

# Functional Analysis

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# Chapter 1

## Balanced Sets

### 1.1 Definitions

**Definition** (Balanced Sets). *Let  $X$  be a vector space over field  $\mathbb{F}$ . Let  $S$  be a subset of  $X$ . We say that  $S$  is **balanced** if*

$$\forall a \in \mathbb{F} : |a| \leq 1, \quad aS \subseteq S.$$

**Definition** (Balanced Hull). *Let  $X$  be a vector space over field  $\mathbb{F}$ . Let  $S$  be a subset of  $X$ . We define the **balanced hull** of  $S$ , denoted by  $\text{balhull}(S)$ , to be the smallest balanced set containing  $S$ .*

**Definition** (Balanced Core). *Let  $X$  be a vector space over field  $\mathbb{F}$ . Let  $S$  be a subset of  $X$ . We define the **balanced core** of  $S$ , denoted by  $\text{balcore}(S)$ , to be the largest balanced set contained in  $S$ .*

### 1.2 Properties

**Proposition 1.2.1.** *Let  $X$  be a vector space over field  $\mathbb{F}$ . Let  $B$  be a balanced subset of  $X$ . Then*

$$\forall a, b \in \mathbb{F} : |a| \leq |b|, \quad aB \subseteq bB.$$

**Proposition 1.2.2.** *Balanced sets are path connected.*

**Proposition 1.2.3** (Act on Other Properties). • *The balanced hull of a compact set is compact.*

- *The balanced hull of a totally bounded set is totally bounded.*
- *The balanced hull of a bounded set is bounded.*

**Proposition 1.2.4** (Act on Other Properties). • *The balanced core of a closed set is closed.*

**Proposition 1.2.5.** *Let  $X$  be a vector space over field  $\mathbb{F}$ . Let  $a$  be a scalar in field  $\mathbb{F}$ . Then*

$$a \operatorname{balhull}(S) = \operatorname{balhull}(aS).$$

### 1.3 Stability of Balance

**Proposition 1.3.1** (Set Operations). • *The union of balanced sets is also balanced.*

- *The intersection of balanced sets is also balanced.*

**Proposition 1.3.2** (Linear Mappings). • *The scalar multiple of a balanced set is also balanced.*

- *The (Minkowski) sum of two balanced sets is also balanced.*
- *The image of a balanced set under a linear operator is also balanced.*
- *The inverse image of a balanced set under a linear operator is also balanced.*

**Proposition 1.3.3** (Topological Operations). *The closure of a balanced set is also balanced.*

**Proposition 1.3.4.** *The convex hull of a balanced set is also balanced (and also convex).*

### 1.4 Absorbing Sets

**Definition** (Absorbing Sets). *Let  $\mathfrak{X}$  be a vector space over field  $\mathbb{F}$ . Let  $S$  be a subset of  $X$ . We say that  $S$  is **absorbing** if*

$$\forall x \in \mathfrak{X}, \quad \exists r \in \mathbb{R} : r > 0, \quad \forall c \in \mathbb{F} : |c| \geq r, \quad x \in cS.$$

*i.e.,*

$$\bigcup_{n \in \mathbb{N}} nS = \mathfrak{X}.$$

**Proposition 1.4.1.** *Every absorbing set contains the origin.*



## Chapter 2

# Normed Linear Spaces

### 2.1 Definitions

**Definition** (Seminorm). Let  $\mathfrak{X}$  be a vector space over field  $\mathbb{F}$ . We define a **seminorm** on  $\mathfrak{X}$ , denoted by  $\nu$ , to be a map from  $\mathfrak{X}$  to  $\mathbb{R}$  that satisfies the following conditions.

(1)  $\forall x \in \mathfrak{X}, \quad \nu(x) \geq 0.$

(2)  $\forall \lambda \in \mathbb{F}, \forall x \in \mathfrak{X}, \quad \nu(\lambda x) = \lambda \nu(x).$

(3) *Triangle Inequality.*

$$\forall x, y \in \mathfrak{X}, \quad \nu(x + y) \leq \nu(x) + \nu(y).$$

*The idea behind the seminorm is that we are trying to give our vector space a notion of “length” of vectors.*

**Definition** (Norm). Let  $\mathfrak{X}$  be a vector space over field  $\mathbb{F}$ . We define a **norm** on  $\mathfrak{X}$ , denoted by  $\nu$ , to be a seminorm on  $\mathfrak{X}$  that satisfies the additional condition:

$$\forall x \in \mathfrak{X}, \quad \nu(x) = 0 \iff x = 0.$$

### 2.2 Properties

**Proposition 2.2.1.** Let  $(V, \|\cdot\|_V)$  be a normed vector space over field  $\mathbb{F}$ . Then  $(V, \|\cdot\|)$  is complete if and only if  $(\overline{B(0,1)}, \|\cdot\|_V)$  is complete.

*Proof.*

For one direction, assume that  $(V, \|\cdot\|)$  is complete.

We are to prove that  $(\overline{B(0,1)}, \|\cdot\|_V)$  is complete.

Since  $(\overline{B(0,1)}, \|\cdot\|_V)$  is a closed subspace of  $(V, \|\cdot\|)$  and  $(V, \|\cdot\|)$  is complete,  $(\overline{B(0,1)}, \|\cdot\|_V)$  is also complete.

For the reverse direction, assume that  $(\overline{B(0,1)}, \|\cdot\|_V)$  is complete.

We are to prove that  $(V, \|\cdot\|_V)$  is complete.

Let  $\{x_i\}_{i \in \mathbb{N}}$  be an arbitrary Cauchy sequence in  $(V, \|\cdot\|_V)$ .

Since  $\{x_i\}_{i \in \mathbb{N}}$  is Cauchy in  $(V, \|\cdot\|_V)$ ,  $\{x_i\}_{i \in \mathbb{N}}$  is bounded in  $(V, \|\cdot\|_V)$ .

Let  $\lambda$  be a positive upper bound for  $\{\|x_i\|_V\}_{i \in \mathbb{N}}$ .

Since  $\{x_i\}_{i \in \mathbb{N}}$  is Cauchy in  $(V, \|\cdot\|_V)$ ,  $\{x_i/\lambda\}_{i \in \mathbb{N}}$  is Cauchy in  $(\overline{B(0,1)}, \|\cdot\|_V)$ .

Since  $\{x_i/\lambda\}_{i \in \mathbb{N}}$  is Cauchy in  $(\overline{B(0,1)}, \|\cdot\|_V)$  and  $(\overline{B(0,1)}, \|\cdot\|_V)$  is complete,  $\{x_i/\lambda\}_{i \in \mathbb{N}}$  converges in  $(\overline{B(0,1)}, \|\cdot\|_V)$ .

Since  $\{x_i/\lambda\}_{i \in \mathbb{N}}$  converges in  $(\overline{B(0,1)}, \|\cdot\|_V)$ ,  $\{x_i\}_{i \in \mathbb{N}}$  converges in  $(V, \|\cdot\|_V)$ .

Since any Cauchy sequence in  $(V, \|\cdot\|_V)$  converges in  $(V, \|\cdot\|_V)$ ,  $(V, \|\cdot\|_V)$  is complete. ■

**Proposition 2.2.2.** *Proper subspaces of a normed linear space has empty interior.*

*Proof.* Let  $\mathfrak{X}$  be a normed linear space. Let  $\mathcal{M}$  be a proper subspace of  $\mathfrak{X}$ . Assume for the sake of contradiction that  $\mathcal{M}$  has non-empty interior. Then  $\exists x_0 \in \mathcal{M}$  and  $\exists r > 0$  such that  $\text{ball}(x_0, r) \subseteq \mathcal{M}$  where  $\text{ball}(x_0, r)$  denotes the open ball centered at point  $x_0$  with radius  $r$ . Let  $x$  be an arbitrary point in  $\mathfrak{X}$ . Define a point  $y(x)$  as  $y(x) := x_0 + \frac{r}{2\|x\|}x$ . Then  $x = \frac{2\|x\|}{r}(y - x_0)$ . It is easy to verify that  $\|y - x_0\| = \frac{r}{2} < r$ . So  $y \in \text{ball}(x_0, r)$ . So  $y \in \mathcal{M}$ . Since  $y, x_0 \in \mathcal{M}$  and  $\mathcal{M}$  is a subspace, we get  $\frac{2\|x\|}{r}(y - x_0) \in \mathcal{M}$ . That is,  $x \in \mathcal{M}$ . So  $\forall x \in \mathfrak{X}, x \in \mathcal{M}$ . So  $\mathcal{M} = \mathfrak{X}$ . This contradicts to the assumption that  $\mathcal{M}$  is a proper subspace of  $\mathfrak{X}$ . So  $\mathcal{M}$  has empty interior. ■

**Proposition 2.2.3.** *Closed proper subspaces of a normed linear space are nowhere dense.*

*Proof.* Let  $\mathfrak{X}$  be a normed linear space. Let  $\mathcal{M}$  be a closed proper subspace of  $\mathfrak{X}$ . Since  $\mathcal{M}$  is closed,  $\text{cl}(\mathcal{M}) = \mathcal{M}$ . So  $\text{cl}(\mathcal{M}) = \mathcal{M}$  is a closed proper subspace of  $\mathfrak{X}$ . Since  $\text{cl}(\mathcal{M})$  is a proper subspace of  $\mathfrak{X}$ ,  $\text{int}(\text{cl}(\mathcal{M})) = \emptyset$ . So  $\mathcal{M}$  is nowhere dense. ■

**Proposition 2.2.4.** *Finite dimensional subspace of a normed linear space is closed.*

**Proposition 2.2.5.** *Finite-dimensional normed linear spaces are complete.*

## 2.3 Equivalence of Norms

**Definition** (Equivalence of Norms). *Let  $\mathfrak{X}$  be a vector space over field  $\mathbb{F}$ . Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two norms on  $V$ . We say that  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are **equivalent** if*

$$\exists c_1, c_2 > 0, \quad \forall v \in \mathfrak{X}, \quad c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_1.$$

Or equivalently,

$$c_1 \|v\|_2 \leq \|v\|_1 \leq c_2 \|v\|_2.$$

**Proposition 2.3.1.** *The equivalence of norms is an equivalence relation.*

**Theorem 1.** *Let  $V$  be a finite dimensional vector space over field  $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$ . Then any two norms on  $V$  are equivalent.*

*Proof.*

Let  $\|\cdot\|_p$  be an arbitrary  $p$ -norm on  $V$  and  $\|\cdot\|$  be an arbitrary norm on  $V$ .

Let  $\mathcal{B}$  be the standard basis for  $V$ . Say  $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$ .

Let  $v$  be an arbitrary vector in  $V$ .

$$\begin{aligned} \|v\| &= \left\| \sum_{i=1}^n v_i e_i \right\| \\ &\leq \sum_{i=1}^n |v_i| \|e_i\| \\ &\leq \left( \sum_{i=1}^n |v_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n \|e_i\|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \\ &= \left( \sum_{i=1}^n \|e_i\|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \|v\|_p \\ &:= c_1 \|v\|_p. \end{aligned}$$

■

**Proposition 2.3.2.** *Let  $X$  be a vector space. Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two norms on  $X$ . Then  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent if and only if they generate the same metric topology.*

*Proof.* Convergence to 0 is equivalent under either  $\|\cdot\|_1$  or  $\|\cdot\|_2$ . i.e., equivalent norms give rise to the same set of sequences that are convergent. Convergence of sequences defines the topology. ■

**Proposition 2.3.3.** *Let  $\mathfrak{X}$  be a vector space. Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two norms on  $\mathfrak{X}$ . Let  $\iota$  be the identity map from  $(\mathfrak{X}, \|\cdot\|_1)$  to  $(\mathfrak{X}, \|\cdot\|_2)$ . Then if  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent,  $\iota$  is continuous, and in fact, a homeomorphism between  $(\mathfrak{X}, \|\cdot\|_1)$  and  $(\mathfrak{X}, \|\cdot\|_2)$ .*

## 2.4 Dual Norms

**Definition (Dual Norm).** *Let  $(V, \|\cdot\|)$  be an normed vector space. We define the **dual norm** of  $\|\cdot\|$ , denoted by  $\|\cdot\|_\circ$ , to be a function given by*

$$\|v\|_\circ := \max_{\|w\|=1} v \cdot w = \max_{\|w\| \neq 0} \frac{|v \cdot w|}{\|w\|}.$$

**Proposition 2.4.1.** *Dual norms of norms are indeed norms.*

**Proposition 2.4.2.** *Let  $(V, \|\cdot\|)$  be a normed vector space. Let  $v, w$  be vectors in the space. Then*

$$|v \cdot w| \leq \|v\| \cdot \|w\|.$$

## 2.5 $p$ -norms

**Definition ( $p$ -norm).** *Let  $V$  be a finite-dimensional normed vector space over field  $\mathcal{F}$ . Let  $\mathcal{B} = \{b_1, \dots, b_n\}$  be a basis for  $V$  where  $n = \dim(V)$ . Let  $v$  be a vector in a normed vector space. For  $p \in [1, +\infty)$ , we define the  **$p$ -norm** of  $v$ , denoted by  $\|v\|_p$ , to be the number given by*

$$\|v\|_p = \left( \sum_{i=1}^n |(v_{\mathcal{B}})_i|^p \right)^{\frac{1}{p}}.$$

**Definition (Infinity Norm - 1).** *Let  $\mathfrak{X} = \mathbb{K}^n$  where  $\mathbb{K}$  is a field and  $n \in \mathbb{N}$ . We define the **infinity norm** on  $\mathfrak{X}$ , denoted by  $\|\cdot\|_\infty$ , to be a function given by*

$$\|v\|_\infty := \max\{|v_i|\}_{i=1}^n.$$

**Definition (Infinity Norm - 2).** *Let  $\mathfrak{X} = \mathbb{K}^{\mathbb{N}}$ . We define the **infinity norm** on  $\mathfrak{X}$ , denoted by  $\|\cdot\|_\infty$ , to be a function given by*

$$\|v\|_\infty := \sup_{i \in \mathbb{N}} |v_i|.$$

**Definition (Infinity Norm - 3).** *Let  $\mathfrak{X} = \mathcal{C}([0, 1], \mathbb{C})$ . We define the **infinity norm** on  $\mathfrak{X}$ , denoted by  $\|\cdot\|_\infty$ , to be a function given by*

$$\nu(f) := \sup_{x \in [0, 1]} |f(x)|.$$

**Proposition 2.5.1.** *Let  $\mathfrak{X} := \mathcal{C}([0, 1], \mathbb{C})$ . Let  $x$  be an arbitrary number in  $[0, 1]$ . Define a function  $\nu_x$  on  $\mathfrak{X}$  by  $\nu_x(f) := |f(x)|$ . Define a function  $\nu$  on  $\mathfrak{X}$  by  $\nu(f) := \sup_{x \in [0, 1]} |f(x)|$ . Then  $\nu_x$  is a seminorm on  $\mathfrak{X}$  for each  $x$  and  $\nu$  is a norm on  $\mathfrak{X}$  and we have  $\nu = \sup_{x \in [0, 1]} \nu_x$ .*

**Proposition 2.5.2.**  *$p$ -norms are indeed norms.*

**Proposition 2.5.3.** *For any vector  $v$  in  $\mathbb{R}^n$ , we have*

$$\lim_{p \rightarrow \infty} \|v\|_p = \|v\|_\infty.$$

*i.e.,*

$$\lim_{p \rightarrow \infty} \left( \sum_{i=1}^n |v_i|^p \right)^{\frac{1}{p}} = \max\{|v_i|\}_{i=1}^n.$$

*Proof.* Let  $p$  be an arbitrary number in  $[1, +\infty)$ . Let  $k$  be an arbitrary index in  $\{1, \dots, n\}$ . Then

$$|v_k| \leq \left( \sum_{i=1}^n |v_i|^p \right)^{1/p} = \|v\|_p.$$

So

$$\max\{|v_k|\} = \|v\|_\infty \leq \|v\|_p.$$

So

$$\lim_{p \rightarrow \infty} \|v\|_p \geq \|v\|_\infty. \quad (1)$$

On the other hand, note that

$$\left( \sum_{i=1}^n |v_i|^p \right) / \|v\|_\infty^p = \sum_{i=1}^n \left( \frac{|v_i|}{\|v\|_\infty} \right)^p$$

decreases as  $p$  increases. So it is bounded above. Say

$$\left( \sum_{i=1}^n |v_i|^p \right) / \|v\|_\infty^p \leq C$$

for some  $C \in \mathbb{R}$ . Then

$$\left( \sum_{i=1}^n |v_i|^p \right)^{1/p} = \|v\|_p \leq C^{1/p} \|v\|_\infty.$$

So

$$\lim_{p \rightarrow \infty} \|v\|_p \leq \lim_{p \rightarrow \infty} C^{1/p} \|v\|_\infty = \|v\|_\infty. \quad (2)$$

From (1) and (2) we get

$$\lim_{p \rightarrow \infty} \|v\|_p = \|v\|_\infty.$$

■

**Proposition 2.5.4.** *Let  $p$  be an arbitrary number in  $[1, +\infty)$ . Then the dual norm of the  $p$ -norm  $\|\cdot\|_p$  is the  $q$ -norm  $\|\cdot\|_q$  where  $q$  is such that satisfies*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

**Proposition 2.5.5.** *Let  $p$  and  $q$  be numbers in  $[1, +\infty]$ . Let  $v$  be a vector in  $\mathbb{R}^n$ . Then if  $p \leq q$ ,*

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \cdot \|x\|_q.$$

**Proposition 2.5.6.** *Let  $w$  and  $z$  be vectors in  $\mathbb{E}^d$ . Then*

$$\|w + z\|_2^2 + \|w - z\|_2^2 = 2(\|w\|_2^2 + \|z\|_2^2).$$



## Chapter 3

# Inner Product Spaces

### 3.1 Inner Products

#### 3.1.1 Definitions

**Definition** (Inner Product). *Let  $V$  be a vector space over field  $\mathbb{F}$ . We define an **inner product** on  $V$ , denoted by  $\langle \cdot, \cdot \rangle$ , to be a scalar-valued function defined on  $V \times V$  such that*

(1) *Positive Definiteness:*

$$\forall x \in V, \langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \iff x = 0.$$

(2) *Sesqui-Linearity:*

$$\begin{aligned} \forall x, y, z, w \in V, \quad \langle x + y, z + w \rangle &= \langle x, z \rangle + \langle y, z \rangle + \langle x, w \rangle + \langle y, w \rangle, \text{ and} \\ \forall a, b \in \mathbb{F}, \forall x, y \in V, \quad \langle ax, by \rangle &= a\bar{b}\langle x, y \rangle. \end{aligned}$$

(3) *Conjugate Symmetry:*

$$\forall x, y \in V, \quad \langle x, y \rangle = \overline{\langle y, x \rangle}.$$

**Definition** (Induced Norm). *Let  $\mathfrak{X}$  be an inner product space over field  $\mathbb{K}$ . We define the **norm induced by  $\langle \cdot, \cdot \rangle$** , denoted by  $\| \cdot \|$ , to be a function from  $\mathfrak{X}$  to  $\mathbb{R}_+$  given by*

$$\|x\| := \sqrt{\langle x, x \rangle}$$

#### 3.1.2 Examples of Inner Products

**Definition** (Standard Inner Product). *For  $V = \mathbb{F}^n$ , we define the **standard inner product** by*

$$\langle x, y \rangle := \sum_{i=1}^n x_i \bar{y}_i.$$

**Definition** (Frobenius Inner Product). For  $V = \mathbb{F}^{n \times n}$ , we define the **Frobenius inner product** by

$$\langle M_1, M_2 \rangle := \text{tr}(M_2^* M_1).$$

**Definition.** Let  $V$  be the space of continuous scalar-valued functions on  $[0, 2\pi]$ . We define the inner product on  $V$  by

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} dx.$$

### 3.1.3 Properties

**Proposition 3.1.1.** Let  $V$  be a finite dimensional inner product space. Let  $\mathcal{B}$  be a basis for  $V$ . Let  $x$  and  $y$  be vectors in  $V$ . Then

$$x = y \iff \forall b \in \mathcal{B}, \quad \langle x, b \rangle = \langle y, b \rangle.$$

## 3.2 Inner Product Space

**Definition** (Inner Product Space). Let  $\mathfrak{X}$  be a vector space over field  $\mathbb{F}$ . Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $\mathfrak{X}$ . We define an **inner product space** to be the pair  $(\mathfrak{X}, \langle \cdot, \cdot \rangle)$ .

## 3.3 Inequalities

**Theorem 2** (Minkowski).

$$\left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}$$

**Proposition 3.3.1** (Cauchy-Schwarz Inequality). Let  $V$  be an inner product space. Then

$$\forall x, y \in V, \quad |\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

**Proposition 3.3.2** (Triangle Inequality). Let  $V$  be an inner product space. Then

$$\forall x, y \in V, \quad \|x + y\| \leq \|x\| + \|y\|$$

**Proposition 3.3.3** (Parallelogram Law). Let  $\mathfrak{X}$  be an inner product space. Then

$$\forall x, y \in \mathfrak{X}, \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

*Proof.*

$$\|x + y\|^2 + \|x - y\|^2 = \langle x + y, x + y \rangle + \langle x - y, x - y \rangle$$



$$\begin{aligned} &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &\quad + \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= 2\langle x, x \rangle + 2\langle y, y \rangle \\ &= 2\|x\|^2 + 2\|y\|^2. \end{aligned}$$

That is,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

■



## Chapter 4

# Orthogonality

### 4.1 Orthogonal Sets

**Definition** (Orthogonality). *Let  $V$  be an inner product space. We say that points  $x$  and  $y$  in  $V$  are **orthogonal** if  $\langle x, y \rangle = 0$ .*

**Definition** (Orthogonal Set). *Let  $\mathfrak{X}$  be an inner product space. Let  $S$  be a subset of  $\mathfrak{X}$ . We say that  $S$  is **orthogonal** if*

$$\forall x, y \in S : x \neq y, \quad \langle x, y \rangle = 0.$$

*i.e., if any two distinct vectors are orthogonal.*

**Definition** (Orthonormal Set). *Let  $\mathfrak{X}$  be an inner product space. Let  $S$  be a set in the space. We say that  $S$  is **orthonormal** if  $S$  is orthogonal and  $\forall x \in S, \|x\| = 1$  where  $\|\cdot\|$  is the norm induced by the inner product.*

**Proposition 4.1.1.** *Orthogonal sets are linearly independent.*

### 4.2 Orthogonal Bases

**Definition** (Orthogonal Basis). *Let  $V$  be an inner product space. Let  $S$  be a set in the space. We say that  $S$  is an **orthogonal basis** for  $V$  if it is an ordered basis for  $V$  and orthogonal.*

**Definition** (Orthonormal Basis). *Let  $\mathfrak{X}$  be an inner product space. Let  $S$  be a set in the space. We say that  $S$  is an **orthonormal basis** for  $\mathfrak{X}$  if it is the maximal, with respect to inclusion, in the collection of all orthonormal sets in the space.*

**Proposition 4.2.1.** *Let  $V$  be an inner product space. Let  $S = \{v_1, \dots, v_n\}$  be an orthogonal subset of  $V$  where each  $v_i$  is non-zero. Then*

$$\forall y \in \text{span}(S), \quad y = \sum_{i=1}^n \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

**Theorem 3** (Gram-Schmidt Process). *Let  $V$  be an inner product space. Let  $S = \{x_0, \dots, x_n\}$  be a linearly independent subset of  $V$ . Then the set  $T = \{y_0, \dots, y_n\}$  given by  $y_0 := x_0$  and*

$$\forall i \in \{1, \dots, n\}, \quad y_i := x_i - \sum_{j=1}^{i-1} \frac{\langle x_i, y_j \rangle}{\|y_j\|} y_j$$

*is an orthogonal subset of  $V$  consisting of non-zero vectors such that  $\text{span}(S) = \text{span}(S')$ .*

**Proposition 4.2.2.** *Let  $V$  be an inner product space and  $S = \{v_0, v_1, \dots, v_n\}$  be an orthogonal subset of  $V$ . Then the set  $S'$  derived from the Gram-Schmidt process is exactly  $S$ .*

**Theorem 4** (Parseval's Identity). *Let  $V$  be a finite-dimensional inner product space. Let  $\mathcal{B} = \{v_1, \dots, v_n\}$  be an orthogonal basis for  $V$ . Then*

$$\forall x, y \in V, \quad \langle x, y \rangle = \sum_{i=1}^n \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

**Proposition 4.2.3.** *Let  $\mathcal{H}$  be a Hilbert space. Let  $\mathcal{E}$  be an orthonormal set in  $\mathcal{H}$ . Then  $\mathcal{E}$  is an orthonormal basis for  $\mathcal{H}$  if and only if*

$$\forall x \in \mathcal{H}, \quad x = \sum_{e \in \mathcal{E}} \langle x, e \rangle e.$$

### 4.3 Orthogonal Complements

**Definition** (Orthogonal Complement). *Let  $\mathfrak{X}$  be an inner product space. Let  $S$  be a non-empty subset of  $V$ . We define the **orthogonal complement** of  $S$ , denoted by  $S^\perp$ , to be a set given by*

$$S^\perp := \{x \in \mathfrak{X} : \forall s \in S, \langle x, s \rangle = 0\}.$$

*i.e., the set of all points in  $\mathfrak{X}$  that are orthogonal to all vectors in  $S$ .*

**Proposition 4.3.1.** *Let  $V$  be a finite-dimensional inner product space. Then*

$$(1) \quad V^\perp = \{0_V\}$$

$$(2) \quad \{0_V\}^\perp = V$$

**Proposition 4.3.2.** *Orthogonal complements are always linear subspaces.*

**Proposition 4.3.3.** *Let  $V$  be an inner product space and  $W$  be a subspace of  $V$  with basis  $\beta$ . Then a vector in  $V$  is also in  $W^\perp$  if and only if it is orthogonal to all vectors in  $\beta$ .*

**Proposition 4.3.4** (Extension). *Let  $V$  be an  $n$ -dimensional inner product space and  $S = \{v_1, v_2, \dots, v_k\}$  be an orthogonal subset of  $V$ . Then  $S$  can be extended to an orthogonal basis  $B = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$  for  $V$ .*

**Proposition 4.3.5.** *Let  $V$  be an inner product space. Then*

- (1)  $S \subseteq T$  implies  $T^\perp \subseteq S^\perp$  for any subsets  $S$  and  $T$  of  $V$ .
- (2)  $S \subseteq (S^\perp)^\perp$  for any subset  $S$  of  $V$ .

**Proposition 4.3.6.** *Let  $V$  be a finite-dimensional inner product space and  $W$  be a subspace of  $V$ . Then*

- (1)  $W = (W^\perp)^\perp$
- (2)  $V = W \oplus W^\perp$

**Proposition 4.3.7.** *Let  $V$  be a finite-dimensional inner product space and  $W_1$  and  $W_2$  be subspaces of  $V$ . Then*

- (1)  $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$
- (2)  $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

## 4.4 Orthogonal Projection

**Definition** (Orthogonal Projection). *Let  $V$  be a vector space. Let  $W$  be a finite-dimensional subspace of  $V$ . Let  $x$  be a vector in  $V$ . We define the **orthogonal projection** of  $x$  on  $W$ , denoted by  $(x)$ , to be the vector  $u$  in  $W$  such that  $x = u + v$  where  $v$  is another vector in  $W^\perp$ .*

## 4.5 Inequalities in Hilbert Spaces

**Theorem 5** (Bessel's Inequality). *Let  $\mathcal{H}$  be a Hilbert space. Let  $\mathcal{E}$  be an orthonormal set in the space. Then*

$$\forall x \in \mathcal{H}, \quad \sum_{e \in \mathcal{E}} |\langle x, e \rangle|^2 \leq \|x\|^2.$$

**Proposition 4.5.1.** *Let  $\mathcal{H}$  be a Hilbert space. Let  $\mathcal{E}$  be an orthonormal set in  $\mathcal{H}$ . Let  $x$  be a point in the space. Then the net  $\sum_{e \in \mathcal{E}} \langle x, e \rangle e$  converges in  $\mathcal{H}$ .*

*Proof.* Let  $\mathcal{F}$  be the collection of all finite subsets of  $\mathcal{E}$ , partially ordered by inclusion. Define for each  $F \in \mathcal{F}$  a vector  $y_F$  as  $y_F := \sum_{e \in F} \langle x, e \rangle e$ . Let  $\varepsilon$  be an arbitrary positive number. Since  $\mathcal{E}$  is an orthonormal set, the set  $\{e \in \mathcal{E} : \langle x, e \rangle \neq 0\}$  is countable. Let  $\{e_i\}_{i \in \mathbb{N}}$  denote the set. By the Bessel's inequality,  $\exists N \in \mathbb{N}$  such that  $\sum_{i=N+1}^{\infty} |\langle x, e_i \rangle|^2 < \varepsilon^2$ . Define a set  $F_0$  as  $F_0 := \{e_1, \dots, e_N\}$ . Let  $F$  and  $G$  be arbitrary elements in  $\mathcal{F}$  such that  $F_0 \leq F$  and  $F_0 \leq G$ . Then

$$\begin{aligned}
 \|y_F - y_G\|^2 &= \left\| \sum_{e \in F \setminus G} \langle x, e \rangle e - \sum_{e \in G \setminus F} \langle x, e \rangle e \right\|^2 \\
 &= \sum_{e \in F \cup G \setminus F \cap G} |\langle x, e \rangle|^2 \\
 &= \sum_{e \in F \cup G} |\langle x, e \rangle|^2 - \sum_{e \in F \cap G} |\langle x, e \rangle|^2 \\
 &\leq \sum_{i=N+1}^{\infty} |\langle x, e_i \rangle|^2 \\
 &< \varepsilon^2.
 \end{aligned}$$

So  $\{y_F\}_{F \in \mathcal{F}}$  is Cauchy. Since  $\mathcal{H}$  is complete and  $\{y_F\}_{F \in \mathcal{F}}$  is Cauchy,  $\{y_F\}_{F \in \mathcal{F}}$  converges. ■

## Chapter 5

# Sequence Spaces

### 5.1 $\ell^p$ Space

**Definition** ( $\ell^p$  Space). We define the  $\ell^p$  space to be the set of all sequences  $x$  such that  $\|x\|_p$  is finite, equipped with the  $p$ -norm  $\|\cdot\|_p$ .

**Definition** (Weighted  $\ell^p$  Space). Let  $(r_i)_{i \in \mathbb{N}}$  be a sequence of positive integers. We define the **weighted  $\ell^p$**  space to be the set given by

$$\ell^p := \left\{ (x_i)_{i \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sum_{i \in \mathbb{N}} r_i |x_i|^p < +\infty \right\}.$$

**Proposition 5.1.1.** For  $p \in [1, +\infty)$ ,  $(\ell^p, \|\cdot\|_p)$  is complete.

*Proof.*

Let  $\{x_n\}_{n \in \mathbb{N}}$  be an arbitrary Cauchy sequence in  $\ell^p$ .

Since  $\{x_n\}_{n \in \mathbb{N}}$  is Cauchy in  $\ell^p$ ,  $\forall \varepsilon > 0$ ,  $\exists N(\varepsilon) \in \mathbb{N}$  such that  $\forall m, n > N$ , we have  $\|x_m - x_n\|_p < \varepsilon$ .

Since  $\|x_m - x_n\|_p < \varepsilon$  and  $|x_m^{(i)} - x_n^{(i)}| \leq \|x_m - x_n\|_p$  for any  $i \in \mathbb{N}$ ,  $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$  for any  $i \in \mathbb{N}$ .

Since for any  $i \in \mathbb{N}$  and any positive number  $\varepsilon$ , there exists an integer  $N(\varepsilon)$  such that for any indices  $m, n > N$ , we have  $|x_m^{(i)} - x_n^{(i)}| < \varepsilon$ , by definition,  $\{x_n^{(i)}\}_{n \in \mathbb{N}}$  is Cauchy in  $\mathbb{F}$ .

Since  $\{x_n^{(i)}\}_{n \in \mathbb{N}}$  is Cauchy in  $\mathbb{F}$  and  $\mathbb{F}$  is complete,  $\{x_n^{(i)}\}_{n \in \mathbb{N}}$  converges.

Let  $x_0^{(i)} = x_n^{(i)}$ . Let  $x_0 = \{x_0^{(i)}\}_{i \in \mathbb{N}}$ .

$$\|x_0\|_p = \left( \sum_{i=1}^{\infty} |x_0^{(i)}|^p \right)^{\frac{1}{p}}$$

■

## 5.2 $c_0$ Space and $c_{00}$ Space

**Definition** ( $c_0$  Space). We define  $c_0$  to be

$$c_0 := \left\{ \{x_n\}_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \lim_{n \rightarrow \infty} x_n = 0 \right\}.$$

**Definition** ( $c_{00}$  Space). We define  $c_{00}$  to be

$$c_{00} := \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}, \forall n > N, x_n = 0 \right\}.$$

i.e., the set of all eventually zero sequences of real numbers. i.e., the set of all sequences with finite support.

**Proposition 5.2.1.** The  $c_{00}$  is not complete in  $(\ell_1, \|\cdot\|_1)$ .

*Proof.* Define a sequence of vectors  $(\mathbf{r}_i)_{i \in \mathbb{N}}$  by  $\mathbf{r}_i^j := \frac{1}{j^2}$  for  $j \in \{1..i\}$  and  $\mathbf{r}_i^j := 0$  for  $j > i$ . Then  $(\mathbf{r}_i)_{i \in \mathbb{N}}$  converges to something that is not in  $c_{00}$ . ■

**Proposition 5.2.2.** The closure of  $c_{00}$  in the space  $(\mathbb{R}^{\omega}, d_1)$  is  $\ell_1$ .

*Proof.* For one direction, we are to prove that  $\text{cl}(c_{00}) \subseteq \ell_1$ . Let  $x$  be an arbitrary element in  $\text{cl}(c_{00})$ . Since  $x \in \text{cl}(c_{00})$ , there exists another element  $y \in c_{00}$  such that  $d_1(x, y) < 1$ . Let  $N \in \mathbb{N}$  be such that  $\forall n > N, y_n = 0$ . Then

$$\begin{aligned} & d_1(x, y) < 1 \\ \iff & \sum_{n \in \mathbb{N}} |x_n - y_n| < 1 \\ \iff & \sum_{n=1}^N |x_n - y_n| + \sum_{n>N} |x_n - y_n| < 1 \\ \iff & \sum_{n=1}^N |x_n - y_n| + \sum_{n>N} |x_n| < 1 \\ \implies & \sum_{n=1}^N \big| |x_n| - |y_n| \big| + \sum_{n>N} |x_n| < 1 \\ \implies & \sum_{n=1}^N (|x_n| - |y_n|) + \sum_{n>N} |x_n| < 1 \\ \implies & \sum_{n=1}^N |x_n| - \sum_{n=1}^N |y_n| + \sum_{n>N} |x_n| < 1 \\ \iff & \sum_{n \in \mathbb{N}} |x_n| - \sum_{n=1}^N |y_n| < 1 \\ \iff & \sum_{n \in \mathbb{N}} |x_n| < 1 + \sum_{n=1}^N |y_n|. \end{aligned}$$



Since  $\sum_{n \in \mathbb{N}} |x_n|$  is bounded,  $x \in \ell_1$ .

For the reverse direction, we are to prove that  $\ell_1 \subseteq \text{cl}(c_{00})$ . Let  $x$  be an arbitrary element in  $\ell_1$ . For  $i \in \mathbb{N}$ , define  $x^i = \{x_j^i\}_{j \in \mathbb{N}}$  as  $x_j^i = x_j$  for  $j \leq i$  and  $x_j^i = 0$  for  $j > i$ . Then  $\forall i \in \mathbb{N}, x^i \in c_{00}$ . Then

$$\begin{aligned} & \lim_{i \in \mathbb{N}} d_1(x^i, x) \\ &= \lim_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |x_j^i - x_j| \\ &= \lim_{i \in \mathbb{N}} \sum_{j > i} |x_j^i - x_j| \\ &= \lim_{i \in \mathbb{N}} \sum_{j > i} |x_j| \\ &= 0. \end{aligned}$$

That is,  $\lim_{i \in \mathbb{N}} d_1(x^i, x) = 0$ . So  $\lim_{i \in \mathbb{N}} x^i = x$ . So  $x \in \text{cl}(c_{00})$ . ■

**Proposition 5.2.3.** *The closure of  $c_{00}$  in the space  $(\mathbb{R}^\omega, d_\infty)$  is  $c_0$ .*

*Proof.* For one direction, we are to prove that  $\text{cl}(c_{00}) \subseteq c_0$ . Let  $x$  be an arbitrary element in  $\text{cl}(c_{00})$ . Let  $\varepsilon$  be an arbitrary positive real number. Since  $x \in \text{cl}(c_{00})$ , there exists another element  $y$  in  $c_{00}$  such that  $d_\infty(x, y) < \varepsilon$ . That is,  $\forall j \in \mathbb{N}, |x_j - y_j| < \varepsilon$ . Since  $y \in c_{00}$ ,  $\exists N \in \mathbb{N}$  such that  $\forall j > N, y_j = 0$ . So  $\forall j > N, |x_j| < \varepsilon$ . That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon.$$

By definition of convergence of limits,  $\lim_{j \in \mathbb{N}} x_j = 0$ . So  $x \in c_0$ .

For the reverse direction, we are to prove that  $c_0 \subseteq \text{cl}(c_{00})$ . Let  $x$  be an arbitrary element in  $c_0$ . For  $i \in \mathbb{N}$ , define  $x^i$  as  $x_j^i = x_j$  for  $j \leq i$  and  $x_j^i = 0$  for  $j > i$ . Then  $\forall i \in \mathbb{N}, x^i \in c_{00}$ . Let  $\varepsilon$  be an arbitrary positive real number. Since  $x \in c_0$ ,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad |x_j| < \varepsilon/2.$$

Let  $i > N$ . Then

$$\begin{aligned} & d_\infty(x^i, x) \\ &= \sup_{j \in \mathbb{N}} |x_j^i - x_j| \\ &= \sup_{j > i} |x_j^i - x_j| \\ &= \sup_{j > i} |x_j| \\ &\leq \varepsilon/2 < \varepsilon. \end{aligned}$$

That is,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall i > N, \quad d_\infty(x^i, x) < \varepsilon.$$

By definition of convergence of sequences,  $\lim_{i \in \mathbb{N}} x^i = x$ . So  $x \in \text{cl}(c_{00})$ . ■

**Proposition 5.2.4.** *Let  $A := \{\{x_n\}_{n \in \mathbb{N}} \in c_{00} : \sum_{n \in \mathbb{N}} x_n = 0\}$ . Then  $A$  is a subset of  $\ell^1$  and is closed in  $(\ell^1, d_1)$ . i.e.  $\text{cl}(A) = A$  in  $(\ell^1, d_1)$ .*

*Proof.* Let  $x = \{x^i\}_{i \in \mathbb{N}}$  be a sequence in  $\ell^1$ , where each  $x^i = \{x_j^i\}_{j \in \mathbb{N}}$  is an element in  $A$ , that converges in  $(\ell^1, d_1)$ . Say  $\lim_{i \rightarrow \infty} x^i = x^\infty$ .

First I claim that  $x^\infty \in c_{00}$ .

Now I claim that  $\sum_{j \in \mathbb{N}} x_j^\infty = 0$ . i.e.  $x^\infty \in A$ . Since  $x^\infty \in c_{00}$ ,

$$\exists N \in \mathbb{N}, \quad \forall j > N, \quad x_j^\infty = 0.$$

Define  $y_i := \sum_{j=1}^N x_j^i$ . Define  $y_\infty := \sum_{j=1}^N x_j^\infty$ . It is easy to see that  $\lim_{i \in \mathbb{N}} y_i = y_\infty$ . Assume for the sake of contradiction that  $y_\infty \neq 0$ . i.e.  $\{y_i\}_{i \in \mathbb{N}}$  does not converge to 0. Then

$$\exists \varepsilon_0 > 0, \quad \forall M \in \mathbb{N}, \quad \exists i_0 > M, \quad |y_{i_0} - 0| = |y_{i_0}| \geq \varepsilon_0. \quad (1)$$

Since  $\lim_{i \rightarrow \infty} x^i = x^\infty$ ,

$$\exists M_0 \in \mathbb{N}, \quad \forall i > M_0, \quad d_1(x^i, x^\infty) < \varepsilon_0. \quad (2)$$

Consider statement (1) for a particular  $M, M_0$ , we have

$$\exists i_0 > M_0, \quad |y_{i_0}| \geq \varepsilon_0. \quad (3)$$

That is,

$$\left| \sum_{j=1}^N x_j^{i_0} \right| \geq \varepsilon_0. \quad (3')$$

Consider statement (2) for a particular  $i, i_0$ , we have

$$d_1(x^{i_0}, x^\infty) < \varepsilon_0. \quad (4)$$

From statement (4) we can derive:

$$\begin{aligned} & d_1(x^{i_0}, x^\infty) < \varepsilon_0 \\ \iff & \sum_{j \in \mathbb{N}} |x_j^{i_0} - x_j^\infty| < \varepsilon_0 \\ \iff & \sum_{j=1}^N |x_j^{i_0} - x_j^\infty| + \sum_{j>N} |x_j^{i_0} - x_j^\infty| < \varepsilon_0 \end{aligned}$$

$$\begin{aligned}
&\implies \sum_{j>N} |x_j^{i_0} - x_j^\infty| < \varepsilon_0 \\
&\iff \sum_{j>N} |x_j^{i_0} - 0| < \varepsilon_0 \\
&\iff \sum_{j>N} |x_j^{i_0}| < \varepsilon_0 \\
&\implies \left| \sum_{j>N} x_j^{i_0} \right| < \varepsilon_0 \\
&\implies \left| \sum_{j>N} x_j^{i_0} \right| < \varepsilon_0 \\
&\iff \left| \sum_{j \in \mathbb{N}} x_j^{i_0} - \sum_{j=1}^N x_j^{i_0} \right| < \varepsilon_0 \\
&\iff \left| 0 - \sum_{j=1}^N x_j^{i_0} \right| < \varepsilon_0 \\
&\iff \left| \sum_{j=1}^N x_j^{i_0} \right| < \varepsilon_0.
\end{aligned}$$

This contradicts to statement (3'). So the original assumption that  $y_\infty \neq 0$  is false. i.e.  $y_\infty = 0$ . It follows that  $\sum_{j \in \mathbb{N}} x_j^\infty = 0$ . This completes the proof. ■

### 5.3 Hölder's Inequality

**Theorem 6** (Hölder's Inequality). *Let  $\mathfrak{X} = \mathbb{R}^n$  for some  $n \in \mathbb{N}$ . Let  $x = (x_i)_{i=1}^n$  and  $y = (y_i)_{i=1}^n$  be vectors in  $\mathfrak{X}$ . Then  $\forall p, q \in (1, +\infty) : 1/p + 1/q = 1$ ,  $\|xy\|_1 \leq \|x\|_p \|y\|_q$ . i.e.,*

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \left( \sum_{i=1}^n |y_i|^q \right)^{1/q}.$$




## Chapter 6

# Function Spaces

### 6.1 The $\mathcal{L}^p$ Norm

$$\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

 the instructors' answer, where instructors collectively construct a single answer

In the sup norm, convergence coincides with uniform convergence. Moreover,  $C[a, b]$  is complete in this norm. It is not complete in any of the  $L^p$  norms for  $1 \leq p < \infty$ . The completion in these norms is called  $L^p(a, b)$ .

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Updated 1 day ago by Kenneth Davidson



# Chapter 7

## Banach Space

### 7.1 Definition

**Definition** (Banach Space). Let  $(\mathfrak{X}, \|\cdot\|)$  be a normed linear space. Let  $d$  be the metric induced by  $\|\cdot\|$ . We say that  $\mathfrak{X}$  is a **Banach space** if  $(\mathfrak{X}, d)$  is a complete metric space. i.e., we define a Banach space to be a complete normed linear space.

### 7.2 Properties

**Proposition 7.2.1.** Let  $(\mathfrak{X}, \|\cdot\|)$  be a normed vector space over field  $\mathbb{F}$ . Then  $(\mathfrak{X}, \|\cdot\|)$  is a Banach space if and only if every absolutely summable series in  $X$  is summable.

**Proposition 7.2.2.** Any Banach space with a Schauder basis has to be separable.

### 7.3 Examples of Banach Space

**Example 7.3.1.**  $(\mathcal{C}([0, 1], \mathbb{F}), \|\cdot\|_\infty)$  is a Banach space.

**Example 7.3.2** (Disc Algebra). Define  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ . Define  $\mathcal{A}(\mathbb{D}) := \{f \in \mathcal{C}(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is holomorphic}\}$ . Define  $\|\cdot\|_\infty$  by  $\|f\|_\infty := \sup_{z \in \overline{\mathbb{D}}} |f(z)|$ . Then  $(\mathcal{A}(\mathbb{D}), \|\cdot\|_\infty)$  is a Banach space.

**Example 7.3.3.** Let  $(X, \Omega, \mu)$  be a measure space. Let  $p$  be a number in  $[1, +\infty)$ . Define

$$\mathcal{L}^p(X, \mu) := \text{span}\{f : X \rightarrow [0, +\infty] \mid f \text{ is measurable and } \int_X |f|^p < +\infty\}.$$

Define an equivalence relation on  $\mathcal{L}^p(X, \mu)$  by  $f \equiv g$  if and only if

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

Define a space  $L^p(X, \mu) := \mathcal{L}^p(X, \mu) / \equiv$ . Then  $L^p(X, \mu)$  is a Banach space when equipped with the norm

$$\|[f]\|_p := \left( \int_X |f|^p \right)^{1/p}.$$

**Example 7.3.4.** Let  $\mathcal{P}_{\mathbb{C}}[0, 1]$  denote the set of all polynomials with complex coefficients. For each  $p \in [1, +\infty)$ , define a norm

$$\|f\|_p := \left( \int_0^1 |f|^p \right)^{1/p}.$$

For  $p = +\infty$ , define a norm

$$\|f\|_{\infty} := \sup_{x \in [0, 1]} |f(x)|.$$

## 7.4 Construction of Banach Spaces

**Definition.** Let  $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$  and  $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$  be two Banach spaces over field  $\mathbb{K}$ . Let  $p \in [1, +\infty)$ . We define

$$\mathfrak{X} \oplus_p \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x, y)\|_p := (\|x\|_{\mathfrak{X}}^p + \|y\|_{\mathfrak{Y}}^p)^{1/p}.$$

For  $p = +\infty$ , we define

$$\mathfrak{X} \oplus_{\infty} \mathfrak{Y} := \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$$

and

$$\|(x, y)\|_{\infty} := \max(\|x\|_{\mathfrak{X}}, \|y\|_{\mathfrak{Y}}).$$

- Note that the norms behave similarly to what the  $p$  norm would do.
- We can similarly define the direct sum of finitely many Banach spaces.

**Proposition 7.4.1.**  $\|\cdot, \cdot\|_p$  is a norm on  $\mathfrak{X} \oplus_p \mathfrak{Y}$ .

**Proposition 7.4.2.**  $\mathfrak{X} \oplus_p \mathfrak{Y}$  is complete with respect to  $\|\cdot, \cdot\|_p$ .



## Chapter 8

# Hilbert Space

### 8.1 Definition

**Definition** (Hilbert Space). We define a **Hilbert space**, denoted by  $\mathcal{H}$ , to be a complete inner product space.

### 8.2 Examples of Hilbert Space

**Example 8.2.1.** Let  $(X, \mu)$  be a measure space. Then  $L^2(X, \mu)$  is a Hilbert space with inner product given by

$$\langle f, g \rangle := \int_X f(x) \overline{g(x)} d\mu(x).$$

**Example 8.2.2.**  $\ell^2(\mathbb{K}) = \{(x_i)_{i \in \mathbb{N}} \in \mathbb{K}^\infty : \sum_{i \in \mathbb{N}} |x_i|^2 < +\infty\}$  is a Hilbert space with inner product given by

$$\langle (x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}} \rangle := \sum_{i \in \mathbb{N}} x_i \overline{y_i}.$$

### 8.3 Properties of Hilbert Space

**Proposition 8.3.1.** Let  $\mathcal{H}$  be a Hilbert space. Let  $S$  be a non-empty set in the space. Then  $S^{\perp\perp} = \text{clspan}(S)$ .

*Proof.* For one direction, we are to prove that  $\text{clspan}(S) \subseteq S^{\perp\perp}$ .

For the reverse direction, we are to prove that  $S^{\perp\perp} \subseteq \text{clspan}(S)$ . Assume for the sake of contradiction that  $\exists x \in S^{\perp\perp}$  with  $x \neq 0$  such that  $x \notin \text{clspan}(S)$ . Say  $x = m_1 + m_2$  for some  $m_1 \in \text{clspan}(S)$  and some  $m_2 \in \text{clspan}(S)^\perp$ . Note that  $\text{clspan}(S)^\perp = S^\perp$ . So  $m_2 \in S^\perp$ . Since  $x \in S^{\perp\perp}$  and  $m_2 \in S^\perp$ , we should have  $\langle x, m_2 \rangle = 0$ . However,

$$\langle x, m_2 \rangle = \langle m_1 + m_2, m_2 \rangle$$

$$\begin{aligned}
&= \langle m_1, m_2 \rangle + \langle m_2, m_2 \rangle \\
&= 0 + \langle m_2, m_2 \rangle \\
&> 0, \text{ since } m_2 \neq 0.
\end{aligned}$$

This leads to a contradiction. So  $S^{\perp\perp} \subseteq \text{clspan}(S)$ . ■

**Theorem 7** (The Riesz Representation Theorem). *Let  $\mathcal{H}$  be a Hilbert space over field  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . Suppose that  $\mathcal{H} \neq \{0\}$ . Then for any  $\varphi \in \mathcal{H}^*$ ,  $\exists y \in \mathcal{H}$  such that*

$$\forall x \in \mathcal{H}, \quad \varphi(x) = \langle x, y \rangle.$$

*Proof.* Define for each  $y \in \mathcal{H}$  a function  $\beta_y \in \mathcal{H}^*$  by  $\beta_y(x) := \langle x, y \rangle$ . We are to prove that  $\mathcal{H}^* = \{\beta_y : y \in \mathcal{H}\}$ . It is easy to verify that each  $\beta_y$  is linear and bounded. So  $\forall y \in \mathcal{H}$ ,  $\beta_y \in \mathcal{H}^*$ . i.e.,  $\{\beta_y : y \in \mathcal{H}\} \subseteq \mathcal{H}^*$ . Define a map  $\Theta$  from  $\mathcal{H}$  to  $\mathcal{H}^*$  as  $\Theta(y) := \beta_y$ . It is easy to verify that  $\Theta$  is linear.

$$\begin{aligned}
\|\Theta(y)\| &= \|\beta_y\| = \sup\{\beta_y(x) : \|x\| = 1\} \\
&= \sup\{\langle x, y \rangle : \|x\| = 1\} \\
&\leq \sup\{\|x\|\|y\| : \|x\| = 1\} \\
&= \|y\|.
\end{aligned}$$

That is,  $\|\Theta(y)\| \leq \|y\|$ . So  $\|\Theta\| \leq 1$ . On the other hand, consider an arbitrary point  $y_0 \in \mathcal{H}$  with  $y_0 \neq 0$ :

$$\begin{aligned}
\|\Theta\| &= \sup \left\{ \frac{\|\Theta(y)\|}{\|y\|} : y \neq 0 \right\} \\
&\geq \frac{\|\Theta(y)\|}{\|y\|} \Big|_{y=y_0} \\
&= \frac{\|\Theta(y_0)\|}{\|y_0\|} \\
&= \frac{\|\beta_{y_0}\|}{\|y_0\|} \\
&= \frac{1}{\|y_0\|} \sup \left\{ \frac{\|\beta_{y_0}(y)\|}{\|y\|} : y \neq 0 \right\} \\
&\geq \frac{1}{\|y_0\|} \frac{\|\beta_{y_0}(y_0)\|}{\|y_0\|} \\
&\geq \frac{1}{\|y_0\|} \frac{\|y_0\|^2}{\|y_0\|} \\
&= 1.
\end{aligned}$$

That is,  $\|\Theta\| \geq 1$ . So  $\|\Theta\| = 1$ . So  $\Theta$  is isometric. It immediately follows that  $\Theta$  is injective. Now it remains to prove that  $\Theta$  is surjective. Let  $\varphi \in \mathcal{H}^*$ . If  $\varphi = 0$ , then  $\varphi = \Theta(0)$  and

we are done. Otherwise, let  $\mathcal{M} := \ker(\varphi)$ . Then we have  $\text{codim } \mathcal{M} = \dim \mathcal{M}^\perp = 1$ . Take  $e \in \mathcal{M}^\perp$  such that  $\|e\| = 1$ . Let  $P$  denote the orthogonal projection onto  $\mathcal{M}$ . Then  $1 - P$  is the orthogonal projection onto  $\mathcal{M}^\perp$ .

$$x = Px + (1 - P)x = Px + \langle x, e \rangle e.$$

So for  $x \in \mathcal{H}$ ,

$$\varphi(x) = \varphi(Px) + \langle x, e \rangle \varphi(e) = \langle x, \overline{\varphi(e)} e \rangle = \beta_y(x)$$

where  $y := \overline{\varphi(e)}e$ . Hence  $\varphi = \beta_y$ . So  $\Theta$  is surjective. This completes the proof. ■



## Chapter 9

# Operators

### 9.1 Bounded Operators

**Definition** (Bounded Operator). *Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be normed linear spaces. Let  $T$  be a linear map from  $\mathfrak{X}$  to  $\mathfrak{Y}$ . We say that  $T$  is a **bounded operator** if*

$$\exists k \in \mathbb{R}, \quad \forall x \in \mathfrak{X}, \quad \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}.$$

**Definition** (Operator Norm). *Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be normed linear spaces. Let  $T$  be a bounded operator from  $\mathfrak{X}$  to  $\mathfrak{Y}$ . We define the **operator norm** of  $T$ , denoted by  $\|T\|$ , to be the number given by*

$$\|T\| := \inf\{k \in \mathbb{R} : \forall x \in \mathfrak{X}, \|Tx\|_{\mathfrak{Y}} \leq k\|x\|_{\mathfrak{X}}\}.$$

**Proposition 9.1.1.**

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{X}, \|x\|_{\mathfrak{X}} = 1\}.$$

**Proposition 9.1.2.** *Let  $X$  and  $Y$  be normed linear spaces. Let  $T$  be a linear map from  $X$  to  $Y$ . Then  $T$  is bounded if and only if  $T$  is continuous.*

### 9.2 Examples of Bounded Operators

**Example 9.2.1** (The Multiplication Operator). *Let  $\mathfrak{X} = (\mathcal{C}([0, 1], \mathbb{C}), \|\cdot\|_{\infty})$ . Let  $f$  be a function in  $\mathfrak{X}$ . We define the **multiplication operator** on  $\mathfrak{X}$ , w.r.t.  $f$ , denoted by  $M_f$ , as*

$$M_f(g) = fg.$$

*Then  $M_f$  is bounded and  $\|M_f\| = \|f\|_{\infty}$ .*

*Proof.* Let  $g$  be an arbitrary function in  $\mathfrak{X}$ . Then

$$\|M_f g\|_{\infty} = \|fg\|_{\infty}$$

$$\begin{aligned}
&= \sup_{x \in [0,1]} |f(x)g(x)| \\
&= \sup_{x \in [0,1]} |f(x)| |g(x)| \\
&\leq \sup_{x \in [0,1]} |f(x)| \sup_{x \in [0,1]} |g(x)| \\
&= \|f\|_\infty \|g\|_\infty.
\end{aligned}$$

That is,  $\|M_f g\|_\infty \leq \|f\|_\infty \|g\|_\infty$ . So  $\|f\|_\infty$  is an element of the set  $S = \{k \in \mathbb{R} : \forall g \in \mathfrak{X}, \|M_f g\|_\infty \leq k \|g\|_\infty\}$ . So  $\|M_f\| = \inf(S) \leq \|f\|_\infty$ . Consider  $g_0$  given by  $g_0(x) = 1$ . Then  $g_0$  in  $\mathfrak{X}$ . Then

$$\|M_f g_0\|_\infty = \|f g_0\|_\infty = \|f\|_\infty = \|f\|_\infty \|g_0\|_\infty.$$

Let  $k$  be an arbitrary element in  $S$ . Assume for the sake of contradiction that  $k < \|f\|_\infty$ . Then

$$\begin{aligned}
\|f\|_\infty \|g_0\|_\infty &= \|M_f g_0\|_\infty \\
&\leq k \|g_0\|_\infty \\
&< \|f\|_\infty \|g_0\|_\infty.
\end{aligned}$$

This leads to a contradiction. So  $\forall k \in S, k \geq \|f\|_\infty$ . So  $\|f\|_\infty$  is a lower bound for the set  $S$ . So  $\|M_f\| = \inf(S) \geq \|f\|_\infty$ . Since  $\|M_f\| \leq \|f\|_\infty$  and  $\|M_f\| \geq \|f\|_\infty$ , we get  $\|M_f\| = \|f\|_\infty$ . ■

**Example 9.2.2** (The Volterra Operator). Let  $\mathfrak{X} = (\mathcal{C}([0, 1], \mathbb{C}), \|\cdot\|_\infty)$ . Define

$$Vf := x \mapsto \int_0^x f(t) dt.$$

Then the Volterra Operator is bounded and  $\|V\| \leq 1$ .

*Proof.* Let  $f$  be an arbitrary function in  $\mathfrak{X}$  with  $\|f\|_\infty = 1$ . Then  $\forall x \in [0, 1]$ ,

$$\begin{aligned}
|Vf(x)| &= \left| \int_0^x f(t) dt \right| \\
&\leq \int_0^x |f(t)| dt \\
&\leq \int_0^x \sup_{t \in [0,1]} |f(t)| dt \\
&= \int_0^x \|f\|_\infty dt \\
&= \int_0^x 1 dt \\
&= x.
\end{aligned}$$

That is,  $\forall x \in [0, 1]$ ,  $|Vf(x)| \leq 1$ . So  $\|Vf\|_\infty \leq 1$ . Since  $\forall f \in \mathfrak{X} : \|f\|_\infty = 1$ ,  $\|Vf\|_\infty \leq 1$ , we get  $\|V\| \leq 1$ . ■

**Example 9.2.3** (The Diagonal Operator). Let  $\mathfrak{X} = \ell^2(\mathbb{N})$ . Let

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & \ddots \end{bmatrix}.$$

Then  $D$  is bounded if and only if  $(d_i)_{i \in \mathbb{N}}$  is bounded and  $\|D\| = \|(d_i)_{i \in \mathbb{N}}\|_\infty$ .

*Proof.* Case 1.

$$\begin{aligned} \|Dx\|_2^2 &= \sum_{i \in \mathbb{N}} |d_i x_i|^2 \\ &\leq \sum_{i \in \mathbb{N}} \|(d_j)_{j \in \mathbb{N}}\|_\infty |x_i|^2 \\ &= \|(d_j)_{j \in \mathbb{N}}\|_\infty \sum_{i \in \mathbb{N}} |x_i|^2 \\ &= \|(d_j)_{j \in \mathbb{N}}\|_\infty \|x\|_2^2. \end{aligned}$$

Case 2.

If  $(d_i)_{i \in \mathbb{N}} \notin \ell^\infty$ ,  $\exists (d_{n_i})_{i \in \mathbb{N}} \rightarrow \infty$ .

$$\begin{aligned} \|De_{n_i}\|_2 &= \|d_{n_i} e_{n_i}\|_2 \\ &= |d_{n_i}| \|e_{n_i}\|_2 \\ &= |d_{n_i}|. \end{aligned}$$

So  $\|D\| \geq \|De_{n_i}\|_2 \rightarrow \infty$ . ■

**Example 9.2.4** (Weighted Shifts).

- Let  $\mathcal{H} = \ell^2_{\mathbb{N}}$ . Let  $(w_n)_{n \in \mathbb{N}} \in \ell^\infty_{\mathbb{N}}$ . We define an *unilateral forward weighted shift*  $W$  on  $\mathcal{H}$  as

$$W(x_n) := (0, w_1 x_1, w_2 x_2, w_3 x_3, \dots).$$

i.e.,

$$W = \begin{bmatrix} 0 & & & \\ w_1 & 0 & & \\ & w_2 & 0 & \\ & & w_3 & 0 \\ & & & \ddots & \ddots \end{bmatrix}.$$

Then  $W$  is bounded and  $\|W\| = \sup\{|w_n| : n \in \mathbb{N}\}$ .

- Let  $\mathcal{H} = \ell_{\mathbb{N}}^2$ . Let  $(v_n)_{n \in \mathbb{N}} \in \ell_{\mathbb{N}}^\infty$ . We define an **unilateral backward weighted shift**  $V$  on  $\mathcal{H}$  as

$$V(x_n) := (v_1 x_2, v_2 x_3, v_3 x_4, \dots).$$

Then  $V$  is bounded and  $\|V\| = \sup\{|v_n| : n \in \mathbb{N}\}$ .

- Let  $\mathcal{H} = \ell_{\mathbb{Z}}^2$ . Let  $(u_n)_{n \in \mathbb{Z}} \in \ell_{\mathbb{Z}}^\infty$ . We define a **bilateral weighted shift**  $U$  on  $\mathcal{H}$  as

$$U(x_n) := (u_{n-1} x_{n-1})_{n \in \mathbb{Z}}.$$

Then  $U$  is bounded and  $\|U\| = \sup\{|u_n| : n \in \mathbb{Z}\}$ .

**Example 9.2.5** (The Composition Operators). Let  $\mathfrak{X} = \mathcal{C}([0, 1], \mathbb{C})$ . Let  $\varphi \in \mathcal{C}([0, 1], [0, 1])$ . We define the **composition operator** on  $\mathfrak{X}$ , denoted by  $C_\varphi$  as

$$C_\varphi(f) := f \circ \varphi.$$

Then  $C_\varphi$  is contractive.

*Proof.*

$$\begin{aligned} \|C_\varphi(f)\| &= \sup_{x \in [0, 1]} |(f \circ \varphi)(x)| \\ &\leq \|f\|_\infty. \end{aligned}$$

■

### 9.3 The Space of Bounded Operators

**Proposition 9.3.1.** Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be normed linear spaces. Then  $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$  is a vector space and the operator norm is a norm on  $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ .

**Proposition 9.3.2.** Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be normed linear spaces. Let  $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$  be the space of bounded linear operators from  $\mathfrak{X}$  to  $\mathfrak{Y}$ . Then if  $\mathfrak{Y}$  is complete,  $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$  is complete.

**Proposition 9.3.3.** Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be normed linear spaces. Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two equivalent norms on  $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ . Then  $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}, \|\cdot\|_1)$  if and only if  $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}, \|\cdot\|_2)$ .

### 9.4 Invertible Bounded Operators

**Proposition 9.4.1.** Let  $(\mathfrak{X}, \|\cdot\|_1)$  be a Banach space. Let  $S \in \mathcal{B}(\mathfrak{X})$  be a bounded linear map that is invertible. Define a norm  $\|\cdot\|_2$  on  $\mathfrak{X}$  as

$$\|x\|_2 := \|Sx\|_1.$$

Then  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent.



*Proof.* On one hand, since  $S$  is bounded,  $\exists c_1$  such that  $\forall x \in \mathfrak{X}$ ,  $\|Sx\|_1 \leq c_1\|x\|_1$ . That is,  $\|x\|_2 \leq c_1\|x\|_1$ .

On the other hand, since  $S$  is invertible,  $S^{-1}$  exists and is also bounded. Since  $S^{-1}$  is bounded,  $\exists c_2$  such that  $\forall x \in \mathfrak{X}$ ,  $\|S^{-1}x\|_1 \leq c_2\|x\|_1$ . Consider  $x = Sx$ , we get  $\forall x \in \mathfrak{X}$ ,  $\|S^{-1}Sx\|_1 \leq c_2\|Sx\|_1$ . That is,  $\|x\|_1 \leq c_2\|x\|_2$ .

So  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent. ■

**Proposition 9.4.2.** *Let  $(\mathfrak{X}, \|\cdot\|)$  be a Banach space. Let  $S$  be a map in  $\mathcal{B}(\mathfrak{X})$  that is invertible. Then*

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$

*Proof.*

$$\begin{aligned} \|S^{-1}\| &= \sup\{\|S^{-1}x\| : \|x\| = 1\} \\ &= \sup\left\{\frac{\|S^{-1}x\|}{\|x\|} : \|x\| \neq 0\right\} \\ &= \sup\left\{\frac{\|S^{-1}Sx\|}{\|Sx\|} : \|x\| \neq 0\right\} \\ &= \sup\left\{\frac{\|x\|}{\|Sx\|} : \|x\| \neq 0\right\} \\ &= \sup\left\{\frac{1}{\|Sx\|} : \|x\| = 1\right\} \\ &= (\inf\{\|Sx\| : \|x\| = 1\})^{-1}. \end{aligned}$$

That is,

$$\|S^{-1}\| = (\inf\{\|Sx\| : \|x\| = 1\})^{-1}.$$
■



## Chapter 10

# Dual Space

### 10.1 Definition

**Definition** ((Topological) Dual Space). *Let  $\mathfrak{X}$  be a normed linear space over field  $\mathbb{K}$ . We define the **(topological) dual space** of  $\mathfrak{X}$ , denoted by  $\mathfrak{X}^*$ , to be the space  $\mathcal{B}(\mathfrak{X}, \mathbb{K})$ .*

**Definition** (Linear Functionals). *We call the elements of  $\mathfrak{X}^*$  **linear functionals**.*

**Proposition 10.1.1.** *Let  $X$  be a normed linear space. Then there exists a contractive map from  $X$  to its double dual  $X^{**}$ .*

### 10.2 Properties

### 10.3 Examples of Dual Space

**Example 10.3.1.**  $c_0(\mathbb{N})^*$  is isometrically isomorphic to  $\ell^1(\mathbb{N})$ .

**Example 10.3.2.**  $c_0(\mathbb{N})^*$  is isometrically isomorphic to  $\ell^\infty(\mathbb{N})$ .



# Chapter 11

## Quotient Spaces

### 11.1 Definitions

**Definition.** Let  $\mathfrak{V}$  be a vector space. Let  $\mathfrak{W}$  be a subspace of  $\mathfrak{V}$ . We define a **quotient space**, denoted by  $\mathfrak{V}/\mathfrak{W}$ , to be a set  $\{v + \mathfrak{W} : v \in \mathfrak{V}\}$  with operations

$$(v_1 + \mathfrak{W}) + (v_2 + \mathfrak{W}) := (v_1 + v_2) + \mathfrak{W} \text{ and}$$

$$\kappa(v + \mathfrak{W}) := (\kappa v) + \mathfrak{W}.$$

**Definition** (Quotient Map). Let  $(\mathfrak{X}, \|\cdot\|)$  be a normed linear space. Let  $\mathfrak{M}$  be a linear manifold in  $\mathfrak{X}$ . We define the **quotient map** on  $\mathfrak{X}$  with respect to  $\mathfrak{M}$ , denoted by  $q_{\mathfrak{M}}$ , to be a function from  $\mathfrak{X}$  to  $\mathfrak{X}/\mathfrak{M}$  given by

$$q_{\mathfrak{M}}(x) := x + \mathfrak{M}$$

**Proposition 11.1.1.** Quotient maps are contractive. i.e.,

$$\forall x \in \mathfrak{X}, \quad \|x + \mathfrak{M}\|_{\mathfrak{X}/\mathfrak{M}} \leq \|x\|_{\mathfrak{X}}.$$

**Proposition 11.1.2.** Let  $\mathfrak{X}$  be a normed linear space. Let  $\mathfrak{M}$  be a closed subspace of  $\mathfrak{X}$ . Let  $q$  be the canonical quotient map from  $\mathfrak{X}$  to  $\mathfrak{X}/\mathfrak{M}$ . Then

- $q$  is a continuous map. i.e.,

$$\forall \text{ open set } W \subseteq \mathfrak{X}/\mathfrak{M}, \quad q^{-1}(W) \text{ is open in } \mathfrak{X}.$$

- $q$  is an open map. i.e.,

$$\forall \text{ open set } G \subseteq \mathfrak{X}, \quad q(G) \text{ is open in } \mathfrak{X}/\mathfrak{M}.$$

*Proof.* Since  $q$  is contractive,  $q$  is continuous and hence (1). ■

**Definition** (Seminorm on Quotient Spaces). *Let  $(\mathfrak{X}, \|\cdot\|)$  be a normed linear space. Let  $\mathfrak{M}$  be a linear manifold in  $\mathfrak{X}$ . We define a **seminorm** on  $\mathfrak{X}/\mathfrak{M}$  to be a function from  $\mathfrak{X}/\mathfrak{M}$  to  $\mathbb{R}$  given by*

$$p(x + \mathfrak{M}) := \inf\{\|x + m\| : m \in \mathfrak{M}\}.$$

**Proposition 11.1.3.** *Seminorms on quotient spaces are indeed seminorms.*

**Proposition 11.1.4.** *A seminorm on a quotient space  $\mathfrak{X}/\mathfrak{M}$  is a norm if and only if  $\mathfrak{M}$  is closed.*

## Chapter 12

# Topological Vector Space

### 12.1 Definitions

**Definition** (Compatible). Let  $\mathcal{W}$  be a vector space over field  $\mathbb{K}$ . Let  $\mathcal{T}$  be a topology on  $\mathcal{W}$ . We say that  $\mathcal{T}$  is **compatible** with the vector space structure on  $\mathcal{W}$  if the addition and scalar multiplication operations on  $\mathcal{W}$  are continuous.

**Definition** (Topological Vector Space). We define a **topological vector space** to be a vector space with a compatible Hausdorff topology.

**Proposition 12.1.1** (Stability under Linear Combinations). Let  $X$  be a normed vector space over  $\mathbb{F}$ . Let  $K$  be a compact set in the space. Let  $C$  be a closed set in the space. Then  $\forall \alpha, \beta \in \mathbb{F}$ , the set  $S$  given by  $S := \alpha K + \beta C$  is closed.

*Proof.* The case where  $\beta = 0$  is trivial. I will assume  $\beta \neq 0$ . Let  $\alpha, \beta \in \mathbb{F}$  be arbitrary. Let  $\{s_i\}_{i \in \mathbb{N}}$  be an arbitrary sequence in  $S$  that converges. Say the limit is  $s_\infty$ . Since  $s_i \in S$  for any  $i \in \mathbb{N}$  and  $S = \alpha K + \beta C$ ,  $s_i = \alpha k_i + \beta c_i$  for some  $k_i \in K$  and some  $c_i \in C$ , for any  $i \in \mathbb{N}$ . Since  $\{k_i\}_{i \in \mathbb{N}}$  is a sequence in  $K$  and  $K$  is compact, there exists a convergent subsequence  $\{k_i\}_{i \in I}$  of  $\{k_i\}_{i \in \mathbb{N}}$  in  $K$ . Say  $\{k_i\}_{i \in I}$  converges to  $k_\infty \in K$ . Since  $\{s_i\}_{i \in \mathbb{N}}$  converges to  $s_\infty$ ,  $\{s_i\}_{i \in I}$  also converges to  $s_\infty$ . Since  $s_i = \alpha k_i + \beta c_i$ ,  $c_i = \beta^{-1}(s_i - \alpha k_i)$ . Define  $c_\infty := \beta^{-1}(s_\infty - \alpha k_\infty)$ . Since  $\{s_i\}_{i \in I}$  converges to  $s_\infty$  and  $\{k_i\}_{i \in I}$  converges to  $k_\infty$  and  $c_i = \beta^{-1}(s_i - \alpha k_i)$ ,  $\{c_i\}_{i \in I}$  converges to  $c_\infty$ . Since  $\{c_i\}_{i \in I}$  is a sequence in  $C$  and converges to  $c_\infty$  and  $C$  is closed,  $c_\infty \in C$ . Since  $s_\infty = \alpha k_\infty + \beta c_\infty$  and  $k_\infty \in K$  and  $c_\infty \in C$ ,  $s_\infty \in \alpha K + \beta C$ . Since for any sequence in  $S$  that converges, the limit is also in  $S$ ,  $S$  is closed. ■

**Remark.** The sum of two closed sets may not be closed.

*Proof.* **Counter-example 1**

Consider  $A := \{n : n \in \mathbb{N}\}$  and  $B := \{n + \frac{1}{n} : n \in \mathbb{N}\}$ .

(<https://math.stackexchange.com/questions/124130/sum-of-two-closed-sets-in-mathbb-r-is-closed>)

Their sum contains the sequence  $\{\frac{1}{n}\}_{n \in \mathbb{N}}$  but does not contain 0.

**Counter-example 2**

Consider  $A := \mathbb{R} \times \{0\}$  and  $B := \{(x, y) \in \mathbb{R}^2 : x, y > 0, xy \geq 1\}$ . Their sum is  $\mathbb{R} \times \mathbb{R}_{++}$ . ■

## 12.2 Examples

**Example 12.2.1.** Let  $\mathfrak{X}$  be a normed linear space. Then  $\mathfrak{X}$  is a topological vector space with the topology induced by the norm.

**Proposition 12.2.1.** Normed linear spaces are Hausdorff.

**Example 12.2.2.** Let  $\mathfrak{X}$  be a Banach space. Let  $\mathfrak{X}^*$  denote the dual space of  $\mathfrak{X}$ . Let  $\tau_*$  denote the weak topology on  $\mathfrak{X}^*$  induced by elements of  $\mathfrak{X}$  as

$$\lim_{\alpha} x_{\alpha}^* = x^* \iff \forall x \in \mathfrak{X}, \lim_{\alpha} x_{\alpha}^*(x) = x^*(x).$$

Then  $(\mathfrak{X}^*, \tau_*)$  is a topological vector space.

## 12.3 Neighborhood Improvements

**Proposition 12.3.1.** Let  $(\mathcal{V}, \tau)$  be a topological vector space. Let  $U \in \mathcal{U}_0$  be a neighborhood of 0 in  $\mathcal{V}$ . Then

- $\exists N \in \mathcal{U}_0$  such that  $N + N \subseteq U$ .
- $\exists M \in \mathcal{U}_0$  and  $\exists \varepsilon > 0$  such that  $\forall 0 < |k| < \varepsilon$ , we have  $kM \subseteq U$ .
-



## Chapter 13

# Equicontinuity in Metric Spaces

### 13.1 Definitions

**Definition** ((Pointwise) Equicontinuity). Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $\mathcal{F}$  be a collection of functions from  $X$  to  $Y$ . Let  $x_0$  be a point in  $X$ . We say that  $\mathcal{F}$  is **(pointwise) equicontinuous** at point  $x_0$  if for any positive number  $\varepsilon$ , there exists some number  $\delta(x_0, \varepsilon)$  such that for any function  $f$  in  $\mathcal{F}$  and any point  $x$  in  $X$ , we have

$$d_Y(f(x), f(x_0)) < \varepsilon$$

whenever  $d_X(x, x_0) < \delta(x_0, \varepsilon)$  is satisfied.

**Definition** (Uniform Equicontinuity). Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $\mathcal{F}$  be a collection of functions from  $X$  to  $Y$ . We say that  $\mathcal{F}$  is **uniformly equicontinuous** if for any positive number  $\varepsilon$ , there exists some number  $\delta(\varepsilon)$  such that for any function  $f$  in  $\mathcal{F}$  and any points  $x_1$  and  $x_2$  in  $X$ , we have

$$d_Y(f(x_1), f(x_2)) < \varepsilon$$

whenever  $d_X(x_1, x_2) < \delta(\varepsilon)$  is satisfied.

### 13.2 Sufficient Conditions

**Proposition 13.2.1.** The closure of an equicontinuous family of functions is equicontinuous.

*Proof.*

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces.

Let  $\mathcal{F}$  be an equicontinuous family of functions from  $X$  to  $Y$ .

We are to prove that  $cl(\mathcal{F})$  is equicontinuous.

Let  $x_0$  be an arbitrary point in  $X$ .

Let  $\varepsilon$  be an arbitrary positive number.

Since  $\mathcal{F}$  is equicontinuous at point  $x_0$ , there exists some  $\delta(x_0, \varepsilon)$  such that for any function  $f$  in  $\mathcal{F}$  and any point  $x$  in  $X$  such that  $d_X(x, x_0) < \delta(x_0, \varepsilon)$ , we have  $d_Y(f(x), f(x_0)) < \varepsilon/3$ .

Let  $f$  be an arbitrary function in  $cl(\mathcal{F})$ .

Let  $x$  be an arbitrary point in  $X$  such that  $d_X(x, x_0) < \delta(x_0, \varepsilon)$ .

Since  $f \in cl(\mathcal{F})$ , there exists some function  $f_0 \in \mathcal{F}$  such that  $d_\infty(f, f_0) < \varepsilon/3$ .

Since  $d_\infty(f, f_0) < \varepsilon/3$ ,  $d_Y(f(x), f_0(x)) < \varepsilon/3$  and  $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$ .

Since  $f_0 \in \mathcal{F}$  and  $d_X(x, x_0) < \delta(x_0, \varepsilon)$ ,  $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$ .

Since  $d_Y(f(x), f_0(x)) < \varepsilon/3$  and  $d_Y(f(x_0), f_0(x_0)) < \varepsilon/3$  and  $d_Y(f_0(x), f_0(x_0)) < \varepsilon/3$ ,  
 $d_Y(f(x), f(x_0)) < \varepsilon$ .

Since for any positive number  $\varepsilon$ , there exists some  $\delta(x_0, \varepsilon)$  such that for any function  $f$  in  $cl(\mathcal{F})$  and any point  $x$  in  $X$  such that  $d_X(x, x_0) < \delta(x_0, \varepsilon)$ , we have  $d_Y(f(x), f(x_0)) < \varepsilon$ , by definition of equicontinuous,  $cl(\mathcal{F})$  is equicontinuous at point  $x_0$ .

Since  $cl(\mathcal{F})$  is equicontinuous at point  $x_0$  for any point  $x_0$  in  $X$ ,  $cl(\mathcal{F})$  is equicontinuous. ■

## Chapter 14

# Adjoint Operator

### 14.1 Definitions

**Definition** (Adjoint Matrix). *Let  $A$  be an  $m \times n$  matrix. We define the **adjoint** of  $A$ , denoted by  $A^*$ , to be an  $n \times m$  matrix given by*

$$(A^*)_{ij} := \overline{(A)_{ji}}.$$

**Definition** (Adjoint Operator). *Let  $V$  and  $W$  be inner product spaces. Let  $T$  be a linear map from  $V$  to  $W$ . We define the **adjoint** of  $T$ , denoted by  $T^*$ , to be a map from  $W$  to  $V$  such that*

$$\forall x \in V, \forall y \in W, \quad \langle T(x), y \rangle_W = \langle x, T^*(y) \rangle_V.$$

**Proposition 14.1.1** (Existence). *Let  $V$  be a finite-dimensional inner product space and  $T$  be a linear operator on  $V$ . Then the adjoint of  $T$  exists.*

**Proposition 14.1.2** (Uniqueness). *Let  $V$  be an inner product space and  $T$  be a linear operator on  $V$ . Then the adjoint of  $T$  is unique, provided that it exists.*

### 14.2 Properties of the Adjoint Operator

**Proposition 14.2.1.** *Let  $V$  be an inner product space. Then*

- (1)  $(I_V)^* = I_V$  where  $I_V$  is the identity operator on  $V$ .
- (2)  $T^{**} = T$  for any linear operator  $T$  on  $V$ .

**Proposition 14.2.2.** *Let  $V$  be an inner product space and  $T$  be a linear operator on  $V$ . Then  $T^*$  is also linear.*

**Proposition 14.2.3.** *Let  $V$  be an inner product space. Then*

(1) *For any linear operators  $T$  and  $U$ ,*

$$(T + U)^* = T^* + U^*.$$

(2) *For any linear operator  $T$ ,*

$$(cT)^* = \bar{c} \cdot T^*.$$

(3) *For any linear operators  $T$  and  $U$ ,*

$$(TU)^* = U^*T^*.$$

**Proposition 14.2.4.** *Let  $V$  be a finite-dimensional inner product space and  $T$  be a linear operator on  $V$ . Then if  $T$  is invertible,  $T^*$  is also invertible.*

**Proposition 14.2.5.** *Let  $V$  be an inner product space and  $T$  be an invertible linear operator on  $V$ . Then  $(T^{-1})^* = (T^*)^{-1}$ .*

### 14.3 Normal Operators

**Definition (Normal).** *Let  $V$  be an inner product space and  $T$  be a linear operator on  $V$ . We say that  $T$  is **normal** if  $TT^* = T^*T$ .*

### 14.4 Self-adjoint

## Chapter 15

# Convolution

**Definition** (Convolution). *Let  $f$  and  $g$  be functions from  $\mathbb{R}$  to  $\mathbb{R}$ . We define the **convolution** of  $f$  and  $g$ , denoted by  $f * g$ , to be a function on  $\mathbb{R}$  given by*

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)dt.$$



## Chapter 16

# Coercive Functions

### 16.1 Definitions

**Definition (Coercive).** Let  $f$  be a function from  $\mathbb{R}^d$  to  $\mathbb{R}^*$ . We say that  $f$  is *coercive* if  $\lim_{\|x\| \rightarrow \infty} f(x) = +\infty$ .

### 16.2 Properties

**Proposition 16.2.1.** Let  $f$  be a proper lower semi-continuous function from  $\mathbb{R}^d$  to  $\mathbb{R}^*$ . Let  $K$  be a compact set in  $\mathbb{R}^d$ . Assume  $K \cap \text{dom}(f) \neq \emptyset$ . Then  $f$  attains its minimum over  $K$ .

*Proof.*

Define  $m := \inf_{x \in K} f(x)$ .

Since  $m = \inf_{x \in K} f(x)$ , there exists a sequence  $\{x_i\}_{i \in \mathbb{N}}$  in  $K$  such that  $\lim_{i \rightarrow \infty} f(x_i) = m$ .

Since  $K$  is compact and  $\{x_i\}_{i \in \mathbb{N}} \subseteq K$ , there exists a convergent subsequence  $\{x_i\}_{i \in I}$  in  $K$  where  $I$  is an infinite subset of  $\mathbb{N}$ .

Say the limit is  $x_\infty$  where  $x_\infty \in K$ .

Since  $\lim_{i \rightarrow \infty} f(x_i) = m$ , we get  $\lim_{i \in I, i \rightarrow \infty} f(x_i) = m$ .

Since  $\lim_{i \in I, i \rightarrow \infty} f(x_i) = m$ , we get  $\liminf_{i \in I, i \rightarrow \infty} f(x_i) = m$ .

Since  $f$  is lower semi-continuous and  $\lim_{i \in I, i \rightarrow \infty} x_i = x_\infty$ , we get  $f(x_\infty) \leq \liminf_{i \in I, i \rightarrow \infty} f(x_i)$ .

That is,  $f(x_\infty) \leq m$ .

Since  $m = \inf_{x \in K} f(x)$ , we have  $\forall x \in K, f(x) \geq m$ .

In particular,  $f(x_\infty) \geq m$ .

Since  $f(x_\infty) \geq m$  and  $f(x_\infty) \leq m$ ,  $f(x_\infty) = m$ .

Since  $f$  is proper,  $f(x_\infty) = m \neq -\infty$ .

So  $f$  attains its minimum at point  $x_\infty$ .

■

**Proposition 16.2.2.** *Let  $f$  be a proper, lower semi-continuous, and coercive function from  $\mathbb{R}^d$  to  $\mathbb{R}^*$ . Let  $C$  be a closed subset of  $\mathbb{R}^d$ . Assume  $C \cap \text{dom}(f) \neq \emptyset$ . Then  $f$  attains its minimum over  $C$ .*

*Proof.*

Since  $C \cap \text{dom}(f) \neq \emptyset$ , take  $x \in C \cap \text{dom}(f)$ .

Since  $f$  is coercive,  $\exists R$  such that  $\forall y, \|y\| > R$ , we have  $f(y) \geq f(x)$ .

Since  $x \in C \cap \text{dom}(f)$  and  $\forall y, \|y\| > R$ , we have  $f(y) \geq f(x)$ , the set of minimizers of  $f$  over  $C$  is the same as the set of minimizers of  $f$  over  $C \cap \text{ball}[0, R]$ .

Since  $C$  and  $\text{ball}[0, R]$  are both closed,  $C \cap \text{ball}[0, R]$  is closed.

Since  $\text{ball}[0, R]$  is bounded,  $C \cap \text{ball}[0, R]$  is bounded.

Since  $C \cap \text{ball}[0, R]$  is closed and bounded, by the Heine-Borel Theorem,  $C \cap \text{ball}[0, R]$  is compact.

Since  $f$  is proper and lower semi-continuous and  $C \cap \text{ball}[0, R]$  is compact,  $f$  attains its minimum over  $C \cap \text{ball}[0, R]$ .

So  $f$  attains its minimum over  $C$ .

■



# Chapter 17

## Unclassified Results

**Proposition 17.0.1.** *Let  $(X, d)$  be a compact metric space. Let  $L(X)$  be the set of all Lipschitz functions from  $X$  to  $\mathbb{R}$ . Let  $C(X)$  be the set of all continuous functions from  $X$  to  $\mathbb{R}$ . Then  $L(X)$  is dense in  $C(X)$ .*

**Proposition 17.0.2.** *Let  $(V, \|\cdot\|)$  be a normed vector space. Let  $S$  be a subset of  $V$ . Let  $p$  be a vector in  $V$ . Then we have the followings.*

$$(1) \ p + \text{int}(S) = \text{int}(p + S),$$

$$(2) \ p + \text{cl}(S) = \text{cl}(p + S).$$

*Proof.*

**Proof of (1).**

For one direction, let  $x$  be an arbitrary point in the set  $(p + \text{int}(S))$ .

We are to prove that  $x \in \text{int}(p + S)$ .

Since  $x \in (p + \text{int}(S))$ ,  $(x - p) \in \text{int}(S)$ .

Since  $(x - p) \in \text{int}(S)$ , by definition of interior, there exists a radius  $r$  such that

$$B(x - p, r) \subseteq S.$$

It follows that  $B(x, r) \subseteq p + S$ .

Since there exists a radius  $r$  such that  $B(x, r) \subseteq p + S$ , by definition of interior,

$$x \in \text{int}(p + S).$$

For the reverse direction, let  $x$  be an arbitrary point in  $\text{int}(p + S)$ .

We are to prove that  $x \in p + \text{int}(S)$ .

Since  $x \in \text{int}(p + S)$ , by definition of interior, there exists a radius  $r$  such that

$$B(x, r) \subseteq (p + S).$$

It follows that  $B(x - p, r) \subseteq S$ .

Since there exists a radius  $r$  such that  $B(x - p, r) \subseteq S$ , by definition of interior,

$$(x - p) \in \text{int}(S).$$

Since  $(x - p) \in \text{int}(S)$ , we get  $x \in (p + \text{int}(S))$ .

**Proof of (2).**

For one direction, let  $x$  be an arbitrary point in the set  $(p + \text{cl}(S))$ .

We are to prove that  $x \in \text{cl}(p + S)$ .

Since  $x \in (p + \text{cl}(S))$ , we get  $(x - p) \in \text{cl}(S)$ .

Since  $(x - p) \in \text{cl}(S)$ , by definition of closure, for any radius  $r$ , we have

$$B(x - p, r) \cap S \neq \emptyset.$$

It follows that  $B(x, r) \cap (p + S) \neq \emptyset$ .

Since for any radius  $r$ ,  $B(x, r) \cap (p + S) \neq \emptyset$ , by definition of closure, we get

$$x \in \text{cl}(p + S).$$

For the reverse direction, let  $x$  be an arbitrary point in  $\text{cl}(p + S)$ .

We are to prove that  $x \in (p + \text{cl}(S))$ .

Since  $x \in \text{cl}(p + S)$ , by definition of closure, for any radius  $r$ , we have

$$B(x, r) \cap (p + S) \neq \emptyset.$$

It follows that  $B(x - p, r) \cap S \neq \emptyset$ .

Since for any radius  $r$ ,  $B(x - p, r) \cap S \neq \emptyset$ , by definition of closure, we get

$$(x - p) \in \text{cl}(S).$$

Since  $(x - p) \in \text{cl}(S)$ , we get  $x \in (p + \text{cl}(S))$ .

■

**Proposition 17.0.3.** *Let  $(V, \|\cdot\|)$  be a normed vector space. Let  $S$  be a subset of  $V$ . Let  $\lambda$  be a non-zero real number. Then*

$$(1) \lambda \text{int}(S) = \text{int}(\lambda S).$$

$$(2) \lambda \text{cl}(S) = \text{cl}(\lambda S).$$

*Proof.*

**Proof of (1).**

For one direction, let  $x$  be an arbitrary point in  $\lambda \text{int}(S)$ .

We are to prove that  $x \in \text{int}(\lambda S)$ .

Since  $x \in \lambda \text{int}(S)$ , we get  $x/\lambda \in \text{int}(S)$ .

Since  $x/\lambda \in \text{int}(S)$ , by definition of interior, there exists a radius  $r$  such that

$$B(x/\lambda, r) \subseteq S.$$

Let  $y$  be an arbitrary point in  $B(x, \lambda r)$ .

Since  $y \in B(x, \lambda r)$ , we get  $\|y - x\| \leq \lambda r$ .

Since  $\|y - x\| \leq \lambda r$ , we get  $\|y/\lambda - x/\lambda\| \leq r$ .

Since  $\|y/\lambda - x/\lambda\| \leq r$ , we get  $y/\lambda \in B(x/\lambda, r)$ .

Since  $y/\lambda \in B(x/\lambda, r)$  and  $B(x/\lambda, r) \subseteq S$ , we get  $y/\lambda \in S$ .

Since  $y/\lambda \in S$ , we get  $y \in \lambda S$ .

Since any point in  $B(x, \lambda r)$  is also in  $\lambda S$ , we get  $B(x, \lambda r) \subseteq \lambda S$ .

Since there exists a radius  $r$  such that  $B(x, \lambda r) \subseteq \lambda S$ , by definition of interior, we get

$$x \in \text{int}(\lambda S).$$

For the reverse direction,

■