A Method For Determining the Analytic Solutions to Anharmonic Potentials : Time Coordinates

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1 Abstract

By considering the varying time coordinates, analytic solutions are found for particles in a nonlinear potential of the form λx^n . A method for finding the analytic solution for the Anharmonic Oscillator is presented by first considering the normal quanta of energy, $\hbar\omega$ in free space (no additional potential present) and then comparing the energy spectrum of this to an oscillator under the influence of a general field (first example, a gravitational field). Then the potential associated with the anharmonic oscillator is substituted in place of that of the gravitational field, as a first guess, revealing an analytic solution. It is then shown that an exact solution to the anharmonic oscillator will require a different coordinate metric than that for the gravitational field.

2 Introduction

The quartic anharmonic potential represents a significant challenge in solving. It is non linear, and the usual method of solving it involves uses the WKB approximation to obtain energies which quickly converge and agree closely with the numerical Schrodinger result. Why then would an exact solution hold much value, when the approximate result is close? The reason lies in the fact that the Quartic potential is an important potential that can be used to model many quantum mechanical systems in nature. Secondly, since the Schrodinger equation solution itself is numerical, its solution may deviate significantly from an actual Quartic potential that it is attempting to model in nature, even if it provides a precise mathematical result, because the numerical solution itself is not exact. This means that if we apply either the Schrodinger or WKB method to a Quartic potential system in nature, our energy levels may be off. Finally, a third reason is that if we can develop a new method to solve the Quartic potential exactly, then we might be able to use this method to tackle other quantum mechanical systems for which an exact solution does not exist. For example, low energy QCD systems where we would like to explain the confinement of the quark gluon system. Thus, a formalism that develops a method to calculate the exact energies of the Quartic potential could then be applied to such problems in Quantum Field theory, for which perturbation theory simply will not admit an exact solution. If such a formalism was developed then, it could also provide physical insights into precisely why perturbation theory does not work in these cases, which itself is an important problem in physics. We already know that mathematically the reason ultimately lies in the inability to represent the functions as Taylor expansions, but there could also be physical reasons .

3 A General Relativistic Formulation

In order to solve this class of centralized non linear potentials exactly, we can first look at another non linear potential which has an exact result, the gravitational field of classical field theories. Einstein has famously shown that to gain an accurate description of a field of this nature, we should treat space and time on the same footing, and that we need to integrate over both space and time coordinates when finding our potential, so that both the space and time coordinates play a dynamical role. For example, in a weak gravitational field, the relation between the proper time and the coordinate time is given as $d\tau = \sqrt{-g_{00}}dt$ where $g_{00} = -(1 + 2\frac{\Phi}{c^2})$

If we are moving in such a potential then, we need to account for the change in time coordinates as we move along the full extent of the potential. In ordinary quantum mechanics the time coordinates are fixed and non changing. However, nature has shown itself to be of the former type, rather than the latter, so in some cases it may be necessary to incorporate the relativistic effects in order to obtain an exact solution.

4 WKB Solution

In order to develop the formalism used to solve the Quartic potential, let's first consider a linear triangular potential of the form $V = \lambda|x|$ The turning points are therefore $E = x_0\lambda$ so that $x_0 = \frac{E}{\lambda}$ We then need to solve the integral

$$\int_{-x_0}^{x_0} \sqrt{2m(x_0\lambda - x\lambda)}dx$$

This potential is linear, and does not require any treatment of the time coordinates as varying with position. Let's view the beginning of the displacement and the end of the displacement as representing an overall potential

$$\Phi = X_{final} - X_{initial}$$

At this point, we can find the solution by doing our potential formulation. Let $x_0 - x = \Phi$

so that

$$2\sqrt{2m\lambda}\int_{0}^{x_0} \sqrt{\Phi}dx = (n + \frac{1}{2})\pi\hbar$$

$$(\frac{E}{\lambda})^{\frac{3}{2}} = [\frac{3(n+\frac{1}{2})\pi\hbar}{4\sqrt{2m\lambda}}]^{\frac{2}{3}}$$

$$E = [\frac{3(n+\frac{1}{2})\lambda\pi\hbar}{4\sqrt{2m}}]^{\frac{2}{3}}$$

This is the exact value for the energies of this potential.

The previous method for solving for the Quartic potential, involves using the WKB approximation. This method is as follows:

Find the turning points $\mu x_0^4 = E$ and therefore $x_0 = (\frac{E}{\mu})^4$

$$\sqrt{2mE_n}\int_{-(\frac{E_n}{\lambda})^{\frac{1}{4}}}^{(\frac{E_n}{\lambda})^{\frac{1}{4}}} \sqrt{1 - \frac{\lambda x^4}{En}}dx = (n - \frac{1}{2})\pi\hbar$$

let $z = (\frac{\lambda}{E_n})^{\frac{1}{4}}x$

$$\sqrt{2mE_n}(\frac{E_n}{\lambda})^{\frac{1}{4}}\int_{-1}^{1}\sqrt{1-z^4}dz = (n-\frac{1}{2})\pi\hbar$$

This cannot be solved exactly because

$C = \int_{-1}^{1}\sqrt{1-z^4}dz$ is not analytic and must be integrated numerically. The energies are found to be:

$$E_n = [\frac{\pi\lambda^{\frac{1}{4}}\hbar}{C\sqrt{2m}}(n-\frac{1}{2})]^{\frac{4}{3}}$$

Now, what if we attempt the same method of substituting a potential Φ, but this time for the λx^4 potential?

$E = \lambda x_0^4$ and $V = \lambda x^4$ Then

$$\sqrt{2m\lambda}\int_{-x_0}^{x_0}\sqrt{x_0^4 - x^4}dx = (n-\frac{1}{2})\pi\hbar$$

$$x_0^4 - x^4 = \Phi^4$$

$$2\sqrt{2m\lambda}\int_{0}^{x_0}\sqrt{\Phi^4}dx = (n-\frac{1}{2})\pi\hbar$$

with $x_0 = (\frac{E_n}{\lambda})^{\frac{1}{4}}$

$$2\sqrt{2m\lambda}\frac{2x_0^3}{3} = (n-\frac{1}{2})\pi\hbar$$

$$(\frac{E_n}{\lambda})^{\frac{3}{4}} = \frac{3\pi\hbar(n-\frac{1}{2})}{\sqrt{2m\lambda}}$$

$$E_n = [\pi\hbar\lambda^{\frac{1}{4}}\frac{3(n-\frac{1}{2})}{4\sqrt{2m}}]^{\frac{4}{3}}$$

We get an exact answer and this is close to the value calculated in the original WKB approximation, but not close enough, it differs by a significant amount in the constant. For the value found in the original WKB $C = 1.74804$ so that the overall value is $\frac{1}{1.74804}$ but the value found from the potential method which worked so well for the linear triangular potential, is now deviating from the numerical result, for it gives a constant of $\frac{3}{4}$. However, unlike the absolute value potential, whose solution was exact in both cases, we should remember that this potential is nonlinear.

Let's pause for a minute. We know that in general, non linear potentials are tough to solve. In classical relativistic physics, we know that there is a special kind of field, (the gravitational field) which can be solved exactly in simple cases where we have a very small amount of geometric curvature. What we observe in this case, is that the time coordinates themselves are changed by the field.

But we also know from relativity, that energy is energy. Any energy has an energy momentum tensor $T_{\mu\nu}$ which will bend time coordinates, there is nothing special about a gravitational field. While it is true that in Quantum Mechanics we typically don't think of time being curved, its also true that the energy density of the nucleus of an atom is immense. And if we think about Newton's law of gravitation, we realize that is it not just the mass, but truly the mass density which ultimately determines the strength of the field. So perhaps it is worth our time to take a relativistic approach to solving the Quartic potential, since in reality, it is used as a model for atoms and molecules. If we accept that energy in any form, be it gravitational or Quartic, can slightly bend spacetime, then a non linear potential such as the Quartic can mathematically be taken to be analogous to a nonlinear gravitational field.

Let's assume the simplest possible case where we have a modified metric in the time coordinates, a weak field with the time metric given as $d\tau = \sqrt{(1+\Phi)}dt$

We can obtain this by noting that

$$\frac{d\tau_1 - d\tau_2}{d\tau_2} = \frac{\Phi_1 - \Phi_2}{c^2}$$

So that for a static field we have

$$\frac{\Delta\tau}{\tau} = \frac{\Delta\Phi}{c^2}$$

Making the displacements infinitesimal :

$$\frac{d\tau}{\tau} = \frac{d\Phi}{c^2}$$

We now have an equation that will give us the value of time in terms of the potential.

$$d\tau = \tau \frac{d\Phi}{c^2}$$

Which gives a relationship between the differentials of the field $d\Phi$ and the proper time $d\tau$

Therefore, we have that the proper time integral

$$\int d\tau = \int \frac{\tau}{c^2} d\Phi$$

Then if we think of the passage of time as being different at two different locations in this field, we can choose to write τ in terms of its value at the earlier point in the field

$$\tau = (1 + \frac{\Phi}{c^2})\tau_1$$

We can let $\tau_1 = 1$. Then if the potential were 0, then the total time will just be the same as an ordinary interval of time without a potential

$\tau_1 = \int_0^1 dt$

$$\int d\tau = \int \frac{(1 + \frac{\Phi}{c^2})\tau_1}{c^2} d\Phi$$

or in units where c = 1 and since $\tau_1 = 1$

In order to get this into the proper form for the WKB approximation,

$$\int d\tau = \int \sqrt{(1 + \Phi)} d\Phi$$

The turning points will occur at the two values where the energy equals the potential, so

$$\int d\tau = \int_{-\Phi}^{\Phi} \sqrt{(1 + \Phi)} d\Phi$$

So now we have a full substitution rule for any power law potential which does not have oscillation, i.e.

$$\lambda(x_0 - x)^n = \lambda \int_{-x}^{x} \Phi^n dx \int_{-\Phi}^{\Phi} \sqrt{(1 + \Phi)^n} d\Phi$$

This last expression can be further simplified. We can factor Φ out of the integrand, and choose it to be 1, or let it be any constant that is absorbed into the constant λ so we have :

$$\lambda(x_0 - x)^n = \lambda \int_{-x}^{x} \Phi^n dx \int_{-1}^{1} \sqrt{(1+\Phi)^n} d\Phi$$

Let's rewrite the integral for n=4 which is the quartic potential we looked at before, but this time considering the time coordinates as well. We have:

$$\sqrt{2m\lambda} \int_{-x_0}^{x_0} \sqrt{\Phi^4} dx \int_{-1}^{1} \sqrt{(1+\Phi)^4} dt = (n - \frac{1}{2})\pi\hbar$$

$$\frac{8}{3}\sqrt{2m\lambda} \int_{-x_0}^{x_0} \sqrt{\Phi^4} dx = (n - \frac{1}{2})\pi\hbar$$

$$\frac{16}{3}\sqrt{2m\lambda} \int_{0}^{x_0} \sqrt{\Phi^4} dx = (n - \frac{1}{2})\pi\hbar$$

$$\frac{16}{3}(\frac{1}{3})\sqrt{2m\lambda}(\frac{E_n}{\lambda})^{\frac{3}{4}} = (n - \frac{1}{2})\pi\hbar$$

$$(\frac{E_n}{\lambda})^{\frac{3}{4}} = \frac{9(n - \frac{1}{2})\pi\hbar}{16\sqrt{2m\lambda}}$$

so that

$$E_n = [\frac{9(n - \frac{1}{2})\pi\hbar\lambda^{\frac{1}{4}}}{16\sqrt{2m}}]^{\frac{4}{3}}$$

compare the value of $\frac{9}{16} = 0.5625$ vs $\frac{1}{1.74804} = 0.5747$

5 A New Correspondence

Let's assume we have a very long spring, with a very weak spring constant, that nonetheless obeys a Quartic Potential. This means that the spring is stretched to its full extent and provides a potential energy that is $V = \lambda x^4$ Now let's imagine that instead of a spring, we have a special configuration of mass, which produces precisely the same potential as the Quartic spring. Then, if our mass is in this gravitational field, we will have to consider the time coordinates as well as the spatial ones. Energy of any kind will bend space, it need not be caused by a mass distribution. Therefore, let's now assume that we have no gravitational field, but we again have the Quartic Anharmonic spring. Then in this case, we should still see an effect on the time coordinates. Taking into account this correspondence, we can then produce a new WKB integral that includes the metric for the time coordinates.

6 Derivation: Harmonic Oscillator In a Gravitational Field

Imagine we have a special spring apparatus that discharges light. We can displace the spring a distance Δx and when we do we get a photon of frequency $\hbar\omega$ emitted right afterwards, starting from the location of the displacement. imagine the spring is incredibly rigid, so that even tiny displacements will produce photons. Now imagine we are in free space, floating far away from any gravitational field. The same apparatus can be tested, and a corresponding photon will be produced as we displace the spring at various lengths. Imagine we can dial in our system so that the energy levels will match those of a quantum harmonic oscillator, because the displacement is tiny and incremental enough.
The potential energy of the system would then be

$$V(x) = \frac{1}{2}\omega^2 x^2$$

Now take the same appartus, and put it above a spherically symmetric gravitational field. The potential energy is altered and is now

$$V(x) = \frac{1}{2}\omega^2 x^2 + \Delta\Phi(x)$$

This time, when we displace the oscillator and a quantum bit of light is produced, its frequency will be slightly blueshifted, compared to if the oscillator was not present in the gravitational field. The oscillator is displaced quantum amounts of δx, and a photon of energy $\hbar\omega*$ some additional amount due to the gravitational field is produced each time. In this way, the energy levels of the harmonic oscillator have shifted. The shift is due to the gravitational field, and it increases with each displacement, because a greater change in potential produced Δx a greater change in frequency blue shift. This shift has been experimentally verified$_1$.

The new shifted frequency is related to the old one by the expression:

$$\Delta\omega_{new} = \omega(1 + \frac{\Delta\Phi}{c^2} + O(\frac{\Delta\Phi^2}{c^4}))$$

Now the energy levels associated with the apparatus without the presence of a gravitational field would be $\sum_{n=0}^{\infty}(n + \frac{1}{2}\hbar\omega)$, so in the presence of the field, they become:

$$E_n = (n + \frac{1}{2})\hbar\omega_n(1 + \frac{\Delta\Phi}{c^2} + O(\frac{\Delta\Phi^2}{c^4})) \quad for\ n = 0, 1, 2, ...$$

We can consider this perturbing of the original energies as due to the new metric introduced by the gravitational field, which shifts the energies correspondingly. In this way, the new potential directly interacts with the old energy levels.

So far there is nothing new about this result. Red shift / blue shift due to gravitational fields is predicted by general relativity and well tested by experiment. But what is interesting to now consider, is what happens an extension is made to this method of altering the energy levels of a quantum harmonic oscillator system to other potentials, such as the anharmonic oscillator, and in doing so, obtain an exact solution in terms of the potential field terms altering the original (regular QM harmonic oscillator solution). While the expression for the shift in energies for the first result was given in terms of a gravitational field, this time one could try a potential such as the anharmonic oscillator

$$V(x) = \frac{1}{2}\omega^2 x^2 + \lambda x^4$$

$$E_n = (n + \frac{1}{2})\hbar\omega_n(1 + \lambda\Delta x^4) \quad for\ n = 0, 1, 2, ...$$

In order to accurately model the metric for our new modified coordinates that will in the end, affect the values for $\hbar\omega$, it is convenient to introduce two numerical parameters describing the displacement of the anharmonic oscillator Δx^4 First, we need a term that represents the constant length when the spring is not stretched at all. We can call this Φ_1 = initial oscillator length

Next, we need a term that represents a small displacement Δx of the oscillator, which will produce our actual energy levels. We want to make each displacement the same distance Δx so we will have n Δx displacements. $\Phi_2 = n\Delta x$

so then

$$\Delta x^4 = [\Phi_1 + (n+1)\Phi_2]^4$$

Inserting these and letting $\lambda = 1$

$$E_n = (n + \frac{1}{2})\hbar\omega_n(1 + [\Phi_1 + (n+1)\Phi_2]^4) \quad for\ n = 0, 1, 2, ...$$

This equation is in very close agreement with calculated results for the the anharmonic oscillator from perturbation theory.Here we obtained the result by using a coordinate change that blue shifts the frequency ω by

$$\omega_{new} = \omega(1 + [\Phi_1 + (n+1)\Phi_2]^4) \quad for\ n = 0, 1, 2, ...$$

In the gravitational case if we were in flat coordinate space, there should be no deviation of the frequencies, and the deviation in fact results due to the change in time coordinate separations since

$$\omega \propto \frac{1}{dT}$$

where dT is the proper time coordinate separation,

$$\frac{\Delta\omega}{\omega} = \frac{GM}{c^2 r}$$

and is a direct result of

$$\Delta\tau_2 = (1 + \frac{\Delta\Phi}{c^2})\tau_1$$

Ultimately, it is the time coordinate separations that shift the energy levels for the gravitational potential

$$Gravitational \qquad Solution$$

$$\omega_{new} = \omega(1 + \frac{\Delta\Phi_n}{c^2} + O(\frac{\Delta\Phi_n^2}{c^4})) \quad for\, n = 0, 1, 2, ...$$

In a mathematically analogous way, this same process occurs for the anharmonic oscillator, which can then be exploited to obtain an exact solution, provided that the exact form of the potential is known.

This amounts to understanding how a non linear potential can impact the time coordinates of a quantum mechanical system, directly affecting the energies. The evidence that this is the case is given by the famous redshift result, where the energy level of the photon is directly affected by the gravitational potential(another nonlinear potential.) The gravitational potential stores energy, and a portion of this energy can be exchanged with the quantum particle of light as it experiences a change of gravitational potential. Analogously, the quantized harmonic oscillator energies will also experience an exchange of energy with the anharmonic potential that they become subject to, resulting in a "blueshift" of their energy spectrum that is of order Δx^4

7 Zero Point Energy

For the case of the Harmonic oscillator in a vacuum, the zero point energy will just be the ordinary result obtained from solving the schrodinger equation. However, when this same oscillator is placed in a gravitational field, there will be a corresponding shift to the zero point energy. We can work this out in detail using the uncertainty principle.

The energy time uncertainty principle is the direct result of the quantization of the system. It is generally interpreted as a limit on the certainty with which we can simultaneously determine the energy and time of a particle in a quantum mechanical system.

We have:

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

and $\Delta t = \frac{1}{\omega}$ so we have $E = \frac{1}{2}\hbar\omega$

This is the zero point energy for a harmonic oscillator without any other potential.

We could view this as the zero point for our previously discussed light emitting simple harmonic oscillator device, when the device is present in flat space, far from any gravitational field. Now, if we then place the oscillator in a gravitational field with potential $\Delta\Phi$,

will the zero point energy change? Yes, because the passage of time, Δt is very much influenced by a gravitational field, as previously discussed.

What we are doing then, is comparing the change of time coordinates at the two different locations of the identical oscillators. Location one is in free space and has a passage of time given as ΔTa while the passage of time in the location within the gravitational field has lengthened to be

$$\Delta Tb = \Delta Ta(1 + \frac{\Delta\Phi}{c^2})$$

However, it is always true that

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

$$\Delta T_a \Delta E_a = \frac{\hbar}{2} = \Delta T_b \Delta E_b$$

and we have

$$\Delta E_b = \frac{\Delta T_a \Delta E_a}{\Delta T_b}$$

We thus recognize ΔE_b to be the change in zero point energy of the simple harmonic oscillator due to it now being placed in a gravitational field.

We therefore identify this as our previously discussed potential term in our anharmonic oscillator potential:

$$\Phi_1 = zero\, point\, energy$$

Any additional displacement within the field itself will also lead to a further change in energies. As we displace the oscillator a discrete distance Δx we will therefore get a new location and a new passage of time, thus a new shift in energies given by the expression above, which we can repeat for n displacements.

So we have,

$$\Delta t_{n+1} = \Delta t_n (1 + \frac{\Delta \Phi_n}{c^2})$$

so

$$\Delta t_{n+1}$$

will be the longer time duration. We can therefore identify the other potential term

$$\Phi_2 = change\ in\ energies\ as\ oscillator\ displacement\ in\ field\ increases$$

While the first potential term was obtained by comparing time measurements between two separate coordinate maps, this term occurs because of a change in position within the field itself. We can also identify this term to be the displacement of the quartic potential of the anharmonic oscillator Δx^4

Since our energy shift is really due to a shift in ω we can look at the contribution as such:

$$\omega_{new} = \omega(1 + [\Phi_1 + (n+1)\Phi_2]^4) \quad for\ n = 0, 1, 2, ...$$

In the expression above, $(1 + [\Phi_1 + (n+1)\Phi_2]^4)$ represents the full shift in energy of the quartic potential. The number 1 term that first occurs is due to the original zero point energy of the harmonic oscillator. Then we have the contribution due to the modified zero point, as well as the change of the oscillator postion in the field. Notice that both of these terms are raised to the forth power, because the modified zero point energy will contribute to the overall displacement, which is O^4 . In fact, we can also think of the zero point as due to the position momentum uncertainty relation, where we have $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$ or $\sqrt{\frac{\hbar}{2m\omega}}\Delta p = \frac{\hbar}{2}$ or $\Delta p^2 = \frac{2m\omega\hbar}{4}$ or $\omega = 2\frac{\Delta p^2}{\hbar m}$ which interestingly looks a lot like the kinetic energy term of the Schrodinger Equation. Ultimately, since ω is shifted by the potential, a modified zero point energy, signifying an uncertainty in the location of the particle in the presence of the field, will be associated with the new energy.

For the case of the gravitational field, displacements within the field will give shifts in the energy spectrum of:

$$\frac{\hbar}{2\Delta E_{n+1}} = \frac{\Delta t_n}{(1 + \frac{\Delta \Phi_n}{c^2})}$$

So now we have a system that gives us our energies associated with a given potential.

$$\Delta E_{n+1} = \frac{\hbar(1 + \frac{\Delta\Phi}{c^2})}{2\Delta t_n}$$

which is the same result we obtained earlier for the energies, by considering the gravitational blueshift.

We can read this as saying that the next energy is related to the one before it by

The key thing to remember is that the coordinate separations for time are varying as we move along the potential.

solving for the energies:

$$\Delta E_{n+1} = \frac{\hbar(1 + \frac{\Delta\Phi_n}{c^2})}{2\Delta t_n}$$

Finally, we have

$$\Delta t_n = \frac{\hbar}{2\Delta E_n}$$

and inserting into the previous expression we get

$$\Delta E_{n+1} = \Delta E_n(1 + \frac{\Delta\Phi_n}{c^2})$$

for case of the harmonic oscillator, we will just have a difference of

$$\Delta E = \hbar\Delta\omega$$

But now we have to be concerned with a shift of frequencies, so :

$$\Delta E_{n+1} = \hbar\Delta\omega_{n+1}$$

$$\Delta E_{n+1} = \hbar\Delta\omega_n(1 + \frac{\Delta\Phi}{c^2})$$

So Now I have an exact result for the harmonic oscillator energies if they were perturbed by a gravitational potential. Say there is no gravitational potential, then insert zeros into the potential terms and recover the differences of energies for a normal harmonic oscillator. Note that the energies become evenly spaced again, whereas they are not evenly spaced for the harmonic oscillator perturbed by the gravitational potential.

The energies start off minimal, and are then blue shifted by the perturbation of the gravitational potential, where the amount of blueshift is given by the expression above.

Now in the next step one can insert a potential into the expression and solve for the available set of energies and times. The first potential to try is the anharmonic oscillator potential. This can be solved for a 1-D case, because our levels will scale from top to bottom, just as our time variable does. We have

$$V(x) = \frac{1}{2}\omega^2 x^2 + \lambda x^4$$

Now comes a critical step. Noting the generality of the potential formulation:

$$\Phi(x) = \frac{1}{2}\omega^2 x^2 + \lambda x^4$$

treat λx^4 mathematically as if it is just a field perturbation such as the one we saw for our gravitational field. We will have two additional terms associated with the change in the zero point energy Phi_1 and the change of actual coordinates as the oscillator is displaced a greater distance, giving larger energies Phi_2

8 Solving for the exact energies

Consider now a new situation. We have an atom located at a point in a gravitational potential where its potential energy from gravitational energy is precisely equal to the anharmonic oscillator energy, first excited state. The atom is a acting as a quantum harmonic oscillator, and is excited to its first energy level, and emits the photon of this energy. The photon starts out being equal to simply the energy of a normal harmonic oscillator, but as it goes downward towards the 2nd atom at the bottom of the potential, it gets blueshifted. When it reaches the 2nd atom it is now of the energy of the anharmonic oscillator, so it excites the anharmoic oscillator which absorbs and re emits it. The photon travels back up through the field, becomes redshifted and excites the atom to emit it as a simple harmonic oscillator.

I set this up as an equation, then solve for the value of the potential which gives me the exact value of the anharmonic oscillator energies in terms of the gravitational potential. So I have an exact form of the quantum anharmonic oscillator

Thus, I can equate the WKB form of the anharmonic oscillator, to the blueshifted energy of the emitted photon. I can then create a system where the anharmonic oscillator is excited by this energy, and remits the light, at the anharmonic frequency. The light is now redshifted back to the energy of the harmonic oscillator. In this way, the system continues into infinity. In actuality, the spontanous emission prevents the atom from being directed in precisely the same way. So now let's consider an ensemble of atoms, say an atmosphere of harmonic oscillators floating above at precisely the right gravitational potential height. These oscillators emit and the photons become blueshifted. On the ground below, we have an ensemble of anharmonic oscillators.

the energies of the anharmonic oscillator, found from the WKB method are:

$$E_n = [\frac{\pi\hbar\lambda^{\frac{1}{4}}(n-\frac{1}{2})}{\int_{-1}^{1}\sqrt{1-\frac{\lambda}{E_n}x^4 dx}}]^{\frac{4}{3}}$$

These will be precisely the energies available to the anharmonic oscillator at the bottom of the gravitational potential.

The energies of the emitting atom are simply

$$E_n = (n+\frac{1}{2})\hbar\omega$$

We have placed this harmonic oscillator at the exact height Phi so that when it emits a photon, the photon will be blueshifted by an amount:

$$E_n = [(n+\frac{1}{2})\hbar\omega][1+\frac{\Delta\Phi}{c^2}]$$

$$[(n+\frac{1}{2})\hbar\omega][1+\frac{\Delta\Phi}{c^2}] = [\frac{\pi\hbar\lambda^{\frac{1}{4}}(n-\frac{1}{2})}{\int_{-1}^{1}\sqrt{1-\frac{\lambda}{E_n}xdx}}]^{\frac{4}{3}}$$

$$[\int_{-1}^{1}\sqrt{1-\frac{\lambda}{E_n}x^4 dx}]^{\frac{4}{3}}[1+\frac{\Delta\Phi}{c^2}] = \frac{[\pi\hbar\lambda^{\frac{1}{4}}(n-\frac{1}{2})]^{\frac{4}{3}}}{[(n+\frac{1}{2})\hbar\omega]}$$

we can imagine displacing the spring a tiny amount $delta\Phi$ so that $x^4dx = \delta\Phi d\Phi$

$$[\int_{-1}^{1}(1-\frac{\lambda}{E_n}\delta\Phi)d\Phi]^{\frac{2}{3}}[1+\frac{d\Phi}{c^2}] = \frac{[\pi\hbar\lambda^{\frac{1}{4}}(n-\frac{1}{2})]^{\frac{4}{3}}}{[(n+\frac{1}{2})\hbar\omega]}$$

9 Comparison with Perturbation methods

recall we started with

$$E_n = \hbar\omega[1+\lambda(\Delta x)^4]$$

We can set the parameter $\lambda = 1$ and we have:

$$E_n = \hbar\omega_{new} = \hbar\omega(1+[\Phi_1+(n+1)\Phi_2]^4) \quad for\; n = 0,1,2,.....$$

the two terms representing the potential are

$$\Phi_1 = initial\; uncertainty\; of\; quantum\; spring\; i.e.\; zero\; field\; modified\; zero\, point\; energy$$

$$(n+1)\Phi_2 = displacement\; of\; spring\; for\; n = 1,2,....$$

Now its important to choose actual values and compare with results obtaining using The Dirac Operator Technique.

Set $\Phi_1 = 0.255$ and $\Phi_2 = 0.0138n$ which will increase for 1,2,3,etc. The results, which are designated as E_n are tabulated below and compared for the energies obtained using Dirac operator methods

Energy	Dirac Operator Tech.$_2$	E_n
ground	0.50374	0.50261
first excited	1.51875	1.50956
second excited	2.54875	2.51929
third excited	3.59375	3.532406
fourth excited	4.65375	4.54958
fifth excited	5.72875	5.57161
sixth excited	6.81875	6.59933
seventh excited	7.92375	7.63370
eight excited	8.68125	8.67574
ninth excited	9.72625	9.72661

$Table\ 1. Energies\ calculated\ and\ compared\ to\ operator\ methods\ Dirac\ Operator\ Values_2$

We know that $\Delta x * \Delta p = \frac{\hbar}{2}$ where the De Broglie wavelength $\lambda = \frac{h}{p}$ so if we introduce the uncertainty we will have $\Delta\lambda = \frac{h}{\Delta p}$ and then substitute this relation to get $\Delta x \propto \Delta\lambda$ Consider the zero point energy as being due to some initial displacement (since the uncertainty principle forces the system to have a Δx . Associate the initial displacement with an initial momentum so that $\Delta x_0 \Delta p_0 = \frac{\hbar}{2}$ and then consider the additional change in wavelength $Delta\lambda$ due to the change in frequency caused by the new anharmonic potential. Then $\lambda_0 + \Delta\lambda \propto 4(x + \Delta x)$

so we can identify the zero point energy value from before as being $\Phi_1 \propto \frac{\lambda_0+\Delta\lambda}{4}$ where $lambda$ is the De Broglie wavelength plus the shift due to the change in ω due to the potential, in this case the shift is about 2 percent so $\lambda_0 + \Delta\lambda = 1.02$.

Since $\Phi_2 \propto \Delta x \propto \Delta\lambda$, we can rewrite our formula for the energies of the anharmonic oscillator in terms of the De Broglie wavelenth.

$$E_n = \hbar\omega_{new} = \hbar\omega(1 + [\frac{\lambda_0 + \Delta\lambda}{4} + (n+1)\Delta\lambda]^4) \quad for\ n = 0, 1, 2, ...$$

Energy	Dirac Operator Tech.$_2$	E_n
ground	0.50374	0.502859
first excited	1.51875	1.51136
second excited	2.54875	2.52461
third excited	3.59375	3.54408
fourth excited	4.65375	4.57147
fifth excited	5.72875	5.60876
sixth excited	6.81875	6.65823
seventh excited	7.92375	7.72246
eight excited	8.68125	8.80435
ninth excited	9.72625	9.90716

$Table\ 2. Energies\ calculated\ and\ compared\ to\ operator\ methods\ Dirac\ Operator\ Values_2$

The results of the energies will differ depending on the choice of constants and starting energies chosen for the oscillator.

We can do the same calculation for other energies obtained for the anharmonic oscillator given different values of the constants.

In this case let

$$\Phi_1 = 1.0078$$

$$\Phi_2 = 0.00984684n$$

Energy	numerical	E_n	2nd order	3rd order
0	1.03473	1.03472	1.03422	1.03487
1	3.16723	3.16720	3.16172	3.16937
2	5.41726	5.38681	5.39141	5.39141
3	7.77027	7.69730	7.69141	7.82192

$$Table\ 3. Energies\ Of\ Anharmonic\ Oscillator\ Calculated_3$$

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