

Graph Theory

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Chapter 1

Graph Basics

1.1 Paths

DEFINITION 1.1 (Vertex-Independent Paths). Let $G = (V, E)$ be a finite undirected graph. Let P and Q be two paths in G . We say that P and Q are **vertex-independent** if and only if they do not have any internal vertex in common.

DEFINITION 1.2 (Edge-Independent Paths). Let $G = (V, E)$ be a finite undirected graph. Let P and Q be two paths in G . We say that P and Q are **edge-independent** if and only if they do not have any internal edge in common.

1.2 Others

DEFINITION 1.3 (Spanning Subgraph). Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G . We say that H is a **spanning** subgraph of G if and only if $W = V$. i.e., if H contains all vertices of G .

1.3 Exercises from [Bollobas]

EXERCISE 1.4. Show that every simple graph G on at least two vertices contains

two distinct vertices of equal degree.

Proof. Let $n := |V(G)|$. Then the possible degrees are $\{0, \dots, n-1\}$. Assume for the sake of contradiction that the degrees are distinct. Then $\exists x \in V(G)$ and $y \in V(G)$ such that $\deg(x) = 0$ and $\deg(y) = n-1$. So y is connected to all other vertices in the graph. In particular, there is an edge between x and y . So $\deg(x) \geq 1$. This contradicts to the choice of x that $\deg(x) = 0$. So there are two distinct vertices in the graph of the same degree. \square

Chapter 2

Graph Connectivity

2.1 Definitions

DEFINITION 2.1 (Vertex Cut). Let $G = (V, E)$ be a finite undirected graph. Let a and b be distinct vertices in G . Let $S \subseteq V \setminus \{a, b\}$. We say that S is a **vertex cut** for a and b if and only if the removal of S from G separates a and b into distinct connected components.

DEFINITION 2.2 (Edge Cut). Let $G = (V, E)$ be a finite undirected graph. Let a and b be nonadjacent vertices in G . Let $S \subseteq E$. We say that S is an **edge cut** for a and b if and only if the removal of S from G separates a and b into distinct connected components.

DEFINITION 2.3 (k -Vertex-Connected Graphs). We say a connected graph is **k -vertex-connected** if and only if it has more than k vertices and remains connected whenever (strictly) fewer than k vertices are removed.

DEFINITION 2.4 (k -Edge-Connected Graphs). We say a connected graph is **k -edge-connected** if and only if it has more than k edges and remains connected whenever (strictly) fewer than k edges are removed.

DEFINITION 2.5 (Cut). A **cut** is a partition of the vertices of a graph into two disjoint subsets.

DEFINITION 2.6 (s-t Cut). Let $G = (V, E)$ be a finite undirected graph. Let s and t be two vertices in G . We define an **s-t cut** to be a cut $C = (S, T)$ of V such that $s \in S$ and $t \in T$.

DEFINITION 2.7 (Size/Value of a Cut). Let $G = (V, E)$ be a finite undirected graph. Let $C = (S, T)$ be a cut of V . We define the **size** of C to be the number of edges crossing the cut. In the case that G is weighted, we define the **value** of C to be the sum of the weights of the edges crossing the cut.

DEFINITION 2.8 (Minimum Cut). We say that a cut is **minimum** if the size/value of the cut is the minimum among all cuts in the graph.

DEFINITION 2.9 (Maximum Cut). We say that a cut is **maximum** if the size/value of the cut is the maximum among all cuts in the graph.

DEFINITION 2.10 (Cut-Set). Let $G = (V, E)$ be a finite undirected graph. Let $C = (S, T)$ be a partition of V . We define the **cut-set** of C to be the set given by

$$\{(u, v) \in E : u \in S, v \in T\}.$$

DEFINITION 2.11 (Cut Space). Let $G = (V, E)$ be a finite undirected graph. Let \mathcal{C} denote the collection of all cut-sets in G . Let $\mathbb{F}_2 := \{0, 1\}$ be a two-element finite field of arithmetic modulo two. We define the addition operation in \mathcal{C} , denoted by $+$, to be a function from \mathcal{C}^2 to \mathcal{C} given by $C_1 + C_2 := C_1 \Delta C_2$ where Δ denotes the symmetric difference operation. What about scalar multiplication???

2.2 Properties (bug)

PROPOSITION 2.12 (Graph Theory An Introductory Course, Bollobas). The complement of a disconnected graph is connected.

Proof. Let G be a disconnected graph. Let \bar{G} denote the complement of G . Let $x, y \in V(\bar{G})$ be arbitrary. ... not finished □

2.3 Ear Decomposition

DEFINITION 2.13 (Adding a Path). We say that a graph G is obtained from G_0 by **adding a path** P if and only if the following condition hold:

- G_0 and P are subgraphs of G with $G = G_0 \cup P$;
- $E(G_0) \cap E(P) = \emptyset$;
- $|V(G_0) \cap V(P)| = 2$;
- P is a path in G between the two vertices in $V(G_0) \cap V(P)$.

THEOREM 2.14 (Ear Decomposition for 2-Vertex-Connected Graphs). Let G be a loopless graph with $|V(G)| \geq 3$. Then G is 2-vertex-connected if and only if there exist subgraphs G_0, G_1, \dots, G_k of G such that

- G_0 is a cycle;
- $G_k = G$;
- $\forall i \in \{1, \dots, k\}$, G_i is obtained from G_{i-1} by adding a path.

Such a finite sequence G_0, G_1, \dots, G_k is an ear decomposition of G .

DEFINITION 2.15 (Adding a Cycle). We say that a graph G is obtained from G_0 by **adding a cycle** C if and only if the following conditions hold:

- G_0 and C are subgraphs of G with $G = G_0 \cup C$;
- $E(G_0) \cap E(C) = \emptyset$;
- $|V(G_0) \cap V(C)| = 1$;
- C is a cycle in G .

THEOREM 2.16 (Ear Decomposition for 2-Edge-Connected Graphs). Let G be a graph with $|E(G)| \geq 1$. Then G is 2-edge-connected if and only if there exist subgraphs G_0, G_1, \dots, G_k of G such that

- G_0 is a cycle;
- $G_k = G$;
- $\forall i \in \{1, \dots, k\}$, G_i is obtained from G_{i-1} by adding a path or adding a cycle.

2.4 Menger's Theorem

THEOREM 2.17 (Menger's Theorem - Edge Connectivity). Let G be a finite undirected graph and x and y two distinct vertices. Then the size of the minimum edge cut for x and y is equal to the maximum number of pairwise edge-independent paths from x to y .

THEOREM 2.18 (Menger's Theorem - Vertex Connectivity). Let G be a finite undirected graph and x and y two nonadjacent vertices. Then the size of the minimum vertex cut for x and y is equal to the maximum number of pairwise vertex-independent paths from x to y .

2.5 Blocks of Graphs

DEFINITION 2.19 (Block). Let G be a graph. Let B be a subgraph of G . We say that B is a **block** of G if and only if B is a maximal subgraph of G with the property that either B is 2-connected or $|V(B)| = 1$.

Note that the only single-vertex blocks are isolated vertices.

Chapter 3

Trees

3.1 Definitions

DEFINITION 3.1 (Spanning Tree). Let $G = (V, E)$ be a graph. Let $H = (W, F)$ be a subgraph of G . We say that H is a **spanning tree** if H is a spanning subgraph of G and is a tree.

3.2 Properties

PROPOSITION 3.2. A graph is connected if and only if it has a spanning tree.

PROPOSITION 3.3 (Graph Theory An Introductory Course, Bollobas). $d_1 \leq \dots \leq d_n$ is the degree sequence of a tree if and only if $d_1 \geq 1$ and $\sum_{i=1}^n d_i = 2n - 2$.

PROPOSITION 3.4 (Graph Theory An Introductory Course, Bollobas). Every integer sequence $d_1 \leq \dots \leq d_n$ with $d_1 \geq 1$ and $\sum_{i=1}^n d_i = 2n - 2k$ is the degree sequence of a forest with k components.

Chapter 4

Graph Isomorphism

4.1 Definitions

DEFINITION 4.1 (Isomorphism). Let G and H be two graphs. We define an **isomorphism** from G to H to be a function f from $V(G)$ to $V(H)$ such that

- f is bijective, and that
- for any pair of vertices $v, w \in V(G)$, $f(v)f(w) \in E(H)$ if and only if $vw \in E(G)$.

i.e., a bijective function that both itself and its inverse preserve adjacency.

DEFINITION 4.2 (Isomorphic). Let G and H be two graphs. We say that G and H are **isomorphic**, denoted by $G \simeq H$, if there exists an isomorphism from G to H .

PROPOSITION 4.3. The relation \simeq of isomorphism is an equivalence relation. That is, it is reflexive, symmetric, and transitive.

4.2 Properties

PROPOSITION 4.4. Let G and H be isomorphic graphs with isomorphism f . Then

for any vertex $v \in V(G)$, we have $\deg_G(v) = \deg_H(f(v))$.

Chapter 5

Matchings and Covers

5.1 Matching

DEFINITION 5.1 (Matching). Let $G = (V, E)$ be a graph. Let M be a subset of E . We say that M is a **matching** in G if every vertex in the spanning subgraph (V, M) has degree at most one.

DEFINITION 5.2 (Saturated). Let $G = (V, E)$ be a graph. Let M be a subset of E . Let v be a vertex of G . We say that v is **M -saturated** if $\deg(v) = 1$ in (V, M) .

DEFINITION 5.3 (Maximal Matching). Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximal matching** if it is a matching in G and any other matching is not a superset of it.

DEFINITION 5.4 (Maximum Matching). Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **maximum matching** if it is a matching in G and any other matching contains edges no more than M .

DEFINITION 5.5 (Perfect Matching). Let $G = (V, E)$ be a graph. Let M be a subset of $E(G)$. We say that M is a **perfect matching** if it matches all vertices of the graph. i.e., any vertex in G is incident to some edge in M .

PROPOSITION 5.6. Every maximum matching is maximal.

PROPOSITION 5.7. Every perfect matching is maximum.

PROPOSITION 5.8. Let $G = (V, E)$ be a graph. Let A and B be two maximal matchings of G . Then both $|A| \leq 2|B|$ and $|B| \leq 2|A|$.

5.2 Cover

DEFINITION 5.9 (Cover). Let $G = (V, E)$ be a graph. Let C be a subset of V . We say that C is a **cover** of G if any edge has an end in C .

5.3 Relations Between Matchings and Covers

PROPOSITION 5.10. Let $G = (V, E)$ be a graph. Let M be a matching of G . Let C be a cover of G . Then $|M| \leq |C|$.

Chapter 6

Bipartite Graphs

6.1 Definitions

DEFINITION 6.1 (Bipartition). Let $G = (V, E)$ be a graph. Let A and B be two subsets of V . We say the pair (A, B) is a **bipartition** of G if and only if $A \cap B = \emptyset$, $A \cup B = V$, and A and B are both independent.

DEFINITION 6.2 (Bipartite Graph). Let $G = (V, E)$ be a graph. We say that G is **bipartite** if and only if there exists a bipartition of G .

DEFINITION 6.3 (Balanced Bipartite Graph). Let $G = (V, E)$ be a bipartite graph with bipartition (A, B) . We say that G is **balanced** if and only if $|A| = |B|$.

6.2 Properties of Bipartite Graphs

PROPOSITION 6.4. Let $G = (V, E)$ be a bipartite graph with bipartition (A, B) . Then

$$\sum_{a \in A} \deg(a) = \sum_{b \in B} \deg(b) = |E|.$$

6.3 Characterizations

PROPOSITION 6.5. A graph is bipartite if and only if it has no odd cycles.

PROPOSITION 6.6. A graph is bipartite if and only if it is 2-colorable.

Chapter 7

Graph Planarity

7.1 Definitions

DEFINITION 7.1 (Plane Embedding). Let $G(V, E, B)$ be an undirected multi-graph. A **plane embedding** of G is a pair of sets (\mathcal{P}, Γ) such that

DEFINITION 7.2 (Planar Graphs (Bollobas Book)). Let G be a graph. Let $\{p_i\}_{i=1}^{|V(G)|}$ be distinct points in \mathbb{R}^3 such that no plane in \mathbb{R}^3 contains more than 3 of these points. Define the **realization** $R(G) \subseteq \mathbb{R}^3$ of G as

$$R(G) := \bigcup_{x_i x_j \in E(G)} \text{conv}\{p_i, p_j\}.$$

We say that G is **planar** if and only if $R(G)$ is homeomorphic to a subset of \mathbb{R}^2 .

DEFINITION 7.3 (Homeomorphic Graphs (Bollobas Book)). Let G and H be graphs. We say that G is **homeomorphic** to H if and only if $R(G)$ is homeomorphic to $R(H)$, or equivalently, G and H have isomorphic subdivisions.

7.2 Properties

PROPOSITION 7.4. Every subgraph of a planar graph is planar.

PROPOSITION 7.5. A multi-graph is planar if and only if its simplification is planar.

PROPOSITION 7.6. Let G be a multi-graph and e be an edge in G . Then G is planar if and only if $G \bullet e$ is planar.

THEOREM 7.7. A multi-graph is planar if and only if it does not contain a repeated subdivision of K_5 or $K_{3,3}$ as a subgraph.

PROPOSITION 7.8. A graph has an embedding if and only if it has a polygonal embedding.

THEOREM 7.9 (Fáry's Theorem). A simple graph has an embedding if and only if it has a straight-lines embedding. That is, an embedding under which every edge is embedded as a line segment.

Proof. Let G be a simple graph. Let $n := |V(G)|$. It is trivial that if G has a straight-line embedding, then it has an embedding. Now I will show by induction that every simple planar graph has a straight-lines embedding.

Base Case: $n = 3$. Now G has only three vertices. The simple complete graph on three vertices is a triangle and trivially has a straight-lines embedding.

Inductive Step: Assume that any simple planar graph on $< n$ vertices has a straight-lines embedding. Let G be a simple planar graph on n vertices. Then $\exists v \in V(G)$ such that $\deg(v) < 5$. Let G' be obtained by removing v from G and re-triangulating the new face f formed when removing v . Then $|V(G')| = n - 1$. By the inductive hypothesis, G' has a straight-lines embedding. Now the region bounded by f is a polygon P with at most 5 sides. Place v in P and join v to the vertices of P via straight lines. By the art gallery theorem, there is a way to place v so that the straight lines from v to the vertices of P do not cross. So G has a straight-lines embedding. This completes the proof. \square

7.3 Numerology

DEFINITION 7.10 (Footprint). Let G be a planar graph. Let (\mathcal{P}, Γ) be a plane embedding of the graph. We define the **footprint** of G , denoted by $fp(G)$, to be the union of the points and curves in \mathbb{R}^2 representing the vertices and edges in G .

DEFINITION 7.11 (Degree). Let G be a planar graph. Let (\mathcal{P}, Γ) be a plane embedding of the graph. We define the **degree** of a face to be the sum of the number of edges and the number of bridges in its boundary.

PROPOSITION 7.12. An edge e in a planar multi-graph is a bridge if and only if the two faces on either side of the curve γ_e are the same.

7.4 Face of Planar Graphs

DEFINITION 7.13 (Face). Let G be a planar graph. Let (\mathcal{P}, Γ) be a plane embedding of the graph. We define a **face** of (\mathcal{P}, Γ) to be a connected component of the set $\mathbb{R}^2 \setminus fp(G)$.

PROPOSITION 7.14. Every planar graph has exactly one unbounded face.

7.5 Euler's Formula

Chapter 8

Duality

8.1 Definitions

DEFINITION 8.1 (Dual Graph). Let $G = (V, E, B)$ be a multigraph. Let (\mathcal{P}, Γ) be a plane embedding of G . Let \mathcal{F} be the set of faces of G . We define the **dual graph** of this embedding to be the multigraph $G^* = (V^*, E^*, B^*)$ where $V^* = \mathcal{F}$ and $E^* = \{e^* : e \in E\}$.

PROPOSITION 8.2. Let $G = (V, E, B)$ be a multigraph. Let (\mathcal{P}, Γ) be a plane embedding of G . Let $G^* = (V^*, E^*, B^*)$ be the dual graph of G . Then for any face $f \in \mathcal{F}$, the degree of f as a face of \mathcal{P}, Γ equals the degree of f as a vertex of G^* .

PROPOSITION 8.3. If G is a connected multigraph embedded in the plane, then G^{**} is isomorphic with G .

Chapter 9

Graph Coloring

9.1 Chromatic Number

DEFINITION 9.1 ((Proper) Coloring). Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define a **(proper) X -coloring** of G to be a function $f : V \rightarrow X$ such that if $vw \in E$, then $f(v) \neq f(w)$.

DEFINITION 9.2 (Chromatic Number). Let $G = (V, E)$ be a graph. Let X be a finite set of colors. We define the **chromatic number** of G , denoted by $\chi(G)$, to be the smallest natural number $k \in \mathbb{N}$ for which G has a (proper) k -coloring.

PROPOSITION 9.3. The chromatic number exists and $\chi(G) \leq |V|$.

Proof. Take $X = V$. □

PROPOSITION 9.4. G is complete if and only if $\chi(G) = |V(G)|$.

PROPOSITION 9.5. The only graph with chromatic number zero is the empty graph.

PROPOSITION 9.6. A graph has chromatic number one if and only if it has no edges and at least one vertex.

PROPOSITION 9.7. A graph has chromatic number two if and only if it is bipartite and has at least one edge.

PROPOSITION 9.8. Let G be a graph. Let $d_{\max}(G)$ be the maximum degree of a vertex in G . Then $\chi(G) \leq 1 + d_{\max}(G)$.

9.2 5-color Theorem

THEOREM 9.9. Every planar graph is 5-colorable.

Proof. (1890)

True for $|V| \leq 5$.

Inductively, suppose the theorem holds for planar graphs on $n - 1$ vertices for $n \geq 5$.

Suppose G is a planar graph on n vertices.

Let v be a vertex of degree ≤ 5 in G . This exists by a lemma in our lectures.

Since G is a planar, $G - v$ is planar. By the induction hypothesis, $G - v$ has a 5-coloring.

If some color does not appear on any neighbor of v , we can extend the coloring to a coloring of G .

Otherwise, v has exactly 5 neighbors with different colors.

For each pair i, j of colors, let G_{ij} be the subgraph of $G - v$ induced by the vertices colored i or j .

If the component H of G_{ij} containing x_i does not contain x_j , then we can switch the colors of all vertices in H between i and j to get a coloring of $G - v$ that assigns only 4 colors to neighbors of v , and thus extends to a coloring of G .

So G_{ij} contains a path from x_i to x_j .

Because $G_{2,5}$ and $G_{1,4}$ have disjoint vertex sets, this contradicts the planarity of G .

□

DEFINITION 9.10 (Near-triangulation). Planar drawing of G where the infinite face is bounded by a cycle, and every other face is bounded by a triangle

THEOREM 9.11. Every planar near-triangulation has a 5-coloring.

Theorem 9.11 \implies Theorem 9.9.

DEFINITION 9.12 (List Assignment). A **list assignment** L of G is a function that assigns a set $L(v)$ of colors to each $v \in V$.

DEFINITION 9.13 (L -coloring). An L -coloring of G is a choice of a color in $L(v)$ for each $v \in V$ such that adjacent vertices get different colors.

DEFINITION 9.14 (5-list-colorable). A graph is **5-list-colorable** if for every list assignment L of G with $|L(v)| \geq 5$, G is L -colorable.

THEOREM 9.15. Every planar near-triangulation is 5-list-colorable.

Theorem 9.15 \implies Theorem 9.11 because coloring is a special case of list coloring.

THEOREM 9.16 (Carsten Thomassen, 1993). If G is a near-triangulation and L is a list assignment such that

1. $|L(v)| = 5$ for every non-boundary vertex,
2. $|L(v)| = 3$ for every boundary vertex.

Then G has an L -coloring even if two adjacent boundary vertices have their colors arbitrarily decided in advance.

Proof.

Case 1. There is a "chord" between two boundary vertices.

Let G_1 and G_2 be subgraph of G obtained by "cutting" G along the chord, where G_1 contains the pre-colored vertices.

By applying the inductive hypothesis to G_1 , and then applying it to G_2 with the two ends of the chord pre-colored according to the coloring of G_1 , we get a coloring of G_1 .

Case 2. There is no chord.

Let u and u' be the pre-colored vertices.

Let x, y be the next two vertices occuring in order around the boundary.

□

Theorem 9.16 \implies Theorem 9.15.

Chapter 10

Probability and Edge Density

Q: Let G be a graph on n vertices with no triangles. How many edges can G have?

THEOREM 10.1 (Mantel). If G is triangle-free and has n vertices, then

$$|E| \leq \frac{n^2}{4}.$$

Proof. Let $P_{2,1}$ denote the probability that a pair of distinct vertices chosen uniformly at random, are adjacent.

$$P_{2,1} = |E| / \binom{n}{2}.$$

Let $P_{3,2}$ denote the probability that a randomly chosen triple of vertices contains exactly two edges. Let $P_{3,1}$ denote ... one edge. Let $P_{3,0}$ denote ... no edges. Notice $P_{3,2} + P_{3,1} + P_{3,0} = 1$.

Part 1: Show that $P_{2,1} = \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1}$. Notice that the graph is triangle-free. So $P_{3,3} = 0$. Choosing a pair at random is the same as choosing a triple at random, then choosing a pair at random within that triple.

For a fixed vertex v , let $Q_{v,1}$ denote the probability that a randomly chosen vertex $u \neq v$ is adjacent to v .

$$Q_{v,1} = \frac{\deg(v)}{n-1}.$$

Let $Q_{v,2}$ denote the probability that two distinct randomly chosen vertices other than v are both adjacent to v .

$$Q_{v,2} = \binom{\deg(v)}{2} / \binom{n-1}{2}.$$

Part 2: Show that $Q_{v,1}^2 \approx Q_{v,2}$. Both give (essentially) the probability that a pair x, y of vertices other than v are both adjacent to v . The LHS allows $x = y$. The RHS does not. But $x = y$ occurs with negligible probability.

Part 3: Show that $P_{2,1} = \frac{1}{n} \sum_v Q_{v,1}$. Both the RHS and LHS are just the probability of an edge between two vertices. The RHS calls the first chosen vertex v .

Part 4: Show that $\frac{1}{3}P_{3,2} = \frac{1}{n} \sum_v Q_{v,2}$. Both sides give the probability that, if we choose 3 vertices at random, and then choose one among those 3 and call it v , that v is adjacent to both the others.

Proof of the theorem.

$$\begin{aligned} P_{2,1} &= \frac{2}{3}P_{3,2} + \frac{1}{3}P_{3,1} \geq \frac{2}{3}P_{3,2} \\ &= 2 \left(\frac{1}{n} \sum_v Q_{v,2} \right) \approx 2 \left(\frac{1}{n} \sum_v Q_{v,1}^2 \right) \\ &\geq 2 \left(\frac{1}{n} \sum_v Q_{v,1} \right)^2 = 2P_{2,1}^2. \end{aligned}$$

So $P_{2,1} \leq \frac{1}{2}$. So $|E| \leq \frac{n^2}{4}$. □

Q: If G has n vertices, no K_{t+1} -subgraph, how many edges can G have?

THEOREM 10.2 (Turan). If G is a graph on n vertices with no K_{t+1} -subgraph, then

$$|E| \leq \frac{n^2}{2} \left(1 - \frac{1}{t} \right).$$

THEOREM 10.3 (Erdos-Stone). If H is a graph and G is a graph on n vertices without H as a subgraph, then

$$|E| \leq \frac{n^2}{2} \left(1 - \frac{1}{\chi(H) - 1} + \varepsilon(n) \right)$$

where $\varepsilon(n) \rightarrow 0$ as $n \rightarrow \infty$ and $\chi(H)$ is the chromatic number of H , the fewest number of colors needed to properly color the vertices of H .

Chapter 11

Weird Stuffs

11.1 Geometric Representation of Graphs

DEFINITION 11.1 (Geometric Representation). Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. We define a **geometric representation** of G to be a map from V to \mathbb{R}^d .

DEFINITION 11.2 (Unit Distance Representation). Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \rightarrow \mathbb{R}^d$ be a geometric representation of G . We say that u is a **unit distance representation** of G if and only if $\forall \{i, j\} \in E, \|u(i) - u(j)\|_2 = 1$.

DEFINITION 11.3 (Orthonormal Representation). Let $G = (V, E)$ be a graph. Let $d \in \mathbb{Z}_+$. Let $u : V \rightarrow \mathbb{R}^d$ be a geometric representation of G . We say that u is an **orthonormal representation** of G if and only if

- $\forall i \in V, \|u(i)\|_2 = 1$; and
- $\forall \{i, j\} \in \overline{E}, \langle u(i), u(j) \rangle = 0$ where \overline{E} is the edge set of the complement of G .

DEFINITION 11.4. We define $t_h(G)$ to be the square radius of the smallest hypersphere that contains a unit distance representation of G .

THEOREM 11.5 (CO 471, Spring 2022, Levent Tuncel). Let $G = (V, E)$ be a graph. Then

$$\begin{aligned}
 t_h(G) &= \min_{\text{subject to:}} t \\
 &\quad X_{ii} = t, \forall i \in V \\
 &\quad X_{ii} - 2X_{ij} + X_{jj} = 1, \forall \{i, j\} \in E \\
 &\quad X \in S_+^V
 \end{aligned}$$

PROPOSITION 11.6 (CO 471, Spring 2022, Levent Tuncel). Let $G = (V, E)$ be a graph. Then G is bipartite if and only if $t_h(G) \leq \frac{1}{4}$.

Proof.

□

PROPOSITION 11.7. Let $n \in \mathbb{Z}_{++}$. Let K_n denote the n -clique. Then $t_h(K_n) =$.

Proof.

□

11.2 Stable Sets

DEFINITION 11.8 (Stable Sets). Let $G = (V, E)$ be a graph. Let S be a subset of the vertex set V . We say that S is a **stable set** in G if and only if $\forall \{i, j\} \in E$, at most one of i or j is in S . i.e., S is a set of pairwise non-adjacent vertices.

DEFINITION 11.9 (Stability Number). Let $G = (V, E)$ be a graph. We define the **stability number** of G , denoted by $\alpha(G)$, to be a number given by

$$\alpha(G) := \max\{|S| : S \text{ is stable in } G\}.$$

DEFINITION 11.10 (Stable Set Polytope). Let $G = (V, E)$ be a graph. We define

the **stable set polytope** of G , denoted by $\text{STAB}(G)$, to be a subset of \mathbb{R}^V given by

$$\text{STAB}(G) := \text{conv} \left\{ x \in \{0, 1\}^V : x \text{ is the incidence vector of some stable set in } G \right\}.$$

DEFINITION 11.11 (Fractional Stable Set Polytope). Let $G = (V, E)$ be a graph. We define the **fractional stable set polytope** of G , denoted by $\text{FRAC}(G)$, to be a subset of \mathbb{R}^V given by

$$\text{FRAC}(G) := \left\{ x \in [0, 1]^V : x_i + x_j \leq 1, \forall \{i, j\} \in E \right\}.$$

PROPOSITION 11.12. Let $G = (V, E)$ be a graph. Then

$$\text{STAB}(G) = \text{conv}(\text{FRAC}(G) \cap \{0, 1\}^V).$$

11.3 Clique Polytope

DEFINITION 11.13. Let $A_{clq}(G)$ denote the 0-1 clique-node incidence matrix of G where each row corresponds to a clique.

DEFINITION 11.14 (Clique Polytope). We define the **clique polytope** of G to be

$$\text{CLQ}(G) := \{x \in \mathbb{R}_+^V : A_{clq}(G)x \leq \bar{e}\}$$

11.4 Theta Bodies

DEFINITION 11.15 (Theta Body). Let $G = (V, E)$ be a graph. We define the

theta body of G , denoted by $\text{TH}(G)$, to be a subset of \mathbb{R}_+^V given by

$$\text{TH}(G) := \left\{ x \in \mathbb{R}_+^V : \sum_{i \in V} (c^\top u(i))^2 x_i \leq 1, \begin{array}{l} \forall c \in \mathbb{R}^V : \|c\|_2 = 1, \\ \forall \text{ orth. rep. } u \text{ of } G \end{array} \right\}.$$

THEOREM 11.16 (CO 471, Spring 2022, Levent Tuncel). For every graph $G = (V, E)$, $\text{TH}(G)$ is a nonempty compact convex set such that

$$\text{STAB}(G) \subseteq \text{TH}(G) \subseteq \text{CLQ}(G) \subseteq \text{FRAC}(G).$$

Proof. We already observed $\text{CLQ}(G) \subseteq \text{FRAC}(G)$ for all graphs G .

Part 1: Show that $\text{TH}(G) \subseteq \text{CLQ}(G)$. Let $\mathcal{C} \subseteq V$ be a nonempty clique in G . Let $c \in \mathbb{R}^V$ be any vector with $\|c\|_2 = 1$. Let $u(i) := c$, $\forall i \in \mathcal{C}$. For all $i \in V \setminus \mathcal{C}$, choose $u(i)$'s as an orthonormal system in $\{c\}^\perp$. Note: u is an orthonormal representation of G . The corresponding orthonormal representation constraint is

$$1 \geq \sum_{i \in V} (c^\top u(i))^2 x_i = \sum_{i \in \mathcal{C}} \underbrace{(c^\top u(i))^2}_{=1} x_i + \sum_{i \in V \setminus \mathcal{C}} \underbrace{(c^\top u(i))^2}_{=0} x_i = \sum_{i \in \mathcal{C}} x_i.$$

Also, by definition $\text{TH}(G) \subseteq \mathbb{R}_+^V$; therefore, $\text{TH}(G) \subseteq \text{CLQ}(G)$ for all graph G .

Part 2: Show that $\text{STAB}(G) \subseteq \text{TH}(G)$. We will show that incidence vectors of every stable set in G belongs to $\text{TH}(G)$. Since $\text{TH}(G)$ is a convex set and $\text{STAB}(G)$ is the convex hull of these incidence vectors, this will prove $\text{STAB}(G) \subseteq \text{TH}(G)$. Let $\mathcal{S} \subseteq V$ be a stable set in G . Let $\bar{x} \in \{0, 1\}^V$ be the incidence vector of \mathcal{S} . Clearly $\bar{x} \geq 0$. Let $u : V \rightarrow \mathbb{R}^V$ be any orthonormal representation of G . Let $c \in \mathbb{R}^V$ be any vector such that $\|c\|_2 = 1$. We may assume $\mathcal{S} = \{1, \dots, k\}$. Define $Q^\top := (u(1), \dots, u(k))$. Then by definition of stable sets, Q^\top is orthogonal.

$$\begin{aligned} \sum_{i \in V} (c^\top u(i))^2 \bar{x}_i &= \sum_{i \in \mathcal{S}} (c^\top u(i))^2 \underbrace{\bar{x}_i}_{=1} + \sum_{i \in V \setminus \mathcal{S}} (c^\top u(i))^2 \underbrace{\bar{x}_i}_{=0} = \sum_{i \in \mathcal{S}} (c^\top u(i))^2 \\ &= \|Q^\top c\|_2^2 \leq \|\tilde{Q}^\top c\|_2^2 = \|c\|_2^2 = 1. \end{aligned}$$

where $\tilde{Q}^\top = [Q^\top \mid \text{complete to an orthonormal basis}]$. Since $0 \in \text{TH}(G)$, $\text{TH}(G) \neq \emptyset$. Since $\text{TH}(G) \subseteq \text{FRAC}(G) \subseteq [0, 1]^V$, $\text{TH}(G)$ is bounded. \square

DEFINITION 11.17 (Lovase Theta Function). Let $G = (V, E)$ be a graph. Let $w \in \mathbb{R}_+^V$. We define the **Lovase Theta function**, denoted by θ , to be a function of G

and w given by

$$\theta(G, w) := \max\{w^\top x : x \in \text{TH}(G)\}.$$

DEFINITION 11.18 (Lovase Theta Number). Let $G = (V, E)$ be a graph. We define the **Lovase Theta number** of G , denoted by $\theta(G)$, to be a number given by

$$\theta(G) := \theta(G, \bar{e}) = \max\{\bar{e}^\top x : x \in \text{TH}(G)\}.$$

THEOREM 11.19 (CO 471, Spring 2022, Levent Tuncel). Let $G = (V, E)$ be a graph. Let $w \in \mathbb{R}_+^V$ be a weight vector. Define a matrix $W \in \mathbb{S}^V$ by $W_{ij} := \sqrt{w_i w_j}$, $\forall i, j \in V$. Then the following quantities are the same:

1. $\theta(G, w)$;
2. If $w_i = 0$, define $\frac{w_i}{(c^\top u(i))^2} := 0$,

$$\inf \left\{ \max_{i \in V} \left\{ \frac{w_i}{(c^\top u(i))^2} \right\} : \begin{array}{l} c \in \mathbb{R}^V, \|c\|_2 = 1, \\ u \text{ is an orth. rep. of } G \end{array} \right\};$$

3. $\min\{\eta \in \mathbb{R} : S \in \mathbb{S}^V, \text{diag}(S) = 0, S_{ij} = 0, \forall \{i, j\} \in \overline{E}, \eta I - S \succeq W\}$;
4. $\max\{\text{tr}(WX) : X_{ij} = 0, \forall \{i, j\} \in E, \text{tr}(X) = 1, X \in \mathbb{S}_+^V\}$.

11.5 Product of Graphs

DEFINITION 11.20 (Strong Product). Let $G = (V, E)$ and $H = (W, F)$ be graphs. We define the **strong product** of G and H , denoted by $G \otimes H$, to be a graph given by $G \otimes H = (V(G \otimes H), E(G \otimes H))$ where

$$V(G \otimes H) := V \times W \text{ and}$$

$$E(G \otimes H) := \left\{ \left\{ (i, u), (j, v) \right\} : \begin{array}{l} (\{i, j\} \in E \text{ and } \{u, v\} \in F) \text{ or} \\ (\{i, j\} \in E \text{ and } u = v \in W) \text{ or} \\ (i = j \in V \text{ and } \{u, v\} \in F) \end{array} \right\}.$$

PROPOSITION 11.21. Let $G = (V, E)$ and $H = (W, F)$ be graphs. Then

$$\theta(G \otimes H) \leq \theta(G) \times \theta(H).$$

DEFINITION 11.22 (Shannon Capacity). Let $G = (V, E)$ be a graph. We define the **Shannon capacity** of G , denoted by $\Theta(G)$, to be a number given by

$$\Theta(G) := \limsup_{k \rightarrow +\infty} (\alpha(G^{\otimes k}))^{1/k}$$

where $\alpha(G^{\otimes k})$ denotes the stability number of $G^{\otimes k}$.

11.6 Lift-and-Project Operators

DEFINITION 11.23 (Lift-and-Project Operators). Let P be a convex subset of $[0, 1]^d$. Define a subset K_P of \mathbb{R}^{1+d} by $K_P := \text{cone}(1 \oplus P)$. Define a subset $M_+(P)$ of \mathbb{S}_+^{1+d} by

$$M_+(P) := \left\{ Y \in \mathbb{S}_+^{1+d} : \text{diag}(Y) = Y e_0, \begin{array}{l} Y e_i \in K_P, \forall i \in \{1, \dots, d\}, \\ Y(e_0 - e_i) \in K_P, \forall i \in \{1, \dots, d\} \end{array} \right\}.$$

We define the **lift-and-project operator**, denoted by LS_+ , to be a function from $\mathcal{P}(\mathbb{R}^d)$ to $\mathcal{P}(\mathbb{R}^d)$ given by

$$\text{LS}_+(P) := \left\{ x \in \mathbb{R}^d : \begin{pmatrix} 1 \\ x \end{pmatrix} = Y e_0 \text{ for some } Y \in M_+(P) \right\}.$$