Functional Analysis

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Chapter 1

Linear Space

1.1 Definitions

PROPOSITION 1.1.1. The singleton set consisting of the zero vector is a linear subspace.

1.2 Linear Subspaces

PROPOSITION 1.2.1. The intersection of a collection of subspaces is a subspace.

PROPOSITION 1.2.2. The sum of two subspaces is a subspace.

Proof. Let $\mathcal{V}$ be a vector space over field $\mathbb{F}$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of $\mathcal{V}$. I will show that $\mathcal{W}_1 + \mathcal{W}_2$ is a subspace of $\mathcal{V}$. Since $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces, $\mathcal{W}_1 \neq \emptyset$ and $\mathcal{W}_2 \neq \emptyset$. So $\mathcal{W}_1 + \mathcal{W}_2 \neq \emptyset$. There remains only to show that $\mathcal{W}_1 + \mathcal{W}_2$ is closed under addition and scalar multiplication.

Closed Under Addition:

Let x and y be arbitrary elements of $\mathcal{W}_1 + \mathcal{W}_2$. Then $x = x_1 + x_2$ for some $x_1 \in \mathcal{W}_1$ and some $x_2 \in \mathcal{W}_2$; and $y = y_1 + y_2$ for some $y_1 \in \mathcal{W}_1$ and some $y_2 \in \mathcal{W}_2$. Since $\mathcal{W}_1$ is a subspace, it is closed under addition. So $x_1 + y_1 \in \mathcal{W}_1$. Since $\mathcal{W}_2$ is a subspace, it is closed under addition. So $x_2 + y_2 \in \mathcal{W}_2$. So

$$x + y = x_1 + x_2 + y_1 + y_2 = (x_1 + y_1) + (x_2 + y_2) \in \mathcal{W}_1 + \mathcal{W}_2.$$

So $\mathcal{W}_1 + \mathcal{W}_2$ is closed under addition.

Closed Under Scalar Multiplication:

Let k be an arbitrary element of $\mathbb{F}$. Let x be an arbitrary element of $\mathcal{W}_1 + \mathcal{W}_2$. Then $x = x_1 + x_2$ for some $x_1 \in \mathcal{W}_1$ and some $x_2 \in \mathcal{W}_2$. Since $\mathcal{W}_1$ is a subspace, it is closed under scalar multiplication. So $kx_1 \in \mathcal{W}_1$. Since $\mathcal{W}_2$ is a subspace, it is closed under scalar multiplication. So $kx_2 \in \mathcal{W}_2$. So

$$kx = k(x_1 + x_2) = kx_1 + kx_2 \in \mathcal{W}_1 + \mathcal{W}_2.$$

So $\mathcal{W}_1 + \mathcal{W}_2$ is closed under scalar multiplication. ■

1.3 Linear Span

PROPOSITION 1.3.1. Linear spans are linear subspaces.

Proof. Let $\mathcal{V}$ be a vector space over field $\mathbb{F}$. Let $S = \{x_1, ..., x_k\}$ be a subset of $\mathcal{V}$ where $k \in \mathbb{N}$. I will show that $\operatorname{span}(S)$ is a subspace of $\mathcal{V}$. Clearly $\operatorname{span}(S) \subseteq \mathcal{V}$. So there remains only to show the following two facts.

Closed Under Addition:

Let a and b be arbitrary elements of $\operatorname{span}(S)$. Then $a = \sum_{i=1}^{k} \lambda_i x_i$ for some $\lambda_1, ..., \lambda_k \in \mathbb{F}$; and $b = \sum_{i=1}^{k} \mu_i x_i$ for some $\mu_1, ..., \mu_k \in \mathbb{F}$. Then

$$a + b = \sum_{i=1}^{k} \lambda_i x_i + \sum_{i=1}^{k} \mu_i x_i = \sum_{i=1}^{k} (\lambda_i + \mu_i) x_i.$$

So $a + b \in \operatorname{span}(S)$. So $\operatorname{span}(S)$ is closed under addition.

Closed Under Scalar Multiplication:

Let a be an arbitrary element of $\operatorname{span}(S)$. Let k be an arbitrary element of $\mathbb{F}$. Then $a = \sum_{i=1}^{k} \lambda_i x_i$ for some $\lambda_1, ..., \lambda_k \in \mathbb{F}$. Then

$$ka = k \sum_{i=1}^{k} \lambda_i x_i = \sum_{i=1}^{k} (k\lambda_i) x_i.$$

So $ka \in \operatorname{span}(S)$. So $\operatorname{span}(S)$ is closed under scalar multiplication. ■

PROPOSITION 1.3.2 (The Linear Span Operator)**.** Let $\mathcal{V}$ be a vector space over field $\mathbb{F}$. Then the linear span operator span has the following properties.

(1) (Extensive) Let S be a subset of $\mathcal{V}$. Then

$$S \subseteq \operatorname{span}(S).$$

(2) (Monotonic) Let S and T be subsets of $\mathcal{V}$. Suppose that $S \subseteq T$. Then

$$\operatorname{span}(S) \subseteq \operatorname{span}(T).$$

(3) (Idempotent) Let S be a subset of $\mathcal{V}$. Then

$$\operatorname{span}(\operatorname{span}(S)) = \operatorname{span}(S).$$

Proof of (1). Let x be an arbitrary element of S. Define $n := 1 \in \mathbb{N}$, $\lambda_1 := 1 \in \mathbb{F}$, and $x_1 := x \in S$. Then $x = 1 \cdot x = \lambda_1 x_1 = \sum_{i=1}^{n} \lambda_i x_i$. So $x \in \operatorname{span}(S)$. ■

Proof of (3). From (1) we know that $\operatorname{span}(S) \subseteq \operatorname{span}(\operatorname{span}(S))$. So there remains only to show that $\operatorname{span}(\operatorname{span}(S)) \subseteq \operatorname{span}(S)$. Let x be an arbitrary element of $\operatorname{span}(\operatorname{span}(S))$. Then $\exists n \in \mathbb{N}$, $\lambda_1, ..., \lambda_n \in \mathbb{F}$, and $v_1, ..., v_n \in \operatorname{span}(S)$ such that $x = \sum_{i=1}^{n} \lambda_i v_i$. For each $i \in \{1, ..., n\}$, since $v_i \in \operatorname{span}(S)$, $\exists n_i \in \mathbb{N}$, $\exists \lambda_{i,1}, ..., \lambda_{i,n_i} \in \mathbb{F}$, and $\exists v_{i,1}, ..., v_{i,n_i} \in S$ such that $v_i = \sum_{j=1}^{n_i} \lambda_{i,j} v_{i,j}$. So

$$x = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n_i} \lambda_{i,j} v_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} (\lambda_i \lambda_{i,j}) v_{i,j}.$$

Note that $\forall i, j \in \{1, ..., n\}$, $\lambda_i \lambda_{i,j} \in \mathbb{F}$ and $v_{i,j} \in S$. So $x \in \operatorname{span}(S)$. So $\operatorname{span}(\operatorname{span}(S)) \in \operatorname{span}(S)$. This completes the proof. ■

1.4 Linear Independence

DEFINITION (Linear Dependence and Linear Independence)**.** Let $\mathcal{V}$ be a vector space over field $\mathbb{F}$. Let S be a subset of $\mathcal{V}$. We say that S is **linearly dependent** if

$$\exists x \in S, \exists n \in \mathbb{N}, \exists x_1, ..., x_n \in S \setminus \{x\}, \exists k_1, ..., k_n \in \mathbb{F} \text{ such that } x = \sum_{i=1}^{n} k_i x_i.$$

We say that S is **linearly independent** if S is not linearly dependent.

PROPOSITION 1.4.1. A set that contains the zero vector is linearly dependent.

1.5 Basis

PROPOSITION 1.5.1. All bases have equal cardinality.

1.6 Dimension

DEFINITION (Dimension - 1)**.** Let $\mathcal{V}$ be a vector space. Let $\mathcal{B}$ be a basis for $\mathcal{V}$. We define the **dimension** of $\mathcal{V}$, denoted by $\dim(\mathcal{V})$, to be the cardinality of $\mathcal{B}$.

DEFINITION (Dimension - 2)**.** Let $\mathcal{V}$ be a vector space. We define the **dimension** of $\mathcal{V}$, denoted by $\dim(\mathcal{V})$, to be the cardinality of the maximal linearly independent set in the space.

DEFINITION (Dimension - 3)**.** Let $\mathcal{V}$ be a vector space. We define the **dimension** of $\mathcal{V}$, denoted by $\dim(\mathcal{V})$, to be the cardinality of the minimal spanning set of the space.

PROPOSITION 1.6.1. Definitions 2 and 3 of dimension are equivalent.

Proof. Let $\mathcal{V}$ be a vector space. Let S be a subset of $\mathcal{V}$.
Forward Direction:

Assume that S is a minimal spanning set. I will show that S is a maximal linearly independent set. Assume for the sake of contradiction that S is not linearly independent. Then $\exists x \in S$ such that $x \in \operatorname{span}(S \backslash \{x\})$. So $\operatorname{span}(S \backslash \{x\}) = \operatorname{span}(S) = \mathcal{V}$. This contradicts to the assumption that S is minimal. So S is linearly independent. Assume for the sake of contradiction that S is not maximal. Then $\exists y \in \mathcal{V} \setminus S$ such that $S \cup \{y\}$ is linearly independent. So $y \notin \operatorname{span}(S)$. This contradicts to the assumption that S is spanning. So S is maximal.
Backward Direction:

Assume that S is a maximal linearly independent set. I will show that S is a minimal spanning set. Assume for the sake of contradiction that S is not spanning. Then $\exists x \in \mathcal{V}$ such that $x \notin \operatorname{span}(S)$. So $S \cup \{x\}$ is linearly independent. This contradicts to the assumption that S is maximal. So S is spanning. Assume for the sake of contradiction that S is not minimal. Then $\exists y \in S$ such that $S \setminus \{y\}$ is spanning. So $y \in \operatorname{span}(S \setminus \{y\})$. This contradicts to the assumption that S is linearly independent. So S is minimal. ■

PROPOSITION 1.6.2. Let $\mathcal{V}$ be a vector space. Let $\mathcal{W}$ be a subspace of $\mathcal{V}$. Then

(1) $\dim(\mathcal{W}) \leq \dim(\mathcal{V})$.

(2) If $\dim(\mathcal{W}) = \dim(\mathcal{V})$ and are both finite, then $\mathcal{W} = \mathcal{V}$.

Proof of (1). Any linearly independent subset of $\mathcal{W}$ is a linearly independent subset of $\mathcal{V}$. Any spanning set of $\mathcal{V}$ is a spanning set of $\mathcal{W}$. ■

PROPOSITION 1.6.3. Let $\mathcal{V}$ be a vector space. Suppose $\dim(\mathcal{V}) = n$ where $n \in \mathbb{N}$. Let S be a subset of $\mathcal{V}$. Then

(1) If $|S| > n$, then S is linearly dependent.

(2) If $|S| < n$, then S does not span $\mathcal{V}$.

(3) If $|S| = n$, then S is linearly independent if and only if $\operatorname{span}(S) = \mathcal{V}$.

Proof of (1). Suppose that $|S| > n$. Assume for the sake of contradiction that S is linearly independent. Then $\dim(\mathcal{V}) \geq |S|$. So $\dim(\mathcal{V}) \geq |S| > n$. This contradicts to the assumption that $\dim(\mathcal{V}) = n$. So S is linearly dependent. ■

Proof of (2). Suppose that $|S| < n$. Assume for the sake of contradiction that S spans $\mathcal{V}$. Then $\dim(\mathcal{V}) \leq |S|$. So $\dim(\mathcal{V}) \leq |S| < n$. This contradicts to the assumption that $\dim(\mathcal{V}) = n$. So S does not span $\mathcal{V}$. ■

Proof of (3). Suppose that $|S| = n$.

Forward Direction:

Assume that S is linearly independent. I will show that $\operatorname{span}(S) = \mathcal{V}$. It is clear that $\dim(\operatorname{span}(S)) = n$. So $\dim(\operatorname{span}(S)) = \dim(\mathcal{V})$ and are both finite. So $\operatorname{span}(S) = \mathcal{V}$.

Backward Direction:

Assume that $\operatorname{span}(S) = \mathcal{V}$. I will show that S is linearly independent. Assume for the sake of contradiction that S is linearly dependent. Then $\exists S' \subsetneq S$ that is linearly independent. Apply (3) on S', we get that S does not span $\mathcal{V}$. This contradicts to the assumption that $\operatorname{span}(S) = \mathcal{V}$. So S is linearly independent. This completes the proof. ■

PROPOSITION 1.6.4. Let $\mathcal{V}$ be a vector space over field $\mathbb{F}$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be finite dimensional subspaces of $\mathcal{V}$. Then we have

$$\dim(\mathcal{W}_1 + \mathcal{W}_2) = \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2) - \dim(\mathcal{W}_1 \cap \mathcal{W}_2).$$

Proof. Since $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of $\mathcal{V}$, $\mathcal{W}_1 \cap \mathcal{W}_2$ is a subspace of $\mathcal{V}$. Let $k := \dim(\mathcal{W}_1 \cap \mathcal{W}_2)$. Then $k \leq \min\{\dim(\mathcal{W}_1), \dim(\mathcal{W}_2)\}$. Let $m := \dim(\mathcal{W}_1) - k$ and $n := \dim(\mathcal{W}_2) - k$. Let $\mathcal{B} = \{v_1, ..., v_k\}$ be a basis for $\mathcal{W}_1 \cap \mathcal{W}_2$. Extend $\mathcal{B}$ to a basis $\mathcal{B}_1 = \{v_1, ..., v_k, x_1, ..., x_m\}$ for $\mathcal{W}_1$. Extend $\mathcal{B}$ to a basis $\mathcal{B}_2 = \{v_1, ..., v_k, y_1, ..., y_n\}$ for $\mathcal{W}_2$.

Part 1: Show that $\mathcal{B}_1 \cup \mathcal{B}_2 = \{v_1, ..., v_k, x_1, ..., x_m, y_1, ..., y_n\}$.

Assume for the sake of contradiction that $\{x_1, ..., x_m\} \cap \{y_1, ..., y_n\} \neq \emptyset$. Say $\exists v \in \{x_1, ..., x_m\} \cap \{y_1, ..., y_n\}$. Then $\{v_1, ..., v_k, v\}$ is a linearly independent subset of $\mathcal{W}_1 \cap \mathcal{W}_2$. However, this contradicts to the assumption that $k = \dim(\mathcal{W}_1 \cap \mathcal{W}_2)$. So $\{x_1, ..., x_m\} \cap \{y_1, ..., y_n\} = \emptyset$. So

$$\mathcal{B}_1 \cup \mathcal{B}_2 = \{v_1, ..., v_k, x_1, ..., x_m, y_1, ..., y_n\}.$$

Part 2: Show that $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent.

Let $\lambda_1, ..., \lambda_k,\ \alpha_1, ..., \alpha_m,\ \beta_1, ..., \beta_n \in \mathbb{F}$ be such that

$$\sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{m} \alpha_i x_i + \sum_{i=1}^{n} \beta_i y_i = 0.$$

I will show that $\lambda_1, ..., \lambda_k,\ \alpha_1, ..., \alpha_m,\ \beta_1, ..., \beta_n = 0$. From the above we get

$$\sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{m} \alpha_i x_i = \sum_{i=1}^{n} (-\beta_i) y_i.$$

Note that the LHS $\in \operatorname{span}(\mathcal{B}_1) = \mathcal{W}_1$. So RHS $=$ LHS $\in \mathcal{W}_1$. Note that RHS $\in \operatorname{span}(\mathcal{B}_2) = \mathcal{W}_2$. So RHS $\in \mathcal{W}_1 \cap \mathcal{W}_2 = \operatorname{span}(\mathcal{B})$. So $\exists \mu_1, ..., \mu_k \in \mathbb{F}$ such that

$$\text{RHS} = \sum_{i=1}^{n} (-\beta_i) y_i = \sum_{i=1}^{k} \mu_i v_i.$$

That is,

$$\sum_{i=1}^{k} \mu_i v_i + \sum_{i=1}^{n} \beta_i y_i = 0.$$

Since $\mathcal{B}_2$ is linearly independent, we get $\mu_1, ..., \mu_k,\ \beta_1, ..., \beta_n = 0$. So

$$\sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{m} \alpha_i x_i = \sum_{i=1}^{n} (-\beta_i) y_i = 0.$$

Since $\mathcal{B}_1$ is linearly independent, we get $\lambda_1, ..., \lambda_k,\ \alpha_1, ..., \alpha_m = 0$. So $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent.

Part 3: Show that $\mathcal{B}_2 \cup \mathcal{B}_2$ is a spanning set for $\mathcal{W}_1 + \mathcal{W}_2$.

Let x be an arbitrary element of $\mathcal{W}_1 + \mathcal{W}_2$. Then $x = x_1 + x_2$ for some $x_1 \in \mathcal{W}_1$ and some $x_2 \in \mathcal{W}_2$. Then $\exists \lambda_1, ..., \lambda_k, \alpha_1, ..., \alpha_m \in \mathbb{F}$ and $\exists \mu_1, ..., \mu_k, \beta_1, ..., \beta_n \in \mathbb{F}$ such that

$$x_1 = \sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{m} \alpha_i x_i \quad \text{and} \quad x_2 = \sum_{i=1}^{k} \mu_i v_i + \sum_{i=1}^{n} \beta_i y_i.$$

So

$$\begin{aligned} x = x_1 + x_2 &= \sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{m} \alpha_i x_i + \sum_{i=1}^{k} \mu_i v_i + \sum_{i=1}^{n} \beta_i y_i \\ &= \sum_{i=1}^{k} (\lambda_i + \mu_i) v_i + \sum_{i=1}^{m} \alpha_i x_i + \sum_{i=1}^{n} \beta_i y_i \in \operatorname{span}(\mathcal{B}_1 \cup \mathcal{B}_2). \end{aligned}$$

So $\mathcal{B}_1 \cup \mathcal{B}_2$ is a spanning set for $\mathcal{W}_1 + \mathcal{W}_2$.

Part 4:

By part 2 and part 3, $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $\mathcal{W}_1 + \mathcal{W}_2$. So

$$\dim(\mathcal{W}_1 + \mathcal{W}_2) = |\mathcal{B}_1 \cup \mathcal{B}_2| = k + m + n.$$

On the other hand, we have

$$\begin{aligned} &\dim(\mathcal{W}_1) + \dim(\mathcal{W}_2) - \dim(\mathcal{W}_1 \cap \mathcal{W}_2) \\ &= |\mathcal{B}_1| + |\mathcal{B}_2| - |\mathcal{B}| \\ &= (k + m) + (k + n) - k = k + m + n \end{aligned}$$

So

$$\dim(\mathcal{W}_1 + \mathcal{W}_2) = \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2) - \dim(\mathcal{W}_1 \cap \mathcal{W}_2),$$

as desired. ■

Chapter 2

Normed Linear Space

2.1 Definitions

DEFINITION (Seminorm)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **seminorm** on $\mathfrak{X}$, denoted by ν, to be a map from $\mathfrak{X}$ to $\mathbb{R}$ that satisfies the following conditions.

(1) $\forall x \in \mathfrak{X}, \quad \nu(x) \geq 0.$

(2) $\forall \lambda \in \mathbb{F}, \forall x \in \mathfrak{X}, \quad \nu(\lambda x) = |\lambda| \nu(x).$

(3) $\forall x, y \in \mathfrak{X}, \quad \nu(x + y) \leq \nu(x) + \nu(y).$

The idea behind the seminorm is that we are trying to give our vector space a notion of "length" of vectors.

DEFINITION (Norm)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **norm** on $\mathfrak{X}$, denoted by ν, to be a seminorm on $\mathfrak{X}$ that satisfies the additional condition:

$$\forall x \in \mathfrak{X}, \quad \mu(x) = 0 \iff x = 0.$$

2.2 Properties

PROPOSITION 2.2.1. Proper subspaces of a normed linear space has empty interior.

Proof. Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a proper subspace of $\mathfrak{X}$. Assume for the sake of contradiction that $\mathcal{M}$ has non-empty interior. Then $\exists x_0 \in \mathcal{M}$ and $\exists r > 0$ such that $\text{ball}(x_0, r) \subseteq \mathcal{M}$ where $\text{ball}(x_0, r)$ denotes the open ball centered at point x_0 with radius r. Let x be an arbitrary point in $\mathfrak{X}$. Define a point $y(x)$ as $y(x) := x_0 + \frac{r}{2\|x\|}x$. Then $x = \frac{2\|x\|}{r}(y - x_0)$. It is easy to verify that $\|y - x_0\| = \frac{r}{2} < r$. So $y \in \text{ball}(x_0, r)$. So $y \in \mathcal{M}$. Since $y, x_0 \in \mathcal{M}$ and $\mathcal{M}$ is a subspace, we get $\frac{2\|x\|}{r}(y - x_0) \in \mathcal{M}$. That is, $x \in \mathcal{M}$. So $\forall x \in \mathfrak{X}$, $x \in \mathcal{M}$. So $\mathcal{M} = \mathfrak{X}$. This contradicts to the assumption that $\mathcal{M}$ is a proper subspace of $\mathfrak{X}$. So $\mathcal{M}$ has empty interior. ∎

PROPOSITION 2.2.2. Closed proper subspaces of a normed linear space are nowhere dense.

Proof. Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a closed proper subspace of $\mathfrak{X}$. Since $\mathcal{M}$ is closed, $\text{cl}(\mathcal{M}) = \mathcal{M}$. So $\text{cl}(\mathcal{M}) = \mathcal{M}$ is a closed proper subspace of $\mathfrak{X}$. Since $\text{cl}(\mathcal{M})$ is a proper subspace of $\mathfrak{X}$, $\text{int}(\text{cl}(\mathcal{M})) = \emptyset$. So $\mathcal{M}$ is nowhere dense. ∎

PROPOSITION 2.2.3. Let $\mathfrak{X}$ be a normed linear space over field $\mathbb{F}$. Then $\mathfrak{X}$ is complete if and only if the closed unit ball $\mathfrak{X}_1$ is complete.

Proof. For one direction, assume that $(V, \|\cdot\|)$ is complete. We are to prove that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete. Since $(\overline{B(0,1)}, \|\cdot\|_V)$ is a closed subspace of $(V, \|\cdot\|)$ and $(V, \|\cdot\|)$ is complete, $(\overline{B(0,1)}, \|\cdot\|_V)$ is also complete. For the reverse direction, assume that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete. We are to prove that $(V, \|\cdot\|_V)$ is complete. Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $(V, \|\cdot\|_V)$. Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ is bounded in $(V, \|\cdot\|_V)$. Let λ be a positive upper bound for $\{\|x_i\|_V\}_{i\in\mathbb{N}}$. Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$. Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$ and $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$. Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ converges in $(V, \|\cdot\|_V)$. Since any Cauchy sequence in $(V, \|\cdot\|_V)$ converges in $(V, \|\cdot\|_V)$, $(V, \|\cdot\|_V)$ is complete. ∎

2.3 Equivalence of Norm

DEFINITION (Equivalence of Norm)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are **equivalent** if

$$\exists c_1, c_2 > 0, \quad \forall v \in \mathfrak{X}, \quad c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_2.$$

Or equivalently,

$$c_1\|v\|_2 \leq \|v\|_1 \leq c_2\|v\|_2.$$

PROPOSITION 2.3.1. The equivalence of norms is an equivalence relation.

PROPOSITION 2.3.2. Let X be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if they generate the same metric topology.

Proof. Convergence to 0 is equivalent under either $\|\cdot\|_1$ or $\|\cdot\|_2$. i.e., equivalent norms give rise to the same set of sequences that are convergent. Convergence of sequences defines the topology. ■

PROPOSITION 2.3.3. Let $\mathfrak{X}$ be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathfrak{X}$. Let ι be the identity map from $(\mathfrak{X}, \|\cdot\|_1)$ to $(\mathfrak{X}, \|\cdot\|_2)$. Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent Then the identity map ι is continuous, and in fact, a homeomorphism between $(\mathfrak{X}, \|\cdot\|_1)$ and $(\mathfrak{X}, \|\cdot\|_2)$.

2.4 Finite-Dimensional Normed Linear Space

PROPOSITION 2.4.1. A finite-dimensional linear manifold of a normed linear space is closed.

THEOREM 2.1. Let $\mathfrak{X}$ be a finite-dimensional normed linear space. Let S be a set in $\mathfrak{X}$. Then S is norm-compact if and only if S is closed and bounded.

THEOREM 2.2. All norms on a finite-dimensional normed linear space are equivalent.

Proof Approach 1. Let $\mathfrak{X}$ be a finite-dimensional normed linear space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathfrak{X}$. Let B_1 denote the closed unit ball under norm $\|\cdot\|_1$. Then B_1 is closed and bounded. Since B_1 is closed and bounded and $\mathfrak{X}$ is finite-dimensional, B_1 is compact. Since B_1 is compact and $\|\cdot\|_2$ is continuous, the set $V := \{\|x\|_2 : x \in B_1\} \subseteq \mathbb{R}_+$ is compact. Since V is a compact subset of $\mathbb{R}_+$, it is bounded. So $\exists c_1 > 0$ such that $\forall x \in B_1$, $\|x\|_2 \leq c_1$. i.e., $\forall x \in \mathfrak{X}$, if $\|x\|_1 \leq 1$, then $\|x\|_2 \leq c_1$. So $\|x\|_2 \leq c_1\|x\|_1$. So $\|\cdot\|_2 \leq c_1\|\cdot\|_1$. Similarly, $\exists c_2 > 0$ such that $\|\cdot\|_1 \leq c_2\|\cdot\|_2$. So $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ■

Proof Approach 2. Let $\|\cdot\|_p$ be an arbitrary p-norm on V and $\|\cdot\|$ be an arbitrary norm on V. Let $\mathcal{B}$ be the standard basis for V. Say $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$. Let v be an arbitrary vector in V.

$$\begin{aligned}
\|v\| = \|\sum_{i=1}^{n} v_i e_i\| \leq \sum_{i=1}^{n} \|v_i e_i\| &= \sum_{i=1}^{n} |v_i|\|e_i\| \\
&\leq \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \\
&= \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \|v\|_p \\
&:= c_1\|v\|_p.
\end{aligned}$$

■

LEMMA 2.1 (Riesz's Lemma)**.** Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{Y}$ be a closed proper subspace of $\mathfrak{X}$. Let α be an element of the open interval $(0, 1)$. Then there is some $x_0 \in \mathfrak{X} \setminus \mathfrak{Y}$ such that $\|x_0\| = 1$ and $\forall y \in \mathfrak{Y}$, we have $\|x_0 - y\| \geq \alpha$.

Proof. Since $\mathfrak{Y} \subsetneq \mathfrak{X}$, $\mathfrak{X} \setminus \mathfrak{Y} \neq \emptyset$. Let x be an element of $\mathfrak{X} \setminus \mathfrak{Y}$. Let d denote the distance $d := \inf\{\|x - y\| : y \in \mathfrak{Y}\}$. Since $x \notin \mathfrak{Y}$ and $\mathfrak{Y}$ is closed, $d > 0$. Since $\alpha < 1$, $\alpha^{-1} > 1$. Since $d > 0$ and $\alpha^{-1} > 1$, we get $d\alpha^{-1} > d$. So $\exists y_0 \in \mathfrak{Y}$ such that $\|x - y_0\| \leq d\alpha^{-1}$. Define a point x_0 by $x_0 := \frac{x - y_0}{\|x - y_0\|}$. Since $y_0 \in \mathfrak{Y}$ and $x \in \mathfrak{X} \setminus \mathfrak{Y}$, we get $x_0 \in \mathfrak{X} \setminus \mathfrak{Y}$. It is clear that

$\|x_0\| = 1$. Let y be an arbitrary element of $\mathfrak{Y}$. Then

$$\begin{aligned}\|x_0 - y\| = \|\frac{x-y_0}{\|x-y_0\|} - y\| &= \frac{\|x-(y_0+\|x-y_0\|y)\|}{\|x-y_0\|} \\ &\geq \frac{d}{\|x-y_0\|}, \text{ since } y_0+\|x-y_0\|y \in \mathfrak{Y} \\ &\geq \frac{d}{d\alpha^{-1}}, \text{ since } \|x-y_0\| \leq d\alpha^{-1} \\ &= \alpha.\end{aligned}$$

That is, $\|x_0 - y\| \geq \alpha$. ■

THEOREM 2.3. A normed linear space is finite-dimensional if and only if its closed unit ball is norm-compact.

Proof. Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{X}_1$ denote the closed unit ball in $\mathfrak{X}$.

Forward Direction: Assume that $\mathfrak{X}$ is finite-dimensional. I will show that $\mathfrak{X}_1$ is norm-compact. Since $\mathfrak{X}$ is finite-dimensional, being compact is equivalent to being closed and bounded. Clearly $\mathfrak{X}_1$ is closed and bounded. So $\mathfrak{X}_1$ is compact.

Backward Direction: Assume that $\mathfrak{X}_1$ is norm-compact. I will show that $\mathfrak{X}$ is finite-dimensional. Assume for the sake of contradiction that $\mathfrak{X}$ is infinite-dimensional. Since $\dim(\mathfrak{X}) = \infty$, $\mathfrak{X} \neq \emptyset$. Let x_1 be an element of $\mathfrak{X}$ such that $\|x_1\| = 1$. Define a subspace $\mathfrak{Y}_1$ by $\mathfrak{Y}_1 := \operatorname{span}\{x_1\}$. Then $\mathfrak{Y}_1$ is a finite-dimensional subspace of $\mathfrak{X}$. Since $\dim(\mathfrak{Y}_1) < \infty$, $\mathfrak{Y}_1$ is closed. Since $\dim(\mathfrak{Y}_1) < \infty$ and $\dim(\mathfrak{X}) = \infty$, $\mathfrak{Y}_1 \subsetneq \mathfrak{X}$.. So $\mathfrak{Y}_1$ is a closed proper subspace of $\mathfrak{X}$. By the Riesz's Lemma, there is some $x_2 \in \mathfrak{X} \setminus \mathfrak{Y}_1$ such that $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq \frac{1}{2}$. Define a subspace $\mathfrak{Y}_2$ by $\mathfrak{Y}_2 := \operatorname{span}\{x_1, x_2\}$. Similarly, $\mathfrak{Y}_2$ is a closed proper subspace of $\mathfrak{X}$. By the Riesz's Lemma, there is some $x_3 \in \mathfrak{X} \setminus \mathfrak{Y}_2$ such that $\|x_3\| = 1$ and both $\|x_3 - x_1\| \geq \frac{1}{2}$ and $\|x_3 - x_2\| \geq \frac{1}{2}$. In general, define a subspace $\mathfrak{Y}_n$ by $\mathfrak{Y}_n := \operatorname{span}\{x_i\}_{i=1}^n$. Then $\mathfrak{Y}_n$ is a closed proper subspace of $\mathfrak{X}$. By the Riesz's Lemma, there is some $x_{n+1} \in \mathfrak{X} \setminus \mathfrak{Y}_n$ such that $\|x_{n+1}\| = 1$ and $\forall i = 1..n$, we have $\|x_{n+1} - x_i\| \geq \frac{1}{2}$. Define a sequence $x \in \mathfrak{X}^{\mathbb{N}}$ by $x := (x_n)_{n \in \mathbb{N}}$. Then by the construction of x, we have that $x \in \mathfrak{X}_1$ and that $\forall i, j \in \mathbb{N}$, $\|x_i - x_j\| \geq \frac{1}{2}$. Since $\forall i, j \in \mathbb{N}$, $\|x_i - x_j\| \geq \frac{1}{2}$, x contains no convergent subsequence. Since $x \in \mathfrak{X}_1$ and x contains no convergent subsequence, $\mathfrak{X}_1$ is not compact. This contradicts to the assumption that $\mathfrak{X}_1$ is compact. So $\mathfrak{X}$ is finite-dimensional. ■

2.5 Dual Norms

DEFINITION (Dual Norm)**.** Let $(V, \|\cdot\|)$ be an normed vector space. We define the **dual norm** of $\|\cdot\|$, denoted by $\|\cdot\|_\circ$, to be a function given by

$$\|v\|_\circ := \max_{\|w\|=1} v \cdot w = \max_{\|w\|\neq 0} \frac{|v \cdot w|}{\|w\|}.$$

PROPOSITION 2.5.1. Dual norms of norms are indeed norms.

PROPOSITION 2.5.2. Let $(V, \|\cdot\|)$ be a normed vector space. Let v, w be vectors in the space. Then

$$|v \cdot w| \leq \|v\| \cdot \|w\|_\circ.$$

PROPOSITION 2.5.3. Let p be an arbitrary number in $[1, +\infty)$. Then the dual norm of the p-norm $\|\cdot\|_p$ is the q-norm $\|\cdot\|_q$ where q is such that satisfies

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Chapter 3

Inner Product Space

3.1 Inner Products

3.1.1 Definitions

DEFINITION (Inner Product)**.** Let V be a vector space over field $\mathbb{F}$. We define an ***inner product*** on V, denoted by $\langle \cdot, \cdot \rangle$, to be a scalar-valued function defined on $V \times V$ such that

(1) Positive Definiteness:

$$\forall x \in V, \langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \iff x = 0.$$

(2) Sesqui-Linearity:

$$\begin{aligned} \forall x, y, z, w \in V, \quad & \langle x + y, z + w \rangle = \langle x, z \rangle + \langle y, z \rangle + \langle x, w \rangle + \langle y, w \rangle, \text{ and} \\ \forall a, b \in \mathbb{F}, \forall x, y \in V, \quad & \langle ax, by \rangle = a\bar{b} \langle x, y \rangle. \end{aligned}$$

(3) Conjugate Symmetry:

$$\forall x, y \in V, \quad \langle x, y \rangle = \overline{\langle y, x \rangle}.$$

DEFINITION (Induced Norm)**.** Let $\mathfrak{X}$ be an inner product space over field $\mathbb{K}$. We define the **norm induced by** $\langle \cdot, \cdot \rangle$, denoted by $\| \cdot \|$, to be a function from $\mathfrak{X}$ to $\mathbb{R}_+$ given by

$$\|x\| := \sqrt{\langle x, x \rangle}$$

3.1.2 Examples of Inner Products

DEFINITION (Standard Inner Product)**.** For $V = \mathbb{F}^n$, we define the **standard inner product** by

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i \overline{y_i}.$$

DEFINITION (Frobenius Inner Product)**.** For $V = \mathbb{F}^{n \times n}$, we define the **Frobenius inner product** by

$$\langle M_1, M_2 \rangle := \operatorname{tr}(M_2^* M_1).$$

DEFINITION. Let V be the space of continuous scalar-valued functions on $[0, 2\pi]$. We define the inner product on V by

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)} dx.$$

3.1.3 Properties

PROPOSITION 3.1.1. Let V be a finite dimensional inner product space. Let $\mathcal{B}$ be a basis for V. Let x and y be vectors in V. Then

$$x = y \iff \forall b \in \mathcal{B}, \quad \langle x, b \rangle = \langle y, b \rangle.$$

3.2 Inner Product Space

DEFINITION (Inner Product Space)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{X}$. We define an **inner product space** to be the pair $(\mathfrak{X}, \langle \cdot, \cdot \rangle)$.

3.3 Inequalities

THEOREM 3.1 (Minkowski)**.**

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}$$

PROPOSITION 3.3.1 (Cauchy-Schwarz Inequality)**.** Let V be an inner product space. Then

$$\forall x,y\in V, \quad |\langle x,y\rangle| \le \|x\|\cdot\|y\|$$

PROPOSITION 3.3.2 (Triangle Inequality)**.** Let V be an inner product space. Then

$$\forall x,y\in V, \quad \|x+y\| \le \|x\|+\|y\|$$

PROPOSITION 3.3.3 (Parallelogram Law)**.** Let $\mathfrak{X}$ be an inner product space. Then

$$\forall x,y\in\mathfrak{X}, \quad \|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2.$$

Proof.

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= \langle x+y,x+y\rangle+\langle x-y,x-y\rangle\\ &= \langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle\\ &\quad+\langle x,x\rangle-\langle x,y\rangle-\langle y,x\rangle+\langle y,y\rangle\\ &= 2\langle x,x\rangle+2\langle y,y\rangle\\ &= 2\|x\|^2+2\|y\|^2.\end{aligned}$$

That is,

$$\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2.$$

■

Chapter 4

Orthogonality

4.1 Orthogonal Sets

DEFINITION (Orthogonality)**.** Let V be an inner product space. We say that points x and y in V are **orthogonal** if $\langle x, y \rangle = 0$.

DEFINITION (Orthogonal Set)**.** Let $\mathfrak{X}$ be an inner product space. Let S be a subset of $\mathfrak{X}$. We say that S is **orthogonal** if

$$\forall x, y \in S : x \neq y, \quad \langle x, y \rangle = 0.$$

i.e., if any two distinct vectors are orthogonal.

DEFINITION (Orthonormal Set)**.** Let $\mathfrak{X}$ be an inner product space. Let S be a set in the space. We say that S is **orthonormal** if S is orthogonal and $\forall x \in S$, $\|x\| = 1$ where $\|\cdot\|$ is the norm induced by the inner product.

PROPOSITION 4.1.1. Orthogonal sets are linearly independent.

4.2 Orthogonal Bases

DEFINITION (Orthogonal Basis)**.** Let V be an inner product space. Let S be a set in the space. We say that S is an **orthogonal basis** for V if it is an ordered basis for V and orthogonal.

DEFINITION (Orthonormal Basis)**.** Let $\mathfrak{X}$ be an inner product space. Let S be a set in the space. We say that S is an **orthonormal basis** for $\mathfrak{X}$ if it is the maximal, with respect to inclusion, in the collection of all orthonormal sets in the space.

PROPOSITION 4.2.1. Let V be an inner product space. Let $S = \{v_1, ..., v_n\}$ be an orthogonal subset of V where each v_i is non-zero. Then

$$\forall y \in \operatorname{span}(S), \quad y = \sum_{i=1}^{n} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

THEOREM 4.1 (Gram-Schmidt Process)**.** Let V be an inner product space. Let $S = \{x_0, ..., x_n\}$ be a linearly independent subset of V. Then the set $T = \{y_0, ..., y_n\}$ given by $y_0 := x_0$ and

$$\forall i \in \{1, ..., n\}, \quad y_i := x_i - \sum_{j=1}^{i-1} \frac{\langle x_i, y_j \rangle}{\|y_j\|} y_j$$

is an orthogonal subset of V consisting of non-zero vectors such that $\operatorname{span}(S) = \operatorname{span}(S')$.

PROPOSITION 4.2.2. Let V be an inner product space and $S = \{v_0, v_1, \ldots, v_n\}$ be an orthogonal subset of V. Then the set $S^{'}$ derived from the Gram-Schmidt process is exactly S.

THEOREM 4.2 (Parseval's Identity)**.** Let V be a finite-dimensional inner product space. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthogonal basis for V. Then

$$\forall x, y \in V, \quad \langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.$$

PROPOSITION 4.2.3. Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. Then $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$ if and only if

$$\forall x \in \mathcal{H}, \quad x = \sum_{e \in \mathcal{E}} \langle x, e\rangle e.$$

4.3 Orthogonal Complements

DEFINITION (Orthogonal Complement)**.** Let $\mathfrak{X}$ be an inner product space. Let S be a non-empty subset of V. We define the **orthogonal complement** of S, denoted by $S^\perp$, to be a set given by

$$S^\perp := \{x \in \mathfrak{X} : \forall s \in S, \langle x, s\rangle = 0\}.$$

i.e., the set of all points in $\mathfrak{X}$ that are orthogonal to all vectors in S.

PROPOSITION 4.3.1. Let V be a finite-dimensional inner product space. Then

(1) $V^\perp = \{O_V\}$

(2) $\{O_V\}^\perp = V$

PROPOSITION 4.3.2. Orthogonal complements are always linear subspaces.

PROPOSITION 4.3.3. Let V be an inner product space and W be a subspace of V with basis β. Then a vector in V is also in $W^\perp$ if and only if it is orthogonal to all vectors in β.

PROPOSITION 4.3.4 (Extension)**.** Let V be an n-dimensional inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal subset of V. Then S can be extended to an orthogonal basis $B = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.

PROPOSITION 4.3.5. Let V be an inner product space. Then

(1) $S \subseteq T$ implies $T^{\perp} \subseteq S^{\perp}$ for any subsets S and T of V.

(2) $S \subseteq (S^{\perp})^{\perp}$ for any subset S of V.

PROPOSITION 4.3.6. Let V be a finite-dimensional inner product space and W be a subspace of V. Then

(1) $W = (W^{\perp})^{\perp}$

(2) $V = W \oplus W^{\perp}$

PROPOSITION 4.3.7. Let V be a finite-dimensional inner product space and W_1 and W_2 be subspaces of V. Then

(1) $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$

(2) $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$

4.4 Orthogonal Projection

DEFINITION (Orthogonal Projection)**.** Let V be a vector space. Let W be a finite-dimensional subspace of V. Let x be a vector in V. We define the **orthogonal projection** of x on W, denoted by (x), to be the vector u in W such that $x = u + v$ where v is another vector in $W^{\perp}$.

4.5 Inequalities in Hilbert Spaces

THEOREM 4.3 (Bessel's Inequality)**.** Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in the space. Then

$$\forall x \in \mathcal{H}, \quad \sum_{e \in \mathcal{E}} |\langle x, e_i \rangle|^2 \leq \|x\|^2.$$

PROPOSITION 4.5.1. Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. Let x be a point in the space. Then the net $\sum_{e \in \mathcal{E}} \langle x, e \rangle e$ converges in $\mathcal{H}$.

Proof. Let $\mathcal{F}$ be the collection of all finite subsets of $\mathcal{E}$, partially ordered by inclusion. Define for each $F \in \mathcal{F}$ a vector y_F as $y_F := \sum_{e \in F} \langle x, e \rangle e$. Let ε be an arbitrary positive number. Since $\mathcal{E}$ is an orthonormal set, the set $\{e \in \mathcal{E} : \langle x, e \rangle \neq 0\}$ is countable. Let $\{e_i\}_{i \in \mathbb{N}}$ denote the set. By the Bessel's inequality, $\exists N \in \mathbb{N}$ such that $\sum_{i=N+1}^{\infty} |\langle x, e_i \rangle|^2 < \varepsilon^2$. Define a set F_0 as $F_0 := \{e_1, ..., e_N\}$. Let F and G be arbitrary elements in $\mathcal{F}$ such that $F_0 \leq F$ and $F_0 \leq G$. Then

$$\begin{aligned}
\|y_F - y_G\|^2 &= \left\| \sum_{e \in F \setminus G} \langle x, e \rangle e - \sum_{e \in G \setminus F} \langle x, e \rangle e \right\|^2 \\
&= \sum_{e \in F \cup G \setminus F \cap G} |\langle x, e \rangle|^2 \\
&= \sum_{e \in F \cup G} |\langle x, e \rangle|^2 - \sum_{e \in F \cap G} |\langle x, e \rangle|^2 \\
&\leq \sum_{i=N+1}^{\infty} |\langle x, e_i \rangle|^2 \\
&< \varepsilon^2.
\end{aligned}$$

So $\{y_F\}_{F \in \mathcal{F}}$ is Cauchy. Since $\mathcal{H}$ is complete and $\{y_F\}_{F \in \mathcal{F}}$ is Cauchy, $\{y_F\}_{F \in \mathcal{F}}$ converges. ∎

Chapter 5

Sequence Space

5.1 p-norms

DEFINITION (p-norm)**.** Let V be a finite-dimensional normed vector space over field $\mathcal{F}$. Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis for V where $n = \dim(V)$. Let v be a vector in a normed vector space. For $p \in [1, +\infty)$, we define the **p-norm** of v, denoted by $\|v\|_p$, to be the number given by

$$\|v\|_p = \left(\sum_{i=1}^{n} |(v_{\mathcal{B}})_i|^p \right)^{\frac{1}{p}}.$$

DEFINITION (Infinity Norm - 1)**.** Let $\mathfrak{X} = \mathbb{K}^n$ where $\mathbb{K}$ is a field and $n \in \mathbb{N}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by

$$\|v\|_\infty := \max\{|v_i|\}_{i=1}^{n}.$$

DEFINITION (Infinity Norm - 2)**.** Let $\mathfrak{X} = \mathbb{K}^{\mathbb{N}}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by

$$\|v\|_\infty := \sup_{i \in \mathbb{N}} |v_i|.$$

DEFINITION (Infinity Norm - 3)**.** Let $\mathfrak{X} = \mathcal{C}([0,1],\mathbb{C})$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|.$$

PROPOSITION 5.1.1. Let $\mathfrak{X} := \mathcal{C}([0,1],\mathbb{C})$. Let x be an arbitrary number in $[0,1]$. Define a function ν_x on $\mathfrak{X}$ by $\nu_x(f) := |f(x)|$. Define a function ν on $\mathfrak{X}$ by $\nu(f) := \sup_{x\in[0,1]} |f(x)|$. Then ν_x is a seminorm on $\mathfrak{X}$ for each x and ν is a norm on $\mathfrak{X}$ and we have $\nu = \sup_{x\in[0,1]} \nu$.

PROPOSITION 5.1.2. p-norms are indeed norms.

PROPOSITION 5.1.3. For any vector v in $\mathbb{R}^n$, we have

$$\lim_{p\to\infty} \|v\|_p = \|v\|_\infty.$$

i.e.,

$$\lim_{p\to\infty} \left(\sum_{i=1}^n |v_i|^p\right)^{\frac{1}{p}} = \max\{|v_i|\}_{i=1}^n.$$

Proof. Let p be an arbitrary number in $[1,+\infty)$. Let k be an arbitrary index in $\{1,..,n\}$. Then

$$|v_k| \le (\sum_{i=1}^n |v_k|^p)^{1/p} = \|v\|_p.$$

So

$$\max\{|v_k|\} = \|v\|_\infty \le \|v\|_p.$$

So

$$\lim_{p\to\infty} \|v\|_p \ge \|v\|_\infty. \tag{1}$$

On the other hand, note that

$$(\sum_{i=1}^n |v_i|^p)/\|v\|_\infty^p = \sum_{i=1}^n (\frac{|v_i|}{\|v\|_\infty})^p$$

decreases as p increases. So it is bounded above. Say

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p \leq C$$

for some $C \in \mathbb{R}$. Then

$$(\sum_{i=1}^{n} |v_i|^p)^{1/p} = \|v\|_p \leq C^{1/p}\|v\|_\infty.$$

So

$$\lim_{p\to\infty} \|v\|_p \leq \lim_{p\to\infty} C^{1/p}\|v\|_\infty = \|v\|_\infty. \tag{2}$$

From (1) and (2) we get

$$\lim_{p\to\infty} \|v\|_p = \|v\|_\infty.$$

■

PROPOSITION 5.1.4. Let p and q be numbers in $[1, +\infty]$. Let v be a vector in $\mathbb{R}^n$. Then if $p \leq q$,

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \cdot \|x\|_q.$$

5.2 ℓ^p Space

DEFINITION (ℓ^p Space)**.** We define the ℓ^p space to be the set of all scalar sequences x such that $\|x\|_p$ is finite, equipped with the p-norm $\|\cdot\|_p$.

DEFINITION (Weighted ℓ^p Space)**.** Let $(r_i)_{i\in\mathbb{N}}$ be a sequence of positive integers. We define the **weighted ℓ^p** space to be the set given by

$$\ell^p := \{(x_i)_{i\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sum_{i\in\mathbb{N}} r_i|x_i|^p < +\infty.$$

PROPOSITION 5.2.1. For $p \in [1, +\infty)$, $(\ell^p, \|\cdot\|_p)$ is complete.

Proof.

Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in ℓ^p.

Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in ℓ^p, $\forall\varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $\forall m, n > N$, we have $\|x_m - x_n\|_p < \varepsilon$.

Since $\|x_m - x_n\|_p < \varepsilon$ and $|x_m^{(i)} - x_n^{(i)}| \leq \|x_m - x_n\|_p$ for any $i\in \mathbb{N}$, $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$ for any $i\in \mathbb{N}$.

Since for any $i\in \mathbb{N}$ and any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N$, we have $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$, by definition, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$.

Since $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$ and $\mathbb{F}$ is complete, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ converges.

Let $x_0^{(i)} = x_n^{(i)}$. Let $x_0 = \{x_0^{(i)}\}_{i\in\mathbb{N}}$.

$$\|x_0\|_p = (\sum_{i=1}^{\infty} |x_0^{(i)}|^p)^{\frac{1}{p}}$$

■

5.3 c_0 Space and c_{00} Space

DEFINITION (c_0 Space)**.** We define c_0 to be

$$c_0 := \big\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\big\}.$$

DEFINITION (c_{00} Space)**.** We define c_{00} to be

$$c_{00} := \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}, \forall n > N, x_n = 0\big\}.$$

i.e., the set of all eventually zero sequences of real numbers. i.e., the set of all sequences with finite support.

PROPOSITION 5.3.1. The c_{00} is not complete in $(\ell_1, \|\cdot\|_1)$.

Proof. Define a sequence of vectors $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ by $\mathfrak{x}_i^j := \frac{1}{j^2}$ for $j \in \{1..i\}$ and $\mathfrak{x}_i^j := 0$ for $j > i$. Then $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ converges to something that is not in c_{00}. ■

PROPOSITION 5.3.2. The closure of c_{00} in the space $(\mathbb{R}^\omega, d_1)$ is ℓ_1.

Proof. For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq \ell_1$. Let x be an arbitrary element in $\mathrm{cl}(c_{00})$. Since $x \in \mathrm{cl}(c_{00})$, there exists another element $y \in c_{00}$ such that $d_1(x, y) < 1$. Let $N \in \mathbb{N}$ be such that $\forall n > N$, $y_n = 0$. Then

$$
\begin{aligned}
& d_1(x,y) < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n - y_n| < 1 \\
\iff & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n - y_n| < 1 \\
\iff & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} \big||x_n| - |y_n|\big| + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} \big(|x_n| - |y_n|\big) + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} |x_n| - \sum_{n=1}^{N} |y_n| + \sum_{n>N} |x_n| < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n| - \sum_{n=1}^{N} |y_n| < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n| < 1 + \sum_{n=1}^{N} |y_n|.
\end{aligned}
$$

Since $\sum_{n\in\mathbb{N}} |x_n|$ is bounded, $x \in \ell_1$.

For the reverse direction, we are to prove that $\ell_1 \subseteq \mathrm{cl}(c_{00})$. Let x be an arbitrary element in ℓ_1. For $i \in \mathbb{N}$, define $x^i = \{x^i_j\}_{j\in\mathbb{N}}$ as $x^i_j = x_j$ for $j \leq i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}, x^i \in c_{00}$. Then

$$
\begin{aligned}
& \lim_{i\in\mathbb{N}} d_1(x^i, x) \\
= & \lim_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |x^i_j - x_j| \\
= & \lim_{i\in\mathbb{N}} \sum_{j>i} |x^i_j - x_j| \\
= & \lim_{i\in\mathbb{N}} \sum_{j>i} |x_j|
\end{aligned}
$$

$$= 0.$$

That is, $\lim_{i\in\mathbb{N}} d_1(x^i, x) = 0$. So $\lim_{i\in\mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

∎

PROPOSITION 5.3.3. The closure of c_{00} in the space $(\mathbb{R}^\omega, d_\infty)$ is c_0.

Proof. For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq c_0$. Let x be an arbitrary element in $\mathrm{cl}(c_{00})$. Let ε be an arbitrary positive real number. Since $x \in \mathrm{cl}(c_{00})$, there exists another element y in c_{00} such that $d_\infty(x, y) < \varepsilon$. That is, $\forall j \in \mathbb{N}, |x_j - y_j| < \varepsilon$. Since $y \in c_{00}$, $\exists N \in \mathbb{N}$ such that $\forall j > N$, $y_j = 0$. So $\forall j > N, |x_j| < \varepsilon$. That is,

$$\forall \varepsilon > 0, \quad , \exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon.$$

By definition of convergence of limits, $\lim_{j\in\mathbb{N}} x_j = 0$. So $x \in c_0$.

For the reverse direction, we are to prove that $c_0 \subseteq \mathrm{cl}(c_{00})$. Let x be an arbitrary element in c_0. For $i \in \mathbb{N}$, define x^i as $x^i_j = x_j$ for $j \leq i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}$, $x^i \in c_{00}$. Let ε be an arbitrary positive real number. Since $x \in c_0$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon/2.$$

Let $i > N$. Then

$$\begin{aligned} & d_\infty(x^i, x) \\ = & \sup_{j\in\mathbb{N}} |x^i_j - x_j| \\ = & \sup_{j>i} |x^i_j - x_j| \\ = & \sup_{j>i} |x_j| \\ \leq & \ \varepsilon/2 < \varepsilon. \end{aligned}$$

That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall i > N, \quad d_\infty(x^i, x) < \varepsilon.$$

By definition of convergence of sequences, $\lim_{i\in\mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

∎

PROPOSITION 5.3.4. Let $A := \left\{\{x_n\}_{n\in\mathbb{N}} \in c_{00} : \sum_{n\in\mathbb{N}} x_n = 0\right\}$. Then A is a subset of ℓ^1 and is closed in (ℓ^1, d_1). i.e. $\mathrm{cl}(A) = A$ in (ℓ^1, d_1).

Proof. Let $x = \{x^i\}_{i\in\mathbb{N}}$ be a sequence in ℓ^1, where each $x^i = \{x^i_j\}_{j\in\mathbb{N}}$ is an element in A, that converges in (ℓ^1, d_1). Say $\lim_{i\to\infty} x^i = x^\infty$.

First I claim that $x^\infty \in c_{00}$.

Now I claim that $\sum_{j\in\mathbb{N}} x^\infty_j = 0$. i.e. $x^\infty \in A$. Since $x^\infty \in c_{00}$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad x^\infty_j = 0.$$

Define $y_i := \sum_{j=1}^N x^i_j$. Define $y_\infty := \sum_{j=1}^N x^\infty_j$. It is easy to see that $\lim_{i\in\mathbb{N}} y_i = y_\infty$. Assume for the sake of contradiction that $y_\infty \neq 0$. i.e. $\{y_i\}_{i\in\mathbb{N}}$ does not converge to 0. Then

$$\exists \varepsilon_0 > 0, \quad \forall M \in \mathbb{N}, \quad \exists i_0 > M, \quad |y_{i_0} - 0| = |y_{i_0}| \geq \varepsilon_0. \tag{1}$$

Since $\lim_{i\to\infty} x^i = x^\infty$,

$$\exists M_0 \in \mathbb{N}, \quad \forall i > M_0, \quad d_1(x^i, x^\infty) < \varepsilon_0. \tag{2}$$

Consider statement (1) for a particular M, M_0, we have

$$\exists i_0 > M_0, \quad |y_{i_0}| \geq \varepsilon_0. \tag{3}$$

That is,

$$|\sum_{j=1}^N x^{i_0}_j| \geq \varepsilon_0. \tag{3'}$$

Consider statement (2) for a particular i, i_0, we have

$$d_1(x^{i_0}, x^\infty) < \varepsilon_0. \tag{4}$$

From statement (4) we can derive:

$$\begin{aligned}
& d_1(x^{i_0}, x^\infty) < \varepsilon_0 \\
\iff & \sum_{j\in\mathbb{N}} |x^{i_0}_j - x^\infty_j| < \varepsilon_0 \\
\iff & \sum_{j=1}^N |x^{i_0}_j - x^\infty_j| + \sum_{j>N} |x^{i_0}_j - x^\infty_j| < \varepsilon_0 \\
\implies & \sum_{j>N} |x^{i_0}_j - x^\infty_j| < \varepsilon_0 \\
\iff & \sum_{j>N} |x^{i_0}_j - 0| < \varepsilon_0 \\
\iff & \sum_{j>N} |x^{i_0}_j| < \varepsilon_0 \\
\implies & |\sum_{j>N} x^{i_0}_j| < \varepsilon_0
\end{aligned}$$

$$\begin{aligned}
&\implies \Big|\sum_{j>N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\iff \Big|\sum_{j\in\mathbb{N}} x_j^{i_0} - \sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\iff \Big|0 - \sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\iff \Big|\sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0.
\end{aligned}$$

This contradicts to statement (3'). So the original assumption that $y_\infty \neq 0$ is false. i.e. $y_\infty = 0$. It follows that $\sum_{j\in\mathbb{N}} x_j^\infty = 0$. This completes the proof.

■

5.4 Hölder's Inequality

THEOREM 5.1 (Hölder's Inequality)**.** Let $\mathfrak{X} = \mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ be vectors in $\mathfrak{X}$. Then $\forall p, q \in (1, +\infty) : 1/p + 1/q = 1$, $\|xy\|_1 \leq \|x\|_p\|y\|_q$. i.e.,

$$\sum_{i=1}^n |x_i y_i| \leq (\sum_{i=1}^n |x_i|^p)^{1/p} (\sum_{i=1}^n |y_i|^q)^{1/q}.$$

Chapter 6

Function Spaces

6.1 The $\mathcal{L}^p$ Norm

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

the instructors' answer, *where instructors collectively construct a single answer*

In the sup norm, convergence coincides with uniform convergence. Moreover, $C[a,b]$ is complete in this norm. It is not complete in any of the L^p norms for $1 \le p < \infty$. The completion in these norms is called $L^p(a,b)$.

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Updated 1 day ago by Kenneth Davidson

Chapter 7

Quotient Space

7.1 Definitions

DEFINITION (Quotient Space)**.** Let $\mathcal{V}$ be a vector space. Let $\mathcal{W}$ be a subspace of $\mathcal{V}$. We define a **quotient space** of $\mathcal{V}$ mod $\mathcal{W}$, denoted by $\mathcal{V}/\mathcal{W}$, to be a set given by $\{v + \mathcal{W} : v \in \mathcal{V}\}$ with addition and scalar multiplication defined by

$$(v_1 + \mathcal{W}) + (v_2 + \mathcal{W}) := (v_1 + v_2) + \mathcal{W} \text{ and}$$

$$\kappa(v + \mathcal{W}) := (\kappa v) + \mathcal{W}.$$

DEFINITION (The Canonical Quotient Map)**.** Let $\mathfrak{X}$ be a vector space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define the **canonical quotient map** on $\mathfrak{X}$ with respect to $\mathfrak{M}$, denoted by $q_{\mathfrak{M}}$, to be a function from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$ given by

$$q_{\mathfrak{M}}(x) := [x] = x + \mathfrak{M}$$

7.2 Quotient Spaces with Seminorms

DEFINITION (Seminorm on Quotient Spaces)**.** Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let $\mathfrak{M}$ be a linear manifold of $\mathfrak{X}$. We define a **seminorm** on $\mathfrak{X}/\mathfrak{M}$ to be a function from $\mathfrak{X}/\mathfrak{M}$ to $\mathbb{R}$ given by

$$p(x+\mathfrak{M}) := \inf\{\|x+m\| : m \in \mathfrak{M}\}.$$

PROPOSITION 7.2.1. Seminorms on quotient spaces are indeed seminorms.

PROPOSITION 7.2.2. A seminorm on a quotient space $\mathfrak{X}/\mathfrak{M}$ is a norm if and only if $\mathfrak{M}$ is closed.

PROPOSITION 7.2.3 (Quotient maps are contractive)**.** Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a closed linear subspace of $\mathfrak{X}$. Then

$$\forall x \in \mathfrak{X}, \quad \|x+\mathfrak{M}\|_{\mathfrak{X}/\mathfrak{M}} \le \|x\|_{\mathfrak{X}}.$$

PROPOSITION 7.2.4. Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a closed linear subspace of $\mathfrak{X}$. Let q denote the canonical quotient map from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$. Then q is a continuous under the norm topology.

Proof. Since q is contractive, q is continuous. ■

7.3 Quotient Spaces with Topologies

DEFINITION (Quotient Toplogy)**.** Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. Let $\mathcal{W}$ be a closed subspace of $\mathcal{V}$. We define the **quotient topology** $\mathcal{T}_q$ on the quotient space $\mathcal{V}/\mathcal{W}$ as

$$\mathcal{T}_q := \left\{G \subseteq \mathcal{V}/\mathcal{W} : q^{-1}(G) \in \mathcal{T}\right\}.$$

PROPOSITION 7.3.1. The canonical quotient map is a continuous map.

PROPOSITION 7.3.2. The canonical quotient map is an open map.

Chapter 8

Dual Space

8.1 Definitions

DEFINITION (Linear Functional)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{K}$. We define a **linear functional** on $\mathfrak{X}$ to be a linear map from $\mathfrak{X}$ to $\mathbb{K}$.

DEFINITION (Algebraic Dual)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{K}$. We define the **algebraic dual** of $\mathfrak{X}$, denoted by $\mathfrak{X}^{\#}$. to be the vector space of all linear functionals on $\mathfrak{X}$.

DEFINITION (Topological Dual)**.** Let $\mathfrak{X}$ be a topological vector space over field $\mathbb{K}$. We define the **topological dual** of $\mathfrak{X}$, denoted by $\mathfrak{X}^*$, to be the vector space of all continuous linear functionals on $\mathfrak{X}$.

PROPOSITION 8.1.1. Let $\mathfrak{X}$ be a normed linear space. Then there exists a contractive map from $\mathfrak{X}$ to its double dual $\mathfrak{X}^{**}$.

8.2 Examples

EXAMPLE 8.2.1. $(c_0(\mathbb{N}))^*$ is isometrically isomorphic to $\ell^1(\mathbb{N})$.

EXAMPLE 8.2.2. $(\ell^1(\mathbb{N}))^*$ is isometrically isomorphic to $\ell^\infty(\mathbb{N})$.

8.3 Linear Functionals

PROPOSITION 8.3.1. Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let $g, f_1..f_n \in \mathcal{V}^\#$ where $n \in \mathbb{N}$. Then $g \in \text{span}\{f_i\}_{i=1}^n$ if and only if $\bigcap_{i=1}^n \ker(f_i) \subseteq \ker(g)$.

Proof. **Forward Direction**: Assume that $g \in \text{span}\{f_i\}_{i=1}^n$. I will show that $\bigcap_{i=1}^n \ker(f_i) \subseteq \ker(g)$. Since $g \in \text{span}\{f_i\}_{i=1}^n$, $\exists \lambda_1, ..., \lambda_n \in \mathbb{K}$ such that $g = \sum_{i=1}^n \lambda_i f_i$. Let x be an arbitrary element of $\bigcap_{i=1}^n \ker(f_i)$. Then

$$g(x) = \sum_{i=1}^n \lambda_i f_i(x) = \sum_{i=1}^n \lambda_i \cdot 0 = 0.$$

That is, $g(x) = 0$. So $x \in \ker(g)$. So $\bigcap_{i=1}^n \ker(f_i) \subseteq \ker(g)$.

Backward Direction: Assume that $\bigcap_{i=1}^n \ker(f_i) \subseteq \ker(g)$. I will show that $g \in \text{span}\{f_i\}_{i=1}^n$. Assume without loss of generality that $\{f_i\}_{i=1}^n$ are linearly independent. Define a set $\mathcal{N}$ by $\mathcal{N} := \bigcap_{i=1}^n \ker(f_i)$. Then $\dim(\mathcal{V}/\mathcal{N}) \leq n$. It follows that $\dim(\mathcal{V}/\mathcal{N})^\# \leq n$. Define for each $i = 1..n$ a function $F_i : \mathcal{V}/\mathcal{N} \to \mathbb{K}$ by $F_i(x + \mathcal{N}) := f_i(x)$. Then clearly each F_i is linear and hence $F_i \in (\mathcal{V}/\mathcal{N})^\#$. Since $f_1, ..., f_n$ are linearly independent, $F_1, ..., F_n$ are linearly independent. Since $(\mathcal{V}/\mathcal{N})^\#$ contains a linearly independent set of size n, $\dim(\mathcal{V}/\mathcal{N})^\# \geq n$. So $\dim(\mathcal{V}/\mathcal{N})^\# = n$. So $\{F_i\}_{i=1}^n$ is a basis for $(\mathcal{V}/\mathcal{N})^\#$. Define a function $G : \mathcal{V}/\mathcal{N} \to \mathbb{K}$ by $G(x+\mathcal{N}) := g(x)$. Then clearly, G is linear and hence $G \in (\mathcal{V}/\mathcal{N})^\#$. Since $G \in (\mathcal{V}/\mathcal{N})^\#$ and $\{F_i\}_{i=1}^n$ is a basis for $(\mathcal{V}/\mathcal{N})^\#$, $\exists \lambda_1, ..., \lambda_n \in \mathbb{K}$ such that $G = \sum_{i=1}^n \lambda_i F_i$. Then $\forall x \in \mathcal{V}$, we have

$$g(x) = G(x + \mathcal{N}) = \sum_{i=1}^n \lambda_i F_i(x + \mathcal{N}) = \sum_{i=1}^n \lambda_i f_i(x).$$

So $g = \sum_{i=1}^n \lambda_i f_i$ and hence $g \in \text{span}\{f_i\}_{i=1}^n$.

∎

PROPOSITION 8.3.2. Let $\mathcal{V}$ be a topological vector space over field $\mathbb{K}$. Let $\rho \in \mathcal{V}^{\#}$. Then $\rho \in \mathcal{V}^*$ if and only if $\ker(\rho)$ is a closed set.

Proof. **Forward Direction**: Assume that $\rho \in \mathcal{V}^*$. I will show that $\ker(\rho)$ is closed. Notice $\{0\}$ is closed in $\mathbb{K}$. Since $\rho \in \mathbb{V}^*$, ρ is continuous. So $\rho^{-1}(\{0\})$ is closed. Note that $\rho^{-1}(\{0\}) = \ker(\rho)$. So $\ker(\rho)$ is closed.

Backward Direction: Assume that $\ker(\rho)$ is a closed set. I will show that $\rho \in \mathcal{V}^*$. If $\rho = 0$, then we are done. Otherwise, assume that $\rho \neq 0$. Define a map $\varphi : \mathcal{V}/\ker(\rho) \to \mathbb{K}$ by $\varphi(x + \ker(\rho)) := \rho(x)$. Then clearly φ is linear. Since $\dim(\mathcal{V}/\ker(\rho)) = 1$ and $\dim(\mathbb{K}) = 1$, φ is continuous. Let q denote the canonical quotient map from $\mathcal{V}$ to $\mathcal{V}/\ker(\rho)$. Then q is continuous. Note that $\rho = \varphi \circ q$. So ρ is continuous.

■

8.4 The Dual Space Operator

PROPOSITION 8.4.1. Let $\mathcal{V}$ be a vector space. Suppose that $\mathcal{V}^*$ is separable. Then $\mathcal{V}$ is also separable.

REMARK. Note that $\ell_1(\mathbb{N})$ is separable but its dual $\ell^\infty(\mathbb{N})$ is not. So the converse of the above is false.

8.5 Annihilator

DEFINITION (Annihilator)**.** Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a subset of $\mathfrak{X}$. We define the **annihilator** of $\mathfrak{M}$, denoted by $\mathfrak{M}^0$, to be the subset of $\mathfrak{X}^*$ given by $\mathfrak{M}^0 := \{x^* \in \mathfrak{X}^* : x^*|_{\mathfrak{M}} = 0\}$.

DEFINITION (Pre-Annihilator)**.** Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{N}$ be a subset of $\mathfrak{X}^*$. We define the **pre-annihilator** of $\mathfrak{N}$, denoted by ${}^0\mathfrak{N}$, to be the subset of $\mathfrak{X}$ given by ${}^0\mathfrak{N} := \{x \in \mathfrak{X} : \hat{x}|_{\mathfrak{N}} = 0\}$.

PROPOSITION 8.5.1. The annihilator operator does not distinguish a set from its closure.

PROPOSITION 8.5.2. Annihilators are weakly closed.

PROPOSITION 8.5.3. Pre-annihilators are normed closed.

PROPOSITION 8.5.4. Let $\mathfrak{X}$ be a Banach space. Let $\mathfrak{M}$ be a closed subspace of $\mathfrak{X}$. Let q denote the canonical quotient map. Define a map Θ from $(\mathfrak{X}/\mathfrak{M})^*$ to $\mathfrak{M}^0$ by $\Theta(\xi) := \xi \circ q$. Then Θ is an isometric isomorphism.

PROPOSITION 8.5.5. Let $\mathfrak{X}$ be a Banach space. Let $\mathfrak{M}$ be a closed subspace of $\mathfrak{X}$. Define a map Θ from $\mathfrak{X}^*/\mathfrak{M}^0$ to $\mathfrak{M}^*$ by $\Theta(x^* + \mathfrak{M}^0) := x^*|_{\mathfrak{M}}$.

Chapter 9

Banach Space

9.1 Definition

DEFINITION (Banach Space)**.** Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let d be the metric induced by $\|\cdot\|$. We say that $\mathfrak{X}$ is a **Banach space** if $(\mathfrak{X}, d)$ is a complete metric space. i.e., we define a Banach space to be a complete normed linear space.

9.2 Examples

EXAMPLE 9.2.1. $(\mathcal{C}([0,1], \mathbb{F}), \|\cdot\|_\infty)$ is a Banach space.

EXAMPLE 9.2.2 (Disc Algebra)**.** Define $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Define $\mathcal{A}(\mathbb{D}) := \{f \in \mathcal{C}(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is holomorphic }\}$. Define $\|\cdot\|_\infty$ by $\|f\|_\infty := \sup_{z\in\overline{\mathbb{D}}} |f(z)|$. Then $(\mathcal{A}(\mathbb{D}), \|\cdot\|_\infty)$ is a Banach space.

EXAMPLE 9.2.3. Let (X, Ω, μ) be a measure space. Let p be a number in $[1, +\infty)$. Define

$$\mathcal{L}^p(X,\mu) := \text{span}\{f : X \to [0,+\infty] \mid f \text{ is measurable and } \int_X |f|^p < +\infty\}.$$

Define an equivalence relation on $\mathcal{L}^p(X,\mu)$ by $f \equiv g$ if and only if

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

Define a space $L^p(X,\mu) := \mathcal{L}^p(X,\mu)/\equiv$. Then $L^p(X,\mu)$ is a Banach space when equipped with the norm

$$\|[f]\|_p := \left(\int_X |f|^p\right)^{1/p}.$$

EXAMPLE 9.2.4. Let $\mathcal{P}_{\mathbb{C}}[0,1]$ denote the set of all polynomials with complex coefficients. For each $p \in [1,+\infty)$, define a norm

$$\|f\|_p := \left(\int_0^1 |f|^p\right)^{1/p}.$$

For $p = +\infty$, define a norm

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(z)|.$$

9.3 Properties

PROPOSITION 9.3.1. Let $(\mathfrak{X}, \|\cdot\|)$ be a normed vector space over field $\mathbb{F}$. Then $(\mathfrak{X}, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in $\mathfrak{X}$ is summable.

Proof. **Forward Direction**: Assume that $\mathfrak{X}$ is a Banach space. I will show that any absolutely summable series in $\mathfrak{X}$ is summable. Let $\sum_{n\in\mathbb{N}} x_n$ be an absolutely summable series. i.e., $\sum_{n\in\mathbb{N}} \|x_n\| < +\infty$. Define for each $n \in \mathbb{N}$ a vector y_n as $y_n := \sum_{i=1}^n x_i$. Let $\varepsilon > 0$ be arbitrary. Then $\exists N \in \mathbb{N}$ such that $\forall n > N$, $\sum_{i=n}^\infty \|x_i\| < \varepsilon$. Let $n > m > N$ be arbitrary. Then

$$\begin{aligned}
\|y_n - y_m\| &= \|\sum_{i=1}^n x_i - \sum_{i=1}^m x_i\| = \|\sum_{i=m+1}^n x_i\| \\
&\leq \sum_{i=m+1}^n \|x_i\| < \sum_{i=m+1}^\infty \|x_i\| \\
&< \varepsilon.
\end{aligned}$$

That is, $\|y_n - y_m\| < \varepsilon$. So $(y_n)_{n\in\mathbb{N}}$ is Cauchy. Since $\mathfrak{X}$ is a Banach space and $(y_n)_{n\in\mathbb{N}}$ is Cauchy, it converges. So $\sum_{n\in\mathbb{N}} x_n$ is summable.

Backward Direction: Assume that every absolutely summable series in $\mathfrak{X}$ is summable. I will show that $\mathfrak{X}$ is a Banach space. Let $(y_n)_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\mathfrak{X}$. Then $\forall n \in \mathbb{N}$, $\exists N_n \in \mathbb{N}$ such that $\forall k, l \geq N_n$, $\|y_k - y_l\| < \frac{1}{2^n}$. Assume that $N_1 < N_2 < \ldots$. Define $x_1 := y_{N_1}$. Define for each $n \in \mathbb{N}$ a vector x_{n+1} as $x_{n+1} := y_{N_{n+1}} - y_{N_n}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \|x_n\| &= \|x_1\| + \sum_{n=1}^{\infty} \|y_{N_{n+1}} - y_{N_n}\| < \|x_1\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \|x_1\| + 1 < +\infty.\end{aligned}$$

So $\sum_{n\in\mathbb{N}} x_n$ is absolutely summable. By assumption, it is summable. i.e., $(y_n)_{n\in\mathbb{N}}$ converges. Since any Cauchy sequence in $\mathfrak{X}$ converges, $\mathfrak{X}$ is complete and hence a Banach space.

■

PROPOSITION 9.3.2 (Quotient Spaces of Banach Spaces are Banach Spaces)**.** Let $\mathfrak{X}$ be a Banach space. Let $\mathcal{M}$ be a closed subspace of $\mathcal{M}$. Then the quotient space $\mathfrak{X}/\mathcal{M}$ is also a Banach space.

Proof. **Proof Approach 1**.

Let $(q(x_n))_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\mathfrak{X}/\mathcal{M}$. We are to prove that it converges.

■

Proof. **Proof Approach 2**.

Let q denote the canonical quotient map. Let $\sum_{n\in\mathbb{N}} q(x_n)$ be an arbitrary absolutely summable series in $\mathcal{X}/\mathcal{M}$. Since $\|q(x_n)\|$ is defined to be $\|q(x_n)\| := \inf\{\|x_n + m\| : m \in \mathbb{M}\}$, $\exists m_n \in \mathcal{M}$ such that $\|x_n + m_n\| < \|q(x_n)\| + \frac{1}{2^n}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \|x_n + m_n\| = \sum_{n=1}^{\infty} \left[\|q(x_n)\| + \frac{1}{2^n}\right] &= \sum_{n-1}^{\infty} \|q(x_n)\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ = \sum_{n=1}^{\infty} \|q(x_n)\| + 1 < +\infty.\end{aligned}$$

So $\sum_{n\in\mathbb{N}}(x_n + m_n)$ is absolutely summable. Since $\mathfrak{X}$ is a Banach space, $\sum_{n\in\mathbb{N}}(x_n + m_n)$ is summable. Say $\sum_{n\in\mathbb{N}}(x_n + m_n) = x_\bullet$. Then

$$\sum_{n=1}^{\infty} q(x_n) = \sum_{n=1}^{\infty} q(x_n + m_n) = \lim_{N\to\infty} \sum_{n=1}^{N} q(x_n + m_n) = \lim_{N\to\infty} q(\sum_{n=1}^{N}(x_n + m_n))$$

$$= q(\lim_{N\to\infty}\sum_{n=1}^{N}(x_n + m_n)) = q(x_\bullet).$$

So $\sum_{n\in\mathbb{N}} q(x_n)$ is summable. Since any absolutely summable series in $\mathfrak{X}/\mathcal{M}$ is summable, $\mathcal{X}/\mathcal{M}$ is complete. ■

PROPOSITION 9.3.3. Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a closed subspace of $\mathfrak{X}$. If $\mathcal{M}$ and $\mathfrak{X}/\mathcal{M}$ are both complete, then $\mathfrak{X}$ is a Banach space.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\mathfrak{X}$. We are to prove that it converges. Let q denote the canonical quotient map. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathfrak{X}$, $(q(x_n))_{n\in\mathbb{N}}$ is Cauchy in $\mathfrak{X}/\mathcal{M}$. Since $\mathfrak{X}/\mathcal{M}$ is a Banach space and $(q(x_n))_{n\in\mathbb{N}}$ is Cauchy, $(q(x_n))_{n\in\mathbb{N}}$ converges. Say $\lim_{n\in\mathbb{N}} q(x_n) = q(x_\bullet)$ for some $x_\bullet \in \mathfrak{X}$. By definition of norms in the quotient space, for $n \in \mathbb{N}$, we can choose $m_n \in \mathcal{M}$ such that $\|x_\bullet - x_n - m_n\| \le \|q(x_\bullet) - q(x_n)\| + \frac{1}{n}$. So

$$\lim_{n\in\mathbb{N}} \|x_\bullet - x_n - m_n\| \le \lim_{n\in\mathbb{N}} \|q(x_\bullet) - q(x_n)\| + \lim_{n\in\mathbb{N}} \frac{1}{n} = 0 + 0 = 0.$$

So $(x_n + m_n)_{n\in\mathbb{N}}$ converges to $x_\bullet$. So $(x_n + m_n)_{n\in\mathbb{N}}$ is Cauchy. Since $(x_n)_{n\in\mathbb{N}}$ and $(x_n + m_n)_{n\in\mathbb{N}}$ are both Cauchy, $(m_n)_{n\in\mathbb{N}}$ is Cauchy. Since $\mathcal{M}$ is a Banach space and $(m_n)_{n\in\mathbb{N}}$ is Cauchy, $(m_n)_{n\in\mathbb{N}}$ converges. Say $\lim_{n\in\mathbb{N}} m_N = m_\bullet$. So

$$\begin{aligned}\lim_{n\in\mathbb{N}} x_n &= \lim_{n\in\mathbb{N}}((x_n + m_n) - m_n) = \lim_{n\in\mathbb{N}}(x_n + m_n) - \lim_{n\in\mathbb{N}} m_n \\ &= x_\bullet - m_\bullet.\end{aligned}$$

So $(x_n)_{n\in\mathbb{N}}$ converges. Since any Cauchy sequence in $\mathfrak{X}$ converges, $\mathfrak{X}$ is a Banach space. ■

PROPOSITION 9.3.4 (Dual Spaces of Banach Spaces are Banach Spaces)**.** Let $\mathfrak{X}$ be a Banach space. Then the dual space $\mathfrak{X}^*$ is a also a Banach space.

PROPOSITION 9.3.5. Any Banach space with a Schauder basis has to be separable.

9.4 Direct Sums and Direct Products of Banach Spaces

DEFINITION. Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be two Banach spaces over field $\mathbb{K}$. Let $p \in [1, +\infty)$. We define

$$\mathfrak{X} \oplus_p \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x, y)\|_p := (\|x\|_{\mathfrak{X}}^p + \|y\|_{\mathfrak{Y}}^p)^{1/p}.$$

For $p = +\infty$, we define

$$\mathfrak{X} \oplus_\infty \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x, y)\|_\infty := \max(\|x\|_{\mathfrak{X}}, \|y\|_{\mathfrak{Y}}).$$

- Note that the norms behave similarly to what the p norm would do.
- We can similarly define the direct sum of finitely many Banach spaces.

PROPOSITION 9.4.1. $\|\cdot,\cdot\|_p$ is a norm on $\mathfrak{X} \oplus_p \mathfrak{Y}$.

PROPOSITION 9.4.2. $\mathfrak{X} \oplus_p \mathfrak{Y}$ is complete with respect to $\|\cdot,\cdot\|_p$.

9.5 Unconditional Convergence in Banach Spaces

DEFINITION (Unconditional Convergence)**.** Let $\mathfrak{X}$ be a Banach space. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a set of vectors in $\mathfrak{X}$. Let $\mathcal{F}$ be the collection of all finite subsets of Λ, partially ordered by inclusion. Define a net $(y_F)_{F\in\mathcal{F}}$ on $\mathcal{F}$ by $y_F := \sum_{\lambda\in F} x_\lambda$. We say that the series $\sum_{\lambda\in\Lambda} x_\lambda$ is **unconditional convergent** if the net $(y_F)_{F\in\mathcal{F}}$ converges.

PROPOSITION 9.5.1 (Equivalent Formulations of Unconditional Convergence)**.** Let $\mathfrak{X}$ be a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathfrak{X}$. Then the following conditions are equivalent.

(1) For any permutation π of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges.

(2) For any subsequence indexing $(k_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} x_{k_n}$ converges.

(3) $\forall\varepsilon > 0$, $\exists\mu \in \mathbb{N}$ such that for all finite subsets F of $\mathbb{N} \setminus \{1..\mu\}$, we have $\|\sum_{n\in F} x_n\| < \varepsilon$.

(4) $\exists y \in \mathfrak{X}$ such that $\forall\varepsilon > 0$, there is a finite subset F_0 of $\mathbb{N}$ such that for all finite F such that $F_0 \subseteq F \subseteq \mathbb{N}$, we have $\|\sum_{n\in F} x_n - y\| < \varepsilon$.

(5) For any sequence $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \alpha_n x_n$ converges.

(6) For any bounded sequence $(\beta_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \beta_n x_n$ converges.

Proof. **Proof of (1) $\implies$ (5)**.

Assume that for any permutation π of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges. We are to prove that for any sequence $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \alpha_n x_n$ converges. Assume for the sake of contradiction that there is some $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \alpha_n x_n$ diverges. i.e., $\exists\varepsilon_0 > 0$ such that $\forall N \in \mathbb{N}$, $\exists k_N > l_N > N$ such that

$$\|\sum_{n=k_N}^{l_N} \alpha_n x_n\| \geq \varepsilon_0. \tag{*}$$

For $N = 1$, find k_1 and l_1. For $N = l_1$, find k_2 and l_2. In general, for $N = l_n$, find k_{n+1} and l_{n+1}. Then we have $k_1 < l_1 < k_2 < l_2 < ....$ For each n, there is an $m_n \in [k_n, l_n]$ and a permutation π_n of $[k_n, l_n]$ such that $\pi_n(i) \in [k_n, m_n]$ if $\alpha_i = 1$ and $\pi_n(i) \in (m_n, l_n]$ if $\alpha_i = -1$. Define a permutation π of $\mathbb{N}$ as $\pi(i) := i$ if $\forall n \in \mathbb{N}$, $i \notin [k_n, l_n]$; and $\pi(i) := \pi_n(i)$ if $i \in [k_n, l_n]$. By assumption, for π, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges. So for ε_0, $\exists N \in \mathbb{N}$ such that $\forall j > i > N$, $\|\sum_{n=i}^{j} x_n\| < \varepsilon_0/2$. So

$$\begin{aligned}
\|\sum_{n=k_N}^{l_N} \alpha_n x_n\| &= \|\sum_{n=k_N}^{m_N} \alpha_n x_n + \sum_{n=m_N+1}^{l_N} \alpha_n x_n\| \\
&= \|\sum_{n=k_N}^{m_N} x_n - \sum_{n=m_N+1}^{l_N} x_n\| \\
&\leq \|\sum_{n=k_N}^{m_N} x_n\| + \|\sum_{n=m_N+1}^{l_N} x_n\| \\
&< \varepsilon_0/2 + \varepsilon_0/2 = \varepsilon_0.
\end{aligned}$$

That is,

$$\|\sum_{n=k_N}^{l_N} \alpha_n x_n\| < \varepsilon_0. \tag{**}$$

Notice (*) and (**) contradict. So the assumption that there is some $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \alpha_n x_n$ diverges does not hold. i.e., for any sequence $(\alpha_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \alpha_n x_n$ converges. ■

Proof. **Proof of (5) $\implies$ (2).**

Assume that for any sequence $(\alpha_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} \alpha_n x_n$ converges. We are to prove that for any subsequence indexing $(k_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} x_{k_n}$ converges. Let $(k_n)_{n\in\mathbb{N}}$ be an arbitrary subsequence indexing. Consider $(\alpha_n)_{n\in\mathbb{N}}$ be given by $\alpha_n := 1$ for all $n \in \mathbb{N}$. Then $\sum_{n\in\mathbb{N}} \alpha_n x_n = \sum_{n\in\mathbb{N}} x_n$ converges. Consider $(\alpha_n)_{n\in\mathbb{N}}$ be given by $\alpha_n := 1$ for $n \in \{k_i\}_{i\in\mathbb{N}}$; and $\alpha_n := -1$ for $n \notin \{k_i\}_{i\in\mathbb{N}}$. Then $\sum_{n\in\mathbb{N}} \alpha_n x_n = \sum_{n\in\{k_i\}_{i\in\mathbb{N}}} x_n - \sum_{n\notin\{k_i\}_{i\in\mathbb{N}}} x_n$ converges Notice

$$\sum_{n\in\mathbb{N}} x_{k_n} = \frac{1}{2}\sum_{n\in\mathbb{N}} x_n + \frac{1}{2}\left(\sum_{n\in\{k_i\}_{i\in\mathbb{N}}} x_n - \sum_{n\notin\{k_i\}_{i\in\mathbb{N}}} x_n\right).$$

So $\sum_{n\in\mathbb{N}} x_{k_n}$ converges. ■

Proof. **Proof of (2) $\implies$ (3).**

Assume that for any subsequence indexing $(k_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} x_{k_n}$ converges. We are to prove that $\forall\varepsilon > 0$, $\exists\mu \in \mathbb{N}$ such that for all finite subsets F of $\mathbb{N} \setminus \{1..\mu\}$, we have $\|\sum_{n\in F} x_n\| < \varepsilon$. Assume for the sake of contradiction that $\exists\varepsilon_0 > 0$ such that $\forall\mu \in \mathbb{N}$, there is some finite subset F of $\mathbb{N} \setminus \{1..\mu\}$ such that $\|\sum_{n\in F} x_n \geq \varepsilon_0$. For $\mu = 1$, find $F_1 \subseteq \mathbb{N} \setminus \{1..\mu\}$ finite. For $\mu = \max\{F_1\}$, find $F_2 \subseteq \mathbb{N} \setminus \{1..\mu\}$ finite. In general, for $\mu = \max\{F_n\}$, find $F_{n+1} \subseteq \mathbb{N} \setminus \{1..\mu\}$ finite. Then we have that the F_n's are disjoint. Define a subsequence indexing $(k_n)_{n\in\mathbb{N}}$ as $(k_n)_{n\in\mathbb{N}} := \bigcup_{n\in\mathbb{N}} F_n$. By assumption, for $(k_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} x_{k_n}$ converges. So for ε_0, $\exists N \in \mathbb{N}$ such that $\forall j > i > N$,

$$\|\sum_{n=i}^{j} x_{k_n}\| < \varepsilon_0. \tag{*}$$

So for N, there is some finite subset F of $\mathbb{N} \setminus \{1..\mu\}$ such that

$$\|\sum_{n\in F} x_n\| \geq \varepsilon_0.$$

Notice $F = \{k_n\}_{n=i_N}^{j_N}$ for some i_N and j_N. So (*) and (**) contradict. So the assumption that $\exists\varepsilon_0 > 0$ such that $\forall\mu \in \mathbb{N}$, there is some finite subset F of $\mathbb{N} \setminus \{1..\mu\}$ such that $\|\sum_{n\in F} x_n \geq \varepsilon_0$ does not hold. i.e., $\forall\varepsilon > 0$, $\exists\mu \in \mathbb{N}$ such that for all finite subsets F of $\mathbb{N} \setminus \{1..\mu\}$, we have $\|\sum_{n\in F} x_n\| < \varepsilon$. ■

Proof. **Proof of (3) $\implies$ (1).**

Assume that $\forall\varepsilon > 0$, $\exists\mu \in \mathbb{N}$ such that for all finite subsets F of $\mathbb{N} \setminus \{1..\mu\}$, we have $\|\sum_{n\in F} x_n\| < \varepsilon$. We are to prove that for any permutation π of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$

converges. Assume for the sake of contradiction that there is some permutation π of $\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ diverges. i.e., $\exists\varepsilon_0 > 0$ such that $\forall N \in \mathbb{N}$, $\exists l_N > k_N > N$ such that $\|\sum_{n=k_N}^{l_N} x_{\pi(n)}\| \geq \varepsilon_0$. Let μ be an arbitrary element of $\mathbb{N}$. Define N as $N := \max\{\pi^{-1}(n)\}_{n=1}^{\mu}$. For N, find $l_N > k_N > N$ such that $\|\sum_{n=k_N}^{l_N} x_{\pi(n)}\| \geq \varepsilon_0$. Define a set F as $F := \{\pi(n)\}_{n=k_N}^{l_N}$. So $F \subseteq \mathbb{N} \setminus \{1..\mu\}$. Then $\|\sum_{n\in F} x_n\| = \|\sum_{n=k_N}^{l_N} x_n\| \geq \varepsilon_0$. So $\exists\varepsilon_0 > 0$ such that $\forall\mu \in \mathbb{N}$, there is some finite subset F of $\mathbb{N} \setminus \{1..\mu\}$ such that $\|\sum_{n\in F} x_n\| \geq \varepsilon_0$. This contradicts to the assumption. So the assumption that there is some permutation π of $\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ diverges does not hold. So for any permutation π of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\pi(n)}$ converges. ∎

9.6 The Open Mapping Theorem

(bug)

LEMMA 9.1. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Let T be an element of $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Suppose that $\mathfrak{Y}_1 \subseteq \mathrm{cl}(T\mathfrak{X}_m)$ for some $m \geq 1$. Then $\mathfrak{Y}_1 \subseteq T\mathfrak{X}_{2m}$.

Proof. Let y be an arbitrary element of $\mathfrak{Y}_1$. Then $y \in \mathrm{cl}(T\mathfrak{X}_m)$. So $\exists x_1 \in \mathfrak{X}_m$ such that $\|y - Tx_1\| < 1/2$. So $y - Tx_1 \in \mathfrak{Y}_{1/2}$. Since $\mathfrak{Y}_1 \subseteq \mathrm{cl}(T\mathfrak{X}_m)$, we have $\mathfrak{Y}_{1/2} \subseteq \mathrm{cl}(T\mathfrak{X}_{m/2})$. So $y - Tx_1 \in \mathrm{cl}(T\mathfrak{X}_{m/2})$. So $\exists x_2 \in \mathfrak{X}_{m/2}$ such that $\|y - Tx_1 - Tx_2\| < 1/4$. In general, suppose that we have $y - \sum_{i=1}^{n} Tx_i \in \mathfrak{Y}_{1/2^n}$ for some $n \in \mathbb{N}$. Since $\mathfrak{Y}_1 \subseteq \mathrm{cl}(T\mathfrak{X}_m)$, we have $\mathfrak{Y}_{1/2^n} \subseteq \mathrm{cl}(T\mathfrak{X}_{m/2^n})$. So $y - \sum_{i=1}^{n} Tx_i \in \mathrm{cl}(T\mathfrak{X}_{m/2^n})$. So $\exists x_{n+1} \in \mathfrak{X}_{m/2^n}$ such that $\|y - \sum_{i=1}^{n+1} Tx_i\| < 1/2^{n+1}$. Then $\sum_{n\in\mathbb{N}} Tx_n = y$. Define a sequence $x_\bullet$ in $\mathfrak{X}$ by $x_\bullet := (x_n)_{n\in\mathbb{N}}$. Since $\forall n \in \mathbb{N}$, $x_n \in \mathfrak{X}_{m/2^{n-1}}$, we have $\sum_{n\in\mathbb{N}} \|x_n\| \leq \sum_{n\in\mathbb{N}} \frac{m}{2^{n-1}} = 2m$. So $x_\bullet$ is absolutely summable. Since $\mathfrak{X}$ is a complete space and $x_\bullet$ is absolutely summable, $x_\bullet$ is summable. Define a point x in $\mathfrak{X}$ by $x := \sum_{n\in\mathbb{N}} x_n$. Then

$$\|x\| = \|\sum_{n\in\mathbb{N}} x_n\| = \|\lim_{n\to\infty} \sum_{i=1}^{n} x_i\| = \lim_{n\to\infty} \|\sum_{i=1}^{n} x_i\| \leq \lim_{n\to\infty} \sum_{i=1}^{n} \|x_i\| = \sum_{n\in\mathbb{N}} \|x_n\| \leq 2m.$$

So $x \in \mathfrak{X}_{2m}$. Now

$$Tx = T\sum_{n\in\mathbb{N}} x_n = T\lim_{n\to\infty} \sum_{i=1}^{n} x_i = \lim_{n\to\infty} T\sum_{i=1}^{n} x_i = \lim_{n\to\infty} \sum_{i=1}^{n} Tx_i = \sum_{n\in\mathbb{N}} Tx_n = y.$$

So $\forall y \in \mathfrak{Y}_1$, $\exists x \in \mathfrak{X}_{2m}$ such that $Tx = y$. So $\mathfrak{Y}_1 \subseteq T\mathfrak{X}_{2m}$. ∎

THEOREM 9.1 (The Open Mapping Theorem)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces.

Let T be a <u>surjective</u> bounded linear map from $\mathfrak{X}$ to $\mathfrak{Y}$. Then T is an open map.

Proof. Notice
$$\mathfrak{Y} = T\mathfrak{X} = T\bigcup_{n\in\mathbb{N}}\mathfrak{X}_n = \bigcup_{n\in\mathbb{N}}T\mathfrak{X}_n \subseteq \bigcup_{n\in\mathbb{N}}\operatorname{cl}(T\mathfrak{X}_n).$$
Since $\mathfrak{Y}$ is complete, by the Baire Category Theorem, $\exists m \in \mathbb{N}$ such that $\operatorname{int}(\operatorname{cl}(T\mathfrak{X}_m)) \neq \emptyset$. ■

not finished

THEOREM 9.2 (The Inverse Mapping Theorem)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Let T be a <u>bijective</u> bounded linear map from $\mathfrak{X}$ to $\mathfrak{Y}$. Then T is a homeomorphism.

THEOREM 9.3 (The Closed Graph Theorem)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Let T be a linear map from $\mathfrak{X}$ to $\mathfrak{Y}$. Suppose that the graph $\mathcal{G}(T) := \{(x, Tx) : x \in \mathfrak{X}\}$ is closed in $\mathfrak{X} \oplus_1 \mathfrak{Y}$. Then T is bounded.

Chapter 10

Hilbert Space

10.1 Definition

DEFINITION (Hilbert Space)**.** We define a **Hilbert space**, denoted by $\mathcal{H}$, to be a complete inner product space.

10.2 Examples

EXAMPLE 10.2.1. Let (X, μ) be a measure space. Then $L^2(X, \mu)$ is a Hilbert space with inner product given by

$$\langle f, g \rangle := \int_X f(x)\overline{g(x)} d\mu(x).$$

EXAMPLE 10.2.2. $\ell^2(\mathbb{K}) = \left\{ (x_i)_{i\in\mathbb{N}} \in \mathbb{K}^\infty : \sum_{i\in\mathbb{N}} |x_i|^2 < +\infty \right\}$ is a Hilbert space with inner product given by

$$\langle (x_i)i \in \mathbb{N}, (y_i)_{i\in\mathbb{N}} \rangle := \sum_{i\in\mathbb{N}} x_n \overline{y_n}.$$

10.3 Properties

PROPOSITION 10.3.1. Let $\mathcal{H}$ be a Hilbert space. Let S be a non-empty set in the space. Then $S^{\perp\perp} = \operatorname{clspan}(S)$.

Proof. For one direction, we are to prove that $\operatorname{clspan}(S) \subseteq S^{\perp\perp}$.

For the reverse direction, we are to prove that $S^{\perp\perp} \subseteq \operatorname{clspan}(S)$. Assume for the sake of contradiction that $\exists x \in S^{\perp\perp}$ with $x \neq 0$ such that $x \notin \operatorname{clspan}(S)$. Say $x = m_1 + m_2$ for some $m_1 \in \operatorname{clspan}(S)$ and some $m_2 \in \operatorname{clspan}(S)^{\perp}$. Note that $\operatorname{clspan}(S)^{\perp} = S^{\perp}$. So $m_2 \in S^{\perp}$. Since $x \in S^{\perp\perp}$ and $m_2 \in S^{\perp}$, we should have $\langle x, m_2\rangle = 0$. However,

$$\begin{aligned}\langle x, m_2\rangle &= \langle m_1 + m_2, m_2\rangle \\ &= \langle m_1, m_2\rangle + \langle m_2, m_2\rangle \\ &= 0 + \langle m_2, m_2\rangle \\ &> 0, \text{ since } m_2 \neq 0.\end{aligned}$$

This leads to a contradiction. So $S^{\perp\perp} \subseteq \operatorname{clspan}(S)$. ■

PROPOSITION 10.3.2 (Stability of Hilbert Spaces Under Quotients)**.** Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then the quotient space $\mathcal{H}/\mathcal{M}$ is again a Hilbert space.

10.4 The Riesz Representation Theorem

THEOREM 10.1 (The Riesz Representation Theorem)**.** Let $\mathcal{H}$ be a Hilbert space over field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Suppose that $\mathcal{H} \neq \{0\}$. Then for any $\varphi \in \mathcal{H}^*$, $\exists y \in \mathcal{H}$ such that

$$\forall x \in \mathcal{H}, \quad \varphi(x) = \langle x, y\rangle.$$

Proof. Define for each $y \in \mathcal{H}$ a function $\beta_y \in \mathcal{H}^*$ by $\beta_y(x) := \langle x, y\rangle$. We are to prove that $\mathcal{H}^* = \{\beta_y : y \in \mathcal{H}\}$. It is easy to verify that each β_y is linear and bounded. So $\forall y \in \mathcal{H}$, $\beta_y \in \mathcal{H}^*$. i.e., $\{\beta_y : y \in \mathcal{H}\} \subseteq \mathcal{H}^*$. Define a map Θ from $\mathcal{H}$ to $\mathcal{H}^*$ as $\Theta(y) := \beta_y$. It is easy to verify that Θ is linear.

$$\begin{aligned}\|\Theta(y)\| = \|\beta_y\| &= \sup\{\beta_y(x) : \|x\| = 1\} \\ &= \sup\{\langle x, y\rangle : \|x\| = 1\}\end{aligned}$$

$$\begin{aligned}&\leq \sup\{\|x\|\|y\| : \|x\| = 1\}\\ &= \|y\|.\end{aligned}$$

That is, $\|\Theta(y)\| \leq \|y\|$. So $\|\Theta\| \leq 1$. On the other hand, consider an arbitrary point $y_0 \in \mathcal{H}$ with $y_0 \neq 0$:

$$\begin{aligned}\|\Theta\| &= \sup\left\{\frac{\|\Theta(y)\|}{\|y\|} : y \neq 0\right\}\\ &\geq \left.\frac{\|\Theta(y)\|}{\|y\|}\right|_{y=y_0}\\ &= \frac{\|\Theta(y_0)\|}{\|y_0\|}\\ &= \frac{\|\beta_{y_0}\|}{\|y_0\|}\\ &= \frac{1}{\|y_0\|}\sup\left\{\frac{\|\beta_{y_0}(y)\|}{\|y\|} : y \neq 0\right\}\\ &\geq \frac{1}{\|y_0\|}\frac{\|\beta_{y_0}(y_0)\|}{\|y_0\|}\\ &\geq \frac{1}{\|y_0\|}\frac{\|y_0\|^2}{\|y_0\|}\\ &= 1.\end{aligned}$$

That is, $\|\Theta\| \geq 1$. So $\|\Theta\| = 1$. So Θ is isometric. It immediately follows that Θ is injective. Now it remains to prove that Θ is surjective. Let $\varphi \in \mathcal{H}^*$. If $\varphi = 0$, then $\varphi = \Theta(0)$ and we are done. Otherwise, let $\mathcal{M} := \ker(\varphi)$. Then we have $\operatorname{codim}\mathcal{M} = \dim\mathcal{M}^\perp = 1$. Take $e \in \mathcal{M}^\perp$ such that $\|e\| = 1$. Let P denote the orthogonal projection onto $\mathcal{M}$. Then $1 - P$ is the orthogonal projection onto $\mathcal{M}^\perp$.

$$x = Px + (1-P)x = Px + \langle x, e\rangle e.$$

So for $x \in \mathcal{H}$,

$$\varphi(x) = \varphi(Px) + \langle x, e\rangle\varphi(e) = \langle x, \overline{\varphi(e)}e\rangle = \beta_y(x)$$

where $y := \overline{\varphi(e)}e$. Hence $\varphi = \beta_y$. So Θ is surjective. This completes the proof. ■

Chapter 11

Operators

11.1 Bounded Operators

DEFINITION (Bounded Operator)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let T be a linear map from $\mathfrak{X}$ to $\mathfrak{Y}$. We say that T is a **bounded operator** if

$$\exists k \in \mathbb{R}, \quad \forall x \in \mathfrak{X}, \quad \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}.$$

DEFINITION (Operator Norm)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let T be a bounded operator from $\mathfrak{X}$ to $\mathfrak{Y}$. We define the **operator norm** of T, denoted by $\|T\|$, to be the number given by

$$\|T\| := \inf\{k \in \mathbb{R} : \forall x \in \mathfrak{X}, \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}\}.$$

PROPOSITION 11.1.1.

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{X}, \|x\|_{\mathfrak{X}} = 1\}.$$

PROPOSITION 11.1.2. Let X and Y be normed linear spaces. Let T be a linear map from X to Y. Then T is bounded if and only if T is continuous.

11.2 Examples of Bounded Operators

EXAMPLE 11.2.1 (The Multiplication Operator)**.** Let $\mathfrak{X} = \big(\mathcal{C}([0,1],\mathbb{C}), \|\cdot\|_\infty\big)$. Let f be a function in $\mathfrak{X}$. We define the **multiplication operator** on $\mathfrak{X}$, w.r.t. f, denoted by M_f, as

$$M_f(g) = fg.$$

Then M_f is bounded and $\|M_f\| = \|f\|_\infty$.

Proof. Let g be an arbitrary function in $\mathfrak{X}$. Then

$$\begin{aligned}
\|M_f g\|_\infty &= \|fg\|_\infty \\
&= \sup_{x\in[0,1]} |f(x)g(x)| \\
&= \sup_{x\in[0,1]} |f(x)||g(x)| \\
&\leq \sup_{x\in[0,1]} |f(x)| \sup_{x\in[0,1]} |g(x)| \\
&= \|f\|_\infty \|g\|_\infty.
\end{aligned}$$

That is, $\|M_f g\|_\infty \leq \|f\|_\infty\|g\|_\infty$. So $\|f\|_\infty$ is an element of the set $S = \{k \in \mathbb{R} : \forall g \in \mathfrak{X}, \|M_f g\|_{\mathfrak{Y}} \leq k\|g\|_{\mathfrak{X}}\}$. So $\|M_f\| = \inf(S) \leq \|f\|_\infty$. Consider g_0 given by $g_0(x) = 1$. Then g_0 in $\mathfrak{X}$. Then

$$\|M_f g_0\|_\infty = \|fg_0\|_\infty = \|f\|_\infty = \|f\|_\infty\|g_0\|_\infty.$$

Let k be an arbitrary element in S. Assume for the sake of contradiction that $k < \|f\|_\infty$. Then

$$\begin{aligned}
\|f\|_\infty\|g_0\|_\infty &= \|M_f g_0\|_\infty \\
&\leq k\|g_0\|_\infty \\
&< \|f\|_\infty\|g_0\|_\infty.
\end{aligned}$$

This leads to a contradiction. So $\forall k \in S$, $k \geq \|f\|_\infty$. So $\|f\|_\infty$ is a lower bound for the set S. So $\|M_f\| = \inf(S) \geq \|f\|_\infty$. Since $\|M_f\| \leq \|f\|_\infty$ and $\|M_f\| \geq \|f\|_\infty$, we get $\|M_f\| = \|f\|_\infty$. ∎

EXAMPLE 11.2.2 (The Volterra Operator)**.** Let $\mathfrak{X} = \big(\mathcal{C}([0,1],\mathbb{C}), \|\cdot\|_\infty\big)$. Define

$$Vf := x \mapsto \int_0^x f(t)dt.$$

Then the Volterra Operator is bounded and $\|V\| \leq 1$.

Proof. Let f be an arbitrary function in $\mathfrak{X}$ with $\|f\|_\infty = 1$. Then $\forall x \in [0,1]$,

$$\begin{aligned} |Vf(x)| &= \left|\int_0^x f(t)dt\right| \\ &\leq \int_0^x |f(t)|dt \\ &\leq \int_0^x \sup_{t\in[0,1]} |f(t)|dt \\ &= \int_0^x \|f\|_\infty dt \\ &= \int_0^x 1dt \\ &= x. \end{aligned}$$

That is, $\forall x \in [0,1]$, $|Vf(x)| \leq 1$. So $\|Vf\|_\infty \leq 1$. Since $\forall f \in \mathfrak{X} : \|f\|_\infty = 1$, $\|Vf\|_\infty \leq 1$, we get $\|V\| \leq 1$. ■

EXAMPLE 11.2.3 (The Diagonal Operator)**.** Let $\mathfrak{X} = \ell^2(\mathbb{N})$. Let

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & \ddots \end{bmatrix}.$$

Then D is bounded if and only if $(d_i)_{i\in\mathbb{N}}$ is bounded and $\|D\| = \|(d_i)_{i\in\mathbb{N}}\|_\infty$.

Proof. <u>Case 1.</u>

$$\begin{aligned} \|Dx\|_2^2 &= \sum_{i\in\mathbb{N}} |d_i x_i|^2 \\ &=\leq \sum_{i\in\mathbb{N}} \|(d_j)_{j\in\mathbb{N}}\|_\infty |x_i|^2 \\ &= \|(d_j)_{j\in\mathbb{N}}\|_\infty \sum_{i\in\mathbb{N}} |x_i|^2 \end{aligned}$$

$$= \|(d_j)_{j\in\mathbb{N}}\|_\infty \|x\|_2^2.$$

Case 2.
If $(d_i)_{i\in\mathbb{N}} \notin \ell^\infty$, $\exists (d_{n_i})_{i\in\mathbb{N}} \to \infty$.

$$\begin{aligned}\|De_{n_i}\|_2 &= \|d_{n_i}e_{n_i}\|_2 \\ &= |d_{n_i}|\|e_{n_i}\|_2 \\ &= |d_{n_i}|.\end{aligned}$$

So $\|D\| \geq \|De_{n_i}\|_2 \to \infty$. ■

EXAMPLE 11.2.4 (Weighted Shifts)**.**

- Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(w_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an **unilateral forward weighted shift** W on $\mathcal{H}$ as
$$W(x_n) := (0, w_1x_1, w_2x_2, w_3x_3, ...).$$
i.e.,
$$W = \begin{bmatrix} 0 & & & & \\ w_1 & 0 & & & \\ & w_2 & 0 & & \\ & & w_3 & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.$$
Then W is bounded and $\|W\| = \sup\{|w_n| : n \in \mathbb{N}\}$.
- Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(v_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an **unilateral backward weighted shift** V on $\mathcal{H}$ as
$$V(x_n) := (v_1x_2, v_2x_3, v_3x_4, ...).$$
Then V is bounded and $\|V\| = \sup\{|v_n| : n \in \mathbb{N}\}$.
- Let $\mathcal{H} = \ell^2_{\mathbb{Z}}$. Let $(u_n)_{n\in\mathbb{Z}} \in \ell^\infty_{\mathbb{Z}}$. We define a **bilateral weighted shift** U on $\mathcal{H}$ as
$$U(x_n) := (u_{n-1}x_{n-1})_{n\in\mathbb{Z}}.$$
Then U is bounded and $\|U\| = \sup\{|u_n| : n \in \mathbb{Z}\}$.

EXAMPLE 11.2.5 (The Composition Operators)**.** Let $\mathfrak{X} = \mathcal{C}([0,1], \mathbb{C})$. Let $\varphi \in$

$\mathcal{C}([0,1],[0,1])$. We define the **composition operator** on $\mathfrak{X}$, denoted by C_φ as

$$C_\varphi(f) := f \circ \varphi.$$

Then C_φ is contractive.

Proof.

$$\begin{aligned} \|C_\varphi(f)\| &= \sup_{x\in[0,1]} |(f \circ \varphi)(x)| \\ &\leq \|f\|_\infty. \end{aligned}$$

■

11.3 The Space of Bounded Linear Operators

PROPOSITION 11.3.1. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Then $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ is a vector space and the operator norm is a norm on $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$.

PROPOSITION 11.3.2. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ be the space of bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$. Then if $\mathfrak{Y}$ is complete, $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ is complete.

PROPOSITION 11.3.3. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$. Then $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y},\|\cdot\|_1)$ if and only if $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y},\|\cdot\|_2)$.

11.4 Invertible Bounded Linear Operators

PROPOSITION 11.4.1. Let $(\mathfrak{X},\|\cdot\|_1)$ be a Banach space. Let $S \in \mathcal{B}(\mathfrak{X})$ be a bounded linear map that is invertible. Define a norm $\|\cdot\|_2$ on $\mathfrak{X}$ as

$$\|x\|_2 := \|Sx\|_1.$$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proof. On one hand, since S is bounded, $\exists c_1$ such that $\forall x \in \mathfrak{X}$, $\|Sx\|_1 \le c_1\|x\|_1$. That is, $\|x\|_2 \le c_1\|x\|_1$.

On the other hand, since S is invertible, S^{-1} exists and is also bounded. Since S^{-1} is bounded, $\exists c_2$ such that $\forall x \in \mathfrak{X}$, $\|S^{-1}x\|_1 \le c_2\|x\|_1$. Consider $x = Sx$, we get $\forall x \in \mathfrak{X}$, $\|S^{-1}Sx\|_1 \le c_2\|Sx\|_1$. That is, $\|x\|_1 \le c_2\|x\|_2$.

So $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ■

PROPOSITION 11.4.2. Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space. Let S be a map in $\mathcal{B}(\mathfrak{X})$ that is invertible. Then

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

Proof.

$$\begin{aligned}
\|S^{-1}\| &= \sup\{\|S^{-1}x\| : \|x\| = 1\} \\
&= \sup\left\{\frac{\|S^{-1}x\|}{\|x\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{\|S^{-1}Sx\|}{\|Sx\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{\|x\|}{\|Sx\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{1}{\|Sx\|} : \|x\| = 1\right\} \\
&= (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.
\end{aligned}$$

That is,

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

■

PROPOSITION 11.4.3. Let $\mathfrak{X}$ be a Banach space. Then the set of invertible bounded linear operators from $\mathfrak{X}$ to $\mathfrak{X}$ is an open set.

Chapter 12

Balanced Set

12.1 Definitions

DEFINITION (Balanced Sets)**.** Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We say that S is **balanced** if

$$\forall a \in \mathbb{F} : |a| \leq 1, \quad aS \subseteq S.$$

12.2 Properties

PROPOSITION 12.2.1. Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let B be a balanced set in $\mathfrak{X}$. Then

$$\forall a, b \in \mathbb{F} : |a| \leq |b|, \quad aB \subseteq bB.$$

PROPOSITION 12.2.2. Balanced sets are path connected.

12.3 Stability of Balance

PROPOSITION 12.3.1 (Set Operations)**.**

- The union of balanced sets is balanced.

- The intersection of balanced sets is balanced.

PROPOSITION 12.3.2 (Convex Hull)**.** The convex hull of a balanced set is balanced.

Proof. Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let B be a balanced set in $\mathfrak{X}$. I will show that $\operatorname{conv}(B)$ is balanced. Let k be an arbitrary element of $\mathbb{F}$ such that $|k| \leq 1$. Let x be an arbitrary element of $k \operatorname{conv}(B)$. Then $\exists n \in \mathbb{N}$, $x_1, ..., x_n \in B$, $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that $\lambda_1, ..., \lambda_n \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, and $x = k \sum_{i=1}^{n} \lambda_i x_i$. Since $|k| \leq 1$, B is balanced, and $x_i \in B$, we get $kx_i \in B$. So

$$x = k \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} \lambda_i (kx_i) \in \operatorname{conv}(B).$$

So $k \operatorname{conv}(B) \subseteq \operatorname{conv}(B)$. So $\operatorname{conv}(B)$ is balanced. ∎

PROPOSITION 12.3.3 (Closure)**.** The closure of a balanced set is balanced.

PROPOSITION 12.3.4 (Linear Maps)**.**

- The scalar multiple of a balanced set is also balanced.
- The (Minkowski) sum of two balanced sets is also balanced.
- The image of a balanced set under a linear operator is also balanced.
- The inverse image of a balanced set under a linear operator is also balanced.

12.4 The Balanced Hull Operator

DEFINITION (Balanced Hull)**.** Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We define the **balanced hull** of S, denoted by $\operatorname{balhull}(S)$, to be the smallest balanced set containing S.

PROPOSITION 12.4.1 (Act on Other Properties)**.**

- The balanced hull of a compact set is compact.
- The balanced hull of a totally bounded set is totally bounded.
- The balanced hull of a bounded set is bounded.

PROPOSITION 12.4.2. Let X be a vector space over field $\mathbb{F}$. Let a be a scalar in field $\mathbb{F}$. Then

$$a\,\mathrm{balhull}(S) = \mathrm{balhull}(aS).$$

12.5 The Balanced Core Operator

DEFINITION (Balanced Core)**.** Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We define the **balanced core** of S, denoted by $\mathrm{balcore}(S)$, to be the largest balanced set contained in S.

PROPOSITION 12.5.1 (Act on Other Properties)**.**

- The balanced core of a closed set is closed.

Chapter 13

Topological Vector Space

13.1 Preliminaries: Absorbing Set

DEFINITION (Absorbing Sets)**.** Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let S be a subset of X. We say that S is **absorbing** if

$$\forall x \in \mathfrak{X}, \quad \exists r \in \mathbb{R} : r > 0, \quad \forall c \in \mathbb{F} : |c| \geq r, \quad x \in cS.$$

i.e.,

$$\bigcup_{n \in \mathbb{N}} nS = \mathfrak{X}.$$

PROPOSITION 13.1.1. Every absorbing set contains the origin.

13.2 Definitions

DEFINITION (Compatible)**.** Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let $\mathcal{T}$ be a topology on $\mathcal{V}$. We say that $\mathcal{T}$ is **compatible** with the vector space structure on $\mathcal{V}$ if the addition and scalar multiplication operations on $\mathcal{V}$ are continuous.

DEFINITION (Topological Vector Space)**.** We define a **topological vector space** to be a vector space with a compatible Hausdorff topology.

13.3 Neighborhood Improvements

PROPOSITION 13.3.1. Let $(\mathcal{V}, \tau)$ be a topological vector space. Let $U \in \mathcal{U}_0$ be a neighborhood of 0 in $\mathcal{V}$. Then we have the followings.

(1) $\exists N \in \mathcal{U}_0$ such that $N + N \subseteq U$.

(2) $\exists M \in \mathcal{U}_0$ and $\exists \varepsilon > 0$ such that $\forall 0 < |k| < \varepsilon$, we have $kM \subseteq U$.

(3) $\mathcal{V} = \bigcup_{n \in \mathbb{N}} nU$.

PROPOSITION 13.3.2. Any neighborhood of 0 in a topological vector space is absorbing.

PROPOSITION 13.3.3. Let $\mathcal{V}$ be a topological vector space over field $\mathbb{K}$. Then every neighborhood of 0 contains a open balanced neighborhood of 0.

Proof. Let U be an arbitrary element of $\mathcal{U}_0$. Let μ denote the multiplication operation on $\mathcal{V}$. Then μ is continuous and hence $\mu^{-1}(U)$ is a neighborhood of $(0,0) \in \mathbb{K} \times \mathcal{V}$. So there exist an $r > 0$ and an open neighborhood $\mathcal{N}$ of 0 such that $\text{ball}(0,r) \times \mathcal{N} \subseteq \mu^{-1}(U)$. Define a set $\mathcal{M}$ by $\mathcal{M} := \mu(\text{ball}(0,r), \mathcal{N}) = \bigcup_{k \in \mathbb{K}:|k|<r} k\mathcal{N}$. Then $\mathcal{M} \subseteq U$. Notice $0 \in \mathcal{N}$ and hence $0 \in k\mathcal{N}$ for any $k \in \mathbb{K}$. So $0 \in \mathcal{M}$.

Open: Since $\mathcal{N}$ is open and scalar multiplication is a homeomorphism, we get $k\mathcal{N}$ is open for any $k \in \mathbb{K}$. So $\bigcup_{k \in \mathbb{K}:|k|<r} k\mathcal{N}$ is open. That is, $\mathcal{M}$ is open.

Balanced: Let a be an arbitrary element of $\mathbb{K}$ such that $|a| < 1$. Then

$$\begin{aligned} a\mathcal{M} &= a \bigcup_{k \in \mathbb{K}:|k|<r} k\mathcal{N}, \text{ by definition of } \mathcal{M} \\ &= \bigcup_{k \in \mathbb{K}:|k|<r} ak\mathcal{N} = \bigcup_{k \in \mathbb{K}:|k|<|a|r} k\mathcal{N} \\ &\subseteq \bigcup_{k \in \mathbb{K}:|k|<r} k\mathcal{N}, \text{ since } a < 1 \\ &= \mathcal{M}, \text{ by definition of } \mathcal{M}. \end{aligned}$$

That is, $\forall a \in \mathbb{K} : |a| < 1$, we have $a\mathcal{M} \subseteq \mathcal{M}$. So $\mathcal{M}$ is balanced. ∎

13.4 Properties of Topological Vector Spaces

PROPOSITION 13.4.1 (Normed Linear Spaces are Topological Vector Spaces)**.** Let $\mathfrak{X}$ be a normed linear space over field $\mathbb{K}$. Then $\mathfrak{X}$ is a topological vector space with the topology induced by the norm.

Proof. **Part 1**: Show that the norm topology is compatible with the vector space structure.

Let σ denote the addition operation in $\mathfrak{X}$. Let $((x_\alpha, y_\alpha))_{\alpha\in\Lambda}$ be an arbitrary net in $\mathfrak{X}\times\mathfrak{X}$ that converges to $(x, y) \in \mathfrak{X}\times\mathfrak{X}$.

$$\begin{aligned}\|\sigma(x_\alpha, y_\alpha) - \sigma(x, y)\| = \|(x_\alpha + y_\alpha) - (x + y)\| &= \|(x_\alpha - x) + (y_\alpha - y)\| \\ &\leq \|x_\alpha - x\| + \|y_\alpha - y\|, \text{ by the triangle inequality} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

That is, $\|\sigma(x_\alpha, y_\alpha) - \sigma(x, y)\| < \varepsilon$. So σ is continuous.

Let μ denote the scalar multiplication operation in $\mathfrak{X}$. Let $((k_\alpha, x_\alpha))_{\alpha\in\Lambda}$ be an arbitrary net in $\mathbb{K}\times\mathfrak{X}$ that converges to $(k, x) \in \mathbb{K}\times\mathfrak{X}$.

$$\begin{aligned}\|\mu(k_\alpha, x_\alpha) - \mu(k, x)\| = \|k_\alpha x_\alpha - kx\| &= \|k_\alpha x_\alpha - kx_\alpha + kx_\alpha - kx\| \\ &\leq \|k_\alpha x_\alpha - kx_\alpha\| + \|kx_\alpha - kx\|, \text{ by the triangle inequality} \\ &= |k_\alpha - k|\|x_\alpha\| + |k|\|x_\alpha - x\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

That is, $\|\mu(k_\alpha, x_\alpha) - \mu(k, x)\| < \varepsilon$. So μ is continuous.

Part 2: Show that the norm topology is Hausdorff.

Let x and y be arbitrary elements of $\mathfrak{X}$. Suppose that $x \neq y$. Define a number r by $r := \|x - y\|/2$. Then $\operatorname{ball}(x, r) \in \mathcal{U}_x$ and $\operatorname{ball}(y, r) \in \mathcal{U}_y$ and $\operatorname{ball}(x, r) \cap \operatorname{ball}(y, r) = \emptyset$. So any two distinct points in $\mathfrak{X}$ are separated by the norm topology. So $\mathfrak{X}$ is Hausdorff.

∎

PROPOSITION 13.4.2. Closure of a linear manifold of a topological vector space is a closed linear subspace.

Proof. Let $\mathcal{V}$ be a topological vector space. Let $\mathcal{W}$ be a linear manifold of $\mathcal{V}$. We are to prove that $\operatorname{cl}(\mathcal{W})$ is a linear subspace. Note that $\operatorname{cl}(\mathcal{W})$ is closed. So there remains only to show that $\operatorname{cl}(\mathcal{W})$ is linear.

Let x and y be arbitrary elements of $\mathrm{cl}(\mathcal{W})$. Then there exists a net $(x_\lambda, y_\lambda)_{\lambda\in\Lambda}$ that converges to (x, y). Since the addition operation σ is continuous, we have $\lim_{\lambda\in\Lambda}(x_\lambda + y_\lambda) = x + y$. Since $\mathcal{W}$ is a linear subspace, $x_\lambda + y_\lambda \in \mathcal{W}$. So $x + y \in \mathrm{cl}(\mathcal{W})$.

Let x be an arbitrary element of $\mathrm{cl}(\mathcal{W})$. Let k be an arbitrary element in $\mathbb{K}$. Then there exists a net $(k\lambda, x_\lambda)_{\lambda\in\Lambda}$ that converges to (k, x). Since the scalar multiplication operation μ is continuous, we have $\lim_{\lambda\in\Lambda}(k_\lambda x_\lambda) = kx$. Since $\mathcal{W}$ is a linear subspace, $k_\lambda x_\lambda \in \mathcal{W}$. So $kx \in \mathrm{cl}(\mathcal{W})$. ■

PROPOSITION 13.4.3. A quotient space of a topological vector space by a closed subspace is a topological vector space with the quotient topology .

13.5 Operation on Sets in a Topological Vector Space

PROPOSITION 13.5.1 (Stability under Linear Combinations)**.** Let X be a normed vector space over $\mathbb{F}$. Let K be a compact set in the space. Let C be a closed set in the space. Then $\forall\alpha, \beta \in \mathbb{F}$, the set S given by $S := \alpha K + \beta C$ is closed.

Proof. The case where $\beta = 0$ is trivial. I will assume $\beta \neq 0$. Let $\alpha, \beta \in \mathbb{F}$ be arbitrary. Let $\{s_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence in S that converges. Say the limit is s_∞. Since $s_i \in S$ for any $i \in \mathbb{N}$ and $S = \alpha K + \beta C$, $s_i = \alpha k_i + \beta c_i$ for some $k_i \in K$ and some $c_i \in C$, for any $i \in \mathbb{N}$. Since $\{k_i\}_{i\in\mathbb{N}}$ is a sequence in K and K is compact, there exists a convergent subsequence $\{k_i\}_{i\in I}$ of $\{k_i\}_{i\in\mathbb{N}}$ in K. Say $\{k_i\}_{i\in I}$ converges to $k_\infty \in K$. Since $\{s_i\}_{i\in\mathbb{N}}$ converges to s_∞, $\{s_i\}_{i\in I}$ also converges to s_∞. Since $s_i = \alpha k_i + \beta c_i$, $c_i = \beta^{-1}(s_i - \alpha k_i)$. Define $c_\infty := \beta^{-1}(s_\infty - \alpha k_\infty)$ Since $\{s_i\}_{i\in I}$ converges to s_∞ and $\{k_i\}_{i\in I}$ converges to k_∞ and $c_i = \beta^{-1}(s_i - \alpha k_i)$, $\{c_i\}_{i\in I}$ converges to c_∞. Since $\{c_i\}_{i\in I}$ is a sequence in C and converges to c_∞ and C is closed, $c_\infty \in C$. Since $s_\infty = \alpha k_\infty + \beta c_\infty$ and $k_\infty \in K$ and $c_\infty \in C$, $s_\infty \in \alpha K + \beta C$. Since for any sequence in S that converges, the limit is also in S, S is closed. ■

REMARK. The sum of two closed sets may not be closed.

Proof. **Counter-example 1**

Consider $A := \{n : n \in \mathbb{N}\}$ and $B := \{n + \frac{1}{n} : n \in \mathbb{N}\}$.
(`https://math.stackexchange.com/questions/124130/sum-of-two-closed-sets-in-mathbb-r-is-closed`)
Their sum contains the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ but does not contain 0.

Counter-example 2

Consider $A := \mathbb{R} \times \{0\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x, y > 0, xy \geq 1\}$. Their sum is $\mathbb{R} \times \mathbb{R}_{++}$. ■

PROPOSITION 13.5.2. Let $\mathfrak{X}$ be a normed vector space. Let S be a subset of $\mathfrak{X}$. Let p be a vector in $\mathfrak{X}$. Then we have the followings.

(1) $p + \text{int}(S) = \text{int}(p + S)$,

(2) $p + \text{cl}(S) = \text{cl}(p + S)$.

Proof of (1). For one direction, let x be an arbitrary point in the set $p + \text{int}(S)$. We are to prove that $x \in \text{int}(p + S)$. Since $x \in (p + \text{int}(S))$, $(x - p) \in \text{int}(S)$. Since $(x - p) \in \text{int}(S)$, by definition of interior, there exists a radius r such that

$$B(x - p, r) \subseteq S.$$

It follows that $B(x, r) \subseteq p + S$. Since there exists a radius r such that $B(x, r) \subseteq p + S$, by definition of interior,

$$x \in \text{int}(p + S).$$

For the reverse direction, let x be an arbitrary point in $\text{int}(p+S)$. We are to prove that $x \in p + \text{int}(S)$. Since $x \in \text{int}(p + S)$, by definition of interior, there exists a radius r such that

$$B(x, r) \subseteq (p + S).$$

It follows that $B(x - p, r) \subseteq S$. Since there exists a radius r such that $B(x - p, r) \subseteq S$, by definition of interior,

$$(x - p) \in \text{int}(S).$$

Since $(x - p) \in \text{int}(S)$, we get $x \in (p + \text{int}(S))$. ■

Proof of (2). For one direction, let x be an arbitrary point in the set $p + \text{cl}(S)$. We are to prove that $x \in \text{cl}(p+S)$. Since $x \in (p+\text{cl}(S))$, we get $(x-p) \in \text{cl}(S)$. Since $(x-p) \in \text{cl}(S)$, by definition of closure, for any radius r, we have

$$B(x - p, r) \cap S \neq \emptyset.$$

It follows that $B(x, r) \cap (p + S) \neq \emptyset$. Since for any radius r, $B(x, r) \cap (p + S) \neq \emptyset$, by definition of closure, we get

$$x \in \text{cl}(p + S).$$

For the reverse direction, let x be an arbitrary point in $\mathrm{cl}(p+S)$. We are to prove that $x \in (p + \mathrm{cl}(S))$. Since $x \in \mathrm{cl}(p+S)$, by definition of closure, for any radius r, we have

$$B(x, r) \cap (p + S) \neq \emptyset.$$

It follows that $B(x-p, r) \cap S \neq \emptyset$. Since for any radius r, $B(x-p, r) \cap S \neq \emptyset$, by definition of closure, we get

$$(x - p) \in \mathrm{cl}(S).$$

Since $(x-p) \in \mathrm{cl}(S)$, we get $x \in (p + \mathrm{cl}(S))$. ■

PROPOSITION 13.5.3. Let $(V, \|\cdot\|)$ be a normed vector space. Let S be a subset of V. Let λ be a non-zero real number. Then

(1) $\lambda\,\mathrm{int}(S) = \mathrm{int}(\lambda S)$.

(2) $\lambda\,\mathrm{cl}(S) = \mathrm{cl}(\lambda S)$.

Proof of (1). For one direction, let x be an arbitrary point in $\lambda\,\mathrm{int}(S)$. We are to prove that $x \in \mathrm{int}(\lambda S)$. Since $x \in \lambda\,\mathrm{int}(S)$, we get $x/\lambda \in \mathrm{int}(S)$. Since $x/\lambda \in \mathrm{int}(S)$, by definition of interior, there exists a radius r such that

$$B(x/\lambda, r) \subseteq S.$$

Let y be an arbitrary point in $B(x, \lambda r)$. Since $y \in B(x, \lambda r)$, we get $\|y - x\| \leq \lambda r$. Since $\|y - x\| \leq \lambda r$, we get $\|y/\lambda - x/\lambda\| \leq r$. Since $\|y/\lambda - x/\lambda\| \leq r$, we get $y/\lambda \in B(x/\lambda, r)$. Since $y/\lambda \in B(x/\lambda, r)$ and $B(x/\lambda, r) \subseteq S$, we get $y/\lambda \in S$. Since $y/\lambda \in S$, we get $y \in \lambda S$. Since any point in $B(x, \lambda r)$ is also in λS, we get $B(x, \lambda r) \subseteq \lambda S$. Since there exists a radius r such that $B(x, \lambda r) \subseteq \lambda S$, by definition of interior, we get

$$x \in \mathrm{int}(\lambda S).$$

■

13.6 Finite-Dimensional Topological Vector Spaces

PROPOSITION 13.6.1. Let $\mathcal{V}$ be an n-dimensional topological vector space where

$n \in \mathbb{N}$. Then $\mathcal{V}$ is homeomorphic to $\mathbb{K}^n$ via the map

$$\sum_{i=1}^{n} k_i e_i \mapsto (k_i)_{i=1}^{n}.$$

COROLLARY 13.1. Let $\mathcal{V}$ be a finite-dimensional vector space. Then there is a unique topology $\mathcal{T}$ which makes $\mathcal{V}$ a topological vector space.

COROLLARY 13.2. Linear maps on a finite-dimensional topological vector space are continuous.

Chapter 14

Continuous and Uniformly Continuous Functions

14.1 Definition

DEFINITION (Uniformly Continuous)**.** Let $\mathcal{V}$ and $\mathcal{W}$ be topological vector spaces. Let f be a function from $\mathcal{V}$ to $\mathcal{W}$. We say that f is **uniformly continuous** if $\forall U \in \mathcal{U}_0^{\mathcal{W}}$, $\exists N \in \mathcal{U}_0^{\mathcal{V}}$ such that $\forall x, y \in \mathcal{V} : x - y \in N$, we have $f(x) - f(y) \in U$.

14.2 Extension of Continuous Linear Maps

PROPOSITION 14.2.1. Let $\mathcal{V}$ and $\mathcal{W}$ be topological vector spaces. Suppose that $\mathcal{W}$ is complete. Let $\mathcal{X}$ be a linear manifold of $\mathcal{V}$. Let T_0 be a continuous linear map from $\mathcal{X}$ to $\mathcal{W}$. Then T_0 extends to a continuous linear map T from $\mathrm{cl}(\mathcal{X})$ to $\mathcal{W}$.

PROPOSITION 14.2.2. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Let $\mathfrak{M}$ be a linear manifold of $\mathfrak{X}$. Let T_0 be a bounded linear operator from $\mathfrak{M}$ to $\mathfrak{Y}$. Then T_0 extends to a bounded linear operator from $\mathrm{cl}(\mathfrak{M})$ to $\mathfrak{Y}$ and we have $\|T\| = \|T_0\|$.

14.3 Relation between the two Notions

PROPOSITION 14.3.1. Uniformly continuous functions are continuous.

PROPOSITION 14.3.2. Continuous linear maps are uniformly continuous.

Proof. Let $\mathcal{V}$ and $\mathcal{W}$ be topological vector spaces. Let T be a continuous linear map from $\mathcal{V}$ to $\mathcal{W}$. I will show that T is uniformly continuous. Fix a point $x_0 \in \mathcal{V}$. Let U_0 be an arbitrary element of $\mathcal{U}_0^{\mathcal{W}}$. Define $U := T(x_0) + U_0$. Then $U \in \mathcal{U}_{T(x_0)}^{\mathcal{W}}$. Since T is continuous at x_0, $\exists N \in \mathcal{U}_{x_0}^{\mathcal{V}}$ such that $\forall x \in N$, $T(x) \in U$. Define $N_0 := -x_0 + N$. Then $N_0 \in \mathcal{U}_0^{\mathcal{V}}$. Let x and y be arbitrary elements of $\mathcal{V}$ such that $x - y \in N_0$. Then

$$\begin{aligned}
& x - y \in N_0 \\
\iff & x_0 + x - y \in N, \text{ since } N = x_0 + N_0 \\
\implies & T(x_0 + x - y) \in U, \text{ by continuity of } T \\
\iff & T(x_0) + T(x) - T(y) \in U, \text{ by linearity of } T \\
\iff & T(x) - T(y) \in U_0, \text{ since } U = T(x_0) + U_0.
\end{aligned}$$

So we have $T(x) - T(y) \in U_0$. So T is uniformly continuous. ■

PROPOSITION 14.3.3. Continuous conjugate linear maps are uniformly continuous

PROPOSITION 14.3.4. Continuous linear maps defined on a balanced and convex subset are uniformly continuous.

Chapter 15

Complete Space

15.1 Cauchy Nets

DEFINITION (Cauchy Net)**.** Let $(\mathcal{V}, \tau)$ be a topological vector space. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathcal{V}$. We say that $(x_\lambda)_{\lambda\in\Lambda}$ is a **Cauchy net** if $\forall U \in \mathcal{U}_0$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda_1, \lambda_2 \geq \lambda_0$, we have $x_{\lambda_1} - x_{\lambda_2} \in U$.

PROPOSITION 15.1.1. Convergent nets are Cauchy.

Proof. Let $\mathcal{V}$ be a topological vector space. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a convergent net with limit point x. Let U be an arbitrary element in $\mathcal{U}_0$. Let N be an element in $\mathcal{U}_0$ that is balanced and open and that $N - N \subseteq U$. Since $\lim_{\lambda\in\Lambda} x_\lambda = x$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0$, $x_\lambda - x \in N$. Let λ_1 and λ_2 be arbitrary elements that are $\geq \lambda_0$. Then

$$x_{\lambda_1} - x_{\lambda_2} = (x_{\lambda_1} - x) - (x_{\lambda_2} - x) \in N - N \subseteq U.$$

That is, $\forall U \in \mathcal{U}_0$, $\exists \lambda_0$ such that $\forall \lambda_1, \lambda_2 \geq \lambda_0$, $x_{\lambda_1} - x_{\lambda_2} \in U$. So $(x_\lambda)_{\lambda\in\Lambda}$ is Cauchy. ∎

15.2 Complete Topological Vector Spaces

DEFINITION (Cauchy Complete)**.** Let $(\mathcal{V}, \tau)$ be a topological vector space. We say that $\mathcal{V}$ is **Cauchy complete** if every Cauchy net in $\mathcal{V}$ converges in $\mathcal{V}$.

PROPOSITION 15.2.1. Let $\mathcal{V}$ be a topological vector space. Let $\mathcal{K}$ be a complete set in $\mathcal{V}$. Then $\mathcal{K}$ is closed in $\mathcal{V}$.

Chapter 16

Locally Convex Space

16.1 Preliminaries

16.1.1 Seminorms

PROPOSITION 16.1.1. Let $\mathcal{V}$ be a topological vector space. Let p be a seminorm on $\mathcal{V}$. Then p is continuous on $\mathcal{V}$ if and only if it is bounded above on some neighborhood of 0.

Proof. **Forward Direction**: Assume that p is continuous. I will show that p is bounded above on some neighborhood of 0. Define a set E by $E := \{x \in \mathcal{V} : p(x) < 1\}$. Note that $\text{range}(p) = [0, +\infty)$, $[0, 1)$ is an open subset of $[0, +\infty)$. Since $[0, 1)$ is open, p is continuous, and $E = p^{-1}([0, 1))$, E is open. Note that $p(0) = 0 < 1$. So $0 \in E$. So $E \in \mathcal{U}_0$. By definition of E, p is bounded above by 1 on E.

Backward Direction: Assume that p is bounded above on some neighborhood of 0. I will show that p is continuous. Say p is bounded above by $M \in \mathbb{R}_+$ on $U \in \mathcal{U}_0$. Let $\varepsilon > 0$ be arbitrary. Define a set $N \in \mathcal{U}_0$ by $N := \frac{\varepsilon}{M+1}U$. Let x and y be arbitrary elements of $\mathcal{V}$ such that $x - y \in N$. Then $x - y = \frac{\varepsilon}{M+1}u$ for some $u \in U$. So

$$|p(x) - p(y)| \leq p(x - y) = p(\frac{\varepsilon}{M+1}u) = \frac{\varepsilon}{M+1}p(u) \leq \frac{\varepsilon}{M+1}M < \varepsilon.$$

That is, $|p(x) - p(y)| < \varepsilon$. So p is uniformly continuous on $\mathcal{V}$ and hence continuous on $\mathcal{V}$.

■

16.1.2 Sublinear Functionals

DEFINITION (Sublinear Functional)**.** Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let f be a function from $\mathcal{V}$ to $\mathbb{R}$. We say that f is **sublinear** if it satisfies:

- Subadditivity:
$$\forall x, y \in \mathcal{V}, \quad f(x+y) \leq f(x) + f(y).$$
- Positive Homogeneity:
$$\forall x \in \mathcal{V}, \forall \lambda \geq 0, \quad f(\lambda x) = \lambda f(x).$$

PROPOSITION 16.1.2. Seminorms are sublinear functionals.

16.1.3 The Minkowski Functional

DEFINITION (Minkowski Functional)**.** Let $\mathcal{V}$ be a topological vector space. Let E be a <u>convex</u> neighborhood of 0 in $\mathcal{V}$. We define the **Minkowski functional** for E, denoted by p_E, to be a function from $\mathcal{V}$ to $\mathbb{R}$ given by

$$p_E(x) := \inf\{r > 0 : x \in rE\}.$$

PROPOSITION 16.1.3 (Convex)**.** A Minkowski functional for a <u>convex</u> neighborhood of 0 is a sublinear functional.

Proof. Let $\mathcal{V}$ be a topological vector space over field $\mathbb{K}$. Let E be a convex neighborhood of 0 in $\mathcal{V}$. Let p denote the Minkowski functional for E. I will show that p is sublinear.
Part 1: Show that $\forall x, y \in \mathcal{V}$, we have $p(x+y) \leq p(x) + p(y)$.

Assume for the sake of contradiction that $\exists x, y \in \mathcal{V}$ such that $p(x+y) > p(x) + p(y)$. Define $\varepsilon > 0$ as $\varepsilon := p(x+y) - (p(x) + p(y))$. By definition, $\exists r_x > 0$ such that $x \in r_x E$ and $r_x < p(x) + \varepsilon/2$, and $\exists r_y > 0$ such that $y \in r_y E$ and $r_y < p(y) + \varepsilon/2$. Since $x \in r_x E$ and $y \in r_y E$, we get $x + y \in (r_x + r_y)E$. So $p(x+y) \leq r_x + r_y$. So

$$p(x+y) \leq r_x + r_y < p(x) + \varepsilon/2 + p(y) + \varepsilon/2 = p(x+y).$$

That is, $p(x+y) < p(x+y)$, a contradiction. So $p(x+y) \leq p(x) + p(y)$.
Part 2: Show that $\forall x \in \mathcal{V}$, $\forall k > 0$, we have $p(kx) = kp(x)$.

Let x be an arbitrary element of $\mathcal{V}$. Let k be an arbitrary element of $\mathbb{R}$ such that $k > 0$. Then

$$\begin{aligned} p(kx) &= \inf\{r > 0 : kx \in rE\} = \inf\{kr > 0 : kx \in krE\} \\ &= \inf\{kr > 0 : x \in rE\} = k \inf\{r > 0 : x \in rE\} = kp(x). \end{aligned}$$

That is, $p(kx) = kp(x)$.

■

PROPOSITION 16.1.4 (Balanced and Convex)**.** A Minkowski functional for a balanced and convex neighborhood of 0 is a seminorm.

Proof. Let $\mathcal{V}$ be a topological vector space over field $\mathbb{K}$. Let E be a balanced convex neighborhood of 0 in $\mathcal{V}$. Let p denote the Minkowski functional for E. I will show that p is a seminorm. I have showed that Minkowski functionals for convex sets are subadditive. It is clear that $\forall x \in \mathcal{V}$, $p(x) \geq 0$. Let x be an arbitrary element of $\mathcal{V}$. Let k be an arbitrary element of $\mathbb{K}$. If $k = 0$, then $p(kx) = p(0x) = p(0) = 0 = 0p(x) = |k|p(x)$ and we are done. Otherwise, $k \neq 0$. Then

$$\begin{aligned} p(kx) &= \inf\{r > 0 : kx \in rE\} = \inf\{|k|r > 0 : kx \in |k|rE\} \\ &= \inf\{|k|r > 0 : kx \in krE\}, \text{ since } E \text{ is balanced} \\ &= \inf\{|k|r > 0 : x \in rE\} = |k| \inf\{r > 0 : x \in rE\} = |k|p(x). \end{aligned}$$

That is, $p(kx) = |k|p(x)$. ■

PROPOSITION 16.1.5 (Open and Convex)**.** Let $\mathcal{V}$ be a topological vector space. Let E be an open and convex neighborhood of 0 in $\mathcal{V}$. Let p_E denote the Minkowski functional for E. Then

$$E = \{x \in \mathcal{V} : p_E(x) < 1\}.$$

Proof. Let F denote the set $\{x \in \mathcal{V} : p_E(x) < 1\}$. I will show that $E = F$.
Forward Direction:

Let x be an arbitrary element of E. I will show that $x \in F$. Define a map $f : \mathbb{R} \to \mathcal{V}$ by $f(t) := tx$. Then f is continuous. Since E is open in $\mathcal{V}$ and $f : \mathbb{R} \to \mathcal{V}$ is continuous, we get $f^{-1}(E)$ is open in $\mathbb{R}$. Notice $x = f(1) \in E$. So $1 \in f^{-1}(E)$. Since $f^{-1}(E)$ is open and $1 \in f^{-1}(E)$, $\exists \delta > 0$ such that $1 + \delta \in f^{-1}(E)$. So $f(1 + \delta) \in E$. So $(1 + \delta)x \in E$. So $x \in \frac{1}{1+\delta}E$. So $p_E(x) \leq \frac{1}{1+\delta}$, which further, is < 1. So $x \in F$.

Backward Direction:

Let x be an arbitrary element of F. I will show that $x \in E$. Since $x \in F$, by definition of F, $p_E(x) < 1$. So by definition of the Minkowski functional, $\exists r_0 > 0$ such that $r_0 < 1$ and $x \in r_0 E$. Since $r_0 < 1$, we have $r_0 E \subseteq E$. So $x \in E$.

■

16.1.4 Separating Families

DEFINITION (Separating Family of Seminorms)**.** Let $\mathcal{V}$ be a vector space. Let Γ be a family of seminorms on $\mathcal{V}$. We say that Γ is **separating** if $\forall x \in \mathcal{V}$ such that $x \neq 0$, $\exists p \in \Gamma$ such that $p(x) \neq 0$.

DEFINITION (Separating Family of Linear Functionals)**.** Let $\mathcal{V}$ be a vector space. Let $\mathcal{L}$ be a collection of linear functionals on $\mathcal{V}$. Define for each $\varphi \in \mathcal{L}$ a seminorm τ_φ on $\mathcal{V}$ by $\tau_\varphi(x) := |\varphi(x)|$. We say that $\mathcal{L}$ is **separating** if the set Γ given by $\Gamma := \{\tau_\varphi : \varphi \in \mathcal{L}\}$ is a separating family of seminorms.

16.2 Locally Convex Space

DEFINITION (Locally Convex Space)**.** Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. We say that $\mathcal{T}$ is **locally convex** if each point in $\mathcal{V}$ admits a neighborhood base consisting of only convex sets.

PROPOSITION 16.2.1. Any convex neighborhood of 0 in a topological vector space contains a convex, open, and balanced neighborhood of 0.

Proof. Let $\mathcal{V}$ be a topological vector space. Let U be an arbitrary element of $\mathcal{U}_0^{\mathcal{V}}$ that is convex. Since $U \in \mathcal{U}_0^{\mathcal{V}}$, U contains an open and balanced neighborhood N of 0. Then $\operatorname{conv}(N) \subseteq \operatorname{conv}(U) = U$ and $\operatorname{conv}(N)$ is open and balanced. So $\operatorname{conv}(N)$ is the set desired. This completes the proof. ■

PROPOSITION 16.2.2. A quotient space of a locally convex space by a closed subspace is a locally convex space with the quotient topology.

Proof. Let $\mathcal{V}$ be a locally convex space. Let $\mathcal{W}$ be a closed subspace of $\mathcal{V}$. Let q denote the canonical quotient map from $\mathcal{V}$ to $\mathcal{V}/\mathcal{W}$. Then q is continuous, open, and linear, and $\mathcal{V}/\mathcal{W}$ is a topological vector space with the quotient topology. So there remains only to show that the quotient topology is locally convex. Let U be an arbitrary element of $\mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$. Since $U \in \mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$ and q is continuous, we get $q^{-1}(U) \in \mathcal{U}_0^{\mathcal{V}}$. Since $\mathcal{V}$ is locally convex and $U \in \mathcal{U}_0^{\mathcal{V}}$, $\exists C \in \mathcal{U}_0^{\mathcal{V}}$ such that $C \subseteq q^{-1}(U)$. Since $C \in \mathcal{U}_0^{\mathcal{V}}$ and q is open, we get $q(C) \in \mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$. Since C is convex and q is linear, $q(C)$ is convex. Since $C \subseteq q^{-1}(U)$, $q(C) \subseteq U$. So $q(C)$ is the set desired. This completes the proof. ■

DEFINITION (Generated Topology)**.** Let $\mathcal{V}$ be a vector space. Let Γ be a separating family of seminorms on $\mathcal{V}$. Define for each $x \in \mathcal{V}$, $F \subseteq \Gamma$ finite, and $\varepsilon > 0$ a subset $\mathcal{N}(x, F, \varepsilon)$ of $\mathcal{V}$ by

$$\mathcal{N}(x, F, \varepsilon) := \{y \in \mathcal{V} : \forall p \in F, p(x - y) < \varepsilon\}.$$

Define a collection $\mathcal{B}$ of subsets of $\mathcal{V}$ by

$$\mathcal{B} := \{\mathcal{N}(x, F, \varepsilon) : x \in \mathcal{V}, F \subseteq \mathcal{F} \text{ finite }, \varepsilon > 0\}.$$

We define the **topology generated by** Γ to be the topology generated by $\mathcal{B}$.

THEOREM 16.1. Let $\mathcal{V}$ be a vector space. Let Γ be a separating family of seminorms on $\mathcal{V}$. Then Γ generates a locally convex topology on $\mathcal{V}$.

THEOREM 16.2. Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space. Then there exists a separating family Γ of seminorms on $\mathcal{V}$ that generates the original topology $\mathcal{T}$.

Proof. Since $(\mathcal{V}, \mathcal{T})$ is locally convex, it admits a neighborhood base $\mathcal{B}_0$ at 0 consisting of only convex, balanced, and open sets. Denote the Minkowski functional for each $E \in \mathcal{B}_0$ by p_E. Define $\Gamma := \{p_E : E \in \mathcal{B}_0\}$. Since each E is balanced and convex, each p_E is a seminorm. Now I claim that Γ is separating. Let x be an arbitrary element of $\mathcal{V}$ such that

$x \neq 0$. Since $x \neq 0$ and $\mathcal{T}$ is Hausdorff, $\exists U \in \mathcal{U}_0^{\mathcal{T}}$ such that $x \notin U$. Since $\mathcal{V}$ is locally convex, I can assume without loss of generality that U is convex, balanced, and open. Since U is open, $U = \{x \in \mathcal{V} : p_U(x) < 1\}$. Since $x \notin U$, we get $p_U(x) \geq 1$. So $p_U(x) \neq 0$. So p_U is the function desired. So Γ is separating. Let $\mathcal{T}'$ denote the topology generated by Γ. Now I claim that $\mathcal{T}' = \mathcal{T}$. Let $\mathcal{B}'$ denote the base generated by Γ.

Forward Direction: Show that $\mathcal{T} \subseteq \mathcal{T}'$.

Let E be an arbitrary element of $\mathcal{B}_0$. Then E is convex and open and hence $E = \mathcal{N}(0, p_E, 1) \in \mathcal{B}' \subseteq \mathcal{T}'$. That is, $E \in \mathcal{T}'$. So $\mathcal{T}'$ contains the neighborhood base $\mathcal{B}_0$ at 0 for $\mathcal{T}$. So $\mathcal{T} \subseteq \mathcal{T}'$.

Backward Direction: Show that $\mathcal{T}' \subseteq \mathcal{T}$.

Since $\mathcal{B}'$ is a base for $\mathcal{T}'$, $\mathcal{N}_0 := \{\mathcal{N}(0, F, \varepsilon) : F \subseteq \Gamma \text{ finite }, \varepsilon > 0\}$ is a neighborhood base at 0 for $\mathcal{T}'$. Let $\mathcal{N}$ be an arbitrary element of $\mathcal{N}_0$. Then $\mathcal{N} = \mathcal{N}(0, F, \varepsilon)$ for some $F \subseteq \Gamma$ finite and some $\varepsilon > 0$. Let p_E be an arbitrary element of F. Then E is convex and open and hence p_E is bounded above by 1 on E. So p_E is continuous under $\mathcal{T}$. So $\mathcal{N}(0, p_E, \varepsilon) = p_E^{-1}([0, \varepsilon))$ is open under $\mathcal{T}$. So $\mathcal{N}(0, F, \varepsilon) = \bigcap_{p_E \in F} \mathcal{N}(0, p_E, \varepsilon)$ is open under $\mathcal{T}$. So $\mathcal{T}$ contains the neighborhood base $\mathcal{N}_0$ at 0 for $\mathcal{T}'$. So $\mathcal{T}' \subseteq \mathcal{T}$.

■

REMARK. The above two theorems say that separating families of seminorms on vector spaces give rise to locally convex topologies, and that all locally convex topologies arise in this manner.

EXAMPLE 16.2.1. The norm topology is exactly the locally convex topology generated by $\Gamma = \{\|\cdot\|\}$.

16.3 Relation to Other Topologies

PROPOSITION 16.3.1. A locally convex topology is equivalent to a metric topology if and only if it can be generated by a countable family of seminorms.

PROPOSITION 16.3.2. A locally convex topology is equivalent to a norm topology if and only if it can be generated by a finite family of seminorms.

16.4 Continuity in Locally Convex Spaces

PROPOSITION 16.4.1. Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space. Let Γ be a separating family of seminorms on $\mathcal{V}$ that generates the original topology $\mathcal{T}$. Let p be a seminorm on $\mathcal{V}$. Then p is continuous if and only if $\exists \kappa > 0$ and $p_1..p_m \in \Gamma$ where $m \in \mathbb{N}$ such that

$$\forall x \in \mathcal{V}, \quad p(x) \leq \kappa \max\{p_i(x)\}_{i=1}^m.$$

Proof. **Forward Direction**: Assume that p is continuous. I will show that $\exists \kappa > 0$ and $p_1..p_m \in \Gamma$ where $m \in \mathbb{N}$ such that $\forall x \in \mathcal{V}$, $p(x) \leq \kappa \max\{p_i(x)\}_{i=1}^m$. Notice $[0, 1)$ is open in $[0, +\infty)$. Since p is a continuous function from $\mathcal{V}$ to $[0, +\infty)$ and $[0, 1)$ is open in $[0, +\infty)$, we get $\mathcal{M} := p^{-1}([0, 1))$ is open in $\mathcal{V}$. Note that $p(0) = 0 \in [0, 1)$. So $0 \in \mathcal{M}$. So $\mathcal{M}$ is an open neighborhood of 0 in $\mathcal{V}$. Then $\mathcal{M}$ contains some basic neighborhood $\mathcal{N} := \mathcal{N}(0, \{p_i\}_{i=1}^m, \varepsilon)$ for some $m \in \mathbb{N}$, $p_1..p_m \in \Gamma$, and $\varepsilon > 0$. Consider an arbitrary element x of $\mathcal{V}$. Let r_x denote the number $\max\{p_i(x)\}_{i=1}^m$.

- **Case 1**: $r_x = 0$. Then for any $k > 0$, we have

  $$\forall i = 1..m, \quad p_i(kx) = kp_i(x) \leq kr_x = k \cdot 0 = 0 < \varepsilon.$$

  So $kx \in \mathcal{N}$ and hence $kx \in \mathcal{M}$. So $p(kx) < 1$. So

  $$p(x) = \frac{1}{k} p(kx) < \frac{1}{k} \cdot 1 = \frac{1}{k}.$$

  Since $k > 0$ was chosen arbitrarily, we get $p(x) = 0$. So $p(x) = r_x \leq 1 \cdot r_x$.

- **Case 2**: $r_x > 0$. Then we have

  $$\forall i = 1..m, \quad p_i(\frac{\varepsilon}{2r_x} x) = \frac{\varepsilon}{2r_x} p_i(x) \leq \frac{\varepsilon}{2r_x} r_x = \frac{\varepsilon}{2} < \varepsilon.$$

  So $\frac{\varepsilon}{2r_x} x \in \mathcal{N}$ and hence $\frac{\varepsilon}{2r_x} x \in \mathcal{M}$. So $p(\frac{\varepsilon}{2r_x} x) < 1$. So

  $$p(x) = \frac{2r_x}{\varepsilon} p(\frac{\varepsilon}{2r_x} x) < \frac{2r_x}{\varepsilon} \cdot 1 = \frac{2}{\varepsilon} \cdot r_x.$$

Take $\kappa := \max\{1, \frac{2}{\varepsilon}\}$. Then $p(x) \leq \kappa r_x$. That is,

$$\forall x \in \mathcal{V}, \quad p(x) \leq \kappa \max\{p_i(x)\}_{i=1}^m.$$

Backward Direction: Assume that $\exists \kappa > 0$ and $p_1..p_m \in \Gamma$ where $m \in \mathbb{N}$ such that $\forall x \in \mathcal{V}$, $p(x) \leq \kappa \max\{p_i(x)\}_{i=1}^m$. I will show that p is continuous. Note that $\mathcal{N} :=$

$\mathcal{N}(0, \{p_i\}_{i=1}^m, 1)$ is an open neighborhood of 0 in $\mathcal{V}$. Consider an arbitrary element x of $\mathcal{N}$. Then $\forall i = 1..m$, $p_i(x) < 1$. So

$$p(x) \leq \kappa \max\{p_i(x)\}_{i=1}^m < \kappa \cdot 1 = \kappa.$$

So p is bounded above by κ on an open neighborhood of 0. So p is continuous.

■

PROPOSITION 16.4.2. Let $(\mathcal{V}, \mathcal{T}_{\mathcal{V}})$ and $(\mathcal{W}, \mathcal{T}_{\mathcal{W}})$ be two locally convex spaces. Let $\Gamma_{\mathcal{V}}$ be a separating family of seminorms on $\mathcal{V}$ that generates the topology $\mathcal{T}_{\mathcal{V}}$. Let $\Gamma_{\mathcal{W}}$ be a separating family of seminorms on $\mathcal{W}$ that generates the topology $\mathcal{T}_{\mathcal{W}}$. Let T be a linear map from $\mathcal{V}$ to $\mathcal{W}$. Then T is continuous if and only if

$$\forall q \in \Gamma_{\mathcal{W}}, \quad \exists \kappa > 0, \exists p_1..p_m \in \Gamma_{\mathcal{V}} \text{ where } m \in \mathbb{N} \text{ such that}$$

$$\forall x \in \mathcal{V}, \quad q(Tx) \leq \kappa \max\{p_i(x)\}_{i=1}^m.$$

COROLLARY 16.1. Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space. Let f be a linear functional on $\mathcal{V}$. Then f is continuous if and only if there is a continuous seminorm p on $\mathcal{V}$ such that

$$\forall x \in \mathcal{V}, \quad |f(x)| \leq p(x).$$

16.5 Convergence in Locally Convex Spaces

PROPOSITION 16.5.1. Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space. Let Γ be a separating family of seminorms on $\mathcal{V}$ that generates the original topology $\mathcal{T}$. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in $\mathcal{V}$. Then $(x_\lambda)_{\lambda \in \Lambda}$ converges to a point $x \in \mathcal{V}$ if and only if

$$\forall p \in \Gamma, \quad \lim_{\lambda \in \Lambda} p(x_\lambda - x) = 0.$$

16.6 Strong Operator Topology

16.7 Weak Operator Topology

Chapter 17

The Hahn-Banach Theorem

17.1 Extension Results

THEOREM 17.1 (The Hahn-Banach Theorem - 2)**.** Let $\mathcal{V}$ be a vector space. Let $\mathcal{M}$ be a linear manifold of $\mathcal{V}$. Let p be a seminorm on $\mathcal{V}$. Let f be a linear functional on $\mathcal{M}$. Suppose that $\forall m \in \mathcal{M}$, $|f(m)| \leq p(m)$. Then there exists a linear functional g on $\mathcal{V}$ such that $g|_{\mathcal{M}} = f$ and that $\forall x \in \mathcal{V}$, $|g(x)| \leq p(x)$.

COROLLARY 17.1. Let $\mathcal{V}$ be a locally convex space. Let $\mathcal{M}$ be a linear manifold of $\mathcal{V}$. Let $f \in \mathcal{M}^*$. Then $\exists g \in \mathcal{V}^*$ such that $g|_{\mathcal{M}} = f$.

THEOREM 17.2 (The Hahn-Banach Theorem - 3)**.** Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a linear manifold of $\mathfrak{X}$. Let $f \in \mathcal{M}^*$. Then $\exists g \in \mathfrak{X}^*$ such that $g|_M = f$ and that $\|g\| = \|f\|$.

COROLLARY 17.2. Let $\mathcal{V}$ be a locally convex space. Let $\{x_i\}_{i=1}^m$ be a linearly independent set of vectors in $\mathcal{V}$ where $m \in \mathbb{N}$. Let $k_1..k_m$ be arbitrary elements of $\mathbb{K}$. Then $\exists g \in \mathcal{V}^*$ such that $\forall i = 1..m$, $g(x_i) = k_i$.

COROLLARY 17.3. Let $\mathcal{V}$ be a locally convex space. Let $\mathcal{M}$ be a finite-dimensional linear manifold of $\mathcal{V}$. Then $\mathcal{M}$ is topologically complemented.

Proof. Let $\{m_i\}_{i=1}^n$ be a basis for $\mathcal{M}$ where $n = \dim(\mathcal{M})$. Then $\{m_i\}_{i=1}^n$ is a linearly independent set of vectors in $\mathcal{V}$. By Corollary 17.2, for each $i = 1..n$, $\exists \rho_i \in \mathcal{V}^*$ such that $\rho_i(m_j) = \delta_{i,j}$. Define $\mathcal{Y} := \bigcap_{i=1}^m \ker(\rho_i)$. Since the ρ_i's are continuous, the $\ker(\rho_i)$'s are closed. So $\mathcal{Y}$ is closed. Since $\dim(\mathcal{M}) < \infty$, $\mathcal{M}$ is closed.

Now I will show that $\mathcal{V} = \mathcal{M} + \mathcal{Y}$. Let v be an arbitrary element of $\mathcal{V}$. Define for $i = 1..n$ a scalar k_i as $k_i := \rho_i(v)$. Define a point m as $m := \sum_{i=1}^n k_i m_i$. Then $m \in \mathcal{M}$. Define a point y as $y := v - m$. Then $\forall i = 1..n$, we have

$$\begin{aligned}\rho_i(y) = \rho_i(v-m) = \rho_i(v - \sum_{j=1}^n k_j m_j) = \rho_i(v) - \sum_{j=1}^n k_j \rho_i(m_j)\\ = k_i - \sum_{j=1}^n k_j \delta_{i,j} = k_i - k_i = 0.\end{aligned}$$

That is, $\rho_i(y) = 0$. So $\forall i = 1..n$, $y \in \ker(\rho_i)$. So $y \in \bigcap_{i=1}^n \ker(\rho_i) = \mathcal{Y}$. So $\forall v \in \mathcal{V}$, $v = m+y$ where $m \in \mathcal{M}$ and $y \in \mathcal{Y}$. So $\mathcal{V} = \mathcal{M} + \mathcal{Y}$.

Now I will show that $\mathcal{M} \cap \mathcal{Y} = \{0\}$. Note that $0 \in \mathcal{M} \cap \mathcal{Y}$. Let z be an arbitrary element of $\mathcal{M} \cap \mathcal{Y}$. Since $z \in \mathcal{M}$, there exist scalars $\{r_j\}_{j=1}^n$ such that $z = \sum_{j=1}^n r_j m_j$. On one hand, since $z = \sum_{j=1}^n r_j m_j$, $\forall i = 1..n$, we have

$$\rho_i(z) = \rho_i(\sum_{j=1}^n r_j m_j) = \sum_{j=1}^n r_j \rho_i(m_j) = \sum_{j=1}^n r_j \delta_{i,j} = r_i.$$

That is, $\rho_i(z) = r_i$. On the other hand, since $z \in \mathcal{Y} = \bigcap_{i=1}^n \ker(\rho_i)$, $\forall i = 1..n$, we have $\rho_i(z) = 0$. So $\forall i = 1..n$, $r_i = 0$. So $z = \sum_{j=1}^n r_j m_j = 0$. So $\mathcal{M} \cap \mathcal{Y} = \{0\}$.

So $\mathcal{M}$ is topologically complemented by $\mathcal{Y}$. ∎

COROLLARY 17.4. Let $\mathfrak{X}$ be a normed linear space. Let $x \in \mathfrak{X}$. Then

$$\|x\| = \max\{|x^*(x)| : x^* \in \mathfrak{X}^*, \|x^*\| \leq 1\}.$$

i.e., $\exists x^* \in \mathfrak{X}^*$ with $\|x^*\| = 1$ such that $\|x\| = |x^*(x)|$.

COROLLARY 17.5. Let $\mathfrak{X}$ be a normed linear space. Then the canonical embedding

$\mathfrak{J} : \mathfrak{X} \to \mathfrak{X}^{**}$ is an isometry.

Proof. Let x be an arbitrary element of $\mathfrak{X}$. We are to prove that $\|x\|_{\mathfrak{X}} = \|\mathfrak{J}x\|_{\mathfrak{X}^{**}}$. Let $\hat{x}$ denote $\mathfrak{J}x$. On one hand, for any $y^* \in \mathfrak{X}^*$, we have

$$|\hat{x}(y^*)| = |y^*(x)| \leq \|y^*\|\|x\|.$$

So $\|\hat{x}\| \leq \|x\|$. On the other hand, by Corollary 17.4, there exists $x^* \in \mathfrak{X}^*$ with $\|x^*\| \leq 1$ such that $|x^*(x)| = \|x\|$. So

$$\|\hat{x}\| \geq |\hat{x}(x^*)| = |x^*(x)| = \|x\|.$$

That is, $\|\hat{x}\| \geq \|x\|$. Since $\forall x \in \mathfrak{X}$, $\|x\| = \|\mathfrak{J}x\|$, we have that $\mathfrak{J}$ is an isometry. ■

COROLLARY 17.6. Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{Y}$ be a closed subspace of $\mathfrak{X}$. Let $z \in \mathfrak{X} \setminus \mathfrak{Y}$. Then $\exists x^* \in \mathfrak{X}^*$ with $\|x^*\| = 1$ such that $x^*|_{\mathfrak{Y}} = 0$ and $x^*(z) = d(z, \mathfrak{Y})$.

Proof. Since $z \notin \mathfrak{Y}$, $\mathfrak{Y} \neq z + \mathfrak{Y}$. By Corollary 17.4, $\exists \xi^* \in (\mathfrak{X}/\mathfrak{Y})^*$ with $\|\xi^*\| = 1$ such that $|\xi^*(z + \mathfrak{Y})| = \|z + \mathfrak{Y}\| = d(z, \mathfrak{Y})$. Let q be the canonical quotient map from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{Y}$. Define a map from $\mathfrak{X}$ to $\mathbb{K}$ as $x^* := \xi^* \circ q$.

Show that $x^* \in \mathfrak{X}^*$:

Clearly x^* is linear. Recall that $\|\xi^*\| = 1$ and that q is a contraction map and hence $\|q\| \leq 1$. So $\|x^*\| \leq \|\xi^*\|\|q\| \leq 1$. So $x^* \in \mathfrak{X}^*$.

Show that $\|x^*\| = 1$:

Since $\|\xi^*\| = 1$, we can find a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathfrak{X}/\mathfrak{Y}$ such that $\forall n \in \mathbb{N}$, we have $\|t_n\| \leq 1$ and $1 - \frac{1}{n} < |\xi^*(t_n)| \leq 1$. So $\lim_{n\in\mathbb{N}} |\xi^*(t_n)| = 1$. Define for each $n \in \mathbb{N}$ a point $x_n \in \mathfrak{X}$ to be such that $q(x_n) = \frac{n}{n+1} t_n$. Then $\forall n \in \mathbb{N}$, we have

$$\|x_n + \mathfrak{Y}\| = \|q(x_n)\| = \|\frac{n}{n+1} t_n\| = \frac{n}{n+1}\|t_n\| < \|t_n\| \leq 1.$$

That is, $\|x_n + \mathfrak{Y}\| < 1$. So $\forall n \in \mathbb{N}$, $\exists y_n \in \mathfrak{Y}$ such that $\|x_n + y_n\| < 1$. On the other hand, we have

$$\begin{aligned}
\lim_{n\in\mathbb{N}} |x^*(x_n + y_n)| &= \lim_{n\in\mathbb{N}} |\xi^*(q(x_n + y_n))| = \lim_{n\in\mathbb{N}} |\xi^*(x_n + y_n + \mathfrak{Y})| = \lim_{n\in\mathbb{N}} \|x_n + y_n + \mathfrak{Y}\| \\
&= \lim_{n\in\mathbb{N}} \|x_n + \mathfrak{Y}\| = \lim_{n\in\mathbb{N}} |\xi^*(x_n + \mathfrak{Y})| = \lim_{n\in\mathbb{N}} |\xi^*(q(x_n))| \\
&= \lim_{n\in\mathbb{N}} |\xi^*(\frac{n}{n+1} t_n)| = \lim_{n\in\mathbb{N}} |\frac{n}{n+1} \xi^*(t_n)|, \text{ by linearity of } \xi^* \\
&= \lim_{n\in\mathbb{N}} \frac{n}{n+1} \cdot \lim_{n\in\mathbb{N}} |\xi^*(t_n)| = 1 \cdot 1 = 1.
\end{aligned}$$

That is, $\lim_{n\in\mathbb{N}} |x^*(x_n+y_n)| = 1$. Since $\forall n \in \mathbb{N}$, $\|x_n+y_n\| < 1$ and $\lim_{n\in\mathbb{N}} |x^*(x_n+y_n)| = 1$, we get $\|x^*\| \geq 1$. Recall that we have proved $\|\|x^*\| \leq 1$. So $\|\|x^*\| = 1$.

Show that $x^*|_{\mathfrak{Y}} = 0$:

Let y be an arbitrary element of $\mathfrak{Y}$. Then we have

$$x^*(y) = \xi^*(q(y)) = \xi^*(y+\mathfrak{Y}) = d(y,\mathfrak{Y}) = 0.$$

So $x^*|_{\mathfrak{Y}} = 0$.

Show that $x^*(z) = d(z,\mathfrak{Y})$:

Note that

$$x^*(z) = |\xi^*(q(z))| = |\xi^*(z+\mathfrak{Y})| = d(z,\mathfrak{Y}).$$

That is, $x^*(z) = d(z,\mathfrak{Y})$.

■

17.2 Separation Results

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PROPOSITION 17.2.1. Let $\mathcal{V}$ be a locally convex space over field $\mathbb{K}$. Let G be a non-empty, open, and convex set in $\mathcal{V}$. Suppose that $0 \notin G$. Then there exists a closed hyperplane $\mathcal{M}$ in $\mathcal{V}$ such that $G \cap \mathcal{M} = \emptyset$.

Proof.

Case 1: $\mathbb{K} = \mathbb{R}$.

Since $G \neq \emptyset$, take $x_0 \in G$. Define a set H as $H := x_0 - G$. Then H is non-empty, open, convex, and $0 \in H$. Let p_H denote the Minkowski functional on H. Since H is an open convex neighborhood of 0, $H = \{x \in \mathcal{V} : p_H(x) < 1\}$. Define a set $\mathcal{W}$ by $\mathcal{W} := \mathbb{R}x_0$. Then $\mathcal{W}$ is a linear manifold of $\mathcal{V}$. Define a map $f : \mathcal{W} \to \mathbb{R}$ by $f(kx_0) := kp_H(x_0)$. Then f is a linear functional on $\mathcal{W}$. Note that

$$f(kx_0) = kp_H(x_0) = p_H(kx_0), \text{ for } k \geq 0, \text{ and}$$
$$f(kx_0) = kp_H(x_0) < 0 \leq p_H(kx_0), \text{ for } k < 0.$$

■

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THEOREM 17.3 (The Hahn-Banach Theorem - 4)**.** Let $\mathcal{V}$ be a locally convex space. Let A and B be two non-empty, open, convex, and disjoint sets in $\mathcal{V}$. Then $\exists f \in \mathcal{V}^*$,

$\exists \kappa \in \mathbb{R}$ such that

$$\forall a \in A, b \in B, \quad \Re f(a) > \kappa > \Re f(b).$$

THEOREM 17.4 (The Hahn-Banach Theorem - 5)**.** Let $\mathcal{V}$ be a locally convex space. Let A and B be two non-empty, closed, convex, and disjoint sets in $\mathcal{V}$. Suppose B is compact. Then $\exists f \in \mathcal{V}^*$, $\exists \alpha, \beta \in \mathbb{R}$ such that

$$\forall a \in A, b \in B, \quad \Re f(a) \geq \alpha > \beta \geq \Re f(b).$$

COROLLARY 17.7. Let $\mathcal{V}$ be a locally convex space. Let A be a non-empty set in $\mathcal{V}$. Then the closed convex hull $\text{clconv}(A)$ equals the intersection of all closed half-spaces that contain A.

Proof. Let Ω denote the set of all closed half-spaces that contain A.

Forward Direction:

Note that $\forall \mathcal{S} \in \Omega$, $\mathcal{S}$ is closed and convex. So $\bigcap_{\mathcal{S}\in\Omega} \mathcal{S}$ is closed and convex. Note also that $A \subseteq \bigcap_{\mathcal{S}\in\Omega} \mathcal{S}$. So $\text{clconv}(A) \subseteq \bigcap_{\mathcal{S}\in\Omega} \mathcal{S}$.

Backward Direction:

Let z be an arbitrary element outside $\text{clconv}(A)$. Then $\text{clconv}(A)$ and $\{z\}$ are two non-empty, closed, convex, and disjoint sets and we have that $\{z\}$ is compact. By the Hahn-Banach theorem, version 5, $\exists f \in \mathcal{V}^*$, $\exists \alpha, \beta \in \mathbb{R}$ such that

$$\forall a \in \text{clconv}(A), \quad \Re f(a) \geq \alpha > \beta \geq \Re f(z).$$

Define a set $\mathcal{S}_0$ by $\mathcal{S}_0 := \{x \in \mathcal{V} : \Re f(x) \geq \alpha\}$. Then $\mathcal{S}_0$ is a closed half-space of $\mathcal{V}$ and $z \notin \mathcal{S}_0$. So $z \notin \bigcap_{\mathcal{S}\in\Omega} \mathcal{S}$. So $\bigcap_{\mathcal{S}\in\Omega} \mathcal{S} \subseteq \text{clconv}(A)$.

■

Chapter 18

Reflexive Banach Space

18.1 Definitions

DEFINITION (Reflexive)**.** Let $\mathfrak{X}$ be a Banach space. Let $\mathfrak{J}$ denote the canonical embedding of $\mathfrak{X}$ into $\mathfrak{X}^{**}$. We say that $\mathfrak{X}$ is **reflexive** if $\mathfrak{J}$ is an isometric isomorphism between $\mathfrak{X}$ and $\mathfrak{X}^{**}$.

18.2 Properties

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PROPOSITION 18.2.1 (Closed Subspace)**.** Let $\mathfrak{X}$ be a reflexive Banach space. Let $\mathfrak{Y}$ be a closed subspace of $\mathfrak{X}$. Then $\mathfrak{Y}$ is a reflexive Banach space.

Proof. Since $\mathfrak{X}$ is a Banach space and $\mathfrak{Y}$ is a closed subspace of $\mathfrak{X}$, $\mathfrak{Y}$ is a Banach space. Let $\mathfrak{J}$ denote the canonical embedding of $\mathfrak{Y}$ into $\mathfrak{Y}^{**}$. By the Hahn-Banach Theorem (Corollary 17.5), $\mathfrak{J}$ is an isometry and hence automatically injective.

To see that $\mathfrak{J}$ is surjective, consider an arbitrary element $y^{**} \in \mathfrak{Y}^{**}$. Define a map $x^{**} : \mathfrak{X}^* \to \mathbb{K}$ by $x^{**}(z^*) := y^{**}(z^*|_{\mathfrak{Y}})$. Then it is easy to check that x^{**} is linear and continuous. So $x^{**} \in \mathfrak{X}^{**}$. Since $\mathfrak{X}$ is reflexive, $\exists x \in \mathfrak{X}$ such that $\hat{x} = x^{**}$.

Assume for the sake of contradiction that $x \notin \mathfrak{Y}$. Since $\mathfrak{Y}$ is a closed subspace of $\mathfrak{X}$ and $x \in \mathfrak{X} \setminus \mathfrak{Y}$, By the Hahn-Banach Theorem, $\exists g \in \mathfrak{X}^*$ such that $g|_{\mathfrak{Y}} = 0$ and $g(x) \neq 0$. Now we have $\hat{x}(g) = g(x) \neq 0$ and $x^{**}(g) = y^{**}(g|_{\mathfrak{Y}}) = y^{**}(0) = 0$. So $\hat{x}(g) \neq x^{**}(g)$. However, this contradicts to the previous conclusion that $\hat{x} = x^{**}$. So $x \in \mathfrak{Y}$.

Now I will show that $\mathfrak{J}(x) = y^{**}$. Consider an arbitrary element $w^* \in \mathfrak{Y}^*$. Let $v^* \in \mathfrak{X}^*$ be a Hahn-Banach extension of w^*. Then

$$\begin{aligned}
y^{**}(w^*) &= y^{**}(v^*|_{\mathfrak{Y}}), \text{ since } v^* \text{ is an extension of } w^* \\
&= x^{**}(v^*), \text{ by definition of } x^{**} \\
&= \hat{x}(v^*), \text{ by the choice of } x \\
&= v^*(x), \text{ by definition of } \hat{x} \\
&= w^*(x), \text{ since } v^* \text{ is an extension of } w^* \\
&= \mathfrak{J}(x)(w^*), \text{ by definition of } \mathfrak{J}(x).
\end{aligned}$$

That is, $y^{**}(w^*) = \mathfrak{J}(x)(w^*)$. So $\mathfrak{J}(x) = y^{**}$. So $\mathfrak{J}$ is surjective. This completes the proof. ∎

PROPOSITION 18.2.2 (Dual Space)**.** Let $\mathfrak{X}$ be a Banach space. Then $\mathfrak{X}$ is reflexive if and only if $\mathfrak{X}^*$ is reflexive.

Proof. **Forward Direction**: Assume that $\mathfrak{X}$ is reflexive. I will show that $\mathfrak{X}^*$ is reflexive. ∎

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PROPOSITION 18.2.3 (Image under Isometric Isomorphism)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Suppose that $\mathfrak{X}$ is reflexive and that there is an isometric isomorphism between $\mathfrak{X}$ and $\mathfrak{Y}$.

PROPOSITION 18.2.4. Let $\mathfrak{X}$ be a Banach space. Then $\mathfrak{X}$ is reflexive if and only if $\mathfrak{X}_1$ is weakly compact.

Proof. **Forward Direction**: Assume that $\mathfrak{X}$ is reflexive. I will show that $\mathfrak{X}_1$ is weakly compact. Since $\mathfrak{X}^*$ is a Banach space, by the Banach-Alaoglu Theorem, $\mathfrak{X}_1^{**}$ is weak*-compact. i.e., $\mathfrak{X}_1^{**}$ is compact in the space $(\mathfrak{X}^{**}, \sigma(\mathfrak{X}^{**}, \mathfrak{X}^*))$. Since $\mathfrak{X}$ is reflexive, $\hat{\mathfrak{X}} = \mathfrak{X}^{**}$. So $\mathfrak{X}$ is isometrically isomorphic to $\mathfrak{X}^{**}$. So $\mathfrak{X}_1$ is compact in the space $(\mathfrak{X}, \sigma(\mathfrak{X}, \mathfrak{X}^*))$. i.e., $\mathfrak{X}_1$ is weakly compact.

Backward Direction: Assume that $\mathfrak{X}_1$ is weakly compact. I will show that $\mathfrak{X}$ is reflexive. $\widehat{\mathfrak{X}_1}$ is compact in the space $(\mathfrak{X}^{**}, \sigma(\mathfrak{X}^{**}, \mathfrak{X}^*))$. i.e., $\widehat{\mathfrak{X}_1}$ is weak*-compact. Since the weak* topology is Hausdorff and $\widehat{\mathfrak{X}_1}$ is weak*-compact, $\widehat{\mathfrak{X}_1}$ is weak*-closed. Since $\mathfrak{X}$ is a Banach space, by the Goldstine's Theorem, $\widehat{\mathfrak{X}_1}$ is weak*-dense in $\mathfrak{X}_1^{**}$. Since $\widehat{\mathfrak{X}_1}$ is weak*-closed and weak*-dense in $\mathfrak{X}_1^{**}$, $\widehat{\mathfrak{X}_1} = \mathfrak{X}_1^{**}$. So $\hat{\mathfrak{X}} = \mathfrak{X}^{**}$. So $\mathfrak{X}$ is reflexive.

■

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Chapter 19

Weak Topology

19.1 Definitions

DEFINITION (Dual Pair)**.** Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let $\mathcal{L}$ be a linear manifold of $\mathcal{V}^{\#}$ and a separating family of linear functionals on $\mathcal{V}$. We define a **dual pair** to be the pair $(\mathcal{V}, \mathcal{L})$.

DEFINITION (Topology Generated by Linear Functionals)**.** Let $(\mathcal{V}, \mathcal{L})$ be a dual pair. Define for each $\varphi \in \mathcal{L}$ a function $p_\varphi : \mathcal{V} \to \mathbb{R}$ by $p_\varphi(x) := |\varphi(x)|$. Then each p_φ is a seminorm on $\mathcal{V}$. Define a family $\Gamma_{\mathcal{L}}$ of seminorms on $\mathcal{V}$ by $\Gamma_{\mathcal{L}} := \{p_\varphi : \varphi \in \mathcal{L}\}$. Then $\Gamma_{\mathcal{L}}$ is a separating family of seminorms on $\mathcal{V}$. We define the **topology generated by** $\mathcal{L}$, denoted by $\sigma(\mathcal{V}, \mathcal{L})$, to be the locally convex topology generated by $\Gamma_{\mathcal{L}}$.

DEFINITION (Weak Topology)**.** Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space. Then $\mathcal{V}^*$ is a linear manifold of $\mathcal{V}^{\#}$. By the Hahn-Banach theorem, we get $\mathcal{V}^*$ is separating. So $(\mathcal{V}, \mathcal{V}^*)$ is a dual pair. We define the **weak topology** on $\mathcal{V}$ to be the topology $\sigma(\mathcal{V}, \mathcal{V}^*)$ induced by the family $\mathcal{V}^*$.

DEFINITION (Weak* Topology)**.** Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space. Then $\hat{\mathcal{V}}$ is a linear manifold of $(\mathcal{V}^*)^{\#}$ and a separating family of linear functionals on $\mathcal{V}^*$ So $(\mathcal{V}^*, \hat{\mathcal{V}})$ is a dual pair. We define the **week* topology** on $\mathcal{V}^*$ to be the topology $\sigma(\mathcal{V}^*, \hat{\mathcal{V}})$

induced by the family $\hat{\mathcal{V}}$.

19.2 Properties

PROPOSITION 19.2.1. Let $(\mathcal{V}, \mathcal{L})$ be a dual pair where $\mathcal{V}$ is a vector space over field $\mathbb{K}$ and $\mathcal{L}$ is a linear manifold of $\mathcal{L}^{\#}$. Then $\mathcal{L} = (\mathcal{V}, \sigma(\mathcal{V}, \mathcal{L}))^*$.

Proof. **Forward Direction**: Clearly if $f \in \mathcal{L}$, then f is $\sigma(\mathcal{V}, \mathcal{L})$-continuous and hence $f \in (\mathcal{V}, \sigma(\mathcal{V}, \mathcal{L}))^*$.

Backward Direction: Let $f \in (\mathcal{V}, \sigma(\mathcal{V}, \mathcal{L}))^*$. I will show that $f \in \mathcal{L}$. Define a seminorm $p : \mathcal{V} \to \mathbb{R}$ by $p(x) := |f(x)|$. Since f is $\sigma(\mathcal{V}, \mathcal{L})$-continuous, p is also $\sigma(\mathcal{V}, \mathcal{L})$-continuous. So $\exists n \in \mathbb{N}$, $\exists \rho_1, ..., \rho_n \in \mathcal{L}$, $\exists \kappa \in \mathbb{R}$ such that $\kappa > 0$ and that $\forall x \in \mathcal{V}$, $p(x) \leq \kappa \max\{|\rho_i(x)|\}_{i=1}^n$. Then if $x \in \bigcap_{i=1}^n \ker(\rho_i)$, $|f(x)| = p(x) \leq \kappa \max\{0\}_{i=1}^n = 0$ and hence $x \in \ker(f)$. So $\bigcap_{i=1}^n \ker(\rho_i) \subseteq \ker(f)$. So $f \in \operatorname{span}\{\rho_i\}_{i=1}^n$ and hence $f \in \mathcal{L}$. This completes the proof. ■

PROPOSITION 19.2.2. Let $(\mathcal{V}, \mathcal{T})$ be a topological space. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in $(\mathcal{V}, \mathcal{T})$ that converges to some point x in $\mathcal{V}$. Then $(x_\lambda)_{\lambda \in \Lambda}$ also converges to x in $(\mathcal{V}, \sigma(\mathcal{V}, \mathcal{V}^*))$.

PROPOSITION 19.2.3. Let $(\mathcal{V}, \mathcal{T}_\mathcal{V})$ and $(\mathcal{W}, \mathcal{T}_\mathcal{W})$ be locally convex spaces. Let T be a continuous linear map from $(\mathcal{V}, \mathcal{T}_\mathcal{V})$ to $(\mathcal{W}, \mathcal{T}_\mathcal{W})$. Then T is also a continuous linear map from $(\mathcal{V}, \sigma(\mathcal{V}, \mathcal{V}^*))$ to $(\mathcal{W}, \sigma(\mathcal{W}, \mathcal{T}_\mathcal{W}))$.

PROPOSITION 19.2.4. The weak* topology is a weaker topology than the weak topology.

Proof Idea. The week* topology is a topology induced by the pre-dual and the weak topology is a topology induced by the dual. But the pre-dual sits inside the dual. Therefore it is harder to converge in the weak topology because there are more functionals that have to be continuous. You'll need more open sets to make those extra functionals continuous. ■

19.3 Theory on Banach Spaces

PROPOSITION 19.3.1. Let $\mathfrak{X}$ be a finite-dimensional Banach space. Then the norm, weak, and weak* topologies on $\mathfrak{X}$ all coincide.

PROPOSITION 19.3.2. Let $\mathfrak{X}$ be a Banach space. Let $\mathfrak{X}^*$ denote the dual space of $\mathfrak{X}$. Let τ_* denote the weak topology on $\mathfrak{X}^*$ induced by elements of $\mathfrak{X}$ as

$$\lim_\alpha x^*_\alpha = x^* \iff \forall x \in \mathfrak{X}, \lim_\alpha x^*_\alpha(x) = x^*(x).$$

Then $(\mathfrak{X}^*, \tau_*)$ is a topological vector space.

19.3.1 The Uniform Boundedness Principle

THEOREM 19.1 (The Uniform Boundedness Principle)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Let $\mathfrak{A}$ be a family of bounded linear maps from $\mathfrak{X}$ to $\mathfrak{Y}$. Suppose that $\forall x \in \mathfrak{X}$, we have $M_x := \sup\{\|Tx\|_{\mathfrak{Y}} : T \in \mathfrak{A}\} < \infty$. Then $\sup\{\|T\| : T \in \mathfrak{A}\} < \infty$.

COROLLARY 19.1. Let $\mathfrak{X}$ be a Banach space. Let S be a set in $\mathfrak{X}$. Then S is bounded if and only if

$$\forall x^* \in \mathfrak{X}^*, \quad \sup\{|x^*(s)| : s \in S\} < \infty.$$

COROLLARY 19.2. Let $\mathfrak{X}$ be a Banach space. Let $\mathfrak{S}$ be a subset of $\mathfrak{X}^*$. Then $\mathfrak{S}$ is bounded if and only if

$$\forall x \in \mathfrak{X}, \quad \sup\{|s^*(x) : s^* \in \mathfrak{S}\} < \infty.$$

19.3.2 The Banach-Steinhaus Theorem

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THEOREM 19.2 (The Banach-Steinhaus Theorem)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$. Suppose that $\forall x \in \mathfrak{X}$, $\exists y_x \in \mathfrak{Y}$ such that $\lim_{n\in\mathbb{N}} T_n x = y_x$. Define a map $T : \mathfrak{X} \to \mathfrak{Y}$ by $Tx := y_x$. Then we have the followings:

(1) $\sup_{n\in\mathbb{N}} \|T_n\| < +\infty$.

(2) T is a bounded linear map and $\|T\| \leq \liminf_{n\in\mathbb{N}} \|T_n\|$.

Proof. **Part 1**: Since $\forall x \in \mathfrak{X}$, $\lim_{n\in\mathbb{N}} T_n x = y_x$, we have $M_x := \sup_{n\in\mathbb{N}} \|T_n x\| < \infty$. By the Uniform Boundedness Principle, we get $M := \sup_{n\in\mathbb{N}} \|T_n\| < \infty$.
Part 2: Clearly T is linear. Let x be an arbitrary element of $\mathfrak{X}$. Then

$$\|Tx\| = \|y_x\| = \|\lim_{n\to\infty} T_n x\| \leq \liminf_{n\to\infty} \|T_n\|\|x\| = (\liminf_{n\in\mathbb{N}} \|T_n\|) \cdot \|x\|.$$

So $\|T\| \leq \liminf_{n\in\mathbb{N}} \|T_n\|$. So T is a bounded linear map. ∎

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COROLLARY 19.3. Let $\mathfrak{X}$ be a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence that converges to x in $(\mathfrak{X}, \sigma(\mathfrak{X},\mathfrak{X}^*))$. Then $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ and $\|x\| \leq \liminf_{n\in\mathbb{N}} \|x_n\|$.

COROLLARY 19.4. Let $\mathfrak{X}$ be a Banach space. Let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence that converges to x^* in $(\mathfrak{X}^*, \sigma(\mathfrak{X}^*,\mathfrak{X}))$. Then $\sup_{n\in\mathbb{N}} \|x_n^*\| < \infty$ and $\|x^*\| \leq \liminf_{n\in\mathbb{N}} \|x_n^*\|$.

19.3.3 The Banach-Alaoglu Theorem

THEOREM 19.3 (The Banach-Alaoglu Theorem)**.** Let $\mathfrak{X}$ be a Banach space. Then the closed unit ball $\mathfrak{X}_1^*$ of $\mathfrak{X}^*$ is weak*-compact.

COROLLARY 19.5. Let $\mathfrak{X}$ be a Banach space. Then $\mathfrak{X}$ is isometrically isomorphic to a subspace of $(\mathcal{C}(L,\mathbb{K}), \|\cdot\|_\infty)$ where L is some compact Hausdorff space.

19.3.4 Goldstine's Theorem

THEOREM 19.4 (Goldstine's Theorem)**.** Let $\mathfrak{X}$ be a Banach space. Let $\mathfrak{J}$ denote the canonical embedding of $\mathfrak{X}$ into $\mathfrak{X}^{**}$. Then $\mathfrak{J}(\mathfrak{X}_1)$ is weak*-dense in $\mathfrak{X}_1^{**}$.

19.3.5 Metrizability

PROPOSITION 19.3.3. Let $\mathfrak{X}$ be a Banach space. Then $\mathfrak{X}_1$ is weakly metrizable if and only if $\mathfrak{X}^*$ is separable.

PROPOSITION 19.3.4. Let $\mathfrak{X}$ be a Banach space. Then $\mathfrak{X}_1^*$ is weak*-metrizable if and only if $\mathfrak{X}$ is separable.

Chapter 20

Locally Compact Space

20.1 The F. Riesz's Theorem

THEOREM 20.1 (The F. Riesz's Theorem)**.** Let $\mathfrak{X}$ be a topological vector space over field $\mathbb{K}$. Then $\mathfrak{X}$ is finite-dimensional if and only if it is locally compact.

Chapter 21

Adjoint Operator

21.1 Banach Space Adjoint

DEFINITION (Banach Space Adjoint)**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Let T be a bounded linear operator from $\mathfrak{X}$ to $\mathfrak{Y}$. We define the **Banach space adjoint** of T, denoted by T^*, to be a bounded linear operator from $\mathfrak{Y}^*$ to $\mathfrak{X}^*$ given by

$$T^* y^*(x) := y^* T x.$$

PROPOSITION 21.1.1. $\|T^*\| = \|T\|$.

PROPOSITION 21.1.2. T^* is invertible if and only if T is invertible.

21.2 Hilbert Space Adjoint

DEFINITION (Adjoint Matrix)**.** Let A be an $m \times n$ matrix. We define the **adjoint** of A, denoted by A^*, to be an $n \times m$ matrix given by

$$(A^*)_{ij} := \overline{(A)_{ji}}.$$

DEFINITION (Adjoint Operator)**.** Let V and W be inner product spaces. Let T be a linear map from V to W. We define the **adjoint** of T, denoted by T^*, to be a map from W to V such that

$$\forall x \in V, \forall y \in W, \quad \langle T(x), y \rangle_W = \langle x, T^*(y) \rangle_V.$$

PROPOSITION 21.2.1 (Existence)**.** Let V be a finite-dimensional inner product space and T be a linear operator on V. Then the adjoint of T exists.

PROPOSITION 21.2.2 (Uniqueness)**.** Let V be an inner product space and T be a linear operator on V. Then the adjoint of T is unique, provided that it exists.

21.3 Properties of the Adjoint Operator

PROPOSITION 21.3.1. Let V be an inner product space. Then

(1) $(I_V)^* = I_V$ where I_V is the identity operator on V.

(2) $T^{**} = T$ for any linear operator T on V.

PROPOSITION 21.3.2. Let V be an inner product space and T be a linear operator on V. Then T^* is also linear.

PROPOSITION 21.3.3. Let V be an inner product space. Then

(1) For any linear operators T and U,

$$(T + U)^* = T^* + U^*.$$

(2) For any linear operator T,

$$(cT)^* = \overline{c} \cdot T^*.$$

(3) For any linear operator T and U,

$$(TU)^* = U^*T^*.$$

PROPOSITION 21.3.4. Let V be a finite-dimensional inner product space and T be a linear operator on V. Then if T is invertible, T^* is also invertible.

PROPOSITION 21.3.5. Let V be an inner product space and T be an invertible linear operator on V. Then $(T^{-1})^* = (T^*)^{-1}$.

21.4 Normal Operators

DEFINITION (Normal)**.** Let V be an inner product space and T be a linear operator on V. We say that T is **normal** if $TT^* = T^*T$.

Chapter 22

Convolution

DEFINITION (Convolution)**.** Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. We define the **convolution** of f and g, denoted by $f * g$, to be a function on $\mathbb{R}$ given by

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)dt.$$

Chapter 23

Coercive Functions

23.1 Definitions

DEFINITION (Coercive)**.** Let f be a function from $\mathbb{R}^d$ to $\mathbb{R}^*$. We say that f is **coercive** if $\lim_{\|x\|\to\infty} f(x) = +\infty$.

23.2 Properties

PROPOSITION 23.2.1. Let f be a proper lower semi-continuous function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let K be a compact set in $\mathbb{R}^d$. Assume $K \cap \mathrm{dom}(f) \neq \emptyset$. Then f attains its minimum over K.

Proof.

Define $m := \inf_{x\in K} f(x)$.

Since $m = \inf_{x\in K} f(x)$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in K such that $\lim_{i\to\infty} f(x_i) = m$.

Since K is compact and $\{x_i\}_{i\in\mathbb{N}} \subseteq K$, there exists a convergent subsequence $\{x_i\}_{i\in I}$ in K where I is an infinite subset of $\mathbb{N}$.

Say the limit is x_∞ where $x_\infty \in K$.

Since $\lim_{i\to\infty} f(x_i) = m$, we get $\lim_{i\in I, i\to\infty} f(x_i) = m$.

Since $\lim_{i\in I, i\to\infty} f(x_i) = m$, we get $\liminf_{i\in I, i\to\infty} f(x_i) = m$.

Since f is lower semi-continuous and $\lim_{i\in I, i\to\infty} x_i = x_\infty$, we get $f(x_\infty) \leq \liminf_{i\in I, i\to\infty} x_i$.

That is, $f(x_\infty) \leq m$.

Since $m = \inf_{x \in K} f(x)$, we have $\forall x \in K, f(x) \geq m$.

In particular, $f(x_\infty) \geq m$.

Since $f(x_\infty) \geq m$ and $f(x_\infty) \leq m$, $f(x_\infty) = m$.

Since f is proper, $f(x_\infty) = m \neq -\infty$.

So f attains its minimum at point x_∞.

■

PROPOSITION 23.2.2. Let f be a proper, lower semi-continuous, and coercive function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let C be a closed subset of $\mathbb{R}^d$. Assume $C \cap \operatorname{dom}(f) \neq \emptyset$. Then f attains its minimum over C.

Proof.

Since $C \cap \operatorname{dom}(f) \neq \emptyset$, take $x \in C \cap \operatorname{dom}(f)$.

Since f is coercive, $\exists R$ such that $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$.

Since $x \in C \cap \operatorname{dom}(f)$ and $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$, the set of minimizers of f over C is the same as the set of minimizers of f over $C \cap \operatorname{ball}[0, R]$.

Since C and $\operatorname{ball}[0, R]$ are both closed, $C \cap \operatorname{ball}[0, R]$ is closed.

Since $\operatorname{ball}[0, R]$ is bounded, $C \cap \operatorname{ball}[0, R]$ is bounded.

Since $C \cap \operatorname{ball}[0, R]$ is closed and bounded, by the Heine-Borel Theorem, $C \cap \operatorname{ball}[0, R]$ is compact.

Since f is proper and lower semi-continuous and $C \cap \operatorname{ball}[0, R]$ is compact, f attains its minimum over $C \cap \operatorname{ball}[0, R]$.

So f attains its minimum over C.

■

Chapter 24

Unclassified Results

PROPOSITION 24.0.1. Let (X, d) be a compact metric space. Let $L(X)$ be the set of all Lipschitz functions from X to $\mathbb{R}$. Let $C(X)$ be the set of all continuous functions from X to $\mathbb{R}$. Then $L(X)$ is dense in $C(X)$.