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CONSPECTUS OF VARIABLE CATEGORIES

Ross STREET
School of Mathematics and Physics, Macquarie University, North Ryde, N.S.W. 2113, Australia

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Introduction

Sheaves in geometry are profitably viewed as variable sets [13]. A non-elementary study of variable sets begins with the analysis of the categories $[\mathscr{C}^{op}, \mathbf{Set}]$ of small-set-valued contravariant functors on a small category $\mathscr{C}$. The objects U of $\mathscr{C}$ are 'the stages of knowledge' and the arrows $r: V \to U$ in $\mathscr{C}$ give 'the internal development' for the sets varying over $\mathscr{C}$.

Finer analysis of a geometric problem at hand usually discloses a small set Σ of arrows in $[\mathscr{C}^{op}, \mathbf{Set}]$ stable under pulling back. The pair $(\mathscr{C}, \Sigma)$ is called a *pullback-stable sketch* [2, 23] (or, when Σ consists of monomorphisms, a *site* [1]). A *sheaf* over $(\mathscr{C}, \Sigma)$ is an object X of $[\mathscr{C}^{op}, \mathbf{Set}]$ with the property that, for all $f: A \to C$ in Σ and all $h: A \to X$, there exists a unique $k: C \to X$ with $kf = h$. The full subcategory $[\mathscr{C}^{op}, \mathbf{Set}]_\Sigma$ of $[\mathscr{C}^{op}, \mathbf{Set}]$ consisting of the sheaves satisfies the elementary axioms required on a category in order that it should be a *quasi-topos* [16]: there is an internal notion of power set which allows one to use set theoretic arguments to obtain geometric results.

A natural notion of *variable structure* is provided by a model in a quasi-topos of the appropriate theory. Certainly the theory of categories, and even richer theories, can be modelled in a quasi-topos; yet this gives too restrictive a notion of *variable category*, by denying the possible 2-dimensional aspect of its internal development.

For example, categories in $[\mathscr{C}^{op}, \mathbf{Set}]$ amount to functors X from $\mathscr{C}^{op}$ to $\mathbf{Cat}$. Most naturally occurring arrows in $\mathbf{Cat}$ are only of interest up to isomorphism, so that the condition that X should preserve composition strictly is absurd. What should be considered are the homomorphisms of bicategories [3] from $\mathscr{C}^{op}$ to $\mathbf{Cat}$. When $\mathscr{C}$ has pullbacks, this then includes the basic category ranging over $\mathscr{C}$ which is the category $\mathscr{C}/U$ at stage U and whose internal development is given by pulling back. Devious contortions do exist for converting the latter basic homomorphism into a functor, however, as this strict approach is developed, further obstacles appear. In particular, problems arise in attempting to force Kan extensions of arrows into such basic

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variable categories to be strict transformations. The natural replacement for $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ is thus the 2-category $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ of homomorphisms, strong transformations, and modifications.

Indeed, we go further and claim that the correct objects of study are the 2-categories $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ where $\mathscr{C}$ is a small bicategory. In the case of variable sets there is no need to allow $\mathscr{C}$ to be anything other than a mere category since a homomorphism $\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ amounts to a mere functor $\pi_* \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ (see (1.4) below). Homomorphisms $\mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ cannot similarly be reduced to category-valued functors.

This possibility of 2-dimensional internal development for variable categories allows the natural inclusion of many examples. A set with an endomorphism can be regarded as a set with extra structure, and so universal algebra applies. While this point of view is important, the observation that a set with an endomorphism is a variable set allows a much more penetrating analysis. A category with a monad on it can be regarded as a category with extra structure (or even as a variable structure in the sense mentioned earlier), but again, to relegate this example to the realm of universal algebra is less penetrating than the observation that it is a fundamental example of a variable category. The bicategory $\mathscr{C}$ for the latter example has one object U and has the simplicial category as its hom-category $\mathscr{C}(U, U)$. Then the 2-category $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ is biequivalent [24] to that sub-2-category of the $\mathrm{Mnd}(\mathbf{Cat})$ of [18] whose objects are all the monads in $\mathbf{Cat}$ but whose arrows are only the strong monad functors. By taking $L_{\mathscr{C}}$ to be as described in [20, Section 4], notice that $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(L_{\mathscr{C}}, A)$ is equivalent to the category of Eilenberg–Moore algebras for the monad corresponding to A.

It may be argued that the 2-categories $\mathrm{Bicat}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ of morphisms of bicategories, transformations, and modifications [24] are the correct objects of study at the outset of a study of variable categories. An unpublished result of Bénabou and the construction of Gray [9, p. 92] indicate that for each bicategory $\mathscr{C}$ there is another bicategory $\mathscr{D}$ such that $\mathrm{Bicat}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ is biequivalent to the 2-category of homomorphisms of bicategories $\mathscr{D}^{\mathrm{op}} \to \mathbf{Cat}$, transformations and modifications. Our belief is that we should look at $\mathrm{Hom}(\mathscr{D}^{\mathrm{op}}, \mathbf{Cat})$, where the arrows are the strong transformations, and use the constructions of [19] to capture the transformations between homomorphisms as strong transformations between appropriately altered homomorphisms.

This brings us to the appropriate generalization of the sheaf condition. Suppose Σ is a set of arrows in a bicategory $\mathscr{K}$. An object X of $\mathscr{K}$ is *pointwise Σ-cocomplete* when, for all $f: A \to C$ in Σ and all $h: A \to X$, there exists a pointwise left extension (see (5.2) below) $k: C \to X$ of h along f. An arrow $t: X \to Y$ in $\mathscr{K}$ is *pointwise Σ-cocontinuous* when it respects these pointwise left extensions. Let $\mathscr{K}_{\mathrm{pw}\Sigma}$ denote the locally full subbicategory of $\mathscr{K}$ consisting of the pointwise Σ-cocomplete objects and the pointwise Σ-cocontinuous arrows. When $(\mathscr{C}, \Sigma)$ is a pullback-stable sketch and $\mathscr{K}$ is the mere category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, then $\mathscr{K}_{\mathrm{pw}\Sigma}$ is precisely the category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]_{\Sigma}$ of sheaves.

Our contention is that the 2-categories $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})_{\mathrm{pw}\Sigma}$, where $\mathscr{C}$ is a small bicategory and Σ is a suitable small set of arrows in $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$, should be the next objects of study in the analysis of variable categories. Notice that the apparent one-sidedness introduced by considering Σ-cocompleteness as opposed to Σ-completeness is resolved by observing that there is an isomorphism $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})^{\mathrm{co}} \cong \mathrm{Hom}(\mathscr{C}^{\mathrm{coop}}, \mathbf{Cat})$ obtained by taking opposite categories at each stage of knowledge.

The search for abstract characterizations of variable categories, in analogy with the elementary axioms for a topos, has resulted in the notion of *Yoneda structure on a 2-category* [27], and the narrower one of *fibrational cosmos* [22, 23]. The latter notion is a little too restrictive for the requirements of this paper, in that it insists that $\mathscr{P}$ should be a 2-functor and that fibrations should be algebras for a certain 2-monad obtained using 2-limits. If we ask only that $\mathscr{P}$ be a homomorphism of bicategories (as in [27]) and that fibrations be algebras for the similarly defined doctrine obtained using bilimits [24], then variable categories in the present sense do form a fibrational cosmos, and the results of [22] carry over with similar changes.

The present paper begins a non-elementary study of variable categories by analysing the 2-categories $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ where $\mathscr{C}$ is a small bicategory. The 2-categories $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})_{\mathrm{pw}\Sigma}$ are considered to some extent herein; a more detailed treatment will appear elsewhere.

Section 1 provides a basic construction of a homomorphism $\# X : \mathscr{C}^{\mathrm{co}} \to \mathbf{Cat}$ from a homomorphism $X : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$.

Fibrations in bicategories were defined in [24]. An independent, more concrete treatment is provided here by first describing fibrations in **Cat** in Section 2, and then in Section 3 describing fibrations in $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ using pointwiseness. Special properties of fibrations for these 2-categories are given. Fibrations in a general bicategory $\mathscr{K}$ can of course then be described by means of the embedding $\mathscr{K} \to \mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathbf{CAT})$ where **CAT** is a suitably large 2-category of categories.

Recall that a fibration from B to A in **Cat** (that is, a fibration over the category $A \times B^{\mathrm{op}}$) is determined up to equivalence by a homomorphism of bicategories $A^{\mathrm{op}} \times B \to \mathbf{Cat}$, or preferably for the present work, by a homomorphism $B \to \mathrm{Hom}(A^{\mathrm{op}}, \mathbf{Cat})$. Bidiscrete fibrations from B to A are determined up to equivalence by a functor $B \to [A^{\mathrm{op}}, \mathbf{Set}]$. In Section 4 these results are generalized from **Cat** to $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$. A bidiscrete fibration from B to A in $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ can be represented up to equivalence by a strong transformation $B \to [(\# A)^{\mathrm{op}}, \mathbf{Set}]$. This suggests that the assignment $A \mapsto [A^{\mathrm{op}}, \mathbf{Set}]$ for constant categories A should be replaced by the assignment $A \mapsto [(\# A)^{\mathrm{op}}, \mathbf{Set}]$ for categories A varying over $\mathscr{C}$. Indeed this assignment is the $\mathscr{P}$ for a Yoneda structure of the (modified) fibrational cosmos kind, as is shown in Section 6.

In Section 5 certain definitions are recalled: notably that of *pointwise extensions*. The *pointwise Σ-cocompletion* is defined to be a left biadjoint for the inclusion of $\mathscr{K}_{\mathrm{pw}\Sigma}$ in $\mathscr{K}$.

In Section 7 we define what it means for an object of Hom($\mathscr{C}^{op}$, **CAT**) to have small colimits. Such objects are shown to admit a wide class of left extensions; this generalizes the usual theorem of Kan [14, p. 236] for constant categories. For an object X of Hom($\mathscr{C}^{op}$, **Cat**) and an object M of Hom($\mathscr{C}^{op}$, **CAT**) which has small colimits, it is deduced in Section 8 that the category of strong transformations $X \to M$ is equivalent to the category of those strong transformations $\mathscr{P}X \to M$ which preserve small colimits in an appropriate sense.

In Section 9 we prove that, for any small set Σ of arrows in Hom($\mathscr{C}^{op}$, **Cat**) the inclusion Hom($\mathscr{C}^{op}$, **Cat**)$_{\mathrm{pw}\Sigma} \to$ Hom($\mathscr{C}^{op}$, **Cat**) has a left biadjoint, the components $X \to \tilde{X}$ of the unit of this biadjunction are fully faithful and dense, and the doctrine generated on Hom($\mathscr{C}^{op}$, **Cat**) is of the Kock–Zöberlein type [24]. Our original proof of this result was a generalization of the proof for the case of constant categories circulated in [25]. Note that it suffices to take Σ to consist only of functors into the terminal category in that case, so that the pointwise left extensions are just colimits. We are grateful to the referee of [25] for his insistence that an easier proof was possible, and to Max Kelly and Bob Walters for their help in finding it. The result for the particular case of constant categories does seem to be known; although, to the author's knowledge, it only appears in the literature with an extra *stability* condition on Σ [11, 28, 29, 30]. {Ehresmann [7] has proved that the forgetful functor from the category of categories with distinguished Σ-colimits and functors which strictly preserve them has a left adjoint. This strict version also follows from the work of Gabriel–Ulmer [8]. The author does not see how to deduce our result from this strict version even for constant categories.}

Prerequisite bicategory theory for this work can be found in Bénabou [3] and the early sections of Street [24].

1. A basic construction for changing variance

1.1. Let **Hom** denote the 2-category of small bicategories, strong transformations, and modifications. For a small bicategory $\mathscr{C}$ and a morphism of bicategories $X : \mathscr{C}^{op} \to \mathbf{Cat}$, we shall describe a homomorphism of bicategories $\int_{\mathscr{C}} X : \mathscr{C}^{co} \to \mathbf{Hom}$. When $\mathscr{C}$ is understood we write $\int X$ for $\int_{\mathscr{C}} X$. The arguments of $\int X$ are written on the left.

1.2. For each object U of $\mathscr{C}$, the bicategory $U\int X$ is described as follows. An object is a triple (x, S, u) where $u : S \to U$ is an arrow in $\mathscr{C}$ and x is an object in XS. An arrow $(\xi, \omega, w) : (x, S, u) \to (x', S', u')$ consists of arrows $w : S \to S'$, $\xi : x \to (Xw)x'$ in $\mathscr{C}$, XS, respectively, and a 2-cell $\omega : u'w \Rightarrow u$ in $\mathscr{C}$. A 2-cell $\rho : (\xi, \omega, w) \to (\xi_1, \omega_1, w_1)$ is a 2-cell $\rho : w \Rightarrow w_1$ in $\mathscr{C}$ such that $\omega = \omega_1 \cdot u'\rho$ and $\xi_1 = (X\rho)x' \cdot \xi$. Composition of arrows in $U\int X$ is performed as in a Kleisli category for a monad; also compare the first basic construction of [19, p. 226].

1.3. For each arrow $r: U \to V$ in $\mathscr{C}$, the homomorphism $r\int X : U\int X \to V\int X$ takes $(\xi, \omega, w):(X, S, u) \to (x', S', u')$ to $(\xi, r\omega, w):(x, S, ru) \to (x', S', ru')$ and a 2-cell ρ to ρ. For each 2-cell $\tau: s \Rightarrow r$ in $\mathscr{C}$, the strong transformation $\tau\int X : r\int X \Rightarrow s\int X$ has component at (x, S, u) equal to the arrow $(1_x, \tau u, 1_S):(x, S, ru) \to (x, S, su)$. The verification that $\int X : \mathscr{C}^{co} \to \mathbf{Hom}$ is a homomorphism is straightforward.

1.4. There is a 2-functor $\pi_* : \mathbf{Hom} \to \mathbf{Cat}$ which takes a bicategory W to the category $\pi_* W$ whose objects w are the objects of W and for which each homset $(\pi_* W)(w, w')$ is the set of path components of the category $W(w, w')$. We write $[\omega]$ for the path component of $\omega : w \to w'$.

1.5. The *cooperative homomorphism* $\# X : \mathscr{C}^{co} \to \mathbf{Cat}$ associated with $X : \mathscr{C}^{op} \to \mathbf{Cat}$ is the composite:

$$\mathscr{C}^{co} \xrightarrow{\int X} \mathbf{Hom} \xrightarrow{\pi_*} \mathbf{Cat}.$$

In order to give the universal property satisfied by this construction, we must give a definition.

1.6. Suppose $X : \mathscr{C}^{op} \to \mathscr{B}$, $Z : \mathscr{C}^{co} \to \mathscr{B}$ are morphisms of bicategories. A *ditransformation* $d : X \to Z$ consists of the data displayed in the following family of diagrams in $\mathscr{B}$:

$$\begin{tikzcd} XV \arrow[r, "d_v"] \arrow[d, "Xr"'] \arrow[dr, phantom, "d_r \Uparrow"] & ZV \\ XU \arrow[r, "d_u"'] & ZU \arrow[u, "Zr"'] \end{tikzcd} \qquad \text{for } r: U \to V \text{ in } \mathscr{C},$$

satisfying the three obvious compatibility conditions arising from identity arrows, composition, and 2-cells in $\mathscr{C}$. There is also an obvious notion of *modification of ditransformations* $\mu : d \to d'$. This gives a category

$$\mathrm{Ditran}(\mathscr{C}^{op}, \mathscr{B})(X, Z)$$

of ditransformations from X to Z. The reader will have no difficulty in establishing the rules for composing ditransformations with opditransformations and with transformations. (Some morphisms must be restricted to being homomorphisms for this.)

1.7. For a morphism $X : \mathscr{C}^{op} \to \mathbf{Cat}$, there is a ditransformation $\partial : X \to \# X$ described as follows. For each object U of $\mathscr{C}$, the functor $\partial_U : XU \to U \# X$ takes $\xi : x \to x'$ to $[\xi, 1, 1]:(x, U, 1_U) \to (x', U, 1_U)$. For each arrow $r: U \to V$ in $\mathscr{C}$, the natural transformation $\partial_r : r \# X \cdot \partial_U \cdot Xr \Rightarrow \partial_V$ has component at $y \in XV$ given by $\partial_r y = [1, 1, r]:((Xr)y, U, r) \to (y, V, 1_V)$.

1.8. The Theorem below, whose proof is routine, is in the spirit of the Theorems of Street [19, Section 3]. On the other hand, taking $\mathscr{C}$ to be $\mathscr{A}^{co}$ and X to be the constant morphism at the terminal category, we obtain Street [20, Section 4, Theorem 11]. When **Cat** is replaced by a small bicategorically cocomplete bicategory $\mathscr{B}$, we can construct a $\# X$ satisfying the first sentence of the Theorem using indexed bicolimits in the terminology of Street [24]; for the second sentence of course $\mathscr{B}$ must be a small cocomplete 2-category.

1.9. Theorem. *For each morphism of bicategories* $X : \mathscr{C}^{op} \to \mathbf{Cat}$, *composition with* $\partial : X \to \# X$ *yields an equivalence of categories*

$$\mathrm{Hom}(\mathscr{C}^{co}, \mathbf{Cat})(\# X, Z) \simeq \mathrm{Ditran}(\mathscr{C}^{op}, \mathbf{Cat})(X, Z)$$

for all homomorphisms $Z : \mathscr{C}^{co} \to \mathbf{Cat}$. *If* $\mathscr{C}$ *is a 2-category, then* $\# X$ *is a 2-functor and the above equivalence restricts to an isomorphism*

$$[\mathscr{C}^{co}, \mathbf{Cat}](\# \mathrm{X}, \mathrm{Z}) \cong \mathrm{Ditran}(\mathscr{C}^{op}, \mathbf{Cat})(X, Z)$$

for all 2-functors $Z : \mathscr{C}^{co} \to \mathbf{Cat}$.

1.10. Our interest in this paper is in the case where X is a homomorphism. For an arrow $f : X \to Y$ in $\mathrm{Hom}(\mathscr{C}^{op}, \mathbf{Cat})$, we obtain a ditransformation $\partial f : X \to \# Y$, and so, by the above theorem, we obtain a strong transformation $\# f : \# X \to \# Y$ with $\# f \cdot \partial \cong \partial f$. In fact, we obtain a homomorphism

$$\# : \mathrm{Hom}(\mathscr{C}^{op}, \mathbf{Cat}) \to \mathrm{Hom}(\mathscr{C}^{co}, \mathbf{Cat}).$$

This homomorphism is in fact determined by its value on the representables (1.16).

1.11. There is a 'Yoneda embedding' $\mathscr{Y} : \mathscr{C} \to \mathrm{Hom}(\mathscr{C}^{op}, \mathbf{Cat})$ whose value at U is $\mathscr{C}(\text{-}, U)$. The bicategorical Yoneda lemma (Street [24]), which asserts that evaluation at the identity provides an equivalence of categories

$$\mathrm{Hom}(\mathscr{C}^{op}, \mathbf{Cat})(\mathscr{C}(\text{-}, U), X) \simeq XU,$$

allows us to regard the objects of $\mathscr{C}$ as objects of $\mathrm{Hom}(\mathscr{C}^{op}, \mathbf{Cat})$ by means of $\mathscr{Y}$. We shall often identify strong transformations $\mathscr{C}(\text{-}, U) \to X$ and modifications between them with their images in XU under the above equivalence.

1.12. For objects U, V of $\mathscr{C}$, the bicategory $U \int V$ (or rather, $U \int \mathscr{C}(\text{-}, V)$) is called the *bicategory of spans from* U *to* V in $\mathscr{C}$. The objects (v, S, u) are *spans* from U to V and the arrows are diagrams

$$\begin{tikzcd}
 & S \arrow[dl, "v"'] \arrow[dr, "u"] \arrow[dd, "w"] & \\
V \arrow[r, phantom, "\Downarrow\sigma"] & & U \arrow[l, phantom, "\Uparrow\omega"'] \\
 & S' \arrow[ul, "v'"] \arrow[ur, "u'"'] &
\end{tikzcd}$$

An arrow of spans (σ, w, ω) as above is called *strong* [*strict*] when the 2-cells σ, ω are invertible [identities]; we often denote such an arrow by $w : S \to S'$ when no confusion is likely. Let $\mathrm{Spn}(\mathscr{C})(U, V)$ denote the subbicategory of $U\int_{\mathscr{C}} V$ consisting of all the objects, the strong arrows, and all the 2-cells between these; when $\mathscr{C}$ is understood, we omit it from the notation. When V is terminal, we write $\mathscr{C}/_{\mathbf{b}}U$ for $\mathrm{Spn}(U, V)$.

1.13. There is also the category $U \# V$ whose objects are spans from U to V and whose arrows are path components of arrows in $U\int V$.

1.14. For homomorphisms of bicategories $J : \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$, $N : \mathscr{A} \to \mathscr{K}$, recall from Street [24] that a *J-indexed bicolimit for N* is an object $J * N$ of $\mathscr{K}$ together with a strong transformation $\lambda : J \to \mathscr{K}(N, J * N)$ which induces an equivalence of categories:

$$\mathscr{K}(J * N, K) \simeq \mathrm{Hom}(\mathscr{A}^{\mathrm{op}}, \mathbf{Cat})(J, \mathscr{K}(N, K))$$

for all objects K of $\mathscr{K}$.

1.15. Using 1.10 and 1.11, we obtain a strong transformation

$$\lambda : X \simeq \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(\mathscr{Y}, X) \xrightarrow{\#} \mathrm{Hom}(\mathscr{C}^{\mathrm{co}}, \mathbf{Cat})(\# \mathscr{Y}, \# X)$$

for each X.

1.16. Proposition. *For each homomorphism* $X : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$, *the object* $\# X$ *of* $\mathrm{Hom}(\mathscr{C}^{\mathrm{co}}, \mathbf{Cat})$ *together with the strong transformation* λ *of* 1.15 *provide an X-indexed bicolimit for* $\# \mathscr{Y} : \mathscr{C} \to \mathrm{Hom}(\mathscr{C}^{\mathrm{co}}, \mathbf{Cat})$. *Symbolically:*

$$X * \# \mathscr{Y} \simeq \# X.$$

Proof.

$$\begin{aligned} \mathrm{Ditran}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(X, Z) &\simeq \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(\mathscr{Y}, \mathrm{Ditran}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(X, Z)) \\ &\simeq \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(X, \mathrm{Ditran}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(\mathscr{Y}, Z)) \\ &\simeq \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(X, \mathrm{Hom}(\mathscr{C}^{\mathrm{co}}, \mathbf{Cat})(\# \mathscr{Y}, Z)) \\ &\simeq \mathrm{Hom}(\mathscr{C}^{\mathrm{co}}, \mathbf{Cat})(X * \# \mathscr{Y}, Z). \end{aligned}$$

The result follows from Theorem (1.9). □

1.17. Since indexed bicolimits are constructed pointwise in $\mathrm{Hom}(\mathscr{C}^{\mathrm{co}}, \mathbf{Cat})$, it follows from the above result that we have equivalences of categories:

$$X * (U \# \mathscr{Y}) \simeq U \# X.$$

2. Fibrations for constant categories

Taking the bicategorical view of **Cat**, we were forced in [24] to slightly generalize Grothendieck's notion of fibration. The details will be made explicit in this section.

2.1. The reader will recall the construction of the *comma category* for functors $f: A \to C$, $g: B \to C$; it is the category f/g whose objects are triples (a, γ, b) where a, b are objects of A, B and $\gamma: fa \to gb$ is an arrow of C. There is an associated diagram:

$$\begin{tikzcd} f/g \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

which has a 2-universal property [21, p. 108] determining the span $(d_0, f/g, d_1)$ uniquely up to a (unique) strict span isomorphism. Composing the above diagram with any equivalence of categories $E \simeq f/g$ yields a diagram

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

which has a biuniversal property determining the span (p, E, q) uniquely up to a (unique up to isomorphism) strong span equivalence. Any such category E together with this diagram is called a *bicomma category for* f, g.

2.2. The *pseudo-pullback* $A \,\mathrm{III}_C\, B$ (or, more precisely, $A_f \,\mathrm{III}_g\, B$) of the functors f, g is the full subcategory of f/g consisting of the objects (a, γ, b) for which γ is invertible. The inclusion composes with the comma category square containing λ above to give a square containing an invertible 2-cell. Remarks of the last paragraph regarding universality apply *mutatis mutandis*: replacement of $A \,\mathrm{III}\, B$ by an equivalent category gives a *bipullback*.

2.3. Suppose (p, E, q) is a span from B to A in **Cat**. The fibre of E over $a: G \to A$, $b: K \to B$ is the span aEb from K to G defined by the following diagram in which the squares are pseudo-pullbacks. Note especially the case where G, K are the terminal category 1 so that a, b are objects of A, B. We would like the assignment $a, b \to aEb$ to be the object function for a homomorphism $A^{\mathrm{op}} \times B \to$ **Cat**. This leads us to the requirement that E should be a fibration as now to be explained.

$$\begin{tikzcd}
aEb \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & Eb \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & K \arrow[d, "b"] \\
aE \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & E \arrow[r, "q"'] \arrow[d, "p"] & B \\
G \arrow[r, "a"'] & A &
\end{tikzcd}$$

2.4. An arrow $\chi : e' \to e$ in E is said to be *left cartesian*, when $q\chi$ is invertible and for all $\xi : e'' \to e$ in E and all $\alpha' : pe'' \to pe'$ in A such that $p\xi = p\chi \cdot \alpha'$, there exists a unique $\xi' : e'' \to e'$ in E such that $\xi = \chi\xi'$ and $p\xi' = \alpha'$. The span (p, E, q) is called a *left fibration from B to A*, when, for all objects e of E and arrows $\alpha : a' \to pe$ in A, there exist a left cartesian arrow $\chi : e' \to e$ in E and an invertible 2-cell $\gamma' : a' \cong pe'$ such that $\alpha = p\chi \cdot \gamma'$. A particular choice of the pair e', χ will be denoted by $\alpha^*(e), \chi^\alpha$. A left fibration E from B to A determines a homomorphism of bicategories

$$\text{-}E : A^{\mathrm{op}} \to \mathbf{Cat}/_{\mathrm{b}}B$$

which takes $\alpha : a' \to a$ to the arrow

$$\begin{tikzcd}
aE \arrow[rr] \arrow[dr, "q^a"'] & \arrow[d, phantom, "\cong"] & a'E \arrow[dl, "q^{a'}"] \\
& B &
\end{tikzcd}$$

in $\mathbf{Cat}/_{\mathrm{b}}B$, where $(\alpha E)(\gamma, e) = (\gamma', (\gamma\alpha)^*(e))$ and $q^a(\gamma, e) = qe$.

2.5. Let $\mathrm{Fil}(B, A)$ denote the sub-2-category of $\mathrm{Spn}(B, A)$ consisting of the left fibrations from B to A, the arrows which preserve left cartesian arrows, and all 2-cells between these. In (2.4) we have described the object function of a biequivalence of bicategories:

$$\mathrm{Fil}(B, A) \sim \mathrm{Hom}(A^{\mathrm{op}}, \mathbf{Cat}/_{\mathrm{b}}B).$$

2.6. The reader will easily provide the definitions of *right cartesian arrow* and *right fibration*. In fact, (p, E, q) is a right fibration from B to A if and only if $(q^{\mathrm{op}}, E^{\mathrm{op}}, p^{\mathrm{op}})$ is a left fibration from A^{op} to B^{op}. The 2-category of right fibrations from B to A is denoted by $\mathrm{Fir}(B, A)$, and there is a biequivalence of bicategories:

$$\mathrm{Fir}(B, A) \sim \mathrm{Hom}(B, \mathbf{Cat}/_{\mathrm{b}}A).$$

2.7. Suppose (p, E, q) is both a left and right fibration from B to A. Each arrow $\xi : e \to e_1$ in E factors as $\xi = \chi^{p\xi} \cdot \theta(\xi) \cdot \chi_{q\xi}$ for a unique arrow $\theta(\xi) : (q\xi)_*(e) \to (p\xi)^*(e)$ which is inverted by both p and q.

2.8. A span (p, E, q) from B to A is called a *fibration* when it is both a left and right fibration and, for all left cartesian arrows $\chi^\alpha : e' \to e$ and right cartesian arrows $\chi_\beta : e \to e''$, the arrow $\theta(\chi_\beta\chi^\alpha)$ is invertible. Write $\mathrm{Fib}(B, A)$ for the sub-2-category of $\mathrm{Spn}(B, A)$ consisting of the fibrations from B to A, the arrows which preserve left and right cartesian arrows, and all the 2-cells between these. There is a biequivalence:

$$\mathrm{Fib}(B, A) \sim \mathrm{Hom}(A^{\mathrm{op}} \times B, \mathbf{Cat}).$$

2.9. An object A of a bicategory $\mathcal{K}$ is said to be *bidiscrete*, when, for all objects X of $\mathcal{K}$, the category $\mathcal{K}(X, A)$ is equivalent to a discrete category. This means precisely that, for all arrows $f, g : X \to A$, there is at most one 2-cell $f \Rightarrow g$, and, if there is one, it is invertible. Write $\mathrm{D}\mathcal{K}$ for the category whose objects are the bidiscrete objects of $\mathcal{K}$ and whose arrows are isomorphism classes of arrows in $\mathcal{K}$ between such objects.

2.10. A span (p, E, q) from B to A in **Cat** is bidiscrete if and only if, for all arrows $\xi, \eta : e \to e'$ in E such that $p\xi = q\eta$ and $p\xi, q\xi$ are invertible, it is the case that $\xi = \eta$ and ξ is invertible. Objects of $\mathrm{Fil}(B, A)$, $\mathrm{Fir}(B, A)$, $\mathrm{Fib}(B, A)$ are bidiscrete if and only if they are bidiscrete as spans.

2.11. A span (p, E, q) from B to A is a bidiscrete fibration, if and only if it is a left and right fibration and bidiscrete as a span.

2.12. If $f : E \to E'$ is an arrow in $\mathrm{Spn}(B, A)$, if E, E' are left fibrations, and if E' is bidiscrete, then f automatically preserves left cartesian arrows. The biequivalences of 2.5, 2.6, and 2.8 restrict to equivalences of categories:

$$\mathrm{D\,Fil}(B, A) \simeq [A^{\mathrm{op}}, \mathrm{D}(\mathbf{Cat}/_{\mathrm{b}}B)], \qquad \mathrm{D\,Fir}(B, A) \simeq [B, \mathrm{D}(\mathbf{Cat}/_{\mathrm{b}}A)]$$

$$\mathrm{D\,Fib}(B, A) \simeq [A^{\mathrm{op}} \times B, \mathbf{Set}].$$

2.13. Proposition. *A span (p, E, q) from B to A in* **Cat** *is a bidiscrete fibration, if and only if there exist arrows $f : A \to C$, $g : B \to C$ in* **Cat** *such that (p, E, q) is the span associated with a bicomma category for f, g.*

Proof. Three basic facts from [24] give 'if':

(i) any span equivalent to a bidiscrete fibration is a bidiscrete fibration;

(ii) if $a : G \to A$, $b : K \to B$ are functors and E is a bidiscrete fibration from B to A, then aEb is a bidiscrete fibration from K to G;

(iii) the category $[\mathbf{2}, C]$ of arrows of C is a bidiscrete fibration from C to C.

To verify 'only if', it suffices to take C to be a bicomma category for p, q which in this case has a simple description. Let $T : A^{\mathrm{op}} \times B \to \mathbf{Set}$ be a functor corresponding to E under the third equivalence of 2.12. Let C be the category which contains A, B as disjoint full subcategories in such a way that each object of C is either in A or in B;

for objects a, b of A, B, the homsets $C(a, b), C(b, a)$ are $T(a, b), 0$, respectively; and, composition

$$C(a', a) \times C(a, b) \times C(b, b') \to C(a', b')$$

takes (α, e, β) to $T(\alpha, \beta)e$. The functors $f: A \to C$, $g: B \to C$ are the inclusions. □

2.14. Suppose E is a fibration from B to A and E' is a fibration from C to B. Let $T: A^{\mathrm{op}} \times B \to \mathbf{Cat}$, $T': B^{\mathrm{op}} \times C \to \mathbf{Cat}$ correspond to E, E' under the biequivalences of 2.8. The *fibrational composite* $E \otimes E'$ [24, Section 4] is the fibration from C to A which corresponds under 2.8 to the homomorphism $T \otimes T': A^{\mathrm{op}} \times C \to \mathbf{Cat}$ given by the formula

$$(T \otimes T')(a, c) = \int^{b} T(a, b) \times T'(b, c).$$

3. Fibrations for variable categories

Throughout this section $\mathscr{C}$ will denote a small bicategory and $\mathscr{K}$ will denote the 2-category $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ of homomorphisms of bicategories $\mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$, strong transformations, and modifications. The definitions of the last section for **Cat** can be carried over to $\mathscr{K}$ by a pointwise procedure; the success of this procedure owes itself to the bicategorical Yoneda lemma (1.11).

3.1. If $f: A \to C$, $g: B \to C$ are arrows in $\mathscr{K}$, the *comma object* f/g *for* f, g is the homomorphism from $\mathscr{C}^{\mathrm{op}}$ to **Cat** given on objects by $(f/g)U = fU/gU$ and given on arrows and 2-cells by means of the universal property of the comma categories. Notice that the projections $d_0U: fU/gU \to AU$, $d_1U: fU/gU \to BU$ are in fact components for *strict* transformations $d_0: f/g \to A$, $d_1: f/g \to B$, and we obtain a diagram in $\mathscr{K}$ as in 2.1. An object E of $\mathscr{K}$ equivalent to f/g gives a *bicomma object for* f, g, just as for constant categories (2.1).

3.2. A span E from B to A in $\mathscr{K}$ is called a *fibration*, when, for each object U of $\mathscr{C}$, EU is a fibration from BU to AU, and, for each $r: V \to U$ in $\mathscr{C}$, the arrow $EU \to (Ar)(EV)(Br)$ in $\mathrm{Spn}(BU, AU)$, induced by Er, preserves left and right cartesian arrows. We give this definition as an illustration; the reader will have no difficulty in providing the definitions of *left fibration*, *right fibration*, *left cartesian 2-cell* $X \Downarrow x\ E$, and so on.

3.3. Suppose $h: Y \to X$ is an arrow in $\mathscr{K}$ and E is a fibration from B to A in $\mathscr{K}$. Then $\mathscr{K}(X, E)$ is a fibration from $\mathscr{K}(X, B)$ to $\mathscr{K}(X, A)$, and the arrow $\mathscr{K}(X, E) \to \mathscr{K}(r, A)\mathscr{K}(Y, E)\mathscr{K}(r, B)$ induced by $\mathscr{K}(r, E)$ preserves left and right cartesian arrows.

3.4. Write $\mathrm{Fib}(\mathscr{K})(B, A)$ (or simply $\mathrm{Fib}(B, A)$ when it is understood that A, B are objects of $\mathscr{K}$) for the sub-2-category of $\mathrm{Spn}(\mathscr{K})(B, A)$ (1.12) consisting of the fibrations from B to A, the arrows $f: E \to E'$ such that each $fU: EU \to E'U$ is in $\mathrm{Fib}(BU, AU)$, and all the 2-cells between these. The definitions of $\mathrm{Fil}(\mathscr{K})(B, A)$, $\mathrm{D\,Fib}(\mathscr{K})(B, A)$, and so on, should now be obvious.

3.5. For a left fibration E from B to A in $\mathscr{K}$, there is a homomorphism of bicategories

$$\text{-}E : \mathscr{K}(X, A)^{\mathrm{op}} \to \mathrm{Fil}(B, X)$$

which takes $a: X \to A$ to the fibre aE over a and which takes $\alpha : a' \Rightarrow a$ to the arrow $\alpha E : aE \to a'E$ whose component at U is $\alpha_U(EU): a_U(EU) \to a'_U(EU)$ (2.4). For a right fibration E from B to A, we obtain a homomorphism

$$E\text{-} : \mathscr{K}(Y, B) \to \mathrm{Fir}(Y, A).$$

For a fibration E from B to A, we obtain a homomorphism:

$$\text{-}E\text{-} : \mathscr{K}(X, A)^{\mathrm{op}} \times \mathscr{K}(Y, B) \to \mathrm{Fib}(Y, X).$$

Compare [24, (3.25) to (3.27)].

3.6. Proposition. *A span (p, E, q) from B to A in* $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ *is a bidiscrete fibration, if and only if there exist arrows $f: A \to C$, $g: B \to C$ in* $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ *such that (p, E, q) is the span associated with a bicomma object for f, g.*

Proof. This follows from Proposition 2.13 and the fact that bicomma objects in $\mathscr{K}$ are formed pointwise. □

3.7. Theorem. *Suppose $q: E \to B$ is a fibration from B to 1 in $\mathscr{K} =$* $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$. *The homomorphism of bicategories*

$$E \underset{B}{\mathrm{III}}\text{-} : \mathscr{K}/_{\mathrm{b}}B \to \mathscr{K}/_{\mathrm{b}}E,$$

given by bipullback along q, has a right biadjoint.

Proof. The special case of this result with $\mathscr{C} = 1$ was proved in Street [24]. Adopting the attitude that an 'object' of a variable category A is an arrow in $\mathscr{K}$ from a representable $\mathscr{C}(\text{-}, U)$, we can mimic that special case. We shall only indicate the construction.

First observe that $E\,\mathrm{III}_B\text{-}$ may be taken to be pseudo-pullback along q. For an object (X, u) of $\mathscr{K}/_{\mathrm{b}}E$, we shall describe the value $(\bar{X}, \bar{u})$ of the right biadjoint to $E\,\mathrm{III}_B\text{-}$ at (X, u).

For each U of $\mathscr{C}$, the category $\bar{X}U$ is described as follows. An object of $\bar{X}U$ is a triple (b, x, θ) where $b: \mathscr{C}(\text{-}, U) \to B$, $x: Eb \to X$ are arrows in $\mathscr{K}$, and θ is an invertible 2-cell:

$$\begin{tikzcd} Eb \arrow[rr, "x"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\overset{\theta}{\Rightarrow}"] & & X \arrow[dl, "u"] \\ & E & \end{tikzcd}$$

An arrow $(\beta, \xi):(b, x, \theta)\to(c, y, \phi)$ in $\bar{X}U$ consists of 2-cells $\beta : b \Rightarrow b'$ and

$$\begin{tikzcd} Eb \arrow[rr, "E\beta"] \arrow[dr, "x"'] \arrow[rr, phantom, shift right=4ex, "\overset{\xi}{\Rightarrow}"] & & Ec \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

in $\mathscr{K}$, such that the following holds:

$$\begin{tikzcd} Eb \arrow[r, "E\beta"] \arrow[d] \arrow[dr, "x"] & Ec \arrow[d, "y"] \arrow[dl, phantom, shift left=3ex, "\overset{\xi}{\Rightarrow}"] \\ E \arrow[ur, phantom, shift right=3ex, "\overset{\theta}{\Rightarrow}"] & X \arrow[l, "u"] \end{tikzcd} = \begin{tikzcd} Eb \arrow[r, "E\beta"] \arrow[d] \arrow[dr, phantom, shift right=3ex, "\overset{x_\beta}{\Rightarrow}"] & Ec \arrow[d, "y"] \arrow[dl] \\ E \arrow[ur, phantom, shift left=3ex, "\overset{\phi}{\Rightarrow}"] & X \arrow[l, "u"] \end{tikzcd}$$

For $r : V \to U$ in $\mathscr{C}$, the functor $\bar{X}r : \bar{X}U \to \bar{X}V$ is given by $(\bar{X}r)(b, x, \theta) = (br, x\hat{r}, \theta\hat{r})$ where $\hat{r}$ is the second projection from $\mathscr{C}(-, V)\ \mathrm{III}_{\mathscr{C}(-,U)}\ Eb = E(br)$, and $(\bar{X}r)(\beta, \xi) = (\beta r, \xi\hat{r})$. For a 2-cell $\rho : r \Rightarrow s$ in $\mathscr{C}$, the natural transformation $\bar{X}\rho$ is given by composition with ρ in the obvious way. This defines a homomorphism $\bar{X} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$.

The arrow $\bar{u} : \bar{X} \to B$ is the strong transformation whose component at U takes (b, x, θ) to b and (β, ξ) to β. □

3.8. Corollary. $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ *is a fibrational bicategory in the sense of Street* [24, (4.11)].

3.9. Consequently, fibrational composition in $\mathscr{K} = \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ behaves well. For example, we obtain a bicategory D Fib($\mathscr{K}$) whose objects are the objects of $\mathscr{K}$ and whose arrows are the bidiscrete fibrations in $\mathscr{K}$. Since the composition of fibrations is defined in terms of finite bicategorical limits and colimits [24, Section 4] and since these limits and colimits are calculated pointwise, composition of fibrations in $\mathscr{K}$ is just obtained pointwise from that in **Cat** (2.14).

4. Representation of variable fibrations

Combining the biequivalence of 2.8 with the canonical biequivalence of Street [24, (1.34)], we obtain a biequivalence

$$\mathrm{Fib}(B, A) \sim \mathrm{Hom}(B, \mathrm{Hom}(A^{\mathrm{op}}, \mathbf{Cat}))$$

for small categories A, B. The generalization of this to variable categories will now be developed.

4.1. The dualizing involution $\mathbf{Cat}^{\mathrm{co}} \to \mathbf{Cat}$ whose value at A is A^{op} will have its value at $f: A \to B$ denoted by $f: A^{\mathrm{op}} \to B^{\mathrm{op}}$ in the present work. More generally, for a homomorphism of bicategories $T: \mathscr{C} \to \mathscr{B}$, the symbol T will be used for the induced homomorphisms $\mathscr{C}^{\mathrm{co}} \to \mathscr{B}^{\mathrm{co}}$, $\mathscr{C}^{\mathrm{op}} \to \mathscr{B}^{\mathrm{op}}$, and $\mathscr{C}^{\mathrm{coop}} \to \mathscr{B}^{\mathrm{coop}}$. Thus, for a homomorphism $X: \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$, we write $X^{\mathrm{op}}: \mathscr{C}^{\mathrm{coop}} \to \mathbf{Cat}$ for the composite of $X: \mathscr{C}^{\mathrm{coop}} \to \mathbf{Cat}^{\mathrm{co}}$ with the dualizing involution. (Do not confuse X^{op} with $X: \mathscr{C} \to \mathbf{Cat}^{\mathrm{op}}$!) The reason for taking the biequivalence of the last paragraph as our starting point instead of 2.8, is related to the fact that, for $X: \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$, X^{op} has domain $\mathscr{C}^{\mathrm{coop}}$ and not $\mathscr{C}^{\mathrm{op}}$ (unless of course $\mathscr{C}$ is a mere category).

4.2. As in the last section, we will write $\mathscr{K}$ for $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ where $\mathscr{C}$ is a small bicategory. Also let **CAT** denote the 2-category of categories in some category **SET** of sets large enough to include the set of small sets. Put $\bar{\mathscr{K}} = \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{CAT})$. Also let **HOM** denote the 3-category of bicategories, homomorphisms, strong transformations and modifications in the category **SET** of sets.

4.3. Suppose X is an object of $\mathscr{K}$ and U is an object of $\mathscr{C}$. For each object S of $\mathscr{C}$, there is a homomorphism

$$\Lambda_S : XS \times \mathscr{C}(S, U)^{\mathrm{op}} \to U \textstyle\int X$$

which takes $(\xi, \omega): (x, u) \to (x', u')$ to $(\xi, \omega, 1): (x, S, u) \to (x', S, u')$. For each arrow $w: S \to S'$ in $\mathscr{C}$, there is a strong transformation

$$\begin{tikzcd}
XS' \times \mathscr{C}(S, U)^{\mathrm{op}} \arrow[r, "Xw \times 1"] \arrow[d, "1 \times \mathscr{C}(w,1)"'] \arrow[dr, phantom, "\Downarrow \Lambda_w"] & XS \times \mathscr{C}(S, U)^{\mathrm{op}} \arrow[d, "\Lambda_S"] \\
XS' \times \mathscr{C}(S', U)^{\mathrm{op}} \arrow[r, "\Lambda_{S'}"'] & U \textstyle\int X
\end{tikzcd}$$

whose component at (x', u) is $(1, 1, w): ((Xw)x', S, u) \to (x', S', wu)$. In fact, Λ exhibits $U \int X$ as a 'lax bicoend' in a suitable sense (see Bozapalides [5] for related concepts).

4.4. There is a biequivalence

$$\mathrm{Hom}((U \textstyle\int X)^{\mathrm{op}}, \mathbf{Cat}) \sim \mathrm{Fib}(\mathscr{K})(\mathscr{C}(-, U), \dot{X})$$

which we shall now describe. Each homomorphism $T: (U \int X)^{\mathrm{op}} \to \mathbf{Cat}$ gives a 'lax wedge' $T\Lambda$ where Λ is as in 4.3. Each object S of $\mathscr{C}$ gives a homomorphism

$T\Lambda_S:(XS)^{op}\times\mathscr{C}(S,U)\to\mathbf{Cat}$ which corresponds to a fibration ES from $\mathscr{C}(S,U)$ to XS under 2.8. Each arrow $w:S\to S'$ in $\mathscr{C}$ gives a strong transformation $T\Lambda_w$ which corresponds to a functor $Ew:ES'\to ES$ compatible with the cartesian arrows (3.2). Then E is a fibration from $\mathscr{C}(-,U)$ to X in $\mathscr{K}$. The assignment of E to T is the object function of the biequivalence: the homomorphism is also given on arrows and 2-cells by composing with Λ and applying 2.8 pointwise. That this homomorphism is locally an equivalence and surjective on objects up to equivalence is straightforward.

4.5. Each homomorphism $A:\mathscr{C}^{op}\to\mathbf{Cat}$ yields a homomorphism $\mathrm{Hom}((\int A)^{op},\mathbf{Cat}):\mathscr{C}^{op}\to\mathbf{HOM}$ which is the composite

$$\mathscr{C}^{op}\xrightarrow{\ \int A\ }\mathbf{Hom}^{coop}\xrightarrow{\ \text{dual}\ }\mathbf{Hom}^{op}\xrightarrow{\ \mathrm{Hom}(-,\,\mathbf{Cat})\ }\mathbf{HOM}.$$

4.6. Representation Theorem. *There is a biequivalence*

$$\mathrm{Fib}(\mathrm{Hom}(\mathscr{C}^{op},\mathbf{Cat}))(B,A)\sim\mathrm{Hom}(\mathscr{C}^{op},\mathbf{HOM})(B,\mathrm{Hom}((\textstyle\int A)^{op},\mathbf{Cat}))$$

whose value at E is the strong transformation e whose component $e_U:BU\to$ $\mathrm{Hom}((U\int A)^{op},\mathbf{Cat})$ *takes b to the homomorphism corresponding under* 4.4 *to the fibre of E over $b:\mathscr{C}(-,U)\to B$. This biequivalence restricts to an equivalence of categories*

$$\mathrm{D}\,\mathrm{Fib}(\mathscr{K})(B,A)\simeq\bar{\mathscr{K}}(B,[(\#A)^{op},\mathbf{Set}]).$$

5. The cocompletion problem

5.1. Recall that a diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] \arrow[rr, phantom, shift right=4, "\overset{\kappa}{\Rightarrow}"] & & C \arrow[dl, "k"] \\ & M & \end{tikzcd}$$

in a bicategory $\mathscr{K}$ is said to *exhibit k as a left extension of h along f* (or to have the *left extension property*), when, for all $t:C\to M$, 'pasting on κ' yields a bijection between 2-cells $k\Rightarrow t$ and 2-cells $h\Rightarrow rf$. An arrow $s:M\to N$ is said to *respect* the left extension, when $s\kappa$ exhibits sk as a left extension of sh along f. The left extension is *absolute* when it is respected by every arrow with source M. *Left liftings* are left extensions in $\mathscr{K}^{op}$.

5.2. The diagram above is said to have the *left extension property at $g:B\to C$*, when the diagram obtained by pasting on at f a bicomma object diagram as in 2.1 has the left extension property. The diagram of 5.1 has the left extension property, if and only if it has it at 1_C. The diagram of 5.1 is said to have the *pointwise left extension property*, when it has the left extension property at all arrows into C.

5.3. An object M of $\mathscr{K}$ is said to be [*pointwise*] *cocomplete relative to an arrow* $f: A \to C$, when each arrow $h: A \to M$ admits a [pointwise] left extension along f. An arrow $s: M \to N$ is [*pointwise*] *cocontinuous relative to* f, when it respects all [pointwise] left extensions along f of arrows into M.

5.4. Let Σ be a set of arrows in $\mathscr{K}$. An object in $\mathscr{K}$ is [*pointwise*] *Σ-cocomplete*, when it is [pointwise] cocomplete relative to all arrows in Σ. Similarly we define [*pointwise*] *Σ-cocontinuous*. Write $\mathscr{K}_\Sigma$ [$\mathscr{K}_{\mathrm{pw}\,\Sigma}$] for the locally full subbicategory of $\mathscr{K}$ consisting of the [pointwise] Σ-cocontinuous arrows.

5.5. Proposition. *Let $\mathscr{H}$ denote either $\mathscr{K}_\Sigma$ or $\mathscr{K}_{\mathrm{pw}\,\Sigma}$ and let $I: \mathscr{H} \to \mathscr{K}$ be the inclusion. Suppose $J: \mathscr{A} \to$ **Cat**, $S: \mathscr{A} \to \mathscr{H}$ are homomorphisms of bicategories such that the J-indexed bilimit $\{J, IS\}$ of IS exists in $\mathscr{K}$. Then the J-indexed bilimit $\{J, S\}$ of S exists in $\mathscr{H}$ and $I\{J, S\} \simeq \{J, IS\}$.*

5.6. It follows that if $\mathscr{K}$ has all small indexed bilimits, then so does $\mathscr{H}$ and the inclusion I preserves them. So the only obstruction to the existence of a left biadjoint to this inclusion is a size condition. The value at X of this biadjoint is called the [*pointwise*] *Σ-cocompletion of X*.

5.7. The general problem is to find conditions on $\mathscr{K}$ and Σ under which each object of $\mathscr{K}$ has a [pointwise] Σ-cocompletion. We shall solve the pointwise Σ-cocompletion problem when Σ is a small set of arrows and $\mathscr{K}$ is Hom($\mathscr{C}^{\mathrm{op}}$, **Cat**) where $\mathscr{C}$ is a small bicategory.

6. Yoneda structures for variable categories

The Representation Theorem 4.6 enables us to produce a Yoneda structure on $\bar{\mathscr{K}}$ (4.2) in the sense of Street–Walters [27]. The technique is essentially that of Street [22].

6.1. Each homomorphism $Z: \mathscr{C}^{\mathrm{co}} \to$ **Cat** yields an opditransformation $yZ: Z \to [Z^{\mathrm{op}}, \mathbf{Set}]$ for which the data

$$\begin{tikzcd} ZU \arrow[r, "(yZ)_U"] \arrow[d, "Zr"'] \arrow[dr, phantom, "\Downarrow (yZ)_r"] & {[(ZU)^{\mathrm{op}}, \mathbf{Set}]} \\ ZV \arrow[r, "(yZ)_V"'] & {[(ZV)^{\mathrm{op}}, \mathbf{Set}]} \arrow[u, "{[Zr,1]}"'] \end{tikzcd}$$

are made up of the Yoneda embeddings $(yZ)_U$ and the natural transformations $(yZ)_r$ coming from the effect of Zr on homsets.

6.2. For each object A of $\bar{\mathscr{K}}$, put $\mathscr{P}A = [(\#A)^{\mathrm{op}}, \mathbf{Set}]$ which is another object of $\bar{\mathscr{K}}$. This describes the object function for a homomorphism

$$\mathscr{P} : \bar{\mathscr{K}}^{\mathrm{coop}} \to \bar{\mathscr{K}}.$$

A fibration E from B to A in $\bar{\mathscr{K}}$ will be called *admissible*, when, for all objects U of $\mathscr{C}$ and objects a, b of AU, BU, the fibre $a(EU)b$ of EU over a, b is equivalent to a small category. Applying the Representation Theorem 4.6 for $\bar{\mathscr{K}}$ in place of $\mathscr{K}$, we obtain an equivalence of categories

$$\mathrm{D\,Fib}_{\mathbf{ad}}(B, A) \simeq \bar{\mathscr{K}}(B, \mathscr{P}A),$$

where the left-hand side is the full subcategory of $\mathrm{D\,Fib}(\bar{\mathscr{K}})(B, A)$ consisting of the admissible bidiscrete fibrations from B to A.

6.3. An object A of $\bar{\mathscr{K}}$ will be called *essentially small*, when it is equivalent to an object of $\mathscr{K}$. An arrow $f : A \to B$ in $\bar{\mathscr{K}}$ is called *admissible*, when the pseudo-comma object f/B is an admissible fibration from B to A. An object A of $\bar{\mathscr{K}}$ is *admissible*, when $1 : \mathrm{A} \to \mathrm{A}$ is admissible.

6.4. For each admissible object A of $\bar{\mathscr{K}}$, the composite transformation

$$A \xrightarrow{\partial} \#A \xrightarrow{y(\#A)} \mathscr{P}A$$

is strong. This follows directly from the fact that the function taking $\alpha : a \to (Au)b$ to $[\alpha, 1, u]$ is an isomorphism of sets:

$$(AS)(a, (Au)b) \cong (U\#A)((a, S, u), (b, U, 1)).$$

We call the above composite the *yoneda arrow for* A and denote it $yA : A \to \mathscr{P}A$. In fact, yA corresponds to the admissible bidiscrete fibration $[\mathbf{2}, A]$ from A to A.

6.5. An admissible arrow $f : A \to B$ in $\bar{\mathscr{K}}$ yields an admissible bidiscrete fibration f/B from B to A, and hence (6.2) an arrow $B(f, 1) : B \to \mathscr{P}A$ in $\bar{\mathscr{K}}$. If moreover A is admissible, the canonical arrow $[\mathbf{2}, A] \to f/f$ corresponds to a 2-cell:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "yA"'] & B \arrow[d, "B(f{,}1)"] \arrow[dl, phantom, "\overset{\chi^f}{\Rightarrow}" near end] \\ & \mathscr{P}A \end{tikzcd}$$

in $\bar{\mathscr{K}}$.

6.6. Proposition. (1) *The* 2*-cell* χ^f *exhibits* $B(f, 1)$ *as a pointwise left extension of* yA *along* f.

(2) *The 2-cell χ^f exhibits f as an absolute left lifting of yA through $B(f, 1)$.*

(3) *If a 2-cell $\sigma : B(f, 1) \Rightarrow g$ has the property that when pasted onto χ^f it yields a 2-cell which exhibits f as an absolute left lifting of yA through g, then σ is an isomorphism.*

Proof. (1) This is proved essentially in [22, Theorem 6, p. 145].

(2) Suppose $h : X \to A$ and $\tau : (yA)a \Rightarrow B(f, 1)h$. Corresponding to τ we have an isomorphism class of an arrow of spans $A/a \to f/h$ which gives a 2-cell $fa \Rightarrow h$ which, when pasted onto $\chi^f a$, yields τ. See [22, Theorem 7, p. 145].

(3) Suppose $\sigma : B(f, 1) \Rightarrow g$ has the property of (3). Let G be the bidiscrete fibration from B to A corresponding to g. For any span (u, S, v) from B to A, a strong span arrow $S \to G$ amounts to an arrow $S \to uGv$ and so to a strong span arrow $A/u \to Gv$. But the latter determines a 2-cell $(yA)u \Rightarrow gv$, and so, by the property of σ, a 2-cell $fu \Rightarrow v$. This gives a strong span arrow $S \to f/B$. This describes an equivalence

$$\mathrm{Spn}(B, A)(S, G) \simeq \mathrm{Spn}(B, A)(S, f/B)$$

for all spans S. So $G \simeq f/B$. So $g \cong B(f, 1)$. □

6.7. Corollary. *The data of* (6.3) *and* (6.5) *enrich the 2-category $\bar{\mathcal{K}}$ with a Yoneda structure.*

6.8. Proposition. *For an admissible object A of $\bar{\mathcal{K}}$, the image of $h : B \to \mathcal{P}A$ under the equivalence*

$$\bar{\mathcal{K}}(B, \mathcal{P}A) \simeq \mathrm{D\ Fib}_{\mathrm{ad}}(B, A)$$

of the Representation Theorem 4.6 *is a bicomma object for yA, h (and so is equivalent to yA/h as a span).*

Proof. See [22, Section 3]. □

6.9. For an object B of $\bar{\mathcal{K}}$, the cooperative homomorphism associated with $B^{\mathrm{op}} : \mathscr{C}^{\mathrm{coop}} \to \mathbf{CAT}$ is a homomorphism $\#(B^{\mathrm{op}}) : \mathscr{C} \to \mathbf{CAT}$; so we have $\#(B^{\mathrm{op}})^{\mathrm{op}} : \mathscr{C}^{\mathrm{co}} \to \mathbf{CAT}$. Put $\mathcal{P}^*B = [\#(B^{\mathrm{op}})^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathbf{CAT}$. The calculation

$$\begin{aligned}
\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{CAT})(B, [(\# A)^{\mathrm{op}}, \mathbf{Set}]) &\simeq \mathrm{D\ Fib}(\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{CAT}))_{\mathrm{ad}}(B, A) \\
&\cong \mathrm{D\ Fib}(\mathrm{Hom}(\mathscr{C}^{\mathrm{coop}}, \mathbf{CAT})^{\mathrm{co}}{}_{\mathrm{ad}}(B^{\mathrm{op}}, A^{\mathrm{op}}) \\
&= \mathrm{D\ Fib}(\mathrm{Hom}(\mathscr{C}^{\mathrm{coop}}, \mathbf{CAT}))_{\mathrm{ad}}(A^{\mathrm{op}}, B^{\mathrm{op}}) \\
&\simeq \mathrm{Hom}(\mathscr{C}^{\mathrm{coop}}, \mathbf{CAT})(A^{\mathrm{op}}, [\#(B^{\mathrm{op}})^{\mathrm{op}}, \mathbf{Set}]) \\
&\cong \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{CAT})(A, [\#(B^{\mathrm{op}})^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}})^{\mathrm{op}}
\end{aligned}$$

yields an equivalence of categories

$$\bar{\mathcal{K}}(B, \mathcal{P}A) \simeq \bar{\mathcal{K}}^{\mathrm{coop}}(\mathcal{P}^*B, A)$$

which is a strong transformation in A. This shows:

6.10. Proposition. *The homomorphisms $\mathcal{P} : \bar{\mathcal{K}}^{\mathrm{coop}} \to \bar{\mathcal{K}}$ has a left biadjoint $\mathcal{P}^*$.*

6.11. Theorem. (i) *An object X of $\bar{\mathcal{K}}$ is admissible, if and only if, for all objects U of C, the category XU has small homsets.*

(ii) *An object X of $\bar{\mathcal{K}}$ is essentially small if and only if both X and $\mathcal{P}X$ are admissible.*

Proof. (i) This follows from the isomorphism

$$a([2, X]U)b \cong (XU)(a, b).$$

(ii) From the definition of $\# X$ and 6.4, it is clear that X is essentially small if and only if each $U \# X$ is equivalent to a small category. By a result of Freyd [26], this holds if and only if the categories $U \# X$ and $(\mathcal{P}X)U$ have small homsets. By 6.4 and (i), this holds if and only if X and $\mathcal{P}X$ are admissible. □

6.12. The Yoneda structure (6.7) on $\bar{\mathcal{K}}$ provides a multitude of Yoneda structures on $\mathcal{K}$ by taking full subhomomorphisms of $\mathcal{P}$ restricted to $\mathcal{K}$ which land in $\mathcal{K}$. Suppose $\mathcal{R} : \mathcal{K}^{\mathrm{coop}} \to \mathcal{K}$ is a homomorphism of bicategories and $\iota : \mathcal{R} \to \mathcal{P}$ is a strong transformation between homomorphisms $\mathcal{K}^{\mathrm{coop}} \to \bar{\mathcal{K}}$ such that, for each A of $\mathcal{K}$ and U of $\mathscr{C}$, the functor $(\iota A)U : (\mathcal{R}A)U \to (\mathcal{P}A)U$ is fully faithful. A fibration E from B to A in $\mathcal{K}$ is called *$\mathcal{R}$-admissible,* when it is isomorphic to an object in the image of the functor

$$\mathcal{K}(B, \mathcal{R}A) \xrightarrow{\mathcal{K}(B, \iota A)} \bar{\mathcal{K}}(B, \mathcal{P}A) \xrightarrow{\simeq} \mathrm{D\,Fib}(B, A).$$

An arrow $f : A \to B$ is *$\mathcal{R}$-admissible* when the fibration f/B from B to A is $\mathcal{R}$-admissible. Since ι is a strong transformation, $\mathcal{R}$-admissible arrows form a right ideal in $\mathcal{K}$. If A and $f : A \to B$ are $\mathcal{R}$-admissible, then the 2-cell χ^f of 6.5 factors through ιA to yield a 2-cell:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "yA"'] \arrow[dr, phantom, shift left=4, "\overset{\chi^f}{\Rightarrow}"] & B \arrow[d, "B(f,1)"] \\ & \mathcal{R}A \end{tikzcd}$$

These data enrich $\mathcal{K}$ with a Yoneda structure.

6.13. Suppose $\mathbf{S}$ is a small full subcategory of **Set**. The inclusion $\mathbf{S} \to \mathbf{Set}$ determines a strong transformation

$$\iota A : [(\# A)^{\mathrm{op}}, \mathbf{S}] \to [(\# A)^{\mathrm{op}}, \mathbf{Set}]$$

with fully faithful components for each A of $\mathscr{K}$, and these are the components of a strong transformation $\iota : \mathscr{R} \to \mathscr{P}$. This gives an example of the situation of 6.12.

6.14. A quite different kind of example of the situation of (6.12) will now be given. We have a homomorphism of bicategories

$$\# : \mathscr{C}^{\mathrm{co}} \times \mathscr{C} \to \mathbf{Cat}$$

on restricting $\#$ as in 1.10 to representables (1.13) and transforming. *Suppose $\mathscr{C}$ has bicomma objects.* Then, for arrows r, s in $\mathscr{C}$, the functor $r \# s$ has a right adjoint $r \check{\#} s$. This gives a homomorphism

$$\check{\#} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C}^{\mathrm{coop}} \to \mathbf{Cat}$$

whose value at U, V is $U \# V$ and at r, s is $r \check{\#} s$. The Yoneda embeddings

$$U \# V \to [(U \# V)^{\mathrm{op}}, \mathbf{Set}]$$

are the components of a strong transformation

$$- \check{\#} \sim \to [(- \# \sim)^{\mathrm{op}}, \mathbf{Set}].$$

For each object A of $\mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$, we thus obtain a strong transformation

$$\mathrm{Hom}(\mathscr{C}^{\mathrm{coop}}, \mathbf{Cat})(A^{\mathrm{op}}, \check{\#}) \to \mathrm{Hom}(\mathscr{C}^{\mathrm{coop}}, \mathbf{CAT})(A^{\mathrm{op}}, [(- \# \sim)^{\mathrm{op}}, \mathbf{Set}])$$

whose components are fully faithful. However,

$$\begin{aligned} \mathrm{Hom}(\mathscr{C}^{\mathrm{coop}}, \mathbf{CAT})(A^{\mathrm{op}}, [(U \# \sim)^{\mathrm{op}}, \mathbf{Set}]) &\simeq [(A * (U \# \sim))^{\mathrm{op}}, \mathbf{Set}] \\ &\simeq [(U \# A)^{\mathrm{op}}, \mathbf{Set}] \end{aligned}$$

by 1.17. So, putting

$$\mathscr{B}A = \mathrm{Hom}(\mathscr{C}^{\mathrm{coop}}, \mathbf{Cat})(A^{\mathrm{op}}, \check{\#}),$$

we have a strong transformation

$$\iota A : \mathscr{B}A \to \mathscr{P}A$$

with fully faithful components. Clearly $\mathscr{B}A$ is the value at A of a homomorphism $\mathscr{B} : \mathscr{K}^{\mathrm{coop}} \to \mathscr{K}$ and ιA is the component at A of a strong transformation $\iota : \mathscr{B} \to \mathscr{P}$ so that we are in the situation of 6.12. The theory of the Yoneda structure on $\mathscr{K}$ arising from this is, in the case where $\mathscr{C}$ is a mere category with pullbacks, the theory of *locally internal categories based on $\mathscr{C}$* [12, 17, 4, 6, 23, 15].

7. Cocompleteness for variable categories

Again in this section we shall use the notation of (4.2). A direct consequence of the bicategorical Yoneda lemma (1.11) is the next observation.

7.1. Proposition. *An object M of $\bar{\mathcal{K}}$ is cocomplete relative to an arrow $\mathscr{C}(-,r)$: $\mathscr{C}(-,V)\to\mathscr{C}(-,U)$ between representables, if and only if $Mr:MU\to MV$ has a left adjoint $\hat{M}r:MV\to MU$.*

7.2. When $M:\mathscr{C}^{\mathrm{op}}\to\mathbf{CAT}$ is such that, for all arrows r in $\mathscr{C}$, Mr has a left adjoint $\hat{M}r$, we obtain a homomorphism $\hat{M}:\mathscr{C}^{\mathrm{co}}\to\mathbf{CAT}$ whose value at U is MU and at r is $\hat{M}r$. It is easily seen that a ditransformation $d:M\to Z$ amounts to a (generally not strong) transformation $\hat{M}\to Z$. In fact, we obtain an isomorphism of categories

$$\mathrm{Bicat}(\mathscr{C}^{\mathrm{op}},\mathbf{CAT})(\hat{M},Z)\cong\mathrm{Ditran}(\mathscr{C}^{\mathrm{op}},\mathbf{CAT})(M,Z).$$

Taking $Z=\hat{M}$ this gives a ditransformation $M\to\hat{M}$ and so, by Theorem 1.9, a strong transformation $\ell:\#M\to\hat{M}$; the component of ℓ at U takes (m,S,u) to $(\hat{M}u)m$.

7.3. In order to find conditions under which M is cocomplete relative to more general arrows, the following Lemma can be used. Recall from [24, (1.15)] that a J-indexed bilimit $\{J,S\}$ for $S:\mathscr{A}\to\mathbf{CAT}$ is given by

$$\{J,S\}=\mathrm{Hom}(\mathscr{A},\mathbf{CAT})(J,S).$$

7.4. Lemma. *Suppose $J:\mathscr{A}\to\mathbf{Cat}$ is a homomorphism with small domain.*

(i) *If $X\in\mathbf{CAT}$ is a category with small colimits, $S:\mathscr{A}\to\mathbf{CAT}$ is a homomorphism, and $f:X\to\{J,S\}$ is a functor such that, for all objects A, a of $\mathscr{A}$, JA, the functor $f_a=f(-)_Aa:X\to SA$ has a left adjoint g_a, then f has a left adjoint g.*

(ii) *If X', S', f', g'_a, g' are as X, S, f, g_a, g in (i), if $m:X\to X'$ is a functor which preserves small colimits, and if $s:S\to S'$ is a strong transformation such that $mg_a\cong g'_as_A$ for each A, a then $mg\cong g'\{J,s\}$.*

Proof. (i) Let $h:J\to S$ denote a strong transformation. For arrows $u:A\to B$, $\beta:(Ju)a\to b$, we obtain a diagram

$$g_a(h_Aa)\xleftarrow{\gamma_\beta h_Aa}g_b((Su)h_Aa)\cong g_b(h_B(Ju)a)\xrightarrow{g_b(h_B\beta)}g_b(h_Bb)$$

in X, where the natural transformation $\gamma_\beta:g_b(Su)\to g_a$ corresponds to the composite

$$(Su)f_a\cong f_{(Ju)a}\xrightarrow[f(-)_B\beta]{}f_b$$

under adjunction. Let $g(h)$ denote the joint pushout of the above diagrams in X over all u, β. The arrows $g(h)\to x$ in X are in natural bijection with modifications $h\to f(x)$.

(ii) This is clear from the above construction for g, g'. □

7.5. The reader will observe that Lemma 7.4 remains valid when $\{J,S\}$ is replaced by $\mathrm{Bicat}(\mathscr{A},\mathbf{CAT})(J,S)$; the two isomorphisms appearing in the proof are replaced by arrows. The fact that the 2-cells of $\mathscr{A}$ do not enter into the proof is due to the fact that the modification condition (Kelly–Street [10, p. 82]) does not involve these 2-cells.

This point is well illustrated by the case where $\{J, S\}$ is an inverter of a 2-cell $S_0 \Downarrow S_1$; here one needs no colimits in X since the restriction of the left adjoint of $f_0 : X \to S_0$ to the inverter is already an adjoint for f.

7.6. Corollary. *Suppose $\mathscr{B}$ is a small bicategory, $J : \mathscr{B}^{\mathrm{op}} \to$ **Cat** is a homomorphism, $J * S$ is a J-indexed bicolimit for $S : \mathscr{B} \to \bar{\mathscr{K}}$, and $f : J * S \to C$ is an arrow in $\bar{\mathscr{K}}$. If the object M of $\bar{\mathscr{K}}$ is cocomplete relative to the arrow*

$$f_b : SB \simeq \mathscr{B}(\text{-}, B) * S \xrightarrow[b*S]{} J * S \xrightarrow[f]{} C$$

for all B, b of $\mathscr{B}, JB$, and if the category $\bar{\mathscr{K}}(C, M)$ has small colimits, then M is cocomplete relative to f.

7.7. Corollary. *Suppose $M : \mathscr{C}^{\mathrm{op}} \to$ **CAT** is a homomorphism satisfying the following conditions:*

(i) *for each object U of $\mathscr{C}$, MU has small colimits;*
(ii) *for each arrow r in $\mathscr{C}$, Mr has a left adjoint.*

Then the object M of $\bar{\mathscr{K}}$ is cocomplete relative to each arrow $f : A \to \mathscr{C}(\text{-}, U)$ in $\mathscr{K}$ whose codomain is representable.

Proof. Let $\mathscr{Y} : \mathscr{C} \to \mathscr{K}$ denote the 'Yoneda embedding'. Since $A \simeq A * \mathscr{Y}$ and $MU \simeq \bar{\mathscr{K}}(\mathscr{C}(\text{-}, U), M)$, the result follows from Corollary 7.6 provided M is cocomplete relative to each arrow $\mathscr{Y}V = \mathscr{C}(\text{-}, V) \to \mathscr{C}(\text{-}, U)$; but this follows from Proposition 7.1. □

7.8. Our interest is in pointwise left extensions. Pointwiseness is closely related to the condition used by Chevalley, Beck and Bénabou–Roubaud in their work on descent, and by Lawvere in his work on hyperdoctrines. The condition was modified for 2-categories by Street [22, p. 150]. Here we require a further modification: we require the form of the condition on a homomorphism $M : \mathscr{C}^{\mathrm{op}} \to$ **CAT** which involves no completeness condition on $\mathscr{C}$.

7.9. Arrows $r : V \to U$, $s : W \to U$ in $\mathscr{C}$ determine a functor $\langle r, s\rangle : (W \# V)^{\mathrm{op}} \to$ **Set** whose value at (x, S, u) is $\mathscr{C}(S, U)(rx, su)$. The functor $\langle r, s\rangle$ corresponds to the bidiscrete fibration from $\mathscr{C}(\text{-}, W)$ to $\mathscr{C}(\text{-}, V)$ associated with the pseudo-comma object of $\mathscr{C}(\text{-}, r)$, $\mathscr{C}(\text{-}, s)$.

7.10. Suppose $M : \mathscr{C}^{\mathrm{op}} \to$ **CAT** satisfies condition (ii) of Corollary 7.7. Each object m of MV determines a functor $M^m : W \# V \to MW$ whose value at (x, S, u) is $((\hat{M}u)Mx)m$. There is a natural transformation

$$\nu : \langle r, s\rangle \Rightarrow (MW)(M^m\text{-}, (\hat{M}s)(Mr)m)$$

whose component at (x, S, u) takes $\sigma : rx \Rightarrow su$ to the component at m of the natural transformation $(\hat{M}u)(Mx) \Rightarrow (Ms)(\hat{M}r)$ corresponding under adjunction to the natural transformation

$$(Mx)(Mr) \cong M(rx) \xrightarrow{M\sigma} M(su) \cong (Mu)(Ms).$$

7.11. A homomorphism $M : \mathscr{C}^{\mathrm{op}} \to \mathbf{CAT}$ is said to be a CB-*homomorphism* when it satisfies condition (ii) of Corollary 7.7 and the following Chevalley–Beck condition:

For all arrows $r : V \to U$, $s : W \to U$ in $\mathscr{C}$ and all objects m of MV, the natural transformation ν of 7.10 *exhibits $(Ms)(\hat{M}r)m$ as an $\langle r, s\rangle$-indexed colimit of $M^m : W \# V \to MW$.*

Symbolically, when MW has the appropriate copowers:

$$\int^{(x,S,u)} \mathscr{C}(S, U)(rx, su) \otimes (Mu)(Mx)m \cong (Ms)(\hat{M}r)m.$$

Notice that when $\mathscr{C}$ has bicomma objects, $\langle r, s\rangle$ is represented by the span (p, E, q) associated with a bicomma object for r, s; so the above condition amounts to the requirement that each canonical natural transformation

$$(\hat{M}q)(Mp) \Rightarrow (Ms)(\hat{M}r)$$

should be invertible. Compare Street [22, p. 150].

7.12. Proposition. *An object M of $\bar{\mathscr{K}}$ is a* CB-*homomorphism, if and only if, for all $r : V \to U$ in $\mathscr{C}$, each arrow $\mathscr{C}(-, V) \to M$ has a pointwise left extension along $\mathscr{C}(-, r) : \mathscr{C}(-, V) \to \mathscr{C}(-, U)$.*

Proof. By Street [22, p. 141], a left extension in $\bar{\mathscr{K}}$ is pointwise if and only if it is a left extension at each arrow from a representable. One easily sees then that the two conditions of the proposition amount to the requirement that, for all $r : V \to U$, $s : W \to U$ in $\mathscr{C}$ and all m of MV, the diagram:

$$\begin{tikzcd}
\mathscr{C}(-,r)/\mathscr{C}(-,s) \arrow[rr, "d_1"] \arrow[d, "d_0"'] \arrow[drr, phantom, "\overset{\lambda}{\Rightarrow}"] & & \mathscr{C}(-,W) \arrow[d, "\mathscr{C}(-,s)"] \\
\mathscr{C}(-,V) \arrow[rr, "\mathscr{C}(-,r)"'] \arrow[dr, "m"'] & \Rightarrow & \mathscr{C}(-,U) \arrow[dl, "(\hat{M}r)m"] \\
& M &
\end{tikzcd}$$

should exhibit $(Ms)(\hat{M}r)m$ as a left extension of md_0 along d_1. □

7.13. Proposition. *The following three conditions on an object M of $\bar{\mathscr{K}}$ are equivalent:*

(i) *for all small categories $\mathscr{R}$, the diagonal arrow $M \to [\mathscr{R}, M]$ has a left adjoint;*

(ii) *for all small categories* $\mathcal{R}$ *and objects* U *of* $\mathscr{C}$, *each arrow* $\mathcal{R} \times \mathscr{C}(-, U) \to M$ *has a pointwise left extension along the projection* $\mathcal{R} \times \mathscr{C}(-, U) \to \mathscr{C}(-, U)$;

(iii) *for each object* U *of* $\mathscr{C}$, *the category* MU *has small colimits and, for each arrow* $r: V \to U$, *the functor* $Mr: MU \to MV$ *preserves small colimits.*

Proof. (i) $\Leftrightarrow$ (iii) That each MU has small colimits is precisely the condition that each diagonal functor $MU \to [\mathcal{R}, MU]$ should have a left adjoint. That each Mr should preserve small colimits is precisely the condition that these adjoints should give a strong transformation $[\mathcal{R}, M] \to M$.

(ii) $\Leftrightarrow$ (iii) Arrows $h: \mathcal{R} \times \mathscr{C}(-, U) \to M$ correspond to functors $h': \mathcal{R} \to MU$; furthermore, h has a left extension along the projection into $\mathscr{C}(-, U)$ precisely when h' has a colimit. Since this projection is a right fibration, it follows from [2, p. 130; 22, p. 141] that for pointwiseness we need only test that each diagram

$$\begin{tikzcd}
\mathcal{R} \times \mathscr{C}(-, V) \arrow[rr, "\mathrm{proj}"] \arrow[d, "1 \times \mathscr{C}(-, r)"'] & & \mathscr{C}(-, V) \arrow[d, "\mathscr{C}(-, r)"] \\
\mathcal{R} \times \mathscr{C}(-, U) \arrow[rr, "\mathrm{proj}"] \arrow[dr, "h"'] & \Rightarrow & \mathscr{C}(-, U) \arrow[dl, "\mathrm{col}\, h'"] \\
& M &
\end{tikzcd}$$

has the left extension property. So pointwiseness amounts precisely to the condition $\mathrm{col}((Mr)h') \cong (Mr)\mathrm{col}\, h'$ for all r. □

7.14. Proposition. *In* $\bar{\mathscr{K}}$, *an arrow* $h: A \to M$ *admits a pointwise left extension along* $f: A \to C$, *if and only if, for each object* U *of* $\mathscr{C}$ *and* $c: \mathscr{C}(-, U) \to C$, *the arrow* $hd_0: f/c \to M$ *admits a pointwise left extension along* $d_1: f/c \to \mathscr{C}(-, U)$.

Proof. The definition of pointwise gives 'only if'. On the other hand, if we have a pointwise left extension $k_u(c)$ of hd_0 along d_1 for each c, this assignment gives the components

$$\bar{\mathscr{K}}(\mathscr{C}(-, U), C) \to \bar{\mathscr{K}}(\mathscr{C}(-, U), M)$$

of a transformation in U which is strong by pointwiseness. By 1.11 this gives a strong transformation $k: C \to M$ which is a pointwise left extension of h along f by [22, p. 141]. □

7.15. Proposition. *Suppose* (p, E, q) *is a bidiscrete fibration from* $\mathscr{C}(-, U)$ *to* A, *which corresponds* (6.2) *to the functor* $T: (U \# A)^{\mathrm{op}} \to \mathbf{Set}$. *Suppose* M *is an object of* $\bar{\mathscr{K}}$ *which* *satisfies condition* (ii) *of Corollary* 7.7. *Suppose* $h: A \to M$ *is an arrow in* $\bar{\mathcal{K}}$ *and* $\ell_U : U \# M \to \hat{M}U$ *is as in* 7.2. *A left extension* k *of* hp *along* q *corresponds* (1.11) *precisely to a* T*-indexed colimit for* $\ell_U(U \# h): U \# A \to \hat{M}U$. *If the appropriate copowers exist in* MU, *this means*:

$$k_U(1_U) \cong \int^{(a,S,u)} T(a, S, u) \otimes (\hat{M}u)h_S(a).$$

Proof. Take $m: \mathcal{C}(\text{-}, U) \to M$. The objects of $(h/m)U$ are triples $(a, h_U(a) \to (Mu)m, u)$ where $a \in AU$ and $u: V \to U$. An arrow $h_U(a) \to (Mu)m$ corresponds to an arrow $(Mu)h_U(a) \to m$. It can be seen from this that the functor $(U \# A)^{\text{op}} \to \mathbf{Set}$ corresponding to the fibration h/m from $\mathcal{C}(\text{-}, U)$ to A is none other than $(MU)(\ell_U(U \# h)\text{-}, m)$. Now a 2-cell $hd_0 \Rightarrow md_1$ corresponds to the isomorphism class of a strong span arrow $f/c \to h/m$, and hence to a natural transformation $T \to (MU)(\ell_U(U \# h)\text{-}, m)$. □

7.16. An object M of $\bar{\mathcal{K}}$ will be said to *have small colimits*, when it is a CB-homomorphism (7.11) and satisfies the condition (iii) of Proposition 7.13. For such an object we have all pointwise left extensions into it along arrows $\mathcal{C}(\text{-}, V) \to \mathcal{C}(\text{-}, U)$ and along arrows $\mathcal{R} \times \mathcal{C}(\text{-}, U) \to \mathcal{C}(\text{-}, U)$, where $\mathcal{R}$ is a small category. In fact we have much more:

7.17. Proposition. *If* M *is an object of* $\bar{\mathcal{K}}$ *with small colimits and* A *is an essentially small object of* $\mathcal{K}$, *then each arrow* $h: A \to M$ *has a pointwise left extension* $k: \mathcal{P}A \to M$ *along* $yA: A \to \mathcal{P}A$. *The formula for* k *is*:

$$k_U(T) \cong \mathrm{col}(T, \ell_U(U \# h)).$$

Proof. By Propositions 7.15 and 6.8, the formula given must be correct provided k exists. The formula is functorial in T and gives the component at U for a transformation $k: \mathcal{P}A \to M$. The difficulty is to show that this transformation is strong. For this we need the formula

$$\begin{aligned}&\int^{(x,W,y)} (V \# A)((b, R, v), ((Ax)a, W, y) \times \mathcal{C}(W, U)(ux, ry)) \cong \\ &\quad \cong (U \# A)((b, R, rv), (a, S, u)),\end{aligned} \tag{7.18}$$

which can be verified directly or from general principles related to 1.17. Then the calculation proceeds as follows:

$$(Mr)k_U(T) \cong \int^{(a,S,u)} T(a, S, u) \otimes (Mr)(\hat{M}u)h_S(a)$$

since Mr preserves small colimits,

$$\cong \int^{(a,S,u)} \int^{(x,W,y)} (T(a,S,u) \times \mathscr{C}(W,U)(ux,ry)) \otimes (\hat{M}y)(Mx)h_S(a)$$

using the Chevalley–Beck condition 7.11,

$$\cong \int^{(a,S,u)} \int^{(x,W,y)} \int^{(b,R,v)} (V \# A)((b,R,v),((Ax)a,W,y)) \times T(a,S,u)$$
$$\times \mathscr{C}(W,U)(ux,ry) \times (\hat{M}v)h_R(b)$$

using the Yoneda lemma and that h is strong,

$$\cong \int^{(b,R,v)} \int^{(a,S,u)} (U \# A)((b,R,rv),(a,S,u)) \times T(a,S,u) \otimes (\hat{M}v)h_R(b)$$

using Fubini and (7.18),

$$\cong \int^{(b,R,v)} T(b,R,rv) \times (\hat{M}v)h_R(b)$$

using the Yoneda lemma,

$$\cong k_V((Mr)T).$$

With this verified, the result follows from [22, p. 141] and Proposition 7.15. □

7.19. Theorem. *An object M of $\bar{\mathscr{K}}$ has small colimits, if and only if, for all admissible* (6.3) *arrows $f: A \to C$ in $\bar{\mathscr{K}}$ with essentially small* (6.3) *domain A, each arrow $h: A \to M$ has a pointwise left extension k along f. The formula for k in these circumstances is*:

$$k_U(c) \cong \operatorname{col}(C(f,c), \ell_U(U \# h)).$$

Proof. Propositions 7.12 and 7.13 give 'if'.

'Only if' and the formula follow from Proposition 7.17 and [22, Theorem 16, p. 154]. □

7.20. The line of analysis of (7.4)–(7.7) also leads to a proof of 'only if' in the above Theorem. By 7.14, it suffices to consider the case where f is a right fibration into a representable $\mathscr{C}(-,U)$. By Corollary 7.7, a left extension of h along f exists; the only question is as to its pointwiseness. This follows from the pointwiseness of left extensions along the arrows $\mathscr{C}(-,V) \to \mathscr{C}(-,U)$ (7.12) by means of [12, p. 130; 22, p. 141], Lemma 7.4 (ii).

7.21. Proposition. *Suppose $M: \mathscr{C}^{\mathrm{op}} \to$ **CAT** is a* CB-*homomorphism such that, for each arrow r in $\mathscr{C}$, the functor Mr has a right adjoint $\check{M}r$. Then $M^{\mathrm{op}}: \mathscr{C}^{\mathrm{coop}} \to$ **CAT** is a* CB-*homomorphism.*

Proof. When $\mathscr{C}$ has bicomma objects, in the notation of 7.11, the canonical $(\hat{M}q)(Mp) \Rightarrow (Ms)(\hat{M}r)$ is invertible, if and only if the corresponding $(Mr)(\check{M}s) \Rightarrow (\check{M}p)(Mq)$ is invertible. The general case is left to the reader. □

7.22. An object M of $\bar{\mathscr{K}}$ will be said to *have small limits*, when M^{op} has small colimits as an object of Hom$(\mathscr{C}^{\mathrm{coop}}, \mathbf{CAT})$. Applying Theorem 7.19 in the latter 2-category and interpreting back in $\bar{\mathscr{K}}$, we obtain lots of pointwise *right* extensions of arrows into such an M.

8. The gross cocompletion

In this section we shall examine the objects $\mathscr{P}X$. The notation will again be that of 4.2.

8.1. Theorem. *For each object X of $\mathscr{K}$, the objects $\mathscr{P}X$ and $\mathscr{P}^*X$ of $\bar{\mathscr{K}}$ have both small colimits* (7.14) *and small limits* (7.17).

Proof. Each $(\mathscr{P}X)U = [(U \# X)^{\mathrm{op}}, \mathbf{Set}]$ has small colimits and small limits which are given pointwise. Each $(\mathscr{P}X)r$ has both a left and a right adjoint given by left and right Kan extension along $r \# A$. To complete the proof that $\mathscr{P}X$ has small colimits and small limits, it suffices by Proposition 7.21 to see that $\mathscr{P}X$ satisfies the Chevalley-Beck condition 7.11. To see this, apply [22, Theorem 14, p. 151] to the arrows $\mathscr{C}(-, s) : \mathscr{C}(-, W)^{\mathrm{op}} \to \mathscr{C}(-, U)^{\mathrm{op}}$. Then apply the homomorphism represented by X^{op} and use the results of 4.6 and 1.11. The result for $\mathscr{P}^*X$ follows by duality. □

8.2. It follows from Theorem 7.19 that lots of pointwise left extensions into $\mathscr{P}X$ exist. Theorem 7.19 can be used in the other direction to give another proof of Theorem 8.1, since the pointwise left extensions into $\mathscr{P}X$ can be produced by means of fibrational composition (3.9). The relationship between these pointwise left extensions and fibrational composition is as follows.

8.3. Proposition. *Suppose A, X are objects of $\mathscr{K}$, and B is an object of $\bar{\mathscr{K}}$. Suppose $h : A \to \mathscr{P}X$, $h' : B \to \mathscr{P}A$ are arrows corresponding to bidiscrete fibrations E from A to X, E' from B to A under the Representation Theorem* 4.6. *Let k denote a pointwise left extension of h along yA. Then $kh' : B \to \mathscr{P}X$ corresponds to $E \otimes E'$ under the Representation Theorem.*

8.4. Suppose M, N are objects of $\bar{\mathscr{K}}$ which have small colimits. An arrow $t : M \to N$ is said to *preserve small colimits*, when it satisfies the following two conditions:

(i) for each object U of $\mathscr{C}$, the functor $t_U : MU \to NU$ preserves small colimits;

(ii) for each arrow $r : V \to U$ in $\mathscr{C}$, the natural transformation $(\hat{N}r)t_V \Rightarrow t_U(\hat{M}r)$ corresponding to $t_V(Mr) \cong (Nr)t_U$ is invertible.

After the work of Section 7 it is clear that t preserves small colimits if and only if t respects all left extensions into M along admissible arrows with essentially small domains. Write Cocts(M, N) for the full subcategory of $\bar{\mathscr{K}}(M, N)$ consisting of the arrows t which preserve small colimits.

8.5. Theorem. *Suppose X is an essentially small object of $\bar{\mathscr{K}}$ and M is an object which has small colimits. Then left extension along $yX : X \to \mathscr{P}X$ provides an equivalence of categories*:

$$\bar{\mathscr{K}}(X, M) \simeq \mathrm{Cocts}(\mathscr{P}X, M).$$

Proof. Pointwise left extension along yX exists by Theorem 7.19 and provides a fully faithful functor $\bar{\mathscr{K}}(X, M) \to \bar{\mathscr{K}}(\mathscr{P}X, M)$ since $yX/yX \simeq [\mathbf{2}, X]$ by Proposition 6.8. The left extension of $h : X \to M$ along yX preserves small colimits since it has a right adjoint $M(h, 1) : M \to \mathscr{P}X$. It follows from Proposition 6.8 that $1 : \mathscr{P}X \to \mathscr{P}X$ is the pointwise left extension of yX along yX. So, if $k : \mathscr{P}X \to M$ preserves small colimits, k is the pointwise left extension of $k(yX)$ along yX. This gives the desired equivalence. □

8.6. The above theorem yields that $\mathscr{P}X$ is a pointwise cocompletion of X relative to admissible arrows with essentially small domains. Any such pointwise left extensions which already exist in X are not in general respected by $yX : X \to \mathscr{P}X$. Pointwise *right* extensions in X *are* respected by yX; we give the dual of this.

8.7. Proposition. *Suppose M is an admissible object of $\bar{\mathscr{K}}$. The coyoneda arrow $y^*M : M \to \mathscr{P}^*M$ (which corresponds to yM under the equivalence of* 6.9) *respects pointwise left extensions along admissible arrows with essentially small domain.*

Proof. One can use the formula of [22, Corollary 4, p. 142]. □

9. On the existence of pointwise cocompletions

9.1. Suppose Σ is a small set of arrows in $\mathscr{K} = \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ where $\mathscr{C}$ is a small bicategory. Let Σ_0 denote the small set of objects of $\mathscr{K}$ which occur as domains of elements of Σ. For each A in Σ_0 and each object U of $\mathscr{C}$, let $(\Gamma A)U$ denote the full subcategory of D Fib$(\mathscr{C}(\text{-}, U), A)$ consisting of those bidiscrete fibrations which arise as bicomma objects of arrows $f : A \to C$ in Σ and $c : \mathscr{C}(\text{-}, U) \to C$ in $\mathscr{K}$. For each $r : V \to U$ in $\mathscr{C}$, we have pseudo-pullbacks:

$$\begin{tikzcd}
f/(Cr)c \arrow[r, "d_1"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathscr{C}(\text{-}, V) \arrow[d, "\mathscr{C}(\text{-}, r)"] \\
f/c \arrow[r, "d_1"'] & \mathscr{C}(\text{-}, U)
\end{tikzcd}$$

It follows that $(\Gamma A)U$ is the value at U of a homomorphism $\Gamma A : \mathscr{C}^{\mathrm{op}} \to \mathbf{CAT}$, and that, using the Representation Theorem 4.6, we have a strong transformation

$$\iota A : \Gamma A \to \mathscr{P}A = [(\# A)^{\mathrm{op}}, \mathbf{Set}]$$

whose components are all fully faithful. Notice that ΓA is an essentially small object of $\tilde{\mathscr{K}}$.

9.2. Notice that an object M of $\tilde{\mathscr{K}}$ is pointwise Σ-cocomplete if and only if, for all $A \in \Sigma_0$, $U \in \mathscr{C}$, $(p, E, q) \in (\Gamma A)U$, $h : A \to M$, there exists a pointwise left extension of hp along q (Proposition 7.14). An arrow $M \to N$ is pointwise Σ-cocontinuous precisely when it respects these pointwise left extensions.

9.3. Theorem. *Suppose $\mathscr{C}$ is a small bicategory and Σ is a small set of arrows in* $\mathscr{K} = \mathrm{Hom}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$. *For each object X of $\mathscr{K}$, there exist a pointwise Σ-cocomplete object $\tilde{X}$ of $\mathscr{K}$ and a dense fully faithful arrow $n : X \to \tilde{X}$ such that, for all pointwise Σ-cocomplete objects M of $\mathscr{K}$, pointwise left extension along n provides an equivalence of categories*:

$$\mathscr{K}(X, M) \simeq \mathscr{K}_{\mathrm{pw}\,\Sigma}(\tilde{X}, M).$$

Proof. Without loss of generality it suffices to find $\tilde{X}$ essentially small in $\tilde{\mathscr{K}}$ instead of an object of $\mathscr{K}$. We use the notation of 9.1.

For each ordinal α, we shall define by transfinite recursion arrows $i_\alpha : X_\alpha \to \mathscr{P}X$ in $\tilde{\mathscr{K}}$ whose components are inclusions of full subcategories. For $\alpha = 0$, the component at U of $i_0 : X_0 \to \mathscr{P}X$ is the full image of $(yX)U : XU \to (\mathscr{P}X)U$. For an ordinal α, suppose $i_\alpha : X_\alpha \to \mathscr{P}X$ is already defined. Let $X_{\alpha+1}U$ denote the full subcategory of $(\mathscr{P}X)U$ containing all the objects of $X_\alpha U$ and each $w \in (\mathscr{P}X)U$ for which there exist $A \in \Sigma_0$, $(p, E, q) \in (\Gamma A)U$, $v : A \to X_\alpha$ and a 2-cell $ivp \Rightarrow wq$ which exhibits w as a (necessarily pointwise) left extension of ivp along q. We have strong transformations

$$j_\alpha : X_\alpha \to X_{\alpha+1}, \qquad i_{\alpha+1} : X_{\alpha+1} \to \mathscr{P}X$$

whose components are inclusions. Furthermore, we have pointwise left extension diagrams

$$\begin{tikzcd} E \arrow[rr, "q"] \arrow[d, "p"'] \arrow[drr, phantom, "\Rightarrow"] & & \mathscr{C}(-, U) \arrow[d, "w"] \\ A \arrow[r, "v"'] & X_\alpha \arrow[r, "j_\alpha"'] & X_{\alpha+1} \end{tikzcd} \qquad E \in (\Gamma A)U \tag{9.4}$$

respected by $i_{\alpha+1}$. For a limit ordinal λ, $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ and $i_\lambda = \bigcup_{\alpha<\lambda} i_\alpha$.

Let θ denote a regular cardinal which exceeds the cardinalities of the bicategory $\mathscr{C}$, the set Σ, the homomorphism X, and, for each $f : A \to C$ in Σ, the homomorphisms

A, C. From the formula of Proposition 7.17 and the construction above, the objects of each $X_\alpha U$ are iterated θ-colimits of representables of the form $(U \# X)(-, \partial_U x)$ in $(\mathcal{P}X)U = [(U \# X)^{\mathrm{op}}, \mathbf{Set}]$. Since each such iterated θ-colimit can be obtained as a coequalizer of a pair of arrows between coproducts of such representables indexed by sets of cardinality $<\theta$, each $X_\alpha U$ has a small skeleton and $j_\alpha U : X_\alpha U \to X_{\alpha+1} U$ is an equivalence for α of cardinality $\geq \theta$. Taking γ to be an ordinal of cardinality θ, we obtain an *essentially small* X_γ and an equivalence $j_\gamma : X_\gamma \to X_{\gamma+1}$. The latter implies (see 9.2 and (9.4)) that *X_γ is pointwise Σ-cocomplete and $i_\gamma : X_\gamma \to \mathcal{P}X$ is pointwise Σ-cocontinuous.*

Let $z : X \to X_\gamma$ be the dense fully faithful arrow defined by $i_\gamma z = y_X$. We shall show that z has the property stated for n in the theorem.

Suppose N is an object of $\bar{\mathcal{K}}$ which has small colimits. Take $h : X \to N$ and let $h_\alpha : X_\alpha \to N$ be a pointwise left extension of h along the fully faithful $X \to X_\alpha$ and let $k : \mathcal{P}X \to N$ be a pointwise left extension of h along yX; these exist by Theorem 7.19. Then k is a pointwise left extension of h_α along i_α. Since i_α is fully faithful, $h_\alpha \cong k i_\alpha$ for all α. Since k has a right adjoint $N(h, 1)$ and since $i_{\alpha+1}$ respects the pointwise left extensions (9.4), *$h_{\alpha+1}$ also respects the pointwise left extensions* (9.4). Also, since i_γ is pointwise Σ-cocontinuous, *h_γ is pointwise Σ-cocontinuous.*

Suppose $t : X_\gamma \to N$ is pointwise Σ-cocontinuous. Put $h = tz$ and produce h_α from h as above. Let $t_\alpha : X_\alpha \to N$ denote the composite of t with the inclusion $X_\alpha \to X_\gamma$. Since the inclusion $X_{\alpha+1} \to X_\gamma$ respects pointwise left extensions of the form (9.4) and t is pointwise Σ-cocontinuous, $t_{\alpha+1} : X_{\alpha+1} \to N$ respects left extensions of the form (9.4). From the construction of the X_α, we see by induction that the canonical 2-cell $h_\alpha \Rightarrow t_\alpha$ is invertible for $0 \leq \alpha \leq \gamma$. So $t \cong h_\gamma$.

A little more generally than in the statement of the theorem, take M to be a pointwise Σ-cocomplete, admissible object of $\bar{\mathcal{K}}$. We shall show that pointwise left extension along $z : X \to X_\gamma$ provides an equivalence:

$$\bar{\mathcal{K}}(X, M) \simeq \bar{\mathcal{K}}_{\mathrm{pw}\,\Sigma}(X_\gamma, M).$$

Since z is fully faithful, pointwise left extension along z is too. So what we must show is:

(a) that each arrow $m : X \to M$ has a pointwise left extension m_γ along z;

(b) that m_γ is pointwise Σ-cocontinuous;

(c) that each pointwise Σ-cocontinuous $s : X_\gamma \to M$ is isomorphic to some m_γ.

Take $m : X \to M$. Put $N = \mathcal{P}^* M$, which has small colimits by Theorem 8.1. Recall from Proposition 8.7 that the fully faithful arrow $y^*M : M \to N$ respects pointwise left extensions along arrows in $\mathcal{K}$. Put $h = (y^*M)m : X \to N$ and construct h_α as above. Since M is pointwise Σ-cocomplete and $h_{\alpha+1}$ preserves left extensions of the form (9.4.), an inductive argument involving the construction of the X_α shows that each h_α factors (up to isomorphism) through y^*M via $m_\alpha : X_\alpha \to M$. Then m_α is a pointwise left extension of m along the fully faithful $X \to X_\alpha$. In particular, we have (a). Since h_γ is pointwise Σ-cocontinuous, so too is m_γ, which proves (b). Given s as in (c), put $t = (y^*M)s$, which is also pointwise Σ-cocontinuous. Our earlier argument shows that $t \cong h_\gamma$ where $h = tz = (y^*M)sz$. Put $m = sz$, and we see from the above that $h_\gamma \cong (y^*M)m_\gamma$. So $(y^*M)s \cong (y^*M)m_\gamma$, from which (c) follows since y^*M is fully faithful. $\square$

9.5. Corollary. *For $\mathscr{C}$, Σ, $\mathscr{K}$ as in Theorem* 9.3, *the inclusion $\mathscr{K}_{\mathrm{pw}\,\Sigma} \to \mathscr{K}$ is* KZ-*doctrinal* [24, (2.27)].

Proof. The inclusion has a left biadjoint whose value at X is $\tilde{X}$ by Theorem 9.3. To see that this generates a KZ-doctrine on $\mathscr{K}$, take M to be pointwise Σ-cocomplete; we must show that the pointwise left extension $m : \tilde{M} \to M$ of 1_M along $n : M \to \tilde{M}$ is a left adjoint for n. Since n is fully faithful, $1_M \Rightarrow mn$ is invertible; the inverse provides a candidate for a counit. Since n is dense, $1_{\tilde{M}}$ is a left extension of n along n. So $n \Rightarrow nmn$ induces $1_{\tilde{M}} \Rightarrow nm$, which is a candidate for a unit, and we obtain one of the adjunction equations. The other adjunction equation follows from the denseness of n.

To see that $\mathscr{K}_{\mathrm{pw}\,\Sigma}$ is the bicategory of algebras (up to biequivalence), suppose X is an algebra for the doctrine. Then $n : X \to \tilde{X}$ has a left adjoint m. Suppose $f : A \to C$ is in Σ. Since n is fully faithful, a pointwise left extension of $h : A \to X$ along f is mk, where k is a pointwise left extension of nh along f. So X is pointwise Σ-cocomplete. Similarly, the strong morphisms of algebras are pointwise Σ-cocontinuous. $\square$

9.6. Let Σ denote a small set of small categories. We identify Σ with the set of arrows $A \to \mathbf{1}$ in **Cat** with $A \in \Sigma$. Each left extension along an arrow into $\mathbf{1}$ is automatically pointwise and amounts to a colimit for the functor being extended. So in Theorem 9.3 with $\mathscr{C} = \mathbf{1}$, $\mathscr{K}_{\mathrm{pw}\,\Sigma}$ is the locally full sub-2-category $\mathbf{Cat}_\Sigma$ of **Cat** consisting of the categories with Σ-colimits and the functors which preserve Σ-colimits. It follows that *the inclusion* $\mathbf{Cat}_\Sigma \to \mathbf{Cat}$ *is* KZ-*doctrinal*. Inter alia this means that Σ-cocompletions exist.

9.7. Let $\mathscr{C}$, $\mathscr{K}$ be as in Theorem 9.3 and let Σ denote the set of arrows $\mathscr{C}(-, V) \to \mathscr{C}(-, U)$ between representables. Then $\mathscr{K}_{\mathrm{pw}\,\Sigma}$ *is the locally full sub-*2*-category of* $\mathscr{K}$ *consisting of the* CB-*homomorphisms* (7.12) and the strong transformations which satisfy 8.4 (ii). Theorem 9.3 yields a universal construction of a CB-homomorphism $\tilde{X}$ from an arbitrary homomorphism X. When $\mathscr{C}$ has bicomma objects, $\tilde{X}$ is equivalent to the homomorphism whose value at U is $U \# X$ and whose value at r is a right adjoint for $r \# X$. It follows from Corollary 9.5 that X is a CB-homomorphism precisely when $X \to \tilde{X}$ has a left adjoint. For the case where $\mathscr{C}$ is a mere category with pullbacks this was also observed by Bénabou [4, p. 899].

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