Measure Theory

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Contents

Chapter 1

Algebras and Sigma-Algebras

1.1 Definitions and Properties

Definition (Algebra)**.** *Let $\mathcal{A}$ be a non-empty collection of subsets of X. We say that $\mathcal{A}$ is an **algebra** on X if it satisfies all of the following conditions.*

(1) $\emptyset, X \in \mathcal{A}$

(2) $\mathcal{A}$ is closed under complement.

(3) $\mathcal{A}$ is closed under formation of finite union and intersection.

Definition (σ-algebra)**.** *Let $\mathcal{A}$ be a non-empty collection of subsets of X. We say that $\mathcal{A}$ is a σ-**algebra** on X if it satisfies all of the following conditions.*

(1) $\emptyset, X \in \mathcal{A}$

(2) $\mathcal{A}$ is closed under complement.

(3) $\mathcal{A}$ is closed under formation of countable union and intersection.

Proposition 1.1.1. *The set of all subsets of X and the set $\{\emptyset, X\}$ are the largest and smallest algebras on X, respectively; and also the largest and smallest σ-algebras on X.*

Proposition 1.1.2 (Set Operation)**.** *The intersection of a non-empty collection of algebras is again an algebra.*

Proof.
Let $\{\mathcal{A}_\alpha\}_{\alpha\in A}$ be a collection of algebras on X and let $\mathcal{A}$ denote $\bigcap_{\alpha\in A}\mathcal{A}_\alpha$.
Part 1: $\emptyset$ and X.
Since each $\mathcal{A}_\alpha$ is an algebra, by definition, $\emptyset \in \mathcal{A}_\alpha$ and $X \in \mathcal{A}_\alpha$ for each $\alpha \in A$.
Since $\emptyset \in \mathcal{A}_\alpha$ for each $\alpha \in A$, $\emptyset \in \mathcal{A}$.
Since $X \in \mathcal{A}_\alpha$ for each $\alpha \in A$, $X \in \mathcal{A}$.
Part 2: complements.
Let A be an arbitrary element in $\mathcal{A}$. Then A is in each of $\mathcal{A}_k$.
By definition of σ-algebra, A^c is also in each of $\mathcal{A}_k$. Then A^c is also in $\mathcal{A}$.
Thus $\mathcal{A}$ is closed under complement. (**)
Let $\{A_k\}_{k=1}^\infty$ be an arbitrary sequence of elements in $\mathcal{A}$. Then $\{A_k\}$ is in each of $\mathcal{A}_k$.
By definition of σ-algebra, the union and intersection of $\{A_k\}$ are also in each of $\mathcal{A}_k$. Then the union and intersection are also in $\mathcal{A}$.
Thus $\mathcal{A}$ is closed under countable union and intersection. (***)
From statements (*) ˜ (***), $\mathcal{A}$ is a σ-algebra.

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1.2 Generated Algebras and Generated sigma-algebras

Definition (Generated Algebra)**.** *Let $\mathcal{S}$ be a collection of subsets of X. We define the algebra generated by $\mathcal{S}$ to be the smallest algebra on X that contains $\mathcal{S}$, or equivalently, the intersection of all algebras on X that contains $\mathcal{S}$.*

Proposition 1.2.1. *Let $\mathcal{S}$ be a collection of subsets of X. Then there exists uniquely a smallest algebra on X containing $\mathcal{S}$.*

Proof.
Part 1: existence
Let $\mathcal{C}$ be the set of all algebras containing $\mathcal{S}$.
By definition, the set of all subsets $\mathcal{P}(X)$ of X is an algebra and contains S.
Thus $\mathcal{P}(X)$ belongs to $\mathcal{C}$ and $\mathcal{C}$ is not empty.
Let $\mathcal{A}$ be the intersection of all algebras in $\mathcal{C}$.
Part 2: the intersection is the smallest one
Let $\mathcal{A}'$ be an arbitrary algebra on X containing $\mathcal{S}$.
By our choice of $\mathcal{A}$, $\mathcal{A}$ is contained in $\mathcal{A}'$.

Since $\mathcal{A}'$ is arbitrary, $\mathcal{A}$ is the smallest algebra on X containing $\mathcal{S}$.

Part 3: uniqueness

Let $\mathcal{A}'$ be also a smallest algebra on X containing $\mathcal{S}$.

Since $\mathcal{A}$ is the smallest, $\mathcal{A} \subseteq \mathcal{A}'$.

Since $\mathcal{A}'$ is the smallest, $\mathcal{A}' \subseteq \mathcal{A}$.

It follows that $\mathcal{A} = \mathcal{A}'$.

■

Definition (Generated σ-algebra)**.** *Let $\mathcal{S}$ be a collection of subsets of X. We define the σ-algebra generated by $\mathcal{S}$ to be the smallest σ-algebra on X that contains $\mathcal{S}$, or equivalently, the intersection of all σ-algebras on X that contains $\mathcal{S}$.*

Proposition 1.2.2. *Let $\mathcal{F}$ be a set of subsets of X. Then there exists uniquely a smallest σ-algebra on X containing $\mathcal{F}$.*

Proof.

Proof of Existence

Let $\mathcal{C}$ be the set of all σ-algebras that contains $\mathcal{F}$.

By definition of σ-algebra, the set of all subsets of X is a σ-algebra.

Note that this σ-algebra contains $\mathcal{F}$. Thus $\mathcal{C}$ is non-empty.

By Proposition 2.1, the intersection of all sets in $\mathcal{C}$, denote by $\mathcal{A}$ is also a σ-algebra.

Proof of Minimum

Let $\mathcal{A}'$ be an arbitrary σ-algebra on X that contains $\mathcal{F}$.

By the choice of $\mathcal{C}$, $\mathcal{A}'$ is in $\mathcal{C}$.

By the choice of $\mathcal{A}$, $\mathcal{A}$ is a subset of $\mathcal{A}'$.

Thus $\mathcal{A}$ is the smallest σ-algebra that contains $\mathcal{F}$.

Proof of Uniqueness

Assume that $\mathcal{A}''$ is another smallest σ-algebra on X that contains $\mathcal{F}$.

Since $\mathcal{A}$ is a smallest σ-algebra, we get $\mathcal{A} \subseteq \mathcal{A}''$.

Since $\mathcal{A}''$ is a smallest σ-algebra, we get $\mathcal{A}'' \subseteq \mathcal{A}$.

Thus $\mathcal{A} = \mathcal{A}''$.

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Example 1.2.1. *Let X be a non-empty set and $\mathcal{A}$ be the set of all subsets A of X that either A or A^c is countable. Then $\mathcal{A}$ is the σ-algebra generated by the set of singleton sets $\mathcal{S} = \{\{x\} : x \in X\}$.*

1.3 Borel Algebras

Definition (Borel σ-algebra)**.** *We define the **Borel σ-algebra** on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, to be the σ-algebra on $\mathbb{R}^n$ generated by the collection of all open subsets of $\mathbb{R}^n$. We say a set is a **Borel subset** of $\mathbb{R}^n$ if it is an element in the Borel σ-algebra.*

(1) The Borel σ-algebra on $\mathbb{R}^n$ can be generated by any of the collections of sets listed below.

(a) The collection of all closed subsets of $\mathbb{R}^n$

$$\{(x_1, x_2, \ldots, x_n) : x_{j_0} \leq c\}$$

$$\{(x_1, x_2, \ldots, x_n) : a < x_{j_0} \leq b\}$$

$$\{(x_1, x_2, \ldots, x_n) : a_j < x_j \leq b_j \ (j = 1, 2, \ldots, n)\}$$

Proof.
Proof Part 1
By definition of Borel σ-algebra, $\mathcal{B}(\mathbb{R}^n)$ contains all open subsets of $\mathbb{R}^n$.
By definition of σ-algebra, $\mathcal{B}(\mathbb{R}^n)$ is closed under complement and hence contains all closed subsets of $\mathbb{R}^n$.
Thus all sets in collection (1) are contained in $\mathcal{B}(\mathbb{R}^n)$.
It follows that the σ-algebra generated by collection (1), denote by $\mathcal{B}_1$ is contained in $\mathcal{B}(\mathbb{R}^n)$.
Proof Part 2
Note that closed half-spaces in $\mathbb{R}^n$ are closed subsets of $\mathbb{R}^n$.
Thus all sets in collection (2) are contained in $\mathcal{B}_1$.
It follows that the σ-algebra generated by collection (2), denote by $\mathcal{B}_2$, is contained in $\mathcal{B}_1$.
Proof Part 3
Define sets

$$A = \{(x_1, x_2, \ldots, x_n) : a < x_{j_0} \leq b\}$$

$$B = \{(x_1, x_2, \ldots, x_n) : x_{j_0} \leq a\}$$

$$C = \{(x_1, x_2, \dots, x_n) : x_{j_0} \leq b\}$$

Note that B and C are contained in $\mathcal{B}_2$ and $A = B^c \cap C$.
Thus all sets in collection (3) are contained in $\mathcal{B}_2$.
It follows that the σ-algebra generated by collection (3), denote by $\mathcal{B}_3$, is contained in $\mathcal{B}_2$.
Proof Part 4
Define sets

$$A = \{(x_1, x_2, \dots, x_n) : a_j < x_j \leq b_j \ (j = 1, 2, \dots, n)\}$$

$$A_k = \{(x_1, x_2, \dots, x_n) : a_k < x_k \leq b_k\}$$

Note that every A_k is contained in $\mathcal{B}_3$ and $A = \bigcap_{k=1}^{n} A_k$.
Thus all sets in collection (4) are contained in $\mathcal{B}_3$.
It follows that the σ-algebra generated by collection (4), denote by $\mathcal{B}_4$, is contained in $\mathcal{B}_3$.
Proof Part 5
Note that any open subset of $\mathbb{R}^n$ can be written as a countable union of open rectangles and any open rectangle can be written as a countable union of rectangles in collection (4).
Thus all open subsets in $\mathbb{R}^n$ are contained in $\mathcal{B}_4$.
It follows that $\mathcal{B}(\mathbb{R}^n)$ is contained in $\mathcal{B}_4$.
Conclusion
We have proved that $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \mathcal{B}_4 \subseteq \mathcal{B}(\mathbb{R}^n)$.
Thus the σ-algebra generated by each collection of sets is the Borel σ-algebra.

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Notation ($\mathcal{G}$, $\mathcal{F}$) Let $\mathcal{G}$ denote the set of all open subsets of $\mathbb{R}^n$ and $\mathcal{F}$ denote the set of all closed subsets of $\mathbb{R}^n$.
Notation ($\mathcal{F}_\delta$, $\mathcal{F}_\sigma$) $\mathcal{F}_\delta$ is the set of all intersections of collection of sets in $\mathcal{F}$ and $\mathcal{F}_\sigma$ is the set of all unions of collection of sets in $\mathcal{F}$.

(1) Let $\mathcal{S}$ be a non-empty collection of sets and let $\mathcal{A}$ be an algebra generated by $\mathcal{S}$. Then for any set A in $\mathcal{A}$, there exists a sub-collection $\mathcal{S}^{'}(A)$ of $\mathcal{S}$ such that A is also in the algebra generated by $\mathcal{S}^{'}$.

(2) Let $\mathcal{S}$ be a non-empty collection of sets and let $\mathcal{A}$ be an σ-algebra generated by $\mathcal{S}$. Then for any set A in $\mathcal{A}$, there exists a sub-collection $\mathcal{S}^{'}(A)$ of $\mathcal{S}$ such that A is also in the σ-algebra generated by $\mathcal{S}^{'}$.

(3) (a) Any closed subset of $\mathbb{R}^n$ is the intersection of some collection of open sets in $\mathbb{R}^n$.

(b) Any open subset of $\mathbb{R}^n$ is the union of some collection of closed sets in $\mathbb{R}^n$.

(4) Let $\mathcal{A}$ be an algebra on X. Then $\mathcal{A}$ is also a σ-algebra if it satisfies any of the conditions listed below

(a) $\mathcal{A}$ is closed under the formation of the union of any increasing sequence of sets.

(b) $\mathcal{A}$ is closed under the formation of the intersection of any decreasing sequence of sets.

(c) $\mathcal{A}$ is closed under the formation of the union of any sequence of disjoint sets.

Proof.

Proof of (1)

Let $\{A_k\}$ be an arbitrary sequence of sets in $\mathcal{A}$.
Construct another sequence of sets $\{B_k\}$ by $B_n = \bigcup_{k=1}^n A_k$.
Then $\{B_k\}$ is increasing and we have $\bigcup_{k=1}^n B_k = \bigcup_{k=1}^n A_k$.
By assumption, $\bigcup_{k\in\mathbb{N}} B_k$ is in $\mathcal{A}$.
It follows that $\bigcup_{k\in\mathbb{N}} A_k$ is also in $\mathcal{A}$.
Thus $\mathcal{A}$ is closed under countable union.
By definition, $\mathcal{A}$ is a σ-algebra.

Proof of (2)

Let $\{A_k\}$ be an arbitrary sequence of sets in $\mathcal{A}$.
Construct another sequence of sets $\{B_k\}$ by $B_n = \bigcap_{k=1}^n A_k$.
Then $\{B_k\}$ is decreasing and we have $\bigcap_{k=1}^n B_k = \bigcap_{k=1}^n A_k$.
By assumption, $\bigcap_{k=1}^n B_k$ is in $\mathcal{A}$.
It follows that $\bigcap_{k=1}^n A_k$ is also in $\mathcal{A}$.
Thus $\mathcal{A}$ is closed under countable intersection.
By definition, $\mathcal{A}$ is a σ-algebra.

Proof of (3)

Let $\{A_k\}$ be an arbitrary sequence of sets in $\mathcal{A}$.

Construct another sequence of sets $\{B_k\}$ by $B_1 = A_1$ and $B_k = A_k - A_{k-1}$ ($k \geq 2$).
Then $\{B_k\}$ is disjoint and we have $\bigcup_{k\in\mathbb{N}} B_k = \bigcup_{k\in\mathbb{N}} A_k$.
By assumption, $\bigcup_{k\in\mathbb{N}} B_k$ is in $\mathcal{A}$.
It follows that $\bigcup_{k\in\mathbb{N}} A_k$ is also in $\mathcal{A}$.
Thus $\mathcal{A}$ is closed under countable union.
By definition $\mathcal{A}$ is a σ-algebra.

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Chapter 2

Additive Set Functions and Measures

2.1 Additive Set Functions

Definition (Additive Set Functions). *Let $\mathcal{A}$ be an algebra over some set X. Let ν be a set function on $\mathcal{A}$. We say that ν is **additive** if for any disjoint sets A and B in $\mathcal{A}$,*

$$\nu(A \cup B) = \nu(A) + \nu(B)$$

Proposition 2.1.1. *Let ν is an additive set function, then $\nu(\emptyset) = 0$.*

Definition (Countably Additive Set Functions). *Let $\mathcal{A}$ be an algebra on X. Let ν be a set function on $\mathcal{A}$. We say that ν is* **countably additive** *if it satisfies all of the conditions listed below.*

(1) $\nu(\emptyset) = 0$

(2)
$$\nu(\bigcup_{k \in \mathbb{N}} S_k) = \sum_{k \in \mathbb{N}} \nu(S_k)$$

Proposition 2.1.2. *Additive set functions cannot take on both $+\infty$ and $-\infty$ as values.*

Proof.
Let $\mathcal{A}$ be an algebra on X and ν be an additive set function on $\mathcal{A}$.

Assume for the sake of contradiction that there exist sets A and B in $\mathcal{A}$ such that

$$\nu(A) = +\infty$$

$$\nu(B) = -\infty$$

Define $S_1 = A - B$, $S_2 = A \cap B$, and $S_3 = B - A$. Then S_1, S_2, and S_3 are mutually disjoint.
By definition of additive set functions, we have

$$\nu(A) = \nu(S_1) + \nu(S_2) = +\infty\#(1)$$

$$\nu(B) = \nu(S_2) + \nu(S_3) = -\infty\#(2)$$

Case 1: $\nu(S_2)$ is finite. Say $\nu(S_2) = c$.
From equations (1) and (2), we get

$$\nu(S_1) = \nu(A) - \nu(S_2) = (+\infty) - c = +\infty\#(3)$$

$$\nu(S_3) = \nu(B) - \nu(S_2) = (-\infty) - c = -\infty\#(4)$$

Since S_1 and S_3 are disjoint, we should get

$$\nu(S_1 \cup S_3) = \nu(S_1) + \nu(S_3)$$

However, the RHS is $(+\infty) + (-\infty)$ and is not defined.
Thus a contradiction has occurred.
Case 2: $\nu(S_2)$ is infinite. Assume without loss of generality that $\nu(S_2) = +\infty$.
It follows from equation (4) that

$$\nu(S_3) = \nu(B) - \nu(S_2) = (-\infty) - (+\infty) = -\infty$$

Since S_2 and S_3 are disjoint, we should get

$$\nu(B) = \nu(S_2) + \nu(S_3)$$

However, the RHS is $(+\infty) + (-\infty)$ and is not defined.
Thus again a contradiction has occurred.
Thus ν cannot take on values $+\infty$ and $-\infty$ simultaneously.

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2.2 Measures

Definition. *(Signed Measures) Let $\mathcal{M}$ be a σ-algebra on X. We define a* **signed measure** *on $\mathcal{M}$ to be a set function μ on $\mathcal{M}$ that satisfies all of the conditions listed below.*

(1) $\mu(\emptyset) = 0$.

(2)

$$\mu(\bigcup_{k\in\mathbb{N}} S_k) = \sum_{k\in\mathbb{N}} \mu(S_k)$$

Definition. *(Measures) Let $\mathcal{M}$ be a σ-algebra on X. We define a* **measure** *on $\mathcal{M}$ to be a set function μ on $\mathcal{M}$ that satisfies all of the conditions listed below.*

(1) (Non-negative) For any set S in $\mathcal{M}$, we have $\mu(S) \geq 0$

(2) $\mu(\emptyset) = 0$.

Proposition 2.2.1.

$$\mu(\bigcup_{k\in\mathbb{N}} S_k) = \sum_{k\in\mathbb{N}} \mu(S_k)$$

$$\mu(B - A) = \mu(B) - \mu(A)$$

Proof.
Note that $B = A \cup (B - A)$ and that A and $(B - A)$ are disjoint.
By definition of measures, μ is countably additive and hence $\mu(B) = \mu(A) + \mu(B - A)$.
By definition of measures, μ is non-negative and hence $\mu(B - A) \geq 0$.
It follows that $\mu(A) \leq \mu(B)$.
If $\mu(A) \neq +\infty$, then we are allowed to subtract $\mu(A)$ from both sides.
Subtracting gives $\mu(B - A) = \mu(B) - \mu(A)$.

∎

$$\mu(\bigcup_{k\in\mathbb{N}} A_k) \leq \sum_{k\in\mathbb{N}} \mu(A_k)$$

Proof.
Construct another sequence of sets $\{B_k\}$ by $B_1 = A_1$ and $B_n = A_n - \bigcup_{k=1}^{n-1} A_k$ $(n \geq 2)$.
Then $\{B_k\}$ is disjoint and we have

$$\bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k \#(1)$$

$$B_k \subseteq A_k \#(2)$$

From (1), we automatically get

$$\mu(\bigcup_{k\in\mathbb{N}} B_k) = \mu(\bigcup_{k\in\mathbb{N}} A_k)\#(3)$$

From (2), by the monotonicity of measures, we get

$$\mu(B_k) \leq \mu(A_k)$$

It follows that

$$\sum_{k\in\mathbb{N}} \mu(B_k) \leq \sum_{k\in\mathbb{N}} \mu(A_k)\#(4)$$

By definition of measures, μ is countably additive and hence

$$\mu(\bigcup_{k\in\mathbb{N}} B_k) = \sum_{k\in\mathbb{N}} \mu(B_k)\#(5)$$

From (3) ˜ (5), we get

$$\mu(\bigcup_{k\in\mathbb{N}} A_k) \leq \sum_{k\in\mathbb{N}} \mu(A_k)$$

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(1) Let $(X, \mathcal{M}, \mu)$ be a measure space. Then for any sets A and B in $\mathcal{A}$, we have $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$

Proof.
Note that $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$.
By the countable additivity of measures, we get

$$\mu(A \cup B) = \mu(A - B) + \mu(A \cap B) + \mu(B - A)\#(1)$$

Note that $A = (A - B) \cup (A \cap B)$ and $B = (B - A) \cup (A \cap B)$.
By the countable additivity of measures, we get

$$\mu(A) = \mu(A - B) + \mu(A \cap B)\#(2)$$

$$\mu(B) = \mu(B - A) + \mu(A \cap B)\#(3)$$

From (1) ˜ (3), we get

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

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Definition. *(Point Mass) Let $\mathcal{A}$ be a σ-algebra on X and x be an element in X. We define a* **point mass** *concentrated at point x, denoted by δ_x, to be a set function on $\mathcal{A}$ defined by*

$$\delta_x(A) = \{\begin{matrix} 1, & x \in A \\ 0, & x \notin A \end{matrix}$$

(1) Point masses are measures.

Proof.
By definition of point masses, they are non-negative.
For any element x in X, we have $x \notin \emptyset$. Thus $\delta_x(\emptyset) = 0$.
Let $\{A_k\}$ be an arbitrary sequence of disjoint sets in $\mathcal{A}$.
Let x be an element in $\bigcup_{k \in \mathbb{N}} A_k$. Then x is in exactly one of A_k.
It follows that

$$\delta_x(\bigcup_{k \in \mathbb{N}} A_k) = \sum_{k \in \mathbb{N}} \mu(A_k) = 1$$

■

Chapter 3

Limits Theorems

Definition. *(Limits of Monotone Sequences of Sets) Let $\{A_k\}$ be a sequence of subsets of X.*

(1) *If $\{A_k\}$ is increasing, we define the limit by*

$$A_k = \bigcup_{k \in \mathbb{N}} A_k$$

(2) *If $\{A_k\}$ is decreasing, we define the limit by*

$$A_k = \bigcap_{k=1}^{\infty} A_k$$

Definition. *(Limit Superior and Limit Inferior) Let $\{A_k\}$ be a sequence of subsets of X.*

$$A_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k$$

$$A_k = \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} A_k$$

Definition. *(Convergence of Sequences of Sets) Let $\{A_k\}$ be a sequence of subsets of X. We say that $\{A_k\}$* **converges** *if $A_k = A_k$. In this case, we denote the common set by A and say that $\{A_k\}$ converges to A or symbolically $A_k = A$.*

Proposition 3.0.1. *Let $\{A_k\}$ be a sequence of subsets of X. Then*

(1) $A_k = \{x : x \in A_k$ *for infinitely many* $k\}$

(2) $A_k = \{x : x \in A_k$ *for all but finitely many* $k\}$

Proof.

Proof of (1)

For one direction, let x be an arbitrary element in A_k.

By definition of limit superior, x is in each of the sets $S_m = \bigcup_{k=m}^{\infty} A_k$.

For $m = 1$, since $x \in S_1 = \bigcup_{k\in\mathbb{N}} A_k$, there exists an index $k_1 \geq 1$ such that $x \in A_{k_1}$.

For $m = k_1 + 1$, since $x \in S_{k_1} = \bigcup_{k=k_1}^{\infty} A_k$, there exists an index $k_2 > k_1$ such that $x \in A_{k_2}$.

Repeat and get a subsequence $\{A_{n_k}\}$ of $\{A_k\}$ such that x is in each A_{n_k}.

For the reverse direction, let x be an arbitrary element such that $x \in A_k$ for infinitely many k.

Assume for the sake of contradiction that $x \notin A_k$.

Then there exists an integer m_0 such that $x \notin S_{m_0} = \bigcup_{k=m_0}^{\infty} A_k$.

It follows that x is not in each A_k for $k \geq m_0$.

This contradicts to the fact that $x \in A_k$ for infinitely many k.

Proof of (2)

For one direction, let x be an arbitrary element in A_k.

By definition of limit inferior, there exists an integer m_0 such that $x \in \bigcap_{k=m_0}^{\infty} A_k$.

It follows that x is in each A_k for $k \geq m_0$.

For the reverse direction, let x be an arbitrary element such that $x \in A_k$ for all but finitely many k.

Then there exists an integer m_0 such that $x \in A_k$ for each $k \geq m_0$.

It follows that $x \in \bigcap_{k=m_0}^{\infty} A_k$.

It follows that $x \in A_k$.

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Proposition 3.0.2. *Let $(X, \mathcal{M}_\mu, \mu)$ be a measure space and $\{A_k\}$ be a monotone sequence of sets in $\mathcal{M}_\mu$. Then*

(1)

$$\mu(A_k) = \mu(A_k)$$

(2)

$$\mu(A_k) = \mu(A_k)$$

Proof.

Proof of (1)

Construct another sequence of sets $\{B_k\}$ by $B_1 = A_1$ and $B_n = A_n - \bigcup_{k=1}^{n-1} A_k$.
Then $\{B_k\}$ is disjoint and we have

$$\bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k\in\mathbb{N}} B_k\#(1)$$

$$A_n = \bigcup_{k=1}^{n} B_k\#(2)$$

From (1), by the countable additivity of measures, we get

$$\mu(\bigcup_{k\in\mathbb{N}} A_k) = \sum_{k\in\mathbb{N}} \mu(B_k)\#(3)$$

From (2), by the countable additivity of measures again, we get

$$\mu(A_n) = \sum_{k=1}^{n} \mu(B_k)$$

It follows that

$$\mu(A_n) = \sum_{k\in\mathbb{N}} \mu(B_k)\#(4)$$

From (3) and (4), we get

$$\mu(\bigcup_{k\in\mathbb{N}} A_k) = \mu(A_n)$$

Proof of (2)

Note that $\{A_k\}$ is decreasing. Thus we can assume without loss of generality that $N = 1$.
Construct another sequence of sets $\{B_k\}$ by $B_k = A_1 - A_k$.
Then $\{B_k\}$ is increasing and we have

$$\bigcap_{k=1}^{\infty} A_k = A_1 - \bigcup_{k\in\mathbb{N}} B_k\#(1)$$

$$A_n = A_1 - B_n\#(2)$$

From (1), by the monotonicity of measures, we get

$$\mu(\bigcap_{k=1}^{\infty} A_k) = \mu(A_1) - \mu(\bigcup_{k\in\mathbb{N}} B_k)$$

Since $\{B_k\}$ is increasing, it follows that

$$\mu(\bigcap_{k=1}^{\infty} A_k) = \mu(A_1) - \mu(B_n)\#(3)$$

From (2), by the monotonicity of measures again, we get

$$\mu(A_n) = \mu(A_1) - \mu(B_n)$$

It follows that

$$\mu(A_n) = \mu(A_1) - \mu(B_n)\#(4)$$

From (3) and (4), we get

$$\mu(\bigcap_{k=1}^{\infty} A_k) = \mu(A_n)$$

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Proposition 3.0.3. *Let* $(X, \mathcal{M}_\mu, \mu)$ *be a measure space and* $\{A_k\}$ *be an arbitrary sequence of sets in* $\mathcal{M}_\mu$. *Then*

(1) $\mu(A_k) \leq \mu(A_k)$.

(2) $\mu(A_k) \geq \mu(A_k)$ *provided that* $\mu(\bigcup_{k\in\mathbb{N}} A_k) < \infty$.

(3) $\mu(A_k) = \mu(A_k)$ *provided that the sequence converges and* $\mu(\bigcup_{k\in\mathbb{N}} A_k) < \infty$.

Proof.

Proof of (1)

Define set S_m by $S_m = \bigcap_{k=m}^{\infty} A_k$. Then the sequence $\{S_m\}$ is non-decreasing and $S_m \subseteq A_m$ for each index m.

$$\mu(\bigcup_{m=1}^{\infty} S_m) = \mu(S_m) = \mu(S_k)\#(1)$$

$$\mu(S_m) \leq \mu(A_m)$$

$$\mu(S_k) \leq \mu(A_k)\#(2)$$

$$\mu(\bigcup_{m=1}^{\infty} S_m) \leq \mu(A_k)$$

$$\mu(A_k) \leq \mu(A_k)$$

Proof of (2)
Define set S_m by $S_m = \bigcup_{k=m}^{\infty} A_k$. Then the sequence $\{S_m\}$ is non-increasing with $\mu(S_1) < \infty$ and $A_m \subseteq S_m$ for each index m.

$$\mu(\bigcap_{m=1}^{\infty} S_m) = \mu(S_m) = \mu(S_k)\#(1)$$

$$\mu(S_m) \geq \mu(A_m)$$

$$\mu(S_k) \geq \mu(A_k)\#(2)$$

$$\mu(\bigcap_{m=1}^{\infty} S_m) \geq \mu(A_k)$$

$$\mu(A_k) \geq \mu(A_k)$$

■

Proposition 3.0.4. *Let $(X, \mathcal{A})$ be a measurable space and μ be a finitely additive measure. Then μ is also a measure if it satisfies any of the conditions listed below.*

(1)

$$\mu(\bigcup_{k\in\mathbb{N}} A_k) = \mu(A_n)$$

(2)

$$\mu(A_k) = 0$$

Proof.
Proof of (1)
Let $\{B_k\}$ be a sequence of disjoint sets in $\mathcal{A}$.
Construct another sequence of sets $\{A_k\}$ by $A_n = \bigcup_{k=1}^{n} B_k$.
Then $\{A_k\}$ is increasing and we have

$$\bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k\in\mathbb{N}} B_k\#(1)$$

$$A_n = \bigcup_{k=1}^{n} B_k \#(2)$$

From (1), we get

$$\mu(\bigcup_{k\in\mathbb{N}} A_k) = \mu(\bigcup_{k\in\mathbb{N}} B_k)\#(3)$$

From (2), by the finite additivity of μ, we get

$$\mu(A_n) = \sum_{k=1}^{n} \mu(B_k)$$

It follows that

$$\mu(A_n) = \sum_{k\in\mathbb{N}} \mu(B_k)\#(4)$$

Apply condition (1) to $\{A_n\}$, we get

$$\mu(\bigcup_{k\in\mathbb{N}} A_k) = \mu(A_n)\#(5)$$

From (3) ˜ (5), we get

$$\mu(\bigcup_{k\in\mathbb{N}} B_k) = \sum_{k\in\mathbb{N}} \mu(B_k)$$

Thus μ is countably additive.

By definition of measures, μ is a countably additive measure.

Proof of (2)

$\blacksquare$

Chapter 4

Variations and Decompositions

4.1 Variations

Definition. *(Variations) Let $\mathcal{A}$ be an algebra on X and ν be an additive set function on $\mathcal{A}$.*

$$\overline{V}(\nu, S) = \sup\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\}$$

$$(\nu, S) = \inf\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\}$$

$$V(\nu, S) = \overline{V}(\nu, S) - (\nu, S)$$

Proposition 4.1.1.

(1) Positive variations are non-negative.

(2) Negative variations are non-positive.

Proof.
Let $\mathcal{A}$ be an algebra on X and ν be an additive set function on $\mathcal{A}$.
Proof of (1)
Let S be a set in $\mathcal{A}$.
Since $\emptyset \subseteq S$, $\nu(\emptyset) \in \{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\}$.

$$\nu(\emptyset) \leq \sup\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\}\#(*)$$

By definition of additive set functions, $\nu(\emptyset) = 0$.
By definition of positive variations, $\sup\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\} = \overline{V}(\nu, S)$.
Substitution gives $\overline{V}(\nu, S) \geq 0$.
Proof of (2)
Let S be a set in $\mathcal{A}$.
Since $\emptyset \subseteq S$, $\nu(\emptyset) \in \{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\}$.

$$\nu(\emptyset) \geq \inf\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\}\#(*)$$

By definition of additive set functions, $\nu(\emptyset) = 0$.
By definition of negative variations, $(\nu, S) = \inf\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\}$.
Substitution gives $(\nu, S) \leq 0$.

∎

(1) Variations of additive set functions are also additive.

Proof. Let $\mathcal{A}$ be an algebra on X and ν be an additive set function on $\mathcal{A}$.
Part 1: values at empty set

$$\overline{V}(\nu, \emptyset) = \sup\{\nu(\emptyset)\} = \nu(\emptyset) = 0 \backslash n(\nu, \emptyset) = \inf\{\nu(\emptyset)\} = \nu(\emptyset) = 0$$

Part 2: additivity

$$\overline{V}(\nu, A \cup B) \leq \overline{V}(\nu, A) + \overline{V}(\nu, B)\#(*) \backslash n\overline{V}(\nu, A \cup B) \geq \overline{V}(\nu, A) + \overline{V}(\nu, B)\#(**)$$

Let S be an arbitrary subset of $A \cup B$.
Since $S \subseteq A \cup B$ and A and B are disjoint, S can be written as $S = (S \cap A) \cup (S \cap B)$ and the sets $S \cap A$ and $S \cap B$ are disjoint.

$$\nu(S) = \nu(S \cap A) + \nu(S \cap B)$$

$$\nu(S \cap A) \leq \overline{V}(\nu, A) \backslash n\nu(S \cap B) \leq \overline{V}(\nu, B)$$

$$\nu(S) \leq \overline{V}(\nu, A) + \overline{V}(\nu, B)$$

$$\overline{V}(\nu, A \cup B) \leq \overline{V}(\nu, A) + \overline{V}(\nu, B)\#(*)$$

Let ε be an arbitrary positive number.

$$\nu(A^{'}) > \overline{V}(\nu, A) - \varepsilon/2$$

$$\nu(B^{'}) > \overline{V}(\nu, B) - \varepsilon/2$$

Define set S_0 by $S_0 = A^{'} \cup B^{'}$.
Since $A^{'} \subseteq A$, $B^{'} \subseteq B$, and A and B are disjoint, $A^{'}$ and $B^{'}$ are disjoint.

$$\nu(S_0) = \nu(A^{'}) + \nu(B^{'})$$

Since $A^{'} \subseteq A$, $B^{'} \subseteq B$, and $S = A^{'} \cup B^{'}$, $S \subseteq A \cup B$.

$$\overline{V}(\nu, A \cup B) \geq \nu(S_0)$$

$$\overline{V}(\nu, A \cup B) \geq \overline{V}(\nu, A) + \overline{V}(\nu, B) - \varepsilon$$

$$\overline{V}(\nu, A \cup B) \geq \overline{V}(\nu, A) + \overline{V}(\nu, B)\#(**)$$

■

(1) Variations of signed measures are still signed measures.

Proof.
Proof part (1): values at empty set
Let $\mathcal{A}$ be a σ-algebra on set X and ν be a signed measure on $\mathcal{A}$.
Consider the empty set. Then the only subset is the empty set itself.

$$\overline{V}(\nu, \emptyset) = \sup\{\nu(\emptyset)\} = 0 \backslash n$$

Proof Part (2): countably additivity
Let $\{S_k\}_{k=1}^{\infty}$ be an arbitrary sequence of disjoint sets in $\mathcal{A}$ and let S denote their union.

$$\overline{V}(\nu, S) \leq \sum_{k \in \mathbb{N}} \overline{V}(\nu, S_k)\#(*)$$

$$\overline{V}(v,S) \geq \textstyle\sum_{k\in\mathbb{N}} \overline{V}(v,S_k)\#(**)$$

Let S' be an arbitrary subset of S that is in $\mathcal{A}$ and $S_k^{'}$ be an arbitrary subset of S_k that is in $\mathcal{A}$ for each k.

$$S' = \bigcup_{k\in\mathbb{N}} (S' \cap S_k)$$

$$\nu(S' \cap S_k) \leq \nu(S_k)$$

Note that S_k is a subset of itself and is in $\mathcal{A}$.

$$\nu(S_k) \leq \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\}$$

$$\nu(S' \cap S_k) \leq \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\}$$

$$\sum_{k\in\mathbb{N}} \nu(S' \cap S_k) \leq \sum_{k\in\mathbb{N}} \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\}$$

$$\nu(S') = \sum_{k\in\mathbb{N}} \nu(S' \cap S_k)$$

$$\nu(S') \leq \sum_{k\in\mathbb{N}} \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\}$$

i.e., the RHS is an upper bound for the set $\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\}$.

$$\sup\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\} \leq \sum_{k\in\mathbb{N}} \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\}$$

$$\overline{V}(\nu,S) \leq \textstyle\sum_{k\in\mathbb{N}} \overline{V}(\nu,S_k)\#(*)$$

Let ε be an arbitrary positive number.

$$\nu(S_k^{'}) > \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\} - \varepsilon/2^k$$

$$\sum_{k\in\mathbb{N}} \nu(S_k^{'}) > \sum_{k\in\mathbb{N}} \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\} - \varepsilon$$

Note that $\{S_k^{'}\}_{k=1}^{\infty}$ are disjoint. Let S denote their union.

$$\nu(S') = \sum_{k\in\mathbb{N}} \nu(S_k^{'})$$

$$\nu(S') > \sum_{k\in\mathbb{N}} \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\} - \varepsilon$$

$$\sup\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\} \geq \ \nu(S')$$

$$\sup\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\} > \sum_{k\in\mathbb{N}} \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\} - \varepsilon$$

$$\sup\{\nu(S') : S' \in \mathcal{A}, S' \subseteq S\} \geq \sum_{k\in\mathbb{N}} \sup\{\nu(S_k^{'}) : S_k^{'} \in \mathcal{A}, S_k^{'} \subseteq S_k\}$$

$$\overline{V}(v, S) \geq \textstyle\sum_{k\in\mathbb{N}} \overline{V}(v, S_k) \#(**)$$

■

4.2 Hahn Decomposition

(1) (Hahn Decomposition) Let $\mathcal{M}$ be a σ-algebra on X and let ν be a signed measure on $\mathcal{M}$. Then there exist sets P and N in $\mathcal{M}$ such that

   (a) $P \cup N = X$ and $P \cap N = \emptyset$.

   (b) For any set S with $S \in \mathcal{M}$ and $S \subseteq P$, we have $\nu(S) \geq 0$.

   (c) For any set S with $S \in \mathcal{M}$ and $S \subseteq N$, we have $\nu(S) \leq 0$.

Definition. *(Hahn Decomposition) We call the set P a positive set for ν, the set N a negative set for ν, and the set pair (P, N) a Hahn decomposition for ν.*

4.3 Jordan Decomposition

$$\nu(S) = \overline{V}(\nu, S) + (\nu, S)$$

Chapter 5

Outer Measures

Definition. *(Outer Measure) Let $\mathcal{P}(X)$ be the set of all subsets of X. We define an **outer measure** on X, denoted by μ^*, to be the set function on $\mathcal{P}$ that satisfies*

(1) μ^* is non-negative.

(2) $\mu^*(\emptyset) = 0$.

(3) μ^* is monotone.

(4) μ^* is countably sub-additive.

Definition. *(Lebesgue Outer Measure) Let A be a subset of $\mathbb{R}^n$ and $\mathcal{C}_A$ be the set of all sequences $\{R_k\}$ of bounded open n-dimensional intervals such that $A \subseteq \bigcup_{k\in\mathbb{N}} R_k$. We define the **Lebesgue outer measure** of A, denoted by $\lambda^*(A)$, by*

$$\lambda^*(A) = \inf\{\sum_{k\in\mathbb{N}} \operatorname{vol}(R_k) : \{R_k\} \in \mathcal{C}_A\}$$

Proposition 5.0.1. *The Lebesgue outer measure is an outer measure.*

Proof.
Proof Part (1)
Since each of $\operatorname{vol}(R_k)$ is non-negative, $\lambda^*(A)$ is non-negative.
Proof Part (2)
Let $\varepsilon > 0$ be arbitrary.
Construct a sequence $\{R_k\}$ of bounded open n-dimensional intervals by

$$R_k = \{(x_1, x_2, \ldots, x_n) : 0 < x_j < \sqrt[n]{\frac{\varepsilon}{2^k}} \ (j = 1, 2, \ldots, n)\}$$

Then $\emptyset \subseteq \bigcup_{k\in\mathbb{N}} R_k$ and we have

$$\sum_{k\in\mathbb{N}} \mathrm{vol}(R_k) = \sum_{k\in\mathbb{N}} \frac{\varepsilon}{2^k} = \varepsilon$$

By definition of infimum, we get $\lambda^*(\emptyset) = 0$.

Proof Part (3): Monotonicity

Let A and B be arbitrary subsets of $\mathbb{R}^n$ with $A \subseteq B$.

Then every sequence of open n-dimensional intervals that covers B also covers A.

It follows that $\mathcal{C}_B \subseteq \mathcal{C}_A$.

Then

$$\{\sum_{k\in\mathbb{N}} \mathrm{vol}(R_k) : \{R_k\} \subseteq \mathcal{C}_B\} \subseteq \{\sum_{k\in\mathbb{N}} \mathrm{vol}(R_k) : \{R_k\} \subseteq \mathcal{C}_A\}$$

Then

$$\lambda^*(A) = \inf\{\sum_{k\in\mathbb{N}} \mathrm{vol}(R_k) : \{R_k\} \subseteq \mathcal{C}_A\} \leq \lambda^*(B) = \{\sum_{k\in\mathbb{N}} \mathrm{vol}(R_k) : \{R_k\} \subseteq \mathcal{C}_B\}$$

Proof Part (4): Countable Sub-additivity

Let $\{A_k\}$ be an arbitrary sequence of subsets of $\mathbb{R}^n$ and let $\varepsilon > 0$ be arbitrary.

For each A_k, construct a sequence of open n-dimensional intervals $\{R_{k,j}\}_{j=1}^{\infty}$ that covers A_k and that

$$\lambda^*(A_k) \leq \sum_{j=1}^{\infty} \mathrm{vol}(R_{k,j}) < \lambda^*(A_k) + \frac{\varepsilon}{2^k}$$

Consider the union of the sequences $\{R_{k,j}\}_{k,j}$. Then it covers $\bigcup_{k\in\mathbb{N}} A_k$ and we have

$$\sum_{k,j} \mathrm{vol}(R_{k,j}) < \sum_{k\in\mathbb{N}} \lambda^*(A_k) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\sum_{k,j} \mathrm{vol}(R_{k,j}) \leq \sum_{k\in\mathbb{N}} \lambda^*(A_k)$$

By the definition of infimum, we get

$$\inf\{\sum_{k\in\mathbb{N}} \mathrm{vol}(R_k) : \{R_k\}\in \mathcal{C}(\bigcup_{k\in\mathbb{N}} A_k)\} \leq \sum_{k,j} \mathrm{vol}(R_{k,j})$$

It follows that

$$\lambda^*(\bigcup_{k\in\mathbb{N}} A_k) \leq \sum_{k\in\mathbb{N}} \lambda^*(A_k)$$

■

(1) The Lebesgue outer measure on $\mathbb{R}^n$ assigns to each n-dimensional interval its volume.

Definition (Outer Measurable Sets)**.** *Let μ^* be an outer measure on X. Let M be a subset of X. We say that M is measurable with respect to μ^* if for any subset A of X, we have*

$$\mu^*(A) = \mu^*(A\cap M) + \mu^*(A\cap M^c)$$

Proposition 5.0.2. *Let μ^* be an outer measure on X. Let S be a subset of X. Then S is μ^*-measurable if either $\mu^*(S) = 0$ or $\mu^*(S^c) = 0$.*

Proof.
Let A be an arbitrary subset of X.
Note that $A = (A\cap S)\cup(A\cap S^c)$.
By the sub-additivity of μ^*, we get

$$\mu^*(A) \leq \mu^*(A\cap S) + \mu^*(A\cap S^c)\#(*)$$

By the monotonicity of μ^*, we get

$$\mu^*(A\cap S) \leq \min\{\mu^*(A), \mu^*(S)\}$$

$$\mu^*(A\cap S^c) \leq \min\{\mu^*(A), \mu^*(S^c)\}$$

Adding both sides gives

$$\mu^*(A\cap S) + \mu^*(A\cap S^c) \leq \min\{\mu^*(A), \mu^*(S)\} + \min\{\mu^*(A), \mu^*(S^c)\}$$

Rearranging gives

$$\mu^*(A \cap S) + \mu^*(A \cap S^c) \leq \mu^*(A) + \min\{\mu^*(S), \mu^*(S^c)\}$$

By assumption, we get $\min\{\mu^*(S), \mu^*(S^c)\} = 0$. Substituting gives

$$\mu^*(A \cap S) + \mu^*(A \cap S^c) \leq \mu^*(A)\#(**)$$

From inequations (*) and (**), we get

$$\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^c)$$

By definition of outer measurable, S is μ^*-measurable.

■

(1) The sets $\emptyset$ and X are outer measurable for any outer measure on X.

(2) Let μ^* be an outer measure on X and S be a subset of X. If S is μ^*-measurable, then S^c is also μ^*-measurable.

(1) Let μ^* be an outer measure on X and let $\mathcal{M}_{\mu^*}$ be the set of all μ^*-measurable subsets of X. Then

  (a) $\mathcal{M}_{\mu^*}$ is a σ-algebra.

  (b) The restriction of μ^* to $\mathcal{M}_{\mu^*}$ is a measure on $\mathcal{M}_{\mu^*}$.

Proof of (1)
By Proposition 4.4, the sets $\emptyset$ and X are in $\mathcal{M}_{\mu}$.
By Proposition 4.5, $\mathcal{M}_{\mu^*}$ is closed under complement.

Definition (Lebesgue Measurable Sets)**.** *We define the **Lebesgue measurable sets** to be the Lebesgue outer measurable subsets of $\mathbb{R}^n$.*

Definition (Lebesgue Measure on $(\mathbb{R}^n, \mathcal{M}_{\lambda^*})$)**.** *We define the **Lebesgue measure** on $(\mathbb{R}^n, \mathcal{M}_{\lambda^*})$, denoted by λ_n, to be the Lebesgue outer measure on $\mathbb{R}^n$, restricted to the set of Lebesgue measurable subsets of $\mathbb{R}^n$.*

Proposition 5.0.3. *A subset S of $\mathbb{R}$ is Lebesgue measurable if and only if for any open subinterval I if $\mathbb{R}$, we have $\lambda^*(I) = \lambda^*(I \cap S) + \lambda^*(I \cap S^c)$.*

Proposition 5.0.4. *Borel subsets of $\mathbb{R}^n$ are Lebesgue measurable.*

Definition (Lebesgue Measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$)**.** *We define the* **Lebesgue measure** *on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, denoted also by λ_n, to be the Lebesgue outer measure on $\mathbb{R}^n$, restricted to Borel subsets of $\mathbb{R}^n$.*

Proposition 5.0.5. *Let μ be a finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let F_μ be a function $F_\mu \colon \mathbb{R} \to \mathbb{R}$ defined by $F_\mu(x) = \mu((-\infty, x))$. Then F_μ is bounded, non-decreasing, and right-continuous and satisfies $F_\mu(x) = 0$.*

Proposition 5.0.6. *For any bounded, non-decreasing, and right-continuous function $F \colon \mathbb{R} \to \mathbb{R}$ that satisfies $F(x) = 0$, there exists a unique finite measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $F(x) = \mu((-\infty, x))$.*

Proposition 5.0.7. *Let μ be a finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and F_μ be a function from $\mathbb{R}$ to $\mathbb{R}$ defined by $F_\mu(x) = \mu((-\infty, x))$. Then we have the followings.*

(1) $\mu((-\infty, c)) = F_\mu(c^-)$

(2) $\mu(\{c\}) = F_\mu(c) - F_\mu(c^-)$

(3) $\mu((a, b)) = F_\mu(b) - F_\mu(a)$

(4) $\mu((a, b)) = F_\mu(b^-) - F_\mu(a^-)$

(5) $\mu((a, b)) = F_\mu(b^-) - F_\mu(a)$

(6) $\mu((a, b)) = F_\mu(b) - F_\mu(a^-)$

Chapter 6

Lebesgue Measure

6.1 Lebesgue Measure on the Line

$$\lambda^*(S) = \inf\{\lambda(S')\}.\backslash n$$

$$\lambda_*(S) = \sup\{\lambda(S')\}.\backslash n$$

Definition. *(Lebesgue Measurable, Lebesgue Measure) Let S be a subset of $\mathbb{R}$. If S is bounded, we say that S is* **Lebesgue measurable** *if $\lambda^*(S) = \lambda_*(S)$. If S is unbounded, we say that S is Lebesgue measurable if the set $S \cap I$ is measurable for any interval I. In this case, we define the* **Lebesgue measure** *of S, denoted by $\lambda(S)$, to be the common number.*

Proposition 6.1.1.

$$\lambda_*(S) + \lambda^*((a,b) - S) = b - a.$$

Proposition 6.1.2 (Monotonicity)**.** *Both the Lebesgue outer measure and the Lebesgue inner measure are monotonic.*

Proposition 6.1.3 (Translation Invariant)**.** *Both the Lebesgue outer measure and the Lebesgue inner measure are translation invariant.*

Proposition 6.1.4. *(1) Open subsets of $\mathbb{R}$ are Lebesgue measurable.*

(2) Closed and bounded subsets of $\mathbb{R}$ are Lebesgue measurable.

Proposition 6.1.5 (Regularity)**.** *Let A be a Lebesgue measurable subset of $\mathbb{R}^n$. Then*

(1) $\lambda(A) = \inf\{\lambda(U)\}$ *where the infimum is taken over all open sets* U *that contains* A*.*

(2) $\lambda(A) = \sup\{\lambda(K)\}$ *where the supremum is taken over all compact sets* K *that is contained in* A*.*

Proof.

Proof of (1)

By the monotonicity of Lebesgue measure, for any open set U that contains A, we have

$$\lambda(U) \geq \lambda(A)\#(1)$$

Let ε be an arbitrary positive number.
By definition of infimum, there exists a sequence $\{R_k\}$ of open n-dimensional intervals that covers A and that

$$\textstyle\sum_{k\in\mathbb{N}} \mathrm{vol}(R_k) < \lambda(A) + \varepsilon\#(2)$$

Define set U_0 to be the union of $\{R_k\}$. Then U_0 is open and U_0 contains A.
By the sub-additivity of measures, we have

$$\lambda(U_0) = \lambda(\textstyle\bigcup_{k\in\mathbb{N}} R_k) \leq \sum_{k\in\mathbb{N}} \lambda(R_k)\#(3)$$

By definition of Lebesgue measure, we have

$$\lambda(R_k) = \mathrm{vol}(R_k)\#(4)$$

From (in)equations (2) ˜ (4), we get

$$\lambda(U_0) < \lambda(A) + \varepsilon\#(5)$$

From inequations (1) and (5), by definition of infimum, we get

$$\lambda(A) = \inf\{\lambda(U)\}$$

Proof of (2)

By the monotonicity of Lebesgue measure, we get

$$\lambda(K) \leq \lambda(A)$$

∎

(1) The Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is the only measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ that assigns to each n-dimensional intervals its volume.

(2) The Lebesgue outer measure on $\mathbb{R}^n$ is translation invariant.

(3) Let μ be a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that μ is finite on bounded Borel subsets of $\mathbb{R}^n$ and is translation invariant. Then there exists a positive number c such that for any set Borel subset B of $\mathbb{R}^n$, we have $\mu(B) = c\lambda(B)$.

(1) There exists a subset of $(0, 1)$ that is not Lebesgue measurable.

Chapter 7

Completeness

Definition. *(Completeness) Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that μ is complete and that $(X, \mathcal{M}, \mu)$ is a complete measure space if for any zero-measure set S in $\mathcal{M}$, any subset S' is also in $\mathcal{M}$.*

Definition. *(Completion) Let $(X, \mathcal{M}, \mu)$ be a measure space. Define sets $\mathcal{Z}$ and $\overline{\mathcal{M}}$ by*

$$\mathcal{Z} = \{Z \in \mathcal{M} : \exists N \in \mathcal{M}, Z \subseteq N, \mu(N) = 0\}$$

$$\overline{\mathcal{M}} = \{M \cup Z : M \in \mathcal{M}, Z \in \mathcal{Z}\}$$

Then we define the **completion** *$\overline{\mu}$ of μ to be a set function on $\overline{\mathcal{M}}$ given by*

$$\overline{\mu}(M \cup Z) = \mu(M)$$

(1) (a) $\overline{\mathcal{M}}$ is a σ-algebra.

(b) $\overline{\mu}$ is a measure.

(c) $\overline{\mu}$ is complete.

Proof.

Proof of (1)

By definition of set $\mathcal{Z}$, one can easily prove that the sets $\emptyset$ and X are in $\mathcal{Z}$.
By definition of measures, the sets $\emptyset$ and X are also in $\mathcal{M}$.
It follows from the definition of set $\overline{\mathcal{M}}$ that the sets $\emptyset$ and X are in $\overline{\mathcal{M}}$.
Let S be an arbitrary set in $\overline{\mathcal{M}}$. Then S can be written as $S = M \cup Z$.

■

Chapter 8

Measurable Functions

Definition. *(Measurable Functions) Let X be a non-empty set and f be a function $f : X \to \mathbb{R}^*$. We say that f is measurable if it satisfies any of the 4 equivalent conditions listed below.*

(1) $f^{-1}((-\infty, c))$ is a measurable set for any real number c.

(2) $f^{-1}((-\infty, c))$ is a measurable set for any real number c.

(3) $f^{-1}((c, +\infty))$ is a measurable set for any real number c.

(4) $f^{-1}((c, +\infty))$ is a measurable set for any real number c.

Notations For a real-valued function f, we define

$$f^+(x) = \max\{f(x), 0\}$$

$$f^-(x) = -\min\{f(x), 0\}$$

$$|f|(x) = |f(x)|$$

(1) (a) Constant functions are measurable.

(b) If f is measurable, then the inverse image of any interval is measurable.

(c) If f is measurable, then the inverse image of any open subset of $\mathbb{R}^*$ is measurable.

(2) If f is measurable, then the functions f^+, f^-, and $|f|$ are measurable.

(3) Let f be a measurable function. Then af is measurable for any real number a.

(4) Let f and g be measurable functions. Then $f + g$ is measurable provided that the sum $f(x) + g(x)$ is everywhere defined.

(5) (Measurability of Products) Let f and g be measurable functions. Then fg is measurable.

(6) Let $\{f_k\}$ be a sequence of measurable functions. Then the functions listed below are all measurable.

(a)