

STAT 333 Stochastic Processes 1

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Chapter 1

Review of Elementary Probability

Chapter 2

Conditional Distributions and Conditional Expectation

Example 2.1. Suppose that W , X , and Y are independent continuous random variables on $(0, +\infty)$. If $Z = X \mid (X < Y)$, then show that $(W, X) \mid (W < X < Y)$ and $(W, Z) \mid (W < Z)$ are identically distributed.

Proof. To show that the two random variables have the same distribution, it suffices to show that they have the same CDF.

Part 1.

Let us first consider the CDF $G(w, x)$ of $(W, X) \mid (W < X < Y)$. For $w, x \geq 0$,

$$\begin{aligned} G(w, x) &= \mathbb{P}(W \leq w, X \leq x \mid W < X < Y) \\ &= \frac{\mathbb{P}(W \leq w, X \leq x, W < X < Y)}{\mathbb{P}(W < X < Y)} \\ &= \frac{\mathbb{P}(W \leq w, X \leq x, W < X, X < Y)}{\mathbb{P}(W < X, X < Y)}. \end{aligned}$$

Conditioning on the random variable X and noting that W , X , and Y are independent random variables, we get

$$\begin{aligned} \mathbb{P}(W \leq w, X \leq x, W < X, X < Y) &= \int_0^{+\infty} \mathbb{P}(W \leq w, X \leq x, W < X, X < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq x, W < s, s < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq x, W < s, s < Y) f_X(s) ds \\ &= \int_0^x \mathbb{P}(W \leq w, W < s, s < Y) f_X(s) ds \\ &= \int_0^x \mathbb{P}(W \leq \min\{w, s\}, s < Y) f_X(s) ds \\ &= \int_0^x \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(W < X, X < Y) &= \int_0^{+\infty} \mathbb{P}(W < X, X < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W < s, s < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W < s, s < Y) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds. \end{aligned}$$

So

$$G(w, x) = \int_0^x \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds \bigg/ \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds.$$

Part 2.

To get the CDF of $(W, Z) \mid (W < Z)$ we first need the distribution of Z . The CDF of Z is calculated by:

$$\begin{aligned}
 \mathbb{P}(Z \leq z) &= \mathbb{P}(X \leq z \mid X < Y) \\
 &= \frac{\mathbb{P}(X \leq z, X < Y)}{\mathbb{P}(X < Y)} \\
 &= \frac{1}{\mathbb{P}(X < Y)} \int_0^{+\infty} \mathbb{P}(X \leq z, X < Y \mid X = s) f_X(s) ds \\
 &= \frac{1}{\mathbb{P}(X < Y)} \int_0^{+\infty} \mathbb{P}(s \leq z, s < Y \mid X = s) f_X(s) ds \\
 &= \frac{1}{\mathbb{P}(X < Y)} \int_0^{+\infty} \mathbb{P}(s \leq z, s < Y) f_X(s) ds \\
 &= \frac{1}{\mathbb{P}(X < Y)} \int_0^z \mathbb{P}(s < Y) f_X(s) ds.
 \end{aligned}$$

and so the PDF of Z is given by

$$\begin{aligned}
 h_Z(z) &= \frac{d}{dz} \mathbb{P}(Z \leq z) \\
 &= \frac{d}{dz} \frac{1}{\mathbb{P}(X < Y)} \int_0^z \mathbb{P}(s < Y) f_X(s) ds \\
 &= \frac{1}{\mathbb{P}(X < Y)} \frac{d}{dz} \int_0^z \mathbb{P}(s < Y) f_X(s) ds \\
 &= \frac{1}{\mathbb{P}(X < Y)} \mathbb{P}(z < Y) f_X(z).
 \end{aligned}$$

Part 3.

Now the joint conditional CDF of $(W, Z) \mid (W < Z)$ is given by

$$H(w, z) = \mathbb{P}(W \leq w, Z \leq z \mid W < Z) = \frac{\mathbb{P}(W \leq w, Z \leq z, W < Z)}{\mathbb{P}(W < Z)}.$$

Due to the independence of W with X and Y , we get the numerator is:

$$\begin{aligned}
 \mathbb{P}(W \leq w, Z \leq z, W < Z) &= \int_0^{+\infty} \mathbb{P}(W \leq w, Z \leq z, W < Z \mid Z = s) h_Z(s) ds \\
 &= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq z, W < s \mid Z = s) h_Z(s) ds \\
 &= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq z, W < s) h_Z(s) ds \\
 &= \int_0^z \mathbb{P}(W \leq w, W < s) h_Z(s) ds \\
 &= \int_0^z \mathbb{P}(W \leq \min\{w, s\}) \frac{1}{\mathbb{P}(X < Y)} \mathbb{P}(s < Y) f_X(s) ds \\
 &= \frac{1}{\mathbb{P}(X < Y)} \int_0^z \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds
 \end{aligned}$$

and the denominator is:

$$\begin{aligned}
 \mathbb{P}(W < Z) &= \int_0^{+\infty} \mathbb{P}(W < Z \mid Z = s) f_Z(s) ds \\
 &= \int_0^{+\infty} \mathbb{P}(W < s \mid Z = s) f_Z(s) ds \\
 &= \int_0^{+\infty} \mathbb{P}(W < s) f_Z(s) ds \\
 &= \int_0^{+\infty} \mathbb{P}(W < s) \frac{1}{\mathbb{P}(X < Y)} \mathbb{P}(s < Y) f_X(s) ds \\
 &= \frac{1}{\mathbb{P}(X < Y)} \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds.
 \end{aligned}$$

So

$$H(w, z) = \int_0^z \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds \Big/ \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds.$$

Part 4.

Notice $G(w, x)$ and $H(w, z)$ are identical. This implies that $(W, X) \mid (W < X < Y) \sim (W, Z) \mid (W < Z)$. ■

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