# Functional Analysis

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# Contents

# Chapter 1

# Normed Linear Spaces

## 1.1 Definitions

**Definition** (Seminorm)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **seminorm** on $\mathfrak{X}$, denoted by $\nu$, to be a map from $\mathfrak{X}$ to $\mathbb{R}$ that satisfies the following conditions.*

*(1)* $\forall x \in \mathfrak{X}, \quad \nu(x) \geq 0.$

*(2)* $\forall \lambda \in \mathbb{F}, \forall x \in \mathfrak{X}, \quad \nu(\lambda x) = \lambda \nu(x).$

*(3) Triangle Inequality.*

$$\forall x, y \in \mathfrak{X}, \quad \nu(x+y) \leq \nu(x) + \nu(y).$$

*The idea behind the seminorm is that we are trying to give our vector space a notion of "length" of vectors.*

**Definition** (Norm)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **norm** on $\mathfrak{X}$, denoted by $\nu$, to be a seminorm on $\mathfrak{X}$ that satisfies the additional condition:*

$$\forall x \in \mathfrak{X}, \quad \mu(x) = 0 \iff x = 0.$$

## 1.2 Properties

**Proposition 1.2.1.** *Let $(V, \|\cdot\|_V)$ be a normed vector space over field $\mathbb{F}$. Then $(V, \|\cdot\|)$ is complete if and only if $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.*

*Proof.*

For one direction, assume that $(V, \|\cdot\|)$ is complete.

We are to prove that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

Since $(\overline{B(0,1)}, \|\cdot\|_V)$ is a closed subspace of $(V, \|\cdot\|)$ and $(V, \|\cdot\|)$ is complete, $(\overline{B(0,1)}, \|\cdot\|_V)$ is also complete.

For the reverse direction, assume that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

We are to prove that $(V, \|\cdot\|_V)$ is complete.

Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $(V, \|\cdot\|_V)$.

Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ is bounded in $(V, \|\cdot\|_V)$.

Let $\lambda$ be a positive upper bound for $\{\|x_i\|_V\}_{i\in\mathbb{N}}$.

Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$ and $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ converges in $(V, \|\cdot\|_V)$.

Since any Cauchy sequence in $(V, \|\cdot\|_V)$ converges in $(V, \|\cdot\|_V)$, $(V, \|\cdot\|_V)$ is complete.

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**Proposition 1.2.2.** *Proper subspaces of a normed linear space has empty interior.*

*Proof.* Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a proper subspace of $\mathfrak{X}$. Assume for the sake of contradiction that $\mathcal{M}$ has non-empty interior. Then $\exists x_0 \in \mathcal{M}$ and $\exists r > 0$ such that $\text{ball}(x_0, r) \subseteq \mathcal{M}$ where $\text{ball}(x_0, r)$ denotes the open ball centered at point $x_0$ with radius $r$. Let $x$ be an arbitrary point in $\mathfrak{X}$. Define a point $y(x)$ as $y(x) := x_0 + \frac{r}{2\|x\|}x$. Then $x = \frac{2\|x\|}{r}(y - x_0)$. It is easy to verify that $\|y - x_0\| = \frac{r}{2} < r$. So $y \in \text{ball}(x_0, r)$. So $y \in \mathcal{M}$. Since $y, x_0 \in \mathcal{M}$ and $\mathcal{M}$ is a subspace, we get $\frac{2\|x\|}{r}(y - x_0) \in \mathcal{M}$. That is, $x \in \mathcal{M}$. So $\forall x \in \mathfrak{X}$, $x \in \mathcal{M}$. So $\mathcal{M} = \mathfrak{X}$. This contradicts to the assumption that $\mathcal{M}$ is a proper subspace of $\mathfrak{X}$. So $\mathcal{M}$ has empty interior. ■

**Proposition 1.2.3.** *Closed proper subspaces of a normed linear space are nowhere dense.*

*Proof.* Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a closed proper subspace of $\mathfrak{X}$. Since $\mathcal{M}$ is closed, $\text{cl}(\mathcal{M}) = \mathcal{M}$. So $\text{cl}(\mathcal{M}) = \mathcal{M}$ is a closed proper subspace of $\mathfrak{X}$. Since $\text{cl}(\mathcal{M})$ is a proper subspace of $\mathfrak{X}$, $\text{int}(\text{cl}(\mathcal{M})) = \emptyset$. So $\mathcal{M}$ is nowhere dense. ■

**Proposition 1.2.4.** *Finite dimensional subspace of a normed linear space is closed.*

**Proposition 1.2.5.** *Finite-dimensional normed linear spaces are complete.*

## 1.3 Equivalence of Norms

**Definition** (Equivalence of Norms)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $V$. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are **equivalent** if*

$$\exists c_1, c_2 > 0, \quad \forall v \in \mathfrak{X}, \quad c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_2.$$

*Or equivalently,*

$$c_1\|v\|_2 \le \|v\|_1 \le c_2\|v\|_2.$$

**Proposition 1.3.1.** *The equivalence of norms is an equivalence relation.*

**Theorem 1.** *Let $V$ be a finite dimensional vector space over field $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$. Then any two norms on $V$ are equivalent.*

*Proof.*

Let $\|\cdot\|_p$ be an arbitrary $p$-norm on $V$ and $\|\cdot\|$ be an arbitrary norm on $V$.

Let $\mathcal{B}$ be the standard basis for $V$. Say $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$.

Let $v$ be an arbitrary vector in $V$.

$$\begin{aligned}
\|v\| &= \|\sum_{i=1}^{n} v_i e_i\| \\
&\le \sum_{i=1}^{n} |v_i|\|e_i\| \\
&\le \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \\
&= \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \|v\|_p \\
&:= c_1\|v\|_p.
\end{aligned}$$

■

**Proposition 1.3.2.** *Let $X$ be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $X$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if they generate the same metric topology.*

*Proof.* Convergence to 0 is equivalent under either $\|\cdot\|_1$ or $\|\cdot\|_2$. i.e., equivalent norms give rise to the same set of sequences that are convergent. Convergence of sequences defines the topology. ■

**Proposition 1.3.3.** *Let $\mathfrak{X}$ be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathfrak{X}$. Let $\iota$ be the identity map from $(\mathfrak{X}, \|\cdot\|_1)$ to $(\mathfrak{X}, \|\cdot\|_2)$. Then if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, $\iota$ is continuous, and in fact, a homeomorphism between $(\mathfrak{X}, \|\cdot\|_1)$ and $(\mathfrak{X}, \|\cdot\|_2)$.*

## 1.4 Dual Norms

**Definition** (Dual Norm)**.** *Let $(V, \|\cdot\|)$ be an normed vector space. We define the* ***dual norm*** *of $\|\cdot\|$, denoted by $\|\cdot\|_\circ$, to be a function given by*

$$\|v\|_\circ := \max_{\|w\|=1} v \cdot w = \max_{\|w\|\neq 0} \frac{|v \cdot w|}{\|w\|}.$$

**Proposition 1.4.1.** *Dual norms of norms are indeed norms.*

**Proposition 1.4.2.** *Let $(V, \|\cdot\|)$ be a normed vector space. Let $v, w$ be vectors in the space. Then*

$$|v \cdot w| \leq \|v\| \cdot \|w\|_\circ.$$

## 1.5 $p$-norms

**Definition** ($p$-norm)**.** *Let $V$ be a finite-dimensional normed vector space over field $\mathcal{F}$. Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis for $V$ where $n = \dim(V)$. Let $v$ be a vector in a normed vector space. For $p \in [1, +\infty)$, we define the $p$-**norm** of $v$, denoted by $\|v\|_p$, to be the number given by*

$$\|v\|_p = \left( \sum_{i=1}^{n} |(v_\mathcal{B})_i|^p \right)^{\frac{1}{p}}.$$

**Definition** (Infinity Norm - 1)**.** *Let $\mathfrak{X} = \mathbb{K}^n$ where $\mathbb{K}$ is a field and $n \in \mathbb{N}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by*

$$\|v\|_\infty := \max\{|v_i|\}_{i=1}^{n}.$$

**Definition** (Infinity Norm - 2)**.** *Let $\mathfrak{X} = \mathbb{K}^\mathbb{N}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by*

$$\|v\|_\infty := \sup_{i \in \mathbb{N}} |v_i|.$$

**Definition** (Infinity Norm - 3)**.** *Let $\mathfrak{X} = \mathcal{C}([0,1], \mathbb{C})$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by*

$$\nu(f) := \sup_{x \in [0,1]} |f(x)|.$$

**Proposition 1.5.1.** *Let $\mathfrak{X} := \mathcal{C}([0,1], \mathbb{C})$. Let $x$ be an arbitrary number in $[0,1]$. Define a function $\nu_x$ on $\mathfrak{X}$ by $\nu_x(f) := |f(x)|$. Define a function $\nu$ on $\mathfrak{X}$ by $\nu(f) := \sup_{x \in [0,1]} |f(x)|$. Then $\nu_x$ is a seminorm on $\mathfrak{X}$ for each $x$ and $\nu$ is a norm on $\mathfrak{X}$ and we have $\nu = \sup_{x \in [0,1]} \nu$.*

**Proposition 1.5.2.** *$p$-norms are indeed norms.*

**Proposition 1.5.3.** *For any vector $v$ in $\mathbb{R}^n$, we have*

$$\lim_{p \to \infty} \|v\|_p = \|v\|_\infty.$$

*i.e.,*

$$\lim_{p \to \infty} \left( \sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}} = \max\{|v_i|\}_{i=1}^{n}.$$

*Proof.* Let $p$ be an arbitrary number in $[1, +\infty)$. Let $k$ be an arbitrary index in $\{1, .., n\}$. Then

$$|v_k| \leq (\sum_{i=1}^{n} |v_k|^p)^{1/p} = \|v\|_p.$$

So

$$\max\{|v_k|\} = \|v\|_\infty \leq \|v\|_p.$$

So

$$\lim_{p\to\infty} \|v\|_p \geq \|v\|_\infty. \tag{1}$$

On the other hand, note that

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p = \sum_{i=1}^{n} (\frac{|v_i|}{\|v\|_\infty})^p$$

decreases as $p$ increases. So it is bounded above. Say

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p \leq C$$

for some $C \in \mathbb{R}$. Then

$$(\sum_{i=1}^{n} |v_i|^p)^{1/p} = \|v\|_p \leq C^{1/p}\|v\|_\infty.$$

So

$$\lim_{p\to\infty} \|v\|_p \leq \lim_{p\to\infty} C^{1/p}\|v\|_\infty = \|v\|_\infty. \tag{2}$$

From (1) and (2) we get

$$\lim_{p\to\infty} \|v\|_p = \|v\|_\infty.$$

■

**Proposition 1.5.4.** *Let $p$ be an arbitrary number in $[1, +\infty)$. Then the dual norm of the $p$-norm $\|\cdot\|_p$ is the $q$-norm $\|\cdot\|_q$ where $q$ is such that satisfies*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

**Proposition 1.5.5.** *Let $p$ and $q$ be numbers in $[1, +\infty]$. Let $v$ be a vector in $\mathbb{R}^n$. Then if $p \leq q$,*

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \cdot \|x\|_q.$$

**Proposition 1.5.6.** *Let $w$ and $z$ be vectors in $\mathbb{E}^d$. Then*

$$\|w + z\|_2^2 + \|w - z\|_2^2 = 2(\|w\|_2^2 + \|z\|_2^2).$$

# Chapter 2

# Inner Product Spaces

## 2.1 Inner Products

### 2.1.1 Definitions

**Definition** (Inner Product)**.** *Let $V$ be a vector space over field $\mathbb{F}$. We define an* **inner product** *on $V$, denoted by $\langle \cdot, \cdot \rangle$, to be a scalar-valued function defined on $V \times V$ such that*

*(1) Positive Definiteness:*

$$\forall x \in V, \langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \iff x = 0.$$

*(2) Sesqui-Linearity:*

$$\begin{aligned} \forall x, y, z, w \in V, \quad & \langle x + y, z + w \rangle = \langle x, z \rangle + \langle y, z \rangle + \langle x, w \rangle + \langle y, w \rangle, \text{ and} \\ \forall a, b \in \mathbb{F}, \forall x, y \in V, \quad & \langle ax, by \rangle = a\bar{b}\langle x, y \rangle. \end{aligned}$$

*(3) Conjugate Symmetry:*

$$\forall x, y \in V, \quad \langle x, y \rangle = \overline{\langle y, x \rangle}.$$

**Definition** (Induced Norm)**.** *Let $\mathfrak{X}$ be an inner product space over field $\mathbb{K}$. We define the* ***norm induced by*** *$\langle \cdot, \cdot \rangle$, denoted by $\| \cdot \|$, to be a function from $\mathfrak{X}$ to $\mathbb{R}_+$ given by*

$$\|x\| := \sqrt{\langle x, x \rangle}$$

### 2.1.2 Examples of Inner Products

**Definition** (Standard Inner Product)**.** *For $V = \mathbb{F}^n$, we define the* ***standard inner product*** *by*

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i \overline{y_i}.$$

**Definition** (Frobenius Inner Product)**.** *For $V = \mathbb{F}^{n\times n}$, we define the* ***Frobenius inner product*** *by*

$$\langle M_1, M_2\rangle := \operatorname{tr}(M_2^* M_1).$$

**Definition.** *Let $V$ be the space of continuous scalar-valued functions on $[0, 2\pi]$. We define the inner product on $V$ by*

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx.$$

### 2.1.3 Properties

**Proposition 2.1.1.** *Let $V$ be a finite dimensional inner product space. Let $\mathcal{B}$ be a basis for $V$. Let $x$ and $y$ be vectors in $V$. Then*

$$x = y \iff \forall b \in \mathcal{B}, \quad \langle x, b\rangle = \langle y, b\rangle.$$

## 2.2 Inner Product Space

**Definition** (Inner Product Space)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{X}$. We define an* ***inner product space*** *to be the pair $(\mathfrak{X}, \langle\cdot,\cdot\rangle)$.*

## 2.3 Inequalities

**Theorem 2** (Minkowski)**.**

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$$

**Proposition 2.3.1** (Cauchy-Schwarz Inequality)**.** *Let $V$ be an inner product space. Then*

$$\forall x, y \in V, \quad |\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

**Proposition 2.3.2** (Triangle Inequality)**.** *Let $V$ be an inner product space. Then*

$$\forall x, y \in V, \quad \|x + y\| \leq \|x\| + \|y\|$$

**Proposition 2.3.3** (Parallelogram Law)**.** *Let $\mathfrak{X}$ be an inner product space. Then*

$$\forall x, y \in \mathfrak{X}, \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

*Proof.*

$$\|x + y\|^2 + \|x - y\|^2 = \langle x + y, x + y\rangle + \langle x - y, x - y\rangle$$

$$\begin{aligned}
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&\quad + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\
&= 2\langle x, x\rangle + 2\langle y, y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

That is,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

■

# Chapter 3

# Orthogonality

## 3.1 Orthogonal Sets

**Definition** (Orthogonality). *Let $V$ be an inner product space. We say that points $x$ and $y$ in $V$ are **orthogonal** if $\langle x, y \rangle = 0$.*

**Definition** (Orthogonal Set). *Let $\mathfrak{X}$ be an inner product space. Let $S$ be a subset of $\mathfrak{X}$. We say that $S$ is **orthogonal** if*

$$\forall x, y \in S : x \neq y, \quad \langle x, y \rangle = 0.$$

*i.e., if any two distinct vectors are orthogonal.*

**Definition** (Orthonormal Set). *Let $\mathfrak{X}$ be an inner product space. Let $S$ be a set in the space. We say that $S$ is **orthonormal** if $S$ is orthogonal and $\forall x \in S$, $\|x\| = 1$ where $\|\cdot\|$ is the norm induced by the inner product.*

**Proposition 3.1.1.** *Orthogonal sets are linearly independent.*

## 3.2 Orthogonal Bases

**Definition** (Orthogonal Basis). *Let $V$ be an inner product space. Let $S$ be a set in the space. We say that $S$ is an **orthogonal basis** for $V$ if it is an ordered basis for $V$ and orthogonal.*

**Definition** (Orthonormal Basis). *Let $\mathfrak{X}$ be an inner product space. Let $S$ be a set in the space. We say that $S$ is an **orthonormal basis** for $\mathfrak{X}$ if it is the maximal, with respect to inclusion, in the collection of all orthonormal sets in the space.*

**Proposition 3.2.1.** *Let $V$ be an inner product space. Let $S = \{v_1, ..., v_n\}$ be an orthogonal subset of $V$ where each $v_i$ is non-zero. Then*

$$\forall y \in \operatorname{span}(S), \quad y = \sum_{i=1}^{n} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

**Theorem 3** (Gram-Schmidt Process)**.** *Let $V$ be an inner product space. Let $S = \{x_0, ..., x_n\}$ be a linearly independent subset of $V$. Then the set $T = \{y_0, ..., y_n\}$ given by $y_0 := x_0$ and*

$$\forall i \in \{1, ..., n\}, \quad y_i := x_i - \sum_{j=1}^{i-1} \frac{\langle x_i, y_j \rangle}{\|y_j\|} y_j$$

*is an orthogonal subset of $V$ consisting of non-zero vectors such that* $\operatorname{span}(S) = \operatorname{span}(S')$.

**Proposition 3.2.2.** *Let $V$ be an inner product space and $S = \{v_0, v_1, \ldots, v_n\}$ be an orthogonal subset of $V$. Then the set $S^{'}$ derived from the Gram-Schmidt process is exactly $S$.*

**Theorem 4** (Parseval's Identity)**.** *Let $V$ be a finite-dimensional inner product space. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthogonal basis for $V$. Then*

$$\forall x, y \in V, \quad \langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

**Proposition 3.2.3.** *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. Then $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$ if and only if*

$$\forall x \in \mathcal{H}, \quad x = \sum_{e \in \mathcal{E}} \langle x, e \rangle e.$$

## 3.3 Orthogonal Complements

**Definition** (Orthogonal Complement)**.** *Let $\mathfrak{X}$ be an inner product space. Let $S$ be a non-empty subset of $V$. We define the* ***orthogonal complement*** *of $S$, denoted by $S^{\perp}$, to be a set given by*

$$S^{\perp} := \{x \in \mathfrak{X} : \forall s \in S, \langle x, s \rangle = 0\}.$$

*i.e., the set of all points in $\mathfrak{X}$ that are orthogonal to all vectors in $S$.*

**Proposition 3.3.1.** *Let $V$ be a finite-dimensional inner product space. Then*

*(1)* $V^{\perp} = \{O_V\}$

*(2)* $\{O_V\}^{\perp} = V$

**Proposition 3.3.2.** *Orthogonal complements are always linear subspaces.*

**Proposition 3.3.3.** *Let $V$ be an inner product space and $W$ be a subspace of $V$ with basis $\beta$. Then a vector in $V$ is also in $W^\perp$ if and only if it is orthogonal to all vectors in $\beta$.*

**Proposition 3.3.4** (Extension)**.** *Let $V$ be an $n$-dimensional inner product space and $S = \{v_1, v_2, \dots, v_k\}$ be an orthogonal subset of $V$. Then $S$ can be extended to an orthogonal basis $B = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ for $V$.*

**Proposition 3.3.5.** *Let $V$ be an inner product space. Then*

*(1) $S \subseteq T$ implies $T^\perp \subseteq S^\perp$ for any subsets $S$ and $T$ of $V$.*

*(2) $S \subseteq (S^\perp)^\perp$ for any subset $S$ of $V$.*

**Proposition 3.3.6.** *Let $V$ be a finite-dimensional inner product space and $W$ be a subspace of $V$. Then*

*(1) $W = (W^\perp)^\perp$*

*(2) $V = W \oplus W^\perp$*

**Proposition 3.3.7.** *Let $V$ be a finite-dimensional inner product space and $W_1$ and $W_2$ be subspaces of $V$. Then*

*(1) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$*

*(2) $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$*

## 3.4 Orthogonal Projection

**Definition** (Orthogonal Projection)**.** *Let $V$ be a vector space. Let $W$ be a finite-dimensional subspace of $V$. Let $x$ be a vector in $V$. We define the **orthogonal projection** of $x$ on $W$, denoted by $(x)$, to be the vector $u$ in $W$ such that $x = u + v$ where $v$ is another vector in $W^\perp$.*

## 3.5 Inequalities in Hilbert Spaces

**Theorem 5** (Bessel's Inequality)**.** *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in the space. Then*

$$\forall x \in \mathcal{H}, \quad \sum_{e \in \mathcal{E}} |\langle x, e_i \rangle|^2 \leq \|x\|^2.$$

**Proposition 3.5.1.** *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. Let $x$ be a point in the space. Then the net $\sum_{e \in \mathcal{E}} \langle x, e \rangle e$ converges in $\mathcal{H}$.*

*Proof.* Let $\mathcal{F}$ be the collection of all finite subsets of $\mathcal{E}$, partially ordered by inclusion. Define for each $F \in \mathcal{F}$ a vector $y_F$ as $y_F := \sum_{e\in F}\langle x, e\rangle e$. Let $\varepsilon$ be an arbitrary positive number. Since $\mathcal{E}$ is an orthonormal set, the set $\{e \in \mathcal{E} : \langle x, e\rangle \neq 0\}$ is countable. Let $\{e_i\}_{i\in\mathbb{N}}$ denote the set. By the Bessel's inequality, $\exists N \in \mathbb{N}$ such that $\sum_{i=N+1}^{\infty} |\langle x, e_i\rangle|^2 < \varepsilon^2$. Define a set $F_0$ as $F_0 := \{e_1, ..., e_N\}$. Let $F$ and $G$ be arbitrary elements in $\mathcal{F}$ such that $F_0 \leq F$ and $F_0 \leq G$. Then

$$\begin{aligned}
\|y_F - y_G\|^2 &= \left\| \sum_{e\in F\setminus G} \langle x, e\rangle e - \sum_{e\in G\setminus F} \langle x, e\rangle e \right\|^2 \\
&= \sum_{e\in F\cup G\setminus F\cap G} |\langle x, e\rangle|^2 \\
&= \sum_{e\in F\cup G} |\langle x, e\rangle|^2 - \sum_{e\in F\cap G} |\langle x, e\rangle|^2 \\
&\leq \sum_{i=N+1}^{\infty} |\langle x, e_i\rangle|^2 \\
&< \varepsilon^2.
\end{aligned}$$

So $\{y_F\}_{F\in\mathcal{F}}$ is Cauchy. Since $\mathcal{H}$ is complete and $\{y_F\}_{F\in\mathcal{F}}$ is Cauchy, $\{y_F\}_{F\in\mathcal{F}}$ converges. ∎

# Chapter 4

# Sequence Spaces

## 4.1 $\ell^p$ Space

**Definition** ($\ell^p$ Space)**.** *We define the $\ell^p$ space to be the set of all sequences $x$ such that $\|x\|_p$ is finite, equipped with the p-norm $\|\cdot\|_p$.*

**Definition** (Weighted $\ell^p$ Space)**.** *Let $(r_i)_{i\in\mathbb{N}}$ be a sequence of positive integers. We define the **weighted $\ell^p$** space to be the set given by*

$$\ell^p := \{(x_i)_{i\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sum_{i\in\mathbb{N}} r_i|x_i|^p < +\infty.$$

**Proposition 4.1.1.** *For $p \in [1, +\infty)$, $(\ell^p, \|\cdot\|_p)$ is complete.*

*Proof.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\ell^p$.

Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $\ell^p$, $\forall\varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $\forall m, n > N$, we have $\|x_m - x_n\|_p < \varepsilon$.

Since $\|x_m - x_n\|_p < \varepsilon$ and $|x_m^{(i)} - x_n^{(i)}| \leq \|x_m - x_n\|_p$ for any $i\in\mathbb{N}$, $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$ for any $i\in\mathbb{N}$.

Since for any $i\in\mathbb{N}$ and any positive number $\varepsilon$, there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N$, we have $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$, by definition, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$.

Since $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$ and $\mathbb{F}$ is complete, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ converges.

Let $x_0^{(i)} = x_n^{(i)}$. Let $x_0 = \{x_0^{(i)}\}_{i\in\mathbb{N}}$.

$$\|x_0\|_p = (\sum_{i=1}^{\infty} |x_0^{(i)}|^p)^{\frac{1}{p}}$$

■

## 4.2 $c_0$ Space and $c_{00}$ Space

**Definition** ($c_0$ Space)**.** *We define $c_0$ to be*

$$c_0 := \big\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\big\}.$$

**Definition** ($c_{00}$ Space)**.** *We define $c_{00}$ to be*

$$c_{00} := \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}, \forall n > N, x_n = 0\big\}.$$

*i.e., the set of all eventually zero sequences of real numbers. i.e., the set of all sequences with finite support.*

**Proposition 4.2.1.** *The $c_{00}$ is not complete in $(\ell_1, \|\cdot\|_1)$.*

*Proof.* Define a sequence of vectors $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ by $\mathfrak{x}_i^j := \frac{1}{j^2}$ for $j \in \{1..i\}$ and $\mathfrak{x}_i^j := 0$ for $j > i$. Then $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ converges to something that is not in $c_{00}$. ∎

**Proposition 4.2.2.** *The closure of $c_{00}$ in the space $(\mathbb{R}^\omega, d_1)$ is $\ell_1$.*

*Proof.* For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq \ell_1$. Let $x$ be an arbitrary element in $\mathrm{cl}(c_{00})$. Since $x \in \mathrm{cl}(c_{00})$, there exists another element $y \in c_{00}$ such that $d_1(x, y) < 1$. Let $N \in \mathbb{N}$ be such that $\forall n > N$, $y_n = 0$. Then

$$\begin{aligned}
& d_1(x,y) < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n - y_n| < 1 \\
\iff & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n - y_n| < 1 \\
\iff & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} \big||x_n| - |y_n|\big| + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} \big(|x_n| - |y_n|\big) + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} |x_n| - \sum_{n=1}^{N} |y_n| + \sum_{n>N} |x_n| < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n| - \sum_{n=1}^{N} |y_n| < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n| < 1 + \sum_{n=1}^{N} |y_n|.
\end{aligned}$$

Since $\sum_{n\in\mathbb{N}} |x_n|$ is bounded, $x \in \ell_1$.

For the reverse direction, we are to prove that $\ell_1 \subseteq \mathrm{cl}(c_{00})$. Let $x$ be an arbitrary element in $\ell_1$. For $i \in \mathbb{N}$, define $x^i = \{x^i_j\}_{j\in\mathbb{N}}$ as $x^i_j = x_j$ for $j \leq i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}, x^i \in c_{00}$. Then

$$\begin{aligned}
&\lim_{i\in\mathbb{N}} d_1(x^i, x)\\
=&\lim_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |x^i_j - x_j|\\
=&\lim_{i\in\mathbb{N}} \sum_{j>i} |x^i_j - x_j|\\
=&\lim_{i\in\mathbb{N}} \sum_{j>i} |x_j|\\
=&\,0.
\end{aligned}$$

That is, $\lim_{i\in\mathbb{N}} d_1(x^i, x) = 0$. So $\lim_{i\in\mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

■

**Proposition 4.2.3.** *The closure of $c_{00}$ in the space $(\mathbb{R}^\omega, d_\infty)$ is $c_0$.*

*Proof.* For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq c_0$. Let $x$ be an arbitrary element in $\mathrm{cl}(c_{00})$. Let $\varepsilon$ be an arbitrary positive real number. Since $x \in \mathrm{cl}(c_{00})$, there exists another element $y$ in $c_{00}$ such that $d_\infty(x, y) < \varepsilon$. That is, $\forall j \in \mathbb{N}, |x_j - y_j| < \varepsilon$. Since $y \in c_{00}$, $\exists N \in \mathbb{N}$ such that $\forall j > N$, $y_j = 0$. So $\forall j > N, |x_j| < \varepsilon$. That is,

$$\forall \varepsilon > 0, \quad , \exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon.$$

By definition of convergence of limits, $\lim_{j\in\mathbb{N}} x_j = 0$. So $x \in c_0$.

For the reverse direction, we are to prove that $c_0 \subseteq \mathrm{cl}(c_{00})$. Let $x$ be an arbitrary element in $c_0$. For $i \in \mathbb{N}$, define $x^i$ as $x^i_j = x_j$ for $j \leq i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}$, $x^i \in c_{00}$. Let $\varepsilon$ be an arbitrary positive real number. Since $x \in c_0$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon/2.$$

Let $i > N$. Then

$$\begin{aligned}
&d_\infty(x^i, x)\\
=&\sup_{j\in\mathbb{N}} |x^i_j - x_j|\\
=&\sup_{j>i} |x^i_j - x_j|\\
=&\sup_{j>i} |x_j|\\
\leq&\, \varepsilon/2 < \varepsilon.
\end{aligned}$$

That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall i > N, \quad d_\infty(x^i, x) < \varepsilon.$$

By definition of convergence of sequences, $\lim_{i\in\mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

■

**Proposition 4.2.4.** *Let* $A := \big\{\{x_n\}_{n\in\mathbb{N}} \in c_{00} : \sum_{n\in\mathbb{N}} x_n = 0\big\}$. *Then* $A$ *is a subset of* $\ell^1$ *and is closed in* $(\ell^1, d_1)$. *i.e.* $\mathrm{cl}(A) = A$ *in* $(\ell^1, d_1)$.

*Proof.* Let $x = \{x^i\}_{i\in\mathbb{N}}$ be a sequence in $\ell^1$, where each $x^i = \{x^i_j\}_{j\in\mathbb{N}}$ is an element in $A$, that converges in $(\ell^1, d_1)$. Say $\lim_{i\to\infty} x^i = x^\infty$.

First I claim that $x^\infty \in c_{00}$.

Now I claim that $\sum_{j\in\mathbb{N}} x^\infty_j = 0$. i.e. $x^\infty \in A$. Since $x^\infty \in c_{00}$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad x^\infty_j = 0.$$

Define $y_i := \sum_{j=1}^N x^i_j$. Define $y_\infty := \sum_{j=1}^N x^\infty_j$. It is easy to see that $\lim_{i\in\mathbb{N}} y_i = y_\infty$. Assume for the sake of contradiction that $y_\infty \neq 0$. i.e. $\{y_i\}_{i\in\mathbb{N}}$ does not converge to 0. Then

$$\exists \varepsilon_0 > 0, \quad \forall M \in \mathbb{N}, \quad \exists i_0 > M, \quad |y_{i_0} - 0| = |y_{i_0}| \geq \varepsilon_0. \tag{1}$$

Since $\lim_{i\to\infty} x^i = x^\infty$,

$$\exists M_0 \in \mathbb{N}, \quad \forall i > M_0, \quad d_1(x^i, x^\infty) < \varepsilon_0. \tag{2}$$

Consider statement (1) for a particular $M$, $M_0$, we have

$$\exists i_0 > M_0, \quad |y_{i_0}| \geq \varepsilon_0. \tag{3}$$

That is,

$$\Big|\sum_{j=1}^N x^{i_0}_j\Big| \geq \varepsilon_0. \tag{3'}$$

Consider statement (2) for a particular $i$, $i_0$, we have

$$d_1(x^{i_0}, x^\infty) < \varepsilon_0. \tag{4}$$

From statement (4) we can derive:

$$\begin{aligned}
& d_1(x^{i_0}, x^\infty) < \varepsilon_0 \\
\iff & \sum_{j\in\mathbb{N}} |x^{i_0}_j - x^\infty_j| < \varepsilon_0 \\
\iff & \sum_{j=1}^N |x^{i_0}_j - x^\infty_j| + \sum_{j>N} |x^{i_0}_j - x^\infty_j| < \varepsilon_0
\end{aligned}$$

$$
\begin{aligned}
&\implies \sum_{j>N} |x_j^{i_0} - x_j^{\infty}| < \varepsilon_0 \\
&\iff \sum_{j>N} |x_j^{i_0} - 0| < \varepsilon_0 \\
&\iff \sum_{j>N} |x_j^{i_0}| < \varepsilon_0 \\
&\implies \Big|\sum_{j>N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\implies \Big|\sum_{j>N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\iff \Big|\sum_{j\in\mathbb{N}} x_j^{i_0} - \sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\iff \Big|0 - \sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0 \\
&\iff \Big|\sum_{j=1}^{N} x_j^{i_0}\Big| < \varepsilon_0.
\end{aligned}
$$

This contradicts to statement (3'). So the original assumption that $y_\infty \neq 0$ is false. i.e. $y_\infty = 0$. It follows that $\sum_{j\in\mathbb{N}} x_j^{\infty} = 0$. This completes the proof.

■

## 4.3 Hölder's Inequality

**Theorem 6** (Hölder's Inequality)**.** *Let $\mathfrak{X} = \mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ be vectors in $\mathfrak{X}$. Then $\forall p, q \in (1, +\infty) : 1/p + 1/q = 1$, $\|xy\|_1 \leq \|x\|_p \|y\|_q$. i.e.,*

$$\sum_{i=1}^{n} |x_i y_i| \leq (\sum_{i=1}^{n} |x_i|^p)^{1/p} (\sum_{i=1}^{n} |y_i|^q)^{1/q}.$$

# Chapter 5

# Function Spaces

## 5.1 The $\mathcal{L}^p$ Norm

$$\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

**the instructors' answer,** *where instructors collectively construct a single answer*

In the sup norm, convergence coincides with uniform convergence. Moreover, $C[a, b]$ is complete in this norm. It is not complete in any of the $L^p$ norms for $1 \leq p < \infty$. The completion in these norms is called $L^p(a, b)$.

undo thanks | 1

Updated 1 day ago by Kenneth Davidson

# Chapter 6

# Banach Space

## 6.1 Definition

**Definition** (Banach Space)**.** *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let $d$ be the metric induced by $\|\cdot\|$. We say that $\mathfrak{X}$ is a **Banach space** if $(\mathfrak{X}, d)$ is a complete metric space. i.e., we define a Banach space to be a complete normed linear space.*

## 6.2 Properties

**Proposition 6.2.1.** *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed vector space over field $\mathbb{F}$. Then $(\mathfrak{X}, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in $X$ is summable.*

**Proposition 6.2.2.** *Any Banach space with a Schauder basis has to be separable.*

## 6.3 Examples of Banach Space

**Example 6.3.1.** $(\mathcal{C}([0,1], \mathbb{F}), \|\cdot\|_\infty)$ *is a Banach space.*

**Example 6.3.2** (Disc Algebra)**.** *Define $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Define $\mathcal{A}(\mathbb{D}) := \{f \in \mathcal{C}(\overline{\mathbb{D}}) : f|_{\mathbb{D}}$ is holomorphic $\}$. Define $\|\cdot\|_\infty$ by $\|f\|_\infty := \sup_{z \in \overline{\mathbb{D}}} |f(z)|$. Then $(\mathcal{A}(\mathbb{D}), \|\cdot\|_\infty)$ is a Banach space.*

**Example 6.3.3.** *Let $(X, \Omega, \mu)$ be a measure space. Let $p$ be a number in $[1, +\infty)$. Define*

$$\mathcal{L}^p(X, \mu) := \operatorname{span}\{f : X \to [0, +\infty] \mid f \text{ is measurable and } \int_X |f|^p < +\infty\}.$$

*Define an equivalence relation on $\mathcal{L}^p(X, \mu)$ by $f \equiv g$ if and only if*

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

*Define a space $L^p(X,\mu) := \mathcal{L}^p(X,\mu)/\equiv$. Then $L^p(X,\mu)$ is a Banach space when equipped with the norm*

$$\|[f]\|_p := \left(\int_X |f|^p\right)^{1/p}.$$

**Example 6.3.4.** *Let $\mathcal{P}_{\mathbb{C}}[0,1]$ denote the set of all polynomials with complex coefficients. For each $p \in [1,+\infty)$, define a norm*

$$\|f\|_p := \left(\int_0^1 |f|^p\right)^{1/p}.$$

*For $p = +\infty$, define a norm*

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(z)|.$$

## 6.4 Construction of Banach Spaces

**Definition.** *Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be two Banach spaces over field $\mathbb{K}$. Let $p \in [1,+\infty)$. We define*

$$\mathfrak{X} \oplus_p \mathfrak{Y} := \{(x,y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

*and*

$$\|(x,y)\|_p := (\|x\|_{\mathfrak{X}}^p + \|y\|_{\mathfrak{Y}}^p)^{1/p}.$$

*For $p = +\infty$, we define*

$$\mathfrak{X} \oplus_\infty \mathfrak{Y} := \{(x,y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

*and*

$$\|(x,y)\|_\infty := \max(\|x\|_{\mathfrak{X}}, \|y\|_{\mathfrak{Y}}).$$

- *Note that the norms behave similarly to what the p norm would do.*
- *We can similarly define the direct sum of finitely many Banach spaces.*

**Proposition 6.4.1.** *$\|\cdot,\cdot\|_p$ is a norm on $\mathfrak{X} \oplus_p \mathfrak{Y}$.*

**Proposition 6.4.2.** *$\mathfrak{X} \oplus_p \mathfrak{Y}$ is complete with respect to $\|\cdot,\cdot\|_p$.*

# Chapter 7

# Hilbert Space

## 7.1 Definition

**Definition** (Hilbert Space)**.** *We define a **Hilbert space**, denoted by $\mathcal{H}$, to be a complete inner product space.*

## 7.2 Examples of Hilbert Space

**Example 7.2.1.** *Let $(X, \mu)$ be a measure space. Then $L^2(X, \mu)$ is a Hilbert space with inner product given by*

$$\langle f, g \rangle := \int_X f(x)\overline{g(x)}d\mu(x).$$

**Example 7.2.2.** $\ell^2(\mathbb{K}) = \left\{(x_i)_{i\in\mathbb{N}} \in \mathbb{K}^\infty : \sum_{i\in\mathbb{N}} |x_i|^2 < +\infty\right\}$ *is a Hilbert space with inner product given by*

$$\langle (x_i)i \in \mathbb{N}, (y_i)_{i\in\mathbb{N}} \rangle := \sum_{i\in\mathbb{N}} x_n\overline{y_n}.$$

## 7.3 Properties of Hilbert Space

**Proposition 7.3.1.** *Let $\mathcal{H}$ be a Hilbert space. Let $S$ be a non-empty set in the space. Then $S^{\perp\perp} = \operatorname{clspan}(S)$.*

*Proof.* For one direction, we are to prove that $\operatorname{clspan}(S) \subseteq S^{\perp\perp}$.

For the reverse direction, we are to prove that $S^{\perp\perp} \subseteq \operatorname{clspan}(S)$. Assume for the sake of contradiction that $\exists x \in S^{\perp\perp}$ with $x \neq 0$ such that $x \notin \operatorname{clspan}(S)$. Say $x = m_1 + m_2$ for some $m_1 \in \operatorname{clspan}(S)$ and some $m_2 \in \operatorname{clspan}(S)^\perp$. Note that $\operatorname{clspan}(S)^\perp = S^\perp$. So $m_2 \in S^\perp$. Since $x \in S^{\perp\perp}$ and $m_2 \in S^\perp$, we should have $\langle x, m_2 \rangle = 0$. However,

$$\langle x, m_2 \rangle = \langle m_1 + m_2, m_2 \rangle$$

$$
\begin{aligned}
&= \langle m_1, m_2\rangle + \langle m_2, m_2\rangle \\
&= 0 + \langle m_2, m_2\rangle \\
&> 0, \text{ since } m_2 \neq 0.
\end{aligned}
$$

This leads to a contradiction. So $S^{\perp\perp} \subseteq \text{clspan}(S)$. ■

**Theorem 7** (The Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space over field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Suppose that $\mathcal{H} \neq \{0\}$. Then for any $\varphi \in \mathcal{H}^*$, $\exists y \in \mathcal{H}$ such that*

$$\forall x \in \mathcal{H}, \quad \varphi(x) = \langle x, y\rangle.$$

*Proof.* Define for each $y \in \mathcal{H}$ a function $\beta_y \in \mathcal{H}^*$ by $\beta_y(x) := \langle x, y\rangle$. We are to prove that $\mathcal{H}^* = \{\beta_y : y \in \mathcal{H}\}$. It is easy to verify that each $\beta_y$ is linear and bounded. So $\forall y \in \mathcal{H}$, $\beta_y \in \mathcal{H}^*$. i.e., $\{\beta_y : y \in \mathcal{H}\} \subseteq \mathcal{H}^*$. Define a map $\Theta$ from $\mathcal{H}$ to $\mathcal{H}^*$ as $\Theta(y) := \beta_y$. It is easy to verify that $\Theta$ is linear.

$$
\begin{aligned}
\|\Theta(y)\| = \|\beta_y\| &= \sup\{\beta_y(x) : \|x\| = 1\} \\
&= \sup\{\langle x, y\rangle : \|x\| = 1\} \\
&\leq \sup\{\|x\|\|y\| : \|x\| = 1\} \\
&= \|y\|.
\end{aligned}
$$

That is, $\|\Theta(y)\| \leq \|y\|$. So $\|\Theta\| \leq 1$. On the other hand, consider an arbitrary point $y_0 \in \mathcal{H}$ with $y_0 \neq 0$:

$$
\begin{aligned}
\|\Theta\| &= \sup\left\{\frac{\|\Theta(y)\|}{\|y\|} : y \neq 0\right\} \\
&\geq \left.\frac{\|\Theta(y)\|}{\|y\|}\right|_{y=y_0} \\
&= \frac{\|\Theta(y_0)\|}{\|y_0\|} \\
&= \frac{\|\beta_{y_0}\|}{\|y_0\|} \\
&= \frac{1}{\|y_0\|} \sup\left\{\frac{\|\beta_{y_0}(y)\|}{\|y\|} : y \neq 0\right\} \\
&\geq \frac{1}{\|y_0\|} \frac{\|\beta_{y_0}(y_0)\|}{\|y_0\|} \\
&\geq \frac{1}{\|y_0\|} \frac{\|y_0\|^2}{\|y_0\|} \\
&= 1.
\end{aligned}
$$

That is, $\|\Theta\| \geq 1$. So $\|\Theta\| = 1$. So $\Theta$ is isometric. It immediately follows that $\Theta$ is injective. Now it remains to prove that $\Theta$ is surjective. Let $\varphi \in \mathcal{H}^*$. If $\varphi = 0$, then $\varphi = \Theta(0)$ and we are done. Otherwise, let $\mathcal{M} := \ker(\varphi)$. Then we have $\operatorname{codim} \mathcal{M} = \dim \mathcal{M}^{\perp} = 1$. Take $e \in \mathcal{M}^{\perp}$ such that $\|e\| = 1$. Let $P$ denote the orthogonal projection onto $\mathcal{M}$. Then $1 - P$ is the orthogonal projection onto $\mathcal{M}^{\perp}$.

$$x = Px + (1 - P)x = Px + \langle x, e\rangle e.$$

So for $x \in \mathcal{H}$,

$$\varphi(x) = \varphi(Px) + \langle x, e\rangle \varphi(e) = \langle x, \overline{\varphi(e)} e\rangle = \beta_y(x)$$

where $y := \overline{\varphi(e)} e$. Hence $\varphi = \beta_y$. So $\Theta$ is surjective. This completes the proof. ■

# Chapter 8

# Operators

## 8.1 Bounded Operators

**Definition** (Bounded Operator)**.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $T$ be a linear map from $\mathfrak{X}$ to $\mathfrak{Y}$. We say that $T$ is a* ***bounded operator*** *if*

$$\exists k \in \mathbb{R}, \quad \forall x \in \mathfrak{X}, \quad \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}.$$

**Definition** (Operator Norm)**.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $T$ be a bounded operator from $\mathfrak{X}$ to $\mathfrak{Y}$. We define the* ***operator norm*** *of $T$, denoted by $\|T\|$, to be the number given by*

$$\|T\| := \inf\{k \in \mathbb{R} : \forall x \in \mathfrak{X}, \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}\}.$$

**Proposition 8.1.1.**

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{X}, \|x\|_{\mathfrak{X}} = 1\}.$$

**Proposition 8.1.2.** *Let $X$ and $Y$ be normed linear spaces. Let $T$ be a linear map from $X$ to $Y$. Then $T$ is bounded if and only if $T$ is continuous.*

## 8.2 Examples of Bounded Operators

**Example 8.2.1** (The Multiplication Operator)**.** *Let $\mathfrak{X} = \big(\mathcal{C}([0,1], \mathbb{C}), \|\cdot\|_\infty\big)$. Let $f$ be a function in $\mathfrak{X}$. We define the* ***multiplication operator*** *on $\mathfrak{X}$, w.r.t. $f$, denoted by $M_f$, as*

$$M_f(g) = fg.$$

*Then $M_f$ is bounded and $\|M_f\| = \|f\|_\infty$.*

*Proof.* Let $g$ be an arbitrary function in $\mathfrak{X}$. Then

$$\|M_f g\|_\infty = \|fg\|_\infty$$

$$\begin{aligned}
&= \sup_{x\in[0,1]} |f(x)g(x)| \\
&= \sup_{x\in[0,1]} |f(x)||g(x)| \\
&\leq \sup_{x\in[0,1]} |f(x)| \sup_{x\in[0,1]} |g(x)| \\
&= \|f\|_\infty \|g\|_\infty.
\end{aligned}$$

That is, $\|M_f g\|_\infty \leq \|f\|_\infty \|g\|_\infty$. So $\|f\|_\infty$ is an element of the set $S = \{k \in \mathbb{R} : \forall g \in \mathfrak{X}, \|M_f g\|_{\mathfrak{Y}} \leq k\|g\|_{\mathfrak{X}}\}$. So $\|M_f\| = \inf(S) \leq \|f\|_\infty$. Consider $g_0$ given by $g_0(x) = 1$. Then $g_0$ in $\mathfrak{X}$. Then

$$\|M_f g_0\|_\infty = \|f g_0\|_\infty = \|f\|_\infty = \|f\|_\infty \|g_0\|_\infty.$$

Let $k$ be an arbitrary element in $S$. Assume for the sake of contradiction that $k < \|f\|_\infty$. Then

$$\begin{aligned}
\|f\|_\infty \|g_0\|_\infty &= \|M_f g_0\|_\infty \\
&\leq k\|g_0\|_\infty \\
&< \|f\|_\infty \|g_0\|_\infty.
\end{aligned}$$

This leads to a contradiction. So $\forall k \in S$, $k \geq \|f\|_\infty$. So $\|f\|_\infty$ is a lower bound for the set $S$. So $\|M_f\| = \inf(S) \geq \|f\|_\infty$. Since $\|M_f\| \leq \|f\|_\infty$ and $\|M_f\| \geq \|f\|_\infty$, we get $\|M_f\| = \|f\|_\infty$. ■

**Example 8.2.2** (The Volterra Operator)**.** *Let* $\mathfrak{X} = \big(\mathcal{C}([0,1],\mathbb{C}), \|\cdot\|_\infty\big)$. *Define*

$$Vf := x \mapsto \int_0^x f(t)dt.$$

*Then the Volterra Operator is bounded and* $\|V\| \leq 1$.

*Proof.* Let $f$ be an arbitrary function in $\mathfrak{X}$ with $\|f\|_\infty = 1$. Then $\forall x \in [0,1]$,

$$\begin{aligned}
|Vf(x)| &= \left|\int_0^x f(t)dt\right| \\
&\leq \int_0^x |f(t)|dt \\
&\leq \int_0^x \sup_{t\in[0,1]} |f(t)|dt \\
&= \int_0^x \|f\|_\infty dt \\
&= \int_0^x 1dt \\
&= x.
\end{aligned}$$

That is, $\forall x \in [0,1]$, $|Vf(x)| \leq 1$. So $\|Vf\|_\infty \leq 1$. Since $\forall f \in \mathfrak{X} : \|f\|_\infty = 1$, $\|Vf\|_\infty \leq 1$, we get $\|V\| \leq 1$. ■

**Example 8.2.3** (The Diagonal Operator)**.** *Let $\mathfrak{X} = \ell^2(\mathbb{N})$. Let*

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & \ddots \end{bmatrix}.$$

*Then $D$ is bounded if and only if $(d_i)_{i\in\mathbb{N}}$ is bounded and $\|D\| = \|(d_i)_{i\in\mathbb{N}}\|_\infty$.*

*Proof.* Case 1.

$$\begin{aligned} \|Dx\|_2^2 &= \sum_{i\in\mathbb{N}} |d_i x_i|^2 \\ &=\leq \sum_{i\in\mathbb{N}} \|(d_j)_{j\in\mathbb{N}}\|_\infty |x_i|^2 \\ &= \|(d_j)_{j\in\mathbb{N}}\|_\infty \sum_{i\in\mathbb{N}} |x_i|^2 \\ &= \|(d_j)_{j\in\mathbb{N}}\|_\infty \|x\|_2^2. \end{aligned}$$

Case 2.
If $(d_i)_{i\in\mathbb{N}} \notin \ell^\infty$, $\exists (d_{n_i})_{i\in\mathbb{N}} \to \infty$.

$$\begin{aligned} \|De_{n_i}\|_2 &= \|d_{n_i} e_{n_i}\|_2 \\ &= |d_{n_i}| \|e_{n_i}\|_2 \\ &= |d_{n_i}|. \end{aligned}$$

So $\|D\| \geq \|De_{n_i}\|_2 \to \infty$. ■

**Example 8.2.4** (Weighted Shifts)**.**

- *Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(w_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an* ***unilateral forward weighted shift*** *$W$ on $\mathcal{H}$ as*

$$W(x_n) := (0, w_1 x_1, w_2 x_2, w_3 x_3, ...).$$

*i.e.,*

$$W = \begin{bmatrix} 0 & & & & \\ w_1 & 0 & & & \\ & w_2 & 0 & & \\ & & w_3 & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.$$

*Then $W$ is bounded and $\|W\| = \sup\{|w_n| : n \in \mathbb{N}\}$.*

- *Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(v_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an* ***unilateral backward weighted shift*** *$V$ on $\mathcal{H}$ as*

$$V(x_n) := (v_1x_2, v_2x_3, v_3x_4, ...).$$

*Then $V$ is bounded and $\|V\| = \sup\{|v_n| : n \in \mathbb{N}\}$.*

- *Let $\mathcal{H} = \ell^2_{\mathbb{Z}}$. Let $(u_n)_{n\in\mathbb{Z}} \in \ell^\infty_{\mathbb{Z}}$. We define a* ***bilateral weighted shift*** *$U$ on $\mathcal{H}$ as*

$$U(x_n) := (u_{n-1}x_{n-1})_{n\in\mathbb{Z}}.$$

*Then $U$ is bounded and $\|U\| = \sup\{|u_n| : n \in \mathbb{Z}\}$.*

**Example 8.2.5** (The Composition Operators)**.** *Let $\mathfrak{X} = \mathcal{C}([0,1],\mathbb{C})$. Let $\varphi \in \mathcal{C}([0,1],[0,1])$. We define the* ***composition operator*** *on $\mathfrak{X}$, denoted by $C_\varphi$ as*

$$C_\varphi(f) := f \circ \varphi.$$

*Then $C_\varphi$ is contractive.*

*Proof.*

$$\begin{aligned} \|C_\varphi(f)\| &= \sup_{x\in[0,1]} |(f\circ\varphi)(x)| \\ &\leq \|f\|_\infty. \end{aligned}$$

■

## 8.3 The Space of Bounded Operators

**Proposition 8.3.1.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Then $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ is a vector space and the operator norm is a norm on $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$.*

**Proposition 8.3.2.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ be the space of bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$. Then if $\mathfrak{Y}$ is complete, $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ is complete.*

**Proposition 8.3.3.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$. Then $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y},\|\cdot\|_1)$ if and only if $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y},\|\cdot\|_2)$.*

## 8.4 Invertible Bounded Operators

**Proposition 8.4.1.** *Let $(\mathfrak{X},\|\cdot\|_1)$ be a Banach space. Let $S \in \mathcal{B}(\mathfrak{X})$ be a bounded linear map that is invertible. Define a norm $\|\cdot\|_2$ on $\mathfrak{X}$ as*

$$\|x\|_2 := \|Sx\|_1.$$

*Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*

*Proof.* On one hand, since $S$ is bounded, $\exists c_1$ such that $\forall x \in \mathfrak{X}$, $\|Sx\|_1 \leq c_1\|x\|_1$. That is, $\|x\|_2 \leq c_1\|x\|_1$.

On the other hand, since $S$ is invertible, $S^{-1}$ exists and is also bounded. Since $S^{-1}$ is bounded, $\exists c_2$ such that $\forall x \in \mathfrak{X}$, $\|S^{-1}x\|_1 \leq c_2\|x\|_1$. Consider $x = Sx$, we get $\forall x \in \mathfrak{X}$, $\|S^{-1}Sx\|_1 \leq c_2\|Sx\|_1$. That is, $\|x\|_1 \leq c_2\|x\|_2$.

So $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ■

**Proposition 8.4.2.** *Let* $(\mathfrak{X}, \|\cdot\|)$ *be a Banach space. Let* $S$ *be a map in* $\mathcal{B}(\mathfrak{X})$ *that is invertible. Then*

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

*Proof.*

$$\begin{aligned}
\|S^{-1}\| &= \sup\{\|S^{-1}x\| : \|x\| = 1\} \\
&= \sup\left\{\frac{\|S^{-1}x\|}{\|x\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{\|S^{-1}Sx\|}{\|Sx\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{\|x\|}{\|Sx\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{1}{\|Sx\|} : \|x\| = 1\right\} \\
&= (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.
\end{aligned}$$

That is,

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

■

# Chapter 9

# Dual Space

## 9.1 Definition

**Definition** (Linear Functional)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{K}$. We define a **linear functional** on $\mathfrak{X}$ to be a linear map from $\mathfrak{X}$ to $\mathbb{K}$.*

**Definition** (Algebraic Dual)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{K}$. We define the **algebraic dual** of $\mathfrak{X}$, denoted by $\mathfrak{X}^{\#}$. to be the space of all linear functionals on $\mathfrak{X}$.*

**Definition** (Topological Dual)**.** *Let $\mathfrak{X}$ be a topological vector space over field $\mathbb{K}$. We define the **topological dual** of $\mathfrak{X}$, denoted by $\mathfrak{X}^*$, to be the space of all <u>continuous</u> linear functionals on $\mathfrak{X}$.*

**Proposition 9.1.1.** *Let $X$ be a normed linear space. Then there exists a contractive map from $X$ to its double dual $X^{**}$.*

## 9.2 Examples of Dual Space

**Example 9.2.1.** $c_0(\mathbb{N})^*$ *is isometrically isomorphic to* $\ell^1(\mathbb{N})$.

**Example 9.2.2.** $c_0(\mathbb{N})^*$ *is isometrically isomorphic to* $\ell^\infty(\mathbb{N})$.

# Chapter 10

# Quotient Spaces

## 10.1 Definitions

**Definition** (Quotient Space)**.** *Let $\mathfrak{V}$ be a vector space. Let $\mathfrak{W}$ be a subspace of $\mathfrak{V}$. We define a **quotient space**, denoted by $\mathfrak{V}/\mathfrak{W}$, to be a set $\{v + \mathfrak{W} : v \in \mathfrak{V}\}$ with operations*

$$(v_1 + \mathfrak{W}) + (v_2 + \mathfrak{W}) := (v_1 + v_2) + \mathfrak{W} \text{ and}$$

$$\kappa(v + \mathfrak{W}) := (\kappa v) + \mathfrak{W}.$$

**Definition** (Quotient Map)**.** *Let $\mathfrak{X}$ be a vector space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define the **quotient map** on $\mathfrak{X}$ with respect to $\mathfrak{M}$, denoted by $q_{\mathfrak{M}}$, to be a function from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$ given by*

$$q_{\mathfrak{M}}(x) := [x] = x + \mathfrak{M}$$

## 10.2 Quotient Spaces with Seminorms

**Definition** (Seminorm on Quotient Spaces)**.** *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define a **seminorm** on $\mathfrak{X}/\mathfrak{M}$ to be a function from $\mathfrak{X}/\mathfrak{M}$ to $\mathbb{R}$ given by*

$$p(x + \mathfrak{M}) := \inf\{\|x + m\| : m \in \mathfrak{M}\}.$$

**Proposition 10.2.1.** *Seminorms on quotient spaces are indeed seminorms.*

**Proposition 10.2.2.** *A seminorm on a quotient space $\mathfrak{X}/\mathfrak{M}$ is a norm if and only if $\mathfrak{M}$ is closed.*

**Proposition 10.2.3** (Quotient maps are contractive)**.** *Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a closed linear subspace of $\mathfrak{X}$. Then*

$$\forall x \in \mathfrak{X}, \quad \|x + \mathfrak{M}\|_{\mathfrak{X}/\mathfrak{M}} \leq \|x\|_{\mathfrak{X}}.$$

**Proposition 10.2.4.** *Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a closed linear subspace of $\mathfrak{X}$. Let $q$ denote the canonical quotient map from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$. Then $q$ is a continuous under the norm topology.*

*Proof.* Since $q$ is contractive, $q$ is continuous. ∎

## 10.3 Quotient Spaces with Topologies

**Definition** (Quotient Toplogy)**.** *Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. Let $\mathcal{W}$ be a closed subspace of $\mathcal{V}$. We define the **quotient topology** on the quotient space $\mathcal{V}/\mathcal{W}$ as*

$$\left\{G \subseteq \mathcal{V}/\mathcal{W} : q^{-1}(G) \in \mathcal{T}\right\}.$$

**Proposition 10.3.1.** *The quotient topology is compatible with the quotient space.*

**Proposition 10.3.2.** *The quotient topology is Hausdorff.*

**Proposition 10.3.3.** *The quotient map is continuous under the quotient topology.*

**Proposition 10.3.4.** *Then*

- *map. i.e.,*
$$\forall \text{ open set } W \subseteq \mathfrak{X}/\mathfrak{M}, \quad q^{-1}(W) \text{ is open in } \mathfrak{X}.$$

- *$q$ is an open map. i.e.,*
$$\forall \text{ open set } G \subseteq \mathfrak{X}, \quad q(G) \text{ is open in } \mathfrak{X}/\mathfrak{M}.$$

# Chapter 11

# Balanced Sets

## 11.1 Definitions

**Definition** (Balanced Sets)**.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $S$ be a subset of $X$. We say that $S$ is **balanced** if*

$$\forall a \in \mathbb{F} : |a| \leq 1, \quad aS \subseteq S.$$

**Definition** (Balanced Hull)**.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $S$ be a subset of $X$. We define the **balanced hull** of $S$, denoted by* $\operatorname{balhull}(S)$*, to be the smallest balanced set containing $S$.*

**Definition** (Balanced Core)**.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $S$ be a subset of $X$. We define the **balanced core** of $S$, denoted by* $\operatorname{balcore}(S)$*, to be the largest balanced set contained in $S$.*

## 11.2 Properties

**Proposition 11.2.1.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $B$ be a balanced subset of $X$. Then*

$$\forall a, b \in \mathbb{F} : |a| \leq |b|, \quad aB \subseteq bB.$$

**Proposition 11.2.2.** *Balanced sets are path connected.*

**Proposition 11.2.3** (Act on Other Properties)**.**

- *The balanced hull of a compact set is compact.*
- *The balanced hull of a totally bounded set is totally bounded.*
- *The balanced hull of a bounded set is bounded.*

**Proposition 11.2.4** (Act on Other Properties).

- *The balanced core of a closed set is closed.*

**Proposition 11.2.5.** *Let $X$ be a vector space over field $\mathbb{F}$. Let $a$ be a scalar in field $\mathbb{F}$. Then*

$$a\,\mathrm{balhull}(S) = \mathrm{balhull}(aS).$$

## 11.3 Stability of Balance

**Proposition 11.3.1** (Set Operations).

- *The union of balanced sets is also balanced.*
- *The intersection of balanced sets is also balanced.*

**Proposition 11.3.2** (Linear Mappings).

- *The scalar multiple of a balanced set is also balanced.*
- *The (Minkowski) sum of two balanced sets is also balanced.*
- *The image of a balanced set under a linear operator is also balanced.*
- *The inverse image of a balanced set under a linear operator is also balanced.*

**Proposition 11.3.3** (Topological Operations). *The closure of a balanced set is also balanced.*

**Proposition 11.3.4.** *The convex hull of a balanced set is also balanced (and also convex).*

## 11.4 Absorbing Sets

**Definition** (Absorbing Sets). *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $S$ be a subset of $X$. We say that $S$ is **absorbing** if*

$$\forall x \in \mathfrak{X}, \quad \exists r \in \mathbb{R} : r > 0, \quad \forall c \in \mathbb{F} : |c| \geq r, \quad x \in cS.$$

*i.e.,*

$$\bigcup_{n \in \mathbb{N}} nS = \mathfrak{X}.$$

**Proposition 11.4.1.** *Every absorbing set contains the origin.*

# Chapter 12

# Topological Vector Space

## 12.1 Definitions

**Definition** (Compatible)**.** *Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let $\mathcal{T}$ be a topology on $\mathcal{V}$. We say that $\mathcal{T}$ is **compatible** with the vector space structure on $\mathcal{V}$ if the addition and scalar multiplication operations on $\mathcal{V}$ are continuous.*

**Definition** (Topological Vector Space)**.** *We define a **topological vector space** to be a vector space with a compatible Hausdorff topology.*

## 12.2 Examples

**Example 12.2.1.** *Let $\mathfrak{X}$ be a normed linear space. Then $\mathfrak{X}$ is a topological vector space with the topology induced by the norm.*

*Proof.*

$$\begin{aligned}
\|\sigma(x_\alpha, y_\alpha) - \sigma(x, y)\| &= \|(x_\alpha + y_\alpha) - (x + y)\| \\
&= \|(x_\alpha - x) + (y_\alpha - y)\| \\
&\le \|x_\alpha - x\| + \|y_\alpha - y\| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

So $\sigma$ is continuous.

$$\begin{aligned}
\|\mu(k_\alpha, x_\alpha) - \mu(k, x)\| &= \|k_\alpha x_\alpha - kx\| \\
&= \|k_\alpha x_\alpha - kx_\alpha + kx_\alpha - kx\| \\
&\le \|k_\alpha x_\alpha - kx_\alpha\| + \|kx_\alpha - kx\|
\end{aligned}$$

$$= |k_\alpha - k| \|x_\alpha + |k| \|x_\alpha - x\|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So $\mu$ is continuous. ∎

**Proposition 12.2.1.** *Normed linear spaces are Hausdorff.*

**Example 12.2.2.** *Let $\mathfrak{X}$ be a Banach space. Let $\mathfrak{X}^*$ denote the dual space of $\mathfrak{X}$. Let $\tau_*$ denote the weak topology on $\mathfrak{X}^*$ induced by elements of $\mathfrak{X}$ as*

$$\lim_\alpha x^*_\alpha = x^* \iff \forall x \in \mathfrak{X}, \lim_\alpha x^*_\alpha(x) = x^*(x).$$

*Then $(\mathfrak{X}^*, \tau_*)$ is a topological vector space.*

## 12.3 Properties

**Proposition 12.3.1.** *Let $\mathcal{V}$ be a topological vector space. Every neighborhood of 0 contains a balanced open neighborhood of 0.*

*Proof.* Let $U$ be an arbitrary element of $\mathcal{U}_0^{\mathcal{V}}$. Let $\mu$ denote the multiplication operation on $\mathcal{V}$. Then $\mu$ is continuous and hence $\mu^{-1}(U)$ is a neighborhood of $(0,0) \in \mathbb{K} \times \mathcal{V}$. So there exist an $r > 0$ and an element $N \in \mathcal{U}_0^{\mathcal{V}}$ that is open such that $\mathrm{ball}(0,r) \times N \subseteq \mu^{-1}(U)$. Define a set $M$ as $M := \bigcup_{k:0<|k|<r} kN$. Since $\mathrm{ball}(0,r) \times N \subseteq \mu^{-1}(U)$, we have $M \subseteq U$. Since $M = \bigcup_{k:0<|k|<r} kN$ and $N \in \mathcal{T}$, we have $M \in \mathcal{T}$. Since $M \supseteq \frac{r}{2}N$, $\frac{r}{2}N \in \mathcal{T}$, and $0 \in \frac{r}{2}N$, we have $M \in \mathcal{U}_0^{\mathcal{V}}$. Let $a$ be an arbitrary element in $\mathbb{K}$ such that $|a| < 1$. Then

$$aM = a \bigcup_{k:0<|k|<r} kN = \bigcup_{k:0<|k|<r} akN = \bigcup_{k:0<|k|<ar} kN \subseteq \bigcup_{k:0<|k|<r} kN = M.$$

So $M$ is balanced. ∎

**Proposition 12.3.2.** *Closure of a linear subspace is a linear subspace.*

*Proof.* Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. Let $\mathcal{W}$ be a linear subspace of $\mathcal{V}$. We are to prove that $\mathrm{cl}(\mathcal{W})$ is a linear subspace.

Let $x$ and $y$ be arbitrary elements of $\mathrm{cl}(\mathcal{W})$. Then there exists a net $(x_\lambda, y_\lambda)_{\lambda\in\Lambda}$ that converges to $(x, y)$. Since the addition operation $\sigma$ is continuous, we have $\lim_{\lambda\in\Lambda}(x_\lambda + y_\lambda) = x + y$. Since $\mathcal{W}$ is a linear subspace, $x_\lambda + y_\lambda \in \mathcal{W}$. So $x + y \in \mathrm{cl}(\mathcal{W})$.

Let $x$ be an arbitrary element of $\mathrm{cl}(\mathcal{W})$. Let $k$ be an arbitrary element in $\mathbb{K}$. Then there exists a net $(k\lambda, x_\lambda)_{\lambda\in\Lambda}$ that converges to $(k, x)$. Since the scalar multiplication operation $\mu$ is continuous, we have $\lim_{\lambda\in\Lambda}(k_\lambda x_\lambda) = kx$. Since $\mathcal{W}$ is a linear subspace, $k_\lambda x_\lambda \in \mathcal{W}$. So $kx \in \mathrm{cl}(\mathcal{W})$. ∎

## 12.4 Operation on Sets in a Topological Vector Space

**Proposition 12.4.1** (Stability under Linear Combinations)**.** *Let $X$ be a normed vector space over $\mathbb{F}$. Let $K$ be a compact set in the space. Let $C$ be a closed set in the space. Then $\forall \alpha, \beta \in \mathbb{F}$, the set $S$ given by $S := \alpha K + \beta C$ is closed.*

*Proof.* The case where $\beta = 0$ is trivial. I will assume $\beta \neq 0$. Let $\alpha, \beta \in \mathbb{F}$ be arbitrary. Let $\{s_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence in $S$ that converges. Say the limit is $s_\infty$. Since $s_i \in S$ for any $i \in \mathbb{N}$ and $S = \alpha K + \beta C$, $s_i = \alpha k_i + \beta c_i$ for some $k_i \in K$ and some $c_i \in C$, for any $i \in \mathbb{N}$. Since $\{k_i\}_{i\in\mathbb{N}}$ is a sequence in $K$ and $K$ is compact, there exists a convergent subsequence $\{k_i\}_{i\in I}$ of $\{k_i\}_{i\in\mathbb{N}}$ in $K$. Say $\{k_i\}_{i\in I}$ converges to $k_\infty \in K$. Since $\{s_i\}_{i\in\mathbb{N}}$ converges to $s_\infty$, $\{s_i\}_{i\in I}$ also converges to $s_\infty$. Since $s_i = \alpha k_i + \beta c_i$, $c_i = \beta^{-1}(s_i - \alpha k_i)$. Define $c_\infty := \beta^{-1}(s_\infty - \alpha k_\infty)$ Since $\{s_i\}_{i\in I}$ converges to $s_\infty$ and $\{k_i\}_{i\in I}$ converges to $k_\infty$ and $c_i = \beta^{-1}(s_i - \alpha k_i)$, $\{c_i\}_{i\in I}$ converges to $c_\infty$. Since $\{c_i\}_{i\in I}$ is a sequence in $C$ and converges to $c_\infty$ and $C$ is closed, $c_\infty \in C$. Since $s_\infty = \alpha k_\infty + \beta c_\infty$ and $k_\infty \in K$ and $c_\infty \in C$, $s_\infty \in \alpha K + \beta C$. Since for any sequence in $S$ that converges, the limit is also in $S$, $S$ is closed. ■

**Remark.** *The sum of two closed sets may not be closed.*

*Proof.* **Counter-example 1**

Consider $A := \{n : n \in \mathbb{N}\}$ and $B := \{n + \frac{1}{n} : n \in \mathbb{N}\}$.
(`https://math.stackexchange.com/questions/124130/sum-of-two-closed-sets-in-mathbb-r-is-closed`)
Their sum contains the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ but does not contain 0.

**Counter-example 2**

Consider $A := \mathbb{R} \times \{0\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x, y > 0, xy \geq 1\}$. Their sum is $\mathbb{R} \times \mathbb{R}_{++}$. ■

**Proposition 12.4.2.** *Let $\mathfrak{X}$ be a normed vector space. Let $S$ be a subset of $\mathfrak{X}$. Let $p$ be a vector in $\mathfrak{X}$. Then we have the followings.*

*(1)* $p + \mathrm{int}(S) = \mathrm{int}(p + S)$,

*(2)* $p + \mathrm{cl}(S) = \mathrm{cl}(p + S)$.

*Proof of (1).* For one direction, let $x$ be an arbitrary point in the set $p + \mathrm{int}(S)$. We are to prove that $x \in \mathrm{int}(p + S)$. Since $x \in (p + \mathrm{int}(S))$, $(x - p) \in \mathrm{int}(S)$. Since $(x - p) \in \mathrm{int}(S)$, by definition of interior, there exists a radius $r$ such that

$$B(x - p, r) \subseteq S.$$

It follows that $B(x, r) \subseteq p + S$. Since there exists a radius $r$ such that $B(x, r) \subseteq p + S$, by definition of interior,

$$x \in \mathrm{int}(p + S).$$

For the reverse direction, let $x$ be an arbitrary point in $\mathrm{int}(p+S)$. We are to prove that $x \in p + \mathrm{int}(S)$. Since $x \in \mathrm{int}(p+S)$, by definition of interior, there exists a radius $r$ such that

$$B(x, r) \subseteq (p + S).$$

It follows that $B(x-p, r) \subseteq S$. Since there exists a radius $r$ such that $B(x-p, r) \subseteq S$, by definition of interior,

$$(x - p) \in \mathrm{int}(S).$$

Since $(x - p) \in \mathrm{int}(S)$, we get $x \in (p + \mathrm{int}(S))$. ■

*Proof of (2).* For one direction, let $x$ be an arbitrary point in the set $p + \mathrm{cl}(S)$. We are to prove that $x \in \mathrm{cl}(p+S)$. Since $x \in (p+\mathrm{cl}(S))$, we get $(x-p) \in \mathrm{cl}(S)$. Since $(x-p) \in \mathrm{cl}(S)$, by definition of closure, for any radius $r$, we have

$$B(x - p, r) \cap S \neq \emptyset.$$

It follows that $B(x, r) \cap (p + S) \neq \emptyset$. Since for any radius $r$, $B(x, r) \cap (p + S) \neq \emptyset$, by definition of closure, we get

$$x \in \mathrm{cl}(p + S).$$

For the reverse direction, let $x$ be an arbitrary point in $\mathrm{cl}(p+S)$. We are to prove that $x \in (p + \mathrm{cl}(S))$. Since $x \in \mathrm{cl}(p+S)$, by definition of closure, for any radius $r$, we have

$$B(x, r) \cap (p + S) \neq \emptyset.$$

It follows that $B(x-p, r) \cap S \neq \emptyset$. Since for any radius $r$, $B(x-p, r) \cap S \neq \emptyset$, by definition of closure, we get

$$(x - p) \in \mathrm{cl}(S).$$

Since $(x - p) \in \mathrm{cl}(S)$, we get $x \in (p + \mathrm{cl}(S))$. ■

**Proposition 12.4.3.** *Let $(V, \|\cdot\|)$ be a normed vector space. Let $S$ be a subset of $V$. Let $\lambda$ be a non-zero real number. Then*

*(1)* $\lambda int(S) = int(\lambda S)$.

*(2)* $\lambda\,\mathrm{cl}(S) = \mathrm{cl}(\lambda S)$.

*Proof of (1).* For one direction, let $x$ be an arbitrary point in $\lambda int(S)$. We are to prove that $x \in int(\lambda S)$. Since $x \in \lambda int(S)$, we get $x/\lambda \in int(S)$. Since $x/\lambda \in int(S)$, by definition of interior, there exists a radius $r$ such that

$$B(x/\lambda, r) \subseteq S.$$

Let $y$ be an arbitrary point in $B(x, \lambda r)$. Since $y \in B(x, \lambda r)$, we get $\|y - x\| \leq \lambda r$. Since $\|y - x\| \leq \lambda r$, we get $\|y/\lambda - x/\lambda\| \leq r$. Since $\|y/\lambda - x/\lambda\| \leq r$, we get $y/\lambda \in B(x/\lambda, r)$. Since $y/\lambda \in B(x/\lambda, r)$ and $B(x/\lambda, r) \subseteq S$, we get $y/\lambda \in S$. Since $y/\lambda \in S$, we get $y \in \lambda S$. Since any point in $B(x, \lambda r)$ is also in $\lambda S$, we get $B(x, \lambda r) \subseteq \lambda S$. Since there exists a radius $r$ such that $B(x, \lambda r) \subseteq \lambda S$, by definition of interior, we get

$$x \in int(\lambda S).$$

■

## 12.5 Neighborhood Improvements

**Proposition 12.5.1.** *Let $(\mathcal{V}, \tau)$ be a topological vector space. Let $U \in \mathcal{U}_0$ be a neighborhood of 0 in $\mathcal{V}$. Then*

- $\exists N \in \mathcal{U}_0$ *such that* $N + N \subseteq U$.
- $\exists M \in \mathcal{U}_0$ *and* $\exists \varepsilon > 0$ *such that* $\forall 0 < |k| < \varepsilon$, *we have* $kM \subseteq U$.
-

## 12.6 Cauchy Nets

**Definition** (Cauchy Net)**.** *Let $(\mathcal{V}, \tau)$ be a topological vector space. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in $\mathcal{V}$. We say that $(x_\lambda)_{\lambda \in \Lambda}$ is a* ***Cauchy net*** *if $\forall U \in \mathcal{U}_0$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda_1, \lambda_2 \geq \lambda_0$, we have $x_{\lambda_1} - x_{\lambda_2} \in U$.*

**Definition** (Cauchy Complete)**.** *Let $(\mathcal{V}, \tau)$ be a topological vector space. We say that $\mathcal{V}$ is* ***Cauchy complete*** *if every Cauchy net in $\mathcal{V}$ converges in $\mathcal{V}$.*

**Proposition 12.6.1.** *Convergent nets are Cauchy.*

*Proof.* Let $\mathcal{V}$ be a topological vector space. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a convergent net with limit point $x$. Let $U$ be an arbitrary element in $\mathcal{U}_0$. Let $N$ be an element in $\mathcal{U}_0$ that is balanced and open and that $N - N \subseteq U$. Since $\lim_{\lambda \in \Lambda} x_\lambda = x$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0$, $x_\lambda - x \in N$. Let $\lambda_1$ and $\lambda_2$ be arbitrary elements that are $\geq \lambda_0$. Then

$$x_{\lambda_1} - x_{\lambda_2} = (x_{\lambda_1} - x) - (x_{\lambda_2} - x) \in N - N \subseteq U.$$

That is, $\forall U \in \mathcal{U}_0$, $\exists \lambda_0$ such that $\forall \lambda_1, \lambda_2 \geq \lambda_0$, $x_{\lambda_1} - x_{\lambda_2} \in U$. So $(x_\lambda)_{\lambda \in \Lambda}$ is Cauchy. ■

## 12.7 Sublinear Functionals

**Definition** (Sublinear Functional)**.** *Let $\mathcal{V}$ be a vector space over field $\mathbb{K}$. Let $f$ be a function from $\mathcal{V}$ to $\mathbb{R}$. We say that $f$ is* ***sublinear*** *if it satisfies:*

- *Subadditivity:*
$$\forall x, y \in \mathcal{V}, \quad f(x+y) \leq f(x) + f(y).$$
- *Positive Homogeneity:*
$$\forall x \in \mathcal{V}, \forall \lambda \geq 0, \quad f(\lambda x) = \lambda f(x).$$

## 12.8 Finite-Dimensional Topological Vector Spaces

**Proposition 12.8.1.** *Let $\mathcal{V}$ be an $n$-dimensional topological vector space. Then $\mathcal{V}$ is homeomorphic to $\mathbb{K}^n$ via the map*
$$\sum_{i=1}^{n} k_i e_i \mapsto (k_i)_{i=1}^{n}.$$

**Corollary.** *Let $\mathcal{V}$ be a finite-dimensional vector space. Then there is a unique topology $\mathcal{T}$ which makes $\mathcal{V}$ a topological vector space.*

# Chapter 13

# Seminorms and Locally Convex Spaces

## 13.1 Locally Convex

**Definition** (Locally Convex Space)**.** *Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. We say that $\mathcal{T}$ is* ***locally convex*** *if it admits a base consisting of only convex sets.*

**Proposition 13.1.1.** *Let $(\mathcal{V}, \mathcal{T})$ be a locally convex topological vector space. Let $\mathcal{W}$ be a closed subspace of $\mathcal{V}$. Then $\mathcal{V}/\mathcal{W}$ is a locally convex topological vector space in the quotient topology.*

*Proof.* Clearly $\mathcal{V}/\mathcal{W}$ is a topological vector space. It suffices to show that $\mathcal{V}/\mathcal{W}$ admits a neighborhood base at 0 consisting of only convex sets. Let $q := \mathcal{V} \to \mathcal{V}/\mathcal{W}$ denote the canonical quotient map. Then $q$ is linear, continuous and open. Let $U$ be an arbitrary element in $\mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$. Then $q^{-1}(U) \in \mathcal{U}_0^{\mathcal{V}}$. Since $\mathcal{V}$ is locally convex, $\exists N \in \mathcal{U}_0^{\mathcal{V}}$ that is convex and that $N \subseteq q^{-1}(U)$. Define a set $M$ as $M := q(N)$. Since $q$ is open and $N \in \mathcal{U}_0^{\mathcal{V}}$, we have $M \in \mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$. Since $q$ is linear and $N$ is convex, $M$ is convex. Since $N \subseteq q^{-1}(U)$, $M \subseteq U$. So $\forall U \in \mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$, $\exists M \in \mathcal{U}_0^{\mathcal{V}/\mathcal{W}}$ that is convex and that $M \subseteq U$. So $\mathcal{V}/\mathcal{W}$ admits a neighborhood base at 0 consisting of only convex sets. ∎

## 13.2 Separating Family of Seminorms

**Definition** (Separating Family of Seminorms)**.** *Let $\mathcal{V}$ be a vector space. Let $\Gamma$ be a family of seminorms on $\mathcal{V}$. We say that $\Gamma$ is* ***separating*** *if $\forall x \in \mathcal{V}$ such that $x \neq 0$, $\exists p \in \Gamma$ such that $p(x) \neq 0$.*

**Theorem 8.** *Let $\mathcal{V}$ be a vector space. Let $\Gamma$ be a separating family of seminorms on $\mathcal{V}$. Define a set $\mathcal{B}$ as*

$$\mathcal{B} := \{N(x, F, \varepsilon) : x \in \mathcal{V}, \varepsilon > 0, F \subseteq \mathcal{F} \text{ is finite } \}$$

*where $N(x, F, \varepsilon)$ is defined as*

$$N(x, F, \varepsilon) := \{y \in \mathcal{V} : \forall p \in F, p(x - y) < \varepsilon\}.$$

*Then $\mathcal{B}$ is a base for a locally convex topology $\mathcal{T}$ on $\mathcal{V}$. Moreover, each $p \in \Gamma$ is continuous.*

**Theorem 9.** *Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space. Then there exists a separating family $\Gamma$ of seminorms on $\mathcal{V}$ that can generate $\mathcal{T}$.*

**Example 13.2.1.** *The norm topology is exactly the locally convex topology generated by $\Gamma = \{\| \cdot \|\}$.*

## 13.3 Strong Operator Topology

## 13.4 Weak Operator Topology

# Chapter 14

# Equicontinuity in Metric Spaces

## 14.1 Definitions

**Definition** ((Pointwise) Equicontinuity)**.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $\mathcal{F}$ be a collection of functions from $X$ to $Y$. Let $x_0$ be a point in $X$. We say that $\mathcal{F}$ is **(pointwise) equicontinuous** at point $x_0$ if for any positive number $\varepsilon$, there exists some number $\delta(x_0, \varepsilon)$ such that for any function $f$ in $\mathcal{F}$ and any point $x$ in $X$, we have*

$$d_Y(f(x), f(x_0)) < \varepsilon$$

*whenever $d_X(x, x_0) < \delta(x_0, \varepsilon)$ is satisfied.*

**Definition** (Uniform Equicontinuity)**.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $\mathcal{F}$ be a collection of functions from $X$ to $Y$. We say that $\mathcal{F}$ is **uniformly equicontinuous** if for any positive number $\varepsilon$, there exists some number $\delta(\varepsilon)$ such that for any function $f$ in $\mathcal{F}$ and any points $x_1$ and $x_2$ in $X$, we have*

$$d_Y(f(x_1), f(x_2)) < \varepsilon$$

*whenever $d_X(x_1, x_2) < \delta(\varepsilon)$ is satisfied.*

## 14.2 Sufficient Conditions

**Proposition 14.2.1.** *The closure of an equicontinuous family of functions is equicontinuous.*

*Proof.*

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces.

Let $\mathcal{F}$ be an equicontinuous family of functions from $X$ to $Y$.

We are to prove that $cl(\mathcal{F})$ is equicontinuous.

Let $x_0$ be an arbitrary point in $X$.

Let $\varepsilon$ be an arbitrary positive number.

Since $\mathcal{F}$ is equicontinuous at point $x_0$, there exists some $\delta(x_0, \varepsilon)$ such that for any function $f$ in $\mathcal{F}$ and any point $x$ in $X$ such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon/3$.

Let $f$ be an arbitrary function in $cl(\mathcal{F})$.

Let $x$ be an arbitrary point in $X$ such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$.

Since $f \in cl(\mathcal{F})$, there exists some function $f_0 \in \mathcal{F}$ such that $d_\infty(f, f_0) < \varepsilon/3$.

Since $d_\infty(f, f_0) < \varepsilon/3$, $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$.

Since $f_0 \in \mathcal{F}$ and $d_X(x, x_0) < \delta(x_0, \varepsilon)$, $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$.

Since $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$ and $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$, $d_Y(f(x), f(x_0)) < \varepsilon$.

Since for any positive number $\varepsilon$, there exists some $\delta(x_0, \varepsilon)$ such that for any function $f$ in $cl(\mathcal{F})$ and any point $x$ in $X$ such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, by definition of equicontinuous, $cl(\mathcal{F})$ is equicontinuous at point $x_0$.

Since $cl(\mathcal{F})$ is equicontinuous at point $x_0$ for any point $x_0$ in $X$, $cl(\mathcal{F})$ is equicontinuous.

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# Chapter 15

# Adjoint Operator

## 15.1 Definitions

**Definition** (Adjoint Matrix). *Let $A$ be an $m \times n$ matrix. We define the **adjoint** of $A$, denoted by $A^*$, to be an $n \times m$ matrix given by*

$$(A^*)_{ij} := \overline{(A)_{ji}}.$$

**Definition** (Adjoint Operator). *Let $V$ and $W$ be inner product spaces. Let $T$ be a linear map from $V$ to $W$. We define the **adjoint** of $T$, denoted by $T^*$, to be a map from $W$ to $V$ such that*

$$\forall x \in V, \forall y \in W, \quad \langle T(x), y \rangle_W = \langle x, T^*(y) \rangle_V.$$

**Proposition 15.1.1** (Existence). *Let $V$ be a finite-dimensional inner product space and $T$ be a linear operator on $V$. Then the adjoint of $T$ exists.*

**Proposition 15.1.2** (Uniqueness). *Let $V$ be an inner product space and $T$ be a linear operator on $V$. Then the adjoint of $T$ is unique, provided that it exists.*

## 15.2 Properties of the Adjoint Operator

**Proposition 15.2.1.** *Let $V$ be an inner product space. Then*

*(1)* $(I_V)^* = I_V$ *where $I_V$ is the identity operator on $V$.*

*(2)* $T^{**} = T$ *for any linear operator $T$ on $V$.*

**Proposition 15.2.2.** *Let $V$ be an inner product space and $T$ be a linear operator on $V$. Then $T^*$ is also linear.*

**Proposition 15.2.3.** *Let $V$ be an inner product space. Then*

*(1) For any linear operators $T$ and $U$,*

$$(T+U)^* = T^* + U^*.$$

*(2) For any linear operator $T$,*

$$(cT)^* = \overline{c} \cdot T^*.$$

*(3) For any linear operator $T$ and $U$,*

$$(TU)^* = U^* T^*.$$

**Proposition 15.2.4.** *Let $V$ be a finite-dimensional inner product space and $T$ be a linear operator on $V$. Then if $T$ is invertible, $T^*$ is also invertible.*

**Proposition 15.2.5.** *Let $V$ be an inner product space and $T$ be an invertible linear operator on $V$. Then $(T^{-1})^* = (T^*)^{-1}$.*

## 15.3 Normal Operators

**Definition** (Normal)**.** *Let $V$ be an inner product space and $T$ be a linear operator on $V$. We say that $T$ is* ***normal*** *if $TT^* = T^*T$.*

## 15.4 Self-adjoint

# Chapter 16

# Convolution

**Definition** (Convolution)**.** *Let $f$ and $g$ be functions from $\mathbb{R}$ to $\mathbb{R}$. We define the **convolution** of $f$ and $g$, denoted by $f * g$, to be a function on $\mathbb{R}$ given by*

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)dt.$$

# Chapter 17

# Coercive Functions

## 17.1 Definitions

**Definition** (Coercive)**.** *Let $f$ be a function from $\mathbb{R}^d$ to $\mathbb{R}^*$. We say that $f$ is* ***coercive*** *if $\lim_{\|x\|\to\infty} f(x) = +\infty$.*

## 17.2 Properties

**Proposition 17.2.1.** *Let $f$ be a proper lower semi-continuous function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let $K$ be a compact set in $\mathbb{R}^d$. Assume $K \cap \mathrm{dom}(f) \neq \emptyset$. Then $f$ attains its minimum over $K$.*

*Proof.*

Define $m := \inf_{x\in K} f(x)$.

Since $m = \inf_{x\in K} f(x)$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in $K$ such that $\lim_{i\to\infty} f(x_i) = m$.

Since $K$ is compact and $\{x_i\}_{i\in\mathbb{N}} \subseteq K$, there exists a convergent subsequence $\{x_i\}_{i\in I}$ in $K$ where $I$ is an infinite subset of $\mathbb{N}$.

Say the limit is $x_\infty$ where $x_\infty \in K$.

Since $\lim_{i\to\infty} f(x_i) = m$, we get $\lim_{i\in I, i\to\infty} f(x_i) = m$.

Since $\lim_{i\in I, i\to\infty} f(x_i) = m$, we get $\liminf_{i\in I, i\to\infty} f(x_i) = m$.

Since $f$ is lower semi-continuous and $\lim_{i\in I, i\to\infty} x_i = x_\infty$, we get $f(x_\infty) \leq \liminf_{i\in I, i\to\infty} x_i$.

That is, $f(x_\infty) \leq m$.

Since $m = \inf_{x\in K} f(x)$, we have $\forall x \in K, f(x) \geq m$.

In particular, $f(x_\infty) \geq m$.

Since $f(x_\infty) \geq m$ and $f(x_\infty) \leq m$, $f(x_\infty) = m$.

Since $f$ is proper, $f(x_\infty) = m \neq -\infty$.

So $f$ attains its minimum at point $x_\infty$.

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**Proposition 17.2.2.** *Let $f$ be a proper, lower semi-continuous, and coercive function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let $C$ be a closed subset of $\mathbb{R}^d$. Assume $C \cap \mathrm{dom}(f) \neq \emptyset$. Then $f$ attains its minimum over $C$.*

*Proof.*

Since $C \cap \mathrm{dom}(f) \neq \emptyset$, take $x \in C \cap \mathrm{dom}(f)$.

Since $f$ is coercive, $\exists R$ such that $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$.

Since $x \in C \cap \mathrm{dom}(f)$ and $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$, the set of minimizers of $f$ over $C$ is the same as the set of minimizers of $f$ over $C \cap \mathrm{ball}[0, R]$.

Since $C$ and $\mathrm{ball}[0, R]$ are both closed, $C \cap \mathrm{ball}[0, R]$ is closed.

Since $\mathrm{ball}[0, R]$ is bounded, $C \cap \mathrm{ball}[0, R]$ is bounded.

Since $C \cap \mathrm{ball}[0, R]$ is closed and bounded, by the Heine-Borel Theorem, $C \cap \mathrm{ball}[0, R]$ is compact.

Since $f$ is proper and lower semi-continuous and $C \cap \mathrm{ball}[0, R]$ is compact, $f$ attains its minimum over $C \cap \mathrm{ball}[0, R]$.

So $f$ attains its minimum over $C$.

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# Chapter 18

# Unclassified Results

**Proposition 18.0.1.** *Let $(X, d)$ be a compact metric space. Let $L(X)$ be the set of all Lipschitz functions from $X$ to $\mathbb{R}$. Let $C(X)$ be the set of all continuous functions from $X$ to $\mathbb{R}$. Then $L(X)$ is dense in $C(X)$.*