Graph Theory

Daniel Mao

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Chapter 1

Graph Basics

1.1 Paths

DEFINITION 1.1 (Vertex-Independent Paths)**.** Let $G = (V, E)$ be a finite undirected graph. Let P and Q be two paths in G. We say that P and Q are **vertex-independent** if and only if they do not have any internal vertex in common.

DEFINITION 1.2 (Edge-Independent Paths)**.** Let $G = (V, E)$ be a finite undirected graph. Let P and Q be two paths in G. We say that P and Q are **edge-independent** if and only if they do not have any internal edge in common.

1.2 Others

DEFINITION 1.3 (Spanning Subgraph)**.** Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G. We say that H is a **spanning** subgraph of G if and only if $W = V$. i.e., if H contains all vertices of G.

Chapter 2

Connectivity

2.1 Definitions

DEFINITION 2.1 (Vertex Cut)**.** Let $G = (V, E)$ be a finite undirected graph. Let a and b be distinct vertices in G. Let $S \subseteq V \setminus \{a, b\}$. We say that S is a **vertex cut** for a and b if and only if the removal of S from G separates a and b into distinct connected components.

DEFINITION 2.2 (Edge Cut)**.** Let $G = (V, E)$ be a finite undirected graph. Let a and b be nonadjacent vertices in G. Let $S \subseteq E$. We say that S is an **edge cut** for a and b if and only if the removal of S from G separates a and b into distinct connected components.

DEFINITION 2.3 (k-Vertex-Connected Graphs)**.** We say a connected graph is **k-vertex-connected** if and only if it has more than k vertices and remains connected whenever (strictly) fewer than k vertices are removed.

DEFINITION 2.4 (k-Edge-Connected Graphs)**.** We say a connected graph is **k-edge-connected** if and only if it has more than k edges and remains connected whenever (strictly) fewer than k edges are removed.

DEFINITION 2.5 (Cut)**.** A **cut** is a partition of the vertices of a graph into two disjoint subsets.

DEFINITION 2.6 (s-t Cut)**.** Let $G = (V, E)$ be a finite undirected graph. Let s and t be two vertices in G. We define an **s-t cut** to be a cut $C = (S, T)$ of V such that $s \in S$ and $t \in T$.

DEFINITION 2.7 (Size/Value of a Cut)**.** Let $G = (V, E)$ be a finite undirected graph. Let $C = (S, T)$ be a cut of V. We define the **size** of C to be the number of edges crossing the cut. In the case that G is weighted, we define the **value** of C to be the sum of the weights of the edges crossing the cut.

DEFINITION 2.8 (Minimum Cut)**.** We say that a cut is **minimum** if the size/value of the cut is the minimum among all cuts in the graph.

DEFINITION 2.9 (Maximum Cut)**.** We say that a cut is **maximum** if the size/value of the cut is the maximum among all cuts in the graph.

DEFINITION 2.10 (Cut-Set)**.** Let $G = (V, E)$ be a finite undirected graph. Let $C = (S, T)$ be a partition of V. We define the **cut-set** of C to be the set given by

$$\{(u, v) \in E : u \in S, v \in T\}.$$

DEFINITION 2.11 (Cut Space)**.** Let $G = (V, E)$ be a finite undirected graph. Let $\mathcal{C}$ denote the collection of all cut-sets in G. Let $\mathbb{F}_2 := \{0, 1\}$ be a two-element finite field of arithmetic modulo two. We define the addition operation in $\mathcal{C}$, denoted by $+$, to be a function from $\mathcal{C}^2$ to $\mathcal{C}$ given by $C_1 + C_2 := C_1 \Delta C_2$ where Δ denotes the symmetric difference operation. What about scalar multiplication???

THEOREM 2.12 (Menger's Theorem - Edge Connectivity)**.** Let G be a finite undirected graph and x and y two distinct vertices. Then the size of the minimum edge cut for x and y is equal to the maximum number of pairwise edge-independent paths from x to y.

THEOREM 2.13 (Menger's Theorem - Vertex Connectivity)**.** Let G be a finite undirected graph and x and y two nonadjacent vertices. Then the size of the minimum vertex cut for x and y is equal to the maximum number of pairwise vertex-independent paths from x to y.

Chapter 3

Trees

3.1 Definitions

DEFINITION 3.1 (Spanning Tree)**.** Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G. We say that H is a **spanning tree** if H is a spanning subgraph of G and is a tree.

3.2 Properties

PROPOSITION 3.2. A graph is connected if and only if it has a spanning tree.

Chapter 4

Graph Isomorphism

4.1 Definitions

DEFINITION 4.1 (Isomorphism)**.** Let G and H be two graphs. We define an **isomorphism** from G to H to be a function f from $V(G)$ to $V(H)$ such that

- f is bijective, and that
- for any pair of vertices $v, w \in V(G)$, $f(v)f(w) \in E(H)$ if and only if $vw \in E(G)$.

i.e., a bijective function that both itself and its inverse preserve adjacency.

DEFINITION 4.2 (Isomorphic)**.** Let G and H be two graphs. We say that G and H are **isomorphic**, denoted by $G \simeq H$, if there exists an isomorphism from G to H.

PROPOSITION 4.3. The relation $\simeq$ of isomorphism is an equivalence relation. That is, it is reflexive, symmetric, and transitive.

4.2 Properties

PROPOSITION 4.4. Let G and H be isomorphic graphs with isomorphism f. Then for any vertex $v \in V(G)$, we have $\deg_G(v) = \deg_H(f(v))$.

Chapter 5

Matchings and Covers

5.1 Matching

DEFINITION 5.1 (Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of E. We say that M is a **matching** in G if every vertex in the spanning subgraph (V, M) has degree at most one.

DEFINITION 5.2 (Saturated)**.** Let $(G = (V, E)$ be a graph. Let M be a subset of E. Let v be a vertex of G. We say that v is M**-saturated** if $\deg(v) = 1$ in (V, M).

DEFINITION 5.3 (Maximal Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximal matching** if it is a matching in G and any other matching is not a superset of it.

DEFINITION 5.4 (Maximum Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximum matching** if it is a matching in G and any other matching contains edges no more than M.

DEFINITION 5.5 (Perfect Matching)**.** Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **perfect matching** if it matches all vertices of the graph. i.e., any vertex in G is incident to some edge in M.

PROPOSITION 5.6. Every maximum matching is maximal.

PROPOSITION 5.7. Every perfect matching is maximum.

PROPOSITION 5.8. Let $G = (V, E)$ be a graph. Let A and B be two maximal matchings of G. Then both $|A| \leq 2|B|$ and $|B| \leq 2|A|$.

5.2 Cover

DEFINITION 5.9 (Cover)**.** Let $G = (V, E)$ be a graph. Let C be a subset of V. We say that C is a **cover** of G if any edge has an end in C.

5.3 Relations Between Matchings and Covers

PROPOSITION 5.10. Let $G = (V, E)$ be a graph. Let M be a matching of G. Let C be a cover of G. Then $|M| \leq |C|$.

Chapter 6

Bipartite Graphs

6.1 Definitions

DEFINITION 6.1 (Bipartition)**.** Let $G = (V, E)$ be a graph. Let A and B be two subsets of V. We say the pair (A, B) is a **bipartition** of G if and only if $A \cap B = \emptyset$, $A \cup B = V$, and A and B are both independent.

DEFINITION 6.2 (Bipartite Graph)**.** Let $G = (V, E)$ be a graph. We say that G is **bipartite** if and only if there exists a bipartition of G.

DEFINITION 6.3 (Balanced Bipartite Graph)**.** Let $G = (V, E)$ be a bipartite graph with bipartition (A, B). We say that G is **balanced** if and only if $|A| = |B|$.

6.2 Properties of Bipartite Graphs

PROPOSITION 6.4. Let $G = (V, E)$ be a bipartite graph with bipartition (A, B). Then

$$\sum_{a \in A} \deg(a) = \sum_{b \in B} \deg(b) = |E|.$$

6.3 Characterizations

PROPOSITION 6.5. A graph is bipartite if and only if it has no odd cycles.

PROPOSITION 6.6. A graph is bipartite if and only if it is 2-colorable.

Chapter 7

Planar Graphs

7.1 Definitions

DEFINITION 7.1 (Plane Embedding)**.** Let $G(V, E, B)$ be an undirected multi-graph. A **plane embedding** of G is a pair of sets $(\mathcal{P}, \Gamma)$ such that

7.2 Properties

PROPOSITION 7.2. Every subgraph of a planar graph is planar.

PROPOSITION 7.3. A multi-graph is planar if and only if its simplification is planar.

PROPOSITION 7.4. Let G be a multi-graph and e be an edge in G. Then G is planar if and only if $G \bullet e$ is planar.

THEOREM 7.5. A multi-graph is planar if and only if it does not contain a repeated subdivision of K_5 or $K_{3,3}$ as a subgrph.

7.3 Numerology

DEFINITION 7.6 (Footprint)**.** Let $G(V, E, B)$ be a planar multi-graph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define the **footprint** of G, denoted by $fp(G)$, to be the union of the points and curves in $\mathbb{R}^2$ representing the vertices and edges in G.

DEFINITION 7.7 (Face)**.** Let $G(V, E, B)$ be a planar multi-graph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define a **face** of $(\mathcal{P}, \Gamma)$ to be a connected component of the set $\mathbb{R}^2 \setminus fp(G)$.

DEFINITION 7.8 (Degree)**.** Let $G(V, E, B)$ be a planar multi-graph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of the graph. We define the **degree** of a face to be the sum of the number of edges and the number of bridges in its boundary.

PROPOSITION 7.9. An edge e in a planar multi-graph is a bridge if and only if the two faces on either side of the curve γ_e are the same.

Chapter 8

Duality

8.1 Definitions

DEFINITION 8.1 (Dual Graph)**.** Let $G = (V, E, B)$ be a multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of G. Let $\mathcal{F}$ be the set of faces of G. We define the **dual graph** of this embedding to be the multigraph $G^* = (V^*, E^*, B^*)$ where $V^* = \mathcal{F}$ and $E^* = \{e^* : e \in E\}$.

PROPOSITION 8.2. Let $G = (V, E, B)$ be a multigraph. Let $(\mathcal{P}, \Gamma)$ be a plane embedding of G. Let $G^* = (V^*, E^*, B^*)$ be the dual graph of G. Then for any face $f \in \mathcal{F}$, the degree of f as a face of $\mathcal{P}, \Gamma$ equals the degree of f as a vertex of G^*.

PROPOSITION 8.3. If G is a connected multigraph embedded in the plane, then G^{**} is isomorphic with G.

Chapter 9

Graph Coloring

9.1 Chromatic Number

DEFINITION 9.1 ((Proper) Coloring)**.** Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define a **(proper)** X**-coloring** of G to be a function $f : V \to X$ such that if $vw \in E$, then $f(v) \neq f(w)$.

DEFINITION 9.2 (Chromatic Number)**.** Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define the **chromatic number** of G, denoted by $\chi(G)$, to be the smallest natural number $k \in \mathbb{N}$ for which G has a (proper) k-coloring.

PROPOSITION 9.3. The chromatic number exists and $\chi(G) \leq |V|$.

Proof. Take $X = V$. □

PROPOSITION 9.4. G is complete if and only if $\chi(G) = |V(G)|$.

PROPOSITION 9.5. The only graph with chromatic number zero is the empty graph.

PROPOSITION 9.6. A graph has chromatic number one if and only if it has no edges and at least one vertex.

PROPOSITION 9.7. A graph has chromatic number two if and only if it is bipartite and has at least one edge.

PROPOSITION 9.8. Let G be a graph. Let $d_{max}(G)$ be the maximum degree of a vertex in G. Then $\chi(G) \leq 1 + d_{max}(G)$.

9.2 5-color Theorem

THEOREM 9.9. Every planar graph is 5-colorable.

Proof. (1890)

True for $|V| \leq 5$.

Inductively, suppose the theorem holds for planar graphs on $n-1$ vertices for $n \geq 5$. Suppose G is a planar graph on n vertices.

Let v be a vertex of degree ≤ 5 in G. This exists by a lemma in our lectures.

Since G is a planar, $G-v$ is planar. By the induction hypothesis, $G-v$ has a 5-coloring.

If some color does not appear on any neighbor of v, we can extend the coloring to a coloring of G.

Otherwise, v has exactly 5 neighbors with different colors.

For each pair i, j of colors, let G_{ij} be the subgraph of $G-v$ induced by the vertices colored i or j.

If the component H of G_{ij} containing x_i does not contain x_j, then we can switch the colors of all vertices in H between i and j to get a coloring of $G-v$ that assigns only 4 colors to neighbors of v, and thus extends to a coloring of G.

So G_{ij} contains a path from x_i to x_j.

Because $G_{2,5}$ and $G_{1,4}$ have disjoint vertex sets, this contradicts the planarity of G.

$\square$

DEFINITION 9.10 (Near-triangulation)**.** Planar drawing of G where the infinite face is bounded by a cycle, and every other face is bounded by a triangle

THEOREM 9.11. Every planar near-triangulation has a 5-coloring.

Theorem 9.11 $\implies$ Theorem 9.9.

DEFINITION 9.12 (List Assignment)**.** A **list assignment** L of G is a function that assigns a set $L(v)$ of colors to each $v \in V$.

DEFINITION 9.13 (L-coloring)**.** An L-coloring of G is a choice of a color in $L(v)$ for each $v \in V$ such that adjacent vertices get different colors.

DEFINITION 9.14 (5-list-colorable)**.** A graph is **5-list-colorable** if for every list assignment L of G with $|L(v)| \geq 5$, G is L-colorable.

THEOREM 9.15. Every planar near-triangulation is 5-list-colorable.

Theorem 9.15 $\implies$ Theorem 9.11 because coloring is a special case of list coloring.

THEOREM 9.16 (Carsten Thomassen, 1993)**.** If G is a near-triangulation and L is a list assignment such that

1. $|L(v)| = 5$ for every non-boundary vertex,
2. $|L(v)| = 3$ for every boundary vertex.

Then G has an L-coloring even if two adjacent boundary vertices have their colors arbitrarily decided in advance.

Proof.

Case 1. There is a "chord" between two boundary vertices.

Let G_1 and G_2 be subgraph of G obtained by "cutting" G along the chord, where G_1 contains the pre-colored vertices.

By applying the inductive hypothesis to G_1, and then applying it to G_2 with the two ends of the chord pre-colored according to the coloring of G_1, we get a coloring of G_1.

Case 2. There is no chord.

Let u and u' be the pre-colored vertices.

Let x, y be the next two vertices occuring in order around the boundary.

□

Theorem 9.16 $\implies$ Theorem 9.15.

Chapter 10

Probability and Edge Density

Q: Let G be a graph on n vertices with no triangles. How many edges can G have?

THEOREM 10.1 (Mantel)**.** If G is triangle-free and has n vertices, then

$$|E| \leq \frac{n^2}{4}.$$

Proof. Let $P_{2,1}$ denote the probability that a pair of distinct vertices chosen uniformly at random, are adjacent.

$$P_{2,1} = |E| / \binom{n}{2}.$$

Let $P_{3,2}$ denote the probability that a randomly chosen triple of vertices contains exactly two edges. Let $P_{3,1}$ denote ... one edge. Let $P_{3,0}$ denote ... no edges. Notice $P_{3,2}+P_{3,1}+P_{3,0} = 1$.

Part 1: Show that $P_{2,1} = \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1}$. Notice that the graph is triangle-free. So $P_{3,3} = 0$. Choosing a pair at random is the same as choosing a triple at random, then choosing a pair at random within that triple.

For a fixed vertex v, let $Q_{v,1}$ denote the probability that a randomly chosen vertex $u \neq v$ is adjacent to v.

$$Q_{v,1} = \frac{deg(v)}{n-1}.$$

Let $Q_{v,2}$ denote the probability that two distinct randomly chosen vertices other than v are both adjacent to v.

$$Q_{v,2} = \binom{deg(v)}{2} / \binom{n-1}{2}.$$

Part 2: Show that $Q_{v,1}^2 \approx Q_{v,2}$. Both give (essentially) the probability that a pair x, y of vertices other than v are both adjacent to v. The LHS allows $x = y$. The RHS does not. But $x = y$ occurs with negligible probability.

Part 3: Show that $P_{2,1} = \frac{1}{n}\sum_v Q_{v,1}$. Both the RHS and LHS are just the probability of an edge between two vertices. The RHS calls the first chosen vertex v.

Part 4: Show that $\frac{1}{3}P_{3,2} = \frac{1}{n}\sum_v Q_{v,2}$. Both sides give the probability that, if we choose 3 vertices at random, and then choose one among those 3 and call it v, that v is adjacent to both the others.

Proof of the theorem.

$$
\begin{aligned}
P_{2,1} &= \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1} \geq \frac{2}{3}P_{3,2} \\
&= 2\left(\frac{1}{n}\sum_v Q_{v,2}\right) \approx 2\left(\frac{1}{n}\sum_v Q_{v,1}^2\right) \\
&\geq 2\left(\frac{1}{n}\sum_v Q_{v,1}\right)^2 = 2P_{2,1}^2.
\end{aligned}
$$

So $P_{2,1} \leq \frac{1}{2}$. So $|E| \leq \frac{n^2}{4}$. □

Q: If G has n vertices, no K_{t+1}-subgraph, how many edges can G have?

THEOREM 10.2 (Turan)**.** If G is a graph on n vertices with no K_{t+1}-subgraph, then

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{t}\right).$$

THEOREM 10.3 (Erdos-Stone)**.** If H is a graph and G is a graph on n vertices without H as a subgraph, then

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{\chi(H) - 1} + \varepsilon(n)\right)$$

where $\varepsilon(n) \to 0$ as $n \to \infty$ and $\chi(H)$ is the chromatic number of H, the fewest number of colors needed to properly color the vertices of H.

Chapter 11

Weird Stuffs

11.1 Geometric Representation of Graphs

DEFINITION 11.1 (Geometric Representation)**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. We define a **geometric representation** of G to be a map from V to $\mathbb{R}^d$.

DEFINITION 11.2 (Unit Distance Representation)**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \to \mathbb{R}^d$ be a geometric representation of G. We say that u is a **unit distance representation** of G if and only if $\forall \{i, j\} \in E$, $\|u(i) - u(j)\|_2 = 1$.

DEFINITION 11.3 (Orthonormal Representation)**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \to \mathbb{R}^d$ be a geometric representation of G. We say that u is an **orthonormal representation** of G if and only if

- $\forall i \in V$, $\|u(i)\|_2 = 1$; and
- $\forall \{i, j\} \in \overline{E}$, $\langle u(i), u(j) \rangle = 0$ where $\overline{E}$ is the edge set of the complement of G.

DEFINITION 11.4. We define $t_h(G)$ to be the square radius of the smallest hypersphere that contains a unit distance representation of G.

THEOREM 11.5 (CO 471, Spring 2022, Levent Tuncel)**.** Let $G = (V, E)$ be a graph. Then

$$\begin{aligned} t_h(G) \quad = \quad & \min \quad t \\ & \text{subject to:} \quad X_{ii} = t, \forall i \in V \\ & X_{ii} - 2X_{ij} + X_{jj} = 1, \forall \{i, j\} \in E \\ & X \in S_+^V \end{aligned}$$

PROPOSITION 11.6 (CO 471, Spring 2022, Levent Tuncel)**.** Let $G = (V, E)$ be a graph. Then G is bipartite if and only if $t_h(G) \leq \frac{1}{4}$.

Proof. □

PROPOSITION 11.7. Let $n \in \mathbb{Z}_{++}$. Let K_n denote the n-clique. Then $t_h(K_n) =$.

Proof. □

11.2 Stable Sets

DEFINITION 11.8 (Stable Sets)**.** Let $G = (V, E)$ be a graph. Let S be a subset of the vertex set V. We say that S is a **stable set** in G if and only if $\forall \{i, j\} \in E$, at most one of i or j is in S. i.e., S is a set of pairwise non-adjacent vertices.

DEFINITION 11.9 (Stability Number)**.** Let $G = (V, E)$ be a graph. We define the **stability number** of G, denoted by $\alpha(G)$, to be a number given by

$$\alpha(G) := \max\{|S| : S \text{ is stable in } G\}.$$

DEFINITION 11.10 (Stable Set Polytope)**.** Let $G = (V, E)$ be a graph. We define the **stable set polytope** of G, denoted by $\mathrm{STAB}(G)$, to be a subset of $\mathbb{R}^V$ given by

$$\mathrm{STAB}(G) := \mathrm{conv}\left\{x \in \{0,1\}^V : x \text{ is the incidence vector of some stable set in } G\right\}.$$

DEFINITION 11.11 (Fractional Stable Set Polytope)**.** Let $G = (V, E)$ be a graph. We define the **fractional stable set polytope** of G, denoted by $\mathrm{FRAC}(G)$, to be a subset of $\mathbb{R}^V$ given by

$$\mathrm{FRAC}(G) := \left\{x \in [0,1]^V : x_i + x_j \leq 1, \forall \{i,j\} \in E\right\}.$$

PROPOSITION 11.12. Let $G = (V, E)$ be a graph. Then

$$\mathrm{STAB}(G) = \mathrm{conv}(\mathrm{FRAC}(G) \cap \{0,1\}^V).$$

11.3 Clique Polytope

DEFINITION 11.13. Let $A_{clq}(G)$ denote the 0-1 clique-node incidence matrix of G where each row corresponds to a clique.

DEFINITION 11.14 (Clique Polytope)**.** We define the **clique polytope** of G to be

$$\mathrm{CLQ}(G) := \{x \in \mathbb{R}^V_+ : A_{clq}(G)x \leq \bar{e}\}$$

11.4 Theta Bodies

DEFINITION 11.15 (Theta Body)**.** Let $G = (V, E)$ be a graph. We define the

theta body of G, denoted by TH(G), to be a subset of $\mathbb{R}_+^V$ given by

$$\mathrm{TH}(G) := \left\{ x \in \mathbb{R}_+^V : \sum_{i \in V} (c^\top u(i))^2 x_i \le 1, \quad \begin{matrix} \forall c \in \mathbb{R}^V : \|c\|_2 = 1, \\ \forall \text{ orth. rep. } u \text{ of } G \end{matrix} \right\}.$$

THEOREM 11.16 (CO 471, Spring 2022, Levent Tuncel)**.** For every graph $G = (V, E)$, TH(G) is a nonempty compact convex set such that

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{CLQ}(G) \subseteq \mathrm{FRAC}(G).$$

Proof. We already observed CLQ(G) $\subseteq$ FRAC(G) for all graphs G.
Part 1: Show that TH(G) $\subseteq$ CLQ(G). Let $\mathcal{C} \subseteq V$ be a nonempty clique in G. Let $C \in \mathbb{R}^V$ be any vector with $\|c\|_2 = 1$. Let $u(i) := c$, $\forall i \in \mathcal{C}$. For all $i \in V \setminus \mathcal{C}$, choose $u(i)$'s as an orthonormal system in $\{c\}^\perp$. Note: u is an orthonormal representation of G. The corresponding orthonormal representation constraint is

$$1 \ge \sum_{i \in V} (c^\top u(i))^2 x_i = \sum_{i \in \mathcal{C}} \underbrace{(c^\top u(i))^2}_{=1} x_i + \sum_{i \in V \setminus \mathcal{C}} \underbrace{(c^\top u(i))^2}_{=0} x_i = \sum_{i \in \mathcal{C}} x_i.$$

Also, by definition TH(G) $\subseteq \mathbb{R}_+^V$; therefore, TH(G) $\subseteq$ CLQ(G) for all graph G.
Part 2: Show that STAB(G) $\subseteq$ TH(G). We will show that incidence vectors of every stable set in G belongs to TH(G). Since TH(G) is a convex set and STAB(G) is the convex hull of these incidence vectors, this will prove STAB(G) $\subseteq$ TH(G). Let $\mathcal{S} \subseteq V$ be a stable set in G. Let $\bar{x} \in \{0,1\}^V$ be the incidence vector of $\mathcal{S}$. Clearly $\bar{x} \ge 0$. Let $u : V \to \mathbb{R}^V$ be any orthonormal representation of G. Let $c \in \mathbb{R}^V$ be any vector such that $\|c\|_2 = 1$. We may assume $\mathcal{S} = \{1, ..., k\}$. Define $Q^\top := (u(1), ..., u(k))$. Then by definition of stable sets, $Q^\top$ is orthogonal.

$$\begin{aligned} \sum_{i \in V} (c^\top u(i))^2 \bar{x}_i &= \sum_{i \in \mathcal{S}} (c^\top u(i))^2 \underbrace{\bar{x}_i}_{=1} + \sum_{i \in V \setminus \mathcal{S}} (c^\top u(i))^2 \underbrace{\bar{x}_i}_{=0} = \sum_{i \in \mathcal{S}} (c^\top u(i))^2 \\ &= \|Q^\top c\|_2^2 \le \|\tilde{Q}^\top c\|_2^2 = \|c\| = 1. \end{aligned}$$

where $\tilde{Q}^\top = [Q^\top |$ complete to an orthonormal basis]. Since $0 \in$ TH(G), TH(G) $\neq \emptyset$. Since TH(G) $\subseteq$ FRAC(G) $\subseteq [0,1]^V$, TH(G) is bounded. $\square$

DEFINITION 11.17 (Lovase Theta Function)**.** Let $G = (V, E)$ be a graph. Let $w \in \mathbb{R}_+^V$. We define the **Lovase Theta function**, denoted by θ, to be a function of G

and w given by

$$\theta(G,w) := \max\{w^\top x : x \in \mathrm{TH}(G)\}.$$

DEFINITION 11.18 (Lovase Theta Number)**.** Let $G=(V,E)$ be a graph. We define the **Lovase Theta number** of G, denoted by $\theta(G)$, to be a number given by

$$\theta(G) := \theta(G,\bar{e}) = \max\{\bar{e}^\top x : x \in \mathrm{TH}(G)\}.$$

THEOREM 11.19 (CO 471, Spring 2022, Levent Tuncel)**.** Let $G=(V,E)$ be a graph. Let $w \in \mathbb{R}^V_+$ be a weight vector. Define a matrix $W \in \mathbb{S}^V$ by $W_{ij} := \sqrt{w_i w_j}$, $\forall i,j \in V$. Then the following quantities are the same:

1. $\theta(G,w)$;
2. If $w_i = 0$, define $\dfrac{w_i}{(c^\top u(i))^2} := 0$,
$$\inf\left\{\max_{i\in V}\left\{\frac{w_i}{(c^\top u(i))^2}\right\} : \begin{array}{l} c \in \mathbb{R}^V, \|c\|_2 = 1, \\ u \text{ is an orth. rep. of } G \end{array}\right\};$$
3. $\min\{\eta \in \mathbb{R} : S \in \mathbb{S}^V, \mathrm{diag}(S) = 0, S_{ij} = 0, \forall\{i,j\} \in \overline{E}, \eta I - S \succeq W\}$;
4. $\max\{\mathrm{tr}(WX) : X_{ij} = 0, \forall\{i,j\} \in E, \mathrm{tr}(X) = 1, X \in \mathbb{S}^V_+\}$.

11.5 Product of Graphs

DEFINITION 11.20 (Strong Product)**.** Let $G=(V,E)$ and $H=(W,F)$ be graphs. We define the **strong product** of G and H, denoted by $G \otimes H$, to be a graph given by $G \otimes H = (V(G\otimes H), E(G \times H))$ where

$$V(G\otimes H) := V \times W \text{ and}$$
$$E(G\otimes H) := \left\{ \Big\{(i,u),(j,v)\Big\} : \begin{array}{l} (\{i,j\} \in E \text{ and } \{u,v\} \in F) \text{ or} \\ (\{i,j\} \in E \text{ and } u = v \in W) \text{ or} \\ (i = j \in V \text{ and } \{u,v\} \in F) \end{array} \right\}.$$

PROPOSITION 11.21. Let $G = (V, E)$ and $H = (W, F)$ be graphs. Then

$$\theta(G \otimes H) \leq \theta(G) \times \theta(H).$$

DEFINITION 11.22 (Shannon Capacity)**.** Let $G = (V, E)$ be a graph. We define the **Shannon capacity** of G, denoted by $\Theta(G)$, to be a number given by

$$\Theta(G) := \limsup_{k \to +\infty} (\alpha(G^{\otimes k}))^{1/k}$$

where $\alpha(G^{\otimes k})$ denotes the stability number of $G^{\otimes k}$.

11.6 Lift-and-Project Operators

DEFINITION 11.23 (Lift-and-Project Operators)**.** Let P be a convex subset of $[0,1]^d$. Define a subset K_P of $\mathbb{R}^{1+d}$ by $K_P := \operatorname{cone}(1 \oplus P)$. Define a subset $M_+(P)$ of $\mathbb{S}_+^{1+d}$ by

$$M_+(P) := \left\{ Y \in \mathbb{S}_+^{1+d} : \operatorname{diag}(Y) = Ye_0, \begin{array}{r} Ye_i \in K_P, \forall i \in \{1, ..., d\}, \\ Y(e_0 - e_i) \in K_P, \forall i \in \{1, ..., d\} \end{array} \right\}.$$

We define the **lift-and-project operator**, denoted by LS_+, to be a function from $\mathcal{P}(\mathbb{R}^d)$ to $\mathcal{P}(\mathbb{R}^d)$ given by

$$\mathrm{LS}_+(P) := \left\{ x \in \mathbb{R}^d : \begin{pmatrix} 1 \\ x \end{pmatrix} = Ye_0 \text{ for some } Y \in M_+(P) \right\}.$$