STAT 333 Stochastic Processes 1

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Conditional Distributions and Conditional Expectation

Example 2.1. *Suppose that W, X, and Y are independent continuous random variables on $(0, +\infty)$. If $Z = X \mid (X < Y)$, then show that $(W, X) \mid (W < X < Y)$ and $(W, Z) \mid (W < Z)$ are identically distributed.*

Proof. To show that the two random variables have the same distribution, it suffices to show that they have the same CDF.

Part 1.

Let us first consider the CDF $G(w, x)$ of $(W, X) \mid (W < X < Y)$. For $w, x \geq 0$,

$$\begin{aligned} G(w, x) &= \mathbb{P}(W \leq w, X \leq x \mid W < X < Y) \\ &= \frac{\mathbb{P}(W \leq w, X \leq x, W < X < Y)}{\mathbb{P}(W < X < Y)} \\ &= \frac{\mathbb{P}(W \leq w, X \leq x, W < X, X < Y)}{\mathbb{P}(W < X, X < Y)}. \end{aligned}$$

Conditioning on the random variable X and noting that W, X, and Y are independent random variables, we get

$$\begin{aligned} \mathbb{P}(W \leq w, X \leq x, W < X, X < Y) &= \int_0^{+\infty} \mathbb{P}(W \leq w, X \leq x, W < X, X < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq x, W < s, s < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W \leq w, s \leq x, W < s, s < Y) f_X(s) ds \\ &= \int_0^{x} \mathbb{P}(W \leq w, W < s, s < Y) f_X(s) ds \\ &= \int_0^{x} \mathbb{P}(W \leq \min\{w, s\}, s < Y) f_X(s) ds \\ &= \int_0^{x} \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(W < X, X < Y) &= \int_0^{+\infty} \mathbb{P}(W < X, X < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W < s, s < Y \mid X = s) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W < s, s < Y) f_X(s) ds \\ &= \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds. \end{aligned}$$

So

$$G(w, x) = \int_0^{x} \mathbb{P}(W \leq \min\{w, s\}) \mathbb{P}(s < Y) f_X(s) ds \Bigg/ \int_0^{+\infty} \mathbb{P}(W < s) \mathbb{P}(s < Y) f_X(s) ds.$$

Part 2.

To get the CDF of $(W, Z) \mid (W < Z)$ we first need the distribution of Z. The CDF of Z is calculated by:

$$\begin{aligned} \mathbb{P}(Z \leq z) &= \mathbb{P}(X \leq z \mid X < Y) \\ &= \frac{\mathbb{P}(X \leq z, X < Y)}{\mathbb{P}(X < Y)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mathbb{P}(X<Y)} \int_0^{+\infty} \mathbb{P}(X \le z, X < Y \mid X = s) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X<Y)} \int_0^{+\infty} \mathbb{P}(s \le z, s < Y \mid X = s) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X<Y)} \int_0^{+\infty} \mathbb{P}(s \le z, s < Y) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X<Y)} \int_0^{z} \mathbb{P}(s < Y) f_X(s) ds.
\end{aligned}
$$

and so the PDF of Z is given by

$$
\begin{aligned}
h_Z(z) &= \frac{d}{dz}\mathbb{P}(Z \le z) \\
&= \frac{d}{dz}\frac{1}{\mathbb{P}(X<Y)} \int_0^{z} \mathbb{P}(s < Y) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X<Y)} \frac{d}{dz} \int_0^{z} \mathbb{P}(s < Y) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X<Y)} \mathbb{P}(z < Y) f_X(z).
\end{aligned}
$$

Part 3.

Now the joint conditional CDF of $(W, Z) \mid (W < Z)$ is given by

$$
H(w,z) = \mathbb{P}(W \le w, Z \le z \mid W < Z) = \frac{\mathbb{P}(W \le w, Z \le z, W < Z)}{\mathbb{P}(W < Z)}.
$$

Due to the independence of W with X and Y, we get the numerator is:

$$
\begin{aligned}
\mathbb{P}(W \le w, Z \le z, W < Z) &= \int_0^{+\infty} \mathbb{P}(W \le w, Z \le z, W < Z \mid Z = s) h_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W \le w, s \le z, W < s \mid Z = s) h_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W \le w, s \le z, W < s) h_Z(s) ds \\
&= \int_0^{z} \mathbb{P}(W \le w, W < s) h_Z(s) ds \\
&= \int_0^{z} \mathbb{P}(W \le \min\{w,s\}) \frac{1}{\mathbb{P}(X<Y)} \mathbb{P}(s < Y) f_X(s) ds \\
&= \frac{1}{\mathbb{P}(X<Y)} \int_0^{z} \mathbb{P}(W \le \min\{w,s\}) \mathbb{P}(s < Y) f_X(s) ds
\end{aligned}
$$

and the denominator is:

$$
\begin{aligned}
\mathbb{P}(W < Z) &= \int_0^{+\infty} \mathbb{P}(W < Z \mid Z = s) f_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W < s \mid Z = s) f_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W < s) f_Z(s) ds \\
&= \int_0^{+\infty} \mathbb{P}(W < s) \frac{1}{\mathbb{P}(X<Y)} \mathbb{P}(s < Y) f_X(s) ds
\end{aligned}
$$

$$= \frac{1}{\mathbb{P}(X < Y)} \int_0^{+\infty} \mathbb{P}(W < s)\mathbb{P}(s < Y) f_X(s) ds.$$

So

$$H(w, z) = \int_0^z \mathbb{P}(W \leq \min\{w, s\})\mathbb{P}(s < Y) f_X(s) ds \Big/ \int_0^{+\infty} \mathbb{P}(W < s)\mathbb{P}(s < Y) f_X(s) ds.$$

Part 4.

Notice $G(w, x)$ and $H(w, z)$ are identical. This implies that $(W, X) \mid (W < X < Y) \sim (W, Z) \mid (W < Z)$. ■

Example 2.2. *Consider an experiment in which independent trials, each having success probability $p \in (0, 1)$, are performed until k consecutive successes are achieved where $k \in \mathbb{Z}^+$. Determine the expected number of trials needed to achieve k consecutive successes.*

Proof. Let N_k represent the number of trials needed to get k consecutive successes. We wish to determine $\mathbb{E}[N_k]$. For $k = 1$, note that $N_1 \sim \text{GEO}(p)$. So $\mathbb{E}[N_k] = \frac{1}{p}$. For arbitrary $k \geq 2$, let us consider conditioning on the outcome of the first trial, represented by W, such that

$$W = \begin{cases} 0, & \text{if the first trial is a failure} \\ 1, & \text{if the first trail is a success.} \end{cases}$$

Thus,

$$\begin{aligned} \mathbb{E}[N_k] &= \mathbb{E}[\mathbb{E}[N_k \mid W]] \\ &= P(W = 0)\mathbb{E}[N_k \mid W = 0] + P(W = 1)\mathbb{E}[N_k \mid W = 1] \\ &= (1 - p)\mathbb{E}[N_k \mid W = 0] + p\mathbb{E}[N_k \mid W = 1]. \end{aligned}$$

Now, it is clear $N_k \mid W = 0 \sim 1 + N_k$. But unfortunately, we do not have a nice corresponding result for $N_k \mid W = 1$. It does not hold true that $N_k \mid W = 1 \sim 1 + N_{k-1}$.

Idea: Let's try: $\mathbb{E}[N_k] = \mathbb{E}[\mathbb{E}[N_k \mid N_{k-1}]]$. Define a random variable $Y \mid N_{k-1} = n$ as

$$Y \mid N_{k-1} = n := \begin{cases} 0, & \text{if the } (n+1)^{\text{th}} \text{ trial is a failure} \\ 1, & \text{if the } (n+1)^{\text{th}} \text{ trail is a success.} \end{cases}$$

By independence of the trials, we have $P(Y = 0 \mid N_{k-1} = n) = 1 - p$ and $P(Y = 1 \mid N_{k-1} = n) = p$. As a result, we get

$$\begin{aligned} \mathbb{E}[N_k] &= \mathbb{E}[\mathbb{E}[N_k \mid N_{k-1}]] \\ &= \sum_y \mathbb{E}[N_k \mid N_{k-1} = n, Y = y] P(Y = y \mid N_{k-1} = n) \\ &= (1 - p) \cdot \mathbb{E}[N_k \mid N_{k-1} = n, Y = 0] + p \cdot \mathbb{E}[N_k \mid N_{k-1} = n, Y = 1]. \end{aligned}$$

Note that $(N_k \mid N_{k-1} = n, Y = 0) \sim n + 1 + N_k$ and $(N_k \mid N_{k-1} = n, Y = 1) = n + 1$. Therefore,

$$\begin{aligned} \mathbb{E}[N_k \mid N_{k-1} = n] &= (1 - p)(n + 1 + \mathbb{E}[N_k]) + p(n + 1) \\ &= n + 1 + (1 - p)\mathbb{E}[N_k]. \end{aligned}$$

So $\mathbb{E}[N_k \mid N_{k-1}] = N_{k-1} + 1 + (1 - p)\mathbb{E}[E_k]$. Now, we have

$$\begin{aligned} \mathbb{E}[N_k] &= \mathbb{E}[N_{k-1} + 1 + (1 - p)\mathbb{E}[N_k]] \\ &= \mathbb{E}[N_{k-1}] + 1 + (1 - p)\mathbb{E}[N_k]. \end{aligned}$$

So $\mathbb{E}[N_K] = \frac{1}{p}\mathbb{E}[N_{k-1}] + \frac{1}{p}$. So $\mathbb{E}[N_k] = \frac{1}{p}\frac{1 - \frac{1}{p^k}}{1 - \frac{1}{p}} = \frac{p^{-k} - 1}{1 - p}$, for any $k \in \mathbb{N}$. ■

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