Functional Analysis

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Contents

Chapter 1

Balanced Sets

1.1 Definitions

Definition (Balanced Sets)**.** *Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We say that S is **balanced** if*

$$\forall a \in \mathbb{F} : |a| \leq 1, \quad aS \subseteq S.$$

Definition (Balanced Hull)**.** *Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We define the **balanced hull** of S, denoted by* $\operatorname{balhull}(S)$*, to be the smallest balanced set containing S.*

Definition (Balanced Core)**.** *Let X be a vector space over field $\mathbb{F}$. Let S be a subset of X. We define the **balanced core** of S, denoted by* $\operatorname{balcore}(S)$*, to be the largest balanced set contained in S.*

1.2 Properties

Proposition 1.2.1. *Let X be a vector space over field $\mathbb{F}$. Let B be a balanced subset of X. Then*

$$\forall a, b \in \mathbb{F} : |a| \leq |b|, \quad aB \subseteq bB.$$

Proposition 1.2.2. *Balanced sets are path connected.*

Proposition 1.2.3 (Act on Other Properties)**.**

- *The balanced hull of a compact set is compact.*
- *The balanced hull of a totally bounded set is totally bounded.*
- *The balanced hull of a bounded set is bounded.*

Proposition 1.2.4 (Act on Other Properties). • *The balanced core of a closed set is closed.*

Proposition 1.2.5. *Let X be a vector space over field $\mathbb{F}$. Let a be a scalar in field $\mathbb{F}$. Then*

$$a\,\mathrm{balhull}(S) = \mathrm{balhull}(aS).$$

1.3 Stability of Balance

Proposition 1.3.1 (Set Operations). • *The union of balanced sets is also balanced.*

- *The intersection of balanced sets is also balanced.*

Proposition 1.3.2 (Linear Mappings). • *The scalar multiple of a balanced set is also balanced.*

- *The (Minkowski) sum of two balanced sets is also balanced.*
- *The image of a balanced set under a linear operator is also balanced.*
- *The inverse image of a balanced set under a linear operator is also balanced.*

Proposition 1.3.3 (Topological Operations). *The closure of a balanced set is also balanced.*

Proposition 1.3.4. *The convex hull of a balanced set is also balanced (and also convex).*

1.4 Absorbing Sets

Definition (Absorbing Sets). *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let S be a subset of X. We say that S is **absorbing** if*

$$\forall x \in \mathfrak{X}, \quad \exists r \in \mathbb{R} : r > 0, \quad \forall c \in \mathbb{F} : |c| \geq r, \quad x \in cS.$$

i.e.,

$$\bigcup_{n\in\mathbb{N}} nS = \mathfrak{X}.$$

Proposition 1.4.1. *Every absorbing set contains the origin.*

Chapter 2

Normed Linear Spaces

2.1 Definitions

Definition (Seminorm)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **seminorm** on $\mathfrak{X}$, denoted by ν, to be a map from $\mathfrak{X}$ to $\mathbb{R}$ that satisfies the following conditions.*

(1) $\forall x \in \mathfrak{X}, \quad \nu(x) \geq 0.$

(2) $\forall \lambda \in \mathbb{F}, \forall x \in \mathfrak{X}, \quad \nu(\lambda x) = \lambda \nu(x).$

(3) Triangle Inequality.

$$\forall x, y \in \mathfrak{X}, \quad \nu(x+y) \leq \nu(x) + \nu(y).$$

The idea behind the seminorm is that we are trying to give our vector space a notion of "length" of vectors.

Definition (Norm)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. We define a **norm** on $\mathfrak{X}$, denoted by ν, to be a seminorm on $\mathfrak{X}$ that satisfies the additional condition:*

$$\forall x \in \mathfrak{X}, \quad \mu(x) = 0 \iff x = 0.$$

2.2 Properties

Proposition 2.2.1. *Let $(V, \|\cdot\|_V)$ be a normed vector space over field $\mathbb{F}$. Then $(V, \|\cdot\|)$ is complete if and only if $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.*

Proof.

For one direction, assume that $(V, \|\cdot\|)$ is complete.

We are to prove that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

Since $(\overline{B(0,1)}, \|\cdot\|_V)$ is a closed subspace of $(V, \|\cdot\|)$ and $(V, \|\cdot\|)$ is complete, $(\overline{B(0,1)}, \|\cdot\|_V)$ is also complete.

For the reverse direction, assume that $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete.

We are to prove that $(V, \|\cdot\|_V)$ is complete.

Let $\{x_i\}_{i\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $(V, \|\cdot\|_V)$.

Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ is bounded in $(V, \|\cdot\|_V)$.

Let λ be a positive upper bound for $\{\|x_i\|_V\}_{i\in\mathbb{N}}$.

Since $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_V)$, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ is Cauchy in $(\overline{B(0,1)}, \|\cdot\|_V)$ and $(\overline{B(0,1)}, \|\cdot\|_V)$ is complete, $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$.

Since $\{x_i/\lambda\}_{i\in\mathbb{N}}$ converges in $(\overline{B(0,1)}, \|\cdot\|_V)$, $\{x_i\}_{i\in\mathbb{N}}$ converges in $(V, \|\cdot\|_V)$.

Since any Cauchy sequence in $(V, \|\cdot\|_V)$ converges in $(V, \|\cdot\|_V)$, $(V, \|\cdot\|_V)$ is complete. ■

Proposition 2.2.2. *Proper subspaces of a normed linear space has empty interior.*

Proof. Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a proper subspace of $\mathfrak{X}$. Assume for the sake of contradiction that $\mathcal{M}$ has non-empty interior. Then $\exists x_0 \in \mathcal{M}$ and $\exists r > 0$ such that $\text{ball}(x_0, r) \subseteq \mathcal{M}$ where $\text{ball}(x_0, r)$ denotes the open ball centered at point x_0 with radius r. Let x be an arbitrary point in $\mathfrak{X}$. Define a point $y(x)$ as $y(x) := x_0 + \frac{r}{2\|x\|}x$. Then $x = \frac{2\|x\|}{r}(y - x_0)$. It is easy to verify that $\|y - x_0\| = \frac{r}{2} < r$. So $y \in \text{ball}(x_0, r)$. So $y \in \mathcal{M}$. Since $y, x_0 \in \mathcal{M}$ and $\mathcal{M}$ is a subspace, we get $\frac{2\|x\|}{r}(y - x_0) \in \mathcal{M}$. That is, $x \in \mathcal{M}$. So $\forall x \in \mathfrak{X}$, $x \in \mathcal{M}$. So $\mathcal{M} = \mathfrak{X}$. This contradicts to the assumption that $\mathcal{M}$ is a proper subspace of $\mathfrak{X}$. So $\mathcal{M}$ has empty interior. ■

Proposition 2.2.3. *Closed proper subspaces of a normed linear space are nowhere dense.*

Proof. Let $\mathfrak{X}$ be a normed linear space. Let $\mathcal{M}$ be a closed proper subspace of $\mathfrak{X}$. Since $\mathcal{M}$ is closed, $\text{cl}(\mathcal{M}) = \mathcal{M}$. So $\text{cl}(\mathcal{M}) = \mathcal{M}$ is a closed proper subspace of $\mathfrak{X}$. Since $\text{cl}(\mathcal{M})$ is a proper subspace of $\mathfrak{X}$, $\text{int}(\text{cl}(\mathcal{M})) = \emptyset$. So $\mathcal{M}$ is nowhere dense. ■

Proposition 2.2.4. *Finite dimensional subspace of a normed linear space is closed.*

Proposition 2.2.5. *Finite-dimensional normed linear spaces are complete.*

2.3 Equivalence of Norms

Definition (Equivalence of Norms)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are **equivalent** if*

$$\exists c_1, c_2 > 0, \quad \forall v \in \mathfrak{X}, \quad c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_2.$$

Or equivalently,

$$c_1\|v\|_2 \leq \|v\|_1 \leq c_2\|v\|_2.$$

Proposition 2.3.1. *The equivalence of norms is an equivalence relation.*

Theorem 1. *Let V be a finite dimensional vector space over field $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$. Then any two norms on V are equivalent.*

Proof.

Let $\|\cdot\|_p$ be an arbitrary p-norm on V and $\|\cdot\|$ be an arbitrary norm on V.

Let $\mathcal{B}$ be the standard basis for V. Say $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$.

Let v be an arbitrary vector in V.

$$\begin{aligned}
\|v\| &= \|\sum_{i=1}^{n} v_i e_i\| \\
&\leq \sum_{i=1}^{n} |v_i| \|e_i\| \\
&\leq \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \\
&= \left(\sum_{i=1}^{n} \|e_i\|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \|v\|_p \\
&:= c_1\|v\|_p.
\end{aligned}$$

■

Proposition 2.3.2. *Let X be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if they generate the same metric topology.*

Proof. Convergence to 0 is equivalent under either $\|\cdot\|_1$ or $\|\cdot\|_2$. i.e., equivalent norms give rise to the same set of sequences that are convergent. Convergence of sequences defines the topology. ■

Proposition 2.3.3. *Let $\mathfrak{X}$ be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathfrak{X}$. Let ι be the identity map from $(\mathfrak{X}, \|\cdot\|_1)$ to $(\mathfrak{X}, \|\cdot\|_2)$. Then if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, ι is continuous, and in fact, a homeomorphism between $(\mathfrak{X}, \|\cdot\|_1)$ and $(\mathfrak{X}, \|\cdot\|_2)$.*

2.4 Dual Norms

Definition (Dual Norm)**.** *Let $(V, \|\cdot\|)$ be an normed vector space. We define the **dual norm** of $\|\cdot\|$, denoted by $\|\cdot\|_\circ$, to be a function given by*

$$\|v\|_\circ := \max_{\|w\|=1} v \cdot w = \max_{\|w\|\neq 0} \frac{|v \cdot w|}{\|w\|}.$$

Proposition 2.4.1. *Dual norms of norms are indeed norms.*

Proposition 2.4.2. *Let $(V, \|\cdot\|)$ be a normed vector space. Let v, w be vectors in the space. Then*

$$|v \cdot w| \leq \|v\| \cdot \|w\|_\circ.$$

2.5 p-norms

Definition (p-norm)**.** *Let V be a finite-dimensional normed vector space over field $\mathcal{F}$. Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a basis for V where $n = \dim(V)$. Let v be a vector in a normed vector space. For $p \in [1, +\infty)$, we define the **p-norm** of v, denoted by $\|v\|_p$, to be the number given by*

$$\|v\|_p = \left(\sum_{i=1}^{n} |(v_\mathcal{B})_i|^p\right)^{\frac{1}{p}}.$$

Definition (Infinity Norm - 1)**.** *Let $\mathfrak{X} = \mathbb{K}^n$ where $\mathbb{K}$ is a field and $n \in \mathbb{N}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by*

$$\|v\|_\infty := \max\{|v_i|\}_{i=1}^n.$$

Definition (Infinity Norm - 2)**.** *Let $\mathfrak{X} = \mathbb{K}^\mathbb{N}$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by*

$$\|v\|_\infty := \sup_{i \in \mathbb{N}} |v_i|.$$

Definition (Infinity Norm - 3)**.** *Let $\mathfrak{X} = \mathcal{C}([0,1], \mathbb{C})$. We define the **infinity norm** on $\mathfrak{X}$, denoted by $\|\cdot\|_\infty$, to be a function given by*

$$\nu(f) := \sup_{x \in [0,1]} |f(x)|.$$

Proposition 2.5.1. *Let $\mathfrak{X} := \mathcal{C}([0,1], \mathbb{C})$. Let x be an arbitrary number in $[0,1]$. Define a function ν_x on $\mathfrak{X}$ by $\nu_x(f) := |f(x)|$. Define a function ν on $\mathfrak{X}$ by $\nu(f) := \sup_{x \in [0,1]} |f(x)|$. Then ν_x is a seminorm on $\mathfrak{X}$ for each x and ν is a norm on $\mathfrak{X}$ and we have $\nu = \sup_{x \in [0,1]} \nu$.*

Proposition 2.5.2. *p-norms are indeed norms.*

Proposition 2.5.3. *For any vector v in $\mathbb{R}^n$, we have*

$$\lim_{p \to \infty} \|v\|_p = \|v\|_\infty.$$

i.e.,

$$\lim_{p \to \infty} \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}} = \max\{|v_i|\}_{i=1}^n.$$

Proof. Let p be an arbitrary number in $[1,+\infty)$. Let k be an arbitrary index in $\{1,..,n\}$. Then

$$|v_k| \leq (\sum_{i=1}^{n} |v_k|^p)^{1/p} = \|v\|_p.$$

So

$$\max\{|v_k|\} = \|v\|_\infty \leq \|v\|_p.$$

So

$$\lim_{p\to\infty} \|v\|_p \geq \|v\|_\infty. \tag{1}$$

On the other hand, note that

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p = \sum_{i=1}^{n} (\frac{|v_i|}{\|v\|_\infty})^p$$

decreases as p increases. So it is bounded above. Say

$$(\sum_{i=1}^{n} |v_i|^p)/\|v\|_\infty^p \leq C$$

for some $C \in \mathbb{R}$. Then

$$(\sum_{i=1}^{n} |v_i|^p)^{1/p} = \|v\|_p \leq C^{1/p}\|v\|_\infty.$$

So

$$\lim_{p\to\infty} \|v\|_p \leq \lim_{p\to\infty} C^{1/p}\|v\|_\infty = \|v\|_\infty. \tag{2}$$

From (1) and (2) we get

$$\lim_{p\to\infty} \|v\|_p = \|v\|_\infty.$$

■

Proposition 2.5.4. *Let p be an arbitrary number in $[1,+\infty)$. Then the dual norm of the p-norm $\|\cdot\|_p$ is the q-norm $\|\cdot\|_q$ where q is such that satisfies*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proposition 2.5.5. *Let p and q be numbers in $[1,+\infty]$. Let v be a vector in $\mathbb{R}^n$. Then if $p \leq q$,*

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \cdot \|x\|_q.$$

Proposition 2.5.6. *Let w and z be vectors in $\mathbb{E}^d$. Then*

$$\|w+z\|_2^2 + \|w-z\|_2^2 = 2(\|w\|_2^2 + \|z\|_2^2).$$

Chapter 3

Inner Product Spaces

3.1 Inner Products

3.1.1 Definitions

Definition (Inner Product). *Let V be a vector space over field $\mathbb{F}$. We define an* **inner product** *on V, denoted by $\langle \cdot, \cdot \rangle$, to be a scalar-valued function defined on $V \times V$ such that*

(1) Positive Definiteness:

$$\forall x \in V, \langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \iff x = 0.$$

(2) Sesqui-Linearity:

$$\begin{aligned} \forall x, y, z, w \in V, &\quad \langle x + y, z + w \rangle = \langle x, z \rangle + \langle y, z \rangle + \langle x, w \rangle + \langle y, w \rangle, \text{ and} \\ \forall a, b \in \mathbb{F}, \forall x, y \in V, &\quad \langle ax, by \rangle = a\bar{b}\langle x, y \rangle. \end{aligned}$$

(3) Conjugate Symmetry:

$$\forall x, y \in V, \quad \langle x, y \rangle = \overline{\langle y, x \rangle}.$$

Definition (Induced Norm). *Let $\mathfrak{X}$ be an inner product space over field $\mathbb{K}$. We define the* ***norm induced by*** *$\langle \cdot, \cdot \rangle$, denoted by $\| \cdot \|$, to be a function from $\mathfrak{X}$ to $\mathbb{R}_+$ given by*

$$\|x\| := \sqrt{\langle x, x \rangle}$$

3.1.2 Examples of Inner Products

Definition (Standard Inner Product). *For $V = \mathbb{F}^n$, we define the* ***standard inner product*** *by*

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i \overline{y_i}.$$

Definition (Frobenius Inner Product)**.** *For $V = \mathbb{F}^{n\times n}$, we define the **Frobenius inner product** by*

$$\langle M_1, M_2\rangle := \operatorname{tr}(M_2^* M_1).$$

Definition. *Let V be the space of continuous scalar-valued functions on $[0, 2\pi]$. We define the inner product on V by*

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx.$$

3.1.3 Properties

Proposition 3.1.1. *Let V be a finite dimensional inner product space. Let $\mathcal{B}$ be a basis for V. Let x and y be vectors in V. Then*

$$x = y \iff \forall b \in \mathcal{B}, \quad \langle x, b\rangle = \langle y, b\rangle.$$

3.2 Inner Product Space

Definition (Inner Product Space)**.** *Let $\mathfrak{X}$ be a vector space over field $\mathbb{F}$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{X}$. We define an **inner product space** to be the pair $(\mathfrak{X}, \langle\cdot,\cdot\rangle)$.*

3.3 Inequalities

Theorem 2 (Minkowski)**.**

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$$

Proposition 3.3.1 (Cauchy-Schwarz Inequality)**.** *Let V be an inner product space. Then*

$$\forall x, y \in V, \quad |\langle x, y\rangle| \le \|x\| \cdot \|y\|$$

Proposition 3.3.2 (Triangle Inequality)**.** *Let V be an inner product space. Then*

$$\forall x, y \in V, \quad \|x + y\| \le \|x\| + \|y\|$$

Proposition 3.3.3 (Parallelogram Law)**.** *Let $\mathfrak{X}$ be an inner product space. Then*

$$\forall x, y \in \mathfrak{X}, \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Proof.

$$\|x + y\|^2 + \|x - y\|^2 = \langle x + y, x + y\rangle + \langle x - y, x - y\rangle$$

$$\begin{aligned} &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &\quad + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= 2\langle x, x\rangle + 2\langle y, y\rangle \\ &= 2\|x\|^2 + 2\|y\|^2. \end{aligned}$$

That is,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

■

Chapter 4

Orthogonality

4.1 Orthogonal Sets

Definition (Orthogonality)**.** *Let V be an inner product space. We say that points x and y in V are **orthogonal** if $\langle x, y\rangle = 0$.*

Definition (Orthogonal Set)**.** *Let $\mathfrak{X}$ be an inner product space. Let S be a subset of $\mathfrak{X}$. We say that S is **orthogonal** if*

$$\forall x, y \in S : x \neq y, \quad \langle x, y\rangle = 0.$$

i.e., if any two distinct vectors are orthogonal.

Definition (Orthonormal Set)**.** *Let $\mathfrak{X}$ be an inner product space. Let S be a set in the space. We say that S is **orthonormal** if S is orthogonal and $\forall x \in S$, $\|x\| = 1$ where $\|\cdot\|$ is the norm induced by the inner product.*

Proposition 4.1.1. *Orthogonal sets are linearly independent.*

4.2 Orthogonal Bases

Definition (Orthogonal Basis)**.** *Let V be an inner product space. Let S be a set in the space. We say that S is an **orthogonal basis** for V if it is an ordered basis for V and orthogonal.*

Definition (Orthonormal Basis)**.** *Let $\mathfrak{X}$ be an inner product space. Let S be a set in the space. We say that S is an **orthonormal basis** for $\mathfrak{X}$ if it is the maximal, with respect to inclusion, in the collection of all orthonormal sets in the space.*

Proposition 4.2.1. *Let V be an inner product space. Let $S = \{v_1, ..., v_n\}$ be an orthogonal subset of V where each v_i is non-zero. Then*

$$\forall y \in \operatorname{span}(S), \quad y = \sum_{i=1}^{n} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

Theorem 3 (Gram-Schmidt Process)**.** *Let V be an inner product space. Let $S = \{x_0, ..., x_n\}$ be a linearly independent subset of V. Then the set $T = \{y_0, ..., y_n\}$ given by $y_0 := x_0$ and*

$$\forall i \in \{1, ..., n\}, \quad y_i := x_i - \sum_{j=1}^{i-1} \frac{\langle x_i, y_j \rangle}{\|y_j\|} y_j$$

is an orthogonal subset of V consisting of non-zero vectors such that $\operatorname{span}(S) = \operatorname{span}(S')$.

Proposition 4.2.2. *Let V be an inner product space and $S = \{v_0, v_1, \ldots, v_n\}$ be an orthogonal subset of V. Then the set $S^{'}$ derived from the Gram-Schmidt process is exactly S.*

Theorem 4 (Parseval's Identity)**.** *Let V be a finite-dimensional inner product space. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthogonal basis for V. Then*

$$\forall x, y \in V, \quad \langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

Proposition 4.2.3. *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. Then $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$ if and only if*

$$\forall x \in \mathcal{H}, \quad x = \sum_{e \in \mathcal{E}} \langle x, e \rangle e.$$

4.3 Orthogonal Complements

Definition (Orthogonal Complement)**.** *Let $\mathfrak{X}$ be an inner product space. Let S be a non-empty subset of V. We define the* ***orthogonal complement*** *of S, denoted by $S^\perp$, to be a set given by*

$$S^\perp := \{x \in \mathfrak{X} : \forall s \in S, \langle x, s \rangle = 0\}.$$

i.e., the set of all points in $\mathfrak{X}$ that are orthogonal to all vectors in S.

Proposition 4.3.1. *Let V be a finite-dimensional inner product space. Then*

(1) $V^\perp = \{O_V\}$

(2) $\{O_V\}^\perp = V$

Proposition 4.3.2. *Orthogonal complements are always linear subspaces.*

Proposition 4.3.3. *Let V be an inner product space and W be a subspace of V with basis β. Then a vector in V is also in $W^{\perp}$ if and only if it is orthogonal to all vectors in β.*

Proposition 4.3.4 (Extension)**.** *Let V be an n-dimensional inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal subset of V. Then S can be extended to an orthogonal basis $B = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.*

Proposition 4.3.5. *Let V be an inner product space. Then*

(1) $S \subseteq T$ *implies* $T^{\perp} \subseteq S^{\perp}$ *for any subsets S and T of V.*

(2) $S \subseteq (S^{\perp})^{\perp}$ *for any subset S of V.*

Proposition 4.3.6. *Let V be a finite-dimensional inner product space and W be a subspace of V. Then*

(1) $W = (W^{\perp})^{\perp}$

(2) $V = W \oplus W^{\perp}$

Proposition 4.3.7. *Let V be a finite-dimensional inner product space and W_1 and W_2 be subspaces of V. Then*

(1) $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$

(2) $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$

4.4 Orthogonal Projection

Definition (Orthogonal Projection)**.** *Let V be a vector space. Let W be a finite-dimensional subspace of V. Let x be a vector in V. We define the* ***orthogonal projection*** *of x on W, denoted by (x), to be the vector u in W such that $x = u + v$ where v is another vector in $W^{\perp}$.*

4.5 Inequalities in Hilbert Spaces

Theorem 5 (Bessel's Inequality)**.** *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in the space. Then*

$$\forall x \in \mathcal{H}, \quad \sum_{e \in \mathcal{E}} |\langle x, e_i \rangle|^2 \leq \|x\|^2.$$

Proposition 4.5.1. *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. Let x be a point in the space. Then the net $\sum_{e \in \mathcal{E}} \langle x, e \rangle e$ converges in $\mathcal{H}$.*

Proof. Let $\mathcal{F}$ be the collection of all finite subsets of $\mathcal{E}$, partially ordered by inclusion. Define for each $F \in \mathcal{F}$ a vector y_F as $y_F := \sum_{e\in F}\langle x, e\rangle e$. Let ε be an arbitrary positive number. Since $\mathcal{E}$ is an orthonormal set, the set $\{e \in \mathcal{E} : \langle x, e\rangle \neq 0\}$ is countable. Let $\{e_i\}_{i\in\mathbb{N}}$ denote the set. By the Bessel's inequality, $\exists N \in \mathbb{N}$ such that $\sum_{i=N+1}^{\infty} |\langle x, e_i\rangle|^2 < \varepsilon^2$. Define a set F_0 as $F_0 := \{e_1, ..., e_N\}$. Let F and G be arbitrary elements in $\mathcal{F}$ such that $F_0 \leq F$ and $F_0 \leq G$. Then

$$\begin{aligned}
\|y_F - y_G\|^2 &= \left\| \sum_{e\in F\setminus G} \langle x, e\rangle e - \sum_{e\in G\setminus F} \langle x, e\rangle e \right\|^2 \\
&= \sum_{e\in F\cup G\setminus F\cap G} |\langle x, e\rangle|^2 \\
&= \sum_{e\in F\cup G} |\langle x, e\rangle|^2 - \sum_{e\in F\cap G} |\langle x, e\rangle|^2 \\
&\leq \sum_{i=N+1}^{\infty} |\langle x, e_i\rangle|^2 \\
&< \varepsilon^2.
\end{aligned}$$

So $\{y_F\}_{F\in\mathcal{F}}$ is Cauchy. Since $\mathcal{H}$ is complete and $\{y_F\}_{F\in\mathcal{F}}$ is Cauchy, $\{y_F\}_{F\in\mathcal{F}}$ converges. ■

Chapter 5

Sequence Spaces

5.1 ℓ^p Space

Definition (ℓ^p Space)**.** *We define the ℓ^p space to be the set of all sequences x such that $\|x\|_p$ is finite, equipped with the p-norm $\|\cdot\|_p$.*

Definition (Weighted ℓ^p Space)**.** *Let $(r_i)_{i\in\mathbb{N}}$ be a sequence of positive integers. We define the **weighted ℓ^p** space to be the set given by*

$$\ell^p := \{(x_i)_{i\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sum_{i\in\mathbb{N}} r_i|x_i|^p < +\infty.$$

Proposition 5.1.1. *For $p \in [1, +\infty)$, $(\ell^p, \|\cdot\|_p)$ is complete.*

Proof.

Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in ℓ^p.

Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in ℓ^p, $\forall\varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $\forall m, n > N$, we have $\|x_m - x_n\|_p < \varepsilon$.

Since $\|x_m - x_n\|_p < \varepsilon$ and $|x_m^{(i)} - x_n^{(i)}| \leq \|x_m - x_n\|_p$ for any $i\in \mathbb{N}$, $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$ for any $i\in \mathbb{N}$.

Since for any $i\in \mathbb{N}$ and any positive number ε, there exists an integer $N(\varepsilon)$ such that for any indices $m, n > N$, we have $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$, by definition, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$.

Since $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$ and $\mathbb{F}$ is complete, $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ converges.

Let $x_0^{(i)} = x_n^{(i)}$. Let $x_0 = \{x_0^{(i)}\}_{i\in\mathbb{N}}$.

$$\|x_0\|_p = (\sum_{i=1}^{\infty} |x_0^{(i)}|^p)^{\frac{1}{p}}$$

■

5.2 c_0 Space and c_{00} Space

Definition (c_0 Space)**.** *We define c_0 to be*

$$c_0 := \big\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\big\}.$$

Definition (c_{00} Space)**.** *We define c_{00} to be*

$$c_{00} := \big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}, \forall n > N, x_n = 0\big\}.$$

i.e., the set of all eventually zero sequences of real numbers. i.e., the set of all sequences with finite support.

Proposition 5.2.1. *The c_{00} is not complete in $(\ell_1, \|\cdot\|_1)$.*

Proof. Define a sequence of vectors $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ by $\mathfrak{x}_i^j := \frac{1}{j^2}$ for $j \in \{1..i\}$ and $\mathfrak{x}_i^j := 0$ for $j > i$. Then $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ converges to something that is not in c_{00}. ■

Proposition 5.2.2. *The closure of c_{00} in the space $(\mathbb{R}^\omega, d_1)$ is ℓ_1.*

Proof. For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq \ell_1$. Let x be an arbitrary element in $\mathrm{cl}(c_{00})$. Since $x \in \mathrm{cl}(c_{00})$, there exists another element $y \in c_{00}$ such that $d_1(x, y) < 1$. Let $N \in \mathbb{N}$ be such that $\forall n > N$, $y_n = 0$. Then

$$\begin{aligned}
& d_1(x,y) < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n - y_n| < 1 \\
\iff & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n - y_n| < 1 \\
\iff & \sum_{n=1}^{N} |x_n - y_n| + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} \big||x_n| - |y_n|\big| + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} \big(|x_n| - |y_n|\big) + \sum_{n>N} |x_n| < 1 \\
\implies & \sum_{n=1}^{N} |x_n| - \sum_{n=1}^{N} |y_n| + \sum_{n>N} |x_n| < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n| - \sum_{n=1}^{N} |y_n| < 1 \\
\iff & \sum_{n\in\mathbb{N}} |x_n| < 1 + \sum_{n=1}^{N} |y_n|.
\end{aligned}$$

Since $\sum_{n\in\mathbb{N}} |x_n|$ is bounded, $x \in \ell_1$.

For the reverse direction, we are to prove that $\ell_1 \subseteq \mathrm{cl}(c_{00})$. Let x be an arbitrary element in ℓ_1. For $i \in \mathbb{N}$, define $x^i = \{x^i_j\}_{j\in\mathbb{N}}$ as $x^i_j = x_j$ for $j \le i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}, x^i \in c_{00}$. Then

$$\begin{aligned} &\lim_{i\in\mathbb{N}} d_1(x^i, x) \\ =\ &\lim_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |x^i_j - x_j| \\ =\ &\lim_{i\in\mathbb{N}} \sum_{j>i} |x^i_j - x_j| \\ =\ &\lim_{i\in\mathbb{N}} \sum_{j>i} |x_j| \\ =\ &0. \end{aligned}$$

That is, $\lim_{i\in\mathbb{N}} d_1(x^i, x) = 0$. So $\lim_{i\in\mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

■

Proposition 5.2.3. *The closure of c_{00} in the space $(\mathbb{R}^\omega, d_\infty)$ is c_0.*

Proof. For one direction, we are to prove that $\mathrm{cl}(c_{00}) \subseteq c_0$. Let x be an arbitrary element in $\mathrm{cl}(c_{00})$. Let ε be an arbitrary positive real number. Since $x \in \mathrm{cl}(c_{00})$, there exists another element y in c_{00} such that $d_\infty(x, y) < \varepsilon$. That is, $\forall j \in \mathbb{N}, |x_j - y_j| < \varepsilon$. Since $y \in c_{00}$, $\exists N \in \mathbb{N}$ such that $\forall j > N$, $y_j = 0$. So $\forall j > N, |x_j| < \varepsilon$. That is,

$$\forall \varepsilon > 0, \quad , \exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon.$$

By definition of convergence of limits, $\lim_{j\in\mathbb{N}} x_j = 0$. So $x \in c_0$.

For the reverse direction, we are to prove that $c_0 \subseteq \mathrm{cl}(c_{00})$. Let x be an arbitrary element in c_0. For $i \in \mathbb{N}$, define x^i as $x^i_j = x_j$ for $j \le i$ and $x^i_j = 0$ for $j > i$. Then $\forall i \in \mathbb{N}$, $x^i \in c_{00}$. Let ε be an arbitrary positive real number. Since $x \in c_0$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon/2.$$

Let $i > N$. Then

$$\begin{aligned} &d_\infty(x^i, x) \\ =\ &\sup_{j\in\mathbb{N}} |x^i_j - x_j| \\ =\ &\sup_{j>i} |x^i_j - x_j| \\ =\ &\sup_{j>i} |x_j| \\ \le\ &\varepsilon/2 < \varepsilon. \end{aligned}$$

That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall i > N, \quad d_\infty(x^i, x) < \varepsilon.$$

By definition of convergence of sequences, $\lim_{i\in\mathbb{N}} x^i = x$. So $x \in \mathrm{cl}(c_{00})$.

■

Proposition 5.2.4. *Let* $A := \big\{\{x_n\}_{n\in\mathbb{N}} \in c_{00} : \sum_{n\in\mathbb{N}} x_n = 0\big\}$. *Then* A *is a subset of* ℓ^1 *and is closed in* (ℓ^1, d_1)*. i.e.* $\mathrm{cl}(A) = A$ *in* (ℓ^1, d_1)*.*

Proof. Let $x = \{x^i\}_{i\in\mathbb{N}}$ be a sequence in ℓ^1, where each $x^i = \{x^i_j\}_{j\in\mathbb{N}}$ is an element in A, that converges in (ℓ^1, d_1). Say $\lim_{i\to\infty} x^i = x^\infty$.

First I claim that $x^\infty \in c_{00}$.

Now I claim that $\sum_{j\in\mathbb{N}} x^\infty_j = 0$. i.e. $x^\infty \in A$. Since $x^\infty \in c_{00}$,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad x^\infty_j = 0.$$

Define $y_i := \sum_{j=1}^N x^i_j$. Define $y_\infty := \sum_{j=1}^N x^\infty_j$. It is easy to see that $\lim_{i\in\mathbb{N}} y_i = y_\infty$. Assume for the sake of contradiction that $y_\infty \neq 0$. i.e. $\{y_i\}_{i\in\mathbb{N}}$ does not converge to 0. Then

$$\exists \varepsilon_0 > 0, \quad \forall M \in \mathbb{N}, \quad \exists i_0 > M, \quad |y_{i_0} - 0| = |y_{i_0}| \geq \varepsilon_0. \tag{1}$$

Since $\lim_{i\to\infty} x^i = x^\infty$,

$$\exists M_0 \in \mathbb{N}, \quad \forall i > M_0, \quad d_1(x^i, x^\infty) < \varepsilon_0. \tag{2}$$

Consider statement (1) for a particular M, M_0, we have

$$\exists i_0 > M_0, \quad |y_{i_0}| \geq \varepsilon_0. \tag{3}$$

That is,

$$\Big|\sum_{j=1}^N x^{i_0}_j\Big| \geq \varepsilon_0. \tag{3'}$$

Consider statement (2) for a particular i, i_0, we have

$$d_1(x^{i_0}, x^\infty) < \varepsilon_0. \tag{4}$$

From statement (4) we can derive:

$$\begin{aligned}
& d_1(x^{i_0}, x^\infty) < \varepsilon_0 \\
\iff & \sum_{j\in\mathbb{N}} |x^{i_0}_j - x^\infty_j| < \varepsilon_0 \\
\iff & \sum_{j=1}^N |x^{i_0}_j - x^\infty_j| + \sum_{j>N} |x^{i_0}_j - x^\infty_j| < \varepsilon_0
\end{aligned}$$

$$
\begin{aligned}
&\implies \sum_{j>N} |x_j^{i_0} - x_j^{\infty}| < \varepsilon_0 \\
&\iff \sum_{j>N} |x_j^{i_0} - 0| < \varepsilon_0 \\
&\iff \sum_{j>N} |x_j^{i_0}| < \varepsilon_0 \\
&\implies |\sum_{j>N} x_j^{i_0}| < \varepsilon_0 \\
&\implies |\sum_{j>N} x_j^{i_0}| < \varepsilon_0 \\
&\iff |\sum_{j\in\mathbb{N}} x_j^{i_0} - \sum_{j=1}^{N} x_j^{i_0}| < \varepsilon_0 \\
&\iff \left|0 - \sum_{j=1}^{N} x_j^{i_0}\right| < \varepsilon_0 \\
&\iff |\sum_{j=1}^{N} x_j^{i_0}| < \varepsilon_0.
\end{aligned}
$$

This contradicts to statement (3'). So the original assumption that $y_\infty \neq 0$ is false. i.e. $y_\infty = 0$. It follows that $\sum_{j\in\mathbb{N}} x_j^{\infty} = 0$. This completes the proof.

■

5.3 Hölder's Inequality

Theorem 6 (Hölder's Inequality)**.** *Let* $\mathfrak{X} = \mathbb{R}^n$ *for some* $n \in \mathbb{N}$*. Let* $x = (x_i)_{i=1}^n$ *and* $y = (y_i)_{i=1}^n$ *be vectors in* $\mathfrak{X}$*. Then* $\forall p, q \in (1, +\infty) : 1/p + 1/q = 1$, $\|xy\|_1 \leq \|x\|_p \|y\|_q$*. i.e.,*

$$\sum_{i=1}^{n} |x_i y_i| \leq (\sum_{i=1}^{n} |x_i|^p)^{1/p} (\sum_{i=1}^{n} |y_i|^q)^{1/q}.$$

Chapter 6

Function Spaces

6.1 The $\mathcal{L}^p$ Norm

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

the instructors' answer, *where instructors collectively construct a single answer*

In the sup norm, convergence coincides with uniform convergence. Moreover, $C[a, b]$ is complete in this norm. It is not complete in any of the L^p norms for $1 \leq p < \infty$. The completion in these norms is called $L^p(a, b)$.

undo thanks | 1

Updated 1 day ago by Kenneth Davidson

Chapter 7

Banach Space

7.1 Definition

Definition (Banach Space). *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let d be the metric induced by $\|\cdot\|$. We say that $\mathfrak{X}$ is a **Banach space** if $(\mathfrak{X}, d)$ is a complete metric space. i.e., we define a Banach space to be a complete normed linear space.*

7.2 Properties

Proposition 7.2.1. *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed vector space over field $\mathbb{F}$. Then $(\mathfrak{X}, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series in X is summable.*

Proposition 7.2.2. *Any Banach space with a Schauder basis has to be separable.*

7.3 Examples of Banach Space

Example 7.3.1. $(\mathcal{C}([0,1], \mathbb{F}), \|\cdot\|_\infty)$ *is a Banach space.*

Example 7.3.2 (Disc Algebra). *Define $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Define $\mathcal{A}(\mathbb{D}) := \{f \in \mathcal{C}(\overline{\mathbb{D}}) : f|_{\mathbb{D}}$ is holomorphic $\}$. Define $\|\cdot\|_\infty$ by $\|f\|_\infty := \sup_{z \in \overline{\mathbb{D}}} |f(z)|$. Then $(\mathcal{A}(\mathbb{D}), \|\cdot\|_\infty)$ is a Banach space.*

Example 7.3.3. *Let (X, Ω, μ) be a measure space. Let p be a number in $[1, +\infty)$. Define*

$$\mathcal{L}^p(X, \mu) := \operatorname{span}\{f : X \to [0, +\infty] \mid f \text{ is measurable and } \int_X |f|^p < +\infty\}.$$

Define an equivalence relation on $\mathcal{L}^p(X, \mu)$ by $f \equiv g$ if and only if

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

Define a space $L^p(X,\mu) := \mathcal{L}^p(X,\mu)/\equiv$. Then $L^p(X,\mu)$ is a Banach space when equipped with the norm

$$\|[f]\|_p := \left(\int_X |f|^p\right)^{1/p}.$$

Example 7.3.4. *Let $\mathcal{P}_{\mathbb{C}}[0,1]$ denote the set of all polynomials with complex coefficients. For each $p \in [1,+\infty)$, define a norm*

$$\|f\|_p := \left(\int_0^1 |f|^p\right)^{1/p}.$$

For $p = +\infty$, define a norm

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(z)|.$$

7.4 Construction of Banach Spaces

Definition. *Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be two Banach spaces over field $\mathbb{K}$. Let $p \in [1,+\infty)$. We define*

$$\mathfrak{X} \oplus_p \mathfrak{Y} := \{(x,y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x,y)\|_p := (\|x\|_{\mathfrak{X}}^p + \|y\|_{\mathfrak{Y}}^p)^{1/p}.$$

For $p = +\infty$, we define

$$\mathfrak{X} \oplus_\infty \mathfrak{Y} := \{(x,y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x,y)\|_\infty := \max(\|x\|_{\mathfrak{X}}, \|y\|_{\mathfrak{Y}}).$$

- *Note that the norms behave similarly to what the p norm would do.*
- *We can similarly define the direct sum of finitely many Banach spaces.*

Proposition 7.4.1. *$\|\cdot,\cdot\|_p$ is a norm on $\mathfrak{X} \oplus_p \mathfrak{Y}$.*

Proposition 7.4.2. *$\mathfrak{X} \oplus_p \mathfrak{Y}$ is complete with respect to $\|\cdot,\cdot\|_p$.*

Chapter 8

Hilbert Space

8.1 Definition

Definition (Hilbert Space)**.** *We define a **Hilbert space**, denoted by $\mathcal{H}$, to be a complete inner product space.*

8.2 Examples of Hilbert Space

Example 8.2.1. *Let (X, μ) be a measure space. Then $L^2(X, \mu)$ is a Hilbert space with inner product given by*

$$\langle f, g\rangle := \int_X f(x)\overline{g(x)}d\mu(x).$$

Example 8.2.2. $\ell^2(\mathbb{K}) = \left\{(x_i)_{i\in\mathbb{N}} \in \mathbb{K}^\infty : \sum_{i\in\mathbb{N}} |x_i|^2 < +\infty\right\}$ *is a Hilbert space with inner product given by*

$$\langle (x_i)i \in \mathbb{N}, (y_i)_{i\in\mathbb{N}}\rangle := \sum_{i\in\mathbb{N}} x_n\overline{y_n}.$$

8.3 Properties of Hilbert Space

Proposition 8.3.1. *Let $\mathcal{H}$ be a Hilbert space. Let S be a non-empty set in the space. Then $S^{\perp\perp} = \mathrm{clspan}(S)$.*

Proof. For one direction, we are to prove that $\mathrm{clspan}(S) \subseteq S^{\perp\perp}$.

For the reverse direction, we are to prove that $S^{\perp\perp} \subseteq \mathrm{clspan}(S)$. Assume for the sake of contradiction that $\exists x \in S^{\perp\perp}$ with $x \neq 0$ such that $x \notin \mathrm{clspan}(S)$. Say $x = m_1 + m_2$ for some $m_1 \in \mathrm{clspan}(S)$ and some $m_2 \in \mathrm{clspan}(S)^\perp$. Note that $\mathrm{clspan}(S)^\perp = S^\perp$. So $m_2 \in S^\perp$. Since $x \in S^{\perp\perp}$ and $m_2 \in S^\perp$, we should have $\langle x, m_2\rangle = 0$. However,

$$\langle x, m_2\rangle = \langle m_1 + m_2, m_2\rangle$$

$$
\begin{aligned}
&= \langle m_1, m_2\rangle + \langle m_2, m_2\rangle \\
&= 0 + \langle m_2, m_2\rangle \\
&> 0,\ \text{since } m_2 \neq 0.
\end{aligned}
$$

This leads to a contradiction. So $S^{\perp\perp} \subseteq \text{clspan}(S)$. ■

Theorem 7 (The Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space over field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Suppose that $\mathcal{H} \neq \{0\}$. Then for any $\varphi \in \mathcal{H}^*$, $\exists y \in \mathcal{H}$ such that*

$$\forall x \in \mathcal{H}, \quad \varphi(x) = \langle x, y\rangle.$$

Proof. Define for each $y \in \mathcal{H}$ a function $\beta_y \in \mathcal{H}^*$ by $\beta_y(x) := \langle x, y\rangle$. We are to prove that $\mathcal{H}^* = \{\beta_y : y \in \mathcal{H}\}$. It is easy to verify that each β_y is linear and bounded. So $\forall y \in \mathcal{H}$, $\beta_y \in \mathcal{H}^*$. i.e., $\{\beta_y : y \in \mathcal{H}\} \subseteq \mathcal{H}^*$. Define a map Θ from $\mathcal{H}$ to $\mathcal{H}^*$ as $\Theta(y) := \beta_y$. It is easy to verify that Θ is linear.

$$
\begin{aligned}
\|\Theta(y)\| = \|\beta_y\| &= \sup\{\beta_y(x) : \|x\| = 1\} \\
&= \sup\{\langle x, y\rangle : \|x\| = 1\} \\
&\leq \sup\{\|x\|\|y\| : \|x\| = 1\} \\
&= \|y\|.
\end{aligned}
$$

That is, $\|\Theta(y)\| \leq \|y\|$. So $\|\Theta\| \leq 1$. On the other hand, consider an arbitrary point $y_0 \in \mathcal{H}$ with $y_0 \neq 0$:

$$
\begin{aligned}
\|\Theta\| &= \sup\left\{\frac{\|\Theta(y)\|}{\|y\|} : y \neq 0\right\} \\
&\geq \left.\frac{\|\Theta(y)\|}{\|y\|}\right|_{y=y_0} \\
&= \frac{\|\Theta(y_0)\|}{\|y_0\|} \\
&= \frac{\|\beta_{y_0}\|}{\|y_0\|} \\
&= \frac{1}{\|y_0\|} \sup\left\{\frac{\|\beta_{y_0}(y)\|}{\|y\|} : y \neq 0\right\} \\
&\geq \frac{1}{\|y_0\|} \frac{\|\beta_{y_0}(y_0)\|}{\|y_0\|} \\
&\geq \frac{1}{\|y_0\|} \frac{\|y_0\|^2}{\|y_0\|} \\
&= 1.
\end{aligned}
$$

That is, $\|\Theta\| \geq 1$. So $\|\Theta\| = 1$. So Θ is isometric. It immediately follows that Θ is injective. Now it remains to prove that Θ is surjective. Let $\varphi \in \mathcal{H}^*$. If $\varphi = 0$, then $\varphi = \Theta(0)$ and we are done. Otherwise, let $\mathcal{M} := \ker(\varphi)$. Then we have $\operatorname{codim} \mathcal{M} = \dim \mathcal{M}^\perp = 1$. Take $e \in \mathcal{M}^\perp$ such that $\|e\| = 1$. Let P denote the orthogonal projection onto $\mathcal{M}$. Then $1 - P$ is the orthogonal projection onto $\mathcal{M}^\perp$.

$$x = Px + (1-P)x = Px + \langle x, e\rangle e.$$

So for $x \in \mathcal{H}$,

$$\varphi(x) = \varphi(Px) + \langle x, e\rangle \varphi(e) = \langle x, \overline{\varphi(e)}e\rangle = \beta_y(x)$$

where $y := \overline{\varphi(e)}e$. Hence $\varphi = \beta_y$. So Θ is surjective. This completes the proof. ∎

Chapter 9

Operators

9.1 Bounded Operators

Definition (Bounded Operator)**.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let T be a linear map from $\mathfrak{X}$ to $\mathfrak{Y}$. We say that T is a **bounded operator** if*

$$\exists k \in \mathbb{R}, \quad \forall x \in \mathfrak{X}, \quad \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}.$$

Definition (Operator Norm)**.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces. Let T be a bounded operator from $\mathfrak{X}$ to $\mathfrak{Y}$. We define the **operator norm** of T, denoted by $\|T\|$, to be the number given by*

$$\|T\| := \inf\{k \in \mathbb{R} : \forall x \in \mathfrak{X}, \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}\}.$$

Proposition 9.1.1.

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{X}, \|x\|_{\mathfrak{X}} = 1\}.$$

Proposition 9.1.2. *Let X and Y be normed linear spaces. Let T be a linear map from X to Y. Then T is bounded if and only if T is continuous.*

9.2 Examples of Bounded Operators

Example 9.2.1 (The Multiplication Operator)**.** *Let $\mathfrak{X} = \big(\mathcal{C}([0,1], \mathbb{C}), \|\cdot\|_\infty\big)$. Let f be a function in $\mathfrak{X}$. We define the **multiplication operator** on $\mathfrak{X}$, w.r.t. f, denoted by M_f, as*

$$M_f(g) = fg.$$

Then M_f is bounded and $\|M_f\| = \|f\|_\infty$.

Proof. Let g be an arbitrary function in $\mathfrak{X}$. Then

$$\|M_f g\|_\infty = \|fg\|_\infty$$

$$
\begin{aligned}
&= \sup_{x\in[0,1]} |f(x)g(x)| \\
&= \sup_{x\in[0,1]} |f(x)||g(x)| \\
&\leq \sup_{x\in[0,1]} |f(x)| \sup_{x\in[0,1]} |g(x)| \\
&= \|f\|_\infty \|g\|_\infty .
\end{aligned}
$$

That is, $\|M_f g\|_\infty \leq \|f\|_\infty \|g\|_\infty$. So $\|f\|_\infty$ is an element of the set $S = \{k \in \mathbb{R} : \forall g \in \mathfrak{X}, \|M_f g\|_{\mathfrak{Y}} \leq k\|g\|_{\mathfrak{X}}\}$. So $\|M_f\| = \inf(S) \leq \|f\|_\infty$. Consider g_0 given by $g_0(x) = 1$. Then g_0 in $\mathfrak{X}$. Then

$$\|M_f g_0\|_\infty = \|f g_0\|_\infty = \|f\|_\infty = \|f\|_\infty \|g_0\|_\infty .$$

Let k be an arbitrary element in S. Assume for the sake of contradiction that $k < \|f\|_\infty$. Then

$$
\begin{aligned}
\|f\|_\infty \|g_0\|_\infty &= \|M_f g_0\|_\infty \\
&\leq k\|g_0\|_\infty \\
&< \|f\|_\infty \|g_0\|_\infty .
\end{aligned}
$$

This leads to a contradiction. So $\forall k \in S$, $k \geq \|f\|_\infty$. So $\|f\|_\infty$ is a lower bound for the set S. So $\|M_f\| = \inf(S) \geq \|f\|_\infty$. Since $\|M_f\| \leq \|f\|_\infty$ and $\|M_f\| \geq \|f\|_\infty$, we get $\|M_f\| = \|f\|_\infty$. ■

Example 9.2.2 (The Volterra Operator). *Let* $\mathfrak{X} = \big(\mathcal{C}([0,1],\mathbb{C}), \|\cdot\|_\infty\big)$. *Define*

$$Vf := x \mapsto \int_0^x f(t)dt.$$

Then the Volterra Operator is bounded and $\|V\| \leq 1$.

Proof. Let f be an arbitrary function in $\mathfrak{X}$ with $\|f\|_\infty = 1$. Then $\forall x \in [0,1]$,

$$
\begin{aligned}
|Vf(x)| &= \left| \int_0^x f(t)dt \right| \\
&\leq \int_0^x |f(t)|dt \\
&\leq \int_0^x \sup_{t\in[0,1]} |f(t)|dt \\
&= \int_0^x \|f\|_\infty dt \\
&= \int_0^x 1dt \\
&= x.
\end{aligned}
$$

That is, $\forall x \in [0,1]$, $|Vf(x)| \leq 1$. So $\|Vf\|_\infty \leq 1$. Since $\forall f \in \mathfrak{X} : \|f\|_\infty = 1$, $\|Vf\|_\infty \leq 1$, we get $\|V\| \leq 1$. ∎

Example 9.2.3 (The Diagonal Operator). *Let* $\mathfrak{X} = \ell^2(\mathbb{N})$. *Let*

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & \ddots \end{bmatrix}.$$

Then D is bounded if and only if $(d_i)_{i\in\mathbb{N}}$ is bounded and $\|D\| = \|(d_i)_{i\in\mathbb{N}}\|_\infty$.

Proof. Case 1.

$$\begin{aligned} \|Dx\|_2^2 &= \sum_{i\in\mathbb{N}} |d_i x_i|^2 \\ &=\leq \sum_{i\in\mathbb{N}} \|(d_j)_{j\in\mathbb{N}}\|_\infty |x_i|^2 \\ &= \|(d_j)_{j\in\mathbb{N}}\|_\infty \sum_{i\in\mathbb{N}} |x_i|^2 \\ &= \|(d_j)_{j\in\mathbb{N}}\|_\infty \|x\|_2^2. \end{aligned}$$

Case 2.

If $(d_i)_{i\in\mathbb{N}} \notin \ell^\infty$, $\exists (d_{n_i})_{i\in\mathbb{N}} \to \infty$.

$$\begin{aligned} \|De_{n_i}\|_2 &= \|d_{n_i} e_{n_i}\|_2 \\ &= |d_{n_i}| \|e_{n_i}\|_2 \\ &= |d_{n_i}|. \end{aligned}$$

So $\|D\| \geq \|De_{n_i}\|_2 \to \infty$. ∎

Example 9.2.4 (Weighted Shifts).

- *Let $\mathcal{H} = \ell^2_{\mathbb{N}}$. Let $(w_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. We define an* ***unilateral forward weighted shift*** *W on $\mathcal{H}$ as*

  $$W(x_n) := (0, w_1x_1, w_2x_2, w_3x_3, ...).$$

  i.e.,

  $$W = \begin{bmatrix} 0 & & & & \\ w_1 & 0 & & & \\ & w_2 & 0 & & \\ & & w_3 & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.$$

  Then W is bounded and $\|W\| = \sup\{|w_n| : n \in \mathbb{N}\}$.

- *Let* $\mathcal{H} = \ell^2_{\mathbb{N}}$. *Let* $(v_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{N}}$. *We define an* ***unilateral backward weighted shift*** V *on* $\mathcal{H}$ *as*
$$V(x_n) := (v_1x_2, v_2x_3, v_3x_4, ...).$$
Then V *is bounded and* $\|V\| = \sup\{|v_n| : n \in \mathbb{N}\}$.

- *Let* $\mathcal{H} = \ell^2_{\mathbb{Z}}$. *Let* $(u_n)_{n\in\mathbb{Z}} \in \ell^\infty_{\mathbb{Z}}$. *We define a* ***bilateral weighted shift*** U *on* $\mathcal{H}$ *as*
$$U(x_n) := (u_{n-1}x_{n-1})_{n\in\mathbb{Z}}.$$
Then U *is bounded and* $\|U\| = \sup\{|u_n| : n \in \mathbb{Z}\}$.

Example 9.2.5 (The Composition Operators). *Let* $\mathfrak{X} = \mathcal{C}([0,1],\mathbb{C})$. *Let* $\varphi \in \mathcal{C}([0,1],[0,1])$. *We define the* ***composition operator*** *on* $\mathfrak{X}$, *denoted by* C_φ *as*
$$C_\varphi(f) := f \circ \varphi.$$
Then C_φ *is contractive.*

Proof.
$$\begin{aligned}\|C_\varphi(f)\| &= \sup_{x\in[0,1]} |(f\circ\varphi)(x)| \\ &\le \|f\|_\infty.\end{aligned}$$
■

9.3 The Space of Bounded Operators

Proposition 9.3.1. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be normed linear spaces. Then* $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ *is a vector space and the operator norm is a norm on* $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$.

Proposition 9.3.2. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be normed linear spaces. Let* $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ *be the space of bounded linear operators from* $\mathfrak{X}$ *to* $\mathfrak{Y}$. *Then if* $\mathfrak{Y}$ *is complete,* $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$ *is complete.*

Proposition 9.3.3. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be normed linear spaces. Let* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *be two equivalent norms on* $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$. *Then* $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y},\|\cdot\|_1)$ *if and only if* $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y},\|\cdot\|_2)$.

9.4 Invertible Bounded Operators

Proposition 9.4.1. *Let* $(\mathfrak{X},\|\cdot\|_1)$ *be a Banach space. Let* $S \in \mathcal{B}(\mathfrak{X})$ *be a bounded linear map that is invertible. Define a norm* $\|\cdot\|_2$ *on* $\mathfrak{X}$ *as*
$$\|x\|_2 := \|Sx\|_1.$$
Then $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent.*

Proof. On one hand, since S is bounded, $\exists c_1$ such that $\forall x \in \mathfrak{X}$, $\|Sx\|_1 \leq c_1\|x\|_1$. That is, $\|x\|_2 \leq c_1\|x\|_1$.

On the other hand, since S is invertible, S^{-1} exists and is also bounded. Since S^{-1} is bounded, $\exists c_2$ such that $\forall x \in \mathfrak{X}$, $\|S^{-1}x\|_1 \leq c_2\|x\|_1$. Consider $x = Sx$, we get $\forall x \in \mathfrak{X}$, $\|S^{-1}Sx\|_1 \leq c_2\|Sx\|_1$. That is, $\|x\|_1 \leq c_2\|x\|_2$.

So $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ∎

Proposition 9.4.2. *Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space. Let S be a map in $\mathcal{B}(\mathfrak{X})$ that is invertible. Then*

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

Proof.

$$\begin{aligned}
\|S^{-1}\| &= \sup\{\|S^{-1}x\| : \|x\| = 1\} \\
&= \sup\left\{\frac{\|S^{-1}x\|}{\|x\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{\|S^{-1}Sx\|}{\|Sx\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{\|x\|}{\|Sx\|} : \|x\| \neq 0\right\} \\
&= \sup\left\{\frac{1}{\|Sx\|} : \|x\| = 1\right\} \\
&= (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.
\end{aligned}$$

That is,

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

∎

Chapter 10

Dual Space

10.1 Definition

Definition ((Topological) Dual Space)**.** *Let $\mathfrak{X}$ be a normed linear space over field $\mathbb{K}$. We define the* ***(topological) dual space*** *of $\mathfrak{X}$, denoted by $\mathfrak{X}^*$, to be the space $\mathcal{B}(\mathfrak{X}, \mathbb{K})$.*

Definition (Linear Functionals)**.** *We call the elements of $\mathfrak{X}^*$* ***linear functionals****.*

Proposition 10.1.1. *Let X be a normed linear space. Then there exists a contractive map from X to its double dual X^{**}.*

10.2 Properties

10.3 Examples of Dual Space

Example 10.3.1. $c_0(\mathbb{N})^*$ *is isometrically isomorphic to* $\ell^1(\mathbb{N})$.

Example 10.3.2. $c_0(\mathbb{N})^*$ *is isometrically isomorphic to* $\ell^\infty(\mathbb{N})$.

Chapter 11

Quotient Spaces

11.1 Definitions

Definition. *Let $\mathfrak{V}$ be a vector space. Let $\mathfrak{W}$ be a subspace of $\mathfrak{V}$. We define a* ***quotient space****, denoted by $\mathfrak{V}/\mathfrak{W}$, to be a set $\{v + \mathfrak{W} : v \in \mathfrak{V}\}$ with operations*

$$(v_1 + \mathfrak{W}) + (v_2 + \mathfrak{W}) := (v_1 + v_2) + \mathfrak{W} \text{ and}$$

$$\kappa(v + \mathfrak{W}) := (\kappa v) + \mathfrak{W}.$$

Definition (Quotient Map)**.** *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define the* ***quotient map*** *on $\mathfrak{X}$ with respect to $\mathfrak{M}$, denoted by $q_{\mathfrak{M}}$, to be a function from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$ given by*

$$q_{\mathfrak{M}}(x) := x + \mathfrak{M}$$

Proposition 11.1.1. *Quotient maps are contractive. i.e.,*

$$\forall x \in \mathfrak{X}, \quad \|x + \mathfrak{M}\|_{\mathfrak{X}/\mathfrak{M}} \leq \|x\|_{\mathfrak{X}}.$$

Proposition 11.1.2. *Let $\mathfrak{X}$ be a normed linear space. Let $\mathfrak{M}$ be a closed subspace of $\mathfrak{X}$. Let q be the canonical quotient map from $\mathfrak{X}$ to $\mathfrak{X}/\mathfrak{M}$. Then*

- *q is a continuous map. i.e.,*

$$\forall \text{ open set } W \subseteq \mathfrak{X}/\mathfrak{M}, \quad q^{-1}(W) \text{ is open in } \mathfrak{X}.$$

- *q is an open map. i.e.,*

$$\forall \text{ open set } G \subseteq \mathfrak{X}, \quad q(G) \text{ is open in } \mathfrak{X}/\mathfrak{M}.$$

Proof. Since q is contractive, q is continuous and hence (1). ∎

Definition (Seminorm on Quotient Spaces). *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Let $\mathfrak{M}$ be a linear manifold in $\mathfrak{X}$. We define a **seminorm** on $\mathfrak{X}/\mathfrak{M}$ to be a function from $\mathfrak{X}/\mathfrak{M}$ to $\mathbb{R}$ given by*

$$p(x+\mathfrak{M}) := \inf\{\|x+m\| : m \in \mathfrak{M}\}.$$

Proposition 11.1.3. *Seminorms on quotient spaces are indeed seminorms.*

Proposition 11.1.4. *A seminorm on a quotient space $\mathfrak{X}/\mathfrak{M}$ is a norm if and only if $\mathfrak{M}$ is closed.*

Chapter 12

Topological Vector Space

12.1 Definitions

Definition (Compatible). *Let $\mathcal{W}$ be a vector space over field $\mathbb{K}$. Let $\mathcal{T}$ be a topology on $\mathcal{W}$. We say that $\mathcal{T}$ is* ***compatible*** *with the vector space structure on $\mathcal{W}$ if the addition and scalar multiplication operations on $\mathcal{W}$ are continuous.*

Definition (Topological Vector Space). *We define a* ***topological vector space*** *to be a vector space with a compatible Hausdorff topology.*

Proposition 12.1.1 (Stability under Linear Combinations). *Let X be a normed vector space over $\mathbb{F}$. Let K be a compact set in the space. Let C be a closed set in the space. Then $\forall \alpha, \beta \in \mathbb{F}$, the set S given by $S := \alpha K + \beta C$ is closed.*

Proof. The case where $\beta = 0$ is trivial. I will assume $\beta \neq 0$. Let $\alpha, \beta \in \mathbb{F}$ be arbitrary. Let $\{s_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence in S that converges. Say the limit is s_∞. Since $s_i \in S$ for any $i \in \mathbb{N}$ and $S = \alpha K + \beta C$, $s_i = \alpha k_i + \beta c_i$ for some $k_i \in K$ and some $c_i \in C$, for any $i \in \mathbb{N}$. Since $\{k_i\}_{i\in\mathbb{N}}$ is a sequence in K and K is compact, there exists a convergent subsequence $\{k_i\}_{i\in I}$ of $\{k_i\}_{i\in\mathbb{N}}$ in K. Say $\{k_i\}_{i\in I}$ converges to $k_\infty \in K$. Since $\{s_i\}_{i\in\mathbb{N}}$ converges to s_∞, $\{s_i\}_{i\in I}$ also converges to s_∞. Since $s_i = \alpha k_i + \beta c_i$, $c_i = \beta^{-1}(s_i - \alpha k_i)$. Define $c_\infty := \beta^{-1}(s_\infty - \alpha k_\infty)$ Since $\{s_i\}_{i\in I}$ converges to s_∞ and $\{k_i\}_{i\in I}$ converges to k_∞ and $c_i = \beta^{-1}(s_i - \alpha k_i)$, $\{c_i\}_{i\in I}$ converges to c_∞. Since $\{c_i\}_{i\in I}$ is a sequence in C and converges to c_∞ and C is closed, $c_\infty \in C$. Since $s_\infty = \alpha k_\infty + \beta c_\infty$ and $k_\infty \in K$ and $c_\infty \in C$, $s_\infty \in \alpha K + \beta C$. Since for any sequence in S that converges, the limit is also in S, S is closed. ■

Remark. *The sum of two closed sets may not be closed.*

Proof. **Counter-example 1**

Consider $A := \{n : n \in \mathbb{N}\}$ and $B := \{n + \frac{1}{n} : n \in \mathbb{N}\}$.
(`https://math.stackexchange.com/questions/124130/sum-of-two-closed-sets-in-mathbb-r-is-closed`)
Their sum contains the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ but does not contain 0.

Counter-example 2

Consider $A := \mathbb{R} \times \{0\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x, y > 0, xy \geq 1\}$. Their sum is $\mathbb{R} \times \mathbb{R}_{++}$. ■

12.2 Examples

Example 12.2.1. *Let $\mathfrak{X}$ be a normed linear space. Then $\mathfrak{X}$ is a topological vector space with the topology induced by the norm.*

Proposition 12.2.1. *Normed linear spaces are Hausdorff.*

Example 12.2.2. *Let $\mathfrak{X}$ be a Banach space. Let $\mathfrak{X}^*$ denote the dual space of $\mathfrak{X}$. Let τ_* denote the weak topology on $\mathfrak{X}^*$ induced by elements of $\mathfrak{X}$ as*

$$\lim_\alpha x^*_\alpha = x^* \iff \forall x \in \mathfrak{X}, \lim_\alpha x^*_\alpha(x) = x^*(x).$$

Then $(\mathfrak{X}^*, \tau_*)$ is a topological vector space.

12.3 Neighborhood Improvements

Proposition 12.3.1. *Let $(\mathcal{V}, \tau)$ be a topological vector space. Let $U \in \mathcal{U}_0$ be a neighborhood of 0 in $\mathcal{V}$. Then*

- *$\exists N \in \mathcal{U}_0$ such that $N + N \subseteq U$.*
- *$\exists M \in \mathcal{U}_0$ and $\exists \varepsilon > 0$ such that $\forall 0 < |k| < \varepsilon$, we have $kM \subseteq U$.*
-

12.4 Cauchy Nets

Definition (Cauchy Net)**.** *Let $(\mathcal{V}, \tau)$ be a topological vector space. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathcal{V}$. We say that $(x_\lambda)_{\lambda\in\Lambda}$ is a **Cauchy net** if $\forall U \in \mathcal{U}_0$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda_1, \lambda_2 \geq \lambda_0$, we have $x_{\lambda_1} - x_{\lambda_2} \in U$.*

Definition (Cauchy Complete)**.** *Let $(\mathcal{V}, \tau)$ be a topological vector space. We say that $\mathcal{V}$ is **Cauchy complete** if every Cauchy net in $\mathcal{V}$ converges in $\mathcal{V}$.*

Proposition 12.4.1. *Convergent nets are Cauchy.*

Chapter 13

Equicontinuity in Metric Spaces

13.1 Definitions

Definition ((Pointwise) Equicontinuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{F}$ be a collection of functions from X to Y. Let x_0 be a point in X. We say that $\mathcal{F}$ is* ***(pointwise) equicontinuous*** *at point x_0 if for any positive number ε, there exists some number $\delta(x_0, \varepsilon)$ such that for any function f in $\mathcal{F}$ and any point x in X, we have*

$$d_Y(f(x), f(x_0)) < \varepsilon$$

whenever $d_X(x, x_0) < \delta(x_0, \varepsilon)$ is satisfied.

Definition (Uniform Equicontinuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{F}$ be a collection of functions from X to Y. We say that $\mathcal{F}$ is* ***uniformly equicontinuous*** *if for any positive number ε, there exists some number $\delta(\varepsilon)$ such that for any function f in $\mathcal{F}$ and any points x_1 and x_2 in X, we have*

$$d_Y(f(x_1), f(x_2)) < \varepsilon$$

whenever $d_X(x_1, x_2) < \delta(\varepsilon)$ is satisfied.

13.2 Sufficient Conditions

Proposition 13.2.1. *The closure of an equicontinuous family of functions is equicontinuous.*

Proof.

Let (X, d_X) and (Y, d_Y) be metric spaces.

Let $\mathcal{F}$ be an equicontinuous family of functions from X to Y.

We are to prove that $cl(\mathcal{F})$ is equicontinuous.

Let x_0 be an arbitrary point in X.

Let ε be an arbitrary positive number.

Since $\mathcal{F}$ is equicontinuous at point x_0, there exists some $\delta(x_0, \varepsilon)$ such that for any function f in $\mathcal{F}$ and any point x in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon/3$.

Let f be an arbitrary function in $cl(\mathcal{F})$.

Let x be an arbitrary point in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$.

Since $f \in cl(\mathcal{F})$, there exists some function $f_0 \in \mathcal{F}$ such that $d_\infty(f, f_0) < \varepsilon/3$.

Since $d_\infty(f, f_0) < \varepsilon/3$, $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$.

Since $f_0 \in \mathcal{F}$ and $d_X(x, x_0) < \delta(x_0, \varepsilon)$, $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$.

Since $d_Y(f(x), f_0(x)) < \varepsilon/3$ and $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$ and $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$, $d_Y(f(x), f(x_0)) < \varepsilon$.

Since for any positive number ε, there exists some $\delta(x_0, \varepsilon)$ such that for any function f in $cl(\mathcal{F})$ and any point x in X such that $d_X(x, x_0) < \delta(x_0, \varepsilon)$, we have $d_Y(f(x), f(x_0)) < \varepsilon$, by definition of equicontinuous, $cl(\mathcal{F})$ is equicontinuous at point x_0.

Since $cl(\mathcal{F})$ is equicontinuous at point x_0 for any point x_0 in X, $cl(\mathcal{F})$ is equicontinuous.

■

Chapter 14

Adjoint Operator

14.1 Definitions

Definition (Adjoint Matrix). *Let A be an $m \times n$ matrix. We define the **adjoint** of A, denoted by A^*, to be an $n \times m$ matrix given by*

$$(A^*)_{ij} := \overline{(A)_{ji}}.$$

Definition (Adjoint Operator). *Let V and W be inner product spaces. Let T be a linear map from V to W. We define the **adjoint** of T, denoted by T^*, to be a map from W to V such that*

$$\forall x \in V, \forall y \in W, \quad \langle T(x), y \rangle_W = \langle x, T^*(y) \rangle_V.$$

Proposition 14.1.1 (Existence). *Let V be a finite-dimensional inner product space and T be a linear operator on V. Then the adjoint of T exists.*

Proposition 14.1.2 (Uniqueness). *Let V be an inner product space and T be a linear operator on V. Then the adjoint of T is unique, provided that it exists.*

14.2 Properties of the Adjoint Operator

Proposition 14.2.1. *Let V be an inner product space. Then*

(1) $(I_V)^* = I_V$ where I_V is the identity operator on V.

(2) $T^{**} = T$ for any linear operator T on V.

Proposition 14.2.2. *Let V be an inner product space and T be a linear operator on V. Then T^* is also linear.*

Proposition 14.2.3. *Let V be an inner product space. Then*

(1) For any linear operators T and U,

$$(T+U)^* = T^* + U^*.$$

(2) For any linear operator T,

$$(cT)^* = \overline{c} \cdot T^*.$$

(3) For any linear operator T and U,

$$(TU)^* = U^*T^*.$$

Proposition 14.2.4. *Let V be a finite-dimensional inner product space and T be a linear operator on V. Then if T is invertible, T^* is also invertible.*

Proposition 14.2.5. *Let V be an inner product space and T be an invertible linear operator on V. Then $(T^{-1})^* = (T^*)^{-1}$.*

14.3 Normal Operators

Definition (Normal)**.** *Let V be an inner product space and T be a linear operator on V. We say that T is* ***normal*** *if $TT^* = T^*T$.*

14.4 Self-adjoint

Chapter 15

Convolution

Definition (Convolution)**.** *Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. We define the* ***convolution*** *of f and g, denoted by $f * g$, to be a function on $\mathbb{R}$ given by*

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)dt.$$

Chapter 16

Coercive Functions

16.1 Definitions

Definition (Coercive)**.** *Let f be a function from $\mathbb{R}^d$ to $\mathbb{R}^*$. We say that f is **coercive** if $\lim_{\|x\| \to \infty} f(x) = +\infty$.*

16.2 Properties

Proposition 16.2.1. *Let f be a proper lower semi-continuous function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let K be a compact set in $\mathbb{R}^d$. Assume $K \cap \mathrm{dom}(f) \neq \emptyset$. Then f attains its minimum over K.*

Proof.

Define $m := \inf_{x \in K} f(x)$.

Since $m = \inf_{x \in K} f(x)$, there exists a sequence $\{x_i\}_{i \in \mathbb{N}}$ in K such that $\lim_{i \to \infty} f(x_i) = m$.

Since K is compact and $\{x_i\}_{i \in \mathbb{N}} \subseteq K$, there exists a convergent subsequence $\{x_i\}_{i \in I}$ in K where I is an infinite subset of $\mathbb{N}$.

Say the limit is x_∞ where $x_\infty \in K$.

Since $\lim_{i \to \infty} f(x_i) = m$, we get $\lim_{i \in I, i \to \infty} f(x_i) = m$.

Since $\lim_{i \in I, i \to \infty} f(x_i) = m$, we get $\liminf_{i \in I, i \to \infty} f(x_i) = m$.

Since f is lower semi-continuous and $\lim_{i \in I, i \to \infty} x_i = x_\infty$, we get $f(x_\infty) \leq \liminf_{i \in I, i \to \infty} x_i$.

That is, $f(x_\infty) \leq m$.

Since $m = \inf_{x \in K} f(x)$, we have $\forall x \in K, f(x) \geq m$.

In particular, $f(x_\infty) \geq m$.

Since $f(x_\infty) \geq m$ and $f(x_\infty) \leq m$, $f(x_\infty) = m$.

Since f is proper, $f(x_\infty) = m \neq -\infty$.

So f attains its minimum at point x_∞.

■

Proposition 16.2.2. *Let f be a proper, lower semi-continuous, and coercive function from $\mathbb{R}^d$ to $\mathbb{R}^*$. Let C be a closed subset of $\mathbb{R}^d$. Assume $C \cap \mathrm{dom}(f) \neq \emptyset$. Then f attains its minimum over C.*

Proof.

Since $C \cap \mathrm{dom}(f) \neq \emptyset$, take $x \in C \cap \mathrm{dom}(f)$.

Since f is coercive, $\exists R$ such that $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$.

Since $x \in C \cap \mathrm{dom}(f)$ and $\forall y, \|y\| > R$, we have $f(y) \geq f(x)$, the set of minimizers of f over C is the same as the set of minimizers of f over $C \cap \mathrm{ball}[0, R]$.

Since C and $\mathrm{ball}[0, R]$ are both closed, $C \cap \mathrm{ball}[0, R]$ is closed.

Since $\mathrm{ball}[0, R]$ is bounded, $C \cap \mathrm{ball}[0, R]$ is bounded.

Since $C \cap \mathrm{ball}[0, R]$ is closed and bounded, by the Heine-Borel Theorem, $C \cap \mathrm{ball}[0, R]$ is compact.

Since f is proper and lower semi-continuous and $C \cap \mathrm{ball}[0, R]$ is compact, f attains its minimum over $C \cap \mathrm{ball}[0, R]$.

So f attains its minimum over C.

■

Chapter 17

Unclassified Results

Proposition 17.0.1. *Let (X, d) be a compact metric space. Let $L(X)$ be the set of all Lipschitz functions from X to $\mathbb{R}$. Let $C(X)$ be the set of all continuous functions from X to $\mathbb{R}$. Then $L(X)$ is dense in $C(X)$.*

Proposition 17.0.2. *Let $(V, \|\cdot\|)$ be a normed vector space. Let S be a subset of V. Let p be a vector in V. Then we have the followings.*

(1) $p + int(S) = int(p + S)$,

(2) $p + \mathrm{cl}(S) = \mathrm{cl}(p + S)$.

Proof.

Proof of (1).

For one direction, let x be an arbitrary point in the set $(p + int(S))$.

We are to prove that $x \in int(p + S)$.

Since $x \in (p + int(S))$, $(x - p) \in int(S)$.

Since $(x - p) \in int(S)$, by definition of interior, there exists a radius r such that

$$B(x - p, r) \subseteq S.$$

It follows that $B(x, r) \subseteq p + S$.

Since there exists a radius r such that $B(x, r) \subseteq p + S$, by definition of interior,

$$x \in int(p + S).$$

For the reverse direction, let x be an arbitrary point in $int(p + S)$.

We are to prove that $x \in p + int(S)$.

Since $x \in int(p + S)$, by definition of interior, there exists a radius r such that

$$B(x, r) \subseteq (p + S).$$

It follows that $B(x-p,r) \subseteq S$.
Since there exists a radius r such that $B(x-p,r) \subseteq S$, by definition of interior,

$$(x-p) \in int(S).$$

Since $(x-p) \in int(S)$, we get $x \in (p + int(S))$.
Proof of (2).
For one direction, let x be an arbitrary point in the set $(p + cl(S))$.
We are to prove that $x \in cl(p+S)$.
Since $x \in (p + cl(S))$, we get $(x-p) \in cl(S)$.
Since $(x-p) \in cl(S)$, by definition of closure, for any radius r, we have

$$B(x-p,r) \cap S \neq \varnothing.$$

It follows that $B(x,r) \cap (p+S) \neq \varnothing$.
Since for any radius r, $B(x,r) \cap (p+S) \neq \varnothing$, by definition of closure, we get

$$x \in cl(p+S).$$

For the reverse direction, let x be an arbitrary point in $cl(p+S)$.
We are to prove that $x \in (p + cl(S))$.
Since $x \in cl(p+S)$, by definition of closure, for any radius r, we have

$$B(x,r) \cap (p+S) \neq \varnothing.$$

It follows that $B(x-p,r) \cap S \neq \varnothing$.
Since for any radius r, $B(x-p,r) \cap S \neq \varnothing$, by definition of closure, we get

$$(x-p) \in cl(S).$$

Since $(x-p) \in cl(S)$, we get $x \in (p + cl(S))$.

■

Proposition 17.0.3. *Let $(V, \|\cdot\|)$ be a normed vector space. Let S be a subset of V. Let λ be a non-zero real number. Then*

(1) $\lambda int(S) = int(\lambda S)$.

(2) $\lambda \operatorname{cl}(S) = \operatorname{cl}(\lambda S)$.

Proof of (1). For one direction, let x be an arbitrary point in $\lambda int(S)$.
We are to prove that $x \in int(\lambda S)$.
Since $x \in \lambda int(S)$, we get $x/\lambda \in int(S)$.
Since $x/\lambda \in int(S)$, by definition of interior, there exists a radius r such that

$$B(x/\lambda, r) \subseteq S.$$

Let y be an arbitrary point in $B(x, \lambda r)$.
Since $y \in B(x, \lambda r)$, we get $\|y - x\| \leq \lambda r$.
Since $\|y - x\| \leq \lambda r$, we get $\|y/\lambda - x/\lambda\| \leq r$.
Since $\|y/\lambda - x/\lambda\| \leq r$, we get $y/\lambda \in B(x/\lambda, r)$.
Since $y/\lambda \in B(x/\lambda, r)$ and $B(x/\lambda, r) \subseteq S$, we get $y/\lambda \in S$.
Since $y/\lambda \in S$, we get $y \in \lambda S$.
Since any point in $B(x, \lambda r)$ is also in λS, we get $B(x, \lambda r) \subseteq \lambda S$.
Since there exists a radius r such that $B(x, \lambda r) \subseteq \lambda S$, by definition of interior, we get

$$x \in int(\lambda S).$$

For the reverse direction,

■