

Variational Analysis

Daniel Mao

Contents

1	Semi-Continuity	1
1.1	Definitions	1
1.2	Properties	1
2	Subgradients	3
2.1	Definitions and Examples	3
2.2	Basic Properties	4
2.3	Calculus of Sub-Differentials	4
2.4	Subdifferentiation and Differentiation	5
3	Quasigradients	7
3.1	Definitions	7

1

Semi-Continuity

1.1 Definitions

Definition (Lower Semi-Continuous). *Let f be a function from \mathbb{E} to \mathbb{R}^* . Let x_0 be a point in \mathbb{E} . We say that f is **lower semi-continuous** at point x_0 if for any sequence $\{x_i\}_{i \in \mathbb{N}}$ that converges to x_0 , we have $f(x) \leq \liminf_{i \rightarrow \infty} f(x_i)$.*

Definition (Lower Semi-Continuous). *Let f be a function from \mathbb{E} to \mathbb{R}^* . We say that f is **lower semi-continuous** if $\text{epi}(f)$ is closed.*

Proposition 1.1.1. *The two definitions of lower semi-continuity are equivalent.*

Definition (Upper Semi-Continuous). *Let X be a topological space. Let f be a extended real-valued function on X . Let x_0 be a point in X . We say that f is **upper semi-continuous** at point x_0 if for any positive number ε , there exists some neighborhood \mathcal{N} of x_0 such that $f(x) \leq f(x_0) + \varepsilon$ for any $x \in \mathcal{N}$ when $f(x_0) \neq -\infty$; or if $\lim_{x \rightarrow x_0} f(x) = -\infty$ when $f(x_0) = -\infty$.*

1.2 Properties

Proposition 1.2.1 (Supremum). *The supremum of a collection of lower semi-continuous functions is again lower semi-continuous. i.e., Let $\{f_i\}_{i \in I}$ be a collection of lower semi-continuous functions where I is some index set. Then the function F given by $F := \sup_{i \in I} f_i$ is lower semi-continuous.*

Proof.

$$\begin{aligned}
& (x, \alpha) \in \text{epi}(F) \\
& \iff \sup_{i \in I} f_i(x) \leq \alpha \\
& \iff \forall i \in I, f_i(x) \leq \alpha \\
& \iff \forall i \in I, (x, \alpha) \in \text{epi}(f_i) \\
& \iff (x, \alpha) \in \bigcap_{i \in I} \text{epi}(f_i).
\end{aligned}$$

So $\text{epi}(F) = \bigcap_{i \in I} \text{epi}(f_i)$. Since f_i are lower semi-continuous, $\text{epi}(f_i)$ are closed. Since $\text{epi}(f_i)$ are closed, $\bigcap_{i \in I} \text{epi}(f_i)$ is closed. That is, $\text{epi}(F)$ is closed. Since $\text{epi}(F)$ is closed, F is lower semi-continuous. ■

Proposition 1.2.2. *A function is continuous at a point if and only if it is both upper and lower semi-continuous there.*

2

Subgradients

2.1 Definitions and Examples

Definition (Sub-Differential). Let f be a proper function from \mathbb{E} to \mathbb{R}^* . We define the **sub-differential** of f , denoted by ∂f , to be a set-valued function on \mathbb{E} given by

$$\partial f(x) := \{v \in \mathbb{E} : \forall y \in \mathbb{E}, \langle v, y - x \rangle \leq f(y) - f(x)\}.$$

Definition (Subdifferentiable). Let f be a proper function from \mathbb{E} to \mathbb{R}^* . Let x be a point in \mathbb{E} . We say that f is **subdifferentiable** at point x if $\partial f(x) \neq \emptyset$.

Definition (Subgradient). Let f be a proper function from \mathbb{E} to \mathbb{R}^* . We define the **subgradients** of f to be the elements of $\partial f(x)$.

Example 2.1.1. Let C be a non-empty closed convex set in \mathbb{E} . Let x be some point in \mathbb{E} . Then

$$\partial \delta_C(x) = N_C(x)$$

where δ_C denotes the indicator function of C and N_C denotes the normal cone to C .

Proof. If $x \notin C$, then $\partial \delta_C(x) = N_C(x) = \emptyset$. Else, let u be an arbitrary point in \mathbb{E} . Then

$$\begin{aligned} u &\in \partial \delta_C(x) \\ \iff \forall y \in \mathbb{E}, \delta_C(y) - \delta_C(x) &\geq \langle u, y - x \rangle \\ \iff \forall y \in C, \delta_C(y) - \delta_C(x) &\geq \langle u, y - x \rangle \\ \iff \forall y \in C, 0 - 0 &\geq \langle u, y - x \rangle \\ \iff \forall y \in C, \langle u, y - x \rangle &\leq 0 \\ \iff \forall y \in C - x, \langle u, y \rangle &\leq 0 \\ \iff u \in N_C(x). \end{aligned}$$

■

2.2 Basic Properties

Proposition 2.2.1 (Domain of the Subdifferential). *Let f be a proper, convex, and lower semi-continuous function from \mathbb{E} to \mathbb{R}^* .*

- (1) $\text{dom}(\partial f) \subseteq \text{dom}(f)$.
- (2) $\text{ri}(\text{dom}(f)) \subseteq \text{dom}(\partial f)$.
- (3) $\text{ri}(\text{dom}(\partial f)) = \text{ri}(\text{dom}(f))$.
- (4) $\text{cl}(\text{dom}(\partial f)) = \text{cl}(\text{dom}(f))$.

Proof of (1). Let x be an arbitrary point in $\text{dom}(\partial f)$. We are to prove that $x \in \text{dom}(f)$. Assume for the sake of contradiction that $x \notin \text{dom}(f)$. Since $x \notin \text{dom}(f)$, $f(x) = +\infty$. Since f is proper, $\exists y \in \mathbb{E}$ such that $f(y) < +\infty$. Since $f(y) < +\infty$ and $f(x) = +\infty$, we have

$$\forall u \in \mathbb{E}, \quad f(y) - f(x) < \langle u, y - x \rangle.$$

So $\forall u \in \mathbb{E}$, $u \notin \partial f(x)$. i.e. $\partial f(x) = \emptyset$. So $x \notin \text{dom}(\partial f)$. This contradicts to the assumption that $x \in \text{dom}(\partial f)$. So the assumption that $x \notin \text{dom}(f)$ is false. i.e. $x \in \text{dom}(f)$. Since $\forall x \in \text{dom}(\partial f)$, $x \in \text{dom}(f)$, we get

$$\text{dom}(\partial f) \subseteq \text{dom}(f).$$

■

2.3 Calculus of Sub-Differentials

Proposition 2.3.1. *Let f and g be proper functions from \mathbb{E} to \mathbb{R}^* . Then $\forall x \in \mathbb{E}$, $\partial f(x) + \partial g(x) \subseteq \partial(f + g)(x)$.*

Proof.

Let x be an arbitrary point in \mathbb{E} .

Let v be an arbitrary point in $\partial f(x) + \partial g(x)$.

Since $v \in \partial f(x) + \partial g(x)$, $\exists u \in \partial f(x)$, $\exists w \in \partial g(x)$ such that $v = u + w$.

Let y be an arbitrary point in \mathbb{E} .

Since $u \in \partial f(x)$, $f(y) \geq f(x) + \langle u, y - x \rangle$.

Since $w \in \partial g(x)$, $g(y) \geq g(x) + \langle w, y - x \rangle$.

$$\begin{aligned} (f + g)(y) &= f(y) + g(y) \\ &\geq f(x) + \langle u, y - x \rangle + g(x) + \langle w, y - x \rangle \\ &= f(x) + g(x) + \langle u + w, y - x \rangle \\ &= (f + g)(x) + \langle v, y - x \rangle. \end{aligned}$$

That is, $(f + g)(y) \geq (f + g)(x) + \langle v, y - x \rangle$.

This is true for any $y \in \mathbb{E}$.

So $v \in \partial(f + g)(x)$.

This is true for any $v \in \partial f(x) + \partial g(x)$.

So $\partial f(x) + \partial g(x) \subseteq \partial(f + g)$.

■

Theorem 1. *Let f and g be proper convex lower semi-continuous functions from \mathbb{E} to \mathbb{R}^* . Assume that $\text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(g)) \neq \emptyset$. Then $\partial(f + g) = \partial f + \partial g$.*

2.4 Subdifferentiation and Differentiation

Theorem 2. *Let f be a proper convex function from \mathbb{R}^n to \mathbb{R}^* . Let x_0 be a point in $\text{dom}(f)$. Let u be a point in \mathbb{R}^n . Then u is a subgradient of f at point x_0 if and only if*

$$\forall d \in \mathbb{R}^n, f'(x_0; d) \geq \langle u, d \rangle.$$

Proof.

$$\begin{aligned} & u \in \partial f(x_0) \\ \iff & \forall y \in \mathbb{R}^n, & f(y) \geq f(x_0) + \langle u, y - x_0 \rangle \\ \iff & \forall d \in \mathbb{R}^n, \forall \lambda > 0, & f(x_0 + \lambda d) \geq f(x_0) + \langle u, x_0 + \lambda d - x_0 \rangle \\ \iff & \forall d \in \mathbb{R}^n, \forall \lambda > 0, & \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda} \geq \langle u, d \rangle \\ \iff & \forall d \in \mathbb{R}^n, & f'(x_0; d) \geq \langle u, d \rangle. \end{aligned}$$

■

Proposition 2.4.1. *Let f be a proper convex function from \mathbb{R}^n to \mathbb{R}^* . Let x_0 be a point in $\text{dom}(f)$. Assume that f is differentiable at point x_0 . Then $\nabla f(x_0)$ is the unique subgradient of f at point x_0 .*

3

Quasigradients

3.1 Definitions

Definition (Quasigradients). *Let f be a quasiconvex function from \mathbb{E} to \mathbb{R}^* . Let x_0 be a point in \mathbb{E} . We define the **quasigradients** of f at point x_0 to be the vectors v such that*

$$\forall x \in \mathbb{E}, \quad \langle v, x - x_0 \rangle \geq 0 \implies f(x) - f(x_0) \geq 0.$$