

Functional Analysis

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Chapter 1

Balanced Sets

1.1 Definitions

Definition (Balanced Sets). *Let X be a vector space over field \mathbb{F} . Let S be a subset of X . We say that S is **balanced** if*

$$\forall a \in \mathbb{F} : |a| \leq 1, \quad aS \subseteq S.$$

Definition (Balanced Hull). *Let X be a vector space over field \mathbb{F} . Let S be a subset of X . We define the **balanced hull** of S , denoted by $\text{balhull}(S)$, to be the smallest balanced set containing S .*

Definition (Balanced Core). *Let X be a vector space over field \mathbb{F} . Let S be a subset of X . We define the **balanced core** of S , denoted by $\text{balcore}(S)$, to be the largest balanced set contained in S .*

1.2 Properties

Proposition 1.2.1. *Let X be a vector space over field \mathbb{F} . Let B be a balanced subset of X . Then*

$$\forall a, b \in \mathbb{F} : |a| \leq |b|, \quad aB \subseteq bB.$$

Proposition 1.2.2. *Balanced sets are path connected.*

Proposition 1.2.3 (Act on Other Properties).

- *The balanced hull of a compact set is compact.*

- *The balanced hull of a totally bounded set is totally bounded.*
- *The balanced hull of a bounded set is bounded.*

Proposition 1.2.4 (Act on Other Properties). • *The balanced core of a closed set is closed.*

Proposition 1.2.5. *Let X be a vector space over field \mathbb{F} . Let a be a scalar in field \mathbb{F} . Then*

$$a \operatorname{balhull}(S) = \operatorname{balhull}(aS).$$

1.3 Stability of Balance

Proposition 1.3.1 (Set Operations). • *The union of balanced sets is also balanced.*

- *The intersection of balanced sets is also balanced.*

Proposition 1.3.2 (Linear Mappings). • *The scalar multiple of a balanced set is also balanced.*

- *The (Minkowski) sum of two balanced sets is also balanced.*
- *The image of a balanced set under a linear operator is also balanced.*
- *The inverse image of a balanced set under a linear operator is also balanced.*

Proposition 1.3.3 (Topological Operations). *The closure of a balanced set is also balanced.*

Proposition 1.3.4. *The convex hull of a balanced set is also balanced (and also convex).*

1.4 Absorbing Sets

Definition (Absorbing Sets). *Let X be a vector space over field \mathbb{F} . Let S be a subset of X . We say that S is **absorbing** if*

$$\forall x \in X, \quad \exists r \in \mathbb{R} : r > 0, \quad \forall c \in \mathbb{F} : |c| \geq r, \quad x \in cA.$$

Proposition 1.4.1. *Every absorbing set contains the origin.*

Chapter 2

Inner Product Spaces

2.1 Inner Products

2.1.1 Definitions

Definition (Inner Product). *Let V be a vector space over field \mathbb{F} . We define an **inner product** on V , denoted by $\langle \cdot, \cdot \rangle$, to be a scalar-valued function defined on $V \times V$ such that*

(1) *Positive Definiteness*

$$\begin{aligned}\forall x, y \in V, \quad \langle x, x \rangle &\geq 0, \text{ and} \\ \forall x \in V, \quad \langle x, x \rangle &= 0 \iff x = O_V.\end{aligned}$$

(2) *Sesqui-Linearity*

$$\begin{aligned}\forall x, y, z, w \in V, \quad \langle x + y, z + w \rangle &= \langle x, z \rangle + \langle y, z \rangle + \langle x, w \rangle + \langle y, w \rangle, \text{ and} \\ \forall a, b \in \mathbb{F}, \forall x, y \in V, \quad \langle ax, by \rangle &= a\bar{b}\langle x, y \rangle.\end{aligned}$$

(3) *Conjugate Symmetry*

$$\forall x, y \in V, \quad \langle x, y \rangle = \overline{\langle y, x \rangle}.$$

Definition (Norm). *Let V be an inner product space over field \mathbb{F} . We define the **norm**, denoted by $\|\cdot\|$, to be a function from V to \mathbb{R}_+ given by*

$$\|x\| := \sqrt{\langle x, x \rangle}$$

Definition (Orthogonal Vectors). *Let V be an inner product space. Let x and y be vectors in V . We say that x and y are **orthogonal** if $\langle x, y \rangle = 0$.*

Definition (Orthogonal Sets). *Let S be a subset of V . We say that S is **orthogonal** if*

$$\forall x, y \in S, \quad \langle x, y \rangle = 0.$$

2.1.2 Examples

Definition (Standard Inner Product). For $V = \mathbb{F}^n$, we define the **standard inner product** by

$$\langle x, y \rangle := \sum_{i=1}^n x_i \overline{y_i}.$$

Definition (Frobenius Inner Product). For $V = \mathbb{F}^{n \times n}$, we define the **Frobenius inner product** by

$$\langle M_1, M_2 \rangle := \text{tr}(M_2^* M_1).$$

Definition. Let V be the space of continuous scalar-valued functions on $[0, 2\pi]$. We define the inner product on V by

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} dx.$$

2.1.3 Properties

Proposition 2.1.1. Let V be a finite dimensional inner product space. Let \mathcal{B} be a basis for V . Let x and y be vectors in V . Then

$$x = y \iff \forall b \in \mathcal{B}, \quad \langle x, b \rangle = \langle y, b \rangle.$$

2.2 Inequalities

Theorem 1 (Minkowski).

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}$$

Proposition 2.2.1 (Cauchy-Schwarz Inequality). Let V be an inner product space. Then

$$\forall x, y \in V, \quad |\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Proposition 2.2.2 (Triangle Inequality). Let V be an inner product space. Then

$$\forall x, y \in V, \quad \|x + y\| \leq \|x\| + \|y\|$$

Proposition 2.2.3 (Parallelogram Law). Let V be an inner product space. Then

$$\forall x, y \in V, \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

2.3 Orthogonality

2.3.1 Orthogonal Sets

Definition (Orthogonality). *Let V be an inner product space. We say that points x and y in V are **orthogonal** if $\langle x, y \rangle = 0$.*

Definition (Orthogonal Sets). *Let V be an inner product space and S be a subset of V . We say that S is **orthogonal** if any two vectors in S are orthogonal.*

Proposition 2.3.1. *Orthogonal sets are linearly independent.*

2.3.2 Orthogonal Bases

Definition (Orthogonal Basis). *Let V be an inner product space and S be a subset of V . We say that S is an **orthogonal basis** for V if it is an ordered basis for V and orthogonal.*

Proposition 2.3.2. *Let V be an inner product space. Let $S = \{v_1, \dots, v_n\}$ be an orthogonal subset of V where each v_i is non-zero. Then*

$$\forall y \in \text{span}(S), \quad y = \sum_{i=1}^n \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

Theorem 2 (Gram-Schmidt Process). *Let V be an inner product space. Let $S = \{v_0, \dots, v_n\}$ be a linearly independent subset of V . Then the set $S' = \{v'_0, \dots, v'_n\}$ given by $v'_0 := v_0$ and*

$$\forall i \in \{1, \dots, n\}, \quad v'_i := v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, v'_j \rangle}{\|v'_j\|^2} v'_j$$

is an orthogonal subset of V consisting of non-zero vectors. Furthermore, we have $\text{span}(S') = \text{span}(S)$.

Proposition 2.3.3. *Let V be an inner product space and $S = \{v_0, v_1, \dots, v_n\}$ be an orthogonal subset of V . Then the set S' derived from the Gram-Schmidt process is exactly S .*

Theorem 3 (Parseval's Identity). *Let V be a finite-dimensional inner product space. Let $B = \{v_1, \dots, v_n\}$ be an orthogonal basis for V . Then*

$$\forall x, y \in V, \quad \langle x, y \rangle = \sum_{i=1}^n \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

Theorem 4 (Bessel's Inequality). *Let V be a finite-dimensional inner product space. Let $B = \{v_1, \dots, v_n\}$ be an orthogonal subset for V . Then*

$$\forall x \in V, \quad \|x\|^2 \geq \sum_{i=1}^n |\langle x, v_i \rangle|^2.$$

2.3.3 Orthogonal Complements

Definition (Orthogonal Complement). *Let V be an inner product space and S be a non-empty subset of V . We define the **orthogonal complement** of S , denoted by S^\perp , to be the set of all points in V that are orthogonal to all vectors in S .*

Proposition 2.3.4. *Let V be a finite-dimensional inner product space. Then*

$$(1) V^\perp = \{O_V\}$$

$$(2) \{O_V\}^\perp = V$$

Proposition 2.3.5. *Orthogonal complements are always linear subspaces.*

Proposition 2.3.6. *Let V be an inner product space and W be a subspace of V with basis β . Then a vector in V is also in W^\perp if and only if it is orthogonal to all vectors in β .*

Proposition 2.3.7 (Extension). *Let V be an n -dimensional inner product space and $S = \{v_1, v_2, \dots, v_k\}$ be an orthogonal subset of V . Then S can be extended to an orthogonal basis $B = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V .*

2.3.4 Properties of the Orthogonal Complement Operator

Proposition 2.3.8. *Let V be an inner product space. Then*

$$(1) S \subseteq T \text{ implies } T^\perp \subseteq S^\perp \text{ for any subsets } S \text{ and } T \text{ of } V.$$

$$(2) S \subseteq (S^\perp)^\perp \text{ for any subset } S \text{ of } V.$$

Proposition 2.3.9. *Let V be a finite-dimensional inner product space and W be a subspace of V . Then*

$$(1) W = (W^\perp)^\perp$$

$$(2) V = W \oplus W^\perp$$

Proposition 2.3.10. *Let V be a finite-dimensional inner product space and W_1 and W_2 be subspaces of V . Then*

$$(1) (W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$$

$$(2) (W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$$

2.3.5 Orthogonal Projection

Definition (Orthogonal Projection). *Let V be a vector space. Let W be a finite-dimensional subspace of V . Let x be a vector in V . We define the **orthogonal projection** of x on W , denoted by (x) , to be the vector u in W such that $x = u + v$ where v is another vector in W^\perp .*

Chapter 3

Normed Linear Spaces

3.1 Definitions

Definition (Seminorm). Let \mathfrak{X} be a vector space over field \mathbb{F} . We define a **seminorm** on \mathfrak{X} , denoted by ν , to be a map from \mathfrak{X} to \mathbb{R} that satisfies the following conditions.

(1) $\forall x \in \mathfrak{X}, \quad \nu(x) \geq 0.$

(2) $\forall \lambda \in \mathbb{F}, \forall x \in \mathfrak{X}, \quad \nu(\lambda x) = \lambda \nu(x).$

(3) *Triangle Inequality.*

$$\forall x, y \in \mathfrak{X}, \quad \nu(x + y) \leq \nu(x) + \nu(y).$$

The idea behind the seminorm is that we are trying to give our vector space a notion of “length” of vectors.

Definition (Norm). Let \mathfrak{X} be a vector space over field \mathbb{F} . We define a **norm** on \mathfrak{X} , denoted by ν , to be a seminorm on \mathfrak{X} that satisfies the additional condition:

$$\forall x \in \mathfrak{X}, \quad \nu(x) = 0 \iff x = 0.$$

3.2 Properties

Proposition 3.2.1. Let $(V, \|\cdot\|_V)$ be a normed vector space over field \mathbb{F} . Then $(V, \|\cdot\|)$ is complete if and only if $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

Proof.

For one direction, assume that $(V, \|\cdot\|)$ is complete.

We are to prove that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

Since $(\overline{B(0,1)}, \|\cdot\|_V)$ is a closed subspace of $(V, \|\cdot\|)$ and $(V, \|\cdot\|)$ is complete, $(\overline{B(0,1)}, \|\cdot\|_V)$ is also complete.

For the reverse direction, assume that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

We are to prove that $(V, \|\cdot\|_V)$ is complete.

Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary Cauchy sequence in $(V, \|\cdot\|_V)$.

Since $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i\}_{i \in \mathbb{N}}$ is bounded in $(V, \|\cdot\|_V)$.

Let λ be a positive upper bound for $\{\|x_i\|_V\}_{i \in \mathbb{N}}$.

Since $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i/\lambda\}_{i \in \mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i \in \mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$ and $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete, $\{x_i/\lambda\}_{i \in \mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i \in \mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$, $\{x_i\}_{i \in \mathbb{N}}$ converges in $(V, \|\cdot\|_V)$.

Since any Cauchy sequence in $(V, \|\cdot\|_V)$ converges in $(V, \|\cdot\|_V)$, $(V, \|\cdot\|_V)$ is complete. ■

3.3 Equivalence of Norms

Definition (Equivalence of Norms). *Let \mathfrak{X} be a vector space over field \mathbb{F} . Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V . We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are **equivalent** if*

$$\exists c_1, c_2 > 0, \quad \forall v \in \mathfrak{X}, \quad c_1 \|v\|_1 \leq \|v\|_2 \leq c_2 \|v\|_1.$$

Or equivalently,

$$c_1 \|v\|_2 \leq \|v\|_1 \leq c_2 \|v\|_2.$$

Proposition 3.3.1. *The equivalence of norms is an equivalence relation.*

Theorem 5. *Let V be a finite dimensional vector space over field $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$. Then any two norms on V are equivalent.*

Proof.

Let $\|\cdot\|_p$ be an arbitrary p -norm on V and $\|\cdot\|$ be an arbitrary norm on V .

Let \mathcal{B} be the standard basis for V . Say $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$.

Let v be an arbitrary vector in V .

$$\begin{aligned}
\|v\| &= \left\| \sum_{i=1}^n v_i e_i \right\| \\
&\leq \sum_{i=1}^n |v_i| \|e_i\| \\
&\leq \left(\sum_{i=1}^n |v_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|e_i\|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \\
&= \left(\sum_{i=1}^n \|e_i\|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \|v\|_p \\
&:= c_1 \|v\|_p.
\end{aligned}$$

■

Proposition 3.3.2. *Let X be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X . Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if they generate the same metric topology.*

Proof. Convergence to 0 is equivalent under either $\|\cdot\|_1$ or $\|\cdot\|_2$. i.e., equivalent norms give rise to the same set of sequences that are convergent. Convergence of sequences defines the topology. ■

Proposition 3.3.3. *Let \mathfrak{X} be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on \mathfrak{X} . Let i be the identity map from $(\mathfrak{X}, \|\cdot\|_1)$ to $(\mathfrak{X}, \|\cdot\|_2)$. Then if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, i is continuous.*

3.4 Dual Norms

Definition (Dual Norm). *Let $(V, \|\cdot\|)$ be a normed vector space. We define the **dual norm** of $\|\cdot\|$, denoted by $\|\cdot\|_\circ$, to be a function given by*

$$\|v\|_\circ := \max_{\|w\|=1} v \cdot w = \max_{\|w\| \neq 0} \frac{|v \cdot w|}{\|w\|}.$$

Proposition 3.4.1. *The dual norms of norms are indeed norms.*

Proposition 3.4.2. *Let $(V, \|\cdot\|)$ be a normed vector space. Let v, w be vectors in the space. Then*

$$|v \cdot w| \leq \|v\| \cdot \|w\|_\circ.$$

3.5 p -norms

Definition (p -norm). Let V be a finite-dimensional normed vector space over field \mathcal{F} . Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis for V where $n = \dim(V)$. Let v be a vector in a normed vector space. For $p \in [1, +\infty)$, we define the **p -norm** of v , denoted by $\|v\|_p$, to be the number given by

$$\|v\|_p = \left(\sum_{i=1}^n |(v_{\mathcal{B}})_i|^p \right)^{\frac{1}{p}}.$$

Definition (Infinity Norm - 1). Let $\mathfrak{X} = \mathbb{K}^n$ where \mathbb{K} is a field and $n \in \mathbb{N}$. We define the **infinity norm** on \mathfrak{X} , denoted by $\|\cdot\|_\infty$, to be a function given by

$$\|v\|_\infty := \max\{|v_i|\}_{i=1}^n.$$

Definition (Infinity Norm - 2). Let $\mathfrak{X} = \mathbb{K}^{\mathbb{N}}$. We define the **infinity norm** on \mathfrak{X} , denoted by $\|\cdot\|_\infty$, to be a function given by

$$\|v\|_\infty := \sup_{i \in \mathbb{N}} |v_i|.$$

Definition (Infinity Norm - 3). Let $\mathfrak{X} = \mathcal{C}([0, 1], \mathbb{C})$. We define the **infinity norm** on \mathfrak{X} , denoted by $\|\cdot\|_\infty$, to be a function given by

$$\nu(f) := \sup_{x \in [0, 1]} |f(x)|.$$

Proposition 3.5.1. Let $\mathfrak{X} := \mathcal{C}([0, 1], \mathbb{C})$. Let x be an arbitrary number in $[0, 1]$. Define a function ν_x on \mathfrak{X} by $\nu_x(f) := |f(x)|$. Define a function ν on \mathfrak{X} by $\nu(f) := \sup_{x \in [0, 1]} |f(x)|$. Then ν_x is a seminorm on \mathfrak{X} for each x and ν is a norm on \mathfrak{X} and we have $\nu = \sup_{x \in [0, 1]} \nu_x$.

Proposition 3.5.2. p -norms are indeed norms.

Proposition 3.5.3. For any vector v in \mathbb{R}^n , we have

$$\lim_{p \rightarrow \infty} \|v\|_p = \|v\|_\infty.$$

i.e.,

$$\lim_{p \rightarrow \infty} \left(\sum_{i=1}^n |v_i|^p \right)^{\frac{1}{p}} = \max\{|v_i|\}_{i=1}^n.$$

Proof. Let p be an arbitrary number in $[1, +\infty)$. Let k be an arbitrary index in $\{1, \dots, n\}$. Then

$$|v_k| \leq \left(\sum_{i=1}^n |v_i|^p \right)^{1/p} = \|v\|_p.$$

So

$$\max\{|v_k|\} = \|v\|_\infty \leq \|v\|_p.$$

So

$$\lim_{p \rightarrow \infty} \|v\|_p \geq \|v\|_\infty. \quad (1)$$

On the other hand, note that

$$(\sum_{i=1}^n |v_i|^p) / \|v\|_\infty^p = \sum_{i=1}^n \left(\frac{|v_i|}{\|v\|_\infty} \right)^p$$

decreases as p increases. So it is bounded above. Say

$$(\sum_{i=1}^n |v_i|^p) / \|v\|_\infty^p \leq C$$

for some $C \in \mathbb{R}$. Then

$$(\sum_{i=1}^n |v_i|^p)^{1/p} = \|v\|_p \leq C^{1/p} \|v\|_\infty.$$

So

$$\lim_{p \rightarrow \infty} \|v\|_p \leq \lim_{p \rightarrow \infty} C^{1/p} \|v\|_\infty = \|v\|_\infty. \quad (2)$$

From (1) and (2) we get

$$\lim_{p \rightarrow \infty} \|v\|_p = \|v\|_\infty.$$

■

Proposition 3.5.4. *Let p be an arbitrary number in $[1, +\infty)$. Then the dual norm of the p -norm $\|\cdot\|_p$ is the q -norm $\|\cdot\|_q$ where q is such that satisfies*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proposition 3.5.5. *Let p and q be numbers in $[1, +\infty]$. Let v be a vector in \mathbb{R}^n . Then if $p \leq q$,*

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \cdot \|x\|_q.$$

Proposition 3.5.6. *Let w and z be vectors in \mathbb{E}^d . Then*

$$\|w + z\|_2^2 + \|w - z\|_2^2 = 2(\|w\|_2^2 + \|z\|_2^2).$$

3.6 Banach Spaces

Definition (Banach Space). *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let d be the metric induced by $\|\cdot\|$. We say that \mathfrak{X} is a **Banach space** if (\mathfrak{X}, d) is a complete metric space. i.e., we define a Banach space to be a complete normed linear space.*

Proposition 3.6.1. *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed vector space over field \mathbb{F} . Then $(\mathfrak{X}, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in X is summable.*

Example 3.6.1. $(\mathcal{C}([0, 1], \mathbb{F}), \|\cdot\|_\infty)$ is a Banach space.

Example 3.6.2 (Disc Algebra). Define $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Define $\mathcal{A}(\mathbb{D}) := \{f \in \mathcal{C}(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is holomorphic}\}$. Define $\|\cdot\|_\infty$ by $\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|$. Then $(\mathcal{A}(\mathbb{D}), \|\cdot\|_\infty)$ is a Banach space.

Example 3.6.3. Let (X, Ω, μ) be a measure space. Let p be a number in $[1, +\infty)$. Define

$$\mathcal{L}^p(X, \mu) := \text{span}\{f : X \rightarrow [0, +\infty] \mid f \text{ is measurable and } \int_X |f|^p < +\infty\}.$$

Define an equivalence relation on $\mathcal{L}^p(X, \mu)$ by $f \equiv g$ if and only if

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

Define a space $L^p(X, \mu) := \mathcal{L}^p(X, \mu) / \equiv$. Then $L^p(X, \mu)$ is a Banach space when equipped with the norm

$$\|[f]\|_p := \left(\int_X |f|^p \right)^{1/p}.$$

Example 3.6.4. Let $\mathcal{P}_{\mathbb{C}}[0, 1]$ denote the set of all polynomials with complex coefficients. For each $p \in [1, +\infty)$, define a norm

$$\|f\|_p := \left(\int_0^1 |f|^p \right)^{1/p}.$$

For $p = +\infty$, define a norm

$$\|f\|_\infty := \sup_{x \in [0, 1]} |f(x)|.$$

3.7 Construction of Banach Spaces

Definition. Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be two Banach spaces over field \mathbb{K} . Let $p \in [1, +\infty)$. We define

$$\mathfrak{X} \oplus_p \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x, y)\|_p := (\|x\|_{\mathfrak{X}}^p + \|y\|_{\mathfrak{Y}}^p)^{1/p}.$$

For $p = +\infty$, we define

$$\mathfrak{X} \oplus_\infty \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x, y)\|_\infty := \max(\|x\|_{\mathfrak{X}}, \|y\|_{\mathfrak{Y}}).$$

- Note that the norms behave similarly to what the p norm would do.

- *We can similarly define the direct sum of finitely many Banach spaces.*

Proposition 3.7.1. $\|\cdot, \cdot\|_p$ is a norm on $\mathfrak{X} \oplus_p \mathfrak{Y}$.

Proposition 3.7.2. $\mathfrak{X} \oplus_p \mathfrak{Y}$ is complete with respect to $\|\cdot, \cdot\|_p$.

Chapter 4

Quotient Spaces

4.1 Definitions

Definition. Let \mathfrak{V} be a vector space. Let \mathfrak{W} be a subspace of \mathfrak{V} . We define a **quotient space**, denoted by $\mathfrak{V}/\mathfrak{W}$, to be a set $\{v + \mathfrak{W} : v \in \mathfrak{V}\}$ with operations

$$(v_1 + \mathfrak{W}) + (v_2 + \mathfrak{W}) := (v_1 + v_2) + \mathfrak{W} \text{ and}$$

$$\kappa(v + \mathfrak{W}) := (\kappa v) + \mathfrak{W}.$$

Definition (Quotient Map). Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let \mathfrak{M} be a linear manifold in \mathfrak{X} . We define the **quotient map** on \mathfrak{X} with respect to \mathfrak{M} , denoted by $q_{\mathfrak{M}}$, to be a function from \mathfrak{X} to $\mathfrak{X}/\mathfrak{M}$ given by

$$q_{\mathfrak{M}}(x) := x + \mathfrak{M}$$

Proposition 4.1.1. Quotient maps are contractive. i.e.,

$$\forall x \in \mathfrak{X}, \quad \|x + \mathfrak{M}\|_{\mathfrak{X}/\mathfrak{M}} \leq \|x\|_{\mathfrak{X}}.$$

Proposition 4.1.2. Let \mathfrak{X} be a normed linear space. Let \mathfrak{M} be a closed subspace of \mathfrak{X} . Let q be the canonical quotient map from \mathfrak{X} to $\mathfrak{X}/\mathfrak{M}$. Then

- q is a continuous map. i.e.,

$$\forall \text{ open set } W \subseteq \mathfrak{X}/\mathfrak{M}, \quad q^{-1}(W) \text{ is open in } \mathfrak{X}.$$

- q is an open map. i.e.,

$$\forall \text{ open set } G \subseteq \mathfrak{X}, \quad q(G) \text{ is open in } \mathfrak{X}/\mathfrak{M}.$$

Proof. Since q is contractive, q is continuous and hence (1). ■

Definition (Seminorm on Quotient Spaces). *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let \mathfrak{M} be a linear manifold in \mathfrak{X} . We define a **seminorm** on $\mathfrak{X}/\mathfrak{M}$ to be a function from $\mathfrak{X}/\mathfrak{M}$ to \mathbb{R} given by*

$$p(x + \mathfrak{M}) := \inf\{\|x + m\| : m \in \mathfrak{M}\}.$$

Proposition 4.1.3. *Seminorms on quotient spaces are indeed seminorms.*

Proposition 4.1.4. *A seminorm on a quotient space $\mathfrak{X}/\mathfrak{M}$ is a norm if and only if \mathfrak{M} is closed.*

Chapter 5

Topological Vector Spaces

5.1 Definitions

5.2 Topological Vector Spaces

Definition (Vector Topology). *Let X be a vector space over a topological field \mathbb{K} . We define a **vector topology** on X to be a topology on X such that vector addition and scalar multiplication are continuous.*

Proposition 5.2.1 (Stability under Linear Combinations). *Let X be a normed vector space over \mathbb{F} . Let K be a compact set in the space. Let C be a closed set in the space. Then $\forall \alpha, \beta \in \mathbb{F}, S := \alpha K + \beta C$ is closed.*

Proof.

The case where $\beta = 0$ is trivial. I will assume $\beta \neq 0$.

Let $\alpha, \beta \in \mathbb{F}$ be arbitrary.

Let $\{s_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence in S that converges.

Say the limit is s_∞ .

Since $s_i \in S$ for any $i \in \mathbb{N}$ and $S = \alpha K + \beta C$, $s_i = \alpha k_i + \beta c_i$ for some $k_i \in K$ and some $c_i \in C$, for any $i \in \mathbb{N}$.

Since $\{k_i\}_{i \in \mathbb{N}}$ is a sequence in K and K is compact, there exists a convergent subsequence $\{k_i\}_{i \in I}$ of $\{k_i\}_{i \in \mathbb{N}}$ in K .

Say $\{k_i\}_{i \in I}$ converges to $k_\infty \in K$.

Since $\{s_i\}_{i \in \mathbb{N}}$ converges to s_∞ , $\{s_i\}_{i \in I}$ also converges to s_∞ .

Since $s_i = \alpha k_i + \beta c_i$, $c_i = \beta^{-1}(s_i - \alpha k_i)$.

Define $c_\infty := \beta^{-1}(s_\infty - \alpha k_\infty)$

Since $\{s_i\}_{i \in I}$ converges to s_∞ and $\{k_i\}_{i \in I}$ converges to k_∞ and $c_i = \beta^{-1}(s_i - \alpha k_i)$, $\{c_i\}_{i \in I}$ converges to c_∞ .

Since $\{c_i\}_{i \in I}$ is a sequence in C and converges to c_∞ and C is closed, $c_\infty \in C$.

Since $s_\infty = \alpha k_\infty + \beta c_\infty$ and $k_\infty \in K$ and $c_\infty \in C$, $s_\infty \in \alpha K + \beta C$.

Since for any sequence in S that converges, the limit is also in S , S is closed. ■

Remark. *The sum of two closed sets may not be closed.*

Proof.

Counter-example 1

Consider $A := \{n : n \in \mathbb{N}\}$ and $B := \{n + \frac{1}{n} : n \in \mathbb{N}\}$.

(<https://math.stackexchange.com/questions/124130/sum-of-two-closed-sets-in-mathbb-r-is-closed>)

Their sum contains the sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ but does not contain 0.

Counter-example 2

Consider $A := \mathbb{R} \times \{0\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x, y > 0, xy \geq 1\}$.

Their sum is $\mathbb{R} \times \mathbb{R}_{++}$. ■

5.3 Neighborhoods

Chapter 6

Sequence Spaces

6.1 ℓ_p Space

Definition ($\ell_p^{(n)}$ Space). We define the $\ell_p^{(n)}$ space to be the set of all sequences $\{x_i\}_{i=1}^{i=n}$ such that

Definition (ℓ_p Space). We define the ℓ_p space to be the set of all sequences x such that $\|x\|_p$ is finite, equipped with the p -norm $\|\cdot\|_p$.

Proposition 6.1.1. For $p \in [1, +\infty)$, $(\ell_p, \|\cdot\|_p)$ is complete.

Proof.

Let $\{x_n\}_{n \in \mathbb{N}}$ be an arbitrary Cauchy sequence in ℓ_p .

Since $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy in ℓ_p , $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $\forall m, n > N$, we have $\|x_m - x_n\|_p < \varepsilon$.

Since $\|x_m - x_n\|_p < \varepsilon$ and $|x_m^{(i)} - x_n^{(i)}| \leq \|x_m - x_n\|_p$ for any $i \in \mathbb{N}$, $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$ for any $i \in \mathbb{N}$.

Since for any $i \in \mathbb{N}$ and any positive number ε , there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N$, we have $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$, by definition, $\{x_n^{(i)}\}_{n \in \mathbb{N}}$ is Cauchy in \mathbb{F} .

Since $\{x_n^{(i)}\}_{n \in \mathbb{N}}$ is Cauchy in \mathbb{F} and \mathbb{F} is complete, $\{x_n^{(i)}\}_{n \in \mathbb{N}}$ converges.

Let $x_0^{(i)} = x_n^{(i)}$. Let $x_0 = \{x_0^{(i)}\}_{i \in \mathbb{N}}$.

$$\|x_0\|_p = \left(\sum_{i=1}^{\infty} |x_0^{(i)}|^p \right)^{\frac{1}{p}}$$

■

6.2 c_0 Space and c_{00} Space

Definition (c_0 Space). We define c_0 to be

$$c_0 := \left\{ \{x_n\}_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n \rightarrow \infty} x_n = 0 \right\}.$$

Definition (c_{00} Space). We define c_{00} to be

$$c_{00} := \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}, \forall n > N, x_n = 0 \right\}.$$

i.e., the set of all eventually zero sequences of real numbers. i.e., the set of all sequences with finite support.

Proposition 6.2.1. The c_{00} is not complete in $(\ell_1, \|\cdot\|_1)$.

Proof. Define a sequence of vectors $(\mathbf{r}_i)_{i \in \mathbb{N}}$ by $\mathbf{r}_i^j := \frac{1}{j^2}$ for $j \in \{1..i\}$ and $\mathbf{r}_i^j := 0$ for $j > i$. Then $(\mathbf{r}_i)_{i \in \mathbb{N}}$ converges to something that is not in c_{00} . ■

Proposition 6.2.2. The closure of c_{00} in the space $(\mathbb{R}^{\omega}, d_1)$ is ℓ_1 .

Proof. For one direction, we are to prove that $\text{cl}(c_{00}) \subseteq \ell_1$. Let x be an arbitrary element in $\text{cl}(c_{00})$. Since $x \in \text{cl}(c_{00})$, there exists another element $y \in c_{00}$ such that $d_1(x, y) < 1$. Let $N \in \mathbb{N}$ be such that $\forall n > N, y_n = 0$. Then

$$\begin{aligned} & d_1(x, y) < 1 \\ \iff & \sum_{n \in \mathbb{N}} |x_n - y_n| < 1 \\ \iff & \sum_{n=1}^N |x_n - y_n| + \sum_{n>N} |x_n - y_n| < 1 \\ \iff & \sum_{n=1}^N |x_n - y_n| + \sum_{n>N} |x_n| < 1 \\ \implies & \sum_{n=1}^N (|x_n| - |y_n|) + \sum_{n>N} |x_n| < 1 \\ \implies & \sum_{n=1}^N (|x_n| - |y_n|) + \sum_{n>N} |x_n| < 1 \\ \implies & \sum_{n=1}^N |x_n| - \sum_{n=1}^N |y_n| + \sum_{n>N} |x_n| < 1 \\ \iff & \sum_{n \in \mathbb{N}} |x_n| - \sum_{n=1}^N |y_n| < 1 \\ \iff & \sum_{n \in \mathbb{N}} |x_n| < 1 + \sum_{n=1}^N |y_n|. \end{aligned}$$

Since $\sum_{n \in \mathbb{N}} |x_n|$ is bounded, $x \in \ell_1$.

For the reverse direction, we are to prove that $\ell_1 \subseteq \text{cl}(c_{00})$. Let x be an arbitrary element in ℓ_1 . For $i \in \mathbb{N}$, define $x^i = \{x_j^i\}_{j \in \mathbb{N}}$ as $x_j^i = x_j$ for $j \leq i$ and $x_j^i = 0$ for $j > i$. Then $\forall i \in \mathbb{N}, x^i \in c_{00}$. Then

$$\begin{aligned} & \lim_{i \in \mathbb{N}} d_1(x^i, x) \\ &= \lim_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |x_j^i - x_j| \\ &= \lim_{i \in \mathbb{N}} \sum_{j > i} |x_j^i - x_j| \\ &= \lim_{i \in \mathbb{N}} \sum_{j > i} |x_j| \\ &= 0. \end{aligned}$$

That is, $\lim_{i \in \mathbb{N}} d_1(x^i, x) = 0$. So $\lim_{i \in \mathbb{N}} x^i = x$. So $x \in \text{cl}(c_{00})$. ■

Proposition 6.2.3. *The closure of c_{00} in the space $(\mathbb{R}^\omega, d_\infty)$ is c_0 .*

Proof. For one direction, we are to prove that $\text{cl}(c_{00}) \subseteq c_0$. Let x be an arbitrary element in $\text{cl}(c_{00})$. Let ε be an arbitrary positive real number. Since $x \in \text{cl}(c_{00})$, there exists another element y in c_{00} such that $d_\infty(x, y) < \varepsilon$. That is, $\forall j \in \mathbb{N}, |x_j - y_j| < \varepsilon$. Since $y \in c_{00}$, $\exists N \in \mathbb{N}$ such that $\forall j > N, y_j = 0$. So $\forall j > N, |x_j| < \varepsilon$. That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon.$$

By definition of convergence of limits, $\lim_{j \in \mathbb{N}} x_j = 0$. So $x \in c_0$.

For the reverse direction, we are to prove that $c_0 \subseteq \text{cl}(c_{00})$. Let x be an arbitrary element in c_0 . For $i \in \mathbb{N}$, define x^i as $x_j^i = x_j$ for $j \leq i$ and $x_j^i = 0$ for $j > i$. Then $\forall i \in \mathbb{N}, x^i \in c_{00}$. Let ε be an arbitrary positive real number. Since $x \in c_0$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon/2.$$

Let $i > N$. Then

$$\begin{aligned} & d_\infty(x^i, x) \\ &= \sup_{j \in \mathbb{N}} |x_j^i - x_j| \\ &= \sup_{j > i} |x_j^i - x_j| \\ &= \sup_{j > i} |x_j| \\ &\leq \varepsilon/2 < \varepsilon. \end{aligned}$$

That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall i > N, \quad d_\infty(x^i, x) < \varepsilon.$$

By definition of convergence of sequences, $\lim_{i \in \mathbb{N}} x^i = x$. So $x \in \text{cl}(c_{00})$. ■

Proposition 6.2.4. *Let $A := \{\{x_n\}_{n \in \mathbb{N}} \in c_{00} : \sum_{n \in \mathbb{N}} x_n = 0\}$. Then A is a subset of ℓ^1 and is closed in (ℓ^1, d_1) . i.e. $\text{cl}(A) = A$ in (ℓ^1, d_1) .*

Proof. Let $x = \{x^i\}_{i \in \mathbb{N}}$ be a sequence in ℓ^1 , where each $x^i = \{x_j^i\}_{j \in \mathbb{N}}$ is an element in A , that converges in (ℓ^1, d_1) . Say $\lim_{i \rightarrow \infty} x^i = x^\infty$.

First I claim that $x^\infty \in c_{00}$.

Now I claim that $\sum_{j \in \mathbb{N}} x_j^\infty = 0$. i.e. $x^\infty \in A$. Since $x^\infty \in c_{00}$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad x_j^\infty = 0.$$

Define $y_i := \sum_{j=1}^N x_j^i$. Define $y_\infty := \sum_{j=1}^N x_j^\infty$. It is easy to see that $\lim_{i \in \mathbb{N}} y_i = y_\infty$. Assume for the sake of contradiction that $y_\infty \neq 0$. i.e. $\{y_i\}_{i \in \mathbb{N}}$ does not converge to 0. Then

$$\exists \varepsilon_0 > 0, \quad \forall M \in \mathbb{N}, \quad \exists i_0 > M, \quad |y_{i_0} - 0| = |y_{i_0}| \geq \varepsilon_0. \quad (1)$$

Since $\lim_{i \rightarrow \infty} x^i = x^\infty$,

$$\exists M_0 \in \mathbb{N}, \quad \forall i > M_0, \quad d_1(x^i, x^\infty) < \varepsilon_0. \quad (2)$$

Consider statement (1) for a particular M, M_0 , we have

$$\exists i_0 > M_0, \quad |y_{i_0}| \geq \varepsilon_0. \quad (3)$$

That is,

$$\left| \sum_{j=1}^N x_j^{i_0} \right| \geq \varepsilon_0. \quad (3')$$

Consider statement (2) for a particular i, i_0 , we have

$$d_1(x^{i_0}, x^\infty) < \varepsilon_0. \quad (4)$$

From statement (4) we can derive:

$$\begin{aligned}
& d_1(x^{i_0}, x^\infty) < \varepsilon_0 \\
\iff & \sum_{j \in \mathbb{N}} |x_j^{i_0} - x_j^\infty| < \varepsilon_0 \\
\iff & \sum_{j=1}^N |x_j^{i_0} - x_j^\infty| + \sum_{j>N} |x_j^{i_0} - x_j^\infty| < \varepsilon_0 \\
\implies & \sum_{j>N} |x_j^{i_0} - x_j^\infty| < \varepsilon_0 \\
\iff & \sum_{j>N} |x_j^{i_0} - 0| < \varepsilon_0 \\
\iff & \sum_{j>N} |x_j^{i_0}| < \varepsilon_0 \\
\implies & \left| \sum_{j>N} x_j^{i_0} \right| < \varepsilon_0 \\
\implies & \left| \sum_{j>N} x_j^{i_0} \right| < \varepsilon_0 \\
\iff & \left| \sum_{j \in \mathbb{N}} x_j^{i_0} - \sum_{j=1}^N x_j^{i_0} \right| < \varepsilon_0 \\
\iff & \left| 0 - \sum_{j=1}^N x_j^{i_0} \right| < \varepsilon_0 \\
\iff & \left| \sum_{j=1}^N x_j^{i_0} \right| < \varepsilon_0.
\end{aligned}$$

This contradicts to statement (3'). So the original assumption that $y_\infty \neq 0$ is false. i.e. $y_\infty = 0$. It follows that $\sum_{j \in \mathbb{N}} x_j^\infty = 0$. This completes the proof. ■

6.3 Hölder's Inequality

Theorem 6 (Hölder's Inequality). *Let $\mathfrak{X} = \mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ be vectors in \mathfrak{X} . Then $\forall p, q \in (1, +\infty) : 1/p + 1/q = 1$, $\|xy\|_1 \leq \|x\|_p \|y\|_q$. i.e.,*

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \left(\sum_{i=1}^n |y_i|^q \right)^{1/q}.$$

Chapter 7

Function Spaces

7.1 The \mathcal{L}^p Norm

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

i the instructors' answer, where instructors collectively construct a single answer

In the sup norm, convergence coincides with uniform convergence. Moreover, $C[a, b]$ is complete in this norm. It is not complete in any of the L^p norms for $1 \leq p < \infty$. The completion in these norms is called $L^p(a, b)$.

[undo](#) [thanks](#) | 1

Updated 1 day ago by Kenneth Davidson

Chapter 8

Hilbert Space

8.1 Hilbert Spaces

Definition (Hilbert Space). *We define a **Hilbert space** to be a complete inner product space.*

Example 8.1.1. ℓ^2 is a Hilbert space.

Chapter 9

Equicontinuity in Metric Spaces

9.1 Definitions

Definition ((Pointwise) Equicontinuity). Let (X, d_X) and (Y, d_Y) be metric spaces. Let \mathcal{F} be a collection of functions from X to Y . Let x_0 be a point in X . We say that \mathcal{F} is **(pointwise) equicontinuous** at point x_0 if for any positive number ε , there exists some number $\delta(x_0, \varepsilon)$ such that for any function f in \mathcal{F} and any point x in X , we have

$$d_Y(f(x), f(x_0)) < \varepsilon$$

whenever $d_X(x, x_0) < \delta(x_0, \varepsilon)$ is satisfied.

Definition (Uniform Equicontinuity). Let (X, d_X) and (Y, d_Y) be metric spaces. Let \mathcal{F} be a collection of functions from X to Y . We say that \mathcal{F} is **uniformly equicontinuous** if for any positive number ε , there exists some number $\delta(\varepsilon)$ such that for any function f in \mathcal{F} and any points x_1 and x_2 in X , we have

$$d_Y(f(x_1), f(x_2)) < \varepsilon$$

whenever $d_X(x_1, x_2) < \delta(\varepsilon)$ is satisfied.

9.2 Sufficient Conditions

Proposition 9.2.1. The closure of an equicontinuous family of functions is equicontinuous.

Proof.

Let (X, d_X) and (Y, d_Y) be metric spaces.

Let \mathcal{F} be an equicontinuous family of functions from X to Y .

We are to prove that $cl(\mathcal{F})$ is equicontinuous.

Let x_0 be an arbitrary point in X .

Let ε be an arbitrary positive number.

Since \mathcal{F} is equicontinuous at point x_0 , there exists some $\delta(x_0, \varepsilon)$ such that for any function f in \mathcal{F} and any point x in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon/3$.

Let f be an arbitrary function in $cl(\mathcal{F})$.

Let x be an arbitrary point in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$.

Since $f \in cl(\mathcal{F})$, there exists some function $f_0 \in \mathcal{F}$ such that $d_\infty(f, f_0) < \varepsilon/3$.

Since $d_\infty(f, f_0) < \varepsilon/3$, $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$.

Since $f_0 \in \mathcal{F}$ and $d_X(x, x_0) < \delta(x_0, \varepsilon)$, $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$.

Since $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$ and $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$, $d_Y(f(x), f(x_0)) < \varepsilon$.

Since for any positive number ε , there exists some $\delta(x_0, \varepsilon)$ such that for any function f in $cl(\mathcal{F})$ and any point x in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, by definition of equicontinuous, $cl(\mathcal{F})$ is equicontinuous at point x_0 .

Since $cl(\mathcal{F})$ is equicontinuous at point x_0 for any point x_0 in X , $cl(\mathcal{F})$ is equicontinuous. ■

Chapter 10

Operators

10.1 Bounded Operators

Definition (Bounded Operator). *Let \mathfrak{X} and \mathfrak{Y} be normed linear spaces. Let T be a linear map from \mathfrak{X} to \mathfrak{Y} . We say that T is a **bounded operator** if*

$$\exists k \in \mathbb{R}, \quad \forall x \in \mathfrak{X}, \quad \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}.$$

Definition (Operator Norm). *Let \mathfrak{X} and \mathfrak{Y} be normed linear spaces. Let T be a bounded operator from \mathfrak{X} to \mathfrak{Y} . We define the **operator norm** of T , denoted by $\|T\|$, to be the number given by*

$$\|T\| := \inf\{k \in \mathbb{R} : \forall x \in \mathfrak{X}, \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}\}.$$

Proposition 10.1.1.

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{X}, \|x\|_{\mathfrak{X}} = 1\}.$$

Proposition 10.1.2. *Let X and Y be normed linear spaces. Let T be a linear map from X to Y . Then T is bounded if and only if T is continuous.*

Example 10.1.1 (The Multiplication Operator). *Let $\mathfrak{X} = (\mathcal{C}([0, 1], \mathbb{C}), \|\cdot\|_{\infty})$. Let f be a function in \mathfrak{X} . We define the **multiplication operator** on \mathfrak{X} , w.r.t. f , denoted by M_f , as*

$$M_f(g) = fg.$$

Then M_f is bounded and $\|M_f\| = \|f\|_{\infty}$.

Proof. Let g be an arbitrary function in \mathfrak{X} . Then

$$\begin{aligned}
 \|M_f g\|_\infty &= \|fg\|_\infty \\
 &= \sup_{x \in [0,1]} |f(x)g(x)| \\
 &= \sup_{x \in [0,1]} |f(x)| |g(x)| \\
 &\leq \sup_{x \in [0,1]} |f(x)| \sup_{x \in [0,1]} |g(x)| \\
 &= \|f\|_\infty \|g\|_\infty.
 \end{aligned}$$

That is, $\|M_f g\|_\infty \leq \|f\|_\infty \|g\|_\infty$. So $\|f\|_\infty$ is an element of the set $S = \{k \in \mathbb{R} : \forall g \in \mathfrak{X}, \|M_f g\|_\infty \leq k \|g\|_\infty\}$. So $\|M_f\| = \inf(S) \leq \|f\|_\infty$. Consider g_0 given by $g_0(x) = 1$. Then g_0 in \mathfrak{X} . Then

$$\|M_f g_0\|_\infty = \|fg_0\|_\infty = \|f\|_\infty = \|f\|_\infty \|g_0\|_\infty.$$

Let k be an arbitrary element in S . Assume for the sake of contradiction that $k < \|f\|_\infty$. Then

$$\begin{aligned}
 \|f\|_\infty \|g_0\|_\infty &= \|M_f g_0\|_\infty \\
 &\leq k \|g_0\|_\infty \\
 &< \|f\|_\infty \|g_0\|_\infty.
 \end{aligned}$$

This leads to a contradiction. So $\forall k \in S, k \geq \|f\|_\infty$. So $\|f\|_\infty$ is a lower bound for the set S . So $\|M_f\| = \inf(S) \geq \|f\|_\infty$. Since $\|M_f\| \leq \|f\|_\infty$ and $\|M_f\| \geq \|f\|_\infty$, we get $\|M_f\| = \|f\|_\infty$. ■

Example 10.1.2 (The Volterra Operator). Let $\mathfrak{X} = (\mathcal{C}([0,1], \mathbb{C}), \|\cdot\|_\infty)$. Define

$$Vf := x \mapsto \int_0^x f(t) dt.$$

Then the Volterra Operator is bounded and $\|V\| \leq 1$.

Proof. Let f be an arbitrary function in \mathfrak{X} with $\|f\|_\infty = 1$. Then $\forall x \in [0, 1]$,

$$\begin{aligned} |Vf(x)| &= \left| \int_0^x f(t) dt \right| \\ &\leq \int_0^x |f(t)| dt \\ &\leq \int_0^x \sup_{t \in [0,1]} |f(t)| dt \\ &= \int_0^x \|f\|_\infty dt \\ &= \int_0^x 1 dt \\ &= x. \end{aligned}$$

That is, $\forall x \in [0, 1]$, $|Vf(x)| \leq 1$. So $\|Vf\|_\infty \leq 1$. Since $\forall f \in \mathfrak{X} : \|f\|_\infty = 1$, $\|Vf\|_\infty \leq 1$, we get $\|V\| \leq 1$. ■

Example 10.1.3 (The Diagonal Operator). Let $\mathfrak{X} = \ell^2(\mathbb{N})$. Let

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & \ddots \end{bmatrix}.$$

Then D is bounded if and only if $(d_i)_{i \in \mathbb{N}}$ is bounded and $\|D\| = \|(d_i)_{i \in \mathbb{N}}\|_\infty$.

Proof. Case 1.

$$\begin{aligned} \|Dx\|_2^2 &= \sum_{i \in \mathbb{N}} |d_i x_i|^2 \\ &\leq \sum_{i \in \mathbb{N}} \|(d_j)_{j \in \mathbb{N}}\|_\infty |x_i|^2 \\ &= \|(d_j)_{j \in \mathbb{N}}\|_\infty \sum_{i \in \mathbb{N}} |x_i|^2 \\ &= \|(d_j)_{j \in \mathbb{N}}\|_\infty \|x\|_2^2. \end{aligned}$$

Case 2.

If $(d_i)_{i \in \mathbb{N}} \notin \ell^\infty$, $\exists (d_{n_i})_{i \in \mathbb{N}} \rightarrow \infty$.

$$\begin{aligned} \|De_{n_i}\|_2 &= \|d_{n_i} e_{n_i}\|_2 \\ &= |d_{n_i}| \|e_{n_i}\|_2 \\ &= |d_{n_i}|. \end{aligned}$$

So $\|D\| \geq \|De_{n_i}\|_2 \rightarrow \infty$. ■

Example 10.1.4 (Weighted Shifts).

- Let $\mathcal{H} = \ell_{\mathbb{N}}^2$. Let $(w_n)_{n \in \mathbb{N}} \in \ell_{\mathbb{N}}^{\infty}$. We define an **unilateral forward weighted shift** W on \mathcal{H} as

$$W(x_n) := (0, w_1 x_1, w_2 x_2, w_3 x_3, \dots).$$

i.e.,

$$W = \begin{bmatrix} 0 & & & & \\ w_1 & 0 & & & \\ & w_2 & 0 & & \\ & & w_3 & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.$$

Then W is bounded and $\|W\| = \sup\{|w_n| : n \in \mathbb{N}\}$.

- Let $\mathcal{H} = \ell_{\mathbb{N}}^2$. Let $(v_n)_{n \in \mathbb{N}} \in \ell_{\mathbb{N}}^{\infty}$. We define an **unilateral backward weighted shift** V on \mathcal{H} as

$$V(x_n) := (v_1 x_2, v_2 x_3, v_3 x_4, \dots).$$

Then V is bounded and $\|V\| = \sup\{|v_n| : n \in \mathbb{N}\}$.

- Let $\mathcal{H} = \ell_{\mathbb{Z}}^2$. Let $(u_n)_{n \in \mathbb{Z}} \in \ell_{\mathbb{Z}}^{\infty}$. We define a **bilateral weighted shift** U on \mathcal{H} as

$$U(x_n) := (u_{n-1} x_{n-1})_{n \in \mathbb{Z}}.$$

Then U is bounded and $\|U\| = \sup\{|u_n| : n \in \mathbb{Z}\}$.

Example 10.1.5 (The Composition Operators). Let $\mathfrak{X} = \mathcal{C}([0, 1], \mathbb{C})$. Let $\varphi \in \mathcal{C}([0, 1], [0, 1])$. We define the **composition operator** on \mathfrak{X} , denoted by C_{φ} as

$$C_{\varphi}(f) := f \circ \varphi.$$

Then C_{φ} is contractive.

Proof.

$$\begin{aligned} \|C_{\varphi}(f)\| &= \sup_{x \in [0, 1]} |(f \circ \varphi)(x)| \\ &\leq \|f\|_{\infty}. \end{aligned}$$

■

10.2 Space of Bounded Operators

Proposition 10.2.1. *Let \mathfrak{X} and \mathfrak{Y} be normed linear spaces. Then $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is a vector space and the operator norm is a norm on $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$.*

Proposition 10.2.2. *Let \mathfrak{X} and \mathfrak{Y} be normed linear spaces. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Then $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}, \|\cdot\|_1)$ if and only if $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}, \|\cdot\|_2)$.*

Proposition 10.2.3. *Let \mathfrak{X} and \mathfrak{Y} be normed linear spaces. Let $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ be the space of bounded linear operators from \mathfrak{X} to \mathfrak{Y} . Then if \mathfrak{Y} is complete, $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is complete.*

10.3 Dual Spaces

Definition ((Topological) Dual Space). *Let \mathfrak{X} be a normed linear space over field \mathbb{K} . We define the **(topological) dual space** of \mathfrak{X} , denoted by \mathfrak{X}^* , to be the space $\mathcal{B}(\mathfrak{X}, \mathbb{K})$.*

Definition (Linear Functionals). *We call the elements of \mathfrak{X}^* **linear functionals**.*

Proposition 10.3.1. *Let X be a normed linear space. Then there exists a contractive map from X to its double dual X^{**} .*

Chapter 11

Adjoint Operator

11.1 Definitions

Definition (Adjoint Matrix). *Let A be an $m \times n$ matrix. We define the **adjoint** of A , denoted by A^* , to be an $n \times m$ matrix given by*

$$(A^*)_{ij} := \overline{(A)_{ji}}.$$

Definition (Adjoint Operator). *Let V and W be inner product spaces. Let T be a linear map from V to W . We define the **adjoint** of T , denoted by T^* , to be a map from W to V such that*

$$\forall x \in V, \forall y \in W, \quad \langle T(x), y \rangle_W = \langle x, T^*(y) \rangle_V.$$

Proposition 11.1.1 (Existence). *Let V be a finite-dimensional inner product space and T be a linear operator on V . Then the adjoint of T exists.*

Proposition 11.1.2 (Uniqueness). *Let V be an inner product space and T be a linear operator on V . Then the adjoint of T is unique, provided that it exists.*

11.2 Properties of the Adjoint Operator

Proposition 11.2.1. *Let V be an inner product space. Then*

- (1) $(I_V)^* = I_V$ where I_V is the identity operator on V .
- (2) $T^{**} = T$ for any linear operator T on V .

Proposition 11.2.2. *Let V be an inner product space and T be a linear operator on V . Then T^* is also linear.*

Proposition 11.2.3. *Let V be an inner product space. Then*

(1) *For any linear operators T and U ,*

$$(T + U)^* = T^* + U^*.$$

(2) *For any linear operator T ,*

$$(cT)^* = \bar{c} \cdot T^*.$$

(3) *For any linear operators T and U ,*

$$(TU)^* = U^*T^*.$$

Proposition 11.2.4. *Let V be a finite-dimensional inner product space and T be a linear operator on V . Then if T is invertible, T^* is also invertible.*

Proposition 11.2.5. *Let V be an inner product space and T be an invertible linear operator on V . Then $(T^{-1})^* = (T^*)^{-1}$.*

11.3 Normal Operators

Definition (Normal). *Let V be an inner product space and T be a linear operator on V . We say that T is **normal** if $TT^* = T^*T$.*

11.4 Self-adjoint

Chapter 12

Convolution

Definition (Convolution). *Let f and g be functions from \mathbb{R} to \mathbb{R} . We define the **convolution** of f and g , denoted by $f * g$, to be a function on \mathbb{R} given by*

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)dt.$$

Chapter 13

Coercive Functions

13.1 Definitions

Definition (Coercive). Let f be a function from \mathbb{R}^d to \mathbb{R}^* . We say that f is **coercive** if $\lim_{\|x\| \rightarrow \infty} f(x) = +\infty$.

13.2 Properties

Proposition 13.2.1. Let f be a proper lower semi-continuous function from \mathbb{R}^d to \mathbb{R}^* . Let K be a compact set in \mathbb{R}^d . Assume $K \cap \text{dom}(f) \neq \emptyset$. Then f attains its minimum over K .

Proof.

Define $m := \inf_{x \in K} f(x)$.

Since $m = \inf_{x \in K} f(x)$, there exists a sequence $\{x_i\}_{i \in \mathbb{N}}$ in K such that $\lim_{i \rightarrow \infty} f(x_i) = m$.

Since K is compact and $\{x_i\}_{i \in \mathbb{N}} \subseteq K$, there exists a convergent subsequence $\{x_i\}_{i \in I}$ in K where I is an infinite subset of \mathbb{N} .

Say the limit is x_∞ where $x_\infty \in K$.

Since $\lim_{i \rightarrow \infty} f(x_i) = m$, we get $\lim_{i \in I, i \rightarrow \infty} f(x_i) = m$.

Since $\lim_{i \in I, i \rightarrow \infty} f(x_i) = m$, we get $\liminf_{i \in I, i \rightarrow \infty} f(x_i) = m$.

Since f is lower semi-continuous and $\lim_{i \in I, i \rightarrow \infty} x_i = x_\infty$, we get $f(x_\infty) \leq \liminf_{i \in I, i \rightarrow \infty} f(x_i)$.

That is, $f(x_\infty) \leq m$.

Since $m = \inf_{x \in K} f(x)$, we have $\forall x \in K, f(x) \geq m$.

In particular, $f(x_\infty) \geq m$.

Since $f(x_\infty) \geq m$ and $f(x_\infty) \leq m$, $f(x_\infty) = m$.

Since f is proper, $f(x_\infty) = m \neq -\infty$.

So f attains its minimum at point x_∞ .

■

Proposition 13.2.2. *Let f be a proper, lower semi-continuous, and coercive function from \mathbb{R}^d to \mathbb{R}^* . Let C be a closed subset of \mathbb{R}^d . Assume $C \cap \text{dom}(f) \neq \emptyset$. Then f attains its minimum over C .*

Proof.

Since $C \cap \text{dom}(f) \neq \emptyset$, take $x \in C \cap \text{dom}(f)$.

Since f is coercive, $\exists R$ such that $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$.

Since $x \in C \cap \text{dom}(f)$ and $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$, the set of minimizers of f over C is the same as the set of minimizers of f over $C \cap \text{ball}[0, R]$.

Since C and $\text{ball}[0, R]$ are both closed, $C \cap \text{ball}[0, R]$ is closed.

Since $\text{ball}[0, R]$ is bounded, $C \cap \text{ball}[0, R]$ is bounded.

Since $C \cap \text{ball}[0, R]$ is closed and bounded, by the Heine-Borel Theorem, $C \cap \text{ball}[0, R]$ is compact.

Since f is proper and lower semi-continuous and $C \cap \text{ball}[0, R]$ is compact, f attains its minimum over $C \cap \text{ball}[0, R]$.

So f attains its minimum over C . ■

Chapter 14

Unclassified Results

Proposition 14.0.1. *Let (X, d) be a compact metric space. Let $L(X)$ be the set of all Lipschitz functions from X to \mathbb{R} . Let $C(X)$ be the set of all continuous functions from X to \mathbb{R} . Then $L(X)$ is dense in $C(X)$.*

Proposition 14.0.2. *Let $(V, \|\cdot\|)$ be a normed vector space. Let S be a subset of V . Let p be a vector in V . Then we have the followings.*

$$(1) \ p + \text{int}(S) = \text{int}(p + S),$$

$$(2) \ p + \text{cl}(S) = \text{cl}(p + S).$$

Proof.

Proof of (1).

For one direction, let x be an arbitrary point in the set $(p + \text{int}(S))$.

We are to prove that $x \in \text{int}(p + S)$.

Since $x \in (p + \text{int}(S))$, $(x - p) \in \text{int}(S)$.

Since $(x - p) \in \text{int}(S)$, by definition of interior, there exists a radius r such that

$$B(x - p, r) \subseteq S.$$

It follows that $B(x, r) \subseteq p + S$.

Since there exists a radius r such that $B(x, r) \subseteq p + S$, by definition of interior,

$$x \in \text{int}(p + S).$$

For the reverse direction, let x be an arbitrary point in $\text{int}(p + S)$.

We are to prove that $x \in p + \text{int}(S)$.

Since $x \in \text{int}(p + S)$, by definition of interior, there exists a radius r such that

$$B(x, r) \subseteq (p + S).$$

It follows that $B(x - p, r) \subseteq S$.

Since there exists a radius r such that $B(x - p, r) \subseteq S$, by definition of interior,

$$(x - p) \in \text{int}(S).$$

Since $(x - p) \in \text{int}(S)$, we get $x \in (p + \text{int}(S))$.

Proof of (2).

For one direction, let x be an arbitrary point in the set $(p + \text{cl}(S))$.

We are to prove that $x \in \text{cl}(p + S)$.

Since $x \in (p + \text{cl}(S))$, we get $(x - p) \in \text{cl}(S)$.

Since $(x - p) \in \text{cl}(S)$, by definition of closure, for any radius r , we have

$$B(x - p, r) \cap S \neq \emptyset.$$

It follows that $B(x, r) \cap (p + S) \neq \emptyset$.

Since for any radius r , $B(x, r) \cap (p + S) \neq \emptyset$, by definition of closure, we get

$$x \in \text{cl}(p + S).$$

For the reverse direction, let x be an arbitrary point in $\text{cl}(p + S)$.

We are to prove that $x \in (p + \text{cl}(S))$.

Since $x \in \text{cl}(p + S)$, by definition of closure, for any radius r , we have

$$B(x, r) \cap (p + S) \neq \emptyset.$$

It follows that $B(x - p, r) \cap S \neq \emptyset$.

Since for any radius r , $B(x - p, r) \cap S \neq \emptyset$, by definition of closure, we get

$$(x - p) \in \text{cl}(S).$$

Since $(x - p) \in \text{cl}(S)$, we get $x \in (p + \text{cl}(S))$. ■

Proposition 14.0.3. *Let $(V, \|\cdot\|)$ be a normed vector space. Let S be a subset of V . Let λ be a non-zero real number. Then*

$$(1) \lambda \text{int}(S) = \text{int}(\lambda S).$$

$$(2) \lambda \text{cl}(S) = \text{cl}(\lambda S).$$

Proof.

Proof of (1).

For one direction, let x be an arbitrary point in $\lambda \text{int}(S)$.

We are to prove that $x \in \text{int}(\lambda S)$.

Since $x \in \lambda \operatorname{int}(S)$, we get $x/\lambda \in \operatorname{int}(S)$.

Since $x/\lambda \in \operatorname{int}(S)$, by definition of interior, there exists a radius r such that

$$B(x/\lambda, r) \subseteq S.$$

Let y be an arbitrary point in $B(x, \lambda r)$.

Since $y \in B(x, \lambda r)$, we get $\|y - x\| \leq \lambda r$.

Since $\|y - x\| \leq \lambda r$, we get $\|y/\lambda - x/\lambda\| \leq r$.

Since $\|y/\lambda - x/\lambda\| \leq r$, we get $y/\lambda \in B(x/\lambda, r)$.

Since $y/\lambda \in B(x/\lambda, r)$ and $B(x/\lambda, r) \subseteq S$, we get $y/\lambda \in S$.

Since $y/\lambda \in S$, we get $y \in \lambda S$.

Since any point in $B(x, \lambda r)$ is also in λS , we get $B(x, \lambda r) \subseteq \lambda S$.

Since there exists a radius r such that $B(x, \lambda r) \subseteq \lambda S$, by definition of interior, we get

$$x \in \operatorname{int}(\lambda S).$$

For the reverse direction,

■