Game Theory

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Contents

Chapter 1

First Chapter

1.1 First Section

DEFINITION 1.1 (Winning Position)**.** Consider a two-player game. We say that a player has a **winning position** if and only if optimal play by that player guarantees a win.

DEFINITION 1.2 (Losing Position)**.** Consider a two-player game. We say that a player has a **losing position** if and only if an optimal move by their opponent guarantees a loss.

PROPOSITION 1.3.

- From a winning position (player to move), there exists a move that leads to a losing position for the other player.
- From a losing position (player to move), every move leads to a winning position for the other player.

1.2 Groups of Games

DEFINITION 1.4 (Equivalent Games)**.** Let G and H be two impartial games. We say that G and H are **equivalent** if and only if for all impartial games J, $G+J$ is a losing position if and only if $H+J$ is a losing position.

- for all impartial games J, $G+J$ is a losing position if and only if $H+J$ is a losing position.
- for all impartial games J, $G+J$ is a winning position if and only if $H+J$ is a winning position.

PROPOSITION 1.5. Game equivalence is an equivalence relation. That is, "$\equiv$" is:

- Reflexive: $\forall G$, we have $G \equiv G$.
- Symmetric: $\forall G, H$, we have $G \equiv H \iff H \equiv G$.
- Transitive: $\forall G, H, K$, we have $((G \equiv H) \wedge (H \equiv K)) \implies G \equiv K$.

PROPOSITION 1.6. $\forall G, H, J$, we have $G \equiv H \implies G+J \equiv H+J$.

PROPOSITION 1.7. $G \equiv H$ implies that G and H are both winning or both losing.

LEMMA 1.8. G is a losing position if and only if $G \equiv *0$.

Proof. **Backward Direction**: Suppose that $G \equiv *0$. Then $\forall J$, $G+J$ is a losing position if and only if $*0+J$ is a losing position. In particular, take $J := *0$, then $G+*0$ is a losing position if and only if $*0+*0$ is a losing position. Notice $G+*0 = *0$ and $*0+*0 = *0$. So G is a losing position if and only if $*0$ is a losing position. We know that $*0$ is indeed a losing position. So G is a losing position.

Forward Direction: Suppose that G is a losing position. I will show that $G \equiv *0$. Let J be an arbitrary impartial game. Notice $*0+J = J$. So there remains to show that $G+J$ is losing if and only if J is losing.

Suppose that $G + J$ is a losing position. I will show that J is a losing position. Assume for the sake of contradiction that J is not losing. Then J is winning. Let $J \to J'$ be a move such that J' is losing. Since G is losing and J' is losing, we get $G + J'$ is losing. So $G + J$ is winning. However, this contradicts to the assumption that $G + J$ is losing. So J is losing.

Suppose that J is a losing position. I will show that $G + J$ is a losing position. Double strong well-founded induction.

□

G is winning and J is losing, then $G + J$ is winning???

DEFINITION 1.9 (Group of Game)**.** Let $\mathcal{G}$ be a set of games. Let $* : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be a binary operation on $\mathcal{G}$. We say that $(\mathcal{G}, *)$ is a **group** if and only if the following conditions hold:

1. Associativity: $\forall G_1, G_2, G_3 \in \mathcal{G}$, $(G_1 * G_2) * G_3 \equiv G_1 * (G_2 * G_3)$.
2. Identity: $\exists I \in \mathcal{G}$ such that $\forall G \in \mathcal{G}$, $G * I \equiv I * G \equiv G$.
3. Inverse: $\forall G \in \mathcal{G}$, $\exists H \in \mathcal{G}$ such that $G * H \equiv H * G \equiv I$.

LEMMA 1.10. *$G \equiv H$ if and only if $G + H \equiv *0$.*

Proof. **Forward Direction**: Suppose that $G \equiv H$. I will show that $G + H \equiv *0$. Since $G \equiv H$, we get

$$G + H \equiv H + H, \text{ by the}$$

□

LEMMA 1.11. *Let G and H be impartial combinatorial games. Suppose that*

- *For each option G' of G, there exists an option of H which is equivalent to G'.*
- *For each option H' of H, there exists an option of G which is equivalent to H'.*

Then $G \equiv H$.

Proof. Since $G' + H' \equiv *0$, we get $G + H \equiv *0$. □

THEOREM 1.12 (Sum of NIM Heaps)**.** Suppose $n_1, ..., n_k \in \mathbb{Z}_{++}$ are distinct powers of 2. Then we have

$$*(n_1 + ... + n_k) \equiv (*n_1 + ... + *n_k).$$

Proof. **Base Case**: $n = 0$.

Inductive Step: Suppose the theorem holds for all positive integers less than n. Write n as $n = 2^{a_1} + ... + 2^{a_k}$ where $a_1 > ... > a_k$. Define

$$q := n - 2^{a_1} = 2^{a_2} + ... + 2^{a_k}.$$

Note that $q < 2^{a_1} < n$. Apply the induction hypothesis on q, we get

$$*q \equiv *2^{a_1} + ... + *2^{a_k}$$

Now there remains to show that $*n \equiv *2^{a_1} + *q$. Consider the options of $*n$: $\{*(n-1), *(n-2), ..., *0\}$ and the options of $*2^{a_1} + *q$: $\{G + *q, *2^{a_1} + H\}$ where G is some option of $*2^{a_1}$ and H is some option of $*q$.

Consider the set $\{*i + *q : 0 \leq i < 2^{a_1}\}$ of options of $*2^{a_1} + *q$.

Consider the set $\{*2^{a_1} + *i : 0 \leq i < q\}$ of options of $*2^{a_1} + *q$. Write i as $i = 2^{b_1} + 2^{b_2} + ...$ Notice $2^{a_1} + i < 2^{a_1} + q < n$. So by the inductive hypothesis, we get

$$*(2^{a_1} + i) = *(2^{a_1} + 2^{b_1} + 2^{b_2} + ...) = *2^{a_1} + *2^{b_1} + *2^{b_2} + ...$$

So the set of options of $*n$ is equivalent to the set of options for $*2^{a_1} + *q$. So $*n \equiv *2^{a_1} + *2^{a_2} + ...$ □

EXAMPLE 1.13.

$$\begin{aligned}(5, 9, 8) &= *5 + *9 + *8 = *(4 + 1) + *(8 + 1) + *8 \\ &= *4 + *1 + *8 + *1 + *8 = *4.\end{aligned}$$

So the optimal move is to take away the $*4$: $(5, 9, 8) \to (1, 9, 8)$.

DEFINITION 1.14 (Balance, Unbalanced)**.** We say that a NIM position $(a_1, ..., a_q)$ is **balanced** if and only if $a_1 \oplus ... \oplus a_q = 0$. We say that is it **unbalanced** otherwise.

THEOREM 1.15. A NIM position $(a_1, ..., a_q)$ is a losing (winning) position if and only if it is balanced (unbalanced).

DEFINITION 1.16 (Minimum Excludant)**.** Given a subset $S \subsetneq \mathbb{N}$, we define $\text{mex}(S)$ to be the smallest element of $\mathbb{N} \setminus S$.

THEOREM 1.17 (MEX Rule)**.** Let $S \subsetneq \mathbb{N}$. Let G be an impartial game whose options are equivalent to $\{*s : s \in S\}$. Then $G \equiv *(\text{mex}(S))$.

Proof. Let $m := \text{mex}(S)$. By the Generalized Copycat principle, it suffices to show that $G + *m \equiv *0$.

Consider an option of the form $G + *m'$ for some $m' < m$. Since $m = \text{mex}(S)$ and $m' < m$, we have $m' \in S$. Then there exists an option G' of G such that $G' \equiv *m'$. The other player can move to $G' + *m'$. Since $G' \equiv *m'$, the game $G' + *m'$ is a losing position (copycat principle). So $G + *m'$ is winning.

Consider an option of the form $G' + *m$ of $G + *m$. Recall that the options of G are $\{*n : n \in S\}$. Let $k \in S$ be a natural number such that $G' \equiv *k$. Then $G' + *m \equiv *k + *m$. Since $m \notin S$ and $k \in S$, $*k + *m$ is winning. So $G' + *m$ is winning.

Hence all options of $G + *m$ are winning. So $G + *m$ is losing. So $G \equiv *m$. □

COROLLARY 1.18. For every impartial game G, there exists a natural number $n \in \mathbb{N}$ such that $G \equiv *n$.

Proof. We use (well-founded) induction on G.

Base case: If G has no options, then $G \equiv *0$.

Inductive step: Suppose the set of options for G is finite and are $G^1, ..., G^q$. By the induction hypothesis, $\forall i \in \{1, ..., q\}$, we have $G^i \equiv *n_i$ for some $n_i \in \mathbb{N}$. So the set of options of G are equivalent to $\{*n_1, ..., *n_q\}$. Apply the MEX rule with $S := \{n_1, ..., n_q\}$, we have

$$G \equiv *(\text{mex}(S)) = *(\text{mex}(\{n_1, ..., n_q\})).$$

□

Chapter 2

Strategic Games

2.1 Pure Strategies

DEFINITION 2.1 (Extensive Games)**.** Games with game trees are called **extensive games with perfect information**.

DEFINITION 2.2 (Strategy)**.** A **strategy** (for a player) specifies a move for every decision node for that player. i.e., a function that maps each decision node to a move.

DEFINITION 2.3 (Strategy Profile)**.** A **strategy profile** specifies a strategy for every player. We represent a strategy (profile) by concatenating moves.

DEFINITION 2.4 (Strategic Form)**.** The **strategic form** of a game consists of:

- A set $N = \{1, ..., n\}$ of players;
- A set S_i of strategies for $i \in N$;
- A utility function $u_i : S_1 \times ... \times S_n \to \mathbb{R}$, for each $i \in N$.

A strategic form is a $|S_1| \times ... \times |S_n| \times N$ dimensional tensor.

2.2 Nash Equilibrium of Pure Strategies

DEFINITION 2.5 (Nash Equilibrium)**.** Let $N := \{1, ..., n\}$ denote the set of players. Let $\mathcal{S}_i$ denote the set of strategies for player i, for $i \in N$. Let $\mathcal{S} := \mathcal{S}_1 \times ... \times \mathcal{S}_n$. We say that a strategy profile $s^* = (s_1, ..., s_n) \in \mathcal{S}$ is a **Nash equilibrium** if and only if $\forall i \in N$, $\forall s_i' \in \mathcal{S}_i$, we have

$$u_i(s_1, ..., s_i', ..., s_n) \leq u_i(s^*).$$

That is, no one player can improve over their utility in s^* by unilaterally deviating in their strategy.

EXAMPLE 2.6 (Prisoner's Dilemma)**.** The Prisoner's dilemma consists of two players, each with strategies Q and C, with payoffs:

	Q	C
Q	$(2, 2)$	$(0, 3)$
C	$(3, 0)$	$(1, 1)$

- (C, C) is the only Nash equilibrium.
- (C, C) is suboptimal overall.

EXAMPLE 2.7 (Bach-Stravinsky)**.**

	Bach	Stravinsky
Bach	$(2, 1)$	$(0, 0)$
Stravinsky	$(0, 0)$	$(1, 2)$

- (B, B) and (S, S) are both Nash equilibria.

EXAMPLE 2.8 (Matching Pennies).

	Heads	Tails
Heads	$(+1, -1)$	$(-1, +1)$
Tails	$(-1, +1)$	$(+1, -1)$

- Player 1 bets on match; player 2 bets on a mismatch.

- Example of a zero-sum game.
- This game has no Nash equilibrium.
- Later in the course we will see that has a mixed Nash equilibrium.

EXAMPLE 2.9. numbers to be fixed

	L	R
T	$(2,1)$	$(0,0)$
M	$(0,0)$	$(1,2)$
B	$(0,0)$	$(1,2)$

Would player 1 ever choose T?

- No, because M is always better than T.
- In this case, T is strictly dominated by M.

2.3 Domination for Pure Strategies

DEFINITION 2.10 (Strictly Dominate)**.** Let $i \in N$. Let $s_i, s_i' \in \mathcal{S}_i$ be two strategies. Let $\mathcal{S}_{-i} := \bigoplus_{j \neq i} \mathcal{S}_j$. We say that s_i **strictly dominates** s_i' if and only if

$$\forall s_{-i} \in \mathcal{S}_{-i}, \quad u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$

DEFINITION 2.11 (Weakly Dominate)**.** Let $i \in N$. Let $s_i, s_i' \in \mathcal{S}_i$ be two strategies. Let $\mathcal{S}_{-i} := \bigoplus_{j \neq i} \mathcal{S}_j$. We say that s_i **weakly dominates** s_i' if and only if

$$\forall s_{-i} \in \mathcal{S}_{-i}, \quad u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}),$$

and $\exists \bar{s}_{-i}^* \in \mathcal{S}_{-i}$ for which the inequality holds strictly.

DEFINITION 2.12 (Best Response Function)**.** We define the **best response func-**

tion for Player i to be a function $B_i : \bigoplus_{j \neq i} \mathcal{S}_j \to \mathcal{P}(\mathcal{S}_i)$ given by

$$\begin{aligned} B_i(s_{-i}) &:= \{s_i \in \mathcal{S}_i : \forall s_i' \in \mathcal{S}_i, u_i(s_i', s_{-i}) \leq u_i(s_i, s_{-i})\} \\ &= \underset{s_i' \in \mathcal{S}_i}{\operatorname{argmax}} \{u_i(s_i', s_{-i})\}. \end{aligned}$$

In other words, $B_i(s_{-i})$ is the set consisting of all strategies of Player i that yield the maximum payoff against (s_{-i}).

PROPOSITION 2.13. A strategy profile $s^* = (s_1, ..., s_n) \in \mathcal{S}$ is a Nash equilibrium if and only if

$$\forall i \in N, \quad s_i \in B_i(s_{-i}).$$

2.4 Mixed Strategies

DEFINITION 2.14 (Mixed Strategy)**.** Let $\mathcal{S}_i$ denote the set of strategies for player i. We define a **mixed strategy** $x^{(i)}$ over $\mathcal{S}_i$ to be a probability distribution over $\mathcal{S}_i$. That is, $x^{(i)} \in \mathbb{R}^{\mathcal{S}_i}$ is such that $x^{(i)} \geq \mathbb{0}$ and $\mathbb{1}^\top x^{(i)} = 1$.

DEFINITION 2.15 (Mixed Strategy Profile)**.** We define a **mixed strategy profile** to be a vector $\vec{x} = (x^{(1)}, ..., x^{(n)}) \in \mathbb{R}^{\mathcal{S}_1} \times ... \times \mathbb{R}^{\mathcal{S}_n}$ specifying a mixed strategy $x^{(i)} \in \mathbb{R}^{\mathcal{S}_i}$ for each player $i \in N$.

DEFINITION 2.16 (Expected Utility)**.** Let $\vec{x} = (x^{(1)}, ..., x^{(n)})$ denote a mixed strategy profile. We define the **expected utility** of player i in $\vec{x}$, denoted by $u_i(\vec{x})$, to be a number given by

$$u_i(\vec{x}) := \sum_{\vec{s} \in \mathcal{S}} \left[u_i(\vec{s}) \prod_{j \in \{1,...,n\}} x^{(j)}_{s_j} \right] = \sum_{s_i \in \mathcal{S}_i} x^{(i)}_{s_i} \sum_{\vec{s} \in \mathcal{S}, \vec{s}_i = s_i} \left[u_i(\vec{s}) \prod_{j \neq i} x^{(j)}_{s_j} \right].$$

We define the **expected utility of strategy** s_i in $\vec{x}$ to be

$$u_i(s_i, \vec{x}) := \sum_{\vec{s} \in \mathcal{S}, \vec{s}_i = s_i} \left[u_i(\vec{s}) \prod_{j \neq i} x_{s_j}^{(j)} \right].$$

2.5 Nash Equilibrium of Mixed Strategies

DEFINITION 2.17 (Mixed Nash Equilibrium)**.** Let $\bar{x} = (\bar{x}^{(1)}, ..., \bar{x}^{(n)})$ be a mixed strategy. We say that $\bar{x}$ is a **mixed Nash equilibrium** if and ony if $\forall i \in \{1, ..., n\}$, for any mixed strategy $x^{(i)}$ over $\mathcal{S}_i$, we have

$$u_i(\bar{x}) \geq u_i(\bar{x}^1, ..., x^i, ..., \bar{x}^n).$$

DEFINITION 2.18 (Best Response)**.** Given a profile $\bar{x}^{-i} = (\bar{x}^1, ..., \bar{x}^{i-1}, \bar{x}^{i+1}, ..., \bar{x}^n)$ of mixed strategies of players in $N \setminus \{i\}$, the best response for $\bar{x}^{-i}$ is the set $B_i(\bar{x}^{-i})$ of all mixed strategies x^i of player i that maximize the expected utility

$$u.$$

PROPOSITION 2.19. Best response functions are continuous.

THEOREM 2.20. A strategy profile is a mixed Nash equilibrium if and only if it lies on both player's best-response graphs.

Optimization problems:

$$\begin{array}{lll} \text{(P)} & \max & \sum_{s \in \mathcal{S}_i} \bar{x}_s^i \cdot u_i(s, \bar{x}^{-i}) \\ & \text{subject to:} & \sum_{s \in \mathcal{S}_i} \bar{x}_s^i = 1, \\ & & \bar{x}^i \geq 0. \end{array}$$

$$\begin{array}{lll} \text{(D)} & \min & y \\ & \text{subject to:} & y \geq u_i(s, \bar{x}^{-i}), \forall s \in \mathcal{S}_i. \end{array}$$

Conversely, we prove that every mixed strategy that chooses from among locally optimal strategies is an optimal strategy...

THEOREM 2.21 (Support Characterization)**.** Given mixed strategies $\bar{x}^{-i}$ of player in $N \setminus \{i\}$, a mixed strategy $\bar{x}^i$ is in $B_i(\bar{x}^{-i})$ if and only if $\bar{x}^i_s > 0$ implies that $s \in \mathcal{S}_i$ is a strategy of maximum expected payoff (against $\bar{x}^{-i}$).

COROLLARY 2.22. The set $B_i(\bar{x}^{-i})$ is a polyhedron.

Proof. Let $S' \subseteq S$ be the subset consisting of pure strategies s that maximize $u_i(s, \bar{x}^{-1})$. Then

$$B_i(\bar{x}^{-i}) = \{x^i : \text{supp}(x^i) \subseteq S' \text{ and } \sum_{s \in S'} x^i_s = 1\}.$$

□

2.6 Domination for Mixed Strategies

DEFINITION 2.23 (Strictly Dominate)**.** A strategy $s_i \in \mathcal{S}_i$ **strictly dominates** strategy $s'_i \in \mathcal{S}_i$ if and only if

$$\forall j \neq i, \forall s_j \in \mathcal{S}_j, \quad u_i(s_1, ..., s_i, ..., s_n) > u_i(s_1, ..., s'_i, ..., s_n).$$

DEFINITION 2.24. Let x^i be a mixed strategy over $\mathcal{S}_i$. Let $s_i \in \mathcal{S}_i$ be a pure strategy. We say that x^i **strictly dominates** s_i if and only if

$$\forall s_{-i} \in \mathcal{S}_{-i}, \quad u_i(x^i, s_{-i}) > u_i(s_i, s_{-i}).$$

THEOREM 2.25. Let $\bar{x} \in \bigoplus_{i=1}^{n} \mathbb{R}^{S_i}_+$ be a mixed Nash equilibrium. Let $s \in \mathcal{S}_i$ be a pure strategy. Suppose that there exists a mixed strategy $x^i \in \mathbb{R}^{S_i}_+$ over $\mathcal{S}_i$ that strictly dominates s, then $\bar{x}^i_s = 0$.

Proof. Assume for the sake of contradiction that $\bar{x}^i_s > 0$.

□

DEFINITION 2.26 (Zero-Sum Game)**.** We say that a game is a **zero-sum game** if and only if

$$\forall s \in \mathcal{S}, \quad \sum_{i=1}^{n} u_i(s) = 0.$$

Player 1's linear program:

$$\begin{array}{lll} (P_1) & \max & \nu_r \\ & \text{subject to:} & x^{(1)\top} A_{\cdot,j} \geq \nu_r, \quad \forall j \in S_2, \\ & & 1^\top x^{(1)} = 1, x^{(1)} \geq 0. \end{array}$$

Player 2's linear program:

$$\begin{array}{lll} (P_2) & \min & \nu_c \\ & \text{subject to:} & A_{i,\cdot} x^{(2)} \leq \nu_c, \quad \forall i \in S_1, \\ & & 1^\top x^{(2)} = 1, x^{(2)} \geq 0. \end{array}$$

They are duals of each other, both feasible and bounded.

These are equivalent to the following programs:

$$\begin{array}{lll} (P_1') & \max & (0_{|S_1|}^\top, 1) \begin{pmatrix} x^{(1)} \\ \nu_r \end{pmatrix} \\ & \text{subject to} & \begin{pmatrix} A^\top & -1_{|S_2|} \\ 1_{|S_1|}^\top & 0 \\ -1_{|S_1|}^\top & 0 \end{pmatrix} \begin{pmatrix} x^{(1)} \\ \nu_r \end{pmatrix} \geq \begin{pmatrix} 0_{|S_2|} \\ 1 \\ -1 \end{pmatrix}, \\ & & x^{(1)} \geq 0_{|S_1|}. \end{array}$$

$$\begin{array}{lll} (P_2') & \min & (0_{|S_2|}^\top, 1) \begin{pmatrix} x^{(2)} \\ \nu_c \end{pmatrix} \\ & \text{subject to:} & \begin{pmatrix} A & 1_{|S_1|} \\ -1_{|S_2|}^\top & 0 \\ 1_{|S_2|}^\top & 0 \end{pmatrix} \begin{pmatrix} x^{(2)} \\ \nu_c \end{pmatrix} \geq \begin{pmatrix} 0_{|S_1|} \\ 1 \\ -1 \end{pmatrix} \\ & & x^{(2)} \leq 0_{|S_2|}. \end{array}$$

THEOREM 2.27. Every finite strategic game has a mixed Nash equilibrium.

Proof. Let $x \in \prod_{i \in N} \Delta_{|\mathcal{S}_i|}$ be a mixed strategy profile. Define for each $i \in \{1, ..., N\}$ and each $s_i \in \mathbb{S}_i$ a function $\Phi_{s_i}^{(i)} : \prod_{i \in N} \Delta_{|\mathcal{S}_i|} \to \mathbb{R}_+$ by

$$\Phi_{s_i}^{(i)}(x) := \max(0, u_i(s_i, x^{-1}) - u_i(x)).$$

Then

- $\Phi^{(i)}_{s_i}(x)$ is positive only if the pure strategy $s_i \in \mathcal{S}_i$ yields higher expected payoff than the mixed strategy $x^{(i)}$;
- By the Support Characterization theorem, $\Phi^{(i)}_{s_i}(x) = 0$ for all $s_i \in \mathbb{S}_i$ if and only if $x^{(i)}$ is a best response to x^{-i};
- $\Phi^{(i)}_{s_i}$ is not necessarily differentiable, but it is continuous.

Define a function $f : \prod_{i \in N} \Delta_{|\mathcal{S}_i|} \to \prod_{i \in N} \Delta_{|\mathcal{S}_i|}$ by $f(x) := \bar{x}$ where $\bar{x}$ is given by:

$$\bar{x}^{(i)}_{s_i} := .$$

Then

-

Let $i \in \{1, ..., n\}$ be arbitrary. Let $s_i \in \mathcal{S}_i$ such that $\hat{x}^{(i)}_{s_i} > 0$ and $u_i(s_i, \hat{x}^{-1}) \leq u_i(\hat{x})$. Then $\Phi^{(i)}_{s_i}(\hat{x}) = 0$ and

$$\hat{x}^{(i)}_{s_i} = (f(\hat{x}))^{(i)}_{s_i} = \frac{\hat{x}^{(i)}_{s_i} + 0}{1 + \sum_{s \in \mathcal{S}_i} \Phi^{(i)}_s(\hat{x})}.$$

So $\forall s \in \mathcal{S}_i$, we have $\Phi^{(i)}_s(\hat{x}) = 0$. So $\forall i \in \{1, ..., n\}$, $\hat{x}^{(i)}$ is a best response to $\hat{x}^{-i}$. So $\hat{x}$ is a Nash equilibrium. $\square$

THEOREM 2.28 (Daskalakis, Goldberg, Papadimitriou (2008))**.** NASH is polynomial parity argument for directed graphs (PPAD)-complete.

REMARK 2.29. NASH, BROUWER, and BORSUK-ULAM are PPAD-complete.

REMARK 2.30. The following problems are NP-complete:

- Find a Nash equilibrium maximizing total utility.
- Find two Nash equilibria (or determine that only one exists).

...

Chapter 3

Lemke-Homson Algorithm

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the strategies for player 1 and player 2, respectively. Let $A, B \in \mathbb{R}^{\mathcal{S}_1 \times \mathcal{S}_2}$ denote the payoff matrices for player 1 and player 2, respectively. Consider the following system

$$
\begin{array}{llll}
(P) & \min & 0 \\
& \text{subject to:} & \mathbb{1}^\top x^{(i)} = 1, \quad \forall i \in \{1,2\}, \\
& & Ax^{(2)} \leq \mathbb{1} v_1, \\
& & B^\top x^{(1)} \leq \mathbb{1} v_2, \\
& & \sum_{i \in \mathcal{S}_i} x_i^{(1)} (v_1 - A_{i \cdot} x^{(2)}) = 0, \\
& & \sum_{j \in \mathcal{S}_j} x_j^{(2)} (v_2 - B_{\cdot j}^\top x^{(1)}) = 0, \\
& & x^{(1)} \in \mathbb{R}^{\mathcal{S}_1}, x^{(2)} \in \mathbb{R}^{\mathcal{S}_2}, v_1, v_2 \in \mathbb{R}.
\end{array}
$$

Note that this is a feasibility problem.

CLAIM 3.1. A non-negative solution to this system is a mixed Nash equilibrium.

Proof. By the Support Characterization theorem, $x^{(1)}$ and $x^{(2)}$ are best responses to each other. $\square$

DEFINITION 3.2 (Lemke-Homson Algorithm)**.** Define $\bar{x}^{(1)} := x^{(1)}/v_2 \in \mathbb{R}^{\mathcal{S}_1}$ and $\bar{x}^{(2)} := x^{(2)}/v_1 \in \mathbb{R}^{\mathcal{S}_2}$. Add slack variables $\gamma^{(1)} \in \mathbb{R}^{\mathcal{S}_1}$ and $\gamma^{(2)} \in \mathbb{R}^{\mathcal{S}_2}$. Then we get the **Lemke-Homson system**:

$$\begin{array}{lll} (P) & \min & 0 \\ & \text{subject to:} & A\bar{x}^{(2)} + \gamma^{(1)} = \mathbb{1}, \\ & & B^\top \bar{x}^{(1)} + \gamma^{(2)} = \mathbb{1}, \\ & & \sum_{i \in \mathcal{S}_1} \bar{x}_i^{(1)} \gamma_i^{(1)} = 0, \\ & & \sum_{j \in \mathcal{S}_2} \bar{x}_j^{(2)} \gamma_j^{(2)} = 0, \\ & & \bar{x}^{(1)}, \gamma^{(1)} \in \mathbb{R}_+^{\mathcal{S}_1}, \\ & & \bar{x}^{(2)}, \gamma^{(2)} \in \mathbb{R}_+^{\mathcal{S}_2}. \end{array}$$

REMARK 3.3. The first two constraints yield

$$\begin{bmatrix} 0 & A & I & 0 \\ B^\top & 0 & 0 & I \end{bmatrix} \begin{pmatrix} \bar{x}^{(1)} \\ \bar{x}^{(2)} \\ \gamma^{(1)} \\ \gamma^{(2)} \end{pmatrix} = \mathbb{1}.$$

Note that there is a trivial (basic) solution to the above system: $\gamma^{(i)} = \mathbb{1}$ and $\bar{x}^{(i)} = \mathbb{0}$, for $i \in \{1, 2\}$. However, there is no mixed strategy with all entries zero.

REMARK 3.4. Set $v_1 := (\mathbb{1}^\top \bar{x}^{(2)})^{-1}$, $v_2 := (\mathbb{1}^\top \bar{x}^{(1)})^{-1}$, and $x^{(1)} := v_2 \bar{x}^{(1)}$, $x^{(2)} := v_1 \bar{x}^{(2)}$ to get a feasible solution to the original problem.

THEOREM 3.5. For a non-degenerate game, the Lemke-Howson algorithm terminates in a finite number of steps.

Proof Idea. It suffices to show that no basis repeats. □

Chapter 4

Market Models

4.1 Cournot Oligopoly Model

DEFINITION 4.1 (Cournot Oligopoly Model)**.** Let $c \in \mathbb{R}_{++}$ denote the cost of production. Let $\alpha \in \mathbb{R}_{++}$ denote the maximum cost that the buyers are willing to pay. Suppose that $c < \alpha$ and

$$\begin{aligned} C_i(q_i) &:= cq_i, \forall i \in N, \text{ and} \\ P(\vec{q}) &:= \max(\alpha - \sum_{i \in N} q_i, 0). \end{aligned}$$

PROPOSITION 4.2 (Utility Function)**.** The utility for player i, under the Cournot Oligopoly Model, is

$$u_i(\vec{q}) = \begin{cases} q_i(\alpha - c - \sum_{j \in N} q_j), & \text{if } \alpha - \sum_{j \in N} q_j \geq 0 \\ -cq_i, & \text{otherwise.} \end{cases}$$

PROPOSITION 4.3 (Best Response Function)**.** The best response function for player i, under the Cournot Oligopoly Model, is

$$B_i(\vec{q}_{-i}) = \begin{cases} \frac{1}{2}(\alpha - c - \sum_{j \neq i} q_j), & \text{if } \alpha - c - \sum_{j \neq i} q_j \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

PROPOSITION 4.4 (Nash Equilibrium)**.** The Nash equilibrium is $\vec{q}^*$ where $\forall i \in N$,

$$\vec{q}_i^* = \frac{\alpha - c}{n+1}.$$

4.2 Bertrand Oligopoly Model

PROPOSITION 4.5. Let $A := \underset{j \in [n]}{\operatorname{argmin}}\{p_j\}$. Let $m := |A|$. Then the utility function

$$u_i(\vec{p}) = \begin{cases} p_i \frac{D(p_i)}{m} - C_i(\frac{D(p_i)}{m}), & \text{if } i \in A \\ -C_i(0), & \text{otherwise.} \end{cases}$$

4.2.1 Two Player, Linear Cost, Inverse Linear Demand

PROPOSITION 4.6 (Utility Function)**.** Let c denote the cost of production. Let α denote the maximum price that the consumers are willing to pay. Suppose that $n = 2$, $C_i(q_i) = cq_i$, and $D(p) = \max(\alpha - p, 0)$. Then firm i makes a profit of

$$u_i(p_1, p_2) = \begin{cases} (\alpha - p_i)(p_i - c), & \text{if } p_i < p_j \\ \frac{1}{2}(\alpha - p_i)(p_i - c), & \text{if } p_i = p_j \\ 0, & \text{if } p_i > p_j \end{cases}$$

for $i \in \{1, 2\}$ and $j := 3 - i$.

PROPOSITION 4.7 (Best Reponse Function)**.** Let p^* denote the profit-maximizing price in a monopoly. That is, $p^* := \frac{c + \alpha}{2}$ is the value of p that maximizes $(\alpha - p)(p - c)$.

Then the best response function B_i for player i is

$$B_i(p_j) = \begin{cases} \{p_i : p_i > p_j\}, & \text{if } p_j < c \\ \{p_i : p_i \geq c\}, & \text{if } p_j = c \\ \emptyset, & \text{if } c < p_j \leq p^* \\ \{p^*\}, & \text{if } p^* < p_j \end{cases}$$

for $i \in \{1, 2\}$ and $j := 3 - i$.

PROPOSITION 4.8 (Nash Equilibrium)**.** The only point that the graphs of B_1 and B_2 intersect is (c, c).

REMARK 4.9.

- Payoff functions can be discontinuous;
- Best responses can be non-existent;
- Graphs of best response functions can be disconnected.

EXAMPLE 4.10 (Infinite Games with no Nash Equilibrium)**.**

- Non-compact strategy space: $S_1 = S_2 := [0, 1)$, $u_i(s_1, s_2) := s_i$.
- Discontinuous payoff functions: $S_1 = S_2 := [0, 1]$, $u_i(s_1, s_2) := \begin{cases} s_i, & \text{if } s_i < 1 \\ 0, & \text{if } s_i = 1 \end{cases}$.
- Discontinuous pay off functions:

Chapter 5

Routing Games

5.1 Atomic Selfish Routing Game

DEFINITION 5.1 (Atomic Selfish Routing Game)**.** An **atomic selfish routing game** consists of

- A directed graph $G = (V, E)$;
- A set of players $N = \{1, ..., n\}$;
- A source-target pair $(s_i, t_i) \in V \times V$ for each $i \in N$;
- A traffic $r_i \in \mathbb{R}_{++}$ for each $i \in N$;
- A cost function $c_e : \mathbb{R}_{++} \to \mathbb{R}_{++}$ that is continuous and non-decreasing.

REMARK 5.2. Atomic selfish routing game is a special case of finite strategic game. The strategy set $\mathcal{P}_i$ for player i is the set of all $s_i t_i$-paths in G. We assume that $\forall i \in N$, $\mathcal{P}_i \neq \emptyset$. A strategy profile is a vector $\vec{p} = (p_1, ..., p_n)$ of paths. Let $f_e^{\vec{p}}$ denote the total number of units of traffic in $\vec{p}$ on edge e. If $r_i = 1$ for all $i \in N$, then $f_e^{\vec{p}}$ equals the number of occurrences of e in $\vec{p}$. The utility of player i is

$$u_i(\vec{p}) = -\sum_{e \in E(p_i)} c_e(f_e^{\vec{p}}).$$

DEFINITION 5.3 (Flow for Atomic)**.** Let $\mathcal{P} := \bigcup_{i \in N} \mathcal{P}_i$. We define a **fow** to be a function $f : N \times \mathcal{P} \to \mathbb{R}_+$. We say that f is **feasible** if and only if $\forall i \in N$, $\exists p_i \in \mathcal{P}_i$ such that $\forall p \in \mathcal{P}$, we have

$$f(i,p) = \begin{cases} r_i, & \text{if } p = p_i \\ 0, & \text{otherwise.} \end{cases}$$

i.e., each player sets all of its traffic to exactly one path that is available for that player.

DEFINITION 5.4 (Cost for Atomic)**.** We define the **cost of a path** p w.r.t. a flow f, denoted by $c_p(f)$, to be a number given by

$$c_p(f) := \sum_{e \in E(p)} c_e(f_e) \text{ where } f_e := \sum_{q \in \{\mathfrak{q} \in \mathcal{P} : e \in \mathfrak{q}\}} \sum_{i \in N} f(i,q).$$

We define the **cost of a flow** f to be an element of $\mathbb{R}$ given by

$$C(f) := \sum_{e \in E(G)} c_e(f_e) f_e.$$

DEFINITION 5.5 (Equilibrium Flow)**.** We say that a feasible flow f is a **equilibrium flow** if and only if $\forall i \in N$, $\forall p, \tilde{p} \in \mathcal{P}_i$, we have

$$f(i,p) > 0 \implies c_p(f) \leq c_{\tilde{p}}(\tilde{f})$$

where $\tilde{f}$ is the flow identical to f except that $\tilde{f}(i,p) = 0$ and $\tilde{f}(i,\tilde{p}) = r_i$.

5.2 Non-atomic Selfish Routing

DEFINITION 5.6 (Non-atomic Selfish Routing)**.** A **non-atomic selfish routing game** consists of

- A directed graph $G = (V, E)$ (multiple edges are allowed).
- A set of players $N = \{1, ..., n\}$.

- For each player $i \in N$, a source-target pair $(s_i, t_i) \in V \times V$. We assume that $\forall i, j \in N$, $(s_i, t_i) = (s_j, t_j) \implies i = j$.
-

DEFINITION 5.7 (Flow for Non-atomic)**.** Let $\mathcal{P}_i$ denote the set of all $s_i t_i$-paths in G. Let $\mathcal{P} := \bigcup_{i \in N} \mathcal{P}_i$. We define a **flow** to be a function $f : \mathcal{P} \to \mathbb{R}_+$. We say that a flow f is **feasible** if and only if

$$\forall i \in N, \quad \sum_{p \in \mathcal{P}_i} f(p) = r_i.$$

DEFINITION 5.8 (Cost for Non-atomic)**.** Let $f : \mathcal{P} \to \mathbb{R}_+$ be a flow. We define the **cost of a path** p w.r.t. a flow f to be

$$c_p(f) := \sum_{e \in E(p)} c_e(f_e) \text{ where } f_e := \sum_{q \in \mathcal{P} : e \in E(q)} f(q).$$

We define the **cost of a flow** f to be

$$C(f) := \sum_{p \in \mathcal{P}} c_p(f) f(p) = \sum_{e \in E} c_e(f_e) f_e.$$

DEFINITION 5.9 (Equilibrium Flow)**.** We say that a feasible flow is **equilibrium** if and only if

$$\forall i \in N, \forall p, \tilde{p} \in \mathcal{P}_i, \quad f_p > 0 \implies c_p ...$$

THEOREM 5.10. Let $(G, \vec{r}, c)$ be a non-atomic selfish routing instance. Then

1. The instance $(G, \vec{r}, c)$ admits at least one equilibrium flow.
2. If f and $\tilde{f}$ are equilibrium flows for $(G, \vec{r}, c)$, then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge e.

DEFINITION 5.11 (Marginal Cost Functions)**.** Let $e \in E$ and $p \in \mathcal{P}$. We define the **marginal cost functions** to be

$$c_e^*(x) := \frac{d(x \cdot c_e(x))}{dx} = c_e(x) + x \cdot c_e'(x) = \frac{\partial}{\partial f_e} C(f),$$
$$c_p^*(f) := \sum_{e \in E(p)} c_e^*(f_e) = \sum_{e \in E(p)} \frac{\partial}{\partial f_e} C(f).$$

LEMMA 5.12. Let $C : \mathbb{R}^n \to \mathbb{R}$ be a convex differentiable function. Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set. Then a feasible point $x^* \in S$ is optimal for the convex problem

$$\min C(x) \quad s.t. \quad x \in S$$

if and only if

$$\forall x \in S, \quad \nabla C(x^*) \cdot (x - x^*) \geq 0.$$

THEOREM 5.13. Let $(G, \vec{r}, c)$ be a non-atomic selfish routing instance such that for every edge e, the function $x \mapsto x \cdot c_e(x)$ is convex and differentiable. Let c_e^* denote the marginal cost function of the edge e. Then f^* is an optimal flow for $(G, \vec{r}, e)$ if and only if $\forall i \in N$, $\forall p_1, p_2 \in \mathcal{P}_i$ such that $f^*(p_1) > 0$, we have $c_{p_1}^*(f^*) \leq c_{p_2}^*(f^*)$.

Proof. ($\Rightarrow$) Suppose that f^* is optimal. Assume for the sake of contradiction that $\exists i \in N$, $\exists p_1, p_2 \in \mathcal{P}_i$ such that $f^*(p_1) > 0$ and $c_{p_1}^*(f^*) > c_{p_2}^*(f^*)$. Define for each $\varepsilon > 0$ a flow $f : \mathcal{P} \to \mathbb{R}_+$ by

$$f(p) := \begin{cases} f^*(p_1) - \varepsilon, & \text{if } p = p_1 \\ f^*(p_2) + \varepsilon, & \text{if } p = p_2 \\ f^*(p), & \text{otherwise.} \end{cases} \implies (f - f^*)_e = \begin{cases} -\varepsilon, & \text{if } e \in E(p_1) \setminus E(p_2) \\ +\varepsilon, & \text{if } e \in E(p_2) \setminus E(p_1) \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \langle \nabla C(f^*), f - f^* \rangle &= \varepsilon \sum_{e \in E(p_2) \setminus E(p_1)} c_e^*(f_e^*) - \varepsilon \sum_{e \in E(p_1) \setminus E(p_2)} c_e^*(f_e^*) \\ &= \varepsilon \Bigg[\sum_{e \in E(p_2)} c_e^*(f_e^*) - \sum_{e \in E(p_1)} c_e^*(f_e^*) \Bigg] \end{aligned}$$

$$= \varepsilon(c^*_{p_2}(f^*) - c_{p_1}(f^*)) < 0.$$

($\Leftarrow$) Suppose that... I will show that f^* is an optimal flow for $(G, \vec{r}, e)$. Now for any feasible flow $f : \mathcal{P} \to \mathbb{R}_+$ obtained from f^* by shifting ε units...

$$\langle \nabla C(f^*), f - f^* \rangle = \varepsilon(c^*_{p_2}(f^*) - c^*_{p_1}(f^*)) \geq 0.$$

$\square$

Define

$$\Phi(f) := \sum_{e \in E} h_e(f_e) \text{ where } h_e(f_e) := \int_0^{f_e} c_e(x)dx.$$

Then

$$c^*_p(f) = \sum_{e \in E(p)} h'_e(f_e) = \sum_{e \in E(p)} c_e(f_e) = c_p(f).$$

...

THEOREM 5.14. Every non-atomic selfish routing game admits a Nash flow.

Proof. Define

$$h_e(f_e) := \int_0^{f_e} c_e(x)dx.$$

Then h_e is convex and differentiable. Notice that a differentiable function is convex on an interval if and only if its derivative is non-decreasing. So c_e are continuous, non-decreasing, and non-negative. $\square$

Proof. If f and $\tilde{f}$ are equilibrium flows for $(G, \vec{r}, c)$, then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge e. Suppose that f and $\tilde{f}$ are both Nash flows. Then f and $\tilde{f}$ are both minimizers of Φ. So $\Phi(f) = \Phi(\tilde{f})$. Since the feasible set is convex, $\forall \lambda \in [0, 1]$, $\lambda f + (1 - \lambda)\tilde{f}$ is also feasible. Note that $\Phi(f) := \sum_{e \in E} h_e(f_e)$ is a sum of convex function and hence convex. So

$$\Phi(\lambda f + (1 - \lambda)\tilde{f}) \leq \lambda\Phi(f) + (1 - \lambda)\Phi(\tilde{f}) = \Phi(f) = \Phi(\tilde{f}).$$

So $\lambda \mapsto \Phi(\lambda f + (1 - \lambda)\tilde{f})$ is a constant function. For a sum of convex functions to be constant, each summand must be linear. So $h_e(f_e) = \displaystyle\int_0^{f_e} c_e(x)dx$ is linear. So $c_e(x)$ is constant from f to $\tilde{f}$. $\square$

THEOREM 5.15. Suppose $\gamma \geq 1$ satisfies $\forall e \in E$, $\forall x \geq 0$, we have

$$x \cdot c_e(x) \leq \gamma \int_0^x c_e(y)dy.$$

Then the price of anarchy is at most γ.

REMARK 5.16. Note that $\gamma < 1$ is impossible since $\forall e \in E$, $\forall x \geq 0$,

$$\frac{d}{dy}(y \cdot c_e(y)) = c_e(y) + y \cdot c_e'(y) \geq c_e(y)$$
$$\Longrightarrow x \cdot c_e(x) = \int_0^x \frac{d}{dy}(y \cdot c_e(y))dy \geq \int_0^x c_e(y)dy.$$

Proof. By the previous calculation, $\forall f : \mathcal{P} \to \mathbb{R}$, we have

$$C(f) = \sum_{e \in E} f_e \cdot c_e(f_e) \geq \sum_{e \in E} \int_0^{f_e} c_e(x)dx = \Phi(f).$$

So for all feasible flows f and $\tilde{f}$ where f is a Nash flow, we then have

$$C(f) \leq \gamma\Phi(f) \leq \gamma\Phi(\tilde{f}) \leq \gamma C(\tilde{f}).$$

□

EXAMPLE 5.17. Let $c_e(x)$ be given by $c_e(x) = \sum_{i=0}^{d} a_i x^i$ for some $d \in \mathbb{Z}_{++}$ and $a_0, ..., a_d \in \mathbb{R}_{++}$. Then we have

$$x \cdot c_e(x) = \sum_{i=0}^{d} a_i x^{i+1} \text{ and}$$
$$(d+1)\int_0^x c_e(y)dy = (d+1)\sum_{i=0}^{d} \frac{a_i}{i+1} x^{i+1} = \sum_{i=0}^{d} \frac{d+1}{i+1} a_i x^{i+1} \geq x \cdot c_e(x).$$

Hence we can take $\gamma = d+1$ in the theorem. So the price of anarchy is at most $d+1$.

5.3 Potential Function of Atomic Selfish Routing Game

DEFINITION 5.18 (Potential Function)**.** Suppose $r_1 = ... = r_n = r$ for some $r \in \mathbb{R}$. Then there exists a Nash equilibrium. Define $\mathcal{P} := \bigcup_{i=1}^{n} \mathcal{P}_i$. Define for each $e \in E$ a number $f_e^{\vec{p}} \in \mathbb{Z}_+$ by $f_e^{\vec{p}} := |\{i \in N : e \in E(\vec{p}_i)\}|$. We define the **potential function** of an atomic selfish routing game, denoted by Φ, to be a function from $\mathcal{P}_1 \times ... \times \mathcal{P}_n$ to $\mathbb{R}$ given by

$$\Phi(\vec{p}) := \sum_{e \in E} \sum_{i=1}^{f_e^{\vec{p}}} c_e(i).$$

DEFINITION 5.19 (Exact Potential Game)**.** We say that a finite strategic game is an **exact potential game** if and only if there exists a potential function $\Phi : \mathcal{S}_1 \times ... \times \mathcal{S}_n \to \mathbb{R}$ such that $\forall i \in N$, $\forall s_i, s_i' \in \mathcal{S}_i$,

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}).$$

Notice that utilities are negatives of the cost. So if Φ increases, u_i would decrease, and vice versa.

THEOREM 5.20. An atomic selfish routing game is an exact potential game with potential function $\Phi : \mathcal{P}_1 \times ... \times \mathcal{P}_n \to \mathbb{R}$ given by

$$\Phi(\vec{p}) = \sum_{e \in E} \sum_{i=1}^{f_e^{\vec{p}}} c_e(i).$$

Proof. Let $i \in N$, $s_i, s_i' \in \mathcal{S}_i$ be arbitrary. Let $p_i := (s_i, s_{-i})$ and $p_i := (s_i', s_{-i})$. Then

$$\begin{aligned}
\Phi(\vec{p}) - \Phi(\vec{p}') &= \sum_{e \in E} \sum_{i=1}^{f_e^{\vec{p}}} c_e(i) - \sum_{e \in E} \sum_{i=1}^{f_e^{\vec{p}'}} c_e(i) \\
&= \sum_{e \in E(p_i) \setminus E(p_i')} c_e(f_e^{\vec{p}}) - \sum_{e \in E(p_i') \setminus E(p_i)} c_e(f_e^{\vec{p}'}) \\
&= \sum_{e \in E(p_i)} c_e(f_e^{\vec{p}}) - \sum_{e \in E(p_i')} c_e(f_e^{\vec{p}'}) \\
&= (-u_i(\vec{p})) - (-u_i(\vec{p}')) = u_i(\vec{p}') - u_i(\vec{p}).
\end{aligned}$$

$\square$

THEOREM 5.21. Every exact potential game has a Nash equilibrium.

Proof. Notice that the set of strategy profiles $\mathcal{S} = \mathcal{S}_1 \times ... \times \mathcal{S}_n$ is a finite set. Let $\vec{s} \in \mathcal{S}$ be a minimizer of Φ. Assume for the sake of contradiction that $\vec{s}$ is not a Nash equilibrium, then $\exists i \in N$, $\exists s_i' \in \mathcal{S}_i$ such that $u_i(\vec{s}') - u_i(\vec{s}) > 0$. By the preceding theorem we get $\Phi(\vec{s}) - \Phi(\vec{s}') > 0$, which contradicts to the assumption that $\vec{s}$ is a minimizer of Φ. $\square$

DEFINITION 5.22 ((λ, μ)-smooth)**.** Let $\lambda \geq 0$ and $\mu < 1$. Let $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$. We say that f is **$(\boldsymbol{\lambda}, \boldsymbol{\mu})$-smooth** if and only if

$$\forall x, y \in \mathbb{R}_{++}, \quad yf(x) \leq \lambda y f(y) + \mu x f(x).$$

EXAMPLE 5.23. Let $f(x) := ax + b$ for some $a, b \in \mathbb{R}_{++}$. Then f is $(1, 1/4)$-smooth.

Proof. Let $x, y \in \mathbb{R}_{++}$ be arbitrary. Then

$$0 \leq a(\frac{1}{2}x - y)^2 = \frac{1}{4}x^2 - axy + ay^2 \iff axy - ay^2 \leq \frac{1}{4}ax^2$$
$$...$$

$\square$

THEOREM 5.24 (Variational Inequality Characterization)**.** Let f be a feasible flow. Then f is a Nash flow if and only if

$$\forall \text{ feasible flow } f^*, \quad \sum_{e \in E} c_e(f_e) f_e \leq \sum_{e \in E} c_e(f_e) f_e^*.$$

Proof. Define for any feasible flow f^*

$$H(f^*) = \sum_{e \in E} c_e(f_e) f_e^*$$

Then

$$H(f^*) = \sum_{e \in E} c_e(f_e) f_e^* = \sum_{e \in E} c_e(f_e) \sum_{p \in \mathcal{P} : e \in E(p)} f_p^* = \sum_{e \in E} \sum_{p \in \mathcal{P} : e \in E(p)} c_e(f_e) f_p^*$$

$$= \sum_{p \in \mathcal{P}} \sum_{e \in E : e \in E(p)} c_e(f_e) f_p^* = \sum_{p \in \mathcal{P}} c_p(f) f_p^* = \sum_{i=1}^{N} \sum_{p \in \mathcal{P}_i} c_p(f) f_p^*.$$

($\Rightarrow$) Suppose that f is a Nash flow. Then $\forall p \in \mathcal{P}_i,\ f_p > 0 \implies \forall \hat{p} \in \mathcal{P}_i,\ c_p(f) \leq c_{\hat{p}}(f)$. So the summation in $H(f)$ is a weighted average of minimal possible terms, whereas the summation in $H(f^*)$ is a weighted average of possibly larger terms. So $H(f) \leq H(f^*)$.

($\Leftarrow$) Suppose that f is a minimizer of H. Then the summation in $H(f)$ must only assign positive weights to the smallest possible values of $c_p(f)$. So $\forall p \in \mathcal{P}_i,\ f_p > 0 \implies \forall \hat{p} \in \mathcal{P}_i$, $c_p(f) \leq c_{\hat{p}}(f)$. □

THEOREM 5.25. Consider a non-atomic selfish routing game. Suppose that c_e is (λ, μ)-smooth for all $e \in E$ Then

$$C(f) \leq \frac{\lambda}{1 - \mu} C(\hat{f})$$

whenever f is a Nash flow ans $\hat{f}$ is an optimal flow.

Proof.

$$\begin{aligned} C(f) &= \sum_{e \in E} c_e(f_e) f_e \leq \sum_{e \in E} c_e(f_e) \hat{f}_e, \text{ by the above theorem} \\ &\leq \lambda \sum_{e \in E} c_e(\hat{f}_e) \hat{f}_e + \mu \sum_{e \in E} c_e(f_e) f_e, \text{ by smoothness of } c_e \\ &= \lambda C(\hat{f}) + \mu C(f). \end{aligned}$$

□

Chapter 6

Cooperative Game

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6.1 Definitions

DEFINITION 6.1 (Cooperative Game, Worthwhile). A **cooperative game** is consists of a set N of players and a characteristic function $v : \mathcal{P}(N) \to \mathbb{R}$ that assigns a value or worth $v(S)$ to every coalition $S \subseteq N$ of players. For simplicity, we usually assume $v(\emptyset) = 0$ and $v(S) \geq 0$, $\forall S \in \mathcal{P}(N)$. We say that a coalition is **worthwhile** if and only if it contains a majority.

DEFINITION 6.2 (Outcoome, Solution Concept, Efficient, Imputation). An **outcome** of a cooperative game (N, v) consists of

- A partition $\pi = \{C_i\}_{i=1}^k$ of N where $C_i \subseteq N$, $\bigcup_{i=1}^{k} C_i = N$, and $C_i \cap C_j = \emptyset$.
- A vector $(x_i)_{i \in N}$ of payoffs such that $\forall C \in \pi$, $x(C) := \sum_{c \in C} x_c \leq v(C)$.

The vector $(x_i)_{i \in N}$ is called a **solution concept** or **allocation**. We say that (π, x) is **efficient** if and only if $x(C) = v(C)$ for all parts $C \in \pi$. We say that $(x_i)_{i \in N}$ is an **imputation** if and only if $x_i \geq v(\{i\})$ for all $i \in N$.

DEFINITION 6.3 (Monotone)**.** A game (N, v) is a **monotone** if and only if

$$\forall S, T \subseteq N, \quad S \subseteq T \implies v(S) \leq v(T).$$

DEFINITION 6.4 (Super-additive)**.** We say that a game is **super-additive** if and only if

$$\forall A, B \subseteq N : A \cap B = \emptyset, \quad v(A) + v(B) \leq v(A \cup B).$$

PROPOSITION 6.5. A super-additive game is monotone.

DEFINITION 6.6 (Super-additive Cover)**.** Let $G = (n, v)$ be a cooperative game. We define the **super-additive cover** of G to be the cooperative game $G^* = (N, v^*)$ where v^* is given by

$$v^*(S) := \max \left\{ \sum_{C \in \pi} v(C) : \pi \text{ is a partition of } S \right\}.$$

DEFINITION 6.7 (Convex Game)**.** We say that a game is **convex** if and only if its characteristic function satisfies

$$\forall A, B \subseteq N, \quad v(A) + v(B) \leq v(A \cup B) + v(A \cap B).$$

PROPOSITION 6.8. A convex game is super-additive.

DEFINITION 6.9 (Measure Game)**.** A **measure game** is a cooperative game with a characteristic function of the form $v(S) = f(\mu(S))$ where $f : [0, +\infty] \to \mathbb{R}_+$ and $\mu : \mathcal{P}(N) \to [0, +\infty]$ is a measure on $\mathcal{P}(N)$.

PROPOSITION 6.10. For a measure game $G = (N, v)$ with $v(S) = f(\mu(S))$:

1. If f is monotone, then G is monotone;
2. If f is super-additive, then G is super-additive;
3. If f is convex, then G is convex.

DEFINITION 6.11 (Marginal Contribution)**.** Let $\sigma = (\sigma_1, ..., \sigma_n)$ be a permutation of N. We define the **marginal contribution** of player σ_i in the permutation, denoted by $\Delta_\sigma(\sigma_i)$, to be a number given by

$$\Delta_\sigma(\sigma_i) := v(\{\sigma_1, ..., \sigma_i\}) - v(\{\sigma_1, ..., \sigma_{i-1}\}).$$

DEFINITION 6.12 (Shapley Value)**.** We define the **Shapley value** to be a solution concept which attempts to achieve the following properties:

- Fairness: The payoff x_i received by Player $i \in N$ should accurately reflect the amount that Player i contributed to the value $v(N)$ of the grand coalition.
- Stability: No sub-coalition should feel the need to pull away from the grand coalition.

(We assume the grand coalition forms. If it doesn't, then apply the solution concept separately to each coalition that forms.)

DEFINITION 6.13 (Shapley Value)**.** Let $n \in \mathbb{Z}_{++}$ denote the number of players. Let $N := \{1, ..., n\}$. We define the **Shapley value** of Player i, denoted by φ_i, to be a number given by

$$\varphi_i := \frac{1}{n!} \sum_{\sigma \in S_n} \Delta_\sigma(i)$$

where S_n denotes the set of all possible permutations of N.

DEFINITION 6.14 (Dummy Player)**.** Let $i \in N$. We say that player i is a **dummy player** if and only if

$$\forall S \subseteq N, \quad v(S \cup \{i\}) = v(S).$$

DEFINITION 6.15 (Symmetric Players)**.** Let $i, j \in N$ with $i \neq j$. We say that players i and j are **symmetric** if and only if

$$\forall C \subseteq (N \setminus \{i, j\}), \quad v(C \cup \{i\}) = v(C \cup \{j\}).$$

THEOREM 6.16. The Shapley value satisfies the following properties:

1. Efficiency (EFF): $\displaystyle\sum_{i \in N} \varphi_i = v(N)$.
2. Dummy Player (DUM): If $i \in N$ is a dummy player, then $\varphi_i = 0$.
3. Symmetry (SYM): If i and j are symmetric players, then $\varphi_i = \varphi_j$.
4. Linear additivity (ADD): If $(N, v^{(1)})$ and $(N, v^{(2)})$ are cooperative games, then the game $(N, v^{(1)} + v^{(2)})$ has Shapley values $\varphi_i^{(1)} + \varphi_i^{(2)}$, for $i \in N$.

Proof for Efficiency. Notice that $\forall \sigma \in S_n$, we have $\displaystyle\sum_{i \in N} \Delta_\sigma(i) = v(N)$. So

$$\sum_{i \in N} \varphi_i = \sum_{i \in N} \frac{1}{n!} \sum_{\sigma \in S_n} \Delta_\sigma(i) = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{i \in N} \Delta_\sigma(i) = \frac{1}{n!} \sum_{\sigma \in S_n} v(N) = v(N).$$

□

Proof for Dummy Player. Notice that $\Delta_\sigma(i)$ is always 0. □

Proof for Symmetric Players. Let $\tau \in S_n$ be the permutation that transposes i and j. I claim that $\forall \sigma \in S_n$, $\Delta_\sigma(i) = \Delta_{\tau \circ \sigma}(j)$. Suppose that i precedes j in σ. Let C be the predecessors of i under σ. Then

$$\begin{aligned} \Delta_\sigma(i) &= v(C \cup \{i\}) - v(C), \text{ and} \\ \Delta_{\tau \circ \sigma}(j) &= v(C \cup \{j\}) - v(C). \end{aligned}$$

These two values are equal, by symmetry. Now suppose that j precedes i under σ. Let C be the predecessors of i under σ, excluding j. Then

$$\Delta_\sigma(i) = v(C \cup \{i,j\}) - v(C \cup \{j\}), \text{ and}$$
$$\Delta_{\tau\circ\sigma}(j) = v(C \cup \{i,j\}) - v(C \cup \{i\}).$$

These two values are equal by symmetry. Now

$$\varphi_i = \frac{1}{n!}\sum_{\sigma\in S_n}\Delta_\sigma(i) = \frac{1}{n!}\sum_{\sigma\in S_n}\Delta_{\tau\circ\sigma}(j) = \varphi_j$$

□

Proof for Linear. □

DEFINITION 6.17 (Shapley Core)**.** ...

COROLLARY 6.18. For a convex game, the Shapley value is in the Shapley core.

Proof. In a convex game, each of solution concept x^σ is in the core. The shapley value is the average of the x^σ's. The core of any super-additive game is polyhedral, and hence convex. Hence the average of the x^σ's is in the core. □

DEFINITION 6.19. We say that a vector $\lambda \in \mathbb{R}_+^{2^N}$ of non-negative weights for coalitions of N is **balanced** if and only if

$$\forall i \in N, \quad \sum\{\lambda_S : S \subseteq N, i \in S\} = 1.$$

REMARK 6.20. Any partition π of N yields a balanced weight vector λ by setting

$$\lambda_C := \begin{cases} 1, & \text{if } C \in \pi \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 6.21. The core of (N, v) is non-empty if and only if

...

Proof. We may assume that (N, v) is super-additive. Consider the following primal and dual LPs:

$$\text{(LP)} \quad \max \sum_{S \subseteq N} \lambda_S v(S) \quad \text{subject to: } \sum_{\substack{S \subseteq N \\ i \in S}} \lambda_S = 1, \quad \lambda \geq \mathbb{0} \in \mathbb{R}^{2^N}.$$

$$\text{(LD)} \quad \min \; x(N) \quad \text{subject to: } x(S) \geq v(S), \quad \forall S \subseteq N.$$

Claim: The core is non-empty if and only if the optimal value for the (LD) is $v(N)$. Proof: Any core allocation satisfies the constraints and $x(N) = v(N)$. Conversely any allocation satisfying the constraints and $x(N) = v(N)$ is a core allocation.

($\Rightarrow$) Suppose the core is non-empty. Then $\min x(N) = v(N)$. By the Weak Duality Relation for Linear Programming, $\max \sum_{S \subseteq N} \lambda_S v(S) = v(N)$.

($\Leftarrow$) Suppose that the core is empty. Then $\min x(N) > v(N)$. By the Strong Duality Relation for Linear Programming, there is some feasible λ such that $\max \sum_{S \subseteq N} \lambda_S v(S) > v(N)$. □

6.2 Matching Games