A Brief Report on Boolean Algebras

(*previously* A Brief Treatment of Boolean Algebras)

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1 Introduction

The two-element set $\mathbb{F}_2$ con be ordered: $0 \leq 0 \leq 1 \leq 1$, for example. The power set $\mathcal{P}(X)$ of a set X can be ordered by inclusion: if $A, B \subset X$, then we say $A \leq B$ if $A \subset B$. In the first case, we have a totally ordered set. In the second case, the set is merely partially ordered. Being a field, $\mathbb{F}_2$ is endowed with the operations *sum* and *product*. These operations satisfy the following (in)equalities:

$$\begin{aligned}
0 \cdot 0 &= 0 & 0+0 &= 0 & 0+0+0\cdot 0 &= 0 \\
0 \cdot 1 &= 0 & 0+1 &= 1 & 0+1+0\cdot 1 &= 1 \\
1 \cdot 0 &= 0 & 1+0 &= 1 & 1+0+1\cdot 0 &= 1 \\
1 \cdot 1 &= 1 & 1+1 &= 0 & 1+1+1\cdot 1 &= 1
\end{aligned}$$

We therefore define the (unnamed) operations $\wedge$ and $\vee$ on $\mathbb{F}_2$ by

$$\begin{aligned}
a \wedge b &:= a \cdot b \\
a \vee b &:= a + b + a \cdot b \ .
\end{aligned}$$

It then follows that

$$\begin{aligned}
&0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0 \ , \\
&1 \wedge 1 = 1 \ , \\
&0 \vee 0 = 0 \quad \text{and} \\
&0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1 \ .
\end{aligned}$$

In particular, $0 \wedge 1 = 0$ and $0 \vee 1 = 1$.

For the power set of a set X, we have the binary operations $\cap$ and $\cup$. These operations are related to inclusion in the following way:

$$\begin{aligned}
\varnothing \subset A &\subset X \ , \\
A \cap B &\subset A \quad \text{and} \\
A &\subset A \cup B
\end{aligned}$$

for all subsets $A, B \in \mathcal{P}(X)$. If we defined $\wedge$ and $\vee$ on $\mathcal{P}(X)$ to mean, respectively, the operations of intersection and union, then, with respect to complementation

$$\begin{aligned} A \wedge A^{\mathsf{c}} &= \varnothing \quad \text{and} \\ A \vee A^{\mathsf{c}} &= X \ . \end{aligned}$$

In both situations each element x in the corresponding set $\mathbb{F}_2$ or $\mathcal{P}(X)$ has a *complementary* element $\bar{x}$ such that $x \wedge \bar{x}$ equals the smallest element and $x \vee \bar{x}$ equals the greatest element in the partially ordered set.

For another example, consider a set X and, in $\mathcal{P}(X)$, the subcollection of sets $A \in \mathcal{P}(X)$ which are either finite or *cofinite* in X, that is to say, either $\#A < \infty$ or $\#A^{\mathsf{c}} < \infty$. Let $\mathsf{Z}(X)$ denote this collection. Given subsets $A, B \in \mathcal{P}(X)$,

i if A is finite, then A^{c} is cofinite (and if A is cofinite, then A^{c} is finite),

ii if A and B are both finite sets, then both $A \cap B$ and $A \cup B$ are finite sets,

iii if A is finite and B is cofinite in X, then, since

$$\begin{aligned} A \cap B &\subset A \quad \text{and} \\ A \cup B &\supset B \ , \end{aligned}$$

their union is cofinite and their intersection is finite.

This shows that the collection $\mathsf{Z}(X)$ of ssubsets $A \in \mathcal{P}(X)$ which are either finite or cofinite in X, is closed with respect to the operations of pairwise union, pairwise intersection and complementation defined in $\mathcal{P}(X)$. It is also true that both X and $\varnothing$ belong to $\mathsf{Z}(X)$. So $\mathsf{Z}(X)$ shares some struture with $\mathcal{P}(X)$, which can then be said to be inherited from $\mathcal{P}(X)$, in a sense.

2 Definitions

A *partially ordered set* (also *poset*) is defined to be a set together with a binary relation, called the *order relation*, which is *reflexive*, *transitive* and *antisymmetric*. A poset is said to be *totally ordered*, if the order relation is *dychotomic*.

Let $(P, \leq)$ be a poset and let $Q \subset P$ be an arbitrary subset. If we restrict the ordered relation $\leq$ to elements belonging to Q, then a partial order is obtained on Q. The pair $(Q, \leq)$ is then a poset, as well. We shall say in this case that Q is a *subposet* of P or that Q *inherits* the poset structure –or the order relation– from P.

A *chain* in a partially ordered set is a subset which is totally ordered by the order relation inherited from the poset.

Examples So, we see that $\mathbb{F}_2$, $\mathcal{P}(X)$ and $\mathsf{Z}(X)$ are partially ordered sets, if endowed with the order relations defined above. The two-element set $\mathbb{F}_2$ is totally ordered but, in general, $\mathcal{P}(X)$ and $\mathsf{Z}(X)$ are not.

2.1 Joins and Meets

Given a partially ordered set $(P, \leq)$ and a subset $A \subset P$, an *upper bound* for A in P is an element $x \in P$ such that

$$a \leq x$$

for all $a \in A$. We shall say that an element $x \in P$ is a *supremum* (or join) for A in P, if it is a least upper bound for A in P, that is to say

i $a \leq x$ for all $a \in A$ and

ii if b satisfies $a \leq b$ for all $a \in A$, then $x \leq b$.

We denote it by $\bigvee A$. Upper bounds and therefore joins need not exist for arbitrary subsets of a poset P. By the antisymmetry of $\leq$ in P, if a join for a subset A exists, then it must be unique. Given a poset $(P, \leq)$ and a pair of elements $x, y \in P$, the *join* of x and y is the supremum (join) for the set $\{x, y\}$ in P, if it exists. We denote it by $\bigvee \{x, y\} = x \vee y$.

Remark. Let $(P, \leq)$ be a poset. Every element $x \in P$ is, trivially, an upper bound for the empty subset $\varnothing \subset P$. This does not imply, however, that there exists a join for $\varnothing$. If the empty subset admits a join, $x = \bigvee \varnothing$, then this means that $x \leq b$ for every element $b \in P$.

Dually, given a poset $(P, \leq)$ and a subset $A \subset P$, we say that an element $x \in P$ is a *lower bound* for A in P, if

$$a \geq x$$

for all $a \in A$. An *infimum* (or meet) for A is a lower bound for A which greatest among the lower bounds for A, this means

i $a \geq x$ for all $a \in A$ and

ii if $b \in P$ satisfies $a \geq b$ for all $a \in A$, then $x \geq b$.

If such an element exists, we shall denote it by $\bigwedge A$. Lower bounds, as well as meets, need not exist. However, if a meet for a subset A exists, then it must be unique by the antisymmetry property of the order relation $\leq$ in P. Given two elements $x, y \in P$, the *meet* of x and y is, if it exists, the infimum (meet) for the set $\{x, y\}$ in P and is denoted by $\bigwedge \{x, y\} = x \wedge y$.

Remark. Let $(P, \leq)$ be a poset. Every element $x \in P$ is, trivially, an lower bound for the empty subset $\varnothing \subset P$. This does not imply, however, that there exists a meet for $\varnothing$. If the empty subset admits a meet, $x = \bigwedge \varnothing$, then this means that $x \geq b$ for every element $b \in P$.

Example: $\mathcal{P}(X)$ In $\mathcal{P}(X)$, every subset $\mathcal{A} \subset \mathcal{P}(X)$ admits a supremum (as well as an infimum) in $\mathcal{P}(X)$. In particular, chains in $\mathcal{P}(X)$ have upper and lower bounds.

$\mathsf{Example}$: $\mathsf{Z}(X)$ In the subposet $\mathsf{Z}(X)$, not every subset has a supremum and an infimum, although every subset (subset of $\mathsf{Z}(X)$) is bounded by X from above and by $\varnothing$ from below. For instance, if $X = \mathbb{N}$ and

$$\mathcal{A} = \{\{2n\} : n \in \mathbb{N}\} ,$$

then any upper bound must contain every even number and must, in particular, be cofinite (in order to belong to $\mathsf{Z}(\mathbb{N})$). So there are (nontrivial) upper bounds, but there is not a smallest one. Similarly, if

$$\mathcal{A} = \{2 \cdot \mathbb{N} \cup \{n \leq m\} : n \in \mathbb{N}\} ,$$

then this set has no infimum. In $\mathsf{Z}(X)$, however, if $\mathcal{A} \subset \mathsf{Z}(X)$ is finite, then it admits both a join and a meet. As it has been mentioned, the poset $\mathcal{P}(X)$ admits supremums and infimums for every given subset $\mathcal{A} \subset \mathcal{P}(X)$. In particular, it admits joins and meets for every given pair $A, B \in \mathcal{P}(X)$, which are given, respectively, by pairwise union and intersetion of A and B. Since $\mathsf{Z}(X)$ is closed with respect to pairwise unions and intersections, $\mathsf{Z}(X)$ also admits joins and meets. The poset $\mathsf{Z}(X)$ also admits a least element, $\varnothing$, and a greatest element, X.

2.2 Join-semilattices and Meet-semilattices

Let $(P. \leq)$ be a partially ordered set. If every pair of elements $x, y \in P$ has a join $x \vee y$ and P has also an element $0 \in P$ which is less than every other element of P, then the following equalities hold for every $x, y, z \in P$:

$$\begin{aligned} x \vee x &= x , \\ x \vee y &= y \vee x , \\ x \vee (y \vee z) &= (x \vee y) \vee z , \\ x \vee 0 &= x . \end{aligned} \tag{1}$$

Equivalently, we may assume that every finite subset (including the empty subset) admits a join (supremum): if this is true, then every set containing exactly two elements admits a join and the fact that there exists an element $x = \bigvee \varnothing$ means that x is less than every other element of P; conversely, if every pair of elements admits a join in P, then, by induction in the number of elements, every finite subset admits a join, as well, and, since $x \in P$ is less than every other element of P, then, by definition, $x = \bigvee \varnothing$. We summarise this paragraph in the following theorem.

Theorem 2.1. *Let $(P, \leq)$ be a poset. Such that every finite (or empty) set admits a join. Then $(P, \vee, 0)$ has the structure of a commutative monoid in which every element is idempotent, that is to say, equations* (1) *hold.*

Conversely, triples $(P, \vee, 0)$ satisfying (1) can also be characterised in terms of an order relation, as the following theorem shows.

Theorem 2.2. *Let* $(P, \vee, 0)$ *be a commutative monoid in which every element is an idempotent. Then there exists a unique partial order on* P *such that* $x \vee y$ *is the join of* x *and* y*, and* 0 *is the least element.*

Proof. If $\leq$ is such a partial order on P, then $x \leq y$ implies $x \vee y = \mathsf{join}(\{x, y\}) = y$ and, conversely, $x \vee y = y$ implies $\mathsf{join}(\{x, y\}) = y$ and $x \leq y$.

If we define a binary relation $\leq$ on P by $x \leq y$ if $x \vee y = y$, then: *(i)* idempotency of $\vee$ implies reflexivity of $\leq$, *(ii)* commutativity implies antisymmetry and *(iii)* associativity implies transitivity. So $\leq$ is a partial order on P.

Given $x, y \in P$, associativity and idempotency imply $x \vee (x \vee y) = (x \vee x) \vee y = x \vee y$, so $x \leq x \vee y$ by definition of $\leq$. Similarly, by commutativity, $y \leq x \vee y$, so $x \vee y$ is an upper bound for the subset $\{x, y\}$. If $z \in P$ is such that $x, y \leq z$, then $(x \vee y) \vee z = x \vee (y \vee z) = z$. This shows that $x \vee y$ is a least upper bound for $\{x, y\}$, which means that $x \vee y$ is the join of x and y. Finally, since 0 is the neutral element, it is the least element in P. $\square$

A set P together with a binary operation $\vee$ and a distinguished element 0 such that equations (1) hold has the structure of a *(join-)semilattice*. Theorems 2.1 and 2.2 show that there is a correspondence between posets P having all finite joins (in particular, we are assuming that $\bigvee \varnothing$ exists, which is equivalent to P having a least element) and join-semilattices (that is, commutative monoids in which every element is idempotent):

$$\left\{\begin{matrix}\text{posets } (P, \leq) \text{ having} \\ \text{all finite joins}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{join-semilattices} \\ (P, \vee, 0)\end{matrix}\right\}$$

The image via this arrow of a poset $(P, \leq)$ is the triple $(P, \vee, 0)$ consisting of the set P, the binary join operation determined by the order relation $\leq$ and the least element $0 = \bigvee \varnothing$. If, conversely, $(P, \vee, 0)$ is a join-semilattice, then the relation given by $x \leq y$ if $x \vee y = y$ is a partial order on P such that $x \vee y$ is the join of x and y and 0 is the least element in P. This means that the triple $(P, \vee, 0)$ is the image of $(P, \leq)$ by the above arrow. The fact that the order $\leq$ is unique with this property means that this arrow is a correspondence.

However, this map is not an equivalence in the sense that, although a join-semilattice homomorphism must be an order-preserving morphism, there are order-preserving morphisms which do not respect the corresponding semilattice structure: for example, if $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of natural numbers with their usual ordering, and if $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $0 \leq n$ for $n \in \mathbb{N}$, then the inclusion $\mathbb{N} \hookrightarrow \mathbb{N}_0$ is order-preserving, but does not preserve the unit. This implies that the reverse arrow from join-semilattices to posets with all finite joins, although a bijection at the level of objects, is merely an injection at the level of morphisms.

The notions of lower bound, meet and greatest element are dual to those of upper bound, join and least element. By reversing all the inequalities implicit in the previous discusion, we arrive at analogous conclusions.

Theorem 2.3. *Let* $(P, \leq)$ *be a poset such that every finite subset admits a meet. Let* $1 = \bigwedge \varnothing$ *denote the greatest element and let* $x \wedge y$ *denote the meet of elements* x *and* y

in P. Then $(P, \wedge, 1)$ has the structure of a commutative monoid in which every element is idempotent. This means that the equations

$$\begin{aligned} x \wedge x &= x \ , \\ x \wedge y &= y \wedge x \ , \\ x \wedge (y \wedge z) &= (x \wedge y \wedge z) \ , \\ x \wedge 1 &= x \ . \end{aligned} \tag{2}$$

hold for every $x, y, z \in P$.

For the sake of symmetry, we shall call a triple $(P, \wedge, 1)$ such that equations (2) hold a *(meet-)semilattice.*

Theorem 2.4. *Let $(P, \wedge, 1)$ be a meet-semilattice. Then there exists a unique partial order on P such that $x \wedge y$ is the meet of x and y, and 1 is the greates element.*

2.3 Boolean Algebras

A *lattice* is a partially ordered set $(P, \leq)$ such that every pair of elements $x, y \in P$ has a meet and a join $x \wedge y$ and $x \vee y$, respectively, and there is a least and a greatest element 0 and 1, respectively, in P (or, equivalently, every finite (possibly empty) subset has a meet and a join in P); we denote it by $(P, \leq, \vee, \wedge, 0, 1)$. By theorems 2.2 and 2.4, a lattice can be characterised in the following way: let P be a set, $\vee$ and $\wedge$ binary operations on P and 0 and 1 distinguished elements of P such that $(P, \vee, 0)$ is a join-semilattice and $(P, \wedge, 1)$ is a meet-semilattice. Then this defines a lattice structure on P, if and only if, in additon, the induced partial orders are *opposite* to each other, that is to say, $x \vee y = y \Leftrightarrow x \wedge y = x$, for every $x, y \in P$. This additional condition is a necessary condition for P to be a poset when given any of the two partial orders, since in any poset $\mathsf{join}(x, y) = y \Leftrightarrow \mathsf{meet}(x, y) = x$. Conversely, if $\leq$ is the partial order on P induced by $(P, \vee, 0)$, say, then, on the one hand,

$$x \leq y \quad \Leftrightarrow \quad y = \mathsf{join}(x, y) = x \vee y \quad \Leftrightarrow \quad x = x \wedge y \ ,$$

which means that, in the order induced by $(P, \vee, 0)$, $x \wedge y$ equals $\mathsf{meet}(x, y)$; on the other hand, $x \wedge 1 = x$ for all $x \in P$ implies $x \vee 1 = 1$ for all $x \in P$ and, thus, 1 is the greatest element in P. Thus, by uniqueness, the partial order induced by $(P, \vee, 0)$ coincides with the (opposite to the) one induced by $(P, \wedge, 1)$, meaning that P is a poset with join $\vee$, meet $\wedge$, least element 0 and greatest element 1.

Proposition 2.5. *Suppose $(P, \vee, 0)$ and $(P, \wedge, 1)$ are semilattices. Then $(P, \vee, \wedge, 0, 1)$ is a lattice, if and only if the* absorptive laws

$$\begin{aligned} x \wedge (x \vee y) &= x \\ x \vee (x \wedge y) &= x \end{aligned} \tag{3}$$

are satisfied for all $x, y \in P$.

Proof. If the equations (3) hold, then $x \vee y = y \Leftrightarrow x \wedge y = x$. Conversely, in a lattice, the absorptive laws hold (take for instance $P = \mathcal{P}(X)$ and notice that only the poset-with-finite-joins-and-meets structure is needed). $\square$

A *distributive lattice* is a lattice P in which the *distributive law* holds: for every $x, y, z \in P$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{4}$$

Lemma 2.6. *If P is a distributive lattice, then the* dual *of the distributive law* (4) *holds, as well: for all $x, y, z \in P$,*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{5}$$

Conversely, in an arbirary lattice, the identity (5) *implies the identity* (4). *Thus, a distributive lattice is characterised by either of* (4) *or* (5).

Distributive lattices have the following important property.

Proposition 2.7. *Let P be a distributive lattice and let $x, y, z \in P$. Then there exists at most one element $w \in P$ satisfying both*

$$\begin{aligned} w \wedge x &= y \quad \textit{and} \\ w \vee x &= z \ . \end{aligned}$$

Let x be and element of an arbitrary lattice P. A *complement* of x in P is an element $\bar{x} \in P$ satisfying both $\bar{x} \wedge x = 0$ and $\bar{x} \vee x = 1$. Complements need not exist, in general; and, if they do, they need not be unique. Proposition 2.7 says that in a distributive lattice P, if an element x admits a complement, then it must be the only complement in P. A *Boolean algebra* is a distributive lattice P together with an additional unary operation $\bar{\cdot}: P \to P$ such that, for every $x \in P$, the element $\bar{x}$ is a complement of x.

Remark. If map between lattices preserves the lattice structure, that is, joins, meets, least element and greatest element, then it must also preserve the complementation relation. In particular, every lattice morphism is a Boolean algebra morphism.

2.4 Boolean Algebras and Boolean Rings

Let P be a Boolean algebra. The *symmetric difference* operation in P is defined as the following binary operation: if $x, y \in P$, let

$$x + y = (x \wedge \bar{y}) \vee (\bar{x} \wedge y) \ .$$

Appealing to the identities (4) and (5),

$$x + y = (x \vee y) \wedge (\bar{x} \vee \bar{y}) \ ,$$

and by existence and uniqueness of complements in a Boolean algebra,

$$\bar{x} \vee \bar{y} = \overline{(x \wedge y)} \ .$$

So, equivalently,

$$x + y = (x \vee y) \wedge \overline{(x \wedge y)} \ .$$

The proof of this equivalence made use of one of the *De Morgan laws*:

$$\begin{aligned} \overline{(x \wedge y)} &= \bar{x} \vee \bar{y} \ , \\ \overline{(x \vee y)} &= \bar{x} \wedge \bar{y} \ . \end{aligned} \tag{6}$$

Both of these identities hold in a Boolean algebra.

Theorem 2.8. *Let P be a Boolean algebra, let $\vee$, $\wedge$, 0 and 1 be its join, meet, least element and greatest element. Let $+$ denote its symmetric difference. Then, for every $x, y, z \in P$,*

$$\begin{aligned} x \wedge (y + z) &= (x \wedge y) + (x \wedge z) \ , \\ x + (y + z) &= (x + y) + z \ , \\ x + y &= y + x \ , \\ x + x &= 0 \quad \textit{and} \\ x + 0 &= x \ . \end{aligned}$$

These identities, together with

$$\begin{aligned} x \wedge y &= y \wedge x \ , \\ x \wedge 1 &= x \quad \text{and} \\ x \wedge x &= x \ , \end{aligned}$$

show that $(P, +, 0)$ is an Abelian group (in which every element is nilpotent) and that $(P, +, \wedge, 0, 1)$ is a commutative ring with unit in which every element is idempotent.

A *Boolean ring* is a ring with unit $(A, +, \cdot, 0, 1)$ in which every element is idempotent, that is, $x^2 = x \cdot x = x$ for all $x \in A$. So, 2.8 shows there exists an arrow sending each Boolean algebra to an associated Boolean ring:

$$\begin{Bmatrix} \text{Boolean} \\ \text{algebras} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{Boolean} \\ \text{rings} \end{Bmatrix}$$

$$(P, \vee, \wedge, 0, 1, \bar{\cdot}) \longmapsto (P, +, \cdot, 0, 1)$$

where $+$ is defined as the symmetric difference operation and $x \cdot y = x \wedge y$. Since a Boolean algebra morphism preserves the operations $\vee$, $\wedge$ and $\bar{\cdot}$ it must also preserve symmetric difference, and since such morphisms preserve greatest and least elements, they must preserve the neutral elements for addition and product in the associated Boolean rings. In other words, every morphism between Boolean algebras *is, too,* a morphism between the corresponding Boolean rings.

Boolean rings are commutative and every element of a Boolean ring satisfies $x+x=0$. This implies that $(A,\cdot,1)$ satisfies the identities (2). By theorem 2.4 there is a unique partial order on A such that $x\cdot y=\mathsf{meet}(x,y)$ and 1 is the greatest element: this order is defined by $x\le y$ if $x\cdot y=x$.

Lemma 2.9. *Let $(A,+,\cdot,0,1)$ be a Boolean ring and let $\le$ be the unique partial order determined by the meet-semilattice $(A,\cdot,1)$. Explicitly, $x\le y$ if $x\cdot y=x$. For $x,y\in A$, let $x\vee y=x+y+x\cdot y$, and let $x\wedge y=x\cdot y$. Then 0 is the least element in A, the following absorptive law holds:*

$$x\wedge(x\vee y)\ =\ x\ ,$$

the product $\wedge$ distributes over $\vee$ and also $x\vee y=\mathsf{join}(x,y)$.

The previous lemma implies that every Boolean ring has the structure of a distributive lattice and that this structure is unique, if we want $x\cdot y=\mathsf{meet}(x,y)$ and 1 to be the greatest element. Moreover, since $(1+x)\cdot x=0$ and $(1+x)+x+(1+x)\cdot x=1$, every element is complemented. This complement must be unique.

Theorem 2.10. *If $(A,+,\cdot,0,1)$ is a Boolean ring, then, with the partial order defined in 2.9 and the unary operation $x\mapsto\bar{x}=1+x$, the distributive lattice A is a Boolean algebra.*

Theorem 2.10 shows the existence of an arrow in the opposite direction to that of the one described after theorem 2.8:

$$\begin{Bmatrix}\text{Boolean}\\ \text{rings}\end{Bmatrix}\longrightarrow\begin{Bmatrix}\text{Boolean}\\ \text{algebras}\end{Bmatrix}$$

$$(A,+,\cdot,0,1)\longmapsto(A,\vee,\wedge,0,1,\bar{\cdot})$$

where the join is given by $x\vee y=x+y+x\cdot y$, the meet is defined by $x\wedge y=x\cdot y$, and the complement of and element x is $\bar{x}=1+x$. Since a Boolean ring morphism by definition preserves sum, product, zero and one, every such morphism must preserve join, meet, least element, greatest element and complements in the associated Boolean algebras. So, every morphism between Boolean rings *is also* a morphism between the corresponding Boolean algebras. The relation between both arrows is clarified by the following lemma.

Lemma 2.11. *If $(A,+,\cdot,0,1)$ is a Boolean ring and A is endowed with the Boolean algebra structure discribed in 2.10, then the symmetric difference operation is given by:*

$$\begin{aligned}(x\wedge\bar{y})\vee(\bar{x}\wedge y)\ &=\ (x\cdot(1+y))\vee((1+x)\cdot y)\\ &=\ x+y\ .\end{aligned}$$

Thus the two arrows between Boolean algebras and Boolean rings are inverse of each other, and since every Boolean algebra morphism is a Boolean ring morphism and *vice versa*, both categories are isomorphic.

Theorem 2.12. *The category of Boolean algebras is isomorphic to the category of Boolean rings.*

Something that was not mentioned is the fact that $\bar{\bar{x}} = x$ for every element x in a Boolean algebra. This property is related to the concept of a Heyting algebra.

Example Let $\mathbb{N}$ be the set of natural numbers. On $\mathcal{P}(\mathbb{N})$, define the following binary relation: $A \sim B$ if $A \Delta B = (A \smallsetminus B) \cup (B \smallsetminus A)$ is finite. This is an equivalence relation. Let $[A]$ denote the class of an element $A \in \mathcal{P}(\mathbb{N})$ and let X denote the set of classes. In X define the following order relation: $[A] \leq [B]$ if $A \smallsetminus B$ is finite (or empty). This definition does not depend on the choice of representatives A an B for the classes. This says that a class $[B]$ is greater than another class $[A]$, if B contains every element of A, except perhaps a finite number of them (and that it is strictly greater, if B exceeds A by more than just a finite number of elements). The set X together with the partial order $\leq$ is a Boolean algebra.

3 Filters

3.1 Ideals and Filters

Let $(P, \vee, 0)$ be a join-semilattice. An *ideal* in P is a subset I of P such that:

i I is a sub-join-semilattice of P and

ii for every $x \in I$ and $y \in P$, if $y \leq x$ then $y \in I$.

In contrast to the ring-theoretic notion of ideal, it is not necessary to include the axiom $0 \in I$, for any one of *(i)* and *(ii)* force $0 \in I$. The set of ideals in a join-semilattice is nonempty: $\{0\}$ and P are ideals.

The first condition in the above definition means that I is a submonoid of $(P, \vee, 0)$, that is, if $x, y \in I$ then $x \vee y \in I$, too. This is the analogue of being an additive subgroup. The second condition is the analogue of being closed under multiplication by elements of the ring. If P is a Boolean algebra, $y \leq x$ if $y \wedge x = y$, so the second condition is translated as being closed under multiplication by elements of P, where multiplication by an element $y \in P$ means $x \mapsto y \wedge x$ (this is the actual multiplication in the corresponding Boolean ring). If now $I \subset P$ satisfies *(i)* and *(ii)* above, then I is also closed under symmetric difference, that is, if $x, y \in I$ then $x + y \in I$, where $+$ denotes the symmetric difference operation in P (I is an ideal in the corresponding Boolean ring). Conversely, if $I + I \subset I$ and $P \cdot I \subset I$, then *(i)* and *(ii)* also hold. Thus, in a Boolean algebra, the ideals according to this definition are exactly the ideals of the corresponding Boolean ring.

Let P be a join-semilattice with partial order $\leq$. If $x \in P$, the subset of elements below x is an ideal and is the smallest ideal containing x. We call the ideal

$$(x) := \{y \in P \,:\, y \leq x\}$$

the *pricipal ideal* generated by x in P. We may also denote this ideal by $\downarrow(x)$.

A *filter* in a meet-semilattice $(P, \wedge, 1)$ is a subset $F \subset P$ satisfying axioms dual to those defining an ideal:

i if $x, y \in F$ then $x \wedge y \in F$ (F is a sub-meet-semilattice) and

ii for every $x \in F$ and $y \in P$, if $y \geq x$ then $y \in F$.

Actually, for a subset F of P to be a sub-meet-semilattice, both *(i)* and $1 \in F$ must hold. But $1 \in F$ is implied by *(ii)*. Thus, every filter contains the greatest element 1 and is, therefore, nonempty. The set of filters in a meet-semilattice is nonempty: $\{1\}$ and P are filters, the trivial filters, in P. A *principal filter* is a filter of the form $\{y \,:\, y \geq x\}$, this filter is said to be generated by the element x. The principal filter generated by x shall be denoted (x) or $\uparrow(x)$.

Remark. Let P and Q be two join-semilattices and let $f : P \to Q$ be a semilattice morphism. Then the *kernel* of f, the subset $\{x \in P \,:\, fx = 0\}$ is an ideal in P. Every ideal in a join-semilattice P can be recovered as the kernel of a semilattice morphism: given an ideal I in P, define a relation $\sim_I$ on P by $x \sim_I y$ if there exist $i, j \in I$ such that $x \vee i = y \vee j$; this is an equivalence relation and since $x \sim_I y$ implies $x \vee z \sim_I y \vee z$ for every $z \in P$, the set of equivalence classes can be made into a join-semilattice; the canonical projection from P onto the set of equivalence classes is a semilattice morphism with kernel I.

If P is a distributive lattice and I is an ideal in P, then I is equal to the kernel of some *lattice* morphism: we only need to verify that the equivalence relation defined in the previous paragraph respects meets, that is $x \wedge z \sim_I y \wedge z$ for every $x, y \in P$ such that $x \sim_I y$. This follows from distributivity: if $x \vee i$ is equal to $y \vee j$, then

$$\begin{aligned} (x \vee i) \wedge z &= (y \vee j) \wedge z \quad \text{and} \\ (x \wedge z) \vee (i \wedge z) &= (y \wedge z) \vee (j \wedge z)\ . \end{aligned}$$

The set of classes can be made into a lattice, which, being a *quotient* of the distributive lattice P, is, then, distributive.

Remark. Surjective lattice or semilattice morphisms are not determined by their kernels. Take for example the totally ordered $P = \{0, x, 1\}$ with $x \neq 0, 1$ and $0 \leq x \leq 1$ and the two-element lattice $2 = \{0, 1\}$. The map $f : P \to 2$ given by

$$0 \mapsto 0\ , \quad x \mapsto 1\ , \quad 1 \mapsto 1$$

is a lattice homomorphism with kernel $\{0\}$. On the other hand, there are no (Boolean) ring homomorphisms from $\{0, x, 1\}$ to $\{0, 1\}$. But this is not the only surjective lattice homomorphism from P with trivial kernel: the identity morphism $\mathsf{id} : P \to P$ is also such a map. In fact these are the only two such morphisms. Although $\ker(\mathsf{id}) = \ker(f)$, the maps f and id can be distinguished by the inverse image of the element 1:

$$f^{-1}(1) = \{y \in P : f(y) = 1\} \neq \{y \in P : \mathsf{id}(y) = 1\} = \mathsf{id}^{-1}(1) .$$

In general, if $f : P \to Q$ is a meet-semilattice homomorphism, the set $f^{-1}(1)$ is a filter in P.

As seen from the previous remark, the interaction between ideals and filters plays an important part in lattice theory.

Proposition 3.1. *Let $(P, \vee, \wedge, 0, 1)$ be a lattice and let $I \subset P$ be an ideal in P. The following conditions on I are equivalent:*

i the complement of I in P is a filter;

ii the greatest element 1 does not belong to I and $x \wedge y \in I$ implies $x \in I$ or $y \in I$; and

iii I is the kernel of a lattice homomorphism $f : P \to \mathsf{2}$.

Proof. If $F = P \smallsetminus I$ is a filter, then $1 \in F$. In particular, $1 \notin I$. If also $x \wedge y \in I$, then $x \wedge y \notin F$. In particular, it must be true that either $x \notin F$ or that $y \notin F$, so that $x \in I$ or $y \in I$.

If $1 \notin I$ and $x \wedge y \in I$ implies $x \in I$ or $y \in I$, then the map $f : P \to \mathsf{2}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I \end{cases}$$

is a lattice homomorphism. Clearly $f(0) = 0$ and $f(1) = 1$. If $x, y \in I$, then $x \vee y \in I$ and $f(x \vee y) = 0 = f(x) \vee f(y)$. If, on the other hand, $x \notin I$ or $y \notin I$, the inequality $x, y \leq x \vee y$ implies $x \vee y \notin I$ and $f(x) \vee f(y) = 1 = f(x \vee y)$. Thus f respects $\vee$. The additional conditions on I imply that f must also respect $\wedge$.

Finally, if I is the kernel of a lattice homomorphism $f : P \to \mathsf{2}$, then $P \smallsetminus I = f^{-1}(1)$ is a filter. ☐

3.2 Prime Ideals and Prime Filters

Let $(P, \vee, \wedge, 0, 1)$ be a lattice. An ideal $I \subset P$ satisfying any of the equivalent conditions stated in 3.1 is said to be a *prime ideal* and its complement $F = P \smallsetminus I$, which is a filter, a *prime filter*.

Let I be an ideal in the lattice P and let F be a filter in P disjoint from I. Applying Zorn's lemma, to the set of ideals which contain I and are disjoint from F, we see that there exists an ideal M in P which is maximal amongst those ideals containing I and disjoint from F.

In particular, if $I = 0$ and F is any filter, there exists a maximal ideal amongst those disjoint from F.

Theorem 3.2. *Let P be distributive lattice and let F be a filter in P. If I is a maximal ideal amongst those disjoint from F, then I is prime.*

Proof. Since $I \cap F = \varnothing$, $1 \notin I$. Let $x_1, x_2 \in P$ be such that $x_1 \wedge x_2 \in I$. Let $J_k = (I, x_k)$ be the ideal generated by I and x_k in P. The proof consists in checking that J_k is the subset of elements of the form $i \vee (x_k \wedge y)$ with $i \in I$ and $y \in P$ (which is an ideal); and then showing that at least one of J_1 and J_2 must be disjoint from F. Thus $J_k \supset I$ for some k. Maximality of I implies $J_k = I$ and $x_k \in I$. $\square$

Applying theorem 3.2 with $F = \{1\}$, so I is a *maximal proper ideal*, we may conclude that every maximal proper ideal is prime.

Remark. Although $\{1\}$ is a filter and $\{0\}$ is an ideal, the subset $P \setminus \{1\}$ is not an ideal and $P \setminus \{0\}$ is not a filter, unless $\{1\}$ is a prime filter (tautologically) and $\{0\}$ is a prime ideal, respectively.

Proposition 3.3. *Let P be a distributive lattice and let $x, y \in P$ with $x \ngeq y$. Then there exists a lattice homomorphism $f : P \to 2$ with $f(x) = 0$ and $f(y) = 1$.*

Proof. Let $I = \downarrow(x)$ and let $F = \uparrow(y)$. The condition $y \ngeq x$ implies $I \cap F = \varnothing$. By Zorn's lemma, there exists a maximal ideal I' amongst those containing I and disjoint from F. By 3.2, the ideal I' is a prime ideal and by 3.1, I' is the kernel of some lattice homomorphism $f : P \to 2$. Since $x \in I'$ and $y \notin I'$, $f(x) = 0$ and $f(y) = 1$. $\square$

Corollary 3.4. *Any distributive lattice P is isomorphic to a sublattice of the power set $\mathcal{P}(X)$ of some set X.*

Proof. Take X to be the set of homomorphisms $P \to 2$. $\square$

Proposition 3.5. *Let I be an ideal in a Boolean algebra P. The following conditions are equivalent:*

i I is prime;

ii for every $x \in P$, either $x \in I$ xor $\bar{x} \in I$; and

iii I is (proper) maximal.

Proof. Suppose first that I is prime. Since $x \wedge \bar{x} = 0$ and $0 \in I$, at least one of x and $\bar{x}$ must belong to I, but since $x \vee \bar{x} = 1$ and $1 \notin I$, at least one must lie outside I.

Assume now that for every $x \in P$ either $x \in I$ or $\bar{x} \in I$ but not both. Since $0 \in I$ and $1 = \bar{0}$, we have $1 \notin I$ and I is proper. If J is some other ideal strictly containing I, then there exists $x \in J \setminus I$. Since $x \notin I$, it must be the case that $\bar{x} \in I$ and $1 = x \vee \bar{x} \in J$ and $J = P$. Thus, the only ideal containing I strictly is P and I is a maximal proper ideal.

Since a Boolean algebra is, by definition, a distributive lattice, every maximal ideal in P must be prime by 3.2. $\square$

In fact, i implies ii is true in an arbitrary lattice, as long as x admits a complement, and ii implies iii is true in any complemented lattice.

Proposition 3.6. *Let I be a (proper) ideal in a lattice P. Assume I is a prime ideal. If $x \in P$ and z is a complement for x in P (possibly not the only one), then $x \in I$ or $z \in I$, but not both.*

If P is a complemented lattice and I is a (proper) ideal such that, for every $x \in P$, either $x \in I$ or $\bar{x} \in I$, then I is maximal.

3.3 Ultrafilters

The above discussion was centered around the notion of ideal. We have defined prime ideals as those whose complement was a filter and maximal (proper) ideals as those which are not strictly contained in another proper ideal. Applying Zorn's lemma we have arrived to the conclusion that maximal ideals exist. Furthermore, given an ideal I and a filter F disjoint from I, there exists a maximal ideal amongst those containing I and disjoint from F. We have also proved that, in a distributive lattice, every maximal ideal is prime and that, in a Boolean algebra, the notions of prime ideal and of maximal ideal agree and such ideals are characterised by the following condition: let $I \subset P$ be an ideal in a Boolean algebra P, then I is a maximal (equivalently, prime) ideal, if and only if for every $x \in P$, either $x \in I$ or $\bar{x} \in I$, but not both. The only ideal in a Boolean algebra containing both an element x and its complement $\bar{x}$ is the trivial ideal P. In this subsection we develop the analogous notion to that of maximal ideal on the side of filters.

Let $(P, \vee, \wedge, 0, 1)$ be a lattice. Let $F \subset P$ be a filter and let I be an ideal in P disjoint from F. Applying Zorn's lemma to the set of filters containing F and disjoint from I, we may conclude that there exists a maximal filter amongst those extending F and remaining disjoint from I. We shall say that a filter F' *extends* a filter F, if $F \subset F'$. If we take $F = \{1\}$, we arrive at the notion of a maximal filter disjoint from an ideal I. In particular, if $I = \{0\}$, we get maximal (proper) filters. An *ultrafilter* in a lattice P is a filter F in P which is a maximal proper filter.

Remark. Let P be a lattice. We have seen in 3.1 that, if $I \subset P$ is an ideal, then the following two conditions are equivalent:

i the complement of I in P is a filter;

ii $1 \notin I$ and $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

(These can be shown to be equivalent independently of the third condition in 3.1). Such were the prime ideals. A prime filter was defined as the complement of a prime ideal. We may, however, state the analagous proposition proposition for filters. Let $F \subset P$ be a filter and consider the following conditions on F.

i The complement of F in P is an ideal;

ii $0 \notin F$ and $x \vee y \in F$ implies $x \in F$ or $y \in F$.

These two conditions on the filter F are equivalent. Thus, the notion of a prime filter as a filter which satisfies any of the above equivalen conditions agrees with the notion of a prime filter as the complement of a prime ideal.

In a Boolean algebra, the prime ideals are precisely the maximal ideals. So, the notions of prime filter and maximal filter are quite different. Perhaps there is an analogous construction to that of Boolean algebras in which thet notion of prime and maximal filter agree.

Anyhow, the matter of ultrafilters in Boolean algebras is not directly settled by the results in the previous subsections.

We shall restrict our attention to Boolean algebras, although some definitions make sense for arbitrary lattices.

Lemma 3.7. *In a Boolean algebra, $x \wedge \bar{y} = 0$, if and only if $x \leq y$. Dually, in a Boolean algebra, $x \vee \bar{y} = 1$, if and only if $x \geq y$.*

A subset A of a lattice P is said to have the *finite intersection property*, if the meet of any finite subset of A is not equal to 0. In symbols, A has the finite intersection property, if, for every $a_1, \ldots, a_m \in A$,

$$a_1 \wedge \cdots \wedge a_m = \bigwedge \{a_1, \ldots, a_m\} \neq 0$$

Lemma 3.8. *Let P be a Boolean algebra.*

(a) If $A \subset P$ has the finite intesection property, then, for any element $x \in P$, either $A \cup \{x\}$ or $A \cup \{\bar{x}\}$ has the finite intersection property.

(b) Let $\{A_i\}_i$ be a chain of subsets of P totally ordered by inclusion. If every A_i has the finite intersection property, then their union $\bigcup_i A_i$ has the finite intersection property, as well.

Let P be a lattice and let $A \subset P$ be a subset. Define $\langle A \rangle$ to be the set of elements of P greater than some element of A and $|A|$ to be the set consisting of the meets in P of the finite subsets of A:

$$\begin{aligned}
\langle A \rangle &:= \{y \in P : x \leq y \text{ for some } x \in A\} \\
&= \bigcup_{x \in A} \uparrow(x) \; . \\
|A| &:= \{a_1 \wedge \cdots \wedge a_m : a_1, \ldots, a_m \in A\} \\
&= \left\{ \bigwedge A_1 : A_1 \subset A, \, \#A_1 < \infty \right\} \; .
\end{aligned}$$

A *base* for a filter F in P is a subset A such that $\langle A \rangle = F$. A *subbase* for F is a set A such that $|A|$ is a base for F. We shall say that A *generates* F, if $F = \langle |A| \rangle$.

Lemma 3.9. *Let P be a Boolean algebra and let $A \subset P$ be a subset. Then*

(a) The set $\langle |A| \rangle$ is a filter in P, though not necessarily a proper one;

(b) the filter $\langle|A|\rangle$ is the smallest filter containing A; and

(c) $\langle|A|\rangle$ is a proper filter, if and only if A has thet finite intersection property.

We call $\langle|A|\rangle$ the *filter generated* by A. By item *(c)* of 3.9, a subset A in a Boolean algebra can be extended to a proper filter –and, in particular, to an ultrafilter–, if and only if A has the finite intersection property.

Proof. An element $x \in P$ belongs to $\langle|A|\rangle$, if and only if there exists a finite subset $A_1 = \{a_1, \ldots, a_m\} \subset A$ such that

$$\bigwedge A_1 = a_1 \wedge \cdots \wedge a_m \leq x .$$

Given a subset $B \subset P$, the set $|B|$ consisting of all finite meets of elements of B (meets of finite subsets of B), is $\wedge$-closed, since, by associativity,

$$\Big(\bigwedge B_1\Big) \wedge \Big(\bigwedge B_2\Big) = \bigwedge(B_1 \cup B_2) .$$

Thus, $\langle|A|\rangle$ satisfies the axioms for filters.

Let F be a filter containing A. Then, since F is $\wedge$-closed, the set $|A|$ of meets of finite subests of A is contained in F. But if $x \in F$, the filter $\uparrow(x)$ is contained in F. Therefore

$$F \supset \bigcup_{\substack{A_1 \subset A \\ \# A_1 < \infty}} \uparrow\Big(\bigwedge A_1\Big) = \langle|A|\rangle .$$

Finally, if A does not have the finite intersection property, there exists a finite subset $A_1 \subset A$ such that $\bigwedge A_1 = 0$. In particular, 0 lies in $|A| \subset \langle|A|\rangle$ and, being a filter, $\langle|A|\rangle = P$. If, conversely, $\langle|A|\rangle$ is not a proper filter, then $0 \in \langle|A|\rangle$, which, by definition, means there exists a finite subset $A_1 \subset A$ such that $\bigwedge A_1 \leq 0$ and so A does not have the finite intersection property. □

The following proposition is dual to 3.6.

Proposition 3.10. *Let F be a (proper) filter in a lattice P. Assume F is a prime filter. If $x \in P$ and z is a complement for x in P (possibly not the only one), then $x \in F$ or $z \in F$, but not both.*

If P is a complemented lattice and F is a (proper) filter such that, for every $x \in P$, either $x \in F$ or $\bar{x} \in F$, then F is maximal.

The following theorem is dual to 3.2.

Theorem 3.11. *Let P be a distributive lattice and let I be an ideal in P. If F is a maximal filter amongst those disjoint from I, then F is prime.*

Proof. The proof is dual to the proof of 3.2. Since $I \cap F = \varnothing$, $0 \notin F$. Let $x_1, x_2 \in P$ be such that $x_1 \vee x_2 \in F$. Let

$$G_1 = \langle |F, x_1| \rangle .$$

be the filter generated by F and x_1 and let

$$E_1 = \{i \wedge (x_1 \vee y) : i \in F,\, y \in P\} .$$

If $i, j \in F$ and $y, z \in P$,

$$[i \wedge (x_1 \vee y)] \wedge [j \wedge (x_1 \vee z)] = (i \wedge j) \wedge [(x_1 \vee y) \wedge (x_1 \vee z)] = (i \wedge j) \wedge (x_1 \vee (y \wedge z)) .$$

Since $i \wedge j \in F$, we conclude that E_1 is $\wedge$-closed. If $z \geq i \wedge (x_1 \vee y)$, then

$$z = z \vee (i \wedge (x_1 \vee y)) = (z \vee i) \wedge (z \vee (x_1 \vee y)) .$$

Since F is a filter and $i \in F$, $z \vee i \in F$ and $z \in E_1$. Thus we see that E_1 is a filter containing F and x_1. Since G_1 is the smallest such filter, $G_1 \subset E_1$. On the other hand, $E_1 \subset G_1$, for if $i \in F$ and $z := i \wedge (x_1 \vee y) \in E_1$, then $z \geq i \wedge x_1$ and $\uparrow(i \wedge x_1) \subset G_1$. Similarly, $G_2 = E_2$.

We now prove that at least one of G_1 and G_2 is disjoint from I. If not, there exist $i, j \in F$ and $y, z \in P$ such that both $i \wedge (x_1 \vee y)$ and $j \wedge (x_2 \vee z)$ belong to I. Since I is an ideal, the element

$$(i \wedge (x_1 \vee y)) \vee (j \wedge (x_2 \vee z)) = (i \vee j) \wedge (i \vee x_2 \vee z) \wedge (j \vee x_1 \vee y) \wedge (x_1 \vee x_2 \vee y \vee z)$$

belongs to I. But it also belongs to F, for the term on the right is a finite meet of elements of F. This contradicts $I \cap F = \varnothing$. Therefore, at least one of G_1 and G_2 must be disjoint from I and hence $G_k \supset F$ for some k. But F is maximal, so $G_k = F$ and $x_k \in F$ for some k. □

In particular, every filter in a distributive lattice can be extended to an ultrafilter. Since in a Boolean algebra the only subsets which can be extended to proper filters are the subsets with the finite intersection property, we conclude that every subset of a Boolean algebra with the finite intersection property can be extended to an ultrafilter. In particular, if $x \in P$ is an element in a Boolean algebra, there is an ultrafilter containing x. More precisely, the following corollary is true.

Corollary 3.12. *If x, y are distinct elements of a Boolean algebra P, then there exists an ultrafilter containing one but not the other.*

Proof. Since $x \neq y$, either $x \not\leq y$ or $x \not\geq y$. Assume, without loss of generality, that $x \not\geq y$ holds. Then, by 3.7, $x \vee \bar{y} \neq 1$ or, equivalently, $y \wedge \bar{x} \neq 0$. Then the set $\{y, \bar{x}\}$ has the finite intersection property and can be extended to an ultrafilter F in P. Thus, $y \in F$ and $\bar{x} \in F$. Since F is, in particular, a proper filter, it must be te case that $x \notin F$. □

This corollary is, in a sense, similar to 3.3. In a distributive lattice P we can separate distinct elements by lattice homomorphisms into the two-element lattice $2 = \{0, 1\}$. Given a lattice homomorphism $f : P \to 2$ there is a descomposition of P as a disjoint union of an ideal and a filter:

$$P = f^{-1}(0) \cup f^{-1}(1) .$$

Although homomorphisms are typically related to ideals through their kernels, in this case we may as well treat this result as relating to filters. 3.12, however, is more precise in its conclusion: although $f^{-1}(1)$ is a maximal filter amongst those disjoint from $\downarrow(x)$ and containing $\uparrow(y)$, it may not be an ultrafilter. 3.12 concludes that two distinct elements in a Boolean algebra can be *maximally separated*, that is, separated by an ultrafilter.

Finally, we state the following consequence of 3.10 and 3.11, which is dual to 3.5. It says that the notions of prime filter and of maximal filter (ultrafilter) agree in a Boolean algebra.

4 Filters in Topology and the Stone Representation

We begin this section defining the neighbourhood filter.

Let X be a topologial space nad let $x \in X$ be any point. The *neighbourhood filter* at *x on X* is the filter N_x in $\mathcal{P}(X)$ (partially ordered by inclusion) consisting of precisely the open subsets of X containing x.

Lemma 4.1. *Given a point x in a topological space X, the following conditions on a filter F on X are equivalent:*

i $x \in \bigcup \{\overline{A} : A \in F\}$;

ii if $A \in F$ and $U \in N_x$, then $A \cap U \neq \varnothing$; and

iii the union $F \cup N_x$ is contained in a (proper) filter in $\mathcal{P}(X)$.

Proof. The first two are equivalent by definition of closure of a set. *(ii)* is equivalent to $F \cup N_x$ having the finite intersection property, which, by 3.9, is equivalent to *(iii)*. □

If any of the equivalent conditions in lemma 4.1 is satisfied, we say that x is an *adherent point* of the filter F. If, moreover, $F \supset N_x$, then F is said to *converge* to x.

If F converges to a point $x \in X$, then x is an adherent point of F. If F is an ultrafilter and x is an adherent point of F, then N_x must be contained in F, which implies that F converges to x. In other words, an ultrafilter converges to a point, if and only if this point is an adherent point of the ultrafilter.

Corollary 4.2. *Let X be a topological space. Then*

(a) X is Hausdorff, if and only if every filter converges to at most one point. Equivalently, X is Hausdorff, if and only if every ultrafilter has at most one adherent point (converges to at most one point);

(b) X is compact, if and only if each filter has, at least, one adherent point. Equivalently, X is compact, if and only if every ultrafilter has at least one adherent point (converges to at least one point).

4.1 Stone Spaces

Given a countable set X, the Boolean subalgebra $\mathsf{Z}(X)$ of finite and cofinite subsets of X (of the power set Boolean algebra $\mathcal{P}(X)$ partially ordered by inclusion) is *not* isomorphic to any power set Boolean algebra, since it is countable.

We see from 4.2 that, given a compact Hausdorff topological space X, there is an associated map:

$$\left\{\begin{matrix}\text{ultrafilters}\\ \text{on } X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{points}\\ \text{in } X\end{matrix}\right\}$$

So we may wonder what kind of structure does the set of ultrafilters on X possess. In oreder to understand this, we consider the case of an arbitrary Boolean algebra.

Let B be a Boolean algebra and let $\mathsf{S}(B)$ denote the set of ultrafilters in B. Consider the following map u which to each element $x \in B$ asigns the subset of $\mathsf{S}(B)$ of ultrafilters containing x, that is,

$$u(x) = \{\phi \in \mathsf{S}(B) : x \in \phi\} . \tag{7}$$

Given $x\,y \in B$, if $x \neq y$, then, by 3.12, there exists an ultrafilter ϕ containing one but not the other. In particular, the map $u : B \to \mathcal{P}(\mathsf{S}(B))$ is injective.

Proposition 4.3. *Let B be a Boolean algebra, $\mathsf{S}(B)$ be the set of filters in B and let $\mathcal{P}(\mathsf{S}(B))$ be the power set Boolean algebra of subsets of $\mathsf{S}(B)$ ordered by inclusion. Let $u : B \to \mathcal{P}(\mathsf{S}(B))$ be the map defined by (7). Then u is an injective Boolean algebra homomorphism. In particular, B is isomorphic to a subalgebra of $\mathcal{P}(\mathsf{S}(B))$.*

Proof. If F is any filter and $x, y \in B$, then $x, y \in F$, if and only if $x \wedge y \in F$. This shows that

$$u(x \wedge y) = u(x) \cap u(y) ,$$

which is equal to the meet of the elements $u(x)$ and $u(y)$ in $\mathcal{P}(\mathsf{S}(B))$. So, u respects meets. By one of the equivalent conditions of being an ultrafilter, if $\phi \in \mathsf{S}(B)$, for every $x \in B$, $x \in \phi$ xor $\bar{x} \in \phi$. Thus,

$$u(\bar{x}) = \mathsf{S}(B) \smallsetminus u(x) = u(x)^{\mathsf{c}} ,$$

which is the complement of $u(x)$ in $\mathcal{P}(\mathsf{S}(B))$. Since no proper filter contains the least element 0 in B, $u(0)$ equals the empty subset and, since every filter contains the greatest element 1, $u(1)$ equals $\mathsf{S}(B)$. But these are precisely the least and greatest elements in the lattice $\mathcal{P}(\mathsf{S}(B))$. Then u respects $\vee$, as well, and is, therefore, a Boolean algebra homomorphism. $\square$

A *Stone space* is a topological space X which is compact, Hausdorff and totally disconected (equivalently, admits a basis of closed and open subsets). Given an arbitrary topological space X, the collection of all subsets of X which are both closed and open forms a Boolean algebra, a subalgebra of the Boolean algebra $\mathcal{P}(X)$ of subsets of X. We shall call this algebra, the *characteristic algebra* of X and denote it by $\mathcal{C}(X)$.

In a set X, if $\mathcal{A}$ is a subalgebra of $\mathcal{P}(X)$, then $\mathcal{A}$ constitutes a base for a topology on X. With this topology, the elements of $\mathcal{A}$ are both open and closed subsets of X. In particular, if X is a Stone space, then $\mathcal{C}(X)$ forms a basis for the topology on X. The following lemma says that the characteristic algebra of a Stone space is the only Boolean subalgebra of subsets of X with this property.

Lemma 4.4. *Let X be a Stone space and let $\mathcal{A}$ be a subalgebra of the Boolean algebra $\mathcal{P}(X)$. If $\mathcal{A}$ is a basis for the topology on X, then $\mathcal{A} = \mathcal{C}(X)$.*

Proof. The elements of $\mathcal{A}$ are both open and closed, since $\mathcal{A}$ is, by hypothesis, a part of the topology on X and $\mathcal{A}$ is, being a subalgebra, closed under complementation. Thus, $\mathcal{A} \subset \mathcal{C}(X)$.

Now let $V \in \mathcal{C}(X)$. On the one hand, since V is open and $\mathcal{A}$ is a base for the topology on X, given $x \in V$, there exists $U_x \in \mathcal{A}$ such that $x \in U_x \subset V$. On the other hand, since V is closed and X is compact, V is, therefore compact. Combining these two observations, we conclude that V is covered by finitely many U_x, so

$$V = U_{x_1} \cup \cdots \cup U_{x_k} .$$

This shows that V belongs to $\mathcal{A}$. □

Given a Boolean algebra B, we have seen that the homomorphism $u : B \to \mathcal{P}(\mathsf{S}(B))$ determines an isomorphism between B and the subalgebra $u(B)$ of $\mathcal{P}(\mathsf{S}(B))$. Now, the subalgebra $u(B)$ determines a topology on $\mathsf{S}(B)$. The topological space consisting of the set $\mathsf{S}(B)$ together with the topology determined by the Boolean algebra $B \simeq u(B) \subset \mathcal{P}(\mathsf{S}(B))$ is called the *Stone space of the Boolean algebra B*. The following theorem shows, in particular, that the Stone space of a Boolean algebra is a Stone space, a compact, Hausdorff and totally disconnected topological space.

Theorem 4.5. *The Stone space $\mathsf{S}(B)$ of a Boolean algebra B is a Stone space and the map $u : B \to \mathcal{P}(\mathsf{S}(B))$ is an isomorphism from B onto the characteristic algebra $\mathcal{C}(\mathsf{S}(B))$ of $\mathsf{S}(B)$.*

Proof. We show firstly that $\mathsf{S}(B)$ is Hausdorff when given the topology determined by $u(B)$. Let $\phi, \psi \in \mathsf{S}(B)$ be two ultrafilters in B and suppose $\phi \neq \psi$. This means ϕ and ψ are different as subsets of B, so there exists $x \in \phi$ such that $x \notin \psi$ (note that, by maximality, both $\psi \setminus \phi$ and $\phi \setminus \psi$ are nonempty). In particular, $\phi \in u(x)$ and $\psi \notin u(x)$. By maximality of ψ, it follows that $\psi \in u(\bar{x})$. Since $u(x)$ and $u(\bar{x})$ are disjoint open subsets, we conclude that $\mathsf{S}(B)$ is Hausdorff. Note that we have also shown that the space of ultrafilters is totally disconnected.

To show that $\mathsf{S}(B)$ is compact, consider an open cover $\{u(x_i) : i \in I\}$ by elements of the base $u(B)$. Assume this cover does not admit a finite subcover. This is equivalent to assuming the collection of complements $\{u(\bar{x}_i) : i \in I\}$ has the finite intersection property. But then the subset $\{\bar{x}_i : i \in I\}$ has the finite intersection property. By 3.9, there exists an ultrafilter ϕ such $\bar{x}_i \in \phi$ for every $i \in I$. This implies

$$\phi \in \bigcap_{i \in I} u(\bar{x}_i) .$$

Equivalently, $x_i \notin \phi$ for every $i \in I$ and

$$\phi \notin \bigcup_{i \in I} u(x_i) ,$$

contradicting the assumption that $\{u(x_i) : i \in I\}$ was a cover of $\mathsf{S}(B)$.

The last assertion follows from 4.4, since $u(B)$ has just been shown to be a base for the topology of the Stone space $\mathsf{S}(B)$. □

The preivious theorem also showed that every Boolean algebra is the characteristic algebra of a Stone space, especifically, of its Stone space. In the opposite direction, every Stone space is homeomorphic to the Stone space of its characteristic algebra (every characteristic algebra is the characteristic algebra of the Stone space of a Boolean algebra. . .).

Theorem 4.6. *Let X be a Stone space. Then X is homeomorphic to the Stone space $\mathsf{S}(\mathcal{C}(X))$ of its characteristic algebra.*

Proof. Let X be a Stone space. Assume X is not the one-point space $\mathsf{1}$; the proof is trivial, otherwise. So, assume that $X \neq \mathsf{1}$ and define a map from X to the Stone space of its characteristic algebra $v : X \to \mathsf{S}(\mathcal{C}(X))$ by

$$v(\xi) = \{U \in \mathcal{C}(X) : \xi \in U\} .$$

For each fixed point ξ of X, the collection $v(\xi)$ constitutes a filter in $\mathcal{C}(X)$. Since X has more than one point and is a Hausdorff space, $v(\xi)$ must be a proper subset of $\mathcal{C}(X)$. For each open and closed subsets $U \in \mathcal{C}(X)$ and $\xi \in X$, either $\xi \in U$ or $\xi \notin U$. Equivalently, either $\xi \in U$ or $\xi \in X \smallsetminus U = U^{\mathsf{c}}$. Thus, for each fixed $x \in X$, $v(x)$ is an ultrafilter. Thus v determines a well-defined map

$$v : X \to \mathsf{S}(\mathcal{C}(X)) .$$

If $\xi, \upsilon \in X$ are distinct elements of X, then there exist $U, V \in \mathcal{C}(X)$ such that $U \cap V = \varnothing$, $\xi \in U$ and $\upsilon \in V$. In particular, $v(\xi) \neq v(\upsilon)$ (in fact, it is enough to assume X is a T_0 space to conclude that v is injective).

Let $\phi \in \mathsf{S}(\mathcal{C}(X))$ be an arbitrary ultrafilter in $\mathcal{C}(X)$. Then ϕ has the finite intersection property. Since $\mathcal{C}(X)$ is a sublattice of the Boolean algebra $\mathcal{P}(X)$ of subsets of X, ϕ has the finite intersection property as a subset of $\mathcal{P}(X)$ (note the parallelism with invariance of compactness under subspace maps). Thus ϕ extends to an ultrafilter F in $\mathcal{P}(X)$. Since X is a compact topological space, F converge to some point $\xi \in X$. Equivalently, ξ is an adherent point of F. Then $\xi \in \bigcap_{U\in F} \overline{U}$. In particular,

$$\xi \in \bigcap \{\overline{U} : U \in \phi\} = \bigcap \{U : U \in \phi\} .$$

Thus $\xi \in U$ for ever U in ϕ and then $\phi \subset v(\xi)$. By maximality of ϕ and properness of $v(\xi)$, we may conclude that $\phi = v(\xi)$. So, we see v is also surjective.

Finally, $v : X \to \mathsf{S}(\mathcal{C}(X))$ is a bijective map between compact Hausdorff spaces which maps the base $\mathcal{C}(X)$ for the topology on X into the base $\mathcal{C}(\mathsf{S}(\mathcal{C}(X)))$ for the topology on $\mathsf{S}(\mathcal{C}(X))$: if $V \in \mathcal{C}(X)$, by surjectivity of v,

$$\begin{aligned} v(V) &= \{\{U \in \mathcal{C}(X) : \xi \in U\} : \xi \in V\} = \{v(\xi) : \xi \in V\} \\ &= \{\phi \in \mathsf{S}(\mathcal{C}(X)) : V \in \phi\} . \end{aligned}$$

In particular, $v(V) \in \mathcal{C}(\mathsf{S}(\mathcal{C}(X)))$ and $v(V)$ is an element of the base for the Stone space of $\mathcal{C}(X)$. In fact, this shows that v maps the base $\mathcal{C}(X)$ of X onto the base $\mathcal{C}(\mathsf{S}(\mathcal{C}(X)))$. It then follows that v is a homeomorphism. □

5 Concluding Remarks

It is more or less clear that $\mathcal{C}(X)$ can be extended to a functor

$$\mathcal{C}(\cdot) : \mathbf{Stone} \to \mathbf{BoolAlg} .$$

But this is not so for the corresponding map $\mathsf{S}(B)$: for instance, if $f : B \to B'$, although $f(\phi) = \{f(x) : x \in \phi\}$ is closed under finite meets and can, thus, be extended to a filter $\langle f(\phi)\rangle$ in a specified way, it is not clear that this filter is either maximal or, even, proper. It seems there is no obvious way of extending $\mathsf{S}(B)$ to a functor in the direction opposite to that of $\mathcal{C}(\cdot)$. At least not if we wanted a *covariant* functor.

5.1 An Adjunction

Clearly the above paragraph made no sense... Let X, Y be compact Hausdorff and totally disconnected spaces. Let $g : X \to Y$ be a continuous funtion. Then the assignment $f(U) = g^{-1}(U)$ determines a lattice homomorphism in the opposite direction: $f : \mathcal{C}(Y) \to \mathcal{C}(X)$, since

$$\begin{aligned} g^{-1}(U \cap V) &= g^{-1}(U) \cap g^{-1}(V) , \\ g^{-1}(U \cup V) &= g^{-1}(U) \cup g^{-1}(V) , \\ g^{-1}(\varnothing) &= \varnothing \quad \text{and} \\ g^{-1}(Y) &= X . \end{aligned}$$

In particular this lattice homomorphism is a Boolean algebra homomorphism. This map $g \mapsto \mathcal{C}(g) = f$ respects identities and compositions. Thus $\mathcal{C}(\cdot)$ is a contravariant functor from the category of Stone spaces to the category of Boolean algebras.

We now want to extend $\mathsf{S}(B)$ to a functor. As we noted above, it makes no sense to try to extend $\mathsf{S}(B)$ to a covariant functor. We shall define a contravariant functor which is given by $B \mapsto \mathsf{S}(B)$ on objects.

Let A, B be two Boolean algebras and let $f : A \to B$ be a Boolean algebra homomorphism. If $\phi \subset B$ be a proper subset, then $f^{-1}(\phi)$ is a proper subset of A, too. If ϕ is a filter in B, its inverse image $f^{-1}(\phi)$ is a filter in A, for

$$\begin{aligned} f(x \wedge y) &= f(x) \wedge f(y) \quad \text{and} \\ y \geq x &\Rightarrow f(y) \geq f(x) \ . \end{aligned}$$

If ϕ is a prime filter and $x \vee y \in f^{-1}(\phi)$, then $f(x) \vee f(y)$ belongs to ϕ and, thus, either $f(x) \in \phi$ or $f(y) \in \phi$. So, $f^{-1}(\phi)$ is prime, as well. Since A and B are Boolean algebras, the notions of prime filter and of ultrafilter –that is, maximal filter– agree.

Remark. We may wonder if the other equivalent maximality conditions for a filter in a Boolean algebra are preserved under the operation of taking inverse images.

If ϕ is a proper filter such that, for every $z \in B$ either $z \in \phi$ or $\bar{z} \in \phi$, then, given $x \in A$, we have $f(x) \in \phi$, if and only if $f(\bar{x}) \notin \phi$. Thus, $f^{-1}(\phi)$ has the property that, for every $x \in A$, either $x \in f^{-1}(\phi)$ or $\bar{x} \in f^{-1}(\phi)$, but not both. Thus, we see that the two equivalent conditons to being a maximal filter are preserved under the operation of taking inverse image by a lattice homomorphims.

If A and B were arbitrary lattices, it would not be true that the inverse image of a maximal filter is a maximal filter. However, if f were surjective, the "forward" image of a filter would be a filter and it would be true that the nverse image of a maximal filter is a maximal filter.

To summarise, if $f : A \to B$ is a Boolean algebra homomorphism and ϕ is an ultrafilter in B, the function $g(\phi) = f^{-1}(\phi) \subset A$ determines a map $g : \mathsf{S}(B) \to \mathsf{S}(A)$. We check now that this map is continuous.

A base for the topology on $\mathsf{S}(A)$ is given by the sets of the form $u_A(x) = \{\psi \in \mathsf{S}(A) : x \in \psi\}$ with $x \in A$. Let $x \in A$ be an arbitrary element. Then

$$\begin{aligned} g^{-1}(u_A(x)) &= \left\{\phi \in \mathsf{S}(B) : x \in f^{-1}(\phi)\right\} \\ &= \{\phi \in \mathsf{S}(B) : f(x) \in \phi\} = u_B(f(x)) \ . \end{aligned}$$

Thus, the preimage of an element of the base $\mathcal{C}(\mathsf{S}(A))$ for the topology on the Stone space of A is an element of the base $\mathcal{C}(\mathsf{S}(B))$ for the topology on the Stone space of B. In particular, $g : \mathsf{S}(B) \to \mathsf{S}(A)$ is continuous. The map $f \mapsto \mathsf{S}(f) = g$ respects identities and compositions, so it extends to a contravariant functor from the category of Boolean algebras to the category of Stone spaces.

We thus have two contravariant functors

$$\mathbf{Stone} \underset{\mathcal{C}(\cdot)}{\overset{\mathsf{S}(\cdot)}{\leftrightarrows}} \mathbf{BoolAlg}$$

Theorem 4.5 shows that the composition $B \mapsto \mathcal{C}(\mathsf{S}(B))$ is naturally isomorphic to the identity functor on **BoolAlg**. This isomorphism is given by the map $u(x) = \{\phi \in \mathsf{S}(B) : x \in \phi\}$ for each Boolean algebra B. Similarly, theorem 4.6 shows that $X \mapsto \mathsf{S}(\mathcal{C}(X))$ is naturally isomorphic to the identity functor in **Stone**. The isomorphism in this case is given by the corresponding function $v(\xi) = \{U \in \mathcal{C}(X) : \xi \in U\}$.

5.2 The Relation with the Prime Spectrum

Let B be a Boolean algebra. We have already seen that the lattice ideals are in corresppondence with the ring-theoretic ideals of B seen as a Boolean ring. In fact, the correspondence is given by the identity map as subsets of B. The same is true for prime ideals. Therefore, we may conclude that ring-theoretic prime ideals in B are in correspondence with lattice ideals in B (by the identity map $\mathsf{id} : \mathcal{P}(B) \to \mathcal{P}(B)$ on subsets), which are in correspondence with prime filters in B (by taking set-theoretic complements $I \mapsto I^{\mathsf{c}} = B \smallsetminus I$). In particular, ring-theoretic prime ideals are exactly the ring-theoretic maximal ideals in B. All ring-theoretic prime ideals of a Boolean ring B are kernels of ring homomorphisms $f : B \to \mathsf{2}$: the canonical projection $B \to B/I$ is a surjective ring homomorphism onto an integral domain in which every element is idempotent, hence $B/I \simeq \mathsf{2}$. Every lattice homomorphism between Boolean algebras is a ring homomorphism between the corresponding Boolean ring structures.

Conversely, every kernel of a ring or lattice homomorphism $f : B \to \mathsf{2}$ is a prime ideal, since its complement $f^{-1}(1)$ is a filter in B. Ring (or lattice) prime ideals in a Boolean algebra B are, therefore, precisely the kernels $f^{-1}(0)$ of ring (or lattice) homomorphisms $f : B \to \mathsf{2}$ and ultrafilters (equivalently, prime filters) are, precisely, the *shells* $f^{-1}(1)$ of such homomorphisms. Thus, the space of ultrafilters in a Boolean algebra is in correspondence with the *prime spectrum* of the Boolean ring B:

$$\mathsf{S}(B) \xrightarrow{\ \cdot^{\mathsf{c}}\ } \left\{\begin{matrix}\text{prime ideals} \\ \text{in } B\end{matrix}\right\} \xrightarrow{\ \mathsf{id}\ } \mathsf{Spec}(B)$$

$$\phi \longmapsto I = \phi^{\mathsf{c}} = B \smallsetminus \phi \longmapsto I$$

Consider the extremes of the above diagram. On the one hand, we have given $\mathsf{S}(B)$ the topology determined by the base $u(B)$, that is, a basic open (closed) set is of the form

$$u(x) \;=\; \{\phi \in \mathsf{S}(B) : x \in \phi\}\;\; ,$$

and we have proved that, with this topology, $\mathsf{S}(B)$ is a Stone space. We have also proved that B can be "recovered" as the characteristic algebra of closed and open subsets of $\mathsf{S}(B)$. On the other hand, the topology on $\mathsf{Spec}(B)$ is determined by the base formed by the sets

$$D_x \;=\; \{I \subset B : I \text{ is a prime ideal, } x \notin I\} \;=\; \mathsf{Spec}(B) \smallsetminus V(x)\;\; ,$$

where $V(x)$ is the set of (ring-theoretic) prime ideals of B containing x. By the bijection between ring prime ideals, lattice prime ideals and ultrafilters in a Boolean algebra, the sets D_x are equal to

$$\begin{aligned} D_x &= \{\phi^{\mathsf{c}} : \phi \in \mathsf{S}(B), x \in \phi\} \\ &= \{I \in \mathsf{Spec}(X) : I^{\mathsf{c}} \in u(x)\} . \end{aligned} \tag{8}$$

In general, for any commutative ring with unit B, if $x, y \in B$,

$$\begin{aligned} D_x \cap D_y &= D_{x \cdot y} , \\ D_0 &= B \quad \text{and} \\ D_1 &= \varnothing . \end{aligned}$$

So, $\{D_x : x \in B\}$ forms a base for a topology on $\mathsf{Spec}(B)$. Furthermore, in a Boolean ring, we see, by (8), that

$$\begin{aligned} \mathsf{Spec}(B) \smallsetminus D_x &= D_{\bar{x}} \quad \text{and} \\ D_x \cup D_y &= (\mathsf{Spec}(B) \smallsetminus D_{\bar{x}}) \cup (\mathsf{Spec}(B) \smallsetminus D_{\bar{y}}) = D_{\overline{\bar{x} \cdot \bar{y}}}. \end{aligned}$$

Thus, the collection of sets D_x are both open and closed in the topology they generate. In fact, these are the only open and closed subsets of $\mathsf{Spec}(B)$: if $V \subset \mathsf{Spec}(B)$ is an open and closed subset, being open, there exists a collection of basic open sets $\{D_{x_i} : i \in I\}$ such that their union is V:

$$\begin{aligned} V &= \bigcup \{D_{x_i} : i \in I\} \\ &= \left\{ I \in \mathsf{Spec}(B) : I^{\mathsf{c}} \in \bigcup_{i \in I} u(x_i) \right\} , \end{aligned}$$

and, being closed, ther exists a collection $\{D_{x_j} : j \in J\}$ such that their intersection is V:

$$\begin{aligned} V &= \bigcap \{D_{x_i} : i \in I\} \\ &= \left\{ I \in \mathsf{Spec}(B) : I^{\mathsf{c}} \in \bigcap_{j \in J} u(x_j) \right\} . \end{aligned}$$

Since $I \mapsto I^{\mathsf{c}}$ is bijective, the set

$$U := \bigcup_{i \in I} u(x_i) = \bigcap_{j \in J} u(x_j)$$

is both open and closed in $\mathsf{S}(B)$. But since $\mathsf{S}(B)$ is compact and U is closed, U must be compact. Therefore,

$$U = u(x_{i_1}) \cup \cdots \cup u(x_{i_k})$$

and

$$V = D_{x_{i_1}} \cup \cdots \cup D_{x_{i_k}}$$

But then V is equal to D_y for some $y \in B$.

Remark. Another –perhaps shorter– proof of this fact consists in showing first that $\mathsf{Spec}\,(B)$ is, in general, compact (though maybe not Hausdorff). If $V \subset \mathsf{Spec}\,(B)$ is both open and closed, being open it is a union of basic open subsets D_x and, being closed, it is compact and the cover can be replaced by a finite subcover $\{D_{x_1}, \ldots, D_{x_k}\}$. Since a finite union of these basic open subsets is, in a Boolean ring, equal to some basic open subset D_y, we conclude that $V = D_y$ and that the basic open subsets D_x are the only open and closed subsets of $\mathsf{Spec}\,(B)$. This proof is rather similar to the proof of 4.4.

We can then prove the following theorem, analogous to 4.5.

Theorem 5.1. *Let B be a Boolean ring. The prime spectrum of B, $\mathsf{Spec}\,(B)$, with its Zariski topology determined by the sets D_x $x \in B$ is a Stone space, a compact Hausdorff totally disconnected topological space. The map $D : B \to \mathcal{P}\,(\mathsf{Spec}\,(B))$ given by*

$$D(x) = D_x = \{I \in \mathsf{Spec}\,(B) \,:\, x \notin I\}$$

determines a Boolean algebra isomorphism from B onto the characteristic algebra $\mathcal{C}\,(\mathsf{Spec}\,(B))$ of open and closed subsets of the prime spectrum of B.

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