**Example 0.1.** *Suppose that $W$, $X$, and $Y$ are independent continuous random variables on $(0, +\infty)$. If $Z = X \mid (X < Y)$, then show that $(W, X) \mid (W < X < Y)$ and $(W, Z) \mid (W < Z)$ are identically distributed.*

*Proof.* To show that the two random variables have the same distribution, it suffices to show that they have the same CDF.

Part 1.

Let us first consider the CDF $G(w, x)$ of $(W, X) \mid (W < X < Y)$. For $w, x \geq 0$,

$$\begin{aligned} G(w,x) &= \mathbb{P}(W \leq w, X \leq x \mid W < X < Y) \\ &= \frac{\mathbb{P}(W \leq w, X \leq x, W < X < Y)}{\mathbb{P}(W < X < Y)} \\ &= \frac{\mathbb{P}(W \leq w, X \leq x, W < X, X < Y)}{\mathbb{P}(W < X, X < Y)}. \end{aligned}$$

Conditioning on the random variable $X$ and noting that $W$, $X$, and $Y$ are independent random variables, we get

$$\begin{aligned} \mathbb{P}(W \leq w, X \leq x, W < X, X < Y) &= \int_0^{+\infty} \mathbb{P}(W \leq w, X \leq x, W < X, X < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq x, W < s, s < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq x, W < s, s < Y) f_X(s) ds \\ &= \int_0^{x} \mathbb{P}(W \leq w, W < s, s < Y) f_X(s) ds \\ &= \int_0^{x} \mathbb{P}(W \leq \min\{w, s\}, s < Y) f_X(s) ds \\ &= \int_0^{x} \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(W < X, X < Y) &= \int_0^{+\infty} \mathbb{P}(W < X, X < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W < s, s < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W < s, s < Y) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds. \end{aligned}$$

So

$$G(w,x) = \int_0^{x} \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds \Big/ \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds.$$

Part 2.

To get the CDF of $(W, Z) \mid (W < Z)$ we first need the distribution of $Z$. The CDF of $Z$ is calculated by:

$$\begin{aligned}
\mathbb{P}(Z \leq z) &= \mathbb{P}(X \leq z \mid X < Y) \\
&= \frac{\mathbb{P}(X \leq z, X < Y)}{\mathbb{P}(X < Y)} \\
&= \frac{1}{\mathbb{P}(X < Y)} \int_0^{+\infty} \mathbb{P}(X \leq z, X < Y \mid X = s) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X < Y)} \int_0^{+\infty} \mathbb{P}(s \leq z, s < Y \mid X = s) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X < Y)} \int_0^{+\infty} \mathbb{P}(s \leq z, s < Y) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X < Y)} \int_0^{z} \mathbb{P}(s < Y) f_X(s) ds.
\end{aligned}$$

and so the PDF of $Z$ is given by

$$\begin{aligned}
h_Z(z) &= \frac{d}{dz} \mathbb{P}(Z \leq z) \\
&= \frac{d}{dz} \frac{1}{\mathbb{P}(X < Y)} \int_0^{z} \mathbb{P}(s < Y) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X < Y)} \frac{d}{dz} \int_0^{z} \mathbb{P}(s < Y) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X < Y)} \mathbb{P}(z < Y) f_X(z).
\end{aligned}$$

<u>Part 3.</u>

Now the joint conditional CDF of $(W, Z) \mid (W < Z)$ is given by

$$H(w, z) = \mathbb{P}(W \leq w, Z \leq z \mid W < Z) = \frac{\mathbb{P}(W \leq w, Z \leq z, W < Z)}{\mathbb{P}(W < Z)}.$$

Due to the independence of $W$ with $X$ and $Y$, we get the numerator is:

$$\begin{aligned}
\mathbb{P}(W \leq w, Z \leq z, W < Z) &= \int_0^{+\infty} \mathbb{P}(W \leq w, Z \leq z, W < Z \mid Z = s) h_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq z, W < s \mid Z = s) h_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq z, W < s) h_Z(s) ds \\
&= \int_0^{z} \mathbb{P}(W \leq w, W < s) h_Z(s) ds \\
&= \int_0^{z} \mathbb{P}(W \leq \min\{w, s\}) \frac{1}{\mathbb{P}(X < Y)} \mathbb{P}(s < Y) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X < Y)} \int_0^{z} \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds
\end{aligned}$$

and the denominator is:

$$\begin{aligned}
\mathbb{P}(W < Z) &= \int_0^{+\infty} \mathbb{P}(W < Z \mid Z = s) f_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W < s \mid Z = s) f_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W < s) f_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W < s) \frac{1}{\mathbb{P}(X < Y)} \mathbb{P}(s < Y) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X < Y)} \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds.
\end{aligned}$$

So

$$H(w, z) = \int_0^{z} \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds \Big/ \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds.$$

Part 4.

Notice $G(w, x)$ and $H(w, z)$ are identical. This implies that $(W, X) \mid (W < X < Y) \sim (W, Z) \mid (W < Z)$. ■