

# Statistical Theory

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# Chapter 1

## Descriptive Statistics

### 1.1 Population

**DEFINITION 1.1** (Population). We define a **population**, denoted by  $\mathcal{P}$ , to be a finite set of elements.

**DEFINITION 1.2** (Unit). We define a **unit**, denoted by  $u$ , to be an element of some population.

**DEFINITION 1.3** (Invariants and Equivariants). Let  $\mathcal{P}$  be a population. Let  $a$  be an attribute. We say that the attribute  $a$  is

1. **location invariant** if

$$\forall b \in \mathbb{R}, a(\mathcal{P} + b) = a(\mathcal{P}).$$

2. **location equivariant** if

$$\forall b \in \mathbb{R}, a(\mathcal{P} + b) = a(\mathcal{P}) + b.$$

3. **scale invariant** if

$$\forall m \in \mathbb{R}^{\geq 0}, a(m \times \mathcal{P}) = a(\mathcal{P}).$$

4. **scale equivariant** if

$$\forall m \in \mathbb{R}^{\geq 0}, a(m \times \mathcal{P}) = m \times a(\mathcal{P}).$$

5. **location-scale invariant** if

$$\forall b \in \mathbb{R}, m \in \mathbb{R}^{\geq 0}, a(m \times \mathcal{P} + b) = a(\mathcal{P}).$$

6. **location-scale equivariant** if

$$\forall b \in \mathbb{R}, m \in \mathbb{R}^{\geq 0}, a(m \times \mathcal{P} + b) = m \times a(\mathcal{P}) + b.$$

7. **replication invariant** if

$$\forall k \in \mathbb{N}, a(\mathcal{P}^k) = a(\mathcal{P}).$$

8. **replication equivariant** if

$$\forall k \in \mathbb{N}, a(\mathcal{P}^k) = k \times a(\mathcal{P}).$$

**PROPOSITION 1.4.** The population average is location-scale equivariant and replication invariant.

## 1.2 Order, Rank, and Quantiles

**DEFINITION 1.5** (Order Statistic). Let  $\mathcal{P} = \{y_1, \dots, y_N\}$  be a population. We define the **order statistic**, denoted by  $y_{(1)}, \dots, y_{(N)}$ , to be the ordered values of the variate values from  $\mathcal{P}$  such that

$$y_{(1)} \leq \dots \leq y_{(N)}.$$

**DEFINITION 1.6** (Rank Statistic). Let  $\mathcal{P} = \{y_1, \dots, y_N\}$  be a population. We define the **rank statistic**, denoted by  $r_1, \dots, r_N$ , to be the ranks of the variate values from  $\mathcal{P}$ .

So  $\forall u \in \mathcal{P}, y_u = y_{(r_u)}$ .

**DEFINITION 1.7** (Quantiles). **Quantiles that Measure Center:**

- Median:  $Q_y(1/2)$ .

- Mid-hinge:  $\frac{Q_y(1/4) + Q_y(3/4)}{2}$ .
- Mid-range:  $\frac{Q_y(1/N) + Q_y(1)}{2}$ .
- Trimean:  $\frac{Q_y(1/4) + Q_y(1/2) + Q_y(3/4)}{2}$ .

**Quantiles that Measure Spread:**

- Range:  $Q_y(1) - Q_y(1/N)$ .
- Inter-Quartile Range:  $Q_y(3/4) - Q_y(1/4)$ .

### 1.3 Statistical Distance

**DEFINITION 1.8** (Kullback–Leibler (KL) Divergence).

- Discrete Case:

Let  $\mathcal{X}$  be a probability space. Let  $P$  and  $Q$  be discrete probability distributions over  $\mathcal{X}$ . We define the **Kullback–Leibler divergence** from  $Q$  to  $P$ , denoted by  $D_{KL}(P||Q)$ , to be the sum given by

$$D_{KL}(P||Q) := \mathbb{E}_{X \sim P} \left[ \log \frac{P(X)}{Q(X)} \right] = \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}.$$

- Continuous Case:

Let  $P$  and  $Q$  be continuous probability distributions. We define the **Kullback–Leibler divergence** from  $Q$  to  $P$ , denoted by  $D_{KL}(P||Q)$ , to be the integral given by

$$D_{KL}(P||Q) := \mathbb{E}_{X \sim P} \left[ \log \frac{P(X)}{Q(X)} \right] = \int_{\mathbb{R}} P(x) \log \frac{P(x)}{Q(x)} dx.$$



## Chapter 2

# Sampling Theory

### 2.1 Sampling Distribution

**PROPOSITION 2.1.** Let  $X_1, \dots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ . Then

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1).$$

*Proof.* Part 1: show that

$$\sum_{i=1}^n (X_i - \mu)^2 = (n-1)s^2 + n(\bar{X} - \mu)^2.$$

$$\begin{aligned} \sum_{i=1}^n (X_i - \mu)^2 &= \sum_{i=1}^n [(X_i - \bar{X}) + (\bar{X} - \mu)]^2 \\ &= \sum_{i=1}^n [(X_i - \bar{X})^2 + (\bar{X} - \mu)^2 + 2(X_i - \bar{X})(\bar{X} - \mu)] \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (\bar{X} - \mu)^2 + \sum_{i=1}^n 2(X_i - \bar{X})(\bar{X} - \mu) \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 + 2(\bar{X} - \mu) \sum_{i=1}^n (X_i - \bar{X}) \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 + 2(\bar{X} - \mu) \sum_{i=1}^n X_i - 2(\bar{X} - \mu) \sum_{i=1}^n \bar{X} \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 + 2(\bar{X} - \mu)n\bar{X} - 2(\bar{X} - \mu)n\bar{X} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 \\
&= (n-1) \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 \\
&= (n-1)s^2 + n(\bar{X} - \mu)^2
\end{aligned}$$

That is,

$$\sum_{i=1}^n (X_i - \mu)^2 = (n-1)s^2 + n(\bar{X} - \mu)^2,$$

as desired.

### Part 2

Since

$$\sum_{i=1}^n (X_i - \mu)^2 = (n-1)s^2 + n(\bar{X} - \mu)^2,$$

we get

$$\sum_{i=1}^n \left( \frac{X_i - \mu}{\sigma} \right)^2 = \frac{(n-1)s^2}{\sigma^2} + \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2.$$

Since  $X_i \sim N(\mu, \sigma^2)$ , we get

$$\frac{X_i - \mu}{\sigma} \sim N(0, 1).$$

So

$$\left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(1)$$

and hence

$$\sum_{i=1}^n \left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(n).$$

Since  $\bar{X} \sim N(\mu, \sigma^2/n)$ , we get

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

So

$$\left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2 \sim \chi^2(1).$$

Notice  $\frac{(n-1)s^2}{\sigma^2}$  and  $\left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2$  are independent, it follows that

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1).$$

□

## 2.2 Inclusion Probability

### Simple Random Sampling Without Replacement

- inclusion probability:  $\pi_u = \frac{n}{N}$ .

Derivation:

$$\begin{aligned}
 \pi_u &= \sum_{S \in \mathcal{P}_S} P(S) I(u \in S) \\
 &= \frac{1}{\binom{N}{n}} \binom{N-1}{n-1} \\
 &= \frac{1}{\frac{N!}{n!(N-n)!}} \frac{(N-1)!}{(n-1)!(N-n)!} \\
 &= \frac{n}{N}.
 \end{aligned}$$

- joint inclusion probability:  $\pi_{uv} = \frac{n(n-1)}{N(N-1)}$

Derivation:

$$\begin{aligned}
 \pi_{uv} &= \sum_{S \in \mathcal{P}_S} P(S) I(u, v \in S) \\
 &= \frac{1}{\binom{N}{n}} \binom{N-2}{n-2} \\
 &= \frac{1}{\frac{N!}{n!(N-n)!}} \frac{(N-2)!}{(n-2)!(N-n)!} \\
 &= \frac{n(n-1)}{N(N-1)}.
 \end{aligned}$$

## 2.3 Survey Errors

**DEFINITION 2.2** (Response Rate). Define the following notations:

- $I$ : complete interviews;
- $P$ : partial interview;
- $R$ : refusals or break-offs;
- $NC$ : non-contacts;
- $O$ : other eligible non-respondents;
-

**DEFINITION 2.3** (Non-Response Bias). Let 1 denote the respondent group. Let 2 denote the non-respondent group. Let  $\bar{Y}_1$  and  $\bar{Y}_2$  denote the population means of group 1 and group 2, respectively. We define the **non-response bias**

## Chapter 3

# Experimental Design

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### 3.1 Completely Random Design - Model 1

**DEFINITION 3.1** (Completely Random Design - Model 1). Let  $k$  denote the number of treatments. Let  $n_i$  denote the number of units that receive the  $i$ -th treatment. We model the observations as

$$y_{ij} = \mu_i + e_{ij}, \text{ for } i \in \{1, \dots, k\} \text{ and } j \in \{1, \dots, n_i\}$$

where  $e_{ij} \sim \mathcal{N}(0, \sigma^2)$  are assumed to be independent. The total number of parameters in this model is  $k + 1$

#### 3.1.1 Estimation of Mean

**PROPOSITION 3.2.** Let  $y_{ij}$  for  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, n_i\}$  be given. Consider the following optimization problem:

$$\begin{aligned} \text{(P)} \quad & \min \quad f(\mu) := \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 \\ & \text{subject to: } \mu \in \mathbb{R}^k. \end{aligned}$$

Then the minimizer  $\hat{\mu} \in \mathbb{R}^k$  of (P) is given by

$$\hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}, \text{ for } i \in \{1, \dots, k\}.$$

*Proof.* Let  $p \in \{1, \dots, k\}$  be arbitrary. Then

$$\begin{aligned} \frac{\partial}{\partial \mu_p} f(\mu) &= \frac{\partial}{\partial \mu_p} \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 = \sum_{j=1}^{n_p} \frac{\partial}{\partial \mu_p} (y_{pj} - \mu_p)^2 = -2 \sum_{j=1}^{n_p} (y_{pj} - \mu_p), \text{ and} \\ \frac{\partial^2}{\partial \mu_p^2} f(\mu) &= \frac{\partial}{\partial \mu_p} \left[ -2 \sum_{j=1}^{n_p} (y_{pj} - \mu_p) \right] = 2n_p > 0. \end{aligned}$$

Suppose  $\hat{\mu} \in \mathbb{R}^k$  is a minimizer of  $f$ . Then we have  $\nabla f(\hat{\mu}) = 0 \in \mathbb{R}^k$ . So

$$\frac{\partial}{\partial \mu_i} f(\hat{\mu}) = 0 \implies \hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}.$$

Testing the Hessian of  $f$  at point  $\hat{\mu} \in \mathbb{R}^k$  confirms that it is indeed a minimizer of  $f$ .  $\square$

**PROPOSITION 3.3** (Mean of the Mean Estimator). We have

$$\mathbb{E}(\hat{\mu}) = \mu \in \mathbb{R}^k.$$

i.e.,  $\hat{\mu}$  is an unbiased estimator for  $\mu$ .

*Proof.* Recall that  $\forall i \in \{1, \dots, k\}$ ,  $\forall j \in \{1, \dots, n_i\}$ , we have  $y_{ij} \sim \mathcal{N}(\mu_i, \sigma^2)$ . So  $\forall i \in \{1, \dots, k\}$ ,  $\forall j \in \{1, \dots, n_i\}$ ,  $\mathbb{E}(y_{ij}) = \mu_i$ . Now we can compute

$$\mathbb{E}(\hat{\mu}_i) = \mathbb{E}\left(\frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}\right) = \frac{1}{n_i} \sum_{j=1}^{n_i} \mathbb{E}(y_{ij}) = \frac{1}{n_i} \sum_{j=1}^{n_i} \mu_i = \mu_i.$$

That is,  $\mathbb{E}(\hat{\mu}) = \mu$ , as desired.  $\square$

**PROPOSITION 3.4** (Variance of the Mean Estimator). We have

$$\mathbb{V}(\hat{\mu}) = \text{Diag}\left(\frac{\sigma^2}{n_i}\right)_{i=1}^k \in \mathbb{S}_+^k.$$

*Proof.* Recall that  $\forall i \in \{1, \dots, k\}$ ,  $\forall j \in \{1, \dots, n_i\}$ , we have  $y_{ij} \sim \mathcal{N}(\mu_i, \sigma^2)$ . So  $\forall i \in \{1, \dots, k\}$ ,  $\forall j \in \{1, \dots, n_i\}$ ,  $\mathbb{V}(y_{ij}) = \sigma^2$ . Now we can compute

$$\begin{aligned} \mathbb{V}(\hat{\mu}_i) &= \mathbb{V}\left(\frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}\right) = \sum_{j=1}^{n_i} \frac{1}{n_i^2} \mathbb{V}(y_{ij}) = \sum_{j=1}^{n_i} \frac{1}{n_i^2} \sigma^2 = \frac{\sigma^2}{n_i}, \quad \forall i, \text{ and} \\ \text{cov}(\hat{\mu}_p, \hat{\mu}_q) &= \mathbb{E}(\hat{\mu}_p \hat{\mu}_q) - \mathbb{E}(\hat{\mu}_p) \mathbb{E}(\hat{\mu}_q) = \mathbb{E}\left(\left(\frac{1}{n_p} \sum_{j=1}^{n_p} y_{pj}\right) \left(\frac{1}{n_q} \sum_{j=1}^{n_q} y_{qj}\right)\right) - \mathbb{E}(\hat{\mu}_p) \mathbb{E}(\hat{\mu}_q) \\ &= \left(\frac{1}{n_p} \sum_{j=1}^{n_p} \mathbb{E}(y_{pj})\right) \left(\frac{1}{n_q} \sum_{j=1}^{n_q} \mathbb{E}(y_{qj})\right) - \mathbb{E}(\hat{\mu}_p) \mathbb{E}(\hat{\mu}_q), \text{ by independence} \\ &= \left(\frac{1}{n_p} \sum_{j=1}^{n_p} \mu_p\right) \left(\frac{1}{n_q} \sum_{j=1}^{n_q} \mu_q\right) - \mu_p \mu_q, \text{ by above} \\ &= \mu_p \mu_q - \mu_p \mu_q = 0, \quad \forall p, q \in \{1, \dots, k\} : p \neq q. \end{aligned}$$

$\square$

### 3.1.2 Estimation of Variance

In this subsection, we assume that  $\forall i \in \{1, \dots, k\}$ ,  $n_i = n$  for some  $n \in \mathbb{Z}_{++}$ .

**DEFINITION 3.5** (Sum of Squares). We define the following terms:

$$\begin{aligned} \text{SS}_{\text{trt}} &:= n \sum_{i=1}^k (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2, \\ \text{SS}_{\text{err}} &:= \sum_{i=1}^k \sum_{j=1}^n (y_{ij} - \bar{y}_{i\cdot})^2, \\ \text{SS}_{\text{tot}} &:= \sum_{i=1}^k \sum_{j=1}^n (y_{ij} - \bar{y}_{\cdot\cdot})^2. \end{aligned}$$

**PROPOSITION 3.6** (Decomposition of  $\text{SS}_{\text{tot}}$ ). We have

$$\text{SS}_{\text{tot}} = \text{SS}_{\text{trt}} + \text{SS}_{\text{err}}.$$

*Proof.*

$$\begin{aligned} \text{SS}_{\text{tot}} &= \sum_{i=1}^k \sum_{j=1}^n (y_{ij} - \bar{y}_{\cdot\cdot})^2 = \sum_{i=1}^k \sum_{j=1}^n (y_{ij} - \bar{y}_{i\cdot} + \bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 \\ &= \sum_{i=1}^k \sum_{j=1}^n \left[ (y_{ij} - \bar{y}_{i\cdot})^2 + (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 + 2(y_{ij} - \bar{y}_{i\cdot})(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) \right] \\ &= \text{SS}_{\text{trt}} + \text{SS}_{\text{err}} + \sum_{i=1}^k \sum_{j=1}^n 2(y_{ij} - \bar{y}_{i\cdot})(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) \\ &= \text{SS}_{\text{trt}} + \text{SS}_{\text{err}} + 2 \sum_{i=1}^k \sum_{j=1}^n y_{ij} \bar{y}_{i\cdot} - 2 \sum_{i=1}^k \sum_{j=1}^n y_{ij} \bar{y}_{\cdot\cdot} - 2 \sum_{i=1}^k \sum_{j=1}^n \bar{y}_{i\cdot}^2 + 2 \sum_{i=1}^k \sum_{j=1}^n \bar{y}_{i\cdot} \bar{y}_{\cdot\cdot} \\ &= \text{SS}_{\text{trt}} + \text{SS}_{\text{err}} + 2n \sum_{i=1}^k \bar{y}_{i\cdot}^2 - 2n \bar{y}_{\cdot\cdot} \sum_{i=1}^k \bar{y}_{i\cdot} - 2n \sum_{i=1}^k \bar{y}_{i\cdot}^2 + 2n \bar{y}_{\cdot\cdot} \sum_{i=1}^k \bar{y}_{i\cdot} \\ &= \text{SS}_{\text{trt}} + \text{SS}_{\text{err}} + 0 = \text{SS}_{\text{trt}} + \text{SS}_{\text{err}}. \end{aligned}$$

□

**DEFINITION 3.7** (Mean Squares). We define the following estimators for the vari-



ance  $\sigma^2$ .

$$\begin{aligned} \text{MS}_{\text{trt}} &:= \text{SS}_{\text{trt}} / (k - 1), \\ \text{MS}_{\text{err}} &:= \text{SS}_{\text{err}} / (k(n - 1)). \end{aligned}$$

**REMARK 3.8.** In the case of  $k = 2$ ,  $\text{MS}_{\text{err}}$  reduces to

$$\text{MS}_{\text{err}} = \frac{1}{2n - 2} \left[ \sum_{j=1}^n (y_{1j} - \bar{y}_{1\cdot})^2 + \sum_{j=1}^n (y_{2j} - \bar{y}_{2\cdot})^2 \right],$$

which is also called the pooled variance and is denoted by  $s_p^2$ .

**LEMMA 3.9.** We have

$$\begin{aligned} \mathbb{E}(y_{ij}^2) &= \sigma^2 + \mu_i^2, \quad \forall i, j, \\ \mathbb{E}(\bar{y}_{i\cdot}^2) &= \frac{\sigma^2}{n} + \mu_i^2, \quad \forall i, \text{ and} \\ \mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \frac{\sigma^2}{kn} + \left( \frac{1}{k} \sum_{i=1}^k \mu_i \right)^2. \end{aligned}$$

*Proof.* Recall that  $\forall i \in \{1, \dots, k\}$ ,  $\forall j \in \{1, \dots, n\}$ , we have  $y_{ij} \sim \mathcal{N}(\mu_i, \sigma^2)$ . So

$$\begin{aligned} \bar{y}_{i\cdot} &= \frac{1}{n} \sum_{j=1}^n y_{ij} \sim \mathcal{N}\left(\mu_i, \frac{\sigma^2}{n}\right), \quad \forall i, \text{ and} \\ \bar{y}_{\cdot\cdot} &= \frac{1}{kn} \sum_{i=1}^k \sum_{j=1}^n y_{ij} \sim \mathcal{N}\left(\frac{1}{k} \sum_{i=1}^k \mu_i, \frac{\sigma^2}{kn}\right). \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}(y_{ij}^2) &= \mathbb{V}(y_{ij}) + \mathbb{E}^2(y_{ij}) = \sigma^2 + \mu_i^2, \quad \forall i, j, \\ \mathbb{E}(\bar{y}_{i\cdot}^2) &= \mathbb{V}(\bar{y}_{i\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot}) = \frac{\sigma^2}{n} + \mu_i^2, \quad \forall i, \text{ and} \\ \mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \mathbb{V}(\bar{y}_{\cdot\cdot}) + \mathbb{E}^2(\bar{y}_{\cdot\cdot}) = \frac{\sigma^2}{kn} + \left( \frac{1}{k} \sum_{i=1}^k \mu_i \right)^2. \end{aligned}$$

□

**PROPOSITION 3.10** (Mean of  $\text{MS}_{\text{err}}$ ). We have

$$\mathbb{E}(\text{MS}_{\text{err}}) = \sigma^2.$$

i.e.,  $\text{MS}_{\text{err}}$  is an unbiased estimator for  $\sigma^2$ .

*Proof.*

$$\begin{aligned} \mathbb{E}(\text{MS}_{\text{err}}) &= \mathbb{E}(\text{SS}_{\text{err}}/(k(n-1))) = \mathbb{E}\left(\frac{1}{k(n-1)} \sum_{i=1}^k \sum_{j=1}^n (y_{ij} - \bar{y}_{i\cdot})^2\right) \\ &= \mathbb{E}\left(\frac{1}{k(n-1)} \sum_{i=1}^k \sum_{j=1}^n (y_{ij}^2 + \bar{y}_{i\cdot}^2 - 2y_{ij}\bar{y}_{i\cdot})\right), \text{ expand} \\ &= \mathbb{E}\left(\frac{1}{k(n-1)} \left[ \sum_{i=1}^k \sum_{j=1}^n y_{ij}^2 + n \sum_{i=1}^k \bar{y}_{i\cdot}^2 - 2 \sum_{i=1}^k \bar{y}_{i\cdot} \sum_{j=1}^n y_{ij} \right]\right), \text{ separate} \\ &= \mathbb{E}\left(\frac{1}{k(n-1)} \left[ \sum_{i=1}^k \sum_{j=1}^n y_{ij}^2 + n \sum_{i=1}^k \bar{y}_{i\cdot}^2 - 2n \sum_{i=1}^k \bar{y}_{i\cdot}^2 \right]\right), \text{ reduce} \\ &= \mathbb{E}\left(\frac{1}{k(n-1)} \left[ \sum_{i=1}^k \sum_{j=1}^n y_{ij}^2 - n \sum_{i=1}^k \bar{y}_{i\cdot}^2 \right]\right), \text{ combine} \\ &= \frac{1}{k(n-1)} \left[ \sum_{i=1}^k \sum_{j=1}^n \mathbb{E}(y_{ij}^2) - n \sum_{i=1}^k \mathbb{E}(\bar{y}_{i\cdot}^2) \right], \text{ by linearity} \\ &= \frac{1}{k(n-1)} \left[ \sum_{i=1}^k \sum_{j=1}^n (\sigma^2 + \mu_i^2) - n \sum_{i=1}^k \left(\frac{\sigma^2}{n} + \mu_i^2\right) \right], \text{ by Lemma 3.9} \\ &= \frac{1}{k(n-1)} \left[ (kn - k)\sigma^2 + n \sum_{i=1}^k (\mu_i^2 - \mu_i^2) \right] = \sigma^2. \end{aligned}$$

That is,  $\mathbb{E}(\text{MS}_{\text{err}}) = \sigma^2$ , as desired.  $\square$

**PROPOSITION 3.11** (Mean of  $\text{MS}_{\text{trt}}$ ). We have

$$\mathbb{E}(\text{MS}_{\text{trt}}) \geq \sigma^2$$

with equality holds if and only if  $\mu = \mathbb{1}\mu_0 \in \mathbb{R}^k$  for some  $\mu_0 \in \mathbb{R}$ . i.e.,  $\text{MS}_{\text{trt}}$  is an unbiased estimator for  $\sigma^2$  given that  $\mu = \mathbb{1}\mu_0$  for some  $\mu_0$ .

*Proof.*

$$\begin{aligned}
\mathbb{E}(\text{MS}_{\text{trt}}) &= \mathbb{E}(\text{SS}_{\text{trt}}/(k-1)) = \mathbb{E}\left(\frac{n}{k-1} \sum_{i=1}^k (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2\right) \\
&= \mathbb{E}\left(\frac{n}{k-1} \sum_{i=1}^k (\bar{y}_{i\cdot}^2 + \bar{y}_{\cdot\cdot}^2 - 2\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot})\right), \text{ expand} \\
&= \mathbb{E}\left(\frac{n}{k-1} \left[ \sum_{i=1}^k \bar{y}_{i\cdot}^2 + k\bar{y}_{\cdot\cdot}^2 - 2\bar{y}_{\cdot\cdot} \sum_{i=1}^k \bar{y}_{i\cdot} \right]\right), \text{ separate} \\
&= \mathbb{E}\left(\frac{n}{k-1} \left[ \sum_{i=1}^k \bar{y}_{i\cdot}^2 + k\bar{y}_{\cdot\cdot}^2 - 2k\bar{y}_{\cdot\cdot}^2 \right]\right), \text{ reduce} \\
&= \mathbb{E}\left(\frac{n}{k-1} \left[ \sum_{i=1}^k \bar{y}_{i\cdot}^2 - k\bar{y}_{\cdot\cdot}^2 \right]\right), \text{ combine} \\
&= \frac{n}{k-1} \left[ \sum_{i=1}^k \mathbb{E}(\bar{y}_{i\cdot}^2) - k\mathbb{E}(\bar{y}_{\cdot\cdot}^2) \right], \text{ by linearity} \\
&= \frac{n}{k-1} \left[ \sum_{i=1}^k \left( \frac{\sigma^2}{n} + \mu_i^2 \right) - k \left( \frac{\sigma^2}{kn} + \left( \frac{1}{k} \sum_{i=1}^k \mu_i \right)^2 \right) \right], \text{ by Lemma 3.9} \\
&= \frac{n}{k-1} \left[ \left( \frac{k}{n} - \frac{1}{n} \right) \sigma^2 + \frac{1}{2k} \sum_{i,j=1}^k (\mu_i - \mu_j)^2 \right] \geq \sigma^2.
\end{aligned}$$

That is,  $\mathbb{E}(\text{MS}_{\text{trt}}) \geq \sigma^2$ . From the above derivation we can see that equality holds if and only if  $\mu = \mathbf{1}\mu_0 \in \mathbb{R}^k$  for some  $\mu_0 \in \mathbb{R}$ .  $\square$

### 3.1.3 Hypothesis Testing for Completely Randomized Design

**PROPOSITION 3.12.** Consider the cases where  $k = 2$ . We are interested in testing the following hypothesis

$$H_0 : \mu_1 = \mu_2 \text{ vs } H_1 : \mu_1 \neq \mu_2.$$

The  $T$ -statistics are:

$$T_0 := \frac{\bar{y}_{1\cdot} - \bar{y}_{2\cdot}}{\sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim \mathcal{N}(0, 1)$$

in the case where  $\sigma^2$  is known and reject the null if  $|T_0| > z_{\alpha/2}$ , or

$$T_0 := \frac{\bar{y}_{1\cdot} - \bar{y}_{2\cdot}}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim \mathcal{T}(n_1 + n_2 - 2)$$

in the case where  $\sigma^2$  is unknown and is estimated by  $s_p^2$  and reject the null if  $|T_0| > \mathcal{T}_{\alpha/2}(n_1 + n_2 - 2)$ .

**PROPOSITION 3.13.** Consider the cases where  $k = 2$ . We are interested in testing the following hypothesis

$$H_0 : \mu_1 = \mu_2 \text{ vs } H_1 : \mu_1 > \mu_2.$$

We reject the null if  $T_0 > +\mathcal{T}_\alpha(n_1 + n_2 - 2)$ .

**PROPOSITION 3.14.** Consider the cases where  $k = 2$ . We are interested in testing the following hypothesis

$$H_0 : \mu_1 = \mu_2 \text{ vs } H_1 : \mu_1 < \mu_2.$$

We reject the null if  $T_0 < -\mathcal{T}_\alpha(n_1 + n_2 - 2)$ .

**PROPOSITION 3.15.** Consider the cases where  $k = 2$ . We are interested in testing the following hypothesis

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ vs } H_1 : \sigma_1^2 \neq \sigma_2^2.$$

The  $F$ -statistics is:

$$F_0 := s_1^2/s_2^2 \sim \mathcal{F}(n_1 - 1, n_2 - 1).$$

We reject the null if  $F_0 > \mathcal{F}_{\alpha/2}(n_1 - 1, n_2 - 1)$  or  $F_0 < \mathcal{F}_{1-\alpha/2}(n_1 - 1, n_2 - 1)$ .

**DEFINITION 3.16** (ANOVA Table).

Table 3.1: ANOVA Table for Completely Randomized Design

	Sum of Squares	Degree of Freedom	Mean Squares	$F_0$
Treatment	$SS_{\text{trt}}$	$k - 1$	$MS_{\text{trt}}$	$MS_{\text{trt}}/MS_{\text{err}}$
Error	$SS_{\text{err}}$	$k(n - 1)$	$MS_{\text{err}}$	
Total	$SS_{\text{tot}}$	$kn - 1$		

**PROPOSITION 3.17.** We are interested in testing the following hypothesis

$$H_0 : \forall i, j \in \{1, \dots, k\}, \mu_i = \mu_j \text{ vs } H_1 : \exists i, j \in \{1, \dots, k\}, \mu_i \neq \mu_j.$$

The  $F$ -statistics is

$$F_0 := MS_{\text{trt}}/MS_{\text{err}} \sim \mathcal{F}(k - 1, k(n - 1)).$$

We reject the null if  $F_0 > \mathcal{F}_\alpha(k - 1, k(n - 1))$ .

## 3.2 Completely Randomized Design - Model 2

**DEFINITION 3.18** (Completely Randomized Design - Model 2). We model the observations as

$$y_{ij} = \mu + \alpha_i + e_{ij}, \text{ for } i \in \{1, \dots, k\} \text{ and } j \in \{1, \dots, n\}$$

with constraint  $\mathbf{1}^\top \alpha = 0$  and  $e_{ij} \sim \mathcal{N}(0, \sigma^2)$  are assumed to be independent. The number of parameters in the model is  $2 + k$ .

### 3.3 Randomized Block Design - Model 1

In the case where the number of treatments equals 2, this reduces to paired comparison design.

**DEFINITION 3.19** (Randomized Block Design - Model 1). Let  $a \in \mathbb{Z}_{++}$  denote the number of treatments. Let  $b \in \mathbb{Z}_{++}$  denote the number of blocks. We model the observations as

$$y_{ij} = \mu_i + \beta_j + e_{ij}, \text{ for } i \in \{1, \dots, a\} \text{ and } j \in \{1, \dots, b\}$$

where  $e_{ij} \sim \mathcal{N}(0, \sigma^2)$  are assumed to be independent. The total number of parameters in this model is  $a + b + 1$ .

#### 3.3.1 Hypothesis Testing

**PROPOSITION 3.20.** Consider the case where  $k = 2$ . We are interested in testing the following hypothesis

$$H_0 : \mu_1 = \mu_2 \text{ vs } H_1 : \mu_1 \neq \mu_2.$$

The  $T$ -statistics is

$$T_0 := \frac{\bar{y}_{1\cdot} - \bar{y}_{2\cdot}}{s_p \sqrt{\frac{2}{n}}} \sim \mathcal{T}(2n - 2)$$

We reject the null if  $|T_0| > \mathcal{T}_{\alpha/2}(2n - 2)$

### 3.4 Randomized Block Design - Model 2

**DEFINITION 3.21** (Randomized Block Design - Model 2). Let  $a \in \mathbb{Z}_{++}$  denote the number of treatments. Let  $b \in \mathbb{Z}_{++}$  denote the number of blocks. We model the observations as

$$y_{ij} = \mu + \alpha_i + \beta_j + e_{ij}, \text{ for } i \in \{1, \dots, a\} \text{ and } j \in \{1, \dots, b\}$$

with constraints  $\mathbf{1}^\top \alpha = 0$  and  $\mathbf{1}^\top \beta = 0$ , and  $e_{ij} \sim \mathcal{N}(0, \sigma^2)$  are assumed to be independent. The total number of parameters in this model is  $2 + a + b$ .

**LEMMA 3.22.** We have

$$\begin{aligned} \mathbb{E}(y_{ij}^2) &= \sigma^2 + (\mu + \alpha_i + \beta_j)^2, \quad \forall i, j, \\ \mathbb{E}(\bar{y}_{i\cdot}^2) &= \frac{\sigma^2}{b} + (\mu + \alpha_i)^2, \quad \forall i, \\ \mathbb{E}(\bar{y}_{\cdot j}^2) &= \frac{\sigma^2}{a} + (\mu + \beta_j)^2, \quad \forall j, \text{ and} \\ \mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \frac{\sigma^2}{ab} + \mu^2. \end{aligned}$$

*Proof.* Recall that  $\forall i \in \{1, \dots, a\}, \forall j \in \{1, \dots, b\}$ , we have  $y_{ij} \sim \mathcal{N}(\mu + \alpha_i + \beta_j, \sigma^2)$ . So

$$\begin{aligned} \bar{y}_{i\cdot} &= \frac{1}{b} \sum_{j=1}^b y_{ij} \sim \mathcal{N}\left(\mu + \alpha_i, \frac{\sigma^2}{b}\right), \quad \forall i \in \{1, \dots, a\}, \\ \bar{y}_{\cdot j} &= \frac{1}{a} \sum_{i=1}^a y_{ij} \sim \mathcal{N}\left(\mu + \beta_j, \frac{\sigma^2}{a}\right), \quad \forall j \in \{1, \dots, b\}, \text{ and} \\ \bar{y}_{\cdot\cdot} &= \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{ab}\right). \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}(y_{ij}^2) &= \mathbb{V}(y_{ij}) + \mathbb{E}^2(y_{ij}) = \sigma^2 + (\mu + \alpha_i + \beta_j)^2, \quad \forall i, j, \\ \mathbb{E}(\bar{y}_{i\cdot}^2) &= \mathbb{V}(\bar{y}_{i\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot}) = \frac{\sigma^2}{b} + (\mu + \alpha_i)^2, \quad \forall i, \\ \mathbb{E}(\bar{y}_{\cdot j}^2) &= \mathbb{V}(\bar{y}_{\cdot j}) + \mathbb{E}^2(\bar{y}_{\cdot j}) = \frac{\sigma^2}{a} + (\mu + \beta_j)^2, \quad \forall j, \text{ and} \\ \mathbb{E}(\bar{y}_{\cdot\cdot}^2) &= \mathbb{V}(\bar{y}_{\cdot\cdot}) + \mathbb{E}^2(\bar{y}_{\cdot\cdot}) = \frac{\sigma^2}{ab} + \mu^2. \end{aligned}$$

□



## 3.4.1 Estimation of Mean

**PROPOSITION 3.23.** Let  $y_{ij}$  for  $i \in \{1, \dots, a\}$  and  $j \in \{1, \dots, b\}$  be given. Consider the following optimization problem:

$$\begin{aligned} \text{(P)} \quad \min \quad & f(\mu, \alpha, \beta) := \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \mu - \alpha_i - \beta_j)^2 \\ \text{subject to:} \quad & \mu \in \mathbb{R}, \alpha \in \mathbb{R}^a, \beta \in \mathbb{R}^b, \\ & \mathbf{1}^\top \alpha = 0, \mathbf{1}^\top \beta = 0. \end{aligned}$$

Then the minimizer  $(\hat{\mu}, \hat{\alpha}, \hat{\beta}) \in \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b$  of (P) is given by

$$\begin{aligned} \hat{\mu} &= \bar{y}_{..} = \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij}, \\ \hat{\alpha}_i &= \bar{y}_{i.} - \bar{y}_{..} = \frac{1}{b} \sum_{j=1}^b y_{ij} - \bar{y}_{..}, \text{ for } i \in \{1, \dots, a\}, \\ \hat{\beta}_j &= \bar{y}_{.j} - \bar{y}_{..} = \frac{1}{a} \sum_{i=1}^a y_{ij} - \bar{y}_{..}, \text{ for } j \in \{1, \dots, b\}. \end{aligned}$$

*Proof.* Form the Lagrangian function  $\mathcal{L} : \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b \oplus \mathbb{R} \oplus \mathbb{R}$

$$\mathcal{L}(\mu, \alpha, \beta, \xi, \eta) := f(\mu, \alpha, \beta) - \xi \mathbf{1}^\top \alpha - \eta \mathbf{1}^\top \beta.$$

Compute the derivatives:

$$\begin{aligned} \frac{\partial}{\partial \mu} \mathcal{L}(\mu, \alpha, \beta, \xi, \eta) &= \frac{\partial}{\partial \mu} \left[ f(\mu, \alpha, \beta) - \xi \mathbf{1}^\top \alpha - \eta \mathbf{1}^\top \beta \right] \\ &= \frac{\partial}{\partial \mu} \left[ \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \mu - \alpha_i - \beta_j)^2 - \xi \mathbf{1}^\top \alpha - \eta \mathbf{1}^\top \beta \right] \\ &= \sum_{i=1}^a \sum_{j=1}^b \frac{\partial}{\partial \mu} (y_{ij} - \mu - \alpha_i - \beta_j)^2 \\ &= -2 \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \mu - \alpha_i - \beta_j), \\ \frac{\partial}{\partial \alpha_p} \mathcal{L}(\mu, \alpha, \beta, \xi, \eta) &= \frac{\partial}{\partial \alpha_p} \left[ f(\mu, \alpha, \beta) - \xi \mathbf{1}^\top \alpha - \eta \mathbf{1}^\top \beta \right] \\ &= \frac{\partial}{\partial \alpha_p} \left[ \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \mu - \alpha_i - \beta_j)^2 - \xi \mathbf{1}^\top \alpha - \eta \mathbf{1}^\top \beta \right] \\ &= \sum_{i=1}^a \sum_{j=1}^b \frac{\partial}{\partial \alpha_p} (y_{ij} - \mu - \alpha_i - \beta_j)^2 - \xi \frac{\partial}{\partial \alpha_p} \mathbf{1}^\top \alpha \end{aligned}$$

$$\begin{aligned}
&= -2 \sum_{j=1}^b (y_{pj} - \mu - \alpha_p - \beta_j) - \xi, \\
\frac{\partial}{\partial \beta_q} \mathcal{L}(\mu, \alpha, \beta, \xi, \eta) &= \frac{\partial}{\partial \beta_q} \left[ f(\mu, \alpha, \beta) - \xi \mathbf{1}^\top \alpha - \eta \mathbf{1}^\top \beta \right] \\
&= \frac{\partial}{\partial \beta_q} \left[ \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \mu - \alpha_i - \beta_j)^2 - \xi \mathbf{1}^\top \alpha - \eta \mathbf{1}^\top \beta \right] \\
&= \sum_{i=1}^a \frac{\partial}{\partial \beta_q} (y_{ij} - \mu - \alpha_i - \beta_j)^2 - \eta \frac{\partial}{\partial \beta_q} \mathbf{1}^\top \beta \\
&= -2 \sum_{i=1}^a (y_{iq} - \mu - \alpha_i - \beta_q) - \eta, \\
\frac{\partial}{\partial \xi} \mathcal{L}(\mu, \alpha, \beta, \xi, \eta) &= \frac{\partial}{\partial \xi} \left[ f(\mu, \alpha, \beta) - \xi \mathbf{1}^\top \alpha - \eta \mathbf{1}^\top \beta \right] = -\mathbf{1}^\top \alpha, \\
\frac{\partial}{\partial \eta} \mathcal{L}(\mu, \alpha, \beta, \xi, \eta) &= \frac{\partial}{\partial \eta} \left[ f(\mu, \alpha, \beta) - \xi \mathbf{1}^\top \alpha - \eta \mathbf{1}^\top \beta \right] = -\mathbf{1}^\top \beta.
\end{aligned}$$

Let  $(\hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\xi}, \hat{\eta}) \in \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b \oplus \mathbb{R} \oplus \mathbb{R}$  be such that  $\nabla \mathcal{L}(\hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\xi}, \hat{\eta}) = \mathbf{0} \in \mathbb{R}^{a+b+3}$ . Then we get the following system of equations:

$$\left\{ \begin{array}{l} -2 \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j) = 0 \\ -2 \sum_{j=1}^b (y_{pj} - \hat{\mu} - \hat{\alpha}_p - \hat{\beta}_j) - \hat{\xi} = 0, \forall p \\ -2 \sum_{i=1}^a (y_{iq} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_q) - \hat{\eta} = 0, \forall q \\ -\mathbf{1}^\top \hat{\alpha} = 0 \\ -\mathbf{1}^\top \hat{\beta} = 0 \end{array} \right. \implies \left\{ \begin{array}{l} \hat{\mu} = \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij} \\ \hat{\alpha}_i = \bar{y}_{i\cdot} - \hat{\mu}, \forall i \\ \hat{\beta}_j = \bar{y}_{\cdot j} - \hat{\mu}, \forall j \\ \hat{\xi} = 0 \\ \hat{\eta} = 0. \end{array} \right.$$

Testing the Hessian of  $\mathcal{L}$  at point  $(\hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\xi}, \hat{\eta}) \in \mathbb{R} \oplus \mathbb{R}^a \oplus \mathbb{R}^b \oplus \mathbb{R} \oplus \mathbb{R}$  confirms that it is indeed a minimizer of  $\mathcal{L}$ .  $\square$

**PROPOSITION 3.24** (Mean of the Mean Estimator). We have

$$\mathbb{E}(\hat{\mu}) = \mu \in \mathbb{R}, \quad \mathbb{E}(\hat{\alpha}) = \alpha \in \mathbb{R}^a, \quad \text{and} \quad \mathbb{E}(\hat{\beta}) = \beta \in \mathbb{R}^b.$$

i.e.,  $\hat{\mu}$  is an unbiased estimator for  $\mu$ ,  $\hat{\alpha} \in \mathbb{R}^a$  is an unbiased estimator for  $\alpha \in \mathbb{R}^a$ , and  $\hat{\beta} \in \mathbb{R}^b$  is an unbiased estimator for  $\beta \in \mathbb{R}^b$ .

*Proof.* Recall that  $\forall i \in \{1, \dots, a\}$ ,  $\forall j \in \{1, \dots, b\}$ , we have  $y_{ij} \sim \mathcal{N}(\mu + \alpha_i + \beta_j, \sigma^2)$ . So

$$\begin{aligned}\bar{y}_{i\cdot} &= \frac{1}{b} \sum_{j=1}^b y_{ij} \sim \mathcal{N}(\mu + \alpha_i, \frac{\sigma^2}{b}), \quad \forall i \in \{1, \dots, a\}, \\ \bar{y}_{\cdot j} &= \frac{1}{a} \sum_{i=1}^a y_{ij} \sim \mathcal{N}(\mu + \beta_j, \frac{\sigma^2}{a}), \quad \forall j \in \{1, \dots, b\}, \\ \bar{y}_{\cdot\cdot} &= \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij} \sim \mathcal{N}(\mu, \frac{\sigma^2}{ab}).\end{aligned}$$

Now we can compute

$$\begin{aligned}\mathbb{E}(\hat{\mu}) &= \mathbb{E}(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij}) = \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b \mathbb{E}(y_{ij}) = \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b (\mu + \alpha_i + \beta_j) \\ &= \mu + \frac{1}{a} \sum_{i=1}^a \alpha_i + \frac{1}{b} \sum_{j=1}^b \beta_j = \mu, \\ \mathbb{E}(\hat{\alpha}_i) &= \mathbb{E}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) = \mathbb{E}(\bar{y}_{i\cdot}) - \mathbb{E}(\bar{y}_{\cdot\cdot}) = \mu + \alpha_i - \mu = \alpha_i, \quad \forall i, \\ \mathbb{E}(\hat{\beta}_j) &= \mathbb{E}(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) = \mathbb{E}(\bar{y}_{\cdot j}) - \mathbb{E}(\bar{y}_{\cdot\cdot}) = \mu + \beta_j - \mu = \beta_j, \quad \forall j.\end{aligned}$$

That is,  $\mathbb{E}(\hat{\mu}) = \mu$ ,  $\mathbb{E}(\hat{\alpha}) = \alpha$ , and  $\mathbb{E}(\hat{\beta}) = \beta$ , as desired.  $\square$

**PROPOSITION 3.25** (Variance of the Mean Estimator).

$$\begin{aligned}\mathbb{V}(\hat{\mu}) &= \frac{\sigma^2}{ab}, \\ \mathbb{V}(\hat{\alpha}_i) &= \frac{a-1}{ab} \sigma^2, \quad \forall i \in \{1, \dots, a\}, \text{ and} \\ \mathbb{V}(\hat{\beta}_j) &= \frac{b-1}{ab} \sigma^2, \quad \forall j \in \{1, \dots, b\}.\end{aligned}$$

*Proof.*

$$\begin{aligned}\mathbb{V}(\hat{\mu}) &= \mathbb{V}(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij}) = \frac{1}{a^2 b^2} \sum_{i=1}^a \sum_{j=1}^b \mathbb{V}(y_{ij}) = \frac{1}{a^2 b^2} \sum_{i=1}^a \sum_{j=1}^b \sigma^2 = \frac{\sigma^2}{ab}, \\ \mathbb{V}(\hat{\alpha}_p) &= \mathbb{V}(\bar{y}_{p\cdot} - \bar{y}_{\cdot\cdot}) = \mathbb{V}(\bar{y}_{p\cdot} - \frac{1}{a} \sum_{i=1}^a \bar{y}_{i\cdot}) = \mathbb{V}((1 - \frac{1}{a})\bar{y}_{p\cdot} - \frac{1}{a} \sum_{\substack{i=1 \\ i \neq p}}^a \bar{y}_{i\cdot}) \\ &= (1 - \frac{1}{a})^2 \mathbb{V}(\bar{y}_{p\cdot}) + \frac{1}{a^2} \sum_{\substack{i=1 \\ i \neq p}}^a \mathbb{V}(\bar{y}_{i\cdot}) = (1 - \frac{1}{a})^2 \frac{\sigma^2}{b} + \frac{1}{a^2} \sum_{\substack{i=1 \\ i \neq p}}^a \frac{\sigma^2}{b} \\ &= \frac{a-1}{ab} \sigma^2, \quad \forall p \in \{1, \dots, a\}, \text{ and}\end{aligned}$$

$$\begin{aligned}
\text{cov}(\hat{\alpha}_p, \hat{\alpha}_q) &= \mathbb{E}(\hat{\alpha}_p \hat{\alpha}_q) - \mathbb{E}(\hat{\alpha}_p) \mathbb{E}(\hat{\alpha}_q) = \mathbb{E}((\bar{y}_{p\cdot} - \bar{y}_{\cdot\cdot})(\bar{y}_{q\cdot} - \bar{y}_{\cdot\cdot})) - \alpha_p \alpha_q \\
&= \mathbb{E}(\bar{y}_{p\cdot} \bar{y}_{q\cdot}) - \mathbb{E}(\bar{y}_{p\cdot} \bar{y}_{\cdot\cdot}) - \mathbb{E}(\bar{y}_{q\cdot} \bar{y}_{\cdot\cdot}) + \mathbb{E}(\bar{y}_{\cdot\cdot} \bar{y}_{\cdot\cdot}) - \alpha_p \alpha_q \\
&= \mathbb{E}\left(\left(\frac{1}{b} \sum_{j=1}^b y_{pj}\right)\left(\frac{1}{b} \sum_{j=1}^b y_{qj}\right)\right) + \mathbb{E}\left(\left(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij}\right)\left(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij}\right)\right) \\
&\quad - \mathbb{E}\left(\left(\frac{1}{b} \sum_{j=1}^b y_{pj}\right)\left(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij}\right)\right) - \mathbb{E}\left(\left(\frac{1}{b} \sum_{j=1}^b y_{qj}\right)\left(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b y_{ij}\right)\right) - \alpha_p \alpha_q \\
&= \left(\frac{1}{b} \sum_{j=1}^b \mathbb{E}(y_{pj})\right)\left(\frac{1}{b} \sum_{j=1}^b \mathbb{E}(y_{qj})\right) - \alpha_p \alpha_q \\
&\quad + \frac{1}{a^2 b^2} \sum_{i=1}^a \sum_{j=1}^b \mathbb{E}(y_{ij}^2) + \left(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b \mathbb{E}(y_{ij})\right)\left(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b \mathbb{E}(y_{ij})\right) - \frac{1}{a^2 b^2} \sum_{i=1}^a \sum_{j=1}^b \mathbb{E}^2(y_{ij}) \\
&\quad - \frac{1}{ab^2} \sum_{j=1}^b \mathbb{E}(y_{pj}^2) - \left(\frac{1}{b} \sum_{j=1}^b \mathbb{E}(y_{pj})\right)\left(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b \mathbb{E}(y_{ij})\right) + \frac{1}{ab^2} \sum_{j=1}^b \mathbb{E}^2(y_{pj}) \\
&\quad - \frac{1}{ab^2} \sum_{j=1}^b \mathbb{E}(y_{qj}^2) - \left(\frac{1}{b} \sum_{j=1}^b \mathbb{E}(y_{qj})\right)\left(\frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b \mathbb{E}(y_{ij})\right) + \frac{1}{ab^2} \sum_{j=1}^b \mathbb{E}^2(y_{qj}) \\
&= (\mu + \alpha_p)(\mu + \alpha_q) - \alpha_p \alpha_q + \sigma^2 + \mu^2 - (\mu + \alpha_p)\mu - \frac{1}{ab} \sigma^2 - (\mu + \alpha_q)\mu - \frac{1}{ab} \sigma^2 \\
&= \left(1 - \frac{2}{ab}\right) \sigma^2, \quad \forall p, q \in \{1, \dots, a\} : p \neq q,
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}(\hat{\beta}_r) &= \mathbb{V}(\bar{y}_{\cdot r} - \bar{y}_{\cdot\cdot}) = \mathbb{V}(\bar{y}_{\cdot r} - \frac{1}{b} \sum_{j=1}^b \bar{y}_{\cdot j}) = \mathbb{V}\left((1 - \frac{1}{b})\bar{y}_{\cdot r} - \frac{1}{b} \sum_{\substack{j=1 \\ j \neq r}}^b \bar{y}_{\cdot j}\right) \\
&= \left(1 - \frac{1}{b}\right)^2 \mathbb{V}(\bar{y}_{\cdot r}) + \frac{1}{b^2} \sum_{\substack{j=1 \\ j \neq r}}^b \mathbb{V}(\bar{y}_{\cdot j}) = \left(1 - \frac{1}{b}\right)^2 \frac{\sigma^2}{a} + \frac{1}{b^2} \sum_{\substack{j=1 \\ j \neq r}}^b \frac{\sigma^2}{a} \\
&= \frac{b-1}{ab} \sigma^2, \quad \forall r \in \{1, \dots, b\}.
\end{aligned}$$

□

### 3.4.2 Estimation of Variance

**DEFINITION 3.26** (Sum of Squares). We define the following terms:

$$\text{SS}_{\text{trt}} := b \sum_{i=1}^a (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2,$$

$$\text{SS}_{\text{blk}} := a \sum_{j=1}^b (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})^2,$$

$$\begin{aligned} \text{SS}_{\text{err}} &:= \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot})^2, \\ \text{SS}_{\text{tot}} &:= \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \bar{y}_{\cdot\cdot})^2. \end{aligned}$$

**PROPOSITION 3.27** (Decomposition of  $\text{SS}_{\text{tot}}$ ). We have

$$\text{SS}_{\text{tot}} = \text{SS}_{\text{trt}} + \text{SS}_{\text{blk}} + \text{SS}_{\text{err}}.$$

*Proof.*

$$\begin{aligned} \text{SS}_{\text{tot}} &= \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \bar{y}_{\cdot\cdot})^2 = \sum_{i=1}^a \sum_{j=1}^b \left[ (y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot}) + (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) + (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) \right]^2 \\ &= \text{SS}_{\text{trt}} + \text{SS}_{\text{blk}} + \text{SS}_{\text{err}} + 2 \sum_{i=1}^a \sum_{j=1}^b (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) \\ &\quad + 2 \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot})(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) + 2 \sum_{j=1}^b \sum_{i=1}^a (y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot})(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) \\ &= \text{SS}_{\text{trt}} + \text{SS}_{\text{blk}} + \text{SS}_{\text{err}} + 2 \left[ \sum_{i=1}^a (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) \right] \left[ \sum_{j=1}^b (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) \right] \\ &\quad + 2 \sum_{i=1}^a (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) \sum_{j=1}^b (y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot}) + 2 \sum_{j=1}^b (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) \sum_{i=1}^a (y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot}) \\ &= \text{SS}_{\text{trt}} + \text{SS}_{\text{blk}} + \text{SS}_{\text{err}} + 2(a\bar{y}_{\cdot\cdot} - a\bar{y}_{\cdot\cdot})(b\bar{y}_{\cdot\cdot} - b\bar{y}_{\cdot\cdot}) \\ &\quad + 2 \sum_{i=1}^a (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})(b\bar{y}_{i\cdot} - b\bar{y}_{i\cdot} - b\bar{y}_{\cdot\cdot} - b\bar{y}_{\cdot\cdot}) + 2 \sum_{j=1}^b (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})(a\bar{y}_{\cdot j} - a\bar{y}_{\cdot\cdot} - a\bar{y}_{\cdot j} + a\bar{y}_{\cdot\cdot}) \\ &= \text{SS}_{\text{trt}} + \text{SS}_{\text{blk}} + \text{SS}_{\text{err}} + 2 \cdot 0 \cdot 0 + 2 \sum_{i=1}^a (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) \cdot 0 + 2 \sum_{j=1}^b (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}) \cdot 0 \\ &= \text{SS}_{\text{trt}} + \text{SS}_{\text{blk}} + \text{SS}_{\text{err}}. \end{aligned}$$

□

**DEFINITION 3.28** (Mean Squares). We define the following estimators for the variance  $\sigma^2$ .

$$\text{MS}_{\text{trt}} := \text{SS}_{\text{trt}} / (a - 1),$$

$$\text{MS}_{\text{err}} := \text{SS}_{\text{err}} / ((a-1)(b-1)).$$

**PROPOSITION 3.29.** We have

$$\mathbb{E}(\text{MS}_{\text{err}}) = \sigma^2.$$

i.e.,  $\text{MS}_{\text{err}}$  is an unbiased estimator for  $\sigma^2$ .

*Proof.*

$$\begin{aligned} \mathbb{E}(\text{MS}_{\text{err}}) &= \mathbb{E}(\text{SS}_{\text{err}} / ((a-1)(b-1))) = \mathbb{E}\left(\frac{1}{(a-1)(b-1)} \sum_{i=1}^a \sum_{j=1}^b (y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot})^2\right) \\ &= \mathbb{E}\left(\frac{1}{(a-1)(b-1)} \left[ \sum_{i=1}^a \sum_{j=1}^b \begin{array}{cccc} +y_{ij}^2 & -y_{ij}\bar{y}_{i\cdot} & -y_{ij}\bar{y}_{\cdot j} & +y_{ij}\bar{y}_{\cdot\cdot} \\ -\bar{y}_{i\cdot}y_{ij} & +\bar{y}_{i\cdot}^2 & +\bar{y}_{i\cdot}\bar{y}_{\cdot j} & -\bar{y}_{i\cdot}\bar{y}_{\cdot\cdot} \\ -\bar{y}_{\cdot j}y_{ij} & +\bar{y}_{\cdot j}\bar{y}_{i\cdot} & +\bar{y}_{\cdot j}^2 & -\bar{y}_{\cdot j}\bar{y}_{\cdot\cdot} \\ +\bar{y}_{\cdot\cdot}y_{ij} & -\bar{y}_{\cdot\cdot}\bar{y}_{i\cdot} & -\bar{y}_{\cdot\cdot}\bar{y}_{\cdot j} & +\bar{y}_{\cdot\cdot}^2 \end{array} \right]\right) \\ &= \mathbb{E}\left(\frac{1}{(a-1)(b-1)} \left[ \begin{array}{cccc} +\sum_{i=1}^a \sum_{j=1}^b y_{ij}^2 & -\sum_{i=1}^a \sum_{j=1}^b y_{ij}\bar{y}_{i\cdot} & -\sum_{i=1}^a \sum_{j=1}^b y_{ij}\bar{y}_{\cdot j} & +\sum_{i=1}^a \sum_{j=1}^b y_{ij}\bar{y}_{\cdot\cdot} \\ -\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{i\cdot}y_{ij} & +\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{i\cdot}^2 & +\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{i\cdot}\bar{y}_{\cdot j} & -\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{i\cdot}\bar{y}_{\cdot\cdot} \\ -\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{\cdot j}y_{ij} & +\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{\cdot j}\bar{y}_{i\cdot} & +\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{\cdot j}^2 & -\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{\cdot j}\bar{y}_{\cdot\cdot} \\ +\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{\cdot\cdot}y_{ij} & -\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{\cdot\cdot}\bar{y}_{i\cdot} & -\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{\cdot\cdot}\bar{y}_{\cdot j} & +\sum_{i=1}^a \sum_{j=1}^b \bar{y}_{\cdot\cdot}^2 \end{array} \right]\right) \\ &= \mathbb{E}\left(\frac{1}{(a-1)(b-1)} \left[ \begin{array}{cccc} +\sum_{i=1}^a \sum_{j=1}^b y_{ij}^2 & -b \sum_{i=1}^a \bar{y}_{i\cdot}^2 & -a \sum_{j=1}^b \bar{y}_{\cdot j}^2 & +ab\bar{y}_{\cdot\cdot}^2 \\ -b \sum_{i=1}^a \bar{y}_{i\cdot}^2 & +b \sum_{i=1}^a \bar{y}_{i\cdot}^2 & +ab\bar{y}_{\cdot\cdot}^2 & -ab\bar{y}_{\cdot\cdot}^2 \\ -a \sum_{j=1}^b \bar{y}_{\cdot j}^2 & +ab\bar{y}_{\cdot\cdot}^2 & +a \sum_{j=1}^b \bar{y}_{\cdot j}^2 & -ab\bar{y}_{\cdot\cdot}^2 \\ +ab\bar{y}_{\cdot\cdot}^2 & -ab\bar{y}_{\cdot\cdot}^2 & -ab\bar{y}_{\cdot\cdot}^2 & +ab\bar{y}_{\cdot\cdot}^2 \end{array} \right]\right) \\ &= \frac{1}{(a-1)(b-1)} \mathbb{E}\left(\sum_{i=1}^a \sum_{j=1}^b y_{ij}^2 - a \sum_{j=1}^b \bar{y}_{\cdot j}^2 - b \sum_{i=1}^a \bar{y}_{i\cdot}^2 + ab\bar{y}_{\cdot\cdot}^2\right), \text{ combine} \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{(a-1)(b-1)} \left[ \sum_{i=1}^a \sum_{j=1}^b \mathbb{E}(y_{ij}^2) - a \sum_{j=1}^b \mathbb{E}(\bar{y}_{\cdot j}^2) - b \sum_{i=1}^a \mathbb{E}(\bar{y}_{i \cdot}^2) + ab \mathbb{E}(\bar{y}_{\cdot \cdot}^2) \right], \text{ by linearity} \\
&= \frac{1}{(a-1)(b-1)} \left[ \sum_{i=1}^a \sum_{j=1}^b (\sigma^2 + (\mu + \alpha_i + \beta_j)^2) + ab \left( \frac{\sigma^2}{ab} + \mu^2 \right) \right. \\
&\quad \left. - a \sum_{j=1}^b \left( \frac{\sigma^2}{a} + (\mu + \beta_j)^2 \right) - b \sum_{i=1}^a \left( \frac{\sigma^2}{b} + (\mu + \alpha_i)^2 \right) \right], \text{ by Lemma 3.22} \\
&= \frac{1}{(a-1)(b-1)} \left[ (ab + 1 - a - b)\sigma^2 + (ab + ab - ab - ab)\mu^2 \right. \\
&\quad \left. + 0\mu + (b-b) \sum_{i=1}^a \alpha_i^2 + (a-a) \sum_{j=1}^b \beta_j^2 + \sum_{i=1}^a \sum_{j=1}^b \alpha_i \beta_j \right] \\
&= \frac{1}{(a-1)(b-1)} (ab + 1 - a - b)\sigma^2 = \sigma^2.
\end{aligned}$$

That is,  $\mathbb{E}(\text{MS}_{\text{err}}) = \sigma^2$ , as desired.  $\square$

**PROPOSITION 3.30** (Mean of  $\text{MS}_{\text{trt}}$ ). We have

$$\mathbb{E}(\text{MS}_{\text{trt}}) \geq \sigma^2$$

with equality holds if and only if  $\alpha = \mathbf{0} \in \mathbb{R}^a$ . i.e.,  $\text{MS}_{\text{trt}}$  is an unbiased estimator for  $\sigma^2$  given that  $\alpha = \mathbf{0}$ .

*Proof.*

$$\begin{aligned}
\mathbb{E}(\text{MS}_{\text{trt}}) &= \mathbb{E}(\text{SS}_{\text{trt}}/(a-1)) = \mathbb{E}\left(\frac{b}{a-1} \sum_{i=1}^a (\bar{y}_{i \cdot} - \bar{y}_{\cdot \cdot})^2\right) \\
&= \mathbb{E}\left(\frac{b}{a-1} \sum_{i=1}^a (\bar{y}_{i \cdot}^2 + \bar{y}_{\cdot \cdot}^2 - 2\bar{y}_{i \cdot} \bar{y}_{\cdot \cdot})\right), \text{ expand} \\
&= \mathbb{E}\left(\frac{b}{a-1} \left[ \sum_{i=1}^a \bar{y}_{i \cdot}^2 + a\bar{y}_{\cdot \cdot}^2 - 2\bar{y}_{\cdot \cdot} \sum_{i=1}^a \bar{y}_{i \cdot} \right]\right), \text{ separate} \\
&= \mathbb{E}\left(\frac{b}{a-1} \left[ \sum_{i=1}^a \bar{y}_{i \cdot}^2 + a\bar{y}_{\cdot \cdot}^2 - 2a\bar{y}_{\cdot \cdot}^2 \right]\right), \text{ reduce} \\
&= \mathbb{E}\left(\frac{b}{a-1} \left[ \sum_{i=1}^a \bar{y}_{i \cdot}^2 - a\bar{y}_{\cdot \cdot}^2 \right]\right), \text{ combine} \\
&= \frac{b}{a-1} \left[ \sum_{i=1}^a \mathbb{E}(\bar{y}_{i \cdot}^2) - a\mathbb{E}(\bar{y}_{\cdot \cdot}^2) \right], \text{ by linearity}
\end{aligned}$$

$$\begin{aligned}
&= \frac{b}{a-1} \left[ \sum_{i=1}^a \left( \frac{\sigma^2}{b} + (\mu + \alpha_i)^2 \right) - a \left( \frac{\sigma^2}{ab} + \mu^2 \right) \right], \text{ by Lemma 3.22} \\
&= \frac{b}{a-1} \left[ \left( \frac{a}{b} - \frac{1}{b} \right) \sigma^2 + \sum_{i=1}^a \alpha_i^2 \right] \geq \sigma^2.
\end{aligned}$$

That is,  $\mathbb{E}(\text{MS}_{\text{trt}}) \geq \sigma^2$ . From the above derivation we can see that equality holds if and only if  $\alpha = \mathbf{0} \in \mathbb{R}^a$ .  $\square$

### 3.4.3 Hypothesis Testing for Randomized Block Design

**DEFINITION 3.31** (ANOVA Table).

Table 3.2: ANOVA Table for Randomized Block Design

	Sum of Squares	Degrees of Freedom	Mean Squares	$F_0$
Treatment	$\text{SS}_{\text{trt}}$	$a - 1$	$\text{MS}_{\text{trt}}$	$\text{MS}_{\text{trt}}/\text{MS}_{\text{err}}$
Block	$\text{SS}_{\text{blk}}$	$b - 1$	$\text{MS}_{\text{blk}}$	
Error	$\text{SS}_{\text{err}}$	$(a - 1)(b - 1)$	$\text{MS}_{\text{err}}$	
Total	$\text{SS}_{\text{tot}}$	$ab - 1$		

**PROPOSITION 3.32.** We are interested in testing the following hypothesis:

$$H_0 : \alpha = \mathbf{0} \in \mathbb{R}^a \text{ vs } H_1 : \alpha \neq \mathbf{0} \in \mathbb{R}^a.$$

The  $F$ -statistics is

$$F_0 := \text{MS}_{\text{trt}}/\text{MS}_{\text{err}} \sim \mathcal{F}(a - 1, (a - 1)(b - 1)).$$

We reject the null if  $F_0 > \mathcal{F}_\alpha(a - 1, (a - 1)(b - 1))$ .

### 3.5 Two-Way Factorial Design

**DEFINITION 3.33.** Let  $a \in \mathbb{Z}_{++}$  denote the number of treatments of factor  $A$ . Let  $b \in \mathbb{Z}_{++}$  denote the number of treatments of factor  $B$ . Let  $n \in \mathbb{Z}_{++}$  denote the number of repetitions for each combination of treatments. We model the observations as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}, \text{ for } i \in \{1, \dots, a\}, j \in \{1, \dots, b\}, k \in \{1, \dots, n\}$$

with constraints  $\mathbf{1}^\top \alpha = 0$ ,  $\mathbf{1}^\top \beta = 0$ ,  $\gamma^\top \mathbf{1} = \mathbf{0}$ , and  $\gamma \mathbf{1} = \mathbf{0}$ , and  $e_{ijk} \sim \mathcal{N}(0, \sigma^2)$  are assumed to be independent. The total number of parameters in this model is  $2 + a + b + ab$ .

#### 3.5.1 Estimation of Mean

#### 3.5.2 Estimation of Variance

**DEFINITION 3.34** (Sum of Squared Errors). We define the following terms:

$$\begin{aligned} \text{SS}_A &:= bn \sum_{i=1}^a (\bar{y}_{i..} - \bar{y}_{...})^2 \\ \text{SS}_B &:= an \sum_{j=1}^b (\bar{y}_{.j.} - \bar{y}_{...})^2 \\ \text{SS}_{AB} &:= n \sum_{i=1}^a \sum_{j=1}^b (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 \\ \text{SS}_{\text{err}} &:= \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (y_{ijk} - \bar{y}_{ij.})^2 \\ \text{SS}_{\text{tot}} &:= \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (y_{ijk} - \bar{y}_{...})^2. \end{aligned}$$

**REMARK 3.35.** Using triple indexing for vectors in  $\mathbb{R}^{abn}$ , we define vectors  $x_A, x_B, x_{AB}, x_{\text{err}}, x_{\text{tot}} \in \mathbb{R}^{abn}$  by

$$\begin{aligned} (x_A)_{i..} &:= \bar{y}_{i..} - \bar{y}_{...}, & (x_B)_{.j.} &:= \bar{y}_{.j.} - \bar{y}_{...}, & (x_{AB})_{ij.} &:= \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}, \\ (x_{\text{err}})_{ijk} &:= y_{ijk} - \bar{y}_{ij.}, & \text{and } (x_{\text{tot}})_{ijk} &:= y_{ijk} - \bar{y}_{...}. \end{aligned}$$

Then  $\forall I \in \{A, B, AB, \text{err}, \text{tot}\}$ , we have  $\text{SS}_I = \|x_I\|_2^2$ ; and  $\forall I, J \in \{A, B, AB, \text{err}\}$ , we have  $\langle x_I, x_J \rangle = 0$ ; and  $x_{\text{tot}} = x_A + x_B + x_{AB} + x_{\text{err}}$ .

**PROPOSITION 3.36** (Decomposition of  $\text{SS}_{\text{tot}}$ ). We have

$$\text{SS}_{\text{tot}} = \text{SS}_A + \text{SS}_B + \text{SS}_{AB} + \text{SS}_{\text{err}}.$$

*Proof.*

$$\begin{aligned}
\text{SS}_{\text{tot}} &= \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (y_{ijk} - \bar{y}_{...})^2 \\
&= \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n \left[ (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) + (y_{ijk} - \bar{y}_{ij.}) \right]^2 \\
&= \text{SS}_A + \text{SS}_B + \text{SS}_{AB} + \text{SS}_{\text{err}} + 2n \sum_{i=1}^a \sum_{j=1}^b (\bar{y}_{i..} - \bar{y}_{...})(\bar{y}_{.j.} - \bar{y}_{...}) \\
&\quad + 2 \sum_{i=1}^a \sum_{j=1}^b \left[ (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) \right] \sum_{k=1}^n (y_{ijk} - \bar{y}_{ij.}) \\
&\quad + 2n \sum_{i=1}^a (\bar{y}_{i..} - \bar{y}_{...}) \sum_{j=1}^b (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) \\
&\quad + 2n \sum_{j=1}^b (\bar{y}_{.j.} - \bar{y}_{...}) \sum_{i=1}^a (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) \\
&= \text{SS}_A + \text{SS}_B + \text{SS}_{AB} + \text{SS}_{\text{err}} + 2(a\bar{y}_{...} - a\bar{y}_{...})(b\bar{y}_{...} - b\bar{y}_{...}) \\
&\quad + 2 \sum_{i=1}^a \sum_{j=1}^b \left[ (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) \right] (n\bar{y}_{ij.} - n\bar{y}_{ij.}) \\
&\quad + 2 \sum_{k=1}^n \sum_{i=1}^a (\bar{y}_{i..} - \bar{y}_{...})(b\bar{y}_{i..} - b\bar{y}_{i..} - b\bar{y}_{...} + b\bar{y}_{...}) \\
&\quad + 2 \sum_{k=1}^n \sum_{j=1}^b (\bar{y}_{.j.} - \bar{y}_{...})(a\bar{y}_{.j.} - a\bar{y}_{...} - a\bar{y}_{.j.} + a\bar{y}_{...}) \\
&= \text{SS}_A + \text{SS}_B + \text{SS}_{AB} + 0 = \text{SS}_A + \text{SS}_B + \text{SS}_{AB}.
\end{aligned}$$

□

**DEFINITION 3.37** (Variance Estimator). We define the following estimators for the variance  $\sigma^2$ .

$$\begin{aligned} \text{MS}_A &:= \text{SS}_A / (a - 1), \\ \text{MS}_B &:= \text{SS}_B / (b - 1), \\ \text{MS}_{AB} &:= \text{SS}_{AB} / ((a - 1)(b - 1)), \\ \text{MS}_{\text{err}} &:= \text{SS}_{\text{err}} / (ab(n - 1)). \end{aligned}$$

**LEMMA 3.38.** We have

$$\begin{aligned} \mathbb{E}(y_{ijk}^2) &= \sigma^2 + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2, \quad \forall i, j, k \\ \mathbb{E}(\bar{y}_{ij\cdot}^2) &= \frac{\sigma^2}{n} + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2, \quad \forall i, j, \\ \mathbb{E}(\bar{y}_{i\cdot\cdot}^2) &= \frac{\sigma^2}{bn} + \mu^2, \quad \forall i, \text{ and} \\ \mathbb{E}(\bar{y}_{\dots}^2) &= \frac{\sigma^2}{abn} + \mu^2. \end{aligned}$$

*Proof.* Recall that  $\forall i \in \{1, \dots, a\}, \forall j \in \{1, \dots, b\}, \forall k \in \{1, \dots, n\}$ , we have  $y_{ijk} \sim \mathcal{N}(\mu + \alpha_i + \beta_j + \gamma_{ij}, \sigma^2)$ . So

$$\begin{aligned} \bar{y}_{ij\cdot} &= \frac{1}{n} \sum_{k=1}^n y_{ijk} \sim \mathcal{N}(\mu + \alpha_i + \beta_j + \gamma_{ij}, \frac{\sigma^2}{n}), \quad \forall i, j, \\ \bar{y}_{i\cdot\cdot} &= \frac{1}{b} \sum_{j=1}^b \bar{y}_{ij\cdot} \sim \mathcal{N}(\mu + \alpha_i, \frac{\sigma^2}{bn}), \quad \forall i, \\ \bar{y}_{\cdot j\cdot} &= \frac{1}{a} \sum_{i=1}^a \bar{y}_{ij\cdot} \sim \mathcal{N}(\mu + \beta_j, \frac{\sigma^2}{an}), \quad \forall j, \text{ and} \\ \bar{y}_{\dots} &= \frac{1}{a} \sum_{i=1}^a \bar{y}_{i\cdot\cdot} \sim \mathcal{N}(\mu, \frac{\sigma^2}{abn}). \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}(y_{ijk}^2) &= \mathbb{V}(y_{ijk}) + \mathbb{E}^2(y_{ijk}) = \sigma^2 + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2, \quad \forall i, j, k, \\ \mathbb{E}(\bar{y}_{ij\cdot}^2) &= \mathbb{V}(\bar{y}_{ij\cdot}) + \mathbb{E}^2(\bar{y}_{ij\cdot}) = \frac{\sigma^2}{n} + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2, \quad \forall i, j, \\ \mathbb{E}(\bar{y}_{i\cdot\cdot}^2) &= \mathbb{V}(\bar{y}_{i\cdot\cdot}) + \mathbb{E}^2(\bar{y}_{i\cdot\cdot}) = \frac{\sigma^2}{bn} + (\mu + \alpha_i)^2, \quad \forall i, \end{aligned}$$

$$\begin{aligned}\mathbb{E}(\bar{y}_{\cdot j}^2) &= \mathbb{V}(\bar{y}_{\cdot j}) + \mathbb{E}^2(\bar{y}_{\cdot j}) = \frac{\sigma^2}{an} + (\mu + \beta_j)^2, \quad \forall j, \text{ and} \\ \mathbb{E}(\bar{y}_{\dots}^2) &= \mathbb{V}(\bar{y}_{\dots}) + \mathbb{E}^2(\bar{y}_{\dots}) = \frac{\sigma^2}{abn} + \mu^2.\end{aligned}$$

□

**PROPOSITION 3.39.** We have

$$\mathbb{E}(\text{MS}_{\text{err}}) = \sigma^2.$$

i.e.,  $\text{MS}_{\text{err}}$  is an unbiased estimator for  $\sigma^2$ .

*Proof.*

$$\begin{aligned}\mathbb{E}(\text{MS}_{\text{err}}) &= \mathbb{E}\left(\frac{1}{ab(n-1)} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (y_{ijk} - \bar{y}_{ij\cdot})^2\right) \\ &= \mathbb{E}\left(\frac{1}{ab(n-1)} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n \left[y_{ijk}^2 + \bar{y}_{ij\cdot}^2 - 2y_{ijk}\bar{y}_{ij\cdot}\right]\right), \text{ expand} \\ &= \mathbb{E}\left(\frac{1}{ab(n-1)} \left[ \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n y_{ijk}^2 + n \sum_{i=1}^a \sum_{j=1}^b \bar{y}_{ij\cdot}^2 - 2 \sum_{i=1}^a \sum_{j=1}^b \bar{y}_{ij\cdot} \sum_{k=1}^n y_{ijk} \right]\right), \text{ separate} \\ &= \mathbb{E}\left(\frac{1}{ab(n-1)} \left[ \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n y_{ijk}^2 + n \sum_{i=1}^a \sum_{j=1}^b \bar{y}_{ij\cdot}^2 - 2n \sum_{i=1}^a \sum_{j=1}^b \bar{y}_{ij\cdot} \right]\right), \text{ reduce} \\ &= \mathbb{E}\left(\frac{1}{ab(n-1)} \left[ \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n y_{ijk}^2 - n \sum_{i=1}^a \sum_{j=1}^b \bar{y}_{ij\cdot}^2 \right]\right), \text{ combine} \\ &= \frac{1}{ab(n-1)} \left[ \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n \mathbb{E}(y_{ijk}^2) - n \sum_{i=1}^a \sum_{j=1}^b \mathbb{E}(\bar{y}_{ij\cdot}^2) \right], \text{ by linearity} \\ &= \frac{1}{ab(n-1)} \left[ \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (\sigma^2 + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2) \right. \\ &\quad \left. - n \sum_{i=1}^a \sum_{j=1}^b \left( \frac{\sigma^2}{n} + (\mu + \alpha_i + \beta_j + \gamma_{ij})^2 \right) \right], \text{ by Lemma 3.38} \\ &= \frac{1}{ab(n-1)} \left[ abn\sigma^2 - ab\sigma^2 \right] = \sigma^2.\end{aligned}$$

□

**PROPOSITION 3.40** (Mean of  $MS_A$ ). We have

$$\mathbb{E}(MS_A) \geq \sigma^2$$

with equality holds if and only if  $\alpha = \mathbf{0} \in \mathbb{R}^a$ . i.e.,  $MS_A$  is an unbiased estimator for  $\sigma^2$  given that  $\alpha = \mathbf{0}$ .

*Proof.*

$$\begin{aligned} \mathbb{E}(MS_A) &= \mathbb{E}\left(\frac{1}{a-1} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (\bar{y}_{i..} - \bar{y}_{...})^2\right) \\ &= \mathbb{E}\left(\frac{1}{a-1} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (\bar{y}_{i..}^2 + \bar{y}_{...}^2 - 2\bar{y}_{i..}\bar{y}_{...})\right), \text{ expand} \\ &= \mathbb{E}\left(\frac{1}{a-1} \left[ \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n \bar{y}_{i..}^2 + abn\bar{y}_{...}^2 - 2bn\bar{y}_{...} \sum_{i=1}^a \bar{y}_{i..} \right]\right), \text{ separate} \\ &= \mathbb{E}\left(\frac{1}{a-1} \left[ bn \sum_{i=1}^a \bar{y}_{i..}^2 + abn\bar{y}_{...}^2 - 2abn\bar{y}_{...}^2 \right]\right), \text{ reduce} \\ &= \mathbb{E}\left(\frac{1}{a-1} \left[ bn \sum_{i=1}^a \bar{y}_{i..}^2 - abn\bar{y}_{...}^2 \right]\right), \text{ combine} \\ &= \frac{1}{a-1} \left[ bn \sum_{i=1}^a \mathbb{E}(\bar{y}_{i..}^2) - abn\mathbb{E}(\bar{y}_{...}^2) \right], \text{ by linearity} \\ &= \frac{1}{a-1} \left[ bn \sum_{i=1}^a \left( \frac{\sigma^2}{bn} + (\mu + \alpha_i)^2 \right) - abn \left( \frac{\sigma^2}{abn} + \mu^2 \right) \right], \text{ by Lemma 3.38} \\ &= \frac{1}{a-1} \left[ (a-1)\sigma^2 + bn \sum_{i=1}^a \alpha_i^2 \right] \geq \sigma^2. \end{aligned}$$

That is,  $\mathbb{E}(MS_A) \geq \sigma^2$ . From the above derivation we can see that equality holds if and only if  $\alpha = \mathbf{0} \in \mathbb{R}^a$ .  $\square$

**PROPOSITION 3.41** (Mean of  $MS_B$ ). We have

$$\mathbb{E}(MS_B) = \sigma^2$$

with equality holds if and only if  $\beta = \mathbf{0} \in \mathbb{R}^b$ . i.e.,  $MS_B$  is an unbiased estimator for  $\sigma^2$  given that  $\beta = \mathbf{0}$ .

**PROPOSITION 3.42.** Under the assumption that  $\gamma = \mathbf{0} \in \mathbb{R}^{a \times b}$ , we have

$$\mathbb{E}(\text{MS}_{\text{AB}}) = \sigma^2.$$

i.e.,  $\text{MS}_{\text{AB}}$  is an unbiased estimator for  $\sigma^2$  given that  $\gamma = \mathbf{0}$ .

### 3.5.3 Hypothesis Testing for Two-Way Factorial Design

**DEFINITION 3.43** (ANOVA Table).

Table 3.3: ANOVA Table for Two-Way Factorial Design

	Sum of Squares	Degrees of Freedom	Mean Squares	$F_0$
A	$\text{SS}_A$	$a - 1$	$\text{MS}_A$	$\text{MS}_A / \text{MS}_{\text{err}}$
B	$\text{SS}_B$	$b - 1$	$\text{MS}_B$	$\text{MS}_B / \text{MS}_{\text{err}}$
AB	$\text{SS}_{\text{AB}}$	$(a - 1)(b - 1)$	$\text{MS}_{\text{AB}}$	$\text{MS}_{\text{AB}} / \text{MS}_{\text{err}}$
Error	$\text{SS}_{\text{err}}$	$ab(n - 1)$	$\text{MS}_{\text{err}}$	
Total	$\text{SS}_{\text{tot}}$	$abn - 1$		

**PROPOSITION 3.44.** We are interested in testing the following hypothesis

$$H_0 : \alpha = \mathbf{0} \in \mathbb{R}^a \text{ vs } H_1 : \alpha \neq \mathbf{0}.$$

The  $F$ -statistics is

$$F_0 := \text{MS}_A / \text{MS}_{\text{err}} \sim \mathcal{F}(a - 1, ab(n - 1)).$$

We reject the null if  $F_0 > \mathcal{F}_\alpha(a - 1, ab(n - 1))$ .

**PROPOSITION 3.45.** We are interested in testing the following hypothesis

$$H_0 : \beta = \mathbf{0} \in \mathbb{R}^b \text{ vs } H_1 : \beta \neq \mathbf{0}.$$



The  $F$ -statistics is

$$F_0 := \text{MS}_B / \text{MS}_{\text{err}} \sim \mathcal{F}(b-1, ab(n-1)).$$

We reject the null if  $F_0 > \mathcal{F}_\alpha(b-1, ab(n-1))$ .

**PROPOSITION 3.46.** We are interested in testing the following hypothesis

$$H_0 : \gamma = \mathbb{0} \in \mathbb{R}^{a \times b} \text{ vs } H_1 : \gamma \neq \mathbb{0}.$$

The  $F$ -statistics is

$$F_0 := \text{MS}_{AB} / \text{MS}_{\text{err}} \sim \mathcal{F}((a-1)(b-1), ab(n-1)).$$

We reject the null if  $F_0 > \mathcal{F}_\alpha((a-1)(b-1), ab(n-1))$ .

## 3.6 Two-Level Factorial Design

## Chapter 4

# Estimation Theory

### 4.1 Maximum Likelihood Estimation

**DEFINITION 4.1** (Likelihood Function). Let  $\Omega$  be a parameter space. Let  $\theta$  be an unknown parameter in  $\Omega$ . Let  $\mathbf{Y}$  be a random variable and  $\mathbf{y}$  be an observed sample. Let  $f$  denote the probability function of  $Y$ . We define the **likelihood function** for  $\theta$ , denoted by  $\mathcal{L}$ , to be a function from  $\Omega$  to  $\mathbb{R}$  given by

$$\mathcal{L}(\theta; \mathbf{y}) := f(\mathbf{y}; \theta).$$

**DEFINITION 4.2** (Maximum Likelihood Estimate). Let  $\Omega$  be a parameter space. Let  $\theta$  be an unknown parameter in  $\Omega$ . Let  $\mathbf{Y}$  be a random variable and  $\mathbf{y}$  be an observed sample. Let  $\mathcal{L}$  be the likelihood function for  $\theta$ . We define the **maximum likelihood estimate** for  $\theta$ , denoted by  $\hat{\theta}$ , to be a parameter given by

$$\hat{\theta} := \operatorname{argmax}_{\theta \in \Omega} \mathcal{L}(\theta; \mathbf{y}).$$

**DEFINITION 4.3** (Relative Likelihood Function). Let  $\Omega$  be a parameter space. Let  $\theta$  be an unknown parameter in  $\Omega$ . Let  $\mathbf{Y}$  be a random variable and  $\mathbf{y}$  be an observed sample. Let  $\mathcal{L}$  be the likelihood function for  $\theta$ . We define the **relative likelihood**

**function** for  $\theta$ , denoted by  $\mathcal{R}$ , to be a function from  $\Omega$  to  $\mathbb{R}$  given by

$$\mathcal{R}(\theta; \mathbf{y}) := \frac{\mathcal{L}(\theta; \mathbf{y})}{\mathcal{L}(\hat{\theta}; \mathbf{y})}.$$

**DEFINITION 4.4** (Log Likelihood Function). Let  $\Omega$  be a parameter space. Let  $\theta$  be an unknown parameter in  $\Omega$ . Let  $\mathbf{Y}$  be a random variable and  $\mathbf{y}$  be an observed sample. Let  $\mathcal{L}$  be the likelihood function for  $\theta$ . We define the **log likelihood function** for  $\theta$ , denoted by  $\ell$ , to be a function from  $\Omega$  to  $\mathbb{R}$  given by

$$\ell(\theta; \mathbf{y}) := \ln \mathcal{L}(\theta; \mathbf{y}).$$

## 4.2 Examples

**PROPOSITION 4.5** (MLE of Binomial Distribution). Let  $n \in \mathbb{N}$ . Let  $\theta$  be an unknown parameter. Let  $y$  be an observed value from a  $\text{Binomial}(n, \theta)$  distribution. Then the likelihood function for  $\theta$  is

$$\mathcal{L}(\theta) = \binom{n}{y} \theta^y (1 - \theta)^{n-y}.$$

**PROPOSITION 4.6** (MLE of Poisson Distribution). Let  $\theta$  be an unknown parameter. Let  $y_1, \dots, y_n$  be an observed random sample from a  $\text{Poisson}(\theta)$  distribution. Then the likelihood function for  $\theta$  is

$$\mathcal{L}(\theta) = \theta^{n\bar{y}} e^{-n\theta} \text{ for } \theta \in \mathbb{R}_+,$$

and the maximum likelihood estimation of  $\theta$  is

$$\hat{\theta} = \bar{y}.$$

**PROPOSITION 4.7** (MLE of Exponential Distribution). Let  $\theta$  be an unknown parameter. Let  $y_1, \dots, y_n$  be an observed random sample from a  $\text{Exponential}(\frac{1}{\theta})$  distri-

bution. Then the likelihood function for  $\theta$  is

$$\mathcal{L}(\theta) = \theta^{-n} e^{-n\bar{y}/\theta},$$

and the maximum likelihood estimate  $\hat{\theta}$  of  $\theta$  is

$$\hat{\theta} = \bar{y}.$$

### 4.3 Confidence Intervals

**DEFINITION 4.8.** Let  $X$  be a random variable from a probability distribution with statistical parameter  $\theta \in \Omega$ . We define a **confidence interval** for the parameter  $\theta$  with confidence level  $\gamma$  to be an interval with random endpoints  $u(X)$  and  $v(X)$  such that

$$\forall \theta \in \Omega, \quad \mathbb{P}_{\theta}(u(X) < \theta < v(X)) = \gamma.$$

## 4.4 Simple Random Sampling Without Replacement

### 4.4.1 Population Mean Estimator



## Chapter 5

# Hypothesis Testing

### 5.1 $p$ -values

**DEFINITION 5.1.** Let  $H_0$  be a null hypothesis. Let  $D$  be a discrepancy measure. Let  $\mathbf{y}$  be an observed value of the data. We define the  **$p$ -value** of the test of the hypothesis  $H_0$ , using the discrepancy measure  $D$ , to be the probability given by

$$p\text{-value} := \mathbb{P}(D(\mathbf{Y}) \geq D(\mathbf{y}); H_0).$$





## Chapter 6

# Asymptotic Theory

*In statistics: asymptotic theory, or large sample theory, is a framework for assessing properties of estimators and statistical tests. Within this framework, it is often assumed that the sample size  $n$  may grow indefinitely; the properties of estimators and tests are then evaluated under the limit of  $n \rightarrow \infty$ . In practice, a limit evaluation is considered to be approximately valid for large finite sample sizes too. (Wikipedia, 2021-07-11)*

*In probability theory, the law of large numbers (LLN) is a theorem that describes the result of performing the same experiment a large number of times. According to the law, the average of the results obtained from a large number of trials should be close to the expected value and will tend to become closer to the expected value as more trials are performed. The LLN is important because it guarantees stable long-term results for the averages of some random events. For example, while a casino may lose money in a single spin of the roulette wheel, its earnings will tend towards a predictable percentage over a large number of spins. Any winning streak by a player will eventually be overcome by the parameters of the game. It is important to remember that the law only applies (as the name indicates) when a large number of observations is considered. There is no principle that a small number of observations will coincide with the expected value or that a streak of one value will immediately be “balanced” by the others (see the gambler’s fallacy). (Wikipedia, 2021-07-11)*

## 6.1 Law of Large Numbers

**THEOREM 6.1** (The Weak Law of Large Numbers (Khinchin's Law)). Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of independent and identically distributed random variables. Suppose that they all have finite expected value and standard variation. Let  $\mu$  denote their common expected value. Then the sample average converges weakly (in probability) to the common expected value:

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\text{weakly}} \mu.$$

*Proof.* Note that

$$\begin{aligned} \mathbb{E}[\bar{X}_n] &= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{1}{n} \sum_{i=1}^n \mu = \mu \text{ and} \\ \text{var}[\bar{X}_n] &= \text{var}\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n^2} \sum_{i=1}^n \text{var}[X_i] = \frac{1}{n^2} \sum_{i=1}^n \sigma^2 = \frac{\sigma^2}{n}. \end{aligned}$$

So by the Chebyshev's inequality, we get

$$\Pr \left[ |\bar{X}_n - \mathbb{E}[\bar{X}_n]| \geq \frac{1}{n} \varepsilon \right] \leq \frac{n^2 \sigma^2}{\varepsilon^2}.$$

□

**THEOREM 6.2** (The Strong Law of Large Numbers (Kolmogorov's Law)). Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of independent and identically distributed random variables. Suppose that they all have finite expected value and standard variation. Let  $\mu$  denote their common expected value. Then the sample average converges strongly (almost surely) to the common expected value:

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\text{strongly}} \mu.$$

## 6.2 Central Limit Theorem

**THEOREM 6.3** (Central Limit Theorem). Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of independent and identically distributed Lebesgue integrable random variables. Let  $\mu$  denote their

common expectation and  $\sigma$  denote their common standard deviation. Then we have

$$\sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{d} \mathcal{N}(0, 1).$$

*Proof.* When MGF's exist:

$$\begin{aligned} M_n(t) &= \mathbb{E} \left[ \exp \left\{ t \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \right\} \right] \\ &= \mathbb{E} \left[ \exp \left\{ \frac{t}{\sqrt{n}} \sum_{i=1}^n Y_i \right\} \right], \quad Y_i := \frac{X_i - \mu}{\sigma} \\ &= \prod_{i=1}^n \mathbb{E} \left[ \exp \left\{ \frac{t Y_i}{\sqrt{n}} \right\} \right] \\ &= \left[ M_Y \left( \frac{t}{\sqrt{n}} \right) \right]^n \\ &= \left[ M_Y(0) + M_Y'(0) \left( \frac{t}{\sqrt{n}} \right) + \frac{1}{2} M_Y''(0) \left( \frac{t}{\sqrt{n}} \right)^2 + o \left( \frac{t}{\sqrt{n}} \right)^2 \right]^n \\ &= \left[ 1 + \mathbb{E}[Y_i] \left( \frac{t}{\sqrt{n}} \right) + \frac{1}{2} \mathbb{E}[Y_i^2] \left( \frac{t}{\sqrt{n}} \right)^2 + o \left( \frac{t}{\sqrt{n}} \right)^2 \right]^n \\ &= \left[ 1 + \mathbb{E}[Y_i] \left( \frac{t}{\sqrt{n}} \right) + \frac{1}{2} (\text{var}[Y_i] + (\mathbb{E}[Y_i])^2) \left( \frac{t}{\sqrt{n}} \right)^2 + o \left( \frac{t}{\sqrt{n}} \right)^2 \right]^n \\ &= \left[ 1 + 0 \cdot \left( \frac{t}{\sqrt{n}} \right) + \frac{1}{2} (1 + 0) \left( \frac{t}{\sqrt{n}} \right)^2 + o \left( \frac{t}{\sqrt{n}} \right)^2 \right]^n \\ &= \left[ 1 + \frac{t^2/2}{n} + o \left( \frac{t^2}{n} \right) \right]^n \\ &\rightarrow \exp \left\{ \frac{t^2}{2} \right\}. \end{aligned}$$

□



## Chapter 7

# Power Transformation

**DEFINITION 7.1** (Box-Cox Transformation). We define the one-parameter **Box-Cox transformation** of  $y$ , denoted by  $T_\lambda(y)$ , to be a function given by

$$T_\lambda(y) := \begin{cases} \frac{y^\lambda - 1}{\lambda}, & \text{if } \lambda \neq 0 \\ \ln y, & \text{if } \lambda = 0. \end{cases}$$

**PROPOSITION 7.2.** The Box-Cox transformation is monotonic increasing.