Differential Equations

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Chapter 1

Differential Equations

1.1 Global Solution

Definition (Globally Lipschitz)**.** *Let* Ω *be a set i*

Theorem 1 (Global Picard Theorem)**.** *If* $\Phi : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and Lipschitz in* y*, and* $c \in [a, b]$*, then the differential equation*

$$F'(x) = \Phi(x, F'(x)), F(c) = \Gamma$$

has a unique solution.

Proof.

$$TF(x) = \Gamma + \int_c^x \Phi(t, F(t))dt.$$

This is a contraction mapping and has a unique fixed point. ■

1.2 Local Solution

Chapter 2

Laplace Transform

n^{th} Order Constant Coefficient Linear Differential Equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_0 y(t) = f(t)$$

2.1 Definition

Definition (Laplace Transform)**.** *We define the Laplace transform of f, denoted by $\mathcal{L}\{f\}$, to be the function given by*

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)dt.$$

Notation: Y(s)

$$Y(s)\mathcal{L}\{y(t)\}(s)$$

2.2 Basic Laplace Transform Formulas

1. Constant Functions

$$f(t) = 1$$

Derivation

$$\begin{aligned}\mathcal{L}\{f(t)\}(s) &= \int_0^\infty e^{-st} 1 dt \\ &= \left(-\frac{1}{s}e^{-st}\right)|_0^\infty = \frac{1}{s}\end{aligned}$$

That is,

$$\mathcal{L}\{1\}(s) = \frac{1}{s}.$$

2. Power Functions

$$f(t) = t^n$$

Derivation

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} t^n dt$$

$$= (-\frac{1}{s}e^{-st}t^n)|_0^\infty - \int_0^\infty (-\frac{1}{s}e^{-st})(nt^{n-1})dt = \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$

$$= \frac{n}{s}[(-\frac{1}{s}e^{-st}t^{n-1})|_0^\infty - \int_0^\infty (-\frac{1}{s}e^{-st})((n-1)t^{n-2})dt] = \frac{n(n-1)}{s^2}\int_0^\infty e^{-st}t^{n-2}dt$$

$$= \ldots = \frac{n!}{s^n}\int_0^\infty e^{-st}dt = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$$

3. Exponential Functions

$$f(t) = e^{at}$$

Derivation

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} e^{at} dt$$

$$= (-\frac{1}{s-a}e^{-(s-a)t})|_0^\infty = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$$

4. Power-Exponential Functions

$$f(t) = t^n e^{at}$$

Derivation

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} t^n e^{at} dt$$

$$= (-\frac{1}{s-a}e^{-(s-a)t}t^n)|_0^\infty - \int_0^\infty (-\frac{1}{s-a}e^{-(s-a)t})(nt^{n-1})dt = \frac{n}{s-a}\int_0^\infty e^{-(s-a)t}t^{n-1}dt$$

$$= \frac{n}{s-a}[(-\frac{1}{s-a}e^{-(s-a)t}t^{n-1})|_0^\infty - \int_0^\infty (-\frac{1}{s-a}e^{-(s-a)t})((n-1)t^{n-2})dt] = \frac{n(n-1)}{(s-a)^2}\int_0^\infty e^{-(s-a)t}t^{n-2}dt$$

$$= \ldots = \frac{n!}{(s-a)^n}\int_0^\infty e^{-(s-a)t}dt = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}}$$

5. Cosine Functions

$$f(t) = \cos(bt)$$

Derivation

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st}\cos(bt)dt$$

$$= (-\frac{1}{s}e^{-st}\cos(bt))|_0^\infty - \int_0^\infty (-\frac{1}{s}e^{-st})(-b\sin(bt))dt = \frac{1}{s} - \frac{b}{s}\int_0^\infty e^{-st}\sin(bt)dt$$

$$= \frac{1}{s} - \frac{b}{s}[(-\frac{1}{s}e^{-st}\sin(bt))|_0^\infty - \int_0^\infty (-\frac{1}{s}e^{-st})(b\cos(bt))dt] = \frac{1}{s} - \frac{b^2}{s^2}\int_0^\infty e^{-st}\cos(bt)dt$$

$$= \frac{1}{s} - \frac{b^2}{s^2}\mathcal{L}\{f(t)\}(s)$$

Solving for $\mathcal{L}\{f(t)\}(s)$ gives

$$\mathcal{L}\{\cos(bt)\}(s) = \frac{s}{s^2+b^2}$$

6. Sine Function

$$f(t) = \sin(bt)$$

Derivation

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st}\sin(bt)dt$$

$$= (-\frac{1}{s}e^{-st}\sin(bt))|_0^\infty - \int_0^\infty (-\frac{1}{s}e^{-st})(b\cos(bt))dt = \frac{b}{s}\int_0^\infty e^{-st}\cos(bt)dt$$

$$= \frac{b}{s}[(-\frac{1}{s}e^{-st}\cos(bt))|_0^\infty - \int_0^\infty (-\frac{1}{s}e^{-st})(-b\sin(bt))dt] = \frac{b}{s^2} - \frac{b^2}{s^2}\int_0^\infty e^{-st}\sin(bt)dt$$

$$= \frac{b}{s^2} - \frac{b^2}{s^2}\mathcal{L}\{f(t)\}(s)$$

Solving for $\mathcal{L}\{f(t)\}(s)$ gives

$$\mathcal{L}\{\sin(bt)\}(s) = \frac{b}{s^2+b^2}$$

Summary

$f(t)$	$\mathcal{L}\{f(t)\}(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$

Expressing $\mathcal{L}\{\mathbf{f^{(n)}}(\mathbf{t})\}(\mathbf{s})$ in terms of $\mathbf{Y}(\mathbf{s})$

Derivation

$$\mathcal{L}\{f^{(n)}(t)\}(s) = \int_0^\infty e^{-st} f^{(n)}(t)dt$$

$$= (f^{(n-1)}(t)e^{-st})|_0^\infty - \int_0^\infty (f^{(n-1)}(t))(-se^{-st})dt = -f^{(n-1)}(0) + s\int_0^\infty f^{(n-1)}(t)e^{-st}dt$$

$$= -f^{(n-1)}(0) + s[(f^{(n-2)}(t)e^{-st})|_0^\infty - \int_0^\infty (f^{(n-2)}(t))(-se^{-st})dt] = -f^{(n-1)}(0) - sf^{(n-2)}(0) + s^2\int_0^\infty f^{(n-2}$$

$$= \ldots = s^n \int_0^\infty f(t)e^{-st}dt - \sum_{j=0}^{n-1} s^j f^{(n-j-1)}(0)$$

$$= s^n Y(s) - \sum_{j=0}^{n-1} s^j f^{(n-j-1)}(0)$$

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n Y(s) - \sum_{j=0}^{n-1} s^j f^{(n-j-1)}(0)$$

Definition: Exponential Type

We say a function $f : [0, +\infty) \to \mathbb{R}$ is of exponential type with order a if there exists $K \in \mathbb{R}$ such that for all $t \in [0, +\infty)$, we have

$$|f(t)| \leq Ke^{at}$$

Easy Facts

A function is bounded if and only if it is of exponential type with order $a = 0$.
A function is converging to 0 if and only if it is of exponential type with order $a < 0$.

Theorem 2. *The set of functions of exponential type $\mathcal{F}$ is a vector space over $\mathbb{R}$.*

Proof.
First we prove that $\mathcal{F}$ is closed under addition and scalar multiplication.
Let $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ be arbitrary.
Say f is of exponential type with order a and g is of exponential type with order b.
Then there exists K_1 and K_2 such that for all $t \in [0, +\infty)$, we have

$$|f(t)| < K_1 e^{at} \#(1)$$

$$|g(t)| < K_2 e^{bt} \#(2)$$

Then we have

$$|(f+g)(t)| \leq |f(t)| + |g(t)| \#(3)$$

$$|(\mathrm{cf})(t)| \leq |c||f(t)| \#(4)$$

From (1) ˜ (3), we get

$$|(f+g)(t)| \leq K_1 e^{at} + K_2 e^{bt} \leq 2\max\{K_1, K_2\} e^{\max\{a,b\}t} \#(5)$$

$$|(\mathrm{cf})(t)| \leq |c| K_1 e^{at} \#(6)$$

From (5) and (6), we conclude that $\mathcal{F}$ is closed under addition and scalar multiplication.
By definition of addition and scalar multiplication of functions, $\mathcal{F}$ automatically satisfies all 8 properties of vector space.
Now we conclude that $\mathcal{F}$ is a vector space over $\mathbb{R}$.

■

Lemma. *Let $f : [0, +\infty)$ be a function of exponential type with order a. Let $s > a$. Then we have*

$$e^{-st} f(t) = 0$$

Proof.
By definition, there exists $K \in \mathbb{R}^+$ such that for all $t \in [0, +\infty)$, we have

$$-Ke^{at} \leq f(t) \leq Ke^{at}$$

Multiplying by e^{-st} gives

$$-Ke^{at}e^{-st} \leq e^{-st} f(t) \leq Ke^{at}e^{-st}$$

Define $b = a - s$. Then the last inequation can be written as

$$-Ke^{bt} \leq e^{-st}f(t) < -Ke^{bt}\#(1)$$

Note that $b < 0$. Thus

$$\{\begin{matrix}(-Ke^{bt}) = 0 \\ Ke^{bt} = 0\end{matrix}\#(2)$$

From (1) and (2), by the Squeeze Principle, we get

$$e^{-st}f(t) = 0$$

■

Theorem: Sufficient Condition for Existence of the Laplace Transform
Let f be a function of exponential type with order a. Then the Laplace Transform $\mathcal{L}\{f(t)\}(s)$ exists for all $s > a$. Moreover, we have

$$\mathcal{L}\{f(t)\}(s) = 0$$

Proof.
Part 1
Say there exists $K \in \mathbb{R}^+$ such that

$$|f(t)| \leq Ke^{at}$$

Consider the Laplace Transform

$$\text{L }\{|f(t)|\}(s) = \int_0^\infty e^{-st}|f(t)|dt$$

By the Orders Property of Integrals, we get

$$\int_0^\infty (-Ke^a e^{-st})dt \leq \int_0^\infty e^{-st}|f(t)|dt \leq \int_0^\infty Ke^a e^{-st}dt$$

$$(-\frac{-K}{s-a}e^{-(s-a)t})|_0^\infty \leq \int_0^\infty e^{-st}|f(t)|dt \leq (-\frac{K}{s-a}e^{-(s-a)t})|_0^\infty$$

$$-\tfrac{K}{s-a} \leq \textstyle\int_0^\infty e^{-st}|f(t)|dt \leq \tfrac{K}{s-a}\#(1)$$

Thus the integral is bounded.
It follows that the improper integral

$$\int_0^\infty e^{-st}|f(t)|dt$$

exists.
Thus the improper integral

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st}f(t)dt$$

exists.
Part 2
Note that

$$\{ \begin{matrix} (-\frac{K}{s-a}) = 0 \\ \frac{K}{s-a} = 0 \end{matrix} \#(2)$$

From (1) and (2), by the Squeeze Principle, we get

$$\mathcal{L}\{f(t)\}(s) = 0$$

■

Lemma. *Let $f : [0, +\infty) \to \mathbb{R}$ be continuous and of exponential type with order a. Then any antiderivative $F(t) : [0, +\infty) \to \mathbb{R}$ of f is of exponential type. Furthermore, if $a > 0$, then F also has order a; if $a \leq 0$, then F has order b for all $b \in \mathbb{R}^+$.*

Proof.
Say there exists $K \in \mathbb{R}^+$ such that for all $t \in [0, +\infty)$, we have

$$|f(t)| \leq Ke^{at}\#(1)$$

Since f is continuous, its antiderivative F exists.
From (1), by the Orders Property of Integrals, we get

$$\int_{t_0}^{t} (-Ke^{at})dt \leq \int_{t_0}^{t} f(t)dt \leq \int_{t_0}^{t} Ke^{at}dt$$

$$(-\frac{K}{a}e^{at})|_{t_0}^{t} \leq \int_{t_0}^{t} f(t)dt \leq (\frac{K}{a}e^{at})|_{t_0}^{t}$$

$$-(\frac{K}{a}e^{at} + C) \leq F(t) + C^{'} \leq \frac{K}{a}e^{at} + C$$

$$-(\tfrac{K}{a}e^{at} + C) \leq F(t) \leq \tfrac{K}{a}e^{at} + C\#(2)$$

Case 1: $\mathbf{a > 0}$
In this case, we have

$$\tfrac{K}{a}e^{at} + C \leq (\tfrac{K}{a} + C)e^{at}\#(3)$$

From (2) and (3), we get

$$|F(t)| \leq (\frac{K}{a} + C)e^{at}$$

By definition, F is of exponential type with order a.
Case 2: $\mathbf{a \leq 0}$
In this case, we have

$$\tfrac{K}{a}e^{at} + C \leq \tfrac{K}{a} + C\#(4)$$

From (2) and (4), we get

$$|F(t)| \leq \frac{K}{a} + C$$

By definition, F is of exponential type with order b for all $b \in \mathbb{R}^+$.

■

2.3 Arithmetic on Input Space and Output Space

Proposition 2.3.1 (Linearity)**.** *Let $f, g : [0, +\infty) \to \mathbb{R}$ be functions of exponential type. Let $\alpha, \beta \in \mathbb{R}$. Then we have*

$$\mathcal{L}\{(\alpha f + \beta g)(t)\}(s) = \alpha\mathcal{L}\{f(t)\}(s) + \beta\mathcal{L}\{g(t)\}(s)$$

Proof.
By Theorem, the function $(\alpha f + \beta g)$ is of exponential type.
Thus $\mathcal{L}\{(\alpha f + \beta g)(t)\}(s)$ exists.
By the linearity of improper integrals, we get

$$\mathcal{L}\{(\alpha f + \beta g)(t)\}(s) = \int_0^\infty e^{-st}(\alpha f + \beta g)(t)dt$$

$$= \alpha \int_0^\infty e^{-st} f(t)dt + \beta \int_0^\infty e^{-st} g(t)dt$$

$$\alpha\mathcal{L}\{f(t)\}(s) + \beta\mathcal{L}\{g(t)\}(s)$$

■

Theorem: Input First Derivative Principle
Let $f : [0, +\infty) \to \mathbb{R}$ be continuously differentiable. Suppose that $f^{'}(t) : [0, +\infty) \to \mathbb{R}$ is of exponential type with order a. Then we have

$$\mathcal{L}\{f^{'}(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0)$$

Proof.
By Lemma, $f(t)$ is of exponential type.
Thus $\mathcal{L}\{f(t)\}(s)$ exists.
It follows that

$$\mathcal{L}\{f^{'}(t)\}(s) = \int_0^\infty e^{-st} f^{'}(t)dt$$

$$= (f(t)e^{-st})|_0^\infty - \int_0^\infty (f(t))(-se^{-st})dt = -f(0) + s\int_0^\infty f(t)e^{-st}dt$$

$$= s\mathcal{L}\{f(t)\}(s) - f(0)$$

■

Theorem: Input Derivative Principle
Let $f : [0, +\infty) \to \mathbb{R}$ be n times continuously differentiable. Suppose that $f^{(j)}(t) : [0, +\infty) \to \mathbb{R}$ is of exponential type with order a. Then for all $s > a$, we have

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f(t)\}(s) - \sum_{j=0}^{n-1} s^j f^{(n-j-1)}(0)$$

Proof.

$$\mathcal{L}\{f^{(n)}(t)\}(s) = \int_0^\infty e^{-st} f^{(n)}(t)dt$$

$$= (f^{(n-1)}(t)e^{-st})|_0^\infty - \int_0^\infty (f^{(n-1)}(t))(-se^{-st})dt = -f^{(n-1)}(0) + s\int_0^\infty f^{(n-1)}(t)e^{-st}dt$$

$$= -f^{(n-1)}(0) + s[(f^{(n-2)}(t)e^{-st})|_0^\infty - \int_0^\infty (f^{(n-2)}(t))(-se^{-st})dt] = -f^{(n-1)}(0) - sf^{(n-2)}(0) + s^2\int_0^\infty f^{(n-2)}(t)e^{-st}dt$$

$$= \ldots = s^n \int_0^\infty f(t)e^{-st}dt - \sum_{j=0}^{n-1} s^j f^{(n-j-1)}(0)$$

$$= s^n \mathcal{L}\{f(t)\}(s) - \sum_{j=0}^{n-1} s^j f^{(n-j-1)}(0)$$

■

Theorem: First Translation Principle

$$\mathcal{L}\{e^{at} f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)$$

Proof.

$$\mathcal{L}\{e^{at} f(t)\}(s) = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt = \mathcal{L}\{f(t)\}(s-a)$$

■

Theorem: Transform Derivative Principle

$$\mathcal{L}\{-tf(t)\}(s) = \frac{d}{\mathrm{ds}}(\mathcal{L}\{f(t)\}(s))$$

Proof.

$$\mathcal{L}\{-tf(t)\}(s) = \int_0^\infty e^{-st}(-tf(t))dt$$

$$= \int_0^\infty \frac{\partial(e^{-st}f(t))}{\partial s}dt = \frac{d}{\mathrm{ds}}\int_0^\infty e^{-st}f(t)dt$$

$$= \frac{d}{\mathrm{ds}}\mathcal{L}\{f(t)\}(s)$$

■

Theorem: Transform nth Derivative Principle

$$\mathcal{L}\{(-t)^n f(t)\}(s) = \frac{d^n}{ds^n}(\mathcal{L}\{f(t)\}(s))$$

Proof.

$$\mathcal{L}\{(-t)^n f(t)\}(s) = \int_0^\infty e^{-st}(-t)^n f(t)dt$$

$$= \int_0^\infty \frac{\partial^n(e^{-st}f(t))}{\partial s^n}dt = \frac{d^n}{ds^n}\int_0^\infty e^{-st}f(t)dt$$

$$= \frac{d^n}{ds^n}(\mathcal{L}\{f(t)\}(s))$$

■

Theorem: The Dilation Principle

$$\mathcal{L}\{f(bt)\}(s) = \frac{1}{b}\mathcal{L}\{f(t)\}(\frac{s}{b})$$

Proof.

$$\mathcal{L}\{f(bt)\}(s) = \int_0^\infty e^{-st}f(bt)dt$$

$$= \frac{1}{b}\int_0^\infty e^{-\frac{s}{b}(bt)}f(bt)d(bt) = \frac{1}{b}\int_0^\infty e^{-\frac{s}{b}t}f(t)dt$$

$$= \frac{1}{b}\mathcal{L}\{f(t)\}(\frac{s}{b})$$

■

Definition (Inverse Laplace Transform)**.** *Given $F(s)$, we call the function $f(t)$ the* **inverse Laplace transform** *of $F(s)$ if*

$$\mathcal{L}\{f(t)\}(s) = F(s)$$

Proposition 2.3.2 (Uniqueness)**.** *Let $f_1, f_2 : [0, +\infty) \to \mathbb{R}$ be continuous. Then if $\mathcal{L}\{f_1(t)\}(s) = \mathcal{L}\{f_2(t)\}(s)$, $f_1 = f_2$. i.e. if a transform function has a continuous input function, then it can have only one such input function.*

Proposition 2.3.3 (Linearity)**.** *Let $F(s)$ and $G(s)$ be the Laplace transform functions with continuous input functions. Let $\alpha, \beta \in \mathbb{R}$. Then we have*

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\}(t) = \alpha\mathcal{L}^{-1}\{F(s)\}(t) + \beta\mathcal{L}^{-1}\{G(s)\}(t)$$

Proposition 2.3.4 (Inverse First Translation Principle)**.** *Let $F(s)$ be a Laplace transform function with a continuous input function. Then*

$$\mathcal{L}^{-1}\{F(s-a)\}(t) = e^{at}\mathcal{L}\{f(t)\}(s)$$

Proposition 2.3.5 (Reduction of Order)**.** *For $b \neq 0$ and $k \in \mathbb{N}$, we have*

$$\mathcal{L}^{-1}\{\frac{1}{(s^2+b^2)^{k+1}}\}(t) = \frac{-t}{2kb^2}\mathcal{L}^{-1}\{\frac{s}{(s^2+b^2)^k}\}(t) + \frac{2k-1}{2kb^2}\mathcal{L}^{-1}\{\frac{1}{(s^2+b^2)^k}\}(t)$$

$$\mathcal{L}^{-1}\{\frac{s}{(s^2+b^2)^{k+1}}\}(t) = \frac{t}{2k}\mathcal{L}^{-1}\{\frac{1}{(s^2+b^2)^k}\}(t)$$

Proof.
By the Transform Derivative Principle, we have

$$\mathcal{L}\{-t\mathcal{L}^{-1}\{\frac{s}{(s^2+b^2)^k}\}(t)\}(s) = \frac{d}{\mathrm{ds}}\mathcal{L}\{\mathcal{L}^{-1}\{\frac{s}{(s^2+b^2)^k}\}(t)\}(s)$$

$$= \frac{d}{\mathrm{ds}}\frac{s}{(s^2+b^2)^k} = \frac{s^2+b^2-2ks^2}{(s^2+b^2)^{k+1}} = \frac{b^2}{(s^2+b^2)^{k+1}} + (1-2k)\frac{s^2}{(s^2+b^2)^{k+1}}$$

By the Transform Derivative Principle, we have

$$\mathcal{L}\{-t\mathcal{L}^{-1}\{\frac{1}{(s^2+b^2)^k}\}(t)\}(s) = \frac{d}{\mathrm{ds}}\mathcal{L}\{\mathcal{L}^{-1}\{\frac{1}{(s^2+b^2)^k}\}(t)\}(s)$$

$$= \frac{d}{\mathrm{ds}}\frac{1}{(s^2+b^2)^k} = \frac{-2sk}{(s^2+b^2)^{k+1}}$$

Dividing both sides by $-2k$ gives

$$\frac{1}{2k}\mathcal{L}\{t\mathcal{L}^{-1}\{\frac{1}{(s^2+b^2)^k}\}(t)\}(s) = \frac{s}{(s^2+b^2)^{k+1}}$$

Take the inverse Laplace transform of both sides gives

$$\frac{1}{2k}t\mathcal{L}^{-1}\{\frac{1}{(s^2+b^2)^k}\}(t) = \mathcal{L}^{-1}\{\frac{s}{(s^2+b^2)^{k+1}}\}(t)$$

as desired.

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