A NOTE ON FIBRATIONS AND ADJUNCTIONS

DANIEL CICALA AND CHRISTINA VASILAKOPOULOU

Abstract. Fibrations and Adjunctions rock.

Contents

1. Preliminaries

In this section, we present the necessary background for this paper and take this opportunity to set our notation and conventions. One convention we employ is the singular focus on opfibrations instead of their dual, fibrations. We cover two flavors of opfibrations, Grothendieck and Street, and also present a lemma that relates colimits in the fibres to colimits in the total category.

Grothendieck Opfibrations. We recall some basic material from the theory of (Grothendieck) opfibrations; standard references include [2, 4, 7].

Fix a functor $U \colon \mathsf{X} \to \mathsf{A}$. An object x in X is said to **be over** an object a in A if $Ux = a$. The same terminology applies for arrows. An arrow β in X over an arrow $f \colon a \to b$ in A is called **cocartesian** if and only if, for all $g \colon b \to b'$ in A and $\gamma \colon x \to y'$ in X with $U(\gamma) = g \circ f$, there exists a unique $\delta \colon y \to y'$ in X such that $U(\delta) = g$ and $\gamma = \delta \circ \beta$ (see Figure 1)

For any object a in A, let X_a denote the **fibre** of U over a, that is the subcategory of X that consists of the objects x over a and **vertical arrows** (those sent by U to 1_a). The functor $U \colon \mathsf{X} \to \mathsf{A}$ is an **opfibration** if and only if, for all $f \colon a \to b$ in A and x in X_a, there is a cocartesian arrow β with domain x above f; it is called a **cocartesian lifting** of b along f. The category A is called the **base** of the opfibration, and X its **total category**. Of course, this is a dual notion to that of a **fibration**.

For any opfibration $U \colon \mathsf{X} \to \mathsf{A}$, we may use the axiom of choice to select a cocartesian lift $\mathrm{Cocart}(f, x) \colon x \to f_!(x)$ of each $f \colon a \to b$ in A and x in X_a. An opfibration equipped with a choice of cocartesian liftings is called a **cloven opfibration**. Moving forward, we take all opfibration to be cloven. As a special case of the universal property, every arrow in the total category of an opfibration factorizes

$$
\begin{tikzcd}
 & & y' & \text{in } \mathsf{X} \\
x \arrow[r, "\beta"'] \arrow[urr, "\gamma"] \arrow[dd, dotted, no head] & y \arrow[ur, dashed, "\exists!\delta"'] \arrow[dd, dotted, no head] & & \\
 & & b' \arrow[uu, dotted, no head] & \text{in } \mathsf{A} \\
a \arrow[r, "f = U\beta"'] \arrow[urr, "gf = U\gamma"] & b \arrow[ur, "g"'] & &
\end{tikzcd}
$$

FIGURE 1. Cocartesian Lifting

uniquely into a cocartesian arrow followed by a vertical arrow:

$$
\begin{tikzcd}
x \arrow[r, "\gamma"] \arrow[dr, "\mathrm{Cocart}(f,x)"'] \arrow[dd, dotted, no head] & y & \text{in } \mathsf{X} \\
 & f_! x \arrow[u, dashed, "\delta"'] \arrow[d, dotted, no head] & \\
a \arrow[r, "f"'] & b & \text{in } \mathsf{A}
\end{tikzcd}
\tag{1}
$$

The choice of cocartesian liftings in a cloven opfibration induces a **reindexing functor** between the fibre categories

$$f_! \colon \mathsf{X}_a \to \mathsf{X}_b, \tag{2}$$

one for each $f\colon a \to b$ in the base category. It can be verified by the cocartesian lifting property that $(1_a)_! \cong 1_{\mathsf{X}_a}$ and $(f \circ g)_! \cong f_! \circ g_!$.

To relate opfibrations to right adjoints, we use the following lemma which clarifies the relationship between the existence of colimits in the total category of an opfibration to the existence of those colimits in the fibres. For more details and a proof of the following result, see [5, Cor. 3.7].

Lemma 1.1. *Suppose* J *is a small category and* $U\colon \mathsf{X} \to \mathsf{A}$ *is an opfibration whose base* A *has* J*-colimits. The following are equivalent:*

(a) all fibres have J*-colimits, and the reindexing functors preserve them;*
(b) the total category X *has* J*-colimits, and* U *(strictly) preserves them.*

Remark 1.2. This lemma still holds when U merely preserves colimits because, when U is an opfibration, we can change our choice of colimits in the total category so that U does strictly preserve them (Lemma A.4).

When condition (a) holds, we typically say the opfibration has *opfibred* J*-colimits* though, in principle, this does not require a J-cocomplete base.

$$\begin{tikzcd}
x \arrow[rr, "\beta"] \arrow[dd, dotted, no head] & & y \arrow[dd, dotted, no head] & \text{in } \mathsf{X} \\
& & & \\
a \arrow[r, "f"'] \arrow[rr, bend left, "U\beta"] & b \arrow[r, "h"'] & Uy & \text{in } \mathsf{A}
\end{tikzcd}$$

FIGURE 2. Street Opfibration

1.1. **Street Opfibrations.** Typically, we ask for constructions in category theory to transport across equivalent categories. Grothendieck opfibrations, in general, do not have this property. That is, given a Grothendieck opfibration $U\colon \mathsf{X} \to \mathsf{A}$ and equivalence of categories $E\colon \mathsf{X}' \to \mathsf{X}$ or $E'\colon \mathsf{A} \to \mathsf{A}'$, neither UE nor $E'U$ are necessarily Grothendieck opfibrations. Indeed, given an arrow $f\colon a \to b$ in A with cocartesian lifting $\beta\colon x \to y$, a Grothendieck opfibration requires an equality $Uy = b$ which may weaken to an isomorphism after extending U by an equivalence. Street opfibrations avoid this defect. While they were originally defined internally to 2-categories [9], we restrict our attention Cat.

Definition 1.3. A functor $U\colon \mathsf{X} \to \mathsf{A}$ is a **Street opfibration** if, for any arrow $f\colon a \to b$ in A and x over a, there exists a cocartesian arrow $\theta\colon x \to y$ and an isomorphism $h\colon b \cong Uy$ such that $fh = U\theta$ (see Figure 2)

The definition of "cocartesian arrow" here is the same used for Grothendieck opfibrations. The isomorphism h, which replaces an equality in a Grothendieck opfibration, ensures that Street opfibrations do transport across equivalences.

In Section 2, we expose a relationship between Street opfibrations and right adjoints for which we use a generalized version of Lemma 1.1. In order to prove this lemma, we first present several standard results and recall the definition of an isofibration.

Definition 1.4. A functor $U\colon \mathsf{X} \to \mathsf{A}$ is an **isofibration** if, for every isomorphism $f\colon Ux \to a$ in A, there is an isomorphism $\widehat{f}\colon x \to x'$ in X such that $U\widehat{f} = f$.

Lemma 1.5. *If a functor is a Street opfibration and an isofibration, then it is also a Grothendieck fibration.*

Proof. Let $U\colon \mathsf{X} \to \mathsf{A}$ be both a Street opfibration and isofibration. To an arrow $f\colon Ux \to a$, we associate an cocartesian lifting $\theta\colon x \to y$ and isomorphism $h\colon a \to Uy$. Let $\widehat{h^{-1}}$ be the invertible arrow in X over h^{-1}. The cocartesian arrow lift of f is $\theta\widehat{h^{-1}}$. □

Lemma 1.6. *A Street opfibration can be decomposed into a Grothendieck opfibration followed by an equivalence of categories.*

Proof. There is a model structure on Cat whose weak equivalences are equivalences of categories and whose fibrations are isofibrations [8]. Hence, we can decompose a Street opfibration into an isofibration followed by a equivalence of categories. The isofibration is equivalent to a Street opfibration so is itself a Street opfibration and therefore, by Lemma 1.5, a Grothendieck opfibration. □

Lemma 1.7. *Suppose* J *is a small category and* $U\colon \mathsf{X} \to \mathsf{A}$ *is a Street opfibration. If* A *has* J*-colimits, the following are equivalent:*

(a) all fibres have J*-colimits and the reindexing functors preserve them;*

(b) X *has* J*-colimits and* U *preserves them.*

Proof. Per Lemma 1.6, decompose U into HU' where U' is a Street opfibration and H is an equivalence. The result follows by applying Lemma 1.1 to U'. □

1.2. **Coreflectors and ralis.** The main results of this paper are a correspondence between the Grothendieck and Street opfibrations and certain right adjoint functors. In this section, we recall the definitions of these certain right adjoints, ralis and coreflectors. These sorts of right adjoints are closely related, with ralis having the stronger definition.

A **rali**, or right-adjoint-left-inverse, is the right adjoint U portion of an adjunction $L \dashv U\colon \mathsf{X} \to \mathsf{A}$ that satisfies the equation $UL = 1_{\mathsf{A}}$ or, equivalently, has a unit consisting of only identity arrows. If ϵ is the counit of a rali, then $U\epsilon$ and ϵ_L are also identities. Also, the equation $ULU = U$ holds. The term **lari**, or left-adjoint-right-inverse, was introduced by Gray [4].

Just as a Street opfibration is a weaker version of a Grothendieck opfibration, a coreflector is a weaker version of a rali. This next proposition gives defining properties for a coreflector.

Proposition 1.8 ([1, Prop. 3.4.1])**.** *The following are equivalent for an adjunction* $L \dashv U\colon \mathsf{X} \to \mathsf{A}$*:*

(a) the left adjoint L *is full and faithful;*

(b) the unit $\eta\colon 1_{\mathsf{A}} \Rightarrow UL$ *is an isomorphism.*

Under these conditions, $U\epsilon$ *and* ϵ_L *are also isomorphisms.*

An adjuntion $L \dashv U$ satisfiying these properties is called a **coreflection**, the right adjoint U a **coreflector**, and A a **coreflective** subcategory of X.

2. Grothendieck opfibrations and ralis

In this section, we investigate the relationship between Grothendieck opfibrations and right adjoints. We start by finding conditions under which a Grothendieck opfibration is a right adjoint. Next, we find conditions under which a right adjoint is a Grothendieck opfibration. We conclude by exhibiting a correspondence between the two by intersecting the respective conditions.

The following result gives an answer to the question: When is a Grothendieck opfibration a right adjoint? It is dual to a proposition of Gray [4, Prop. 4.4].

Proposition 2.1. *Let* $U\colon \mathsf{A} \to \mathsf{X}$ *be a Grothendieck opfibration. If the fibres of* U *have initial objects and the reindexing functors preserve them, then* U *is a rali.*

Proof. Let $\perp_a$ denote the initial object in fiber X_a. Define a functor $L\colon \mathsf{A} \to \mathsf{X}$ by setting $La = \perp_a$ and $L(f\colon a \to b) = ! \circ \mathrm{Cocart}(f, \perp_a)$, where ! is the canonical map $f_!(\perp_a) \to \perp_b$ induced by the reindexing functors preserving initial objects (see Figure 3). This assignment is functorial due to the universal property of each cocartesian lift.

$$\begin{tikzcd}
\mathsf{A} \arrow[rr, "L"] & & \mathsf{X} & \\
a \arrow[d, "f"'] & & \perp_a \arrow[d, "Lf"'] \arrow[dr, "\mathrm{Cocart}(f,\perp_a)"] & \\
b & & \perp_b & f_!(\perp_a) \arrow[l, "\cong", "!"']
\end{tikzcd}$$

FIGURE 3. Left adjoint for Proposition 2.1

$$\begin{tikzcd}
LUx \arrow[r, "Lf"] \arrow[d, "\epsilon_a"'] & Lb \arrow[d] \\
x \arrow[r, "\mathrm{Cocart}(f,x)"'] & x +_{LUx} Lb \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

FIGURE 4. Defining the cocartesian lift

To show that L is left adjoint to U, we establish a natural bijection

$$\mathsf{X}(La, x) \xrightarrow{\cong} \mathsf{A}(a, Ux) = \mathsf{A}(ULa, Ux)$$

by $k \mapsto Uk$. Uniqueness follows from the unique factorization of k into $! \circ \mathrm{Cocart}(Uk, \perp_a)$ (cf. Diagram 1) and naturality follows from the universal property of opfibrations.

Finally, we show that U is a left inverse of L because the unit of the adjunction $\eta\colon 1_{\mathsf{A}} \to UL$ is the identity natural transformation. Indeed, $ULa = U\perp_a = a$ and moreover, applying the natural bijection coming from the adjunction gives

$$\mathsf{X}(\perp_a, \perp_a) = \mathsf{X}(La, La) \cong \mathsf{A}(a, ULa) = \mathsf{A}(a, a)$$

which ensures that the identity on $\perp_a$ corresponds to the identity on a. □

This result contains a slight surprise. Our opfibration becoming a right adjoint depends on it preserving initial objects. While this proposition uses reindexing functors to relate opfibrations and ralis, we can use Lemma 1.1 shift the hypothesis to a property of U instead.

Corollary 2.2. *An opfibration is a rali if the base category has and the functor strictly preserves the initial object.*

Next, we examine when a rali is an opfibration, which happens when it strictly preserves pushouts. Our pushouts are choosen so that they strictly preserve identities. Of course, strict preservation is a strong condition but it can be safely assumed for a non-trivial class of functors as discussed in Appendix A.

Proposition 2.3. *Suppose that $U\colon \mathsf{X} \to \mathsf{A}$ is a rali with left adjoint L. Then U is an opfibration if X and A have chosen pushouts and U strictly preserves them.*

Proof. Take an arrow $f\colon a \to b$ in A and an object x above a. Define the cocartesian lifting of x along f to be the (chosen) pushout of Lf along the counit ϵ_x.

To verify $\mathrm{Cocart}(f, x)$ is a cocartesian lifting, first of all it must be mapped to f via U. To see this, apply U to the above square. Before doing so, consider serveral facts. Proposition 1.8 implies that $U\epsilon_x = 1_a$ and $ULf = f$. Pushouts in A preserve identity. Also, because U strictly preserves pushouts, we have that

$$U(x +_{LUx} Lb) = Ux +_{ULUx} ULb = a +_a b$$

Now, the image of the above square under U is

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[d, "1_a"'] & & b \arrow[d, "1_b"] \\ a \arrow[r, "f"'] & a +_a Lb = b \arrow[ul, phantom, "\ulcorner"] & \end{tikzcd}$$

Moreover, the universal property (cf. Diagram 1) of the proposed cocartesian lifting follows from universality of pushouts. $\square$

Combining Propositions 2.1 and 2.3, we establish certain conditions under which a Grothendieck opfibration corresponds to a rali and vice versa.

Theorem 2.4. *Suppose that X and A have chosen pushouts and initial objects, and a functor $U\colon \mathsf{X} \to \mathsf{A}$ strictly preserves them. Then U is a right adjoint left inverse if and only if U is an opfibration.*

3. Street opfibrations and coreflectors

In this section, we trod the same path as in Section 2 but shed any strictness around preserving colimits by working with Street opfibrations instead of Grothendieck opfibrations and with reflectors instead of ralis

Generalising Proposition 2.1, we obtain the following.

Proposition 3.1. *Let $U\colon \mathsf{X} \to \mathsf{A}$ be a Street opfibration. Then U is a coreflector if its fibers have initial objects which are preserved by the reindexing functors.*

Proof. We define L exactly as in Proposition 2.1.

The unit of the adjunction $L \dashv U$ is the identity $a \to ULa = U\bot_a = a$. The counit is the initial map $\bot_{Ux} = LUx \to x$. The triangle identities are satisfied as well. The first is the composite

$$La \xrightarrow{L\eta_a} LULa \xrightarrow{\epsilon_{La}} La$$

which is given by $\bot_a \to \bot_a \to \bot_a$. The second is the composite

$$Ux \xrightarrow{\eta_{Ux}} ULUx \xrightarrow{U\eta_x} Ux$$

which is given by $Ux \to Ux \to Ux$.

Because the unit of the adjunction is an isomorphism, L is full and faithful [3, Prop. 1.3] making U a coreflector. $\square$

Applying Lemma 1.7, we can transport the assumptions on the fibers to the base category of our Street fibration.

Corollary 3.2. *A Street opfibration is a reflector if the base category has and the functor preserves the (initial) terminal object.*

Generalising Proposition 2.3, we obtain the following result.

Proposition 3.3. *Suppose that $U\colon \mathsf{X} \to \mathsf{A}$ is a coreflector. Then U is a Street opfibration if X has pushouts and U preserves them.*

Proof. Fix an arrow $f \colon a \to b$ in A and an object x in the fibre of a. We claim that $\widehat{f}$, defined as the pushout of Lf along the counit ϵ_x

$$\begin{tikzcd} LUx \arrow[r, "Lf"] \arrow[d, "\epsilon"'] & Lb \arrow[d] \\ x \arrow[r, "\widehat{f}"] & x +_{LUx} Lb \end{tikzcd}$$

is the cocartesian lift. There is a string of isomorphisms

$$U(x +_{LUx} Lb) \cong Ux +_{ULUx} Lb \cong Ux +_{Ux} b \cong b$$

whose composite we call h. Then

$$\begin{tikzcd} & U(x +_{LUx} Lb) \arrow[d, "h"] \\ a \arrow[ur, "U\widehat{f}"] \arrow[r, "f"'] & b \end{tikzcd}$$

commutes, and so $\widehat{f}$ is an appropriate lift. It remains to show that $\widehat{f}$ is cocartesian.

Consider a X-arrow $g \colon x \to y$ with a A-arrow $\theta \colon U(x +_{LUx} Lb) \to Uy$ so that $\theta U\widehat{f} = Ug$. Can we uniquely lift θ? Set up the diagram

$$\begin{tikzcd} LUx \arrow[r, "Lf"] \arrow[d, "\epsilon_x"'] & Lb \arrow[d] \arrow[ddr, bend left, "g'"] & \\ x \arrow[r, "\widehat{f}"] \arrow[drr, bend right, "g"] & x +_{LUx} Lb & \\ & & \mathrm{y} \end{tikzcd}$$

where $g' \coloneqq \epsilon_y L\theta L h^{-1}$. To show the outer square commutes, it suffices to show that $g'Lf$ and $g\epsilon_x$ have the same image under the adjunction homset correspondence. We have

$$g' \circ Lf = \epsilon_y \circ L\theta \circ Lh \circ Lf \mapsto U\epsilon_y \circ UL\theta \circ ULh \circ ULf \circ \eta_{ux} = \theta \circ h \circ f = \theta \circ U \circ \widehat{f} = Ug$$

and

$$g\epsilon_x \mapsto Ug \circ \eta_{Ux} = Ug$$

□

We now intersect the hypothesis of Proposition 3.1 and Proposition 3.3 to provide the main result of this section.

Theorem 3.4. *Let $U\colon \mathsf{X} \to \mathsf{A}$ be a functor such that A and X have chosen initial objects and pushouts and U preserves them. Then U is a coreflector if and only if U is a Street opfibration.*

Appendix A. Choice of colimits

In what follows, and in particular for our main Theorem 2.4, we often require that certain colimits must be *strictly* preserved. Although the strict preservation of colimits does not adhere to the principle of equivalence, it is required when working with Grothendieck fibrations. Moreover, in Section 3 we examine the non-strict context which then naturally matches to the notion of a Street fibration as discussed above.

In more detail, assuming the axiom of choice we can regard any category with colimits as having *chosen* ones, in the sense of choosing a specific adjoint (from all isomorphic ones) to the constant diagram functor:

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\Delta_{\mathsf{C}}" description] & {[\mathsf{J},\mathsf{C}]} \arrow[l, bend right, "\mathrm{colim}_{\mathsf{J}}"', "\perp"] \arrow[l, bend left, "\mathrm{lim}_{\mathsf{J}}", "\perp"'] \end{tikzcd}$$

Some categories, like Set, even have a canonical choice corresponding to well-known constructions of colimits of sets. In general, a functor $F\colon \mathsf{C} \to \mathsf{D}$ between categories with colimits, for example, preserves them when the following diagram

$$\begin{tikzcd} {[\mathsf{J},\mathsf{C}]} \arrow[r, "\mathrm{colim}"] \arrow[d, "{[\mathsf{J},F]}"'] \arrow[dr, phantom, "\cong"] & \mathsf{C} \arrow[d, "F"] \\ {[\mathsf{J},\mathsf{D}]} \arrow[r, "\mathrm{colim}"'] & \mathsf{D} \end{tikzcd} \tag{3}$$

commutes up to natural isomorphism. The following two lemmas (due to Steve Lack) present two natural settings where functors between categories with chosen colimits strictly preserve them; evidently, such a thing is to be expected only when the colimits in the categories have been both previously constructed from chosen limits in some fixed category.

Lemma A.1. *Suppose C is a category with chosen colimits of any class. Then for any two categories A and B and any functor $F\colon \mathsf{A} \to \mathsf{B}$ between them, the pre-composition functor*

$$\begin{aligned} F^*\colon [\mathsf{B},\mathsf{C}] &\longrightarrow [\mathsf{A},\mathsf{C}] \\ \left(\mathsf{B} \xrightarrow{H} \mathsf{C}\right) &\longmapsto \left(\mathsf{A} \xrightarrow{F} \mathsf{B} \xrightarrow{H} \mathsf{C}\right) \end{aligned}$$

strictly preserves the chosen colimits.

Proof. This follows from the pointwise construction of colimits in functor categories. □

As a particular case, the following result concerns our motivating example.

Corollary A.2. *There is a canonical choice of colimits in Grph inherited from those in Set so that the domain and codomain functors $\mathsf{Grph} \to \mathsf{Set}$ strictly preserve all colimits.*

Proof. The domain and codomain functors, respectively 1^* and 2^*, are built from the functors $1, 2\colon \mathbf{1}=\{\bullet\} \longrightarrow \{1 \rightrightarrows 2\}=\mathbf{2}$. Choosing the canonical colimits in Set, by Lemma A.1 we obtain two functors

$$\mathsf{Grph} = [\mathbf{2}, \mathsf{Set}] \underset{2^*}{\overset{1^*}{\rightrightarrows}} [\mathbf{1}, \mathsf{Set}] = \mathsf{Set}$$

that strictly preserve them. □

The following case again follows from a construction of chosen limits in common ground; colimits adhere to a dual result.

Lemma A.3. *Suppose* C *and* D *have chosen colimits and* $F\colon \mathsf{C} \to \mathsf{D}$ *is an arbitrary functor. Then the comma category* $F \downarrow \mathsf{D}$ *can be equipped with colimits in such a way that both projections* $\mathsf{C} \leftarrow F \downarrow \mathsf{D} \to \mathsf{D}$ *strictly preserve them.*

Proof. This follows from the canonical construction of colimits in comma categories (see [1, §2.16]). □

Finally, opfibrations form another class of functors that is notable when it comes to strictly preserving colimits. In more detail, we do not lose generality by assuming that a colimit preserving opfibration preserves strictly. Such a statement does not apply to arbitrary functors. This is due to the following lemma, which is a special case of a more general fact: the embedding of OpFib in the 2-category $\mathsf{OpFib}_\sim$ where 2-cells are squares filled with isomorphisms, is locally fully faithful and essentially surjective (also for fibs...references...[6]?). We thank Claudio Hermida for these observations.

Lemma A.4. *Suppose* U *and* Q *are opfibrations, and there is a natural isomorphism*

$$\begin{tikzcd} \mathsf{X} \arrow[r, "H"] \arrow[d, "U"'] \arrow[dr, phantom, "\overset{\phi}{\cong}"] & \mathsf{Y} \arrow[d, "Q"] \\ \mathsf{A} \arrow[r, "F"'] & \mathsf{B} \end{tikzcd}$$

Then ϕ *factors as a commutative square composed by an isomorphism* $\widehat{\phi}$*, as in*

$$\begin{tikzcd} \mathsf{X} \arrow[r, bend left, "H"] \arrow[r, bend right, "\widehat{H}"'] \arrow[r, phantom, "\overset{\widehat{\phi}}{\cong}"] \arrow[d, "U"'] & \mathsf{Y} \arrow[d, "Q"] \\ \mathsf{A} \arrow[r, "F"'] & \mathsf{B} \end{tikzcd}$$

As a result, if U is an opfibration that preserves J-colimits for some small J, we can factor the natural isomorphism (3), where the left leg is also a fibration, as

$$\begin{tikzcd} {[\mathsf{J}, \mathsf{X}]} \arrow[r, bend left, "\mathrm{colim}"] \arrow[r, bend right, "\widehat{\mathrm{colim}}"'] \arrow[r, phantom, "\cong"] \arrow[d, "{[\mathsf{J}, U]}"'] & \mathsf{X} \arrow[d, "U"] \\ {[\mathsf{J}, \mathsf{A}]} \arrow[r, "\mathrm{colim}"'] & \mathsf{A} \end{tikzcd}$$

essentially changing the choice of colimits in the total category and establishing that Q now strictly preserves them. In a dual way, we may assume that any fibration strictly preserves chosen limits, if it preserves limits in the ordinary sense.

As a concrete example of the implications of the above result, suppose that $U\colon \mathsf{X} \to \mathsf{A}$ is an opfibration that preserves coproducts. This means that for any $x, y \in \mathsf{X}$, there is a canonical isomorphism $U(x+y) \cong U(x) + U(y)$ in A which we can call k, for some choice of coproducts in X and A. Fixing the ones in the base category A, we can change the choice of coproducts in X so that U strictly preserves them, by considering the cocartesian lifting of $x+y$ along the isomorphism k

$$\begin{tikzcd} x+y \arrow[r, "\mathrm{Cocart}(k,x+y)"] \arrow[d, dotted, no head] & k_!(x+y) \arrow[d, dotted, no head] & \text{in } \mathsf{X} \\ U(x+y) \arrow[r, "\cong", "k"'] & U(x)+U(y) & \text{in } \mathsf{A} \end{tikzcd}$$

We can then verify that $k_!(x+y)$ has the universal property of a coproduct in X.

References

[1] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Basic category theory.

[2] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[3] Peter Gabriel and Michel Zisman. *Calculus of fractions and homotopy theory*, volume 35. Springer Science & Business Media, 2012.

[4] John W. Gray. Fibred and cofibred categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 21–83. Springer, New York, 1966.

[5] Claudio Hermida. On fibred adjunctions and completeness for fibred categories. In *Recent trends in data type specification (Caldes de Malavella, 1992)*, volume 785 of *Lecture Notes in Comput. Sci.*, pages 235–251. Springer, Berlin, 1994.

[6] Claudio Hermida. Some properties of fib as a fibred 2-category. *Journal of Pure and Applied Algebra*, 134(1):83 – 109, 1999.

[7] Claudio Alberto Hermida. *Fibrations, logical predicates and indeterminates*. PhD thesis, University of Edinburgh, 1993.

[8] Charles Rezk. A model category for categories. *Available from the author's web page*, 1996.

[9] Ross Street. Fibrations and Yoneda's lemma in a 2-category. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 104–133. Lecture Notes in Math., Vol. 420, 1974.

UPDATE DEPARTMENTS

Email address: `update emails`