

Semidefinite Optimization

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Chapter 1

First Chapter

1.1 Primal Problem and Dual Problem

DEFINITION (Primal Problem). Let $C \in \mathbb{S}^n$. Let $b \in \mathbb{R}^m$. Let \mathcal{A} be a linear transformation from \mathbb{S}^n to \mathbb{R}^m . We define the **primal problem** to be the following.

$$\begin{aligned} \text{(P)} \quad & \inf \quad \langle C, X \rangle \\ & \text{subject to} \quad \mathcal{A}X = b \\ & \quad \quad \quad X \succeq 0 \end{aligned}$$

DEFINITION (Dual Problem). We define the **dual problem** of the above primal problem to be the following.

$$\begin{aligned} \text{(D)} \quad & \sup \quad b^\top y \\ & \text{subject to} \quad \mathcal{A}^*y + S = C \\ & \quad \quad \quad S \succeq 0 \end{aligned}$$

where \mathcal{A}^* denotes the adjoint of \mathcal{A} .

1.2 Weak Duality

THEOREM 1.1 (The Weak Duality Relation). Let \bar{X} be feasible in (P). Let (\bar{y}, \bar{S})

be feasible in (D). Then

$$\langle C, \bar{X} \rangle - b^\top \bar{y} \geq 0.$$

Proof. Since (\bar{y}, \bar{S}) is feasible, we have

$$C = \mathcal{A}^* y + S. \quad (1)$$

Since \bar{X} is feasible, we have

$$\mathcal{A}X = b. \quad (2)$$

Then

$$\begin{aligned} \langle C, \bar{X} \rangle - b^\top \bar{y} &= \langle \mathcal{A}^* y + S, \bar{X} \rangle - b^\top \bar{y}, \text{ by equation (1)} \\ &= \langle \mathcal{A}^* \bar{y}, \bar{X} \rangle + \langle S, \bar{X} \rangle - b^\top \bar{y}, \text{ by linearity} \\ &= \langle \bar{y}, \mathcal{A} \bar{X} \rangle + \langle \bar{S}, \bar{X} \rangle - b^\top \bar{y}, \text{ by definition of adjoint} \\ &= \langle \bar{y}, b \rangle + \langle \bar{S}, \bar{X} \rangle - b^\top \bar{y}, \text{ by equation (2)} \\ &= \langle \bar{S}, \bar{X} \rangle \\ &\geq 0, \text{ since } X, S \succeq 0. \end{aligned}$$

That is, $\langle C, \bar{X} \rangle - b^\top \bar{y} \geq 0$. ■

COROLLARY 1.1.

- (1) If (P) is unbounded, then (D) is infeasible.
- (2) If (D) is unbounded, then (P) is infeasible.
- (3) If \bar{X} and (\bar{y}, \bar{S}) are feasible in (P) and (D) respectively and $\langle C, \bar{X} \rangle = b^\top \bar{y}$, then \bar{X} is optimal in (P) and (\bar{y}, \bar{S}) is optimal in (D).

Chapter 2

Complementary Slackness

PROPOSITION 2.0.1 (Complementary Slackness). Let $X, S \in \mathbb{S}_+^n$. Then $\langle X, S \rangle = 0$ if and only if $XS = 0$.