Stochastic Process

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Convergence of Random Variables

1.1 Definitions

Definition (Convergence in Distribution). *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let F_n be the cumulative distribution function of X_n. Let X be a random variable. Let F_X be the cumulative distribution function of X. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$* ***converges in distribution*** *to X, denoted by $X_n \xrightarrow{d} X$, if $\forall x$ at which F is continuous,*

$$\lim_{n\to\infty} F_n(x) = F_X(x).$$

In this case, we say F_X is the asymptotic distribution of $\{X_n\}_{n\in\mathbb{N}}$.

Definition (Convergence in Probability). *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let X be a random variable. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$* ***converges in probability*** *to X, denoted by $X_n \xrightarrow{p}$, if*

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} P\big(|X_n - X| \geq \varepsilon\big) = 0.$$

Or equivalently,

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} P\big(|X_n - X| < \varepsilon\big) = 1.$$

Definition (Almost Sure Convergence). *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let X be a random variable. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$* ***converges almost surely*** *to X if*

$$P(\lim_{n\to\infty} X_n = X) = 1.$$

Definition (Sure Convergence)**.** *Let Ω be a sample space of the underlying probability space. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let X be a random variable. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$* ***converges surely*** *to X if*

$$\forall\omega \in \Omega, \quad \lim_{n\to\infty} X_n(\omega) = X(\omega).$$

Definition (Convergence in Mean)**.** *Let $r \geq 1$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables. Let X be a random variable. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$* ***converges in the r^{th} mean*** *to X, denoted by $X_n \xrightarrow{L^r} X$, if the r^{th} absolute moments $\mathbb{E}[|X_n^r|]$ and $\mathbb{E}[|X|^r]$ of X_n and X exists and*

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^r] = 0.$$

1.2 Markov's Inequality

Theorem 1 (Markov's Inequality)**.** *Let X be a random variable. Let k and c be arbitrary positive numbers. Then*

$$P\big(|X| \geq c\big) \leq \frac{\mathbb{E}\big[|X|^k\big]}{c^k}.$$

Corollary.

$$P\Big(|X - \mathbb{E}[X]| > k\sqrt{\mathrm{var}[X]}\Big) \leq \frac{1}{k^2}.$$

1.3 Properties

Proposition 1.3.1. *Convergence in probability implies convergence in distribution.*

Proposition 1.3.2. *Almost sure convergence implies convergence in probability.*

Proposition 1.3.3. *Convergence in the r^{th} mean for $r \geq 1$ implies convergence in probability.*

Proposition 1.3.4. *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of random variables. Let c be a constant. Then $\{X_i\}_{i\in\mathbb{N}}$ converges to c in distribution if and only if $\{X_i\}_{i\in\mathbb{N}}$ converges to c in probability.*

Sketch Proof.

$$\begin{aligned}
P\big(|X_i - c| \geq \varepsilon\big) &= P\big(X_i \geq c + \varepsilon\big) + P\big(X_i \leq c - \varepsilon\big) \\
&= 1 - P\big(X_i < c + \varepsilon\big) + F_i(c - \varepsilon) \\
&\leq 1 - P\big(X_i \leq c + \varepsilon/2\big) + F_i(c - \varepsilon) \\
&= 1 - F_i(c + \varepsilon/2) + F_i(c - \varepsilon)
\end{aligned}$$

$$
\begin{aligned}
&\lim_{i\to\infty}\big[1 - F_i(c+\varepsilon/2) + F_i(c-\varepsilon)\big] \\
&= 1 - F(c+\varepsilon/2) + F(c-\varepsilon) \\
&= 1 - 1 + 0 \\
&= 0.
\end{aligned}
$$

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Proposition 1.3.5 (Continuous Map). *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of random variables. Let g be a continuous function on the X_i's. Then*

(1) *if $X_i \xrightarrow{d} X$, we have $g(X_i) \xrightarrow{d} g(X)$.*

(2) *if $X_i \xrightarrow{p} c$, we have $g(X_i) \xrightarrow{p} g(c)$.*

Proposition 1.3.6 (Slutsky's Theorem). *Let $\{X_i\}_{i\in\mathbb{N}}$ and $\{Y_i\}_{i\in\mathbb{N}}$ be sequences of random variables. Suppose $X_i \xrightarrow{d} X$ for some random variable X and $Y_i \xrightarrow{p} c$ for some constant c. Then*

(1) $X_i + Y_i \xrightarrow{d} X + c$.

(2) $X_iY_i \xrightarrow{d} cX$.

(3) $X_i/Y_i \xrightarrow{d} X/c$.

1.4 Law of Large Numbers

Theorem 2 (Strong Law of Large Nubmers). *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables. Suppose that $\mathbb{E}[X_i] = \mu$ for some $\mu \in \mathbb{R}$ for all $i \in \mathbb{N}$. Then their cumulative average $\bar{X}_n$ converges almost surely to μ. That is,*

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \overset{almost surely}{\longrightarrow} \mu.$$

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Markov Decision Process

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Poisson Process

3.1 Homogeneous Poisson Process

3.1.1 Definitions

Definition (Homogeneous Poisson Process)**.** *We say a counting process is a* ***homogeneous Poisson counting process*** *with rate* $\lambda > 0$ *if it has the following three properties:*

- $N(0) = 0$;
- *it has independent increments; and*
- *the number of events in any interval of length* t *is a Poisson random variable with parameter* λt.

Definition (Homogeneous Poisson Process)**.** *We say a point process is a* ***homogeneous Poisson point process*** *with rate* $\lambda > 0$ *if the following two conditions hold:*

- *The probability* $\mathbb{P}\{N(a, b] = n\}$ *of the number* $N(a, b]$ *of points of the process in the interval* $(a, b]$ *being equal to some counting number* n *is given by*

$$\mathbb{P}\{N(a, b] = n\} = \frac{[\lambda(b-a)]^n}{n!} e^{-\lambda(b-a)}.$$

  i.e. the number of arrivals in each finite interval has a Poisson distribution.

- *For any positive integer* k *and non-overlapping intervals* $(a_1, b_1], ..., (a_k, b_k]$,

$$\mathbb{P}\left\{\bigwedge_{i=1}^{k} N(a_i, b_i] = n_i\right\} = \prod_{i=1}^{k} \mathbb{P}\{N(a_i, b_i] = n_i\}.$$

  i.e. the number of arrivals in disjoint intervals are independent random variables.