 or 1.

Proof. Since A is idempotent, by definition, $A^2 = A$. Let (x, λ) be an arbitrary eigenpair of A. Then

$$Ax = \lambda x \text{ and } A^2x = \lambda^2 x.$$

Since $A^2 = A$ and $A^2x = \lambda^2x$, we get $Ax = \lambda^2x$. Since $Ax = \lambda x$ and $Ax = \lambda^2x$, we get $\lambda x = \lambda^2x$. Since x is an eigenvector of A, $x \neq 0$. Since $\lambda x = \lambda^2x$ and $x \neq 0$, we get $\lambda \in \{0, 1\}$. ■

7.3 Eigenspace

DEFINITION (Eigenspace)**.** Let A be an $m \times n$ matrix over field $\mathbb{F}$. Let λ be an eigenvalue of A. We define the **eigenspace** of A, associated with λ, denoted by E_λ, to be a set given by

$$E_\lambda := \{v \in \mathbb{F}^n : Av = \lambda v\}.$$

i.e., E_λ is the set of all eigenvectors of A with eigenvalue λ and the zero vector.

PROPOSITION 7.3.1. Eigenspaces are linear subspaces.

Chapter 8

Singular Values and Singular Vectors

DEFINITION (Singular Value, Singular Vector)**.** Let M be an $m \times n$ real or complex matrix. We define a **singular value** for M to be a non-negative real number σ such that there exist unit vectors $u \in \mathbb{F}^m$ and $v \in \mathbb{F}^n$ such that $Mv = \sigma u$ and $M^* u = \sigma v$. We call u the **left-singular vector** for σ and v the **right-singular vector** for σ.

8.1 Singular Value Decomposition

DEFINITION (Singular Value Decomposition)**.** Let M be an $m \times n$ real or complex matrix. We define a **singular value decomposition** to be a factorization of the form $M = U\Sigma V^*$ where U is an $m \times m$ unitary matrix, the columns of U are the left-singular vectors of M; V is an $n \times n$ unitary matrix, the columns of V are the right-singular vectors of M; Σ is an $m \times n$ rectangular diagonal matrix, the diagonal entries of Σ are the singular values of M.

Chapter 9

Special Types of Matrices

9.1 Elementary Matrices

PROPOSITION 9.1.1. The inverse of an elementary matrix can be obtained by multiplying its off-diagonal entries by -1.

Unconfirmed...

9.2 Definite Symmetric Matrices

DEFINITION (Definite Symmetric Matrices)**.** Let M be an $n \times n$ Hermitian complex. We say that

- M is **positive definite**, denoted by $M \succ 0$, if

$$\forall x \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad x^* M x > 0.$$

- M is **negative definite**, denoted by $M \prec 0$, if

$$\forall x \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad x^* M x < 0.$$

- M is **positive semi-definite**, denoted by $M \succeq 0$, if

$$\forall x \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad x^* M x \geq 0.$$

- M is **negative semi-definite**, denoted by $M \preceq 0$, if

$$\forall x \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad x^* M x \leq 0.$$

PROPOSITION 9.2.1. Let M be an $n \times n$ Hermitian matrix. Then

- M is positive definite if and only if all of its eigenvalues are positive.
- M is negative definite if and only if all of its eigenvalues are negative.
- M is positive semi-definite if and only if all of its eigenvalues are non-negative.
- M is negative semi-definite if and only if all of its eigenvalues are non-positive.

PROPOSITION 9.2.2. If A is positive definite, then A^{-1} exists and is also positive definite.

Proof Approach 1. Let y be an arbitrary vector. Then there exists some x such that $y = Ax$ since A is invertible. Now

$$y^T A^{-1} y \tag{9.1}$$
$$= x^T A^T A^{-1} A x \tag{9.2}$$
$$= x^T A^T x = \qquad x^T A x > 0. \tag{9.3}$$

Since $\forall y, y^T A^{-1} y > 0$, we get A^{-1} is positive definite. ■

Proof Approach 2. Since A is positive definite, all its eigenvalues are positive. Eigenvalues of A^{-1} are reciprocals of eigenvalues of A. So all eigenvalues of A^{-1} are positive. So A^{-1} is positive definite. ■

9.3 Hermitian Matrix

DEFINITION (Hermitian Matrix)**.** We say that a complex square matrix is **Hermitian**, or **self-adjoint**, if it equals to its complex conjugate.

PROPOSITION 9.3.1. The eigenvalues of a Hermitian matrix are all real.

Proof Approach 1.

Let A be a Hermitian matrix.

Let (λ, v) be an arbitrary eigenpair of A.

Since (λ, v) is an eigenpair, $Av = \lambda v$.

Since $Av = \lambda v$, $v^*Av = v^*\lambda v = \lambda v^*v$.

Since $(v^*Av)^* = v^*A^*v^{**} = v^*Av$, v^*Av is Hermitian.

Since $(v^*v)^* = v^*v^{**} = v^*v$, v^*v is Hermitian.

Say $v^*Av = [a]$ and $v^*v = [b]$.

Since $v^*Av = \lambda v^*v$ and $v^*Av = [a]$ and $v^*v = [b]$, $a = \lambda b$.

Since v^*Av is Hermitian, $a = \overline{a}$.

Since $a = \overline{a}$, a is real.

Since v^*v is Hermitian, $b = \overline{b}$.

Since $b = \overline{b}$, b is real.

Since $a = \lambda b$ and both a and b are real, λ is real.

■

Proof Approach 2.

$$
\begin{aligned}
&\lambda\langle v, v\rangle \\
&= \langle \lambda v, v\rangle \\
&= \langle Av, v\rangle \\
&= \langle v, A^*v\rangle \\
&= \langle v, Av\rangle \\
&= \langle v, \lambda v\rangle \\
&= \overline{\lambda}\langle v, v\rangle.
\end{aligned}
$$

That is, $\lambda\langle v, v\rangle = \overline{\lambda}\langle v, v\rangle$. Since v is an eigenvector, $v \neq \vec{0}$. Since $v \neq \vec{0}$, $\langle v, v\rangle \neq 0$. Since $\langle v, v\rangle \neq 0$ and $\lambda\langle v, v\rangle = \overline{\lambda}\langle v, v\rangle$, $\lambda = \overline{\lambda}$. Since $\lambda = \overline{\lambda}$, λ is real.

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9.4 Triangular Matrix

DEFINITION (Upper Triangular Matrix)**.**

DEFINITION (Lower Triangular Matrix)**.**

PROPOSITION 9.4.1. The product of two upper triangular matrices is also upper triangular. i.e. if U_1 and U_2 are upper triangular matrices with appropriate dimensions, then $U := U_1 U_2$ is also upper triangular.

PROPOSITION 9.4.2. The inverse of an upper triangular matrix is also upper triangular, if it exists. i.e. if U is an invertible upper triangular matrix, then U^{-1} is also upper triangular.

9.5 Normal Matrices

DEFINITION (Normal)**.** Let M be a complex square matrix. We say that M is **normal** if

$$M^*M = MM^*M.$$

9.6 Unitary Matrices

DEFINITION (Unitary - 1)**.** Let U be a complex square matrix. We say that U is **unitary** if $U^*U = I$, or equivalently, $UU^* = I$, where U^* denotes the complex conjugate of U and I denotes the identity matrix.

DEFINITION (Unitary - 2)**.** Let U be a complex square matrix. We say that U is **unitary** if the columns of U form an orthonormal basis for $\mathbb{C}^n$, or equivalently, the rows of U form an orthonormal basis for $\mathbb{C}^n$.

PROPOSITION 9.6.1 (Unitary Matrices Preserve Inner Products)**.** Let U be a complex square matrix. Then U is unitary if and only if

$$\forall x, y \in \mathbb{C}^n, \quad \langle Ux, Uy \rangle = \langle x, y \rangle \, .$$

PROPOSITION 9.6.2. The product of two unitary matrices is still unitary.

Chapter 10

Matrix Diagonalization

10.1 Unitary Diagonalization

10.1.1 Definitions

DEFINITION (Unitarily Similar)**.** Let A and B be complex square matrices of the same dimension. We say that A and B are **unitarily similar** if there exists a unitary matrix U such that

$$U^*AU = B.$$

THEOREM 10.1 (Schur)**.** Any matrix is unitarily similar to an upper triangular matrix.

DEFINITION (Unitarily Diagonalizable)**.** Let M be a complex square matrix. We say that M is **unitarily diagonalizable** if M is unitarily similar to a diagonal matrix.

10.1.2 Properties

PROPOSITION 10.1.1. Unitarily diagonalizable matrices are normal.

10.2 Sufficient Conditions

PROPOSITION 10.2.1. Hermitian matrices are unitarily diagonalizable.

PROPOSITION 10.2.2. Normal matrices are unitarily diagonalizable.

Chapter 11

Matrix Decomposition

11.1 LU Decomposition

THEOREM 11.1. Let A be an $n \times n$ matrix with $\det(A) \neq 0$. Then there exists a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U.

11.2 Eigenvalue Decomposition

DEFINITION (Eigenvalue Decomposition)**.** Let A be an $n \times n$ matrix where $n \in \mathbb{N}$. Let $\{(x_i, \lambda_i)\}_{i=1}^n$ be the eigenpairs of A. We define the **eigenvalue decomposition** of A to be a factorization of A given by

$$A = Q\Lambda Q^{-1}$$

where $Q = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}$ and $\Lambda = \operatorname{diag}(\{\lambda_i\}_{i=1}^n)$.

PROPOSITION 11.2.1. Let A be an $n \times n$ matrix. Then A can be eigendecomposed if and only if A has n linearly independent eigenvectors.