

Lecture 29 — Software Statistics

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The Setting

The Java SDK [Ora14] specifies an interface called `Set`. Its semantics is the mathematical notion of a set — an unordered collection of unique items. You can have a set of integers like { 8, 16, 44, 18, 99, -1 }, but duplicates are not permitted in a set. Several implementations of `Set` are provided by the SDK. Are they all the same? If not, which one should we use?

They are not identical, so we expect that the answer is: “it depends on your application.” However, it is reasonable to expect that there exist some criteria to which we should be able to agree based on which we compare different implementations of `Set`. We want these two operations to be fast:

1. Addition (putting something in the set); and
2. Membership-checking (finding out if an element is a member of the set) to be fast.

These criteria, in turn, provide an objective way of evaluating if one implementation is better than another. Thus, our goal is more modest than the absolute, non-comparative, criterion of “fast”. We are just figuring out if one set-implementation is “better than” (faster than) another in these two operations.

The question now is: how can we evaluate two given implementations in a rigorous and consistent way?

Statistics

We use statistics to try to establish that (if?) one set-implementation is better than another. The application of statistics to this question is not straightforward. To illustrate this via an example, consider a scenario where Alice follows the process below:

1. Randomly choose a bunch of integers (let’s say 100).
2. Use some API for measuring time on the computer. Write Java code to measure the time it takes to add each of those integers to an instance of a set-implementation I_1 , starting at $I_1 = \emptyset$ (the empty set). Compute the average time, a_1 .
3. Repeat for an instance of another set-implementation I_2 to get the average a_2 for it.
4. If $a_1 < a_2$, conclude that the first set-implementation is better than the second from the standpoint of additions to the set.

What is wrong with adopting the above process?

1. How do we know that the random set of integers Alice chooses is representative of all possible integers we may want to add to a set?

2. Do we have to consider the order in which we add the integers? For example, starting with a set S , it is possible that adding integer i_1 then i_2 may result in a different total time for the two additions than adding i_2 and then i_1 .
3. Why is it meaningful to use the average to draw our conclusion?
4. Even if we feel that use of the average is meaningful, how do we know that the averages are *statistically significant*? That is, how do we know that we can make a conclusion based on the fact that $a_1 < a_2$? Is it possible that in a run of this experiment, we ended up with $a_1 < a_2$ purely by chance?
5. Computers demonstrate considerable variations over time in how long they take to perform an operation. That is, if you run the Java code again, the time for adding a particular integer i to the set in the second run may result in a different time-measurement than the first time. How do we know that we can trust the reading which contributes to the average based on which we draw our conclusion?

Ideally, we will have good answers to all of the above questions. Even if we are unable to conclusively address a concern such as the above, it is very important to recognize the concern. And at the minimum articulate an assumption explicitly.

Basics

A (mathematical) *function* is defined as follows. Given two sets, a domain and a range, a function associates a member of the range with every member of the domain.

We can associate the time it takes to add or membership-check an element against a set as a *random variable*. A random variable is a function $f: E \rightarrow \mathbb{R}$, where E is a set of *events*, and \mathbb{R} is the set of real numbers. In our case, the range of a random variable we choose is presumably the time it takes to add the item to the set, or check its membership in the set. We choose the domain based on the assumptions we make. For this work, we choose the domain E as the set of all integers.

That is, we adopt two random variables, a, m , the former for add and the latter for membership-checking. Each maps an integer to a time-value. One of the points of characterizing a, m as random variables is that we typically associate a random variable with a *probability distribution*.

A probability distribution is itself a function. It maps each event in E to a probability that it occurs. For example, it is possible that the probability that we add the integer 15 to a set, is different from the probability that we add 32. Thus, the probability we associate with the event $\langle 15 \rangle$ is different from the probability we associate with the event $\langle 32 \rangle$.

If we can discover this $a(e)$ and $m(e)$ for a “typical” $e \in E$ for two different set-implementations, then we can draw a conclusion. This has two challenges: (1) what is a “typical” $e \in E$, and, (2) how do we compute a and m even if we know this e ?

Characterizing a and m

The domain of a random variable (E , above) is called a *population*. To characterize a random variable, an approach is to determine to what value in the range it maps every member of the population, and then compute a *central tendency* or *centre* of those values. A centre, as its name suggests, represents the “middle” of all possible values. The *average*, the sum of all values divided by the number of values, is a centre. Other examples of centres are the *median* and *trimean*. The choice of a centre is crucial. Not every centre is *robust* to *outliers*. An outlier is a value that occurs infrequently. Robustness is the property that a centre is representative of the “middle” notwithstanding the presence of outliers. We discuss further our choice of the centre below.

It is often impractical or even impossible to compute a centre for the entire population. For example, if the population is all Canadians, determining the value of a random variable for every Canadian may be deemed too expensive. Consequently, we collect data *samples*. A sample is a particular member of the population. The procedure by which we collect samples is important. Otherwise, the samples may not reflect the properties of the population, and thereby, the random variable.

Given a random variable X whose underlying probability distribution is P , a well-accepted property of samples is that they are: (1) *independent and identically distributed* (iid), and, (2) each sample, when viewed as a random variable in its own right, has the same distribution P as X .

With regards to (1), a set of random variables is said to be iid if each random variable from the set has the same probability distribution as all the other random variables in the set. With regards to (2), as an example, if X is the random variable that is the height of a person, then the manner in which we collect samples must reflect the distribution of heights amongst individuals. If the probability that there exists a person of height h in the entire population is ρ , then the probability that height h ends up in our set of samples must be ρ . (Or, the probability that we pick a person of height h to be part of our sample-set must be ρ .)

We now address three issues we must resolve: how do we choose a sample, what is our choice for centre, and how do we compute our choice of centre for the random variables a, m given the samples we choose?

Choice of samples Our discussions above indicate that we should choose samples in a manner that reflects the population. A difficulty for us is that we do not know what the underlying distribution of integers in the population is. That is, we do not know what kinds of integers are added/membership-checked in them. We resolve this issue by assumption.

We would of course like to minimize the number and nature of assumptions in any study. However, practicality may require that we make some. And it is okay to make assumptions, provided we clearly state them, and remember them when the time comes to draw conclusions.

We have done the following in our study. We have restricted ourselves to integer items only. We make the assumption that every integer is equally likely to be added to, or queried for, in a set. This restriction and assumption limits the applicability of our results. So, one needs to be careful in applying or interpreting our results.

Choice of centre Our choice of centre is the average, or mean. We choose this for convenience — it is easy to compute an average, particularly for a continuous (i.e., non-discrete) values.

The mean is not *robust* against outliers. An outlier is an atypical value, that is, a value that occurs with low probability. An outlier can skew the mean in a manner that the mean is no longer a meaningful centre. The median, on the other hand, is robust. The median of a continuous random variable is a value, m , such that the probability that a given value lies above or below m is $1/2$.

In our hypothesis testing below, we assume that the population is distributed normally. In such a case, the mean and median are the same. Note that it is possible to test for normality, rather than simply assume it. We omit a discussion of that, and urge you to take a stats course. Statistics is a big and complex subject and easily takes up a whole course all on its own.

Computing the centre for a, m Given our above assumptions, to compute the centre (average, in our case), we simply pick random integers to add/query, check how long it takes, and compute the average for some large number of such trials.

There are some additional issues we need to consider, however.

- The time for individual additions/queries can be so small that the measurement mechanism we use can be highly unreliable. We resolve this by performing several adds/queries, and measuring the time for those. We have chosen 250,000 adds/queries.
- Java VMs demonstrate strange behaviour when running such experiments. Specifically, there is a “warm up” period for every VM after it starts to run that we need to exclude.

A way that has been proposed to deal with this [Sau14] is to measure a statistic called the coefficient of variance (CoV) across some number, call it k , of measurements. Then, check whether this CoV is below some threshold. This approach is supposedly robust in practice.

We have also run our experiments several times, across several days and times of the day. And established that there is no statistically significant variance across those.

Hypothesis Testing

Now that we have decided on a centre and how to compute it, we can compare the three implementations of set across each of the two axes, add-time and query-time. We do this by comparing two implementations at a time. Given two implementations, say, `TreeSet` and `HashSet`, we ask whether the difference in the centres for them is statistically significant. An effective way to establish this is via a hypothesis test.

In a hypothesis test, we articulate a hypothesis, that is, an assertion that may be true or false. We then ask whether the hypothesis should be accepted or rejected. In statistics, we typically enunciate a *null hypothesis*. It is called that because it is usually a hypothesis of ‘no effect,’ or ‘no difference.’ In our case, an appropriate null hypothesis is that there is no difference in performance between two different implementations of set.

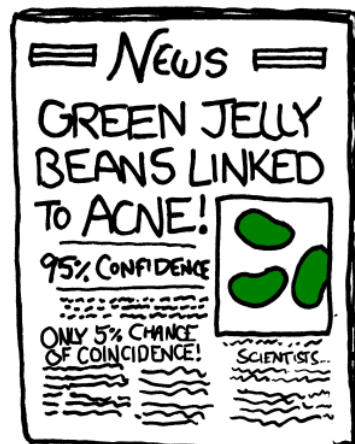
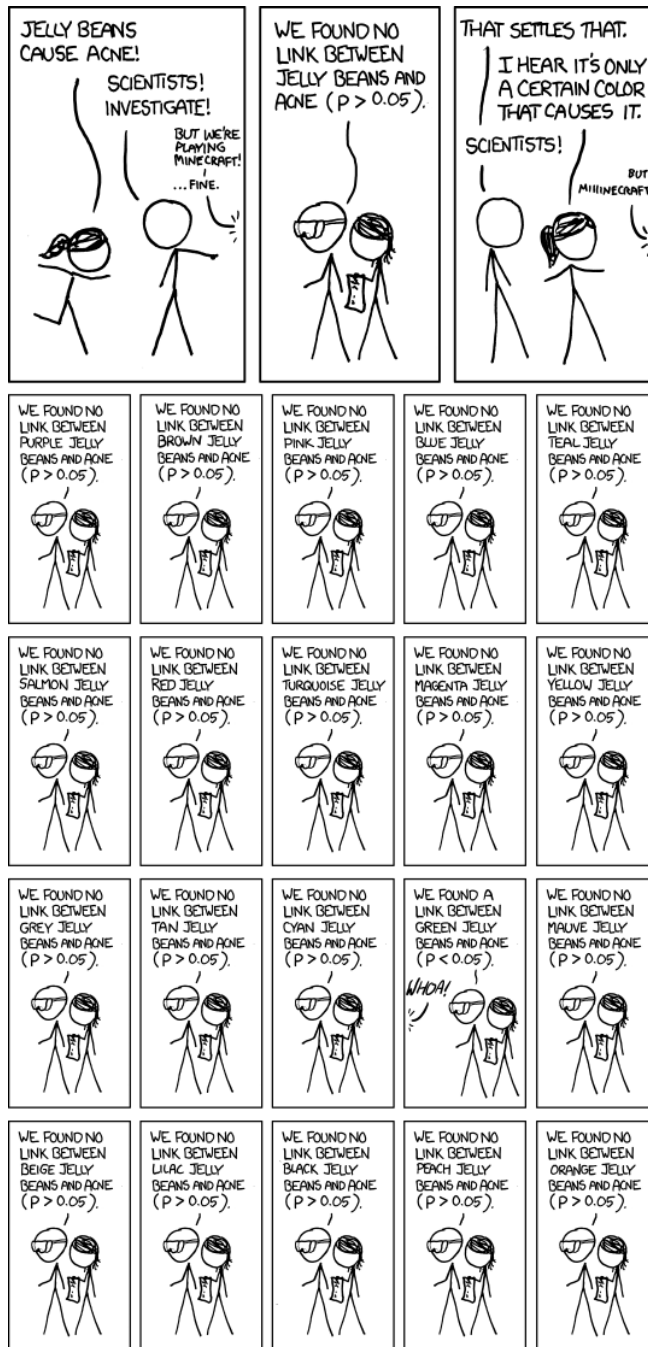
The complement of the null hypothesis is called an *alternate hypothesis*. If we articulate a null hypothesis as above, then the alternate hypothesis is that there is a difference in performance between `HashSet` and `TreeSet`.

It is important to point out that our null hypothesis was posed before we designed the statistical experiments. Posing hypotheses after looking at data is dangerous, as it can compromise objectivity and lead to incorrect inferences. As Sherlock Holmes said, it’s unwise to form the theory before getting the facts; we have a tendency to twist the facts to fit the theory when we should be using the facts to generate the theory.

Once we choose a null hypothesis, we choose a probability p which is the threshold at which we *reject* the null hypothesis and adopt the alternate hypothesis. The meaning of a value for p is:

If the null hypothesis is true, then there is a probability p of observing a difference in the averages as large as we observe.

As you can tell from the above semantics for p , a smaller value for p as threshold is better, i.e., suggests more statistical significance. This is because if the null hypothesis is indeed true, then there is only a small chance of observing the difference in averages that we did observe. Therefore, the fact that we did observe it, and can presumably reproduce the observations if we repeat the experiment, suggests that probabilistically, the null hypothesis is false.



[Mun14]

Choice of test There are a number of tests from which one could choose. The main thing to note here is the prerequisites each tests requires for it to be sound. For example, a number of tests require that the underlying population is distributed normally.

We do not go over the fundamentals of particular tests. You should take a course on statistics for that. But we again warn you that before you use any particular test, you should carefully read about the pre-conditions for the test to be applicable. For example, we use the Student's t-test in our following discussion.

The requirements for the Student's t-test in our context are: (1) the data is continuous, (2) the data follows a normal probability distribution, (3) the samples are independent, and (4) they are simple random samples from the respective populations. We know that (1) is true if we assume that any value for time may occur, and we ensure (3) and (4) from the manner in which we choose the samples. We satisfy (2) by assumption.

Results Following are the average and standard deviation, in seconds, for `HashSet`, `LinkedHashSet` and `TreeSet` for adding 250,000 integers starting from the empty set.

Implementation	Average	Std. Dev.
HashSet	0.183095	0.00124
LinkedHashSet	0.20011	0.000609
TreeSet	0.3975	0.00222

Following are the average and standard deviation, in seconds, for `HashSet`, `LinkedHashSet` and `TreeSet` for querying each of the 250,000 integers that have been added. Thus, our queries are elements we know to be in the set only. We articulate our null hypothesis accordingly to account for not testing for elements that are not in the set.

Implementation	Average	Std. Dev.
HashSet	0.029934	0.000685
LinkedHashSet	0.083366	0.001148
TreeSet	0.097701	0.002612

In reporting such numbers, it is important to identify as much of the setup as possible so there is sufficient information for someone to reproduce these results. Our information is:

the experiments were conducted on an isolated Intel dual core ASUS P5LD2-VM PC with a 3 GHz processor and 3.1 GiB of RAM, running the Ubuntu Linux operating system. The java version was 1.6.0_31, and it was the OpenJDK Runtime Environment. To measure time, we used `System.nanoTime()`. Each of our averages are over 30 iterations after the VM reached steady-state.

Steady-state refers to the VM after warm-up. See our discussions above under, 'Computing the centre...'

Hypothesis tests We now articulate null hypothesis, and a choice for the threshold p value, and conduct our tests. As an example, we have chosen a two-tailed, unpaired Student's t-test. Again, we urge you to take a stats course to understand the test in more depth. We have used GraphPad's QuickCalcs site, <http://www.graphpad.com/quickcalcs/>, to enter our data and run the tests.

As p value, we choose, say, 0.01, i.e., 1%. This choice is arbitrary, and really depends on what your peers will accept. In conducting a hypothesis test in the context of the discovery of Higgs Boson, for example, the p value was chosen as "five-sigma" (i.e., 5 times the standard deviation), which corresponds to a p

threshold value of 3×10^{-7} . (See the discussion at <http://blogs.scientificamerican.com/observations/2012/07/17/five-sigmawhats-that/>.)

We choose a much more conservative threshold of 0.01. So, we reject the null hypothesis if the calculated probability is less than 0.01.

Our first null hypothesis is, “To add 250,000 integers, there is no difference in time between `HashSet` and `LinkedHashSet`.” The Student’s t-test run with our above mean, standard deviation and N (number of samples, which in our case is 30) produces the following output:

The two-tailed P value is less than 0.0001.

That is, the computed probability is less than our threshold probability of 0.01. Therefore, we reject the null hypothesis that there is no difference in performance. There is just one more issue, however. Our alternate hypothesis is that there is a difference in performance. But not that a particular one of those is better than the other. However, we could informally look at our data and conclude that our experiments suggest that `HashSet` performs better than `LinkedHashSet`, given all our assumptions and choices.

We can similarly perform the other two comparisons, `HashSet` vs. `TreeSet`, and `LinkedHashSet` vs. `TreeSet`, and get similar results. Similarly, a Student’s t-test for the null hypothesis, “there is no difference in querying 250,000 members of `LinkedHashSet` and `TreeSet`” using our data from the table above for querying produces the same output as above:

The two-tailed P value is less than 0.0001.

Multiple comparisons Are we done? Unfortunately, not quite. In the above, we have performed a pairwise comparison of three groups. This is an example of a situation that makes us susceptible to a problem known as *multiple comparisons*. The problem is that we posed multiple hypotheses using the same data. Therefore, the chance that we see statistical significance in some pairwise comparison is increased purely by chance.

Consider our situation, in which we have only a probability of 0.01 of erroneously rejecting each null hypothesis if indeed the null hypothesis is true. We have, however, posed three such hypotheses. Therefore, $\Pr\{\geq 1 \text{ erroneous rejection}\} = 1 - \Pr\{\text{no erroneous rejection}\} = 1 - (0.99)^3 \approx 0.03 > 0.01$. (We multiply the three probabilities, 0.99, under the assumption that those probabilities across the three tests are independent.)

How do we address this? A natural way is to correct for the multiple comparisons. A correction is suggested by the example above, and is called the Bonferroni correction. Under the Bonferroni correction, to test k hypotheses with a desired significance level of s , we test each hypothesis at a significance level of s/k . In our case, $k = 3$, and so we need to test each of our pairwise hypotheses at a threshold of 0.0033 so that the family of tests has a threshold of 0.01.

In our case, the computed probability based on the data is less than 0.0033 for each test, so our conclusions are remain valid.

The Bonferroni correction is conservative, i.e., overkill, and therefore safe. A more exact correction can be applied instead. The way we computed the probability above for erroneously rejecting at least one null hypothesis is the way to compute an exact correction, but we omit the details.

Closing This is just an introduction to the use of statistics in evaluating software performance. The techniques are sound but this is a complicated subject and the brief overview in this lecture is by no

means a substitute for a full course on statistics. However, it should serve to alert you to the issues and perhaps whet your appetite to investigate such matters further in the future.

References

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