Game Theory

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Chapter 1

First Chapter

1.1 First Section

DEFINITION 1.1 (Winning Position)**.** Consider a two-player game. We say that a player has a **winning position** if and only if optimal play by that player guarantees a win.

DEFINITION 1.2 (Losing Position)**.** Consider a two-player game. We say that a player has a **losing position** if and only if an optimal move by their opponent guarantees a loss.

PROPOSITION 1.3.

- From a winning position (player to move), there exists a move that leads to a losing position for the other player.
- From a losing position (player to move), every move leads to a winning position for the other player.

1.2 Groups of Games

DEFINITION 1.4 (Equivalent Games)**.** Let G and H be two impartial games. We say that G and H are **equivalent** if and only if for all impartial games J, $G + J$ is a losing position if and only if $H + J$ is a losing position.

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PROPOSITION 1.5. Game equivalence is an equivalence relation.

PROPOSITION 1.6. $G \equiv H$ implies that G and H are both winning or both losing.

LEMMA 1.7. G is a losing position if and only if $G \equiv *0$.

Proof. **Backward Direction**: Suppose that $G \equiv *0$. Then $\forall J$, $G + J$ is a losing position if and only if $*0 + J$ is a losing position. In particular, take $J := *0$, then $G + *0$ is a losing position if and only if $*0 + *0$ is a losing position. Notice $G + *0 = *0$ and $*0 + *0 = *0$. So G is a losing position if and only if $*0$ is a losing position. We know that $*0$ is indeed a losing position. So G is a losing position.

Forward Direction: Suppose that G is a losing position. I will show that $G \equiv *0$. Let J be an arbitrary impartial game. Notice $*0 + J = J$. So there remains to show that $G + J$ is losing if and only if J is losing.

Suppose that $G + J$ is a losing position. I will show that J is a losing position. Assume for the sake of contradiction that J is not losing. Then J is winning. Let $J \to J'$ be a move such that J' is losing. Since G is losing and J' is losing, we get $G + J'$ is losing. So $G + J$ is winning. However, this contradicts to the assumption that $G + J$ is losing. So J is losing.

Suppose that J is a losing position. I will show that $G + J$ is a losing position. Double strong well-founded induction.

$\square$

G is winning and J is losing, then $G + J$ is winning???

DEFINITION 1.8 (Group of Game)**.** Let $\mathcal{G}$ be a set of games. Let $* : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be a binary operation on $\mathcal{G}$. We say that $(\mathcal{G}, *)$ is a **group** if and only if the following conditions hold:

1. Associativity: $\forall G_1, G_2, G_3 \in \mathcal{G}$, $(G_1 * G_2) * G_3 \equiv G_1 * (G_2 * G_3)$.
2. Identity: $\exists I \in \mathcal{G}$ such that $\forall G \in \mathcal{G}$, $G * I \equiv I * G \equiv G$.
3. Inverse: $\forall G \in \mathcal{G}$, $\exists H \in \mathcal{G}$ such that $G * H \equiv H * G \equiv I$.

LEMMA 1.9. $G \equiv H$ if and only if $G + H \equiv *0$.

Proof. **Forward Direction**: Suppose that $G \equiv H$. I will show that $G + H \equiv *0$. Since $G \equiv H$, we get

$$G + H \equiv H + H, \text{ by the}$$

□

LEMMA 1.10. Let G and H be impartial combinatorial games. Suppose that

- For each option G' of G, there exists an option of H which is equivalent to G'.
- For each option H' of H, there exists an option of G which is equivalent to H'.

Then $G \equiv H$.

Proof. Since $G' + H' \equiv *0$, we get $G + H \equiv *0$. □

THEOREM 1.11 (Sum of NIM Heaps)**.** Suppose $n_1, ..., n_k \in \mathbb{Z}_{++}$ are distinct powers of 2. Then we have

$$*(n_1 + ... + n_k) \equiv (*n_1 + ... + *n_k).$$

Proof. **Base Case**: $n = 0$.

Inductive Step: Suppose the theorem holds for all positive integers less than n. Write n as $n = 2^{a_1} + 2^{a_2}...$ where $a_1 > a_2 > ....$ Define $q := n - 2^{a_1} = 2^{a_2} + 2^{a_3} + ...$ Note that $q < n$ and $q < 2^{a_1}$. Apply induction on q. Then we get $*q \equiv *2^{a_1} + *2^{a_2} + ...$ Now there remains to show that $*n \equiv *2^{a_1} + *q$. Consider the options of $*n$: $\{*(n-1), *(n-2), ..., *0\}$ and the options of $*2^{a_1} + *q$: $\{G + *q, *2^{a_1} + H\}$ where G is some option of $*2^{a_1}$ and H is some option of $*q$.

Consider the options of the form $G + *q$ where G is some option for $*2^{a_1}$.

Consider the options of the form $*2^{a_1} + H$ where H is some option of $*q$. The set of options is $\{*2^{a_1} + *i : 0 \leq i < q\}$. Write i as $i = 2^{b_1} + 2^{b_2} + ...$ Notice $2^{a_1} + i < 2^{a_1} + q < n$. So by the inductive hypothesis, we get

$$*(2^{a_1} + i) = *(2^{a_1} + 2^{b_1} + 2^{b_2} + ...) = *2^{a_1} + *2^{b_1} + *2^{b_2} + ...$$

So the set of options of $*n$ is equivalent to the set of options for $*2^{a_1} + *q$. So $*n \equiv *2^{a_1} + *2^{a_2} + ...$ □

EXAMPLE 1.12.

$$\begin{aligned}(5, 9, 8) &= *5 + *9 + *8 = *(4 + 1) + *(8 + 1) + *8 \\ &= *4 + *1 + *8 + *1 + *8 = *4.\end{aligned}$$

So the optimal move is to take away the $*4$: $(5, 9, 8) \to (1, 9, 8)$.

DEFINITION 1.13 (Balance, Unbalanced)**.** We say that a NIM position $(a_1, ..., a_q)$ is **balanced** if and only if $a_1 \oplus ... \oplus a_q = 0$. We say that is it **unbalanced** otherwise.

THEOREM 1.14. A NIM position $(a_1, ..., a_q)$ is a losing (winning) position if and only if it is balanced (unbalanced).