Contents

1 Preliminary topics

1.1 Group theoretical topics

1.1.1 Group action

We could let the group G act on a set X and study the action of G on X defined by

$$G \times X \to X$$

and thereby permuting the elements of X by for any $g \in G$,

$$(g, x) \mapsto g \cdot x = gx,$$

for any $x \in X$.

1.1.2 The symmetric group

Denote by $\mathcal{S}_n$ the set of bijections of $\{1, 2, \dots n\}$, which is a group under composition of bijections. The number of elements in $\mathcal{S}_n$ is $n!$.

Example 1.1. The elements of $\mathcal{S}_3$ are

$$(1), (1, 2), (1, 3), (2, 3), (123), (132).$$

There are $3! = 6$ elements: the identity element (denoted e), three transpositions and two 3-cycles.

Definition 1.2 (Sign of a permutation)**.** Let $\sigma \in \mathcal{S}_n$, then the sign of σ is defined as a function

$$\begin{aligned} \mathrm{sgn} &: \mathcal{S}_n \to \{+1, -1\}, \\ \mathrm{sgn} &: \sigma \mapsto (-1)^k \end{aligned}$$

where k is the number of transpositions required to compose σ. If k is an even integer, then σ is called even, and vice versa for an odd k.

Note that if σ is composed of s transpositions and τ is composed of t transposition, then their composition $\sigma\tau$ is composed of $s + t$ transpositions, in some representation of $\sigma\tau$. Is the sign function still well-defined? That is, can a permutation be expressed both as a composition of even number and of a odd number of transpositions?

Proposition 1.3. The sign of a permutation is well-defined. That is a permutation is either *even* with sign +1 or *odd* with sign -1.

Proof. [Big04, Thm.12.6.1.] □

Recall that two elements $\sigma, \tau \in \mathcal{S}_n$ are conjugate if there exists a $\pi \in \mathcal{S}_n$ such that $\sigma = \pi\tau\pi^{-1}$. The conjugacy classes partition $\mathcal{S}_n$ into disjoint subsets.

Proposition 1.4 (Conjugacy in $\mathcal{S}_n$)**.** Any elements $\sigma, \tau \in \mathcal{S}_n$ are conjugate if and only if they are of the same cycle type. SOURCE+PR

1.2 Linear algebra topics

1.2.1 Trace

The trace of a matrix is defined as the sum along the diagonal, ie. for a $n \times n$ matrix $X = (x_{ij})$ we have

$$\mathrm{Tr}\ X = \sum_{i=1}^{n} x_{ii}.$$

Recall that if two matrices X and Y are conjugate there exists a matrix T such that $XT = TY$, or if T is invertible $X = TYT^{-1}$. If X and Y are conjugate, then they have the same trace??????? EIGENVALUES??????

1.2.2 Definition of kernel and image of a map

Definition 1.5 (Kernel and image of a linear map)**.** Let V and W be two vector spaces and let $\varphi : V \to W$ be a linear map. Then the kernel and the image of the map are defined thusly:

i) $\ker\varphi = \{\mathbf{v} \in V \mid \varphi(\mathbf{v}) = \mathbf{0}\} = \varphi^{-1}(\mathbf{0})$.

ii) $\mathrm{im}\ \varphi = \{\mathbf{w} \in W \mid \exists \mathbf{v} \in V \text{ s.t. } \varphi(\mathbf{v}) = \mathbf{w}\} = \varphi(V)$.

Remark 1.6. The kernel and the image of a linear map are subspaces of the domain and codomain of the map respectively, ie. $\ker\varphi$ is a subspace of V and $\mathrm{im}\ \varphi$ is a subspace of W.

1.2.3 Existence of complementary subspaces

Corollary 1.7. Let V be a vector space and W be a vector subspace of V. Then there exists a complementary vector subspace W' in V such that $W \cap W' = \emptyset$ and $W \cup W' = V$. This is equivalent to saying that V is the direct sum of W and W', denoted as $V = W \oplus W'$.

Proof. La la la □

1.2.4 Tensor algebra

Definition 1.8 (Bilinearity)**.** content...

Let V and W be vector spaces provided with respective bases $(\hat{\mathbf{v}}_i)_{i=1}^m$ and $(\hat{\mathbf{u}}_i)_{i=1}^n$, where $m = \dim V$ and $n = \dim W$. Let $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Also, let $f : \mathrm{GL}(V)$ and $g : \mathrm{GL}(W)$ be linear maps. In the bases provided let the matrices of f and g be $F = (f_{ij})_{m\times m} \in \mathrm{GL}_m(\mathbb{C})$ and $G = (g_{ij})_{n\times n} \in \mathrm{GL}_n(\mathbb{C})$ with eigen values $\{\lambda_i\}$ and $\{\mu_i\}$, then the following vector spaces may be constructed, or rather extended bilinearly, from V and W:

- The *direct sum of* V *and* W, denoted $V \oplus W$.

  It has the basis $(\hat{\mathbf{v}}_i \oplus \hat{\mathbf{w}}_j)_{1\leq i\leq m}^{1\leq j\leq n}$, and is of dimension $m+n$.

  An element of $V \oplus W$ looks like

  $$\mathbf{v} \oplus \mathbf{w} = \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \Big(v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n\Big)^T.$$

  In the basis provided, $F \oplus G \in \mathrm{GL}(V \oplus W)$ is the $m+n \times m+n$ block matrix

  $$\begin{pmatrix} F & \mathbf{0} \\ \mathbf{0} & G \end{pmatrix},$$

and its action on $\mathbf{v} \oplus \mathbf{w}$ is

$$
\begin{aligned}
(F \oplus G) \cdot (\mathbf{v} \oplus \mathbf{w}) &= \begin{pmatrix} F & \mathbf{0} \\ \mathbf{0} & G \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} \\
&= \begin{pmatrix} F\mathbf{v} \\ G\mathbf{w} \end{pmatrix} \\
&= F\mathbf{v} \oplus G\mathbf{w}.
\end{aligned}
$$

The trace of $F \oplus G$ is clearly the sum of the traces of F and G, hence the eigen values of f and g are also eigen values of $f \oplus g$, that is the eigen values are $\{\lambda_i\} \cup \{\mu_j\}$.

- By recursion, the *direct sum of n copies of V*, denoted nV.

Example 1.9. The direct sum of n copies of a field $\mathbb{K}$, is usually denoted $\mathbb{K}^n$, eg. $\mathbb{R}^3$.

- The *tensor product of V and W*, denoted $V \otimes W$.

  It has the basis $(\hat{\mathbf{v}}_i \otimes \hat{\mathbf{w}}_j)_{1 \le i \le m}^{1 \le j \le n}$, and is of dimension mn.

  An element of $V \otimes W$ looks like

$$
\mathbf{v} \otimes \mathbf{w} = \Big(v_1\mathbf{w}, v_2\mathbf{w}, \ldots, v_m\mathbf{w}\Big)^T = \Big(v_i\mathbf{w}\Big)_{mn \times 1}
$$

  In the basis provided, $F \otimes G \in \mathrm{GL}(V \otimes W)$ is the block matrix

$$
(f_{ij}G)_{mn \times mn}
$$

and its action on $\mathbf{v} \oplus \mathbf{w}$ is

$$
\begin{aligned}
(F \otimes G) \cdot (\mathbf{v} \otimes \mathbf{w}) &= \Big(f_{ij}G\Big) \cdot \Big(v_i\mathbf{w}\Big) \\
&= \Big(f_{ij}Gv_i\mathbf{w}\Big) \\
&= \Big(f_{ij}v_iG\mathbf{w}\Big) \\
&= \Big(f_{ij}v_i\Big) \otimes G\mathbf{w} \\
&= F\mathbf{v} \otimes G\mathbf{w}.
\end{aligned}
$$

The trace of $F \otimes G$ is the sum of the traces of the diagonal matrices in the block matrix $(f_{ij}G)$, which is

$$f_{11}\mathrm{Tr}\ G + f_{22}\mathrm{Tr}\ G + \cdots + f_{mm}\mathrm{Tr}\ G = \mathrm{Tr}\ F \cdot \mathrm{Tr}\ G$$

hence the eigen values of $f \otimes g$ are $\{\lambda_i \cdot \mu_j\}$.

- By recursion, the *nth tensor power of* V, denoted $V^{\otimes n}$. By definition, the first tensor power is the space itself, and the zeroth power is the ground field.

- The nth tensor power of V has two subspaces, the symmetric powers $\mathrm{Sym}^n V$ and the alternating powers $\bigwedge^n V$. EEEEEH further definition?? In particular the symmetric and exterior squares are such that

$$V \otimes V = \mathrm{Sym}^2 V \oplus \bigwedge^2 V.$$

need more text/proof

- After fixing the basis $(\hat{\mathbf{e}}_i)_{i=1}^m$ for V, the dual space V^* can be constructed by the duals $\hat{\mathbf{e}}_i^*$ defined by... It is identified with the set of all linear functions from V to $\mathbb{C}$.

need more text/proof

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- Set of homomoprhisms V to W.

2 Representation Theory of Finite Groups

The main purpose of this text is to study linear representations of groups, or in other words, studying groups and identifying them with vector spaces and their elements with maps of those spaces. Only finite groups and finite-dimensional vector spaces will be discussed in this text.

Include note or source!

Let G be a finite group written multiplicatively. Denote its size by $|G|$ and the action of its elements with $\cdot$, for example by $g \cdot x$, sometimes shorted to gx.

Let V be a finite-dimensional vector space over the field of complex numbers $\mathbb{C}$. The group of invertible linear transformations of V is denoted by $\mathrm{GL}(V)$. If V is provided with a basis, usually denoted $(\hat{\mathbf{e}}_i)_{i=1}^n$ where $n = \dim V$, then $\mathrm{GL}(V)$ is identified with the set of linear transformations of V, denoted $\mathrm{GL}_n(\mathbb{C})$[DF04, 18.1].

Definition 2.1 (Representation)**.** [Ser77] A representation of a group G in the vector space V is a homomorphism

$$\rho : G \to \mathrm{GL}(V). \tag{2.1}$$

For every g in G, there is a linear map ρ_g in $\mathrm{GL}(V)$ with an action on the elements of V defined by

$$g \cdot \mathbf{v} = \rho_g(\mathbf{v}).$$

The linearity of the map means that for any $\mathbf{v}, \mathbf{w} \in V$,

$$g \cdot (a\mathbf{v} + b\mathbf{w}) = ag \cdot \mathbf{v} + bg \cdot \mathbf{w}, \tag{2.2}$$

with $a, b \in \mathbb{C}$ and also since it is a homomorphism, for any $g, h \in G$, we have

$$\rho_{gh} = \rho_g \rho_h. \tag{2.3}$$

From the homomorphism of ρ, two consequences follow.

Proposition 2.2. The homomorphism preserves identity and inverses. For the identity element e of G and an arbitrary element g with inverse g^{-1} in G we have,

$$\rho_e = \mathrm{id} \quad \text{and} \quad \rho_g^{-1} = \rho_{g^{-1}}, \tag{2.4}$$

where id is the identity transformation and $\rho_{g^{-1}}$ is the map associated with the inverse of g.

Proof. Take g as an arbitrary element of G. The first identity follows from taking $h = e$ in Equation 2.3:

$$\rho_{eg} = \rho_e \rho_g.$$

Since $eg = g$ for any $g \in G$ we must have that $\rho_g = \rho_e \rho_g$ which is true if and only if $\rho_e = \mathrm{id}$. Now instead taking $h = g^{-1}$ we have:

$$\rho_{gg^{-1}} = \rho_g \rho_{g^{-1}},$$

but $gg^{-1} = e$ and $\rho_e = \mathrm{id}$, implying $\rho_{g^{-1}} = \rho_g^{-1}$. □

Notation. A vector space V provided with such a homomorphism discussed above is said to be a *representation space* of G, allowing us to focus on the space itself[1].

Note. The homomorphism ρ, the set of maps $\{\rho_g\}_{g \in G}$ and the representation space V are interchangeably and abusively called the *representation of* G.

If a basis $(\hat{\mathbf{e}}_i)_{i=1}^n$ is provided for a representation space, then a *matrix representation* can be provided. In this case, ρ_g is the matrix in $\mathrm{GL}_n(\mathbb{C})$ associated with the action of g expressed in the basis provided. A matrix representation is not canonical and is dependent on the basis chosen.

Definition 2.3 (Degree of a representation)**.** Let V be a representation space of G. The dimension of V is referred to as the degree of the representation.

2.1 Examples

This section follows [Ser77, 1.?].

2.1.1 Trivial representations

Example 2.4 (Trivial representation). For any group G there is a trivial degree 1 representation defined by mapping each element of G to 1, ie.

$$\rho_g = 1,$$

[1]In fact, this gives V the structure of a G-module [Sag01, 1.3].

for every $g \in G$. It is clearly a homomorphism since for any $g, h \in G$ we have that

$$\rho_{gh} = 1 = 1 \cdot 1 = \rho_g \rho_h.$$

This could be extended to any vector space by mapping g to the identity transformation of that vector space.

Note. The trivial representation (the mapping is trivial) is not to be confused with the trivial space (the vector space containing only the zero vector).

Example 2.5 (Alternating representation of $\mathcal{S}_n$). Choosing $G = \mathcal{S}_n$, another degree 1 representation can be found by studying the signs, or parities, of the elements of $\mathcal{S}_n$. By Proposition 1.3, the sign of a permutation is well-defined, so for any two permutations σ and τ with respective signs $(-1)^s$ and $(-1)^t$, their composition has the sign

$$\operatorname{sgn}(\sigma\tau) = (-1)^{s+t} = (-1)^s \cdot (-1)^t = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau),$$

so clearly the map $\operatorname{sgn} : \mathcal{S}_n \to \{\pm 1\}$ is a homomorphism and thus a representation of degree 1, where even permutations are mapped to $+1$ and odd to -1.

2.1.2 Degree 1 representations of $\mathcal{C}_n$

Choose $G = \mathcal{C}_n$, and let g be a generator such that

$$\mathcal{C}_n = \{e, g, g^2, \ldots, g^{n-1}\}$$

and $g^n = e$. Consider a map $\rho : \mathcal{C}_n \to \mathbb{C}$ defined as a homomorphism by $\rho_{g^a}\rho_{g^b} = \rho_{g^{a+b}}$, then by Equation 2.4 we have that $\rho_e = 1$, but $g^n = e$ and by induction on Equation 2.3 we have that $\rho_{g^n} = (\rho_g)^n$, so then we must have that ρ_g is mapped to a nth root of unity. In conclusion, for $\mathcal{C}_n$ we have found n representations of degree 1, each mapping g to a nth root of unity and the powers of g to the corresponding powers of the root of unity.

Clarify mapping of g and g^a for different roots of unity

Example 2.6 ($\mathcal{C}_3$). The three third roots of unity are 1, $\omega = \frac{-1+i\sqrt{3}}{2}$ and $w^2 = \frac{-1-i\sqrt{3}}{2}$, so the three representations of $\mathcal{C}_3$ are presented in table 1. Note that $\rho_1^* = \rho_2$.

Example 2.7 ($\mathcal{C}_4$). The four fourth roots of unity are $1, i, -1$ and $-i$ and the four corresponding representations of $\mathcal{C}_4$ are presented in Table 2. Note that $\rho_1^* = \rho_3$.

Example 2.8 ($\mathcal{C}_5$). The five fifth roots of unity are $\exp \frac{2\pi i m}{5}$, $0 \leq m \leq 4$, so five representations of $\mathcal{C}_5$ are presented in table 3. Note that $\rho_1^* = \rho_4$ and $\rho_2^* = \rho_3$.

$\mathcal{C}_3$	e	g	g^2
ρ_0	1	1	1
ρ_1	1	ω	ω^2
ρ_2	1	ω^2	ω

Table 1: Three reprs. of $\mathcal{C}_3$. $\omega = e^{2\pi i/3}$.

$\mathcal{C}_4$	e	g	g^2	g^3
ρ_0	1	1	1	1
ρ_1	1	i	-1	$-i$
ρ_2	1	-1	1	-1
ρ_3	1	$-i$	-1	i

Table 2: Four reprs. of $\mathcal{C}_4$.

$\mathcal{C}_5$	e	g	g^2	g^3	g^4
ρ_0	1	1	1	1	1
ρ_1	1	ω^1	ω^2	ω^3	ω^4
ρ_2	1	ω^2	ω^4	ω^1	ω^3
ρ_3	1	ω^3	ω^1	ω^4	ω^2
ρ_4	1	ω^4	ω^3	ω^2	ω^1

Table 3: Five reprs. of $\mathcal{C}_5$. $\omega = e^{2\pi i/5}$.

2.1.3 Permutation representation

Given a group G, we chose a suitable set X that G in some sense "naturally" acts on by permutation. Let V be a vector space spanned by a orthonormal basis $(\hat{\mathbf{e}}_x)_{x\in X}$, then we have a homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

defined by its action on the basis vectors by, for any $g \in G$,

$$\rho_g : \hat{\mathbf{e}}_x \mapsto \hat{\mathbf{e}}_{gx},$$

that is ρ inherited the group action of G on X. It is a homomorphism since for any $g, h \in G$ we have

$$g(h \cdot \hat{\mathbf{e}}_x) = g \cdot \hat{\mathbf{e}}_{hx} = \hat{\mathbf{e}}_{ghx} = (gh) \cdot \hat{\mathbf{e}}_x$$

for any $x \in X$.

We have imposed a G-module structure on V.

Letting $G = \mathcal{S}_n$, it would be appropriate to choose the set $X = \{1, 2, \ldots, n\}$ and to let any $\sigma \in \mathcal{S}_n$ permute any $1 \leq i \leq n$ by $i \mapsto \sigma(i)$. Choosing a basis $(\hat{\mathbf{e}}_i)_{i=1}^n$ to span a vector space V, we define a homomorphism

$$\rho : \mathcal{S}_n \to \mathrm{GL}(V)$$

defined for any $\sigma \in \mathcal{S}_n$ as

$$\rho_\sigma : \hat{\mathbf{e}}_i \mapsto \hat{\mathbf{e}}_{\sigma(i)}$$

for any $1 \leq i \leq n$. In this basis, the corresponding set of matrix representations are the permutation matrices

$$\rho_\sigma = (r_{ij})_{n\times n}, \text{ where } r_{ij} = \delta_{j,\sigma(i)} = \begin{cases} 1, \text{ if } j = \sigma(i), \\ 0, \text{ otherwise,} \end{cases}$$

in which the ith column has a 1 in the $\sigma(i)$th row, and the rest of the rows have a 0.

Under this action, a vector

$$(a_1, a_2, \ldots, a_n) = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + \cdots + a_n\hat{\mathbf{e}}_n \in V$$

is permuted to

$$(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(n)}) = a_1\hat{\mathbf{e}}_{\sigma(1)} + a_2\hat{\mathbf{e}}_{\sigma(2)} + \cdots + a_n\hat{\mathbf{e}}_{\sigma(n)} \in V.$$

The dimension of V is $|X|$, for example the permutation representation of $\mathcal{S}_n$ is of degree n.

Yet another interpretation i that σ permute the n coordina axes.

Example 2.9 (Permutation representation of $\mathcal{S}_2$). The symmetric group of degree 2 has two elements, $\mathcal{S}_2 = \{(1), (1, 2)\}$ and their matrix representations in $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$-space are presented in Table 4.

$$\rho_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_{(1,2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Table 4: Matrix representations of $\mathcal{S}_2$

Example 2.10 (Permutation representation of $\mathcal{S}_3$). Likewise, representations of $\mathcal{S}_3$ are presented in Table 5.

Example 2.11 (Permutation representation of $\mathcal{S}_4$). Yet again, matrix representations of some elements of $\mathcal{S}_4$ are presented in Table 6. The elements chosen are one representative from every conjugacy class of $\mathcal{S}_4$.

$$\rho_{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_{(1,2,3)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_{(1,3,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\rho_{(1,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_{(1,3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho_{(2,3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Table 5: Matrix representations of $\mathcal{S}_3$

$$rho_{(1)} = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right), \quad \rho_{(1,2)} = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right), \quad \rho_{(1,2)(3,4)} = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right),$$

$$\rho_{(1,2,3)} = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right), \quad \rho_{(1,2,3,4)} = \left(\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right).$$

Table 6: Some matrix representations of $\mathcal{S}_4$

2.1.4 Regular representation

Following the same reasoning as in the previous section, but instead, let G act on itself. The corresponding vector space V is spanned by the basis $(\hat{\mathbf{e}}_g)_{g \in G}$ constructed from the elements of G. A representation of G in V is then a map

$$\rho : G \to \mathrm{GL}(V)$$

defined by

$$g \cdot \hat{\mathbf{e}}_h = \hat{\mathbf{e}}_{gh}$$

We have created a Group Algebra $\mathbb{C}[G]$.

for any $g, h \in G$.

The dimension of V is $|G|$, for example the regular representation of $\mathcal{C}_n$ is of degree n and for $\mathcal{S}_n$ it is of degree $n!$, a number which grows increasingly quick for larger n, however the regular representation will be shown to be important later.

Example 2.12 (Regular representation of $\mathcal{C}_3$). content...

Example 2.13 (Regular representation of $\mathcal{S}_3$). Defining a homomorphism $\rho : S_3 \to \mathrm{GL}(V)$ on the vector space spanned by the basis[2] $(\hat{\mathbf{e}}_\sigma)_{\sigma \in \mathcal{S}_3}$ defined for every σ by $\rho(\sigma) \cdot \hat{\mathbf{e}}_\tau = \hat{\mathbf{e}}_{\sigma \circ \tau} = \hat{\mathbf{e}}_{\sigma\tau}$ for some τ, we arrive at a degree 6 representation of $\mathcal{S}_3$.

Table of 6×6-*matrices :-(*

[2]Explicitly, this is $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_{12}, \hat{\mathbf{e}}_{13}, \hat{\mathbf{e}}_{23}, \hat{\mathbf{e}}_{123}, \hat{\mathbf{e}}_{132}\}$.

2.2 Subrepresentations

As laid out in [FH04], we are looking for representations which are said to be "atomic" and conversely, for any arbitrary representation we wish to find how it is composed of these indecomposable representations. To proceed we need the notion of a vector space map that respects or conserves the group action.

Morphisms of reprs.

Definition 2.14 (G-linear map)**.** [FH04] Let V and W be two representation spaces of a group G. A vector space map

$$\varphi : V \to W$$

is called a G-linear map if it commutes with the group action of G, ie. for any $\mathbf{v} \in V$ and $g \in G$ we have

$$\varphi(g \cdot \mathbf{v}) = g \cdot \varphi(\mathbf{v}),$$

or in terms of the maps $\rho_g^V : G \to \mathrm{GL}(V)$ and $\rho_g^W : G \to \mathrm{GL}(W)$,

$$\varphi \circ \rho_g^V(\mathbf{v}) = \rho_g^W \circ \varphi(\mathbf{v}).$$

Equivalently one can say that the diagram in Figure 1 is commutative for every $g \in G$.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g\cdot"'] & W \arrow[d, "g\cdot"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

Figure 1: Morphism of representations

Consider the case where W is a G-invariant[3] vector subspace of $V$and let ρ be a representation of G in V. Then the restriction of ρ to W, here denoted $\rho|_W$, is a

[3]For W to be a G-invariant space, it means that for every $g \in G$, $g \cdot \mathbf{w} \in W$ for any $\mathbf{w} \in W$ or that $\rho_g(W) \subseteq W$ for any $g \in G$.

isomorphism of W onto itself since W is G-invariant, and thus $\rho_{gh}|_W = \rho_g|_W \cdot \rho_h|_W$, hence $\rho|_W : G \to \mathrm{GL}(W)$ is a representation if G in the subspace W of V, motivating the following definition.

Definition 2.15 (Subrepresentation)**.** Let G be a finite group and ρ be a representation of G in a vector space V. A restriction of ρ to a G-invariant vector subspace W of V is called a subrepresentation of ρ.

In other words, a subrepresentation is a G-invariant vector subspace of a "parent" representation.

Example 2.16 (Trivial subspaces)*.* Any representation V has itself as well as the zero space as subrepresentations.

2.2.1 Some proper and non-trivial subrepresentations

Let G be any group and let us study its permutation representation and regular representation.

Example 2.17 (Trivial representation inside the permutation representation)*.* [Sag01, Example 1.4.3.] Let G act on a set $X = \{x_1, x_2, \dots x_k\}$, where $k = |X|$, and let V be the vector space spanned by the basis $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \dots, \hat{\mathbf{x}}_k)$. Consider the one-dimensional subspace of V spanned by the sum of all basis vectors, ie.

$$W = \mathrm{Span}\{\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2 + \cdots + \hat{\mathbf{x}}_k\}.$$

A vector $\mathbf{w} \in W$ is a scalar multiple of this sum, and for any $g \in G$, the action of g on $\mathbf{w}$ will simply reorder this sum and return the same $\mathbf{w}$, that is for any $g \in G$, we have that $\rho_g|_W = 1$. Thus W is a G-invariant subspace of V and the permutation representation has the trivial representation as a subrepresentation.

Example 2.18 (Trivial representation inside the regular representation)*.* [Sag01, Example 1.4.4.] Similarly to the last example, we span a vector space V by a basis $(\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \dots, \hat{\mathbf{g}}_k)$, where $k = |G|$, and consider the subspace

$$W = \mathrm{Span}\{\hat{\mathbf{g}}_1 + \hat{\mathbf{g}}_2 + \cdots + \hat{\mathbf{g}}_k\}.$$

Completely analogously to the last example, W is shown to be a G-invariant subspace of V and the regular representation also has the trivial representation as a subrepresentation.

Example 2.19 (Alternating representation inside the regular representation of $\mathcal{S}_n$). [Sag01, Example 1.4.4.] Let $G = \mathcal{S}_n$, then for the regular representation, V is spanned by a basis vector for every $\sigma \in \mathcal{S}_n$. Let W be the subspace of V spanned by the sum $\sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma)\hat{\mathbf{e}}_\sigma$. The action of a $\tau \in \mathcal{S}_n$ on a $\mathbf{w} \in W$ is

$$\tau \cdot a \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma)\hat{\mathbf{e}}_\sigma = a \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma)\hat{\mathbf{e}}_{\tau\sigma}$$

where $a \in \mathbb{C}$. Let $\pi = \tau\sigma$, then the sum can instead be indexed over all these $\pi \in \mathcal{S}_n$, hence W is a G-invariant subspace of V and the alternating representation is also found inside the regular representation.

What happens with the sgn?

2.2.2 Subrepresentations as a kernel

From linear algebra, we are familiar with the kernel and image of a map (see Definition 1.5).

Proposition 2.20 (Kernel and image of a G-linear map)**.** Let V and W be representation spaces of a group G and let $\varphi : V \to W$ be a G-linear map. Then the kernel and the image of φ are also representations of G and more specifically, $\ker\varphi$ is a subrepresentation of V and $\operatorname{im}\varphi$ is a subrepresentation of W.

Cokernel??

Proof. i) Let $\mathbf{v} \in \ker\varphi$. Then $g \cdot \varphi(\mathbf{v}) = \mathbf{0}$ since by definition $\varphi(\mathbf{v}) = \mathbf{0}$, but then we must also have $\varphi(g \cdot \mathbf{v}) = \mathbf{0}$ since φ is a G-linear map, implying that $g \cdot \mathbf{v} \in \ker\varphi$. Since this holds for any $g \in G$ and $\mathbf{v} \in \ker\varphi$, the kernel of φ is a G-invariant subspace of V and thus a subrepresentation of V.

ii) For a $\mathbf{w} \in \operatorname{im}\varphi$, by definition there exists a $\mathbf{v} \in V$ such that $\varphi(\mathbf{v}) = \mathbf{w}$. Since $g \cdot \mathbf{v} \in V$, then we have that $\varphi(g \cdot \mathbf{v}) \in W$. This implies that $g \cdot \varphi(\mathbf{v})$ is also in W. Since this holds for any $g \in G$ and $\mathbf{v} \in V$, the image of φ is a G-invariant subspace of W and a thus a subrepresentation of W. □

Recall from linear algebra the existence of complementary vector subspaces of a vector space. Is there a similar property of subrepresentations? Consider the case when W is a subrepresentation of V and let $\pi : V \to W$ be the projection of V onto W. Recall (Corollary 1.7) that for each subspace $W \leq V$ there exists a complementary and disjoint subspace $W' \leq V$ such that $V = W \oplus W'$. This means that every $\mathbf{v} \in V$ is partitioned into $\mathbf{w} \oplus \mathbf{w}'$ where $\mathbf{w} \in W$ and $\mathbf{w}' \in W'$ such that

$$\begin{cases} \pi(\mathbf{v}) = \mathbf{w}, \\ \pi(\mathbf{w}) = \mathbf{w} \text{ and} \\ \pi(\mathbf{w}') = \mathbf{0}. \end{cases}$$

The image of the projection is clearly W and the kernel is W'. But is π a G-linear map?

Proposition 2.21 (Existence of complementary subrepresentations)**.** Let ρ be a representation of a finite group G in a finite-dimensional vector space V. Let W be a G-invariant subspace of V. Then there exists a G-invariant subspace W' of V complementary to W such that $V = W \oplus W'$.

Proof. By Proposition 2.20 we know that a G-linear map from one representation space V to another W has its kernel and image as subrepresentations of V and W respectively. If we can find such a G-linear map from V to W we are done.

The projection π may not be a G-linear map since $\pi(g \cdot \mathbf{v})$ is not generally the same as $g \cdot \pi(\mathbf{v})$. NEEDS A COUNTEREXAMPLE? Instead, consider taking the *average* of π over G,

$$\bar{\pi} = \frac{1}{|G|} \sum_{g \in G} \rho_g \cdot \pi \cdot \rho_g^{-1},$$

and see if it conserves the action of G. Is it still a projection $V \to W$? That is, does $\bar{\pi} \cdot \rho_g = \rho_g \cdot \bar{\pi}$ hold for every $g \in G$? Equivalently, let's consider

$$\begin{aligned} \rho_g \cdot \bar{\pi} \cdot \rho_g^{-1} &= \frac{1}{|G|} \sum_{h \in G} \rho_g \cdot \rho_h \cdot \pi \cdot \rho_h^{-1} \cdot \rho_g^{-1} && \text{(Def. of } \bar{\pi}) \\ &= \frac{1}{|G|} \sum_{h \in G} \rho_{gh} \cdot \pi \cdot \rho_{gh}^{-1} && (\rho \text{ homom., } (gh)^{-1} = h^{-1}g^{-1}) \\ &= \frac{1}{|G|} \sum_{g' \in G} \rho_{g'} \cdot \pi \cdot \rho_{g'}^{-1} && (\text{Let } g' = gh) \\ &= \bar{\pi}. \end{aligned}$$

So, $\bar{\pi} \cdot \rho_g = \rho_g \cdot \bar{\pi}$ for every $g \in G$ and thus $\bar{\pi} : V \to W$ is a G-linear map, which means that its kernel, $\ker \bar{\pi}$, identified to be the complement W' of W in V, is also a G-invariant subspace of V and thereby a subrepresentation of V. □

is proof misses some parts, we need to show that $\bar{\pi}$ also is a proj V->W, implying existence of

In Example 2.17, the trivial representation was found to be a subrepresentation of the permutation representation. Let $G = \mathcal{S}_n$ and let V be the permutation representation and assign it the basis $(\hat{\mathbf{e}}_i)_{i=1}^n$. The one-dimensional subspace W spanned by the sum of all $\hat{\mathbf{e}}_i$ was found to be the trivial representation, then by Proposition 2.21, there exists another subrepresentation W' of V complementary to W. The dimension of W' is $n-1$. Let's introduce to V a inner product $(\cdot|\cdot)$ that fulfills

$$(\hat{\mathbf{e}}_i|\hat{\mathbf{e}}_j) = \delta_{i,j} = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if } i \neq j. \end{cases}$$

for the basis $(\hat{\mathbf{e}}_i)_{i=1}^n$. Then the basis $(\hat{\mathbf{e}}'_j)_{j=1}^{n-1}$ of W' can be constructed by ensuring that each $\hat{\mathbf{e}}'_j$ is orthogonal to the basis vector of W, that is

$$\left(\hat{\mathbf{e}}'_j \,\middle|\, \sum_{i=1}^n \hat{\mathbf{e}}_i\right) = 0$$

for every $1 \leq j \leq n-1$. One can see that one such basis is found by choosing every basis vector $\hat{\mathbf{e}}'_j$ to be the difference of two sequent $\hat{\mathbf{e}}_i$, that is choosing the basis $(\hat{\mathbf{e}}_j - \hat{\mathbf{e}}_{j+1})_{j=1}^{n-1}$ for W'. This is called the *Standard representation of* $\mathcal{S}_n$.

Example 2.22 (Standard representation of $\mathcal{S}_3$). The matrices of the standard representation is found by studying the action inherited from $\mathcal{S}_3$ on the basis

$$\begin{cases} \hat{\mathbf{e}}'_1 = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2, \\ \hat{\mathbf{e}}'_2 = \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3. \end{cases}$$

For example, $(1,2)\cdot(\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2) = \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}'_1$ and $(1,2)\cdot(\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3) = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}'_1 + \hat{\mathbf{e}}'_2$, hence $\rho_{(1,2)} = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The matrices are presented in Table 7.

$$\rho_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_{(1,2,3)} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \rho_{(1,3,2)} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\rho_{(1,2)} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho_{(1,3)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_{(2,3)} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Table 7: Matrices of the standard representations of $\mathcal{S}_3$

2.3 Tensor operations on representations

Building on the vector spaces constructed in Section 1.2.4, given two representations V and W of a group, the following are also representations.

- The *direct sum of* $V \oplus W$ is a representation given by

$$\rho_{V\oplus W}(\mathbf{v} \oplus \mathbf{w}) = (\rho_V \oplus \rho_W)(\mathbf{v} \oplus \mathbf{w}) = \rho_V(\mathbf{v}) \oplus \rho_W(\mathbf{w}).$$

- By recursion, for a positive integer n, the *direct sum of n copies of* V, denoted

$$nV := \bigoplus_{i=1}^{n} V = \underbrace{V \oplus \cdots \oplus V}_{n \text{ times}},$$

  is a representation.

- The *tensor product* $V \otimes W$ is a representation given by

$$\rho_{V\otimes W}(\mathbf{v} \otimes \mathbf{w}) = (\rho_V \otimes \rho_W)(\mathbf{v} \otimes \mathbf{w}) = \rho_V(\mathbf{v}) \otimes \rho_W(\mathbf{w}).$$

- By recursion, for a positive integer n, the *nth tensor power of* V, denoted

$$V^{\otimes n} := \bigotimes_{i=1}^{n} V = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}},$$

  is a representation. The zeroth power is by definition the ground field $\mathbb{C}$ and the first power is V itself.

- The nth tensor power has two subrepresentations, the symmetric powers $\mathrm{Sym}^n V$ and the exterior powers $\bigwedge^n V$. *Needs further explanation*

- After fixing a basis $(\hat{\mathbf{e}}_i)_{i=1}^{k}$ for V, a *dual representation*, denoted V^*, can be defined by the dual space, spanned by the ...

- Set of homomorphisms...

Example 2.23 (Decomposition of the permutation representation of $\mathcal{S}_3$). Letting V be the permutation representation of $\mathcal{S}_3$, it was found to have two complementary subrepresentations W and W'. $\rho^V_{(1,2)} = \rho^W_{(1,2)} \oplus \rho^{W'}_{(1,2)}$.

2.4 Irreducible representations

In Section 2.2 we found that a given representation can be divided into complementary subrepresentations, now we introduce the notion of an "indivisible" representation.

Definition 2.24 (Irreducible representation)**.** If there are no proper and non-trivial G-invariant subspaces of a representation V, it is said to be *irreducible.*

We have already met a few of these.

Example 2.25 (Degree 1 representations are irreducible). [Sag01, Example 1.4.2.] A vector space of dimension 1 has no other subspace other than itself and the zero space, thus it is irreducible. Hence the trivial representation of degree 1 of any group and the alternating representation of $\mathcal{S}_n$ discussed in Section 2.1 and the degree 1 representations of $\mathcal{C}_n$ found in Section 2.1.2 are irreducible representations of their respective groups.

The results from Proposition 2.21 invites the notion of *complete reducibility* of an arbitrary representation. This is presented in the following theorem.

Theorem 2.26 (Maschke's theorem)**.** Let G be a finite group and let V be any representation space of G. Then V is composed as a direct sum of a number of not necessarily distinct subrepresentations W_i of V, or in other words,

$$V = \bigoplus_i W_i$$

Note. The direct sum can be expressed in terms of distinct subrepresentation W_i of V and their multiplicities a_i as

$$V = \bigoplus_i a_i W_i$$

Example 2.27 (Decomposition of permutation representation of $\mathcal{S}_n$). Let $G = \mathcal{S}_n$, and let P be the permutation representation space, S be the standard representation space and T be the trivial representation space, then V is fully decomposed as

$$P = T \oplus V.$$

Theorem 2.28 (Schur's lemma)**.** content...

3 Character Theory

This section is based on [Ser77, Ch.2.].

Let V be a vector space with basis $(\hat{\mathbf{e}}_i)_{i=1}^n$ and let $\rho : G \to \mathrm{GL}(V)$ be a representation. Now, for each $g \in G$, define a function $\chi_\rho : G \to \mathbb{C}$ to be the trace of the matrix of g in V. In other words $\chi_\rho(g) = \mathrm{Tr}\ (\rho_g)$. This function is called the *character* of a representation.

We recall from linear algebra the concept of the trace of a matrix $(a_{ij})_{n\times n}$. It is the sum along the diagonal,

$$\mathrm{Tr}\ (a_{ij}) = \sum_{i=1}^{n} a_{ii} = a_{11} + \cdots + a_{nn},$$

and is also the sum of the eigen values of the matrix and is independent of the basis chosen. Why is it useful when studying representations?

Proposition 3.1. [Ser77, Prop.2.1.] The trace function *characterizes* the representation in some useful ways:

i) The character of the identity element of G is the degree of the representation.

ii) The character of an element in G is the complex conjugate of the character of the inverse element.

iii) The character is constant under conjugation.

Proof. i) By Propostion 2.2, ρ_e is the $n \times n$ identity matrix, hence $\chi(e) = \mathrm{Tr}\ \mathbf{1} = n$.

ii) Let $g \in G$ and let g^{-1} be its inverse. The character of g^{-1} is

$$\mathrm{Tr}\ (\rho_{g^{-1}}) = \mathrm{Tr}\ (\rho_g^{-1})$$

which is the sum of the eigen values of ρ_g^{-1}. If λ is an eigen value of ρ_g, then $1/\lambda$ is an eigen value of ρ_g^{-1}, hence

$$\mathrm{Tr}\ (\rho_g^{-1}) = \sum_i \frac{1}{\lambda_i} = \sum_i \overline{\lambda_i} = \overline{\mathrm{Tr}\ (\rho_g)} = \overline{\chi(g)}$$

iii) It is well known that the trace is conserved under conjugation. □

Definition 3.2.

Definition 3.3. Let V be a representation space of a group G of dimension n and let $\rho_g = (r_{ij})$ be the matrix representation of a $g \in G$. Then the *character* of g in V is defined to be the trace of the matrix representation, that is

$$\chi_V(g) := \mathrm{Tr}\ \rho_g = \sum_{i=1}^{n} r_{ii}.$$

Notation. A "character vector", simply called the group character, of G can be defined as the vector containing the character of every element of G.

Example 3.4. The trivial character is $(1, \dots, 1)$ for any group.

Example 3.5. The permutation character of $\mathcal{S}_3$ is $(3, 0, 0, 1, 1, 1)$.

Remark 3.6. We note that since the trace is fixed under conjugation, any elements of a group that are conjugate have the same character, ie. for a conjugate pair $g, h \in G$ there exists an $a \in G$ such that $h = gag^{-1}$, and their characters are

$$\chi(h) = \chi(gag^{-1}) = \chi(g).$$

Notation. If the group G is partitioned into the conjugacy classes $K_1, K_2, \dots, K_l$, then the group character can be abbreviated to contain one representative k_i from every class K_i, ie. $\chi = (\chi(k_1), \chi(k_2), \dots \chi(k_l))$.

Example 3.7. The group $\mathcal{S}_3$ is partitioned into the classes $[(1)], [(1, 2)]$ and $[(1, 2, 3)]$, and the group character of the permutation representation can be written as $(3, 0, 1)$.

Something on the characters are basis of class function space, hence the number of irreducible representations of a group is the same as the number of conjugacy classes in the group.

Moving on to the toolbox presented in Section 2.3, we propose the following:

Proposition 3.8. Let V and W be representation spaces of a group G. For a $g \in G$, let $\chi_V(g)$ and $\chi_W(g)$ be its characters in those representations. Then we have,

i) The character in $V \oplus W$ is $\chi_V + \chi_W$,

ii) The character in $V \otimes W$ is $\chi_V \cdot \chi_W$

iii) The characters of the symmetric and exterior squares are $\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$ and $\chi_{\bigwedge^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$, compatible with $V \otimes V = \mathrm{Sym}^2 V \oplus \bigwedge^2 V$.

Proof. These topics were treated in Section 1.2.4.

The trace of $\rho_g^V \oplus \rho_g^W$ is clearly the sum of the traces of ρ_g^V and ρ_g^W.

ii) Likewise, the trace of $\rho_g^V \otimes \rho_g^W$ is found to be the product of the traces of ρ_g^V and ρ_g^W.

iii) *To be written...*

□

3.1 Orthogonality relations

Let φ and ψ be the group characters of a group G, then we define the scalar product of characters as

$$(\varphi|\psi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

This is an inner product since it is linear in φ, semilinear in ψ, as well is $(\varphi|\varphi) > 0$ for any $\varphi \neq \mathbf{0}$ and $(\varphi|\varphi) = 0$ if and only if $\varphi = \mathbf{0}$.

Notation. The inner product of a character with itself is sometimes lazily denoted by

$$|\varphi| := (\varphi|\varphi).$$

Theorem 3.9 (Irreducibility criterion)**.** Let $\chi \neq \psi$ be characters of irreducible representations of G (called irreducible characters), then we have that

i) $(\chi|\chi) = 1$, and

ii) $(\chi|\psi) = 0$.

Proof. *To be written...* □

Now, let V be the direct sum of irreducible representations W_i of G such that

$$V = \bigoplus_i W_i.$$

If χ_i is the character of W_i, then by Proposition 3.8 the character of V is

$$\varphi = \sum_i \chi_i.$$

Let χ be the character of an irreducible representation of G, then we have that

$$(\varphi|\chi) = \sum_i (\chi_i|\chi).$$

Since χ and all of the χ_i are irreducible characters, all of the inner products $(\chi_i|\chi)$ is either 1 or 0, depending on if χ and χ_i are of isomorphic representations. Hence, $(\varphi|\chi)$ will return the number of occurences of χ in φ. This number is called the *multiplicity.*

Remark 3.10. This also means that the composition of a representation φ into a direct sum of irreducible subrepresentations χ_i is unique. *Something on two compositions of the same V must be identical.*

Remark 3.11. The converse is also true, if two representations have the same character, then they are isomorphic since they contain the same irreducible representations.

Note. Let V be a representation with the composition

$$V = \bigoplus_{i=1}^{k} m_i W_i = m_1 W_1 \oplus m_2 W_2 \oplus \dots,$$

where the m_i are the multiplicities of the irreducible W_i. If the character of W_i is denoted by χ_i, then the character of V is

$$\varphi = \sum_{i=1}^{k} m_i \chi_i,$$

where $m_i = (\varphi|\chi_i)$. Taking the inner product of φ with itself we have

$$(\varphi|\varphi) = \sum_{i=1}^{k} m_i^2. \tag{3.1}$$

Theorem 3.12. The character φ is irreducible if and only if $(\varphi|\varphi) = 1$.

Proof. If φ is irreducible, then by Theorem 3.9 we have that $(\varphi|\varphi) = 1$. For the converse statement, if the sum in Equation 3.1 to be equal to one, we must have that only one of the m_i is equal to 1, and the rest are 0. Then V is composed of only one irreducible character, hence it is irreducible. □

Using these criteria and relations as a character calculus toolbox, we can study some groups, determine all of their irreducible representations and decompose any arbitrary representation.

3.2 Decomposition of the regular representation

Some representations where painstakingly found in the regular representation (the trivial in any group, along with the alternating in the symmetric group), however in general the following hold for the regular representation:

For a $g \in G$, it will act on a basis vector $\hat{\mathbf{e}}_h$ of the regular representation space V by

$$g \cdot \hat{\mathbf{e}}_h = \hat{\mathbf{e}}_{gh}$$

and the resulting matrix ρ_g can then be constructed by studying the action of this $g \in G$ on every $\hat{\mathbf{e}}_h$, where $h \in G$. However, the trace of this matrix, ie. the character $\varphi(g)$, only depends on the values on the diagonal, which are the fixed points under the action of g. What this means is that the hth column will have an 1 in the h row if and only if $gh = h$, which holds only if $g = e$, hence ρ_g is a $|G| \times |G|$ permuation matrix with trace

$$\varphi(g) = \begin{cases} |G|, \text{ if } g = e, \\ 0, \text{ otherwise,} \end{cases}$$

and the character of the regular representation is

$$\varphi = (|G|, 0, \ldots, 0).$$

Now taking the inner product of it with itself we have,

$$(\varphi|\varphi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\varphi(g)},$$

all summands vanish except $g = e$, and since $\varphi(e) = |G|$ we have that

$$(\varphi|\varphi) = \frac{1}{|G|}|G|^2 = |G|,$$

hence by Equation 3.1, the square sum of the multiplicities of the regular representation is the order of the group. Also, assume that W_i is the family of irreducible representations of a group with respective characters χ_i, then the regular representation is a direct sum of these with (not-necessarily non-zero) multiplicities m_i, then we have that the multiplicity of some irreducible χ_j in φ is

$$(\varphi|\chi_j) = \sum_i (\chi_i|\chi_j) = \sum_i \frac{1}{|G|} \sum_{g\in G} \chi_i(g)\overline{\chi_j(g)}.$$

The χ_i vanish for all $g \neq e$,

$$= \sum_i \frac{1}{|G|} \chi_i(e)\overline{\chi_j(e)} = \sum_i \frac{1}{|G|} |G| \chi_j(e) = \sum_i \chi_j(e)$$

hence the multiplicity of an irreducible representation inside the regular representation is the degree of that irreducible.

some mathematical error somewhe

We arrive at:

Theorem 3.13. The regular representation V of a group is decomposed in term of the irreducible representations of the group W_i as

$$V = \bigoplus_i \dim(W_i) W_i.$$

Definition 3.14. A character table is...

3.3 Characters of $\mathcal{C}_n$

Example 3.15 (Characters of $\mathcal{C}_n$). In section 2.1.2, the n degree 1 representations of any cyclic group $\mathcal{C}_n$ was described, and by theorem **??**, those described are all irreducible representations of $\mathcal{C}_n$. Remember that the trace of a 1×1-matrix is its only element, therefore the Tables 1, 2 and 3 are the character tables of respectively $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_5$.

3.4 Characters of $\mathcal{S}_n$

Theorem 3.16 (Character of the permutation representation)**.** The character of a permutation representation is the number of fixed points of the associated element of the symmetric group.

Proof. The character is the sum of the diagonal elements of the matrix representation. The matrices of the permutation representations are permutation matrices, which have a 1 on the diagonal if that row represents a fixed point and 0 otherwise. Therefore the number of fixed point is the sum along the diagonal. □

The representations of S_3 found so far in sections are presented in table **??**. The group is presented in their conjugacy classes $[\sigma] \subset S_3$ where σ is any representative of that class. Beneath every class is the number of elements in that class. Rows below the dashed horizontal line are not irreducible.

S_3	[1]	[12]	[123]
$\lvert[\sigma]\rvert$	1	3	2
χ_T	1	1	1
χ_A	1	-1	1
χ_S	2	0	-1
χ_P	3	1	0
χ_R	6	0	0

Table 8: Character table of S_3.

Decomposition of P. In section **??**, the permutation representation was decomposed through geometric arguments to the direct sum of the trivial and the standard representation, this is seen in the table as $\chi_T + \chi_S = (1, 1, 1) + (2, 0, -1) = (3, 1, 0) = \chi_P$, confirming that $P = T \oplus S$.

Decomposition of R. In section **??**, the matrices of the regular representation was exhaustively calculated. The character of R thus is $\chi_R = (6, 0, 0)$, since the identity element is the only element in S_3 that leaves every other element fixed. In fact this is true for any S_n:

Corollary 3.17 (Character of the regular representation of S_n)**.** The character of the regular representation for any S_n is $(n!, 0, \dots, 0)$, where $n!$ is the size of S_n.

Proof. The center of S_n is trivial. □

Corollary 3.18 (Character of the regular representation of a group with trivial center)**.** The character of the regular representation for any group G with trivial center ($Z(G) = e$) is $(|G|, 0, \dots, 0)$, where $n!$ is the size of S_n.

Proof. SOURCE □

Counterexample: D_8 has a non-trivial center. The quotient $D_8/Z(D_8)$ is isomorphic to V.

To decompose R, we take the inner product of χ_R with every other irreducible character.

$$(\chi_T|\chi_R) = 1, \quad (\chi_A|\chi_R) = 1, \quad (\chi_S|\chi_R) = 2, \tag{3.2}$$

so $\chi_R = \chi_T + \chi_A + 2\chi_S$ and $R = T \oplus A \oplus 2S$.

Also:

Theorem 3.19. If V_i are the irreducible representations of a group, then the regular representation R is the direct sum of $\dim V_i$ copies of V_i, ie.

$$R = \bigoplus_{i=1}^{k} (\dim V_i) \cdot V_i = \bigoplus_{i=1}^{k} \chi_{V_i}(e) \cdot V_i. \tag{3.3}$$

Proof. It came to me in a dream. □

Decomposition of $S \otimes S$. The character of $S \otimes S$ is $\chi_{S\otimes S} = \chi_S \cdot \chi_S = (4, 0, 1)$, which by a quick glance on the character table is seen to be the sum of all irreducibles, ie. $\chi_{S\otimes S} = \chi_T + \chi_A + \chi_S$ and $S \otimes S = T \oplus A \oplus S$.

Decomposition of $\mathcal{S}^2 S$ and $\bigwedge^2 S$. The character of $\mathcal{S}^2 S$ is $\chi_{\mathcal{S}^2 S} = \frac{1}{2}\left[(4, 0, 1) - (2, 2, -1)\right] = (1, -1, 1) = \chi_A$, so $\mathcal{S}^2 S \cong A$. The character of $\bigwedge^2 S$ is $\chi_{\bigwedge^2 S} = (3, 1, 0) = \chi_P = \chi_T + \chi_S$, so $\bigwedge^2 S \cong P \cong T \oplus S$, confirming $S \otimes S = \mathcal{S}^2 S \oplus \bigwedge^2 S$ since $\chi_{\mathcal{S}^2 S} + \chi_{\bigwedge^2 S} = A \oplus T \oplus S = \chi_{S\otimes S}$. Also, $R = S \otimes S \oplus S$.

Decomposition of $S^{\otimes n}$. [FH04, Exercise 2.7.] To find the decomposition of larger tensor powers than $n = 2$, we study the character $\chi_{S^{\otimes n}} = (2^n, 0, (-1)^n)$ and take the inner product of it with the other irreducibles and find:

$$(\chi_T|\chi_{S^{\otimes n}}) = (\chi_A|\chi_{S^{\otimes n}}) = \frac{1}{6}\left(2^n + (-1)^n\right), \tag{3.4}$$

$$(\chi_S|\chi_{S^{\otimes n}}) = \frac{1}{6}\left(2^{n+1} + (-1)^{n+1}\right), \tag{3.5}$$

so $\chi_{S^{\otimes n}} = a_n(\chi_T + \chi_A) + a_{n+1}\chi_S$ and thus $S^{\otimes n} = a_n(T \oplus A) \oplus a_{n+1} S$ where $a_n = \left(2^n + (-1)^n\right)/6$.

The contributions from the last few paragraphs are to the now complete character table of S_3 is presented in table **??**.

S_3	[1]	[12]	[123]	
$\lvert[\sigma]\rvert$	1	3	2	*Alternate compositions*
χ_T	1	1	1	
χ_A	1	-1	1	$A \cong \mathcal{S}^2 S$
χ_S	2	0	-1	
χ_P	3	1	0	$P \cong T \oplus S \cong \bigwedge^2 S$
$\chi_{S\otimes S}$	4	0	1	$S \otimes S \cong T \oplus A \oplus S$
χ_R	6	0	0	$R \cong T \oplus A \oplus 2S$

Table 9: Complete character table of S_3. The representations above the dashed line are irreducibles, and those below are composed. Some compositions are presented in the right-most column.

3.5 Characters of $\mathcal{S}_4$

This section follows the same methods from the previous section on S_3. So far, we have three irreducible representations of S_4: The trivial and the alternating representation from section **??**; the standard representation from section **??** and also we can surmise that there is a regular representation of S_4 with character $(24, 0, 0, 0, 0)$ from theorem 3.17. Along with the composite permutation representation, the character table of S_4 is presented in table 10.

S_3	[1]	[12]	[123]	[1234]	[(12)(34)]
$\lvert[\sigma]\rvert$	1	6	8	6	3
χ_T	1	1	1	1	1
χ_A	1	-1	1	-1	1
χ_S	3	1	0	-1	-1
χ_P	4	2	1	0	0
χ_R	24	0	0	0	0

Table 10: Character table of S_4.

Recall that there are five conjugacy classes in S_4, then by theorem **??** we can expect two more irreducible representations of S_3. To find them, we compose new representations out of the ones we have found so far, using the toolbox of section **??**.

Decomposition of $A \otimes S$. The character of this composition is $(3, -1, 0, 1, -1)$, an "alternating" version of S, and taking the inner product of it with itself we find $|\chi_{A\otimes S}|^2 = 1$ and thus $A \otimes S$ is an irreducible representation. From this point on it is denoted as S'.

Decomposition of $S \otimes S$. The character of this composition is $(9, 1, 0, 1, 1)$ and it is found to be linearly dependent on χ_T, χ_S and $\chi_{A\otimes S}$ with overlaps of 1 and independent of χ_A, however T, S and S' together are of degrees 1, 3 and 3 and $S \otimes S$ is of degree 9, so there is either another representation of degree 2 or two of degree 1. *SOURCE, theorem om square sum of* $\dim V_i$. Denoting this representation by V, its character is $\chi_V = \chi_{S\otimes S} - \chi_T - \chi_S - \chi_{S'} = (2, 0, -1, 0, 2)$, which is found to be such that $|\chi_V|^2 = 1$, hence V is irreducible and $S \otimes S$ is found to be decomposed to $T \oplus V \oplus S \oplus S'$. We have now found all five irreducible representations of S_4.

Decomposition of R. The character of R is $(24, 0, 0, 0, 0)$ and its overlaps with T, A, V, S and S' are respectively 1, 1, 2, 3 and 3, so $R = T \oplus A \oplus 2V \oplus 3S \oplus 3S'$.

Decomposition of $\mathcal{S}^2 S$ and $\bigwedge^2 S$. The character of $\mathcal{S}^2 S$ is

$$\frac{1}{2}\left[(9, 1, 0, 1, 1) - (3, 3, 0, -1, 3)\right] = (3, -1, 0, 1, -1) \tag{3.6}$$

which is the character of S'. Likewise, the character of $\bigwedge^2 S$ is $(6, 2, 0, 0, 2)$ which is found to be the sum of the characters of T, S and V, hence $\bigwedge^2 S = T \oplus S \oplus V$.

Decomposition of the nth tensor power of V. The character of $V^{\otimes n}$ is

$$\chi_{V^{\otimes n}} = (2^n, 0, (-1)^n, 0, 2^n). \tag{3.7}$$

Its overlap with T, A, V, S and S' are respectively $a_n, a_n, a_{n+1}, 0$ and 0, where

$$a_n := \frac{1}{24}\left(4 \cdot 2^n + 8 \cdot (-1)^n\right), \tag{3.8}$$

so the decomposition is

$$V^{\otimes n} = a_n(T \oplus A) \oplus a_{n+1} V. \tag{3.9}$$

Decomposition of the nth tensor power of S. The character of $S^{\otimes n}$ is

$$(\chi_S)^n = (3^n, 1, 0, (-1)^n, (-1)^n), \tag{3.10}$$

and after projecting it on the characters of the irreducible representations from table 11, we find that

$$S^{\otimes n} = a_n T \oplus b_n A \oplus c_n V \oplus a_{n+1} S \oplus b_{n+1} S', \tag{3.11}$$

where

$$\begin{cases} a_n = \frac{1}{24}(3^n + 9(-1)^n + 6), \\ b_n = \frac{1}{24}(3^n - 3(-1)^n - 6), \text{ and} \\ c_n = \frac{1}{24}(2 \cdot 3^n + 6(-1)^n). \end{cases} \tag{3.12}$$

Decomposition of the nth tensor powers of S'. Since $S' = A \otimes S$, the nth power of S' is the tensor product of n factors of $A \otimes S$. Since the tensor product is a commutative operation, we can rearrange it as

$$(A \otimes S)^{\otimes n} = S^{\otimes n} \otimes A^{\otimes n} \tag{3.13}$$

The nth tensor power of A is T if n is even and A if n is odd, so for even n, $S'^{\otimes n} \cong S^{\otimes n}$, and for odd n, $S'^{\otimes n} \cong S^{\otimes n} \otimes A$. Then the character is

$$\chi_{S'^{\otimes n}} = \begin{cases} \Big(3^n, 1, 0, (-1)^n, (-1)^n\Big), \text{ if } n \text{ is even, or} \\ \Big(3^n, -1, 0, (-1)^{n+1}, (-1)^n\Big), \text{ if } n \text{ is odd.} \end{cases} \tag{3.14}$$

Tensor multiplying equation 3.11 with A, we have:

$$\begin{aligned} S^{\otimes n} \otimes A &= \Big(a_n T \oplus b_n A \oplus c_n V \oplus a_{n+1} S \oplus b_{n+1} S'\Big) \otimes A & (3.15) \\ &= b_n T \oplus a_n A \oplus c_n V \oplus b_{n+1} S \oplus a_{n+1} S', & (3.16) \end{aligned}$$

and then we have found the decomposition of all tensor powers of S'. For even n, it is the same as $S^{\otimes n}$, for odd n, A and T, and S and S' switch multiplicities.

The findings of the last few paragraphs (except those on larger tensor powers) are presented in table 11.

S_4	[1]	[12]	[123]	[1234]	[(12)(34)]	
$\|[\sigma]\|$	1	6	8	6	3	*Alternate compositions*
χ_T	1	1	1	1	1	
χ_A	1	-1	1	-1	1	
χ_V	2	0	-1	0	2	
χ_S	3	1	0	-1	-1	
$\chi_{S'}$	3	-1	0	1	-1	$S' :\cong A \otimes S \cong \mathcal{S}^2 S$
χ_P	4	2	1	0	0	$P \cong T \oplus S$
$\chi_{\bigwedge^2 S}$	6	2	0	0	2	$\bigwedge^2 S \cong T \oplus S \oplus V$
$\chi_{S\otimes S}$	9	1	0	1	1	$S \otimes S \cong T \oplus V \oplus S \oplus S' \cong S' \otimes S'$
χ_R	24	0	0	0	0	$R \cong T \oplus A \oplus 2V \oplus 3S \oplus 3S'$

Table 11: Complete character table of S_4. The representations above the dashed line are irreducibles, and those below are composed. Some compositions are presented in the right-most column.

3.6 Characters of the Klein 4-group V

The Klein 4-group, denoted by[4] V is, along with Z_4, the only group of order 4. It can be described as the set

$$V = \{e, x, y, xy\}, \tag{3.17}$$

where every element is its own inverse and the product of any two distinct non-identity elements is the third one. It is an abelian group, which means that every element has its own conjugacy class (SOURCE).

Since every group has a trivial representation T in any vector space, so do V. The center of V is trivial, so the regular representation R has the character $(4, 0, 0, 0)$ and is clearly not irreducible.

If V instead is considered as a subgroup of S_4, that is

$$V = \{(1), (12)(34), (13)(24), (14)(23)\} < S_4, \tag{3.18}$$

then each entry in the character of the permutation representation P is the number of fixed points of the associated element. Then one sees that the character of the permutation representation is also $(4, 0, 0, 0)$, it isomorphic to the regular representation.

From theorem **??** we expect three additional irreducible representations. Since

[4] V is for vierer, four in german.

the square sum over all representations of the character of e must be equal to the order of the group (SOURCE), they must be of degree 1. We know that they must be parallel to themselves and orthogonal to T and have a 1 as the character of e, so the three additional are alterations of T with two of the ones exchanged with negative ones. Then R is the direct sum of these four irreducibles. These findings are presented in table 12 as the representations A, B and C.

V	e	x	y	xy
T	1	1	1	1
A	1	-1	-1	1
B	1	-1	1	-1
C	1	1	-1	-1
R	4	0	0	0

Table 12: Character table for V.

3.7 Characters of the dihedral group of order 8, D_8

D_8 is a 2-Sylow subgroup of S_4. Restrict the character table of S_4 to D_8.

D_8 has its non-trivial center $Z(\mathrm{D}_8) = \{e, r^2\}$ as a normal subgroup. The quotient $\mathrm{D}_8/\{e, r^2\}$ is isomorphic to the Klein 4-group. The character table of the Klein 4-group is a "subset" of the character table of D_8.

References

[Big04] Norman L. Biggs. *Discrete Mathematics*. 2004.

[DF04] David S. Dummit and Richard M. Foote. *Abstract Algebra*. Wiley, 3 edition, 2004.

[FH04] William Fulton and Joe Harris. *Representation Theory: A First Course*. Springer, 2004.

[HU15] Anders Holst and Victor Ufnarovski. *Matrix Theory*. 2015.

[Sag01] Bruce E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Springer, 2 edition, 2001.

[Ser77] Jean-Pierre Serre. *Linear Representations of Finite Groups*. 1977.