Matrix Theory

Daniel Mao

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Chapter 1

Fundamentals

1.1 Definitions

DEFINITION (Column Space)**.** Let A be an $m \times n$ matrix. We define the **column space** of A, denoted by $\operatorname{col}(A)$, to be the set given by

$$\operatorname{col}(A) := \{Av : v \in \mathbb{R}^n\}.$$

DEFINITION (Row Space)**.** Let A be an $m \times n$ matrix. We define the **row space** of A, denoted by $\operatorname{row}(A)$, to be the set given by

$$\operatorname{row}(A) := \{A^\top v : v \in \mathbb{R}^m\}.$$

DEFINITION (Nullspace)**.** Let A be an $m \times n$ matrix. We define the **nullspace** of A, denoted by $\operatorname{null}(A)$, to be the set given by

$$\operatorname{null}(A) := \{v \in \mathbb{R}^n : Av = \mathbf{0}\}.$$

DEFINITION (Left Nullspace)**.** Let A be an $m \times n$ matrix. We define the **left**

nullspace of A, denoted by $\mathrm{null}(A^\top)$, to be the set given by

$$\mathrm{null}(A^\top) := \left\{ v \in \mathbb{R}^m : A^\top v = \mathbf{0} \right\}.$$

1.2 Main Results

THEOREM 1.1 (The Fundamental Theorem of Linear Algebra)**.** Let A be an $m \times n$ matrix. Then $\mathrm{col}(A)^\perp = \mathrm{null}(A^\top)$ and $\mathrm{row}(A)^\perp = \mathrm{null}(A)$.

Chapter 2

Rank

2.1 Definitions

DEFINITION (Column Rank)**.** Let A be a matrix. We define the **column rank** of A to be the dimension of the column space of A. i.e.

$$\operatorname{colrank}(A) := \dim(\operatorname{col}(A)).$$

DEFINITION (Row Rank)**.** Let A be a matrix. We define the **row rank** of A to be the dimension of the row space of A. i.e.

$$\operatorname{rowrank}(A) := \dim(\operatorname{row}(A)).$$

DEFINITION (Rank)**.** Let A be a matrix. Then the column rank and the row rank are the same. We define the **rank** of A to be this common number.

DEFINITION (Full Rank)**.** Let A be an $m \times n$ matrix. We say that A has **full rank** if $\operatorname{rank}(A) = \min\{m, n\}$.

2.2 Properties

PROPOSITION 2.2.1. Let A be an $m \times n$ matrix. Then

- A is injective if and only if A has full column rank. i.e. $\operatorname{rank}(A) = n$, and
- A is surjective if and only if A has full row rank. i.e. $\operatorname{rank}(A) = m$.

PROPOSITION 2.2.2. Let A and B be matrices with appropriate dimensions. Then

$$\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

PROPOSITION 2.2.3. Let A, B, and C be matrices with appropriate dimensions. Then

- If B has full row rank, then $\operatorname{rank}(AB) = \operatorname{rank}(A)$, and
- If C has full column rank, then $\operatorname{rank}(CA) = \operatorname{rank}(A)$.

PROPOSITION 2.2.4 (Subadditivity)**.** Let A and B be matrices with appropriate dimensions. Then

$$\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

PROPOSITION 2.2.5. Let A be a matrix over $\mathbb{C}$. Let A^- denote the complex conjugate of A. Let $A^\top$ denote the transpose of A. Let A^* denote the conjugate transpose of A. Then

$$\operatorname{rank}(A) = \operatorname{rank}(A^-) = \operatorname{rank}(A^\top) = \operatorname{rank}(A^*) = \operatorname{rank}(AA^*) = \operatorname{rank}(A^*A).$$

Chapter 3

Matrix Inverse

3.1 Definitions

DEFINITION (Invertible)**.** Let A be an $n \times n$ matrix over $\mathbb{C}$. We say that A is **invertible** if there exists another $n \times n$ matrix B over $\mathbb{C}$ such that $AB = BA = I_n$.

PROPOSITION 3.1.1. Let A be an $n \times n$ invertible matrix over $\mathbb{C}$. Then the $n \times n$ matrix B over $\mathbb{C}$ satisfying $AB = BA = I_n$ is unique.

DEFINITION (Inverse)**.** Let A be an $n \times n$ matrix over $\mathbb{C}$. We define the **inverse** of A, denoted by A^{-1}, to be the unique $n \times n$ matrix over $\mathbb{C}$ satisfying $AA^{-1} = A^{-1}A = I_n$.

DEFINITION (Left/Right Inverse)**.** Let A be an $m \times n$ matrix over $\mathbb{C}$. We define

- the **left inverse** of A, to be an $n \times m$ matrix B over $\mathbb{C}$ such that $BA = I_n$.
- the **right inverse** of A, to be an $n \times m$ matrix B over $\mathbb{C}$ such that $AB = I_n$.

3.2 Characterization

PROPOSITION 3.2.1. Let A be an $n \times n$ matrix over field K. Then the following statements are equivalent.

- A is invertible.
- $\dim(\text{row}(A)) = n$.
- $\dim(\text{col}(A)) = n$.
- $\dim(\text{null}(A)) = 0$.

PROPOSITION 3.2.2. Let A be an $n \times n$ matrix over field K. Then the following statements are equivalent.

- A is invertible.
- A is row-equivalent to I_n.
- A is column-equivalent to I_n.
- A can be written as a finite product of elementary matrices.

PROPOSITION 3.2.3. Let A be an $n \times n$ matrix over field K. Then A is invertible if and only if $\det(A) \neq 0$.

PROPOSITION 3.2.4. Let A be an $n \times n$ matrix over field K. Then A is invertible if and only if 0 is not an eigenvalue of A.

3.3 Arithmetic Properties

PROPOSITION 3.3.1. Let A be an invertible matrix. Then

- $(A^{-1})^{-1} = A$.
- $(kA)^{-1} = k^{-1}A^{-1}$.

- $(AB)^{-1} = B^{-1}A^{-1}$.
- $(A^T)^{-1} = (A^{-1})^T$.

3.4 Pseudo-Inverse

DEFINITION (Moore-Penrose Pseudo-Inverse)**.** Let A be an $n \times d$ matrix. We define the **Moore-Penrose pseudo-inverse** of A, denoted by $A^{\dagger}$, to be a $d \times n$ matrix G such that

$$AGA = A, \quad GAG = G, \quad (AG)^{\top} = AG, \quad (GA)^{\top} = GA.$$

Chapter 4

Determinant

4.1 Definitions

DEFINITION (Cofactor)**.** Let M be an $n \times n$ matrix where $n \geq 2$. We define the $(i, j)^{\text{th}}$ **cofactor** of M, denoted by $C_{i,j}(M)$, to be a number given by

$$C_{i,j}(M) := (-1)^{i+j} \det(M(i, j))$$

where $M(i, j)$ denotes the submatrix obtained from M by removing the i^{th} row and the j^{th} column.

DEFINITION (Determinant)**.** Let M be an $n \times n$ matrix where $n \geq 2$. We define the **determinant** of M, denoted by $\det(M)$, to be a number given by

$$\det(M) := \sum_{i=1}^{n} [M]_{i,j} C_{i,j}(M)$$

where j can be anything in $\{1, ..., n\}$, $[M]_{i,j}$ denotes the $(i, j)^{\text{th}}$ entry of M, and $C_{i,j}(M)$ denotes the $(i, j)^{\text{th}}$ cofactor of M. Equivalently,

$$\det(M) := \sum_{j=1}^{n} [M]_{i,j} C_{i,j}(M)$$

where i can be anything in $\{1, ..., n\}$, $[M]_{i,j}$ denotes the $(i, j)^{\text{th}}$ entry of M, and $C_{i,j}(M)$ denotes the $(i, j)^{\text{th}}$ cofactor of M.

We define the determinant of an 1×1 matrix to be the number itself.

4.2 Properties

PROPOSITION 4.2.1. Let A be a matrix. Then

$$\det(A^\top) = \det(A).$$

PROPOSITION 4.2.2. Let A and B be positive semi-definite matrices with appropriate dimensions. Then

$$\det(A + B) \geq \det(A) + \det(B).$$

PROPOSITION 4.2.3. Let A be an $n \times n$ matrix. Let c be some scalar. Then

$$\det(cA) = c^n \det(A).$$

PROPOSITION 4.2.4. Let A be an invertible matrix. Then

$$\det(A^{-1}) = \det(A)^{-1}.$$

PROPOSITION 4.2.5. Let A and B be matrices with appropriate dimensions. Then

$$\det(AB) = \det(A)\det(B).$$

PROPOSITION 4.2.6. The determinant operator is a multi-linear operator on the rows/columns.

4.3 Adjoint of a Matrix

DEFINITION (Adjoint)**.** Let M be an $n \times n$ matrix. We define the **adjoint** of M, denoted by adj(M), to be an $n \times n$ matrix given by

$$(\mathrm{adj}(M))_{ij} = C_{ji}(M),$$

for $i, j = 1, ..., n$.

PROPOSITION 4.3.1. Let M be an $n \times n$ matrix. Then

$$M \,\mathrm{adj}(M) = \mathrm{adj}(M)M = \det(M)I_n.$$

Chapter 5

Trace

DEFINITION. Let A be a square matrix. We define the trace of A, denoted by $\mathrm{tr}(A)$, to be the sum of the entries on the main diagonal of A.

5.1 Basic Properties

PROPOSITION 5.1.1. Trace is a linear operator.

PROPOSITION 5.1.2. The trace of an idempotent matrix is equal to its rank.

PROPOSITION 5.1.3. The trace of a matrix equals the sum of its eigenvalues.

5.2 Invariant Properties

PROPOSITION 5.2.1 (Transpose Invariant)**.** Let $M \in \mathbb{C}^{n\times n}$. Then we have

$$\mathrm{tr}(M) = \mathrm{tr}(M^\top).$$

PROPOSITION 5.2.2 (Cyclical Permutation Invariant)**.** Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then we have
$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

PROPOSITION 5.2.3 (Similarity Invariant)**.** If A is similar to B, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.

Chapter 6

Matrix Norm

DEFINITION. $\|A\| := \sup_{\|x\|=1} \|Ax\|$

6.1 Properties

PROPOSITION 6.1.1. Let A be an $n \times n$ matrix. Then if A is symmetric, we have

$$\|A\| = \max\{\lambda_i\}_{i=1}^n$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues of A.

Chapter 7

Eigenvalues and Eigenvectors

7.1 Definitions

DEFINITION (Eigenvalue and Eigenvector)**.** Let A be a matrix. Let x be a vector. Let λ be a scalar. We say that x is an **eigenvector** of A and that λ is an **eigenvalue** of A if $x \neq 0$ and

$$Ax = \lambda x.$$

7.2 Properties

PROPOSITION 7.2.1. Let A be an invertible matrix. Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of A. Then the eigenvalues of A^{-1} are $\{\lambda_i^{-1}\}_{i=1}^n$.

Proof.

$$\begin{aligned} & Av = \lambda v \\ \iff & A^{-1}Av = A^{-1}\lambda v \\ \iff & v = \lambda A^{-1} v \\ \iff & A^{-1} v = \lambda^{-1} v. \end{aligned}$$

■

PROPOSITION 7.2.2. Let A be an invertible matrix. Let $\{x_i\}_{i=1}^n$ be the eigenvectors of A. Then the eigenvectors of A^{-1} are also $\{x_i\}_{i=1}^n$.

PROPOSITION 7.2.3. Let A be a matrix. Let n be a positive integer. Let (x, λ) be an eigenpair of A. Then

$$A^n x = \lambda^n x.$$

Proof. I will prove by induction on n.

Base Case: $n = 1$.

This is to prove that $Ax = \lambda x$. This holds since (x, λ) is an eigenpair of A.

Inductive Step:

Assume that $A^n x = \lambda^n x$ for some $n \in \mathbb{N}$. We are to prove that $A^{n+1}x = \lambda^{n+1}x$.

$$\begin{aligned} A^{n+1}x &= A^n Ax \\ &= A^n \lambda x \\ &= \lambda A^n x \\ &= \lambda\lambda^n x \text{ by the inductive hypothesis} \\ &= \lambda^{n+1}x. \end{aligned}$$

That is,

$$A^{n+1}x = \lambda^{n+1}x.$$

Summary:

By the principle of mathematical induction,

$$\forall n \in \mathbb{N}, \quad A^n x = \lambda^n x.$$

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PROPOSITION 7.2.4. If a square matrix is idempotent, then its eigenvalues are either 0 or 1.

Proof. Since A is idempotent, by definition, $A^2 = A$. Let (x, λ) be an arbitrary eigenpair of A. Then

$$Ax = \lambda x \text{ and } A^2 x = \lambda^2 x.$$

Since $A^2 = A$ and $A^2x = \lambda^2 x$, we get $Ax = \lambda^2 x$. Since $Ax = \lambda x$ and $Ax = \lambda^2 x$, we get $\lambda x = \lambda^2 x$. Since x is an eigenvector of A, $x \neq 0$. Since $\lambda x = \lambda^2 x$ and $x \neq 0$, we get $\lambda \in \{0, 1\}$. ■

7.3 Eigenspace

DEFINITION (Eigenspace)**.** Let A be an $m \times n$ matrix over field $\mathbb{F}$. Let λ be an eigenvalue of A. We define the **eigenspace** of A, associated with λ, denoted by E_λ, to be a set given by

$$E_\lambda := \{v \in \mathbb{F}^n : Av = \lambda v\}.$$

i.e., E_λ is the set of all eigenvectors of A with eigenvalue λ and the zero vector.

PROPOSITION 7.3.1. Eigenspaces are linear subspaces.

Chapter 8

Singular Values and Singular Vectors

DEFINITION (Singular Value, Singular Vector)**.** Let M be an $m \times n$ real or complex matrix. We define a **singular value** for M to be a non-negative real number σ such that there exist unit vectors $u \in \mathbb{F}^m$ and $v \in \mathbb{F}^n$ such that $Mv = \sigma u$ and $M^*u = \sigma v$. We call u the **left-singular vector** for σ and v the **right-singular vector** for σ.

Chapter 9

Special Types of Matrices

9.1 Elementary Matrices

PROPOSITION 9.1.1. The inverse of an elementary matrix can be obtained by multiplying its off-diagonal entries by -1.

Unconfirmed...

9.2 Triangular Matrix

DEFINITION (Upper Triangular Matrix)**.**

DEFINITION (Lower Triangular Matrix)**.**

PROPOSITION 9.2.1. The product of two upper triangular matrices is also upper triangular. i.e. if U_1 and U_2 are upper triangular matrices with appropriate dimensions, then $U := U_1U_2$ is also upper triangular.

PROPOSITION 9.2.2. The inverse of an upper triangular matrix is also upper triangular, if it exists. i.e. if U is an invertible upper triangular matrix, then U^{-1} is also upper triangular.

9.3 Symmetric and Hermitian Matrices

9.3.1 Equivalent Conditions

DEFINITION (Symmetric Matrix)**.** Let $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ (a real square matrix). We say that M is **symmetric** if

$$M = M^\top$$

where $M^\top$ denotes the transpose of M.

DEFINITION (Hermitian Matrix - 1)**.** Let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$ (a complex square matrix). We say that M is **Hermitian**, or **self-adjoint**, if

$$M = M^*$$

where M^* denotes the conjugate transpose of M.

DEFINITION (Hermitian Matrix - 2)**.** Let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$ (a complex square matrix). We say that M is **Hermitian**, or **self-adjoint**, if

$$\forall x, y \in \mathbb{C}^n, \quad \langle x, Ay\rangle = \langle Ax, y\rangle .$$

DEFINITION (Hermitian Matrix - 3)**.** Let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$ (a complex square matrix). We say that M is **Hermitian**, or **self-adjoint**, if

$$\forall x \in \mathbb{C}^n, \quad \langle x, Ax\rangle \in \mathbb{R}.$$

9.3.2 Stability of Hermitian Matrices

PROPOSITION 9.3.1 (Sum of Two Hermitian Matrices)**.** Let A and B be Hermitian matrices. Then $A + B$ is also Hermitian.

PROPOSITION 9.3.2 (Associative Product)**.** Let A and B be Hermitian matrices. Suppose that $AB = BA$. Then AB is also Hermitian.

PROPOSITION 9.3.3 (Inverse of a Hermitian Matrix)**.** Let M be a Hermitian matrix. Suppose that M is invertible. Then M^{-1} is also Hermitian.

9.3.3 Properties of Hermitian Matrices

PROPOSITION 9.3.4. Hermitian matrices are normal.

PROPOSITION 9.3.5. The determinant of a Hermitian matrix is real.

Proof. Let M be a Hermitian matrix. Then

$$\det(M) = \det(M^*) = \det(\overline{M}^\top) = \det(\overline{M}) = \overline{\det(M)}.$$

That is, $\det(M) = \overline{\det(M)}$. So $\det(M) \in \mathbb{R}$. ■

PROPOSITION 9.3.6 (Eigenvalues)**.** The eigenvalues of a Hermitian matrix are all real.

Proof Approach 1. Let A be a Hermitian matrix. Let (λ, v) be an arbitrary eigenpair of A. Then we have $Av = \lambda v$ and hence

$$v^* A v = v^* \lambda v = \lambda v^* v. \tag{1}$$

Note that v^*Av has size 1×1. So $v^*Av = [a]$ for some $a \in \mathbb{C}$.

$$(v^*Av)^* = v^*A^*v^{**} = v^*Av$$

$$\begin{aligned}&\implies v^*Av \text{ is Hermitian} \iff [a] \text{ is Hermitian}\\&\implies a=\bar{a} \implies a\in\mathbb{R}.\end{aligned}$$

That is,

$$v^*Av = a \in \mathbb{R}. \tag{2}$$

Note that v^*v has size 1×1. So $v^*v = [b]$ for some $b \in \mathbb{C}$.

$$\begin{aligned}&(v^*v)^* = v^*v^{**} = v^*v\\&\implies v^*v \text{ is Hermitian} \iff [b] \text{ is Hermitian}\\&\implies b=\bar{b} \implies b\in\mathbb{R}.\end{aligned}$$

That is,

$$v^*v = b \in \mathbb{R}. \tag{3}$$

From (1), (2), and (3), we get $a = \lambda b$. It follows that $\lambda \in \mathbb{R}$.

■

Proof Approach 2. Let A be a Hermitian matrix. Let (λ, v) be an arbitrary eigenpair of A.

$$\begin{aligned}&\lambda\langle v,v\rangle\\&=\langle\lambda v,v\rangle\\&=\langle Av,v\rangle\\&=\langle v,A^*v\rangle\\&=\langle v,Av\rangle\\&=\langle v,\lambda v\rangle\\&=\overline{\lambda}\langle v,v\rangle.\end{aligned}$$

That is, $\lambda\langle v,v\rangle = \overline{\lambda}\langle v,v\rangle$. Since v is an eigenvector, $v \neq \vec{0}$. Since $v \neq \vec{0}$, $\langle v,v\rangle \neq 0$. Since $\langle v,v\rangle \neq 0$ and $\lambda\langle v,v\rangle = \overline{\lambda}\langle v,v\rangle$, $\lambda = \overline{\lambda}$. Since $\lambda = \overline{\lambda}$, λ is real.

■

9.4 Orthogonal and Unitary Matrices

9.4.1 Equivalent Conditions

DEFINITION (Orthogonal)**.** Let $U \in \mathcal{M}_{n\times n}(\mathbb{R})$ (a real square matrix). We say that U is **orthogonal** if

$$UU^\top = U^\top U = I$$

where $U^\top$ denotes the transpose of U and I denotes the $n \times n$ identity matrix.

DEFINITION (Unitary - 1)**.** Let $U \in \mathcal{M}_{n\times n}(\mathbb{C})$ (a complex square matrix). We say that U is **unitary** if

$$UU^* = U^*U = I$$

where U^* denotes the complex conjugate of U and I denotes the $n \times n$ identity matrix.

DEFINITION (Unitary - 2)**.** Let $U \in \mathcal{M}_{n\times n}(\mathbb{C})$ (a complex square matrix). We say that U is **unitary** if the columns of U form an orthonormal basis for $\mathbb{C}^n$, or equivalently, the rows of U form an orthonormal basis for $\mathbb{C}^n$.

9.4.2 Stability of Unitary Matrices

PROPOSITION 9.4.1. The product of two unitary matrices is still unitary.

9.4.3 Properties of Unitary Matrices

PROPOSITION 9.4.2 (Unitary Matrices Preserve Inner Products)**.** Let U be a complex square matrix. Then U is unitary if and only if

$$\forall x, y \in \mathbb{C}^n, \quad \langle Ux, Uy \rangle = \langle x, y \rangle .$$

PROPOSITION 9.4.3 (Eigenvalues)**.** The eigenvalues of a unitary matrix are all unimodular.

Proof. Let U be a unitary matrix. Let (λ, v) be an arbitrary eigenpair of U. Since U is a unitary matrix, we get

$$\langle Uv, Uv \rangle = \langle v, v \rangle .$$

Since (λ, v) is an eigenpair of U, we get

$$\langle Uv, Uv \rangle = \langle \lambda v, \lambda v \rangle = \lambda^2 \langle v, v \rangle .$$

So $\langle v, v \rangle = \lambda^2 \langle v, v \rangle$. Since v is an eigenvector, $v \neq 0$ and hence $\langle v, v \rangle \neq 0$. So $\lambda^2 = 1$.

■

9.5 Normal Matrices

9.5.1 Equivalent Conditions

DEFINITION (Normal Matrix - 1)**.** Let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$. We say that M is **normal** if

$$MM^* = M^*M,$$

where M^* denotes the conjugate transpose of M.

DEFINITION (Normal Matrix - 2)**.** Let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$. We say that M is **normal** if $\exists \mathcal{B} \subseteq \mathcal{E}(M)$ such that $\mathcal{B}$ is a orthonormal basis for $\mathbb{C}^n$ where $\mathcal{E}(M)$ denotes the set of eigenvectors of M.

PROPOSITION 9.5.1. Definitions (1) and (2) of normal matrices are equivalent.

Proof. Let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$.

Forward Direction Assume that $MM^* = M^*M$. I will show that M has an orthonormal basis of eigenvectors.

■

DEFINITION (Normal Matrix - 3)**.** Let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$. We say that M is **normal** if M is diagonalizable by a unitary matrix.

9.5.2 Stability of Normal Matrices

PROPOSITION 9.5.2. Let A and B be normal matrices. Suppose that $AB = BA$. Then

(1) $A + B$ is also normal.

(2) AB is also normal.

9.5.3 Properties of Normal Matrices

PROPOSITION 9.5.3. Let M be a normal matrix. Then if M is triangular, M is diagonal.

PROPOSITION 9.5.4. Let M be a normal matrix. Then M is Hermitian if and only if $\sigma(M) \subseteq \mathbb{R}$ where $\sigma(M)$ denotes the set of eigenvalues of M.

Proof. **Forward Direction** Assume that M is Hermitian. I will show hat $\sigma(M) \subseteq \mathbb{R}$. Since M is Hermitian, we get $\sigma(M) \subseteq \mathbb{R}$.

Backward Direction Assume that $\sigma(M) \subseteq \mathbb{R}$. I will show that M is Hermitian. Since M is normal, it is diagonalizable by a unitary matrix. Say $M = U^*DU$ where U is unitary and D is diagonal. Then the diagonal entries of D are the eigenvalues of M and hence are real. So $D^* = D$. Then

$$M^* = (U^*DU)^* = U^*D^*U^{**} = U^*D^*U = U^*DU = M.$$

So M is Hermitian.

∎

PROPOSITION 9.5.5. Let M be a normal matrix. Then M is unitary if and only if $\sigma(M) \subseteq \mathbb{T}$ where $\sigma(M)$ denotes the set of eigenvalues of M and $\mathbb{T}$ denotes the unit circle of the complex plane.

9.6 Definite Matrices

DEFINITION (Definite Matrices)**.** Let M be an $n \times n$ Hermitian matrix. We say that

- M is **positive definite**, denoted by $M \succ 0$, if

$$\forall x \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad x^*Mx > 0.$$

- M is **negative definite**, denoted by $M \prec 0$, if

$$\forall x \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad x^* M x < 0.$$

- M is **positive semi-definite**, denoted by $M \succeq 0$, if

$$\forall x \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad x^* M x \geq 0.$$

- M is **negative semi-definite**, denoted by $M \preceq 0$, if

$$\forall x \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad x^* M x \leq 0.$$

PROPOSITION 9.6.1. Let M be an $n \times n$ Hermitian matrix. Then

- M is positive definite if and only if all of its eigenvalues are positive.
- M is negative definite if and only if all of its eigenvalues are negative.
- M is positive semi-definite if and only if all of its eigenvalues are non-negative.
- M is negative semi-definite if and only if all of its eigenvalues are non-positive.

Proof. Assume that M is positive definite. I will show that the eigenvalues of M are all positive. Let (λ, x) be an arbitrary eigenpair of M. Then we have $Mx = \lambda x$. Since M is positive definite, we have $x^* M x > 0$. So $x^* \lambda x = \lambda x^* x > 0$. Note that $x^* x \geq 0$. So $\lambda > 0$. ∎

PROPOSITION 9.6.2. If A is positive definite, then A^{-1} exists and is also positive definite.

Proof Approach 1. Let y be an arbitrary vector. Then there exists some x such that $y = Ax$ since A is invertible. Now

$$y^T A^{-1} y \tag{9.1}$$
$$= x^T A^T A^{-1} A x \tag{9.2}$$
$$= x^T A^T x = \qquad x^T A x > 0. \tag{9.3}$$

Since $\forall y, y^T A^{-1} y > 0$, we get A^{-1} is positive definite. ∎

Proof Approach 2. Since A is positive definite, all its eigenvalues are positive. Eigenvalues of A^{-1} are reciprocals of eigenvalues of A. So all eigenvalues of A^{-1} are positive. So A^{-1} is positive definite. ∎

Chapter 10

Matrix Diagonalization

10.1 Diagonalization in General

DEFINITION (Diagonalizable Matrix)**.** Let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$. We say that M is **diagonalizable** if and only if $P^{-1}MP = D$ for some invertible matrix $P \in \mathcal{M}_{n\times n}(\mathcal{C})$ and some diagonal matrix $D \in \mathcal{M}_{n\times n}(\mathcal{C})$.

PROPOSITION 10.1.1. Let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$. Then M is diagonalizable if and only if $\exists$ eigenpairs $((\lambda_i, v_i))_{i=1}^n$ of M such that the matrix $P = [v_1, ..., v_n]$ is invertible. In this case, we have

$$P^{-1}MP = \mathrm{diag}(\lambda_1, ..., \lambda_n).$$

10.2 Unitary Diagonalization

10.2.1 Definitions

DEFINITION (Unitarily Similar)**.** Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{C})$. We say that A and B are **unitarily similar** if there exists a unitary matrix U such that

$$U^*AU = B.$$

THEOREM 10.1 (Schur)**.** Any matrix is unitarily similar to an upper triangular matrix.

DEFINITION (Unitarily Diagonalizable)**.** Let M be a complex square matrix. We say that M is **unitarily diagonalizable** if M is unitarily similar to a diagonal matrix.

10.2.2 Properties

PROPOSITION 10.2.1. Unitarily diagonalizable matrices are normal.

10.3 Sufficient Conditions

PROPOSITION 10.3.1. Hermitian matrices are unitarily diagonalizable.

PROPOSITION 10.3.2. Normal matrices are unitarily diagonalizable.

Chapter 11

Matrix Decomposition

11.1 Lower-Upper Decomposition

DEFINITION (Lower-Upper (LU) Decomposition)**.** Let A be some square matrix. In the following let L denote lower triangular matrices, U denote upper triangular matrices, P denote permutation matrices, and D denote diagonal matrices. We define the followings:

- **LU decomposition**:
$$A = LU.$$
- **LUP decomposition**:
$$A = LUP.$$
- **PLU decomposition**:
$$A = PLU.$$
- **LDU decomposition**:
$$A = LDU$$
where L and U are required to be unitriangular.

THEOREM 11.1 (Lower-Upper (LU) Decomposition)**.**

- All square matrices admit LUP and PLU decompositions.

LU decomposition can be viewed as the matrix form of Gaussian elimination.

11.2 Cholesky Decomposition

DEFINITION (Cholesky Decomposition)**.** Let A be some square matrix. In the following let L denote lower triangular matrices and D denote diagonal matrices. We define the followings:

- **Cholesky decomposition**:
$$A = LL^*$$
where the diagonal entries of L are real.

- **Square-Root-Free Cholesky (LDL) decomposition**:
$$A = LDL$$
where L is required to be unitriangular.

The diagonal elements of L are required to be 1 at the cost of introducing an additional diagonal matrix D in the decomposition.

THEOREM 11.2 (Existence and Uniqueness)**.**

- All Hermitian positive definite matrices admit a unique Cholesky decomposition and the matrix L has strictly positive real diagonal entries.
- All Hermitian positive semi-definite matrices admit a Cholesky decomposition and the matrix L has non-negative real diagonal entries.

11.3 Eigenvalue Decomposition

DEFINITION (Eigenvalue Decomposition)**.** Let A be an $n \times n$ matrix where $n \in \mathbb{N}$. Let $\{(x_i, \lambda_i)\}_{i=1}^n$ be the eigenpairs of A. We define the **eigenvalue decomposition** of A to be a factorization of A given by

$$A = Q\Lambda Q^{-1}$$

where $Q = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}$ and $\Lambda = \operatorname{diag}(\{\lambda_i\}_{i=1}^n)$.

PROPOSITION 11.3.1. Let A be an $n \times n$ matrix. Then A can be eigendecomposed if and only if A has n linearly independent eigenvectors.

11.4 Singular Value Decomposition

DEFINITION (Singular Value Decomposition)**.** Let M be an $m \times n$ real or complex matrix. We define a **singular value decomposition** to be a factorization of the form $M = U\Sigma V^*$ where U is an $m \times m$ unitary matrix, the columns of U are the left-singular vectors of M; V is an $n \times n$ unitary matrix, the columns of V are the right-singular vectors of M; Σ is an $m \times n$ rectangular diagonal matrix, the diagonal entries of Σ are the singular values of M.