Theorem 0.1. *Let* $\mathbf{D}$ *have pushouts. Given a coreflection* $L \dashv R \colon \mathbf{C} \to \mathbf{D}$ *where* R *preserves pushouts,* R *is a Street opfibration.*

Proof. Fix an arrow $f \colon a \to b$ in $\mathbf{C}$ and an object x such that $Rx \coloneqq a$. Define $\hat{f}$ as the pushout of Lf along the counit ε_x

$$\begin{tikzcd}
LRx \arrow[r, "Lf"] \arrow[d, "\varepsilon"'] & Lb \arrow[d] \\
x \arrow[r, "\hat{f}"] & x +_{LRx} Lb
\end{tikzcd}$$

Observe that $R\hat{f} \colon Rx \to R(x +_{LRx} Lb)$. Also there is a string of isomorphisms

$$R(x +_{LRx} Lb) \to Rx +_{RLRx} RLb \to Rx +_{Rx} b \to b$$

whose composite we call h. Then $R\hat{f} = f.h^{-1}$ as desired. *(note: some details are needed here)*

Now show that $\hat{f}$ is cocartesian. Consider a $\mathbf{D}$-arrow $g \colon x \to y$ with a C-arrow $\theta \colon R(x +_{LRx} Lb) \to Ry$ so that $R\hat{f}.\theta = Rg$. Can we uniquely lift θ? Set up the diagram

$$\begin{tikzcd}
LRx \arrow[r, "Lf"] \arrow[d, "\varepsilon_x"'] & Lb \arrow[d] \arrow[ddr, bend left, "g'"] & \\
x \arrow[r, "\hat{f}"] \arrow[drr, bend right, "g"] & x +_{LRx} Lb & \\
 & & \text{y}
\end{tikzcd}$$

where $g' \coloneqq Lh^{-1}.L\theta.\varepsilon_y$. To show the outer square commutes, it suffices to show that $Lf.g'$ and $\varepsilon_x.g$ have the same image under the adjunction homset correspondence. We have

$$Lf.g' = Lf.Lh^{-1}.L\theta.\varepsilon_y \mapsto \eta_{Rx}.RLf.RLh^{-1}.RL\theta.R\varepsilon_y = f.h^{-1}.\theta = R\hat{f}.\theta = Rg$$

and

$$\varepsilon_x.g \mapsto \eta_{Rx}.Rg = Rg.$$

□

Theorem 0.2. *Given a Street opfibration* $R \colon \mathbf{D} \to \mathbf{C}$*, it is a right adjoint when it is* nice*.*

Let's try to figure out what *nice*could be. Here are some helpful theorems.

Theorem 0.3 (Freyd's general adjoint functor theorem)**.** *A functor* $F \colon X \to Y$ *is a right adjoint if* x *is complete, locally small, and* F *satisfies the* solution set condition*. The latter says, for any* Y*-object* y*, there exists a small set* I *indexing a collection of* X*-objects* x_{i_I} *and* Y*-arrows* $f_i \colon y \to F(x_i)$ *such that every* F*-valued* Y*-arrow* $y \to Rx$ *factors as* $Fg.f_k$ *for* $k \in I$ *and* $g \colon x_k \to x$*.*

Theorem 0.4 (Gabriel-Zisman)**.** *An adjunction $L \dashv R\colon \mathbf{C} \leftrightarrow \mathbf{D}$. TFAE:*

(a) L is full and faithful;
(b) the unit is an isomorphism;
(c) the induced comonad on $\mathbf{D}$ is idempotent, L is conservative, and R is essentially surjective.

Now, let's prove a strong theorem then try to weaken it

Theorem 0.5. *Let $\mathbf{D}$ be locally small and complete. Also let $R\colon \mathbf{D} \to \mathbf{C}$ be a continuous, surjective-on-objects Grothendiek opfibration. Then R is a right adjoint.*

Proof. We use 0.3. Suffice to show the solution set condition holds. Fix a C-objects c. Then the indexing set is $*$, the collection of D-objects consists of a single object x_c over c (which exists by surjective-on-objects assumption), and the collection of $\mathbf{C}$-arrows consists of the identity. Any map $f\colon c \to Rd$ has a cocartesian lifting $\hat{f}\colon x_c \to x_{Rd}$ and $f = R\hat{f}.1_c$. □

Theorem 0.6. *Let $\mathbf{D}$ be locally small and complete. Also let $R\colon \mathbf{D} \to \mathbf{C}$ be a continuous, surjective-on-objects, conservative Street opfibration. Then R is a right adjoint.*

Proof. We use 0.3. Suffice to show the solution set condition holds. Fix a C-objects c. Then the indexing set is $*$, the collection of D-objects consists of a single object x_c over c (which exists by surjective-on-objects assumption), and the collection of $\mathbf{C}$-arrows consists of the identity. For any map $f\colon c \to Rd$, there exists an essential cocartesian lifting $\hat{f}\colon x_c \to d'$ together with a $\mathbf{C}$-isomorphism $h\colon Rd' \to Rd$ such that $f = h.R\hat{f}$. But $f = h.R\hat{f} = R\hat{h}.R\hat{f}.1_c = R(\hat{h}.\hat{f}).1_c$ where $h = R\hat{h}$ because R is conservative. □

Theorem 0.7. *Let $\mathbf{D}$ be locally small and complete. Also let $R\colon \mathbf{D} \to \mathbf{C}$ be a continuous, essentially surjective, conservative Street opfibration. Then R is a right adjoint.*

Proof. We use 0.3. Suffice to show the solution set condition holds. Fix a C-object c. Then the indexing set is $*$, the collection of D-objects consists of a single object d' where $\theta\colon c \cong Rd'$ (which exists by essential surjectivity), and the collection of $\mathbf{C}$-arrows is θ. For any $\mathbf{C}$-arrow $f\colon c \to Rd$, we have (by Street opfibrationness) $f.\theta^{-1}\colon Rd' \to c \to Rd$, a cocartesian essential lifting $f.\hat{\theta}^{-1}\colon d' \to d''$ in $\mathbf{D}$, and an isomorphism $h\colon d \to d''$ (by conservatism) such that $f.\theta^{-1} = Rh.Rf.\hat{\theta}^{-1}$. This implies, as required by GAFT, that $f = R(h\widehat{f.\theta}).\theta$. □

Thoughts about these assumptions. Here are the desired examples I can think of now: **Set** together with **Graph** or **Top**. The enriched over sets and completeness are both there. So is essential surjectivity. And reflection of isomorphisms. The continuity is definitely needed, since it's necessary if R is a right adjoint.

Go back to the right adjoint we had in the "converse" to the above theorem. That is, we have a coreflection $L \dashv R\colon \mathbf{C} \leftrightarrow \mathbf{D}$ where L is left exact. Of course this gives the continuity of R. Essential surjectivity follows from Gabriel-Zisman. Is it a Street opfibration? L is conservative, but is R?

Example 0.8. No, R is not in general conservative. Consider the underlying node functor **Graph** $\to$ **Set**. All **Graph**-endomorphisms on

$$\bullet \rightrightarrows \bullet$$

are sent to the identity on 2.

At this point, we have partial converses:

| Let $\mathbf{D}$ have pushouts. Given a coreflection $L \dashv R\colon \mathbf{C} \to \mathbf{D}$ where R preserves pushouts, R is a Street opfibration. |

| Let $\mathbf{D}$ be locally small and complete. Also let $R\colon \mathbf{D} \to \mathbf{C}$ be a continuous, essentially surjective, conservative Street opfibration. Then R is a right adjoint. |