

Graph Theory

Daniel Mao

Contents

1	Graph Basics	1
2	Trees	3
2.1	Definitions	3
2.2	Properties	3
3	Graph Isomorphism	5
3.1	Definitions	5
3.2	Properties	5
4	Matchings and Covers	7
4.1	Matching	7
4.2	Cover	8
4.3	Relations Between Matchings and Covers	8
5	Bipartite Graphs	9
5.1	Definitions	9
5.2	Properties of Bipartite Graphs	9
5.3	Characterizations	10
6	Planar Graphs	11
6.1	Definitions	11
6.2	Properties	11
6.3	Numerology	12
7	Duality	13
7.1	Definitions	13
8	Graph Coloring	15
8.1	Chromatic Number	15
8.2	5-color Theorem	16

9 Probability and Edge Density	19
10 Weird Stuffs	21
10.1 Geometric Representation of Graphs	21
10.2 Stable Sets	22
10.3 Theta Bodies	23
10.4 Product of Graphs	24

Chapter 1

Graph Basics

DEFINITION (Spanning Subgraph). Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G . We say that H is a **spanning** subgraph of G if and only if $W = V$. i.e., if H contains all vertices of G .

Chapter 2

Trees

2.1 Definitions

DEFINITION (Spanning Tree). Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G . We say that H is a **spanning tree** if H is a spanning subgraph of G and is a tree.

2.2 Properties

PROPOSITION 2.2.1. A graph is connected if and only if it has a spanning tree.

Chapter 3

Graph Isomorphism

3.1 Definitions

DEFINITION (Isomorphism). Let G and H be two graphs. We define an **isomorphism** from G to H to be a function f from $V(G)$ to $V(H)$ such that

- f is bijective, and that
- for any pair of vertices $v, w \in V(G)$, $f(v)f(w) \in E(H)$ if and only if $vw \in E(G)$.

i.e., a bijective function that both itself and its inverse preserve adjacency.

DEFINITION (Isomorphic). Let G and H be two graphs. We say that G and H are **isomorphic**, denoted by $G \simeq H$, if there exists an isomorphism from G to H .

PROPOSITION 3.1.1. The relation \simeq of isomorphism is an equivalence relation. That is, it is reflexive, symmetric, and transitive.

3.2 Properties

PROPOSITION 3.2.1. Let G and H be isomorphic graphs with isomorphism f . Then for any vertex $v \in V(G)$, we have $\deg_G(v) = \deg_H(f(v))$.

Chapter 4

Matchings and Covers

4.1 Matching

DEFINITION (Matching). Let $G = (V, E)$ be a graph. Let M be a subset of E . We say that M is a **matching** in G if every vertex in the spanning subgraph (V, M) has degree at most one.

DEFINITION (Saturated). Let $G = (V, E)$ be a graph. Let M be a subset of E . Let v be a vertex of G . We say that v is **M -saturated** if $\deg(v) = 1$ in (V, M) .

DEFINITION (Maximal Matching). Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximal matching** if it is a matching in G and any other matching is not a superset of it.

DEFINITION (Maximum Matching). Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximum matching** if it is a matching in G and any other matching contains edges no more than M .

DEFINITION (Perfect Matching). Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **perfect matching** if it matches all vertices of the graph. i.e., any vertex in G is incident to some edge in M .

PROPOSITION 4.1.1. Every maximum matching is maximal.

PROPOSITION 4.1.2. Every perfect matching is maximum.

PROPOSITION 4.1.3. Let $G = (V, E)$ be a graph. Let A and B be two maximal matchings of G . Then both $|A| \leq 2|B|$ and $|B| \leq 2|A|$.

4.2 Cover

DEFINITION (Cover). Let $G = (V, E)$ be a graph. Let C be a subset of V . We say that C is a **cover** of G if any edge has an end in C .

4.3 Relations Between Matchings and Covers

PROPOSITION 4.3.1. Let $G = (V, E)$ be a graph. Let M be a matching of G . Let C be a cover of G . Then $|M| \leq |C|$.

Chapter 5

Bipartite Graphs

5.1 Definitions

DEFINITION (Bipartition). Let $G = (V, E)$ be a graph. Let A and B be two subsets of V . We say the pair (A, B) is a **bipartition** of G if and only if $A \cap B = \emptyset$, $A \cup B = V$, and A and B are both independent.

DEFINITION (Bipartite Graph). Let $G = (V, E)$ be a graph. We say that G is **bipartite** if and only if there exists a bipartition of G .

DEFINITION (Balanced Bipartite Graph). Let $G = (V, E)$ be a bipartite graph with bipartition (A, B) . We say that G is **balanced** if and only if $|A| = |B|$.

5.2 Properties of Bipartite Graphs

PROPOSITION 5.2.1. Let $G = (V, E)$ be a bipartite graph with bipartition (A, B) . Then

$$\sum_{a \in A} \deg(a) = \sum_{b \in B} \deg(b) = |E|.$$

5.3 Characterizations

PROPOSITION 5.3.1. A graph is bipartite if and only if it has no odd cycles.

PROPOSITION 5.3.2. A graph is bipartite if and only if it is 2-colorable.

Chapter 6

Planar Graphs

6.1 Definitions

DEFINITION (Plane Embedding). Let $G(V, E, B)$ be an undirected multi-graph. A **plane embedding** of G is a pair of sets (\mathcal{P}, Γ) such that

6.2 Properties

PROPOSITION 6.2.1. Every subgraph of a planar graph is planar.

PROPOSITION 6.2.2. A multi-graph is planar if and only if its simplification is planar.

PROPOSITION 6.2.3. Let G be a multi-graph and e be an edge in G . Then G is planar if and only if $G \bullet e$ is planar.

THEOREM 6.1. A multi-graph is planar if and only if it does not contain a repeated subdivision of K_5 or $K_{3,3}$ as a subgraph.

6.3 Numerology

DEFINITION (Footprint). Let $G(V, E, B)$ be a planar multi-graph. Let (\mathcal{P}, Γ) be a plane embedding of the graph. We define the **footprint** of G , denoted by $fp(G)$, to be the union of the points and curves in \mathbb{R}^2 representing the vertices and edges in G .

DEFINITION (Face). Let $G(V, E, B)$ be a planar multi-graph. Let (\mathcal{P}, Γ) be a plane embedding of the graph. We define a **face** of (\mathcal{P}, Γ) to be a connected component of the set $\mathbb{R}^2 \setminus fp(G)$.

DEFINITION (Degree). Let $G(V, E, B)$ be a planar multi-graph. Let (\mathcal{P}, Γ) be a plane embedding of the graph. We define the **degree** of a face to be the sum of the number of edges and the number of bridges in its boundary.

PROPOSITION 6.3.1. An edge e in a planar multi-graph is a bridge if and only if the two faces on either side of the curve γ_e are the same.

Chapter 7

Duality

7.1 Definitions

DEFINITION (Dual Graph). Let $G = (V, E, B)$ be a multigraph. Let (\mathcal{P}, Γ) be a plane embedding of G . Let \mathcal{F} be the set of faces of G . We define the **dual graph** of this embedding to be the multigraph $G^* = (V^*, E^*, B^*)$ where $V^* = \mathcal{F}$ and $E^* = \{e^* : e \in E\}$.

PROPOSITION 7.1.1. Let $G = (V, E, B)$ be a multigraph. Let (\mathcal{P}, Γ) be a plane embedding of G . Let $G^* = (V^*, E^*, B^*)$ be the dual graph of G . Then for any face $f \in \mathcal{F}$, the degree of f as a face of \mathcal{P}, Γ equals the degree of f as a vertex of G^* .

PROPOSITION 7.1.2. If G is a connected multigraph embedded in the plane, then G^{**} is isomorphic with G .

Chapter 8

Graph Coloring

8.1 Chromatic Number

DEFINITION ((Proper) Coloring). Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define a **(proper) X -coloring** of G to be a function $f : V \rightarrow X$ such that if $vw \in E$, then $f(v) \neq f(w)$.

DEFINITION (Chromatic Number). Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define the **chromatic number** of G , denoted by $\chi(G)$, to be the smallest natural number $k \in \mathbb{N}$ for which G has a (proper) k -coloring.

PROPOSITION 8.1.1. The chromatic number exists and $\chi(G) \leq |V|$.

Proof. Take $X = V$. ■

PROPOSITION 8.1.2. G is complete if and only if $\chi(G) = |V(G)|$.

PROPOSITION 8.1.3. The only graph with chromatic number zero is the empty graph.

PROPOSITION 8.1.4. A graph has chromatic number one if and only if it has no edges and at least one vertex.

PROPOSITION 8.1.5. A graph has chromatic number two if and only if it is bipartite and has at least one edge.

PROPOSITION 8.1.6. Let G be a graph. Let $d_{\max}(G)$ be the maximum degree of a vertex in G . Then $\chi(G) \leq 1 + d_{\max}(G)$.

8.2 5-color Theorem

THEOREM 8.1. Every planar graph is 5-colorable.

Proof. (1890)

True for $|V| \leq 5$.

Inductively, suppose the theorem holds for planar graphs on $n - 1$ vertices for $n \geq 5$.

Suppose G is a planar graph on n vertices.

Let v be a vertex of degree ≤ 5 in G . This exists by a lemma in our lectures.

Since G is a planar, $G - v$ is planar. By the induction hypothesis, $G - v$ has a 5-coloring.

If some color does not appear on any neighbor of v , we can extend the coloring to a coloring of G .

Otherwise, v has exactly 5 neighbors with different colors.

For each pair i, j of colors, let G_{ij} be the subgraph of $G - v$ induced by the vertices colored i or j .

If the component H of G_{ij} containing x_i does not contain x_j , then we can switch the colors of all vertices in H between i and j to get a coloring of $G - v$ that assigns only 4 colors to neighbors of v , and thus extends to a coloring of G .

So G_{ij} contains a path from x_i to x_j .

Because $G_{2,5}$ and $G_{1,4}$ have disjoint vertex sets, this contradicts the planarity of G .

■

DEFINITION (Near-triangulation). Planar drawing of G where the infinite face is bounded by a cycle, and every other face is bounded by a triangle

THEOREM 8.2. Every planar near-triangulation has a 5-coloring.

Theorem 8.2 \implies Theorem 8.1.

DEFINITION (List Assignment). A **list assignment** L of G is a function that assigns a set $L(v)$ of colors to each $v \in V$.

DEFINITION (L -coloring). An L -coloring of G is a choice of a color in $L(v)$ for each $v \in V$ such that adjacent vertices get different colors.

DEFINITION (5-list-colorable). A graph is **5-list-colorable** if for every list assignment L of G with $|L(v)| \geq 5$, G is L -colorable.

THEOREM 8.3. Every planar near-triangulation is 5-list-colorable.

Theorem 8.3 \implies Theorem 8.2 because coloring is a special case of list coloring.

THEOREM 8.4 (Carsten Thomassen, 1993). If G is a near-triangulation and L is a list assignment such that

- (1) $|L(v)| = 5$ for every non-boundary vertex,
- (2) $|L(v)| = 3$ for every boundary vertex.

Then G has an L -coloring even if two adjacent boundary vertices have their colors arbitrarily decided in advance.

Proof.

Case 1. There is a "chord" between two boundary vertices.

Let G_1 and G_2 be subgraph of G obtained by "cutting" G along the chord, where G_1 contains the pre-colored vertices.

By applying the inductive hypothesis to G_1 , and then applying it to G_2 with the two ends of the chord pre-colored according to the coloring of G_1 , we get a coloring of G_1 .

Case 2. There is no chord.

Let u and u' be the pre-colored vertices.

Let x, y be the next two vertices occuring in order around the boundary.

■

Theorem 8.4 \implies Theorem 8.3.

Chapter 9

Probability and Edge Density

Q: Let G be a graph on n vertices with no triangles. How many edges can G have?

THEOREM 9.1 (Mantel). If G is triangle-free and has n vertices, then

$$|E| \leq \frac{n^2}{4}.$$

Proof. Let $P_{2,1}$ denote the probability that a pair of distinct vertices chosen uniformly at random, are adjacent.

$$P_{2,1} = |E| / \binom{n}{2}.$$

Let $P_{3,2}$ denote the probability that a randomly chosen triple of vertices contains exactly two edges. Let $P_{3,1}$ denote ... one edge. Let $P_{3,0}$ denote ... no edges. Notice $P_{3,2} + P_{3,1} + P_{3,0} = 1$.

Part 1: Show that $P_{2,1} = \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1}$. Notice that the graph is triangle-free. So $P_{3,3} = 0$. Choosing a pair at random is the same as choosing a triple at random, then choosing a pair at random within that triple.

For a fixed vertex v , let $Q_{v,1}$ denote the probability that a randomly chosen vertex $u \neq v$ is adjacent to v .

$$Q_{v,1} = \frac{\deg(v)}{n-1}.$$

Let $Q_{v,2}$ denote the probability that two distinct randomly chosen vertices other than v are both adjacent to v .

$$Q_{v,2} = \binom{\deg(v)}{2} / \binom{n-1}{2}.$$

Part 2: Show that $Q_{v,1}^2 \approx Q_{v,2}$. Both give (essentially) the probability that a pair x, y of vertices other than v are both adjacent to v . The LHS allows $x = y$. The RHS does not. But $x = y$ occurs with negligible probability.

Part 3: Show that $P_{2,1} = \frac{1}{n} \sum_v Q_{v,1}$. Both the RHS and LHS are just the probability of an edge between two vertices. The RHS calls the first chosen vertex v .

Part 4: Show that $\frac{1}{3}P_{3,2} = \frac{1}{n} \sum_v Q_{v,2}$. Both sides give the probability that, if we choose 3 vertices at random, and then choose one among those 3 and call it v , that v is adjacent to both the others.

Proof of the theorem.

$$\begin{aligned} P_{2,1} &= \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1} \geq \frac{2}{3}P_{3,2} \\ &= 2 \left(\frac{1}{n} \sum_v Q_{v,2} \right) \approx 2 \left(\frac{1}{n} \sum_v Q_{v,1}^2 \right) \\ &\geq 2 \left(\frac{1}{n} \sum_v Q_{v,1} \right)^2 = 2P_{2,1}^2. \end{aligned}$$

So $P_{2,1} \leq \frac{1}{2}$. So $|E| \leq \frac{n^2}{4}$. ■

Q: If G has n vertices, no K_{t+1} -subgraph, how many edges can G have?

THEOREM 9.2 (Turan). If G is a graph on n vertices with no K_{t+1} -subgraph, then

$$|E| \leq \frac{n^2}{2} \left(1 - \frac{1}{t} \right).$$

THEOREM 9.3 (Erdos-Stone). If H is a graph and G is a graph on n vertices without H as a subgraph, then

$$|E| \leq \frac{n^2}{2} \left(1 - \frac{1}{\chi(H)-1} + \varepsilon(n) \right)$$

where $\varepsilon(n) \rightarrow 0$ as $n \rightarrow \infty$ and $\chi(H)$ is the chromatic number of H , the fewest number of colors needed to properly color the vertices of H .

Chapter 10

Weird Stuffs

10.1 Geometric Representation of Graphs

DEFINITION (Geometric Representation). Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. We define a **geometric representation** of G to be a map from V to \mathbb{R}^d .

DEFINITION (Unit Distance Representation). Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \rightarrow \mathbb{R}^d$ be a geometric representation of G . We say that u is a **unit distance representation** of G if and only if $\forall \{i, j\} \in E, \|u(i) - u(j)\|_2 = 1$.

DEFINITION (Orthonormal Representation). Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \rightarrow \mathbb{R}^d$ be a geometric representation of G . We say that u is an **orthonormal representation** of G if and only if

- $\forall i \in V, \|u(i)\|_2 = 1$; and
- $\forall \{i, j\} \in \overline{E}, \langle u(i), u(j) \rangle = 0$ where \overline{E} is the edge set of the complement of G .

DEFINITION. We define $t_h(G)$ to be the square radius of the smallest hypersphere that contains a unit distance representation of G .

THEOREM 10.1 (CO 471, Spring 2022, Levent Tuncel). Let $G = (V, E)$ be a graph. Then

$$\begin{aligned} t_h(G) &= \min t \\ \text{subject to:} \quad & X_{ii} = t, \forall i \in V \\ & X_{ii} - 2X_{ij} + X_{jj} = 1, \forall \{i, j\} \in E \\ & X \in S_+^V \end{aligned}$$

PROPOSITION 10.1.1 (CO 471, Spring 2022, Levent Tuncel). Let $G = (V, E)$ be a graph. Then G is bipartite if and only if $t_h(G) \leq \frac{1}{4}$.

Proof. ■

PROPOSITION 10.1.2. Let $n \in \mathbb{Z}_{++}$. Let K_n denote the n -clique. Then $t_h(K_n) =$.

Proof. ■

10.2 Stable Sets

DEFINITION (Stable Sets). Let $G = (V, E)$ be a graph. Let S be a subset of the vertex set V . We say that S is a **stable set** in G if and only if $\forall \{i, j\} \in E$, at most one of i or j is in S . i.e., S is a set of pairwise non-adjacent vertices.

DEFINITION (Stability Number). Let $G = (V, E)$ be a graph. We define the **stability number** of G , denoted by $\alpha(G)$, to be a number given by

$$\alpha(G) := \max\{|S| : S \text{ is stable in } G\}.$$

DEFINITION (Stable Set Polytope). Let $G = (V, E)$ be a graph. We define the

stable set polytope of G , denoted by $\text{STAB}(G)$, to be a subset of \mathbb{R}^V given by

$$\text{STAB}(G) := \text{conv}\{x : x \text{ is the incidence vector of some stable set in } G\}.$$

DEFINITION (Fractional Stable Set Polytope). Let $G = (V, E)$ be a graph. We define the **fractional stable set polytope** of G , denoted by $\text{FRAC}(G)$, to be a subset of \mathbb{R}^V given by

$$\text{FRAC}(G) := \{x \in [0, 1]^V : x_i + x_j \leq 1, \forall \{i, j\} \in E\}.$$

PROPOSITION 10.2.1. Let $G = (V, E)$ be a graph. Then

$$\text{STAB}(G) = \text{conv}(\text{FRAC}(G) \cap \{0, 1\}^V).$$

10.3 Theta Bodies

DEFINITION (Theta Body). Let $G = (V, E)$ be a graph. We define the **theta body** of G , denoted by $\text{TH}(G)$, to be a subset of \mathbb{R}_+^V given by

$$\text{TH}(G) := \left\{ x \in \mathbb{R}_+^V : \sum_{i \in V} (c^\top u(i))^2 x_i \leq 1, \begin{array}{l} \forall c \in \mathbb{R}^V : \|c\|_2 = 1, \\ \forall \text{ orth. repr. } u \text{ of } G \end{array} \right\}.$$

DEFINITION (Lovase Theta Function). Let $G = (V, E)$ be a graph. Let $w \in \mathbb{R}_+^V$. We define the **Lovase Theta function**, denoted by θ , to be a function of G and w given by

$$\theta(G, w) := \max\{w^\top x : x \in \text{TH}(G)\}.$$

DEFINITION (Lovase Theta Number). Let $G = (V, E)$ be a graph. We define the **Lovase Theta number** of G , denoted by $\theta(G)$, to be a number given by

$$\theta(G) := \theta(G, \bar{e}) = \max\{\bar{e}^\top x : x \in \text{TH}(G)\}.$$

THEOREM 10.2 (CO 471, Spring 2022, Levent Tuncel). Let $G = (V, E)$ be a graph. Let $w \in \mathbb{R}_+^V$. Then the following quantities are the same:

(1) $\theta(G, w)$;

(2) If $w_i = 0$, define $\frac{w_i}{(c^\top u(i))^2} := 0$,

$$\inf \left\{ \max_{i \in V} \left\{ \frac{w_i}{(c^\top u(i))^2} \right\} : \begin{array}{l} c \in \mathbb{R}^V, \|c\|_2 = 1, \\ u \text{ is an orth. repr. of } G \end{array} \right\};$$

(3) $\min\{\eta \in \mathbb{R} : S \in \mathbb{S}^V, \text{diag}(S) = 0, S_{ij} = 0, \forall \{i, j\} \in \overline{E}, \eta I - S \succeq W\}$;

(4) $\max\{\text{tr}(WX) : X_{ij} = 0, \forall \{i, j\} \in E, \text{tr}(X) = 1, X \in \mathbb{S}_+^V\}$.

where $W \in \mathbb{S}^V$ is given by $W_{ij} = \sqrt{w_i w_j}$, $\forall i, j \in V$.

10.4 Product of Graphs

DEFINITION (Strong Product). Let $G = (V, E)$ and $H = (W, F)$ be graphs. We define the **strong product** of G and H , denoted by $G \otimes H$, to be a graph given by $G \otimes H = (V(G \otimes H), E(G \otimes H))$ where

$$V(G \otimes H) := V \times W \text{ and}$$

$$E(G \otimes H) := \left\{ \{(i, u), (j, v)\} : \begin{array}{l} (\{i, j\} \in E \text{ and } \{u, v\} \in F) \text{ or} \\ (\{i, j\} \in E \text{ and } u = v \in W) \text{ or} \\ (i = j \in V \text{ and } \{u, v\} \in F) \end{array} \right\}.$$

PROPOSITION 10.4.1. Let $G = (V, E)$ and $H = (W, F)$ be graphs. Then

$$\theta(G \otimes H) \leq \theta(G) \times \theta(H).$$

DEFINITION (Shannon Capacity). Let $G = (V, E)$ be a graph. We define the

Shannon capacity of G , denoted by $\Theta(G)$, to be a number given by

$$\Theta(G) := \limsup_{k \rightarrow +\infty} (\alpha(G^{\otimes k}))^{1/k}$$

where $\alpha(G^{\otimes k})$ denotes the stability number of $G^{\otimes k}$.