

Theorem 0.1. *Let \mathbf{D} have pushouts. Given a coreflection $L \dashv R: \mathbf{C} \rightarrow \mathbf{D}$ where R preserves pushouts, R is a Street opfibration.*

Proof. Fix an arrow $f: a \rightarrow b$ in \mathbf{C} and an object x such that $Rx := a$. Define \hat{f} as the pushout of Lf along the counit ε_x

$$\begin{array}{ccc} LRx & \xrightarrow{Lf} & Lb \\ \varepsilon_x \downarrow & & \downarrow \\ x & \xrightarrow{\hat{f}} & x +_{LRx} Lb \end{array}$$

Observe that $R\hat{f}: Rx \rightarrow R(x +_{LRx} Lb)$. Also there is a string of isomorphisms

$$R(x +_{LRx} Lb) \rightarrow Rx +_{RLRx} RLb \rightarrow Rx +_{Rx} b \rightarrow b$$

whose composite we call h . Then $R\hat{f} = f.h^{-1}$ as desired. (*note: some details are needed here*)

Now show that \hat{f} is cocartesian. Consider a \mathbf{D} -arrow $g: x \rightarrow y$ with a \mathbf{C} -arrow $\theta: R(x +_{LRx} Lb) \rightarrow Ry$ so that $R\hat{f}.\theta = Rg$. Can we uniquely lift θ ? Set up the diagram

$$\begin{array}{ccc} LRx & \xrightarrow{Lf} & Lb \\ \varepsilon_x \downarrow & & \downarrow \\ x & \xrightarrow{\hat{f}} & x +_{LRx} Lb \end{array} \quad \begin{array}{c} \nearrow g' \\ \searrow g \end{array} \quad \begin{array}{c} \\ y \end{array}$$

where $g' := Lh^{-1}.L\theta.\varepsilon_y$. To show the outer square commutes, it suffices to show that $Lf.g'$ and $\varepsilon_x.g$ have the same image under the adjunction homset correspondence. We have

$$Lf.g' = Lf.Lh^{-1}.L\theta.\varepsilon_y \mapsto \eta_{Rx}.RLf.RLh^{-1}.RL\theta.R\varepsilon_y = f.h^{-1}.\theta = R\hat{f}.\theta = Rg$$

and

$$\varepsilon_x.g \mapsto \eta_{Rx}.Rg = Rg.$$

□

Theorem 0.2. *Given a Street opfibration $R: \mathbf{D} \rightarrow \mathbf{C}$, it is a right adjoint when it is nice.*

Let's try to figure out what *nice* could be. Here are some helpful theorems.

Theorem 0.3 (Freyd's general adjoint functor theorem). *A functor $F: X \rightarrow Y$ is a right adjoint if x is complete, locally small, and F satisfies the solution set condition. The latter says, for any Y -object y , there exists a small set I indexing a collection of X -objects x_{iI} and Y -arrows $f_i: y \rightarrow F(x_{iI})$ such that every F -valued Y -arrow $y \rightarrow Rx$ factors as $Fg.f_k$ for $k \in I$ and $g: x_k \rightarrow x$.*

Theorem 0.4 (Gabriel-Zisman). *An adjunction $L \dashv R: \mathbf{C} \leftrightarrow \mathbf{D}$. TFAE:*

- (a) *L is full and faithful;*
- (b) *the unit is an isomorphism;*
- (c) *the induced comonad on \mathbf{D} is idempotent, L is conservative, and R is essentially surjective.*

Now, let's prove a strong theorem then try to weaken it

Theorem 0.5. *Let \mathbf{D} be locally small and complete. Also let $R: \mathbf{D} \rightarrow \mathbf{C}$ be a continuous, surjective-on-objects Grothendieck opfibration. Then R is a right adjoint.*

Proof. We use 0.3. Suffice to show the solution set condition holds. Fix a \mathbf{C} -objects c . Then the indexing set is $*$, the collection of \mathbf{D} -objects consists of a single object x_c over c (which exists by surjective-on-objects assumption), and the collection of \mathbf{C} -arrows consists of the identity. Any map $f: c \rightarrow Rd$ has a cocartesian lifting $\hat{f}: x_c \rightarrow x_{Rd}$ and $f = R\hat{f}.1_c$. \square

Theorem 0.6. *Let \mathbf{D} be locally small and complete. Also let $R: \mathbf{D} \rightarrow \mathbf{C}$ be a continuous, surjective-on-objects, conservative Street opfibration. Then R is a right adjoint.*

Proof. We use 0.3. Suffice to show the solution set condition holds. Fix a \mathbf{C} -objects c . Then the indexing set is $*$, the collection of \mathbf{D} -objects consists of a single object x_c over c (which exists by surjective-on-objects assumption), and the collection of \mathbf{C} -arrows consists of the identity. For any map $f: c \rightarrow Rd$, there exists an essential cocartesian lifting $\hat{f}: x_c \rightarrow d'$ together with a \mathbf{C} -isomorphism $h: Rd' \rightarrow Rd$ such that $f = h.R\hat{f}$. But $f = h.R\hat{f} = R\hat{h}.R\hat{f}.1_c = R(\hat{h}.\hat{f}).1_c$ where $h = R\hat{h}$ because R is conservative. \square

Theorem 0.7. *Let \mathbf{D} be locally small and complete. Also let $R: \mathbf{D} \rightarrow \mathbf{C}$ be a continuous, essentially surjective, conservative Street opfibration. Then R is a right adjoint.*

Proof. We use 0.3. Suffice to show the solution set condition holds. Fix a \mathbf{C} -object c . Then the indexing set is $*$, the collection of \mathbf{D} -objects consists of a single object d' where $\theta: c \cong Rd'$ (which exists by essential surjectivity), and the collection of \mathbf{C} -arrows is θ . For any \mathbf{C} -arrow $f: c \rightarrow Rd$, we have (by Street opfibrationness) $f.\theta^{-1}: Rd' \rightarrow c \rightarrow Rd$, a cocartesian essential lifting $f.\theta^{-1}: d' \rightarrow d''$ in \mathbf{D} , and an isomorphism $h: d \rightarrow d''$ (by conservativism) such that $f.\theta^{-1} = Rh.Rf.\hat{\theta}^{-1}$. This implies, as required by GAFT, that $f = R(hf.\hat{\theta}).\theta$. \square

Thoughts about these assumptions. Here are the desired examples I can think of now: **Set** together with **Graph** or **Top**. The enriched over sets and completeness are both there. So is essential surjectivity. And reflection of isomorphisms. The continuity is definitely needed, since it's necessary if R is a right adjoint.

Go back to the right adjoint we had in the “converse” to the above theorem. That is, we have a coreflection $L \dashv R: \mathbf{C} \leftrightarrow \mathbf{D}$ where L is left exact. Of course this gives the continuity of R . Essential surjectivity follows from Gabriel-Zisman. Is it a Street opfibration? L is conservative, but is R ?

Example 0.8. No, R is not in general conservative. Consider the underlying node functor $\mathbf{Graph} \rightarrow \mathbf{Set}$. All **Graph**-endomorphisms on

$$\bullet \xrightarrow{\quad} \bullet$$

are sent to the identity on 2.

At this point, we have partial converses:

Let \mathbf{D} have pushouts. Given a coreflection $L \dashv R: \mathbf{C} \rightarrow \mathbf{D}$ where R preserves pushouts, R is a Street opfibration.

Let \mathbf{D} be locally small and complete. Also let $R: \mathbf{D} \rightarrow \mathbf{C}$ be a continuous, essentially surjective, conservative Street opfibration. Then R is a right adjoint.