On fibred adjunctions and completeness for fibred categories

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Abstract

We show how the completeness and cocompleteness of the total category of a fibration can be inferred from that of the fibre categories and its base. Our results are somewhat stronger than those in [BGT91] and they are obtained as direct consequences of an important property of general fibred adjunctions. Our aim is to show that fibred category theory can provide insight into constructions of relevance in algebraic specifications, *e.g.* limits and colimits of many-sorted algebras, by explaining them at a natural level of abstraction.

1. Introduction

In [BGT91], Tarlecki *et al.* present, in a tutorial fashion, indexed categories as applied in algebraic specifications, in particular in the theory of institutions [GB90]. The examples presented there include many-sorted algebraic signatures, many-sorted algebras and theories and presentations in an institution. An indexed category induces a total category (loosely speaking, the 'disjoint sum' of its fibres) and the authors go on to show sufficient conditions for this category to be (co)complete, thereby obtaining proofs of (co)completeness for some of the examples mentioned before. However the conditions for completeness and cocompleteness are rather asymmetric, *i.e.* duality is not explicit.

In this paper, we present slightly stronger versions of the abovementioned results as consequences of a factorisation property of general fibred adjunctions, *i.e.* an adjunction between fibrations over possibly different bases. Fibrations – also referred to as fibred categories – and indexed categories are essentially equivalent notions, although the former are technically more convenient to work with. The above mentioned asymmetry for completeness and cocompleteness of fibred categories is best understood by looking at cofibrations as well as fibrations, and letting duality do the work – see Corollaries 3.5 and 3.8. As pointed out in [BGT91] the (co)completeness results, in the sharper version we present here, were already contained in [Gra66]. However our proofs are different (and simpler), since the property of general fibred adjunctions we use is not present there.

Besides the applications of indexed categories in algebraic specification presented in [BGT91], it is worth mentioning a few other applications of fibrations of relevance to computer science. [Win90] contains applications of fibrations to the semantics of concurrency via (labelled) transition systems; finite completeness and cocompleteness of the (total) fibred category are put to use in this context as well as (vertical) fibred adjunctions. More significantly, fibrations are the key ingredient in the categorical semantics of (proof-theoretic) logics and type theories; see [Jac91] for a comprehensive account. Actually, we discovered Thm. 3.4 below when analysing logical relations [Mit90] categorically, motivated by [MR91]; such analysis fits within the application of fibred categories to (the semantics of) logic just mentioned. See [Her93] for details.

One further application which fits within the algebraic specification tradition is the description of strong data types, or data types with parameters, as in [Jac93b]. The description essentially relies on initial algebras for endofunctors on a distributive category, and can be used to obtain structural induction principles for the corresponding datatypes, as explained in [Her93].

Since we do not assume familiarity with either indexed categories or fibrations, we include some background material about both in §2. None of the material in that section is original. §3 deals with general fibred adjunctions and completeness. It contains the main technical result of the paper, namely Thm. 3.4, and the above mentioned characterisation of (co)completeness for the total category of a (co)fibration. This material is essentially the contents of Chapter 3 in [Her93], where full proofs can be found. Since the purpose of this paper is mainly to show how to apply Thm. 3.4 to the situation at hand, we omit the tedious technical details of its proof.

The paper is meant to be reasonably self-contained, but we do assume familiarity with basic notions of category theory, which can be found in [Mac71]. There are ocasional references to the concept of 2-category, the paradigmatic example being *Cat*, in which for any two categories, the set of functors is itself a category whose morphisms are natural transformations. However, this level of abstraction is not essential to understand the paper since we express the definitions and results on an elementary level. Anyway, the interested reader may be willing to consult [KS74] for the basics on 2-categories.

2. Basics of fibred and indexed categories

This section reviews some standard basic concepts and properties concerning indexed categories and fibrations. The presentation follows [Jac91, Jac93a, Her93], where the reader may find considerably more information and references. The examples are mostly drawn from [BGT91]. We begin by fixing some notation which will be used throughout. Categories will generally be denoted **A**, **B**, *etc.* *Set* will denote the category of sets and functions (relative to some given universe, as in [Mac71, p.21]) and *Cat* denotes the 2-category of categories, functors and natural transformations.

The notion of fibration or fibred category, introduced in [Gro71], captures the concept of a category varying over (or indexed by) another category. Before giving the actual definition, we recall the analogous situation for sets, which may help convey the intuition about the categorical concept. A family $\{X_i\}_{i\in I}$ of sets indexed by a set I corresponds to a function $X : I \to \mathit{Set}$. We may think of this as a 'set' X varying over I. It can be equivalently presented as a function $p : X \to I$, since such a function gives rise to the family $\{X_i = p^{-1}(i)\}_{i\in I}$ and conversely, given a family $\{X_i\}_{i\in I}$ we get $p : \coprod_{i\in I} X_i \to I$, where $\coprod_{i\in I} X_i$ is the disjoint union of the X_i's and p maps an element in X_i to i. It is clear that such constructions (between morphisms into I and I-indexed families) are inverse to each other. We can summarise this situation by the following isomorphism of categories:

$$\mathit{Set}/I \cong \mathit{Set}^I$$

where Set/I denotes the usual slice category of morphisms into I and commutative triangles, and Set^I is the category of functors from I (regarded as a discrete category) to Set. These two equivalent views of indexed families of sets have their categorical counterparts: a function $X : I \to \mathit{Set}$ is generalised to an indexed category, *cf.* Def. 2.1, while a function $p : X \to I$ is generalised to a fibration, *cf.* Def. 2.6. The isomorphism above becomes then an equivalence between fibred and indexed categories, *cf.* Prop. 2.10 below. Despite this equivalence, the notion of fibration is technically more convenient, as forcibly argued in [Bén85]. We start off with the definition of indexed category, since it seems to be the more intuitive of the two notions.

2.1. Indexed categories

We recall the definitions of indexed categories and their associated morphisms: indexed functors and indexed natural transformations. The purpose is to set up a (2-)category of indexed categories where we can deal with the structure of indexed categories in a similar way as we do with an ordinary category within *Cat*. The standard reference for indexed categories is [PS78].

2.1. DEFINITION (Indexed category). Given a category $\mathbf{B}$, a $\mathbf{B}$-*indexed category* is a pseudo-functor $\mathcal{F} : \mathbf{B}^{op} \to \mathit{Cat}$, *i.e.* it is given by the following data

- For every object $A \in |\mathbf{B}|$, a category $\mathcal{F}A$ (usually called a *fibre*).
- For every arrow $f : A \to B$ in $\mathbf{B}$, a *reindexing* functor $\mathcal{F}(f) = f^* : \mathcal{F}B \to \mathcal{F}A$, together with natural isomorphisms $\gamma_A : 1_{\mathcal{F}A} \cong 1_A^*$ and $\delta_{f,g} : f^* \circ g^* \cong (g \circ f)^*$ satisfying the following coherence conditions: for $u : A \to B$, $v : B \to C$ and $w : D \to A$ in $\mathbf{B}$

$$
\begin{aligned}
\delta_{u,1_B} \circ u^*\gamma_B &= 1_{u^*} \\
\delta_{1_A,u} \circ \gamma_A u^* &= 1_{u^*} \\
\delta_{w,v\circ u} \circ w^*\delta_{u,v} &= \delta_{u\circ w,v} \circ \delta_{w,u}v^* \; : w^* \circ u^* \circ v^* \xrightarrow{\cdot} (v \circ u \circ w)^*
\end{aligned}
$$

2.2. REMARK. The coherence conditions above express associativity and identity laws. Their role will become clear with Prop. 2.10.iii. Quite often these isomorphisms are actual identities, in which case we have a *strict* indexed category, which amounts simply to a functor $\mathcal{F} : \mathbf{B}^{op} \to \mathit{Cat}$. This simpler version is the only one considered in [BGT91]. Even when this case is seemingly standard for the examples in *ibid.*, the more general case is important, at least for the definition of completeness conditions – see §2.3.

2.3. EXAMPLES.

(i) (*Many-sorted sets*) Consider the following functor $SS : \mathit{Set}^{op} \to \mathit{Cat}$

$$
\begin{aligned}
SS(I) &= \mathit{Set}^I \\
SS(f : I \to J) &= (X : J \to \mathit{Set}) \mapsto (X \circ f : I \to \mathit{Set})
\end{aligned}
$$

Note that the objects of a fibre $SS(I)$ correspond to families of sets, as remarked before. The functor $SS(f : I \to J)$ performs reindexing along f. The coherent isomorphisms are simply identities.

(ii) (*Many-sorted algebraic signatures*) Consider the functor $(\_)^+ : \mathit{Set} \to \mathit{Set}$, which assigns to a set S the free semigroup it generates, *i.e.* the set S^+ of all finite non-empty sequences of elements of S. The functor $\mathcal{AS} = \mathcal{SS} \circ ((\_)^+)^{op} : \mathcal{S}et^{op} \to \mathcal{C}at$ is then an indexed category; its fibres $\mathcal{AS}(S)$ correspond to S-sorted algebraic signatures, *i.e.* for every non-empty sequence $s_1, \ldots, s_n \in S^+$ (regarded as arity or rank $s_1, \ldots, s_{n-1} \to s_n$), a set of operation symbols (of that rank). A reindexing functor $\mathcal{AS}(f : S \to S')$ transforms S'-sorted signatures into S-sorted signatures by renaming sorts according to f.

For an ordinary category, we can deal with some of its structure, e.g. (co)limits, in terms of functors and natural transformations; these are required to define the important notion of adjunction. Similarly, in order to talk about structure in an indexed category, we need the notions of indexed functor and indexed natural transformation, so that we can define indexed adjunctions. We will denote adjunctions (standard, indexed and fibred) by $F \dashv G : \mathbf{B} \to \mathbf{A}$ (via η, ϵ), where $F : \mathbf{A} \to \mathbf{B}$ is left adjoint to $G : \mathbf{B} \to \mathbf{A}$ (with unit η and counit ϵ, which we will frequently leave implicit) [Mac71, Thm.2.(v),p.81].

2.4. Definition. Let $\mathcal{F} : \mathbf{B}^{op} \to \mathcal{C}at$ and $\mathcal{G} : \mathbf{B}^{op} \to \mathcal{C}at$ be indexed categories.

- An *indexed functor* $\mathcal{H} : \mathcal{F} \to \mathcal{G}$ consists of:

  (i) For every $A \in |\mathbf{B}|$, a functor $\mathcal{H}(A) : \mathcal{F}(A) \to \mathcal{G}(A)$

  (ii) For every $u : A \to B$, a natural isomorphism $\phi_u : \mathcal{G}(u) \circ \mathcal{H}(B) \dot{\to} \mathcal{H}(A) \circ \mathcal{F}(u)$, satisfying the following coherence conditions: for $u : A \to B$, $v : B \to C$:

$$\begin{aligned} \phi_{1_A} \circ H(A)\gamma_A &= \gamma_A H(A) \\ \phi_{v \circ u} \circ H(A)\,\delta_{u,v} &= \delta_{u,v} H(C) \circ u^* \phi_v \circ \phi_u v^* \end{aligned}$$

- An *indexed natural transformation* $\alpha : \mathcal{H} \Rightarrow \mathcal{H}'$, for indexed functors $\mathcal{H}, \mathcal{H}' : \mathcal{F} \to \mathcal{G}$, consists of a natural transformation $\alpha_A : \mathcal{H}(A) \dot{\to} \mathcal{H}'(A)$ (for every object $A \in |\mathbf{B}|$) such that for every $u : A \to B$, $\mathcal{G}(u)\alpha_B = \alpha_A \mathcal{F}(u)$ (modulo the ϕ_u's)

2.5. Remark. Having defined indexed functors and indexed natural transformations, the notion of indexed adjunction is then analogous to the standard notion of adjunction between categories. We can give the following description, which the reader might find more intuitive: given indexed functors $\mathcal{H} : (\mathcal{F} : \mathbf{B}^{op} \to \mathcal{C}at) \to (\mathcal{G} : \mathbf{B}^{op} \to \mathcal{C}at)$ and $\mathcal{H}' : (\mathcal{G} : \mathbf{B}^{op} \to \mathcal{C}at) \to (\mathcal{F} : \mathbf{B}^{op} \to \mathcal{C}at)$, $\mathcal{H}$ is an indexed left adjoint to $\mathcal{H}'$ iff:

- For every $A \in |\mathbf{B}|$, $\mathcal{H}_A \dashv \mathcal{H}'_A$.
- For every $u : A \to B$, the pair $(\mathcal{F}(u), \mathcal{G}(u))$ preserves the adjunctions, *i.e.* it is a (pseudo-)map of adjunctions from $\mathcal{H}_B \dashv \mathcal{H}'_B$ to $\mathcal{H}_A \dashv \mathcal{H}'_A$ similarly to [Mac71, p.97].

Thus, indexed categories over a given category $\mathbf{B}$, indexed functors and indexed natural transformations form a (2-)category $\mathcal{IC}at(\mathbf{B})$, with the evident 'fibrewise' notions of composition and identities, inherited from $\mathcal{C}at$.

2.2. Fibrations

Since the primary focus of this paper will be on fibrations instead of indexed categories, we now introduce fibred categories and show their equivalence to indexed categories, so as to set the scene for work within the 'fibred' context.

2.6. Definition (Fibrations and cofibrations). Consider a functor $p : \mathbf{E} \to \mathbf{B}$. We have:

(i) A morphism $f : X \to Y$ in $\mathbb{E}$ is (p-)*cartesian* (over a morphism $u : A \to B$ in $\mathbb{B}$) if $p\,f = u$ and, for every $f' : X' \to Y$ with $p\,f' = u \circ v$ in $\mathbb{B}$, there exists a unique morphism $\phi_{f'} : X' \to X$ such that $p\,\phi_{f'} = v$ and $f' = f \circ \phi_f$. Diagrammatically,

$$
\begin{tikzcd}
\mathbb{E} \arrow[dd, dotted, "p"] & X' \arrow[d, dashed, "\exists!\phi_{f'}"'] \arrow[dr, "f'"] & \\
 & X \arrow[r, "f"'] & Y \\
\mathbb{B} & pX' \arrow[d, "v"'] \arrow[dr, "pf'"] & \\
 & A \arrow[r, "u"'] & B
\end{tikzcd}
$$

Thus, we may say a cartesian f is a 'terminal lifting' of u. We call such f a cartesian *lifting* of u.

(ii) Dually, a morphism $g : X \to Y$ is (p-)*cocartesian* (over a morphism $u : A \to B$ in $\mathbb{B}$) if $p\,g = u$ and, for every $g' : X \to Y'$ with $p\,g' = w \circ u$ in $\mathbb{B}$, there exists a unique morphism $\psi_{g'} : Y \to Y'$ such that $p\,\phi_{g'} = w$ and $g' = \psi_{g'} \circ g$.

(iii) The functor $p : \mathbb{E} \to \mathbb{B}$ is called a *fibration* if for every $X \in |\mathbb{E}|$ and $u : A \to p\,X$ in $\mathbb{B}$, there is a cartesian morphism with codomain X above u, *i.e.* such that its image along p is u. $\mathbb{B}$ is then called the *base* of the fibration and $\mathbb{E}$ its *total category*. Dually, p is a *cofibration* if $p^{op} : \mathbb{E}^{op} \to \mathbb{B}^{op}$ is a fibration, *i.e.* for every $X \in |\mathbb{E}|$ and $u : pX \to B$ in $\mathbb{B}$, there is a cocartesian morphism with domain X above u. If p is both a fibration and a cofibration, it is called a *bifibration*.

(iv) For $A \in |\mathbb{B}|$, $\mathbb{E}_A$, the *fibre* over A, denotes the subcategory of $\mathbb{E}$ whose objects are above A and its morphisms, called (p-)*vertical*, are above 1_A.

2.7. Example (Family fibration). The following standard construction of a fibration over *Set* is described in [Bén85]. It provides a simple way of understanding the origin of (the terminology for) some fibred concepts. Every category $\mathbb{C}$ gives rise to a family fibration $f(\mathbb{C}) : Fam(\mathbb{C}) \to Set$. Objects of $Fam(\mathbb{C})$ are families $\{X_i\}_{i\in I}$ of $\mathbb{C}$-objects, i.e. $X : I \to |\mathbb{C}|$; morphisms $(u, \{f_i\}_{i\in I}) : \{X_i\}_{i\in I} \to \{Y_j\}_{j\in J}$ are pairs consisting of a function $u : I \to J$ (in *Set*) and a family of morphisms such that $f_i : X_i \to Y_{u(i)}$ in $\mathbb{C}$. $f(C)$ takes a family of objects to its indexing set and a morphism to its first component. $(u, \{f_i\}_{i\in I})$ is cartesian when every f_i is an isomorphism. $f(C)$ is then a fibration since given $u : I \to J$ and $\{Y_j\}_{j\in J}$, $(u, \{1_{Y_{u(i)}}\}) : \{Y_{u(i)}\}_{i\in I} \to \{Y_j\}_{j\in J}$ is cartesian above u.

If a functor $p : \mathbb{E} \to \mathbb{B}$ is a fibration, we will frequently display it as $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$. A choice of cartesian lifting for every appropriate morphism in $\mathbb{B}$ is called a *cleavage* for p (which is then a *cloven* fibration) and denoted by $\overline{(\_)}$, so that for $u : A \to pX$ in $\mathbb{B}$, $\overline{u}(X) : u^*(X) \to X$ is a cartesian morphism above u. Actually any fibration can be turned into a cloven one, using the axiom of choice to obtain a cleavage. In case we have for any pair of composable morphisms u and v that $\overline{u \circ v}(X) = \overline{u}(X) \circ \overline{v}(u^*(X))$ and $\overline{1_A}(X) = 1_X$, the cleavage is a *splitting* and the fibration is *split*. Dually, if p is a cofibration, given $X \in \mathbf{E}_A$ and $u : A \to B$ in $\mathbf{B}$, we denote $\underline{(u)} : X \to u_!(X)$ a cocartesian lifting of u (at X).

A cleavage for $\begin{array}{c}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{array}$ induces, for every $u : A \to B$ in $\mathbf{B}$, a *reindexing functor* $u^* : \mathbf{E}_B \to \mathbf{E}_A$ as follows: for $X \in |\mathbf{E}_B|$ let $u^*(X)$ be the domain of the cartesian morphism over u given by the cleavage, and for $f : X \to Y$ in $\mathbf{E}_B$ let $u^*(f) : u^*X \to u^*Y$ be the unique vertical morphism such that $f \circ \overline{u}(X) = \overline{u}(Y) \circ u^*(f)$.

Just as we did before with indexed categories, we will need notions of functors and natural transformations between fibrations.

2.8. Definition. Consider two fibrations $\begin{array}{c}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{array}$ and $\begin{array}{c}\mathbf{D}\\ \downarrow q\\ \mathbf{A}\end{array}$.

- A morphism $(H, K) : p \to q$ is given by a commutative square

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "H"] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\
\mathbf{B} \arrow[r, "K"'] & \mathbf{A}
\end{tikzcd}$$

  where H preserves cartesian morphisms, meaning that if f is p-cartesian, Hf is q-cartesian. (H, K) is called a *fibred 1-cell*; it determines a collection $\{H\,|_A : \mathbf{E}_A \to \mathbf{D}_{KA}\}_{A \in \mathbf{B}}$.

  When p and q are fibrations with the same base category $\mathbf{B}$, we may consider fibred 1-cells of the form $(H, 1_B) : p \to q$, which are then given by functor H such that $q \circ H = p$ and preserves cartesian morphisms. Such an H will be called a *fibred functor*, written $H : p \to q$.

- Given fibred 1-cells $(H, K), (G, L) : p \to q$, a *fibred 2-cell* from (H, K) to (G, L) is a pair of natural transformations $(\tau : H \dot\to G, \sigma : K \dot\to L)$ with τ above σ, meaning that $q\tau_X = \sigma_{pX}$ for every $X \in \mathbf{E}$. We will display such a fibred 2-cell as follows

$$\begin{tikzcd}
\mathbf{E} \arrow[r, bend left, "H"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\
\mathbf{B} \arrow[r, bend left, "K"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbf{A}
\end{tikzcd}$$

  and we will write it as $(\tau, \sigma) : (H, K) \Rightarrow (G, L)$. When p and q are fibrations with the same base category $\mathbf{B}$, we may consider 'vertical' 2-cells of the form $(\tau, 1_{1_B}) : (H, 1_B) \Rightarrow (G, 1_B)$ between fibred functors H and G, which amount then to natural transformations $\tau : H \dot\to G$ such that $q\tau = 1_p$. Such a 2-cell will be called a *fibred natural transformation*.

We thus obtain a (2-)category $\mathcal{F}ib(\mathbf{B})$ consisting of fibrations over $\mathbf{B}$, fibred functors between them (and fibred natural transformations), with compositions and identities inherited from $\mathcal{C}at$. More generally, considering fibrations over an arbitrary base category, fibred 1-cells (and fibred 2-cells) we have a (2-)category $\mathcal{F}ib$. For instance, the 'model part' of an institution morphism [GB90] is a fibred 1-cell, that is, a morphism in $\mathcal{F}ib$. Dually, we have (2-)categories $\mathcal{C}o\mathcal{F}ib$(resp. $\mathcal{C}o\mathcal{F}ib(\mathbf{B})$ of cofibrations (over $\mathbf{B}$) and (vertical) cofibred 1-cells and 2-cells.

2.9. Remark. Similarly to Remark 2.5, the definition of fibred 1-cells and 2-cells (resp. that of fibred functors and fibred natural transformations) determines what an adjunction between fibrations (resp. fibred adjunctions) is. We spell out fibred adjunctions: given fibrations $\begin{smallmatrix}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbf{D}\\ \downarrow q\\ \mathbf{B}\end{smallmatrix}$, an adjunction between them is given by a pair of fibred functors $F : p \to q$ and $G : q \to p$ together with fibred natural transformations $\eta : 1_{\mathbf{E}} \dot{\to} GF$ and $\epsilon : FG \dot{\to} 1_{\mathbf{D}}$ such that $F \dashv G$ (in the standard sense, with unit η and counit ϵ).

We will now show the correspondence between (cloven) fibrations and indexed categories, due to Grothendieck, which amounts to an equivalence between the (2-)categories $\mathcal{F}ib(\mathbf{B})$ and $\mathcal{I}\mathcal{C}at(\mathbf{B})$.

2.10. Proposition.
(i) *Every cloven fibration* $\begin{smallmatrix}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{smallmatrix}$ *gives rise to an indexed category* $\mathcal{F}_p : \mathbf{B}^{op} \to \mathcal{C}at$.

(ii) *Every indexed category* $\mathcal{F} : \mathbf{B}^{op} \to \mathcal{C}at$ *gives rise to a fibration* $p_{\mathcal{F}} : \mathcal{G}\mathcal{F} \to \mathbf{B}$ (**Grothendieck construction**).

(iii) *The above correspondences yield an equivalence of (2-)categories*

$$\mathcal{I}\mathcal{C}at(\mathbf{B}) \simeq \mathcal{F}ib(\mathbf{B})$$

so that $\mathcal{F}_{p_{\mathcal{F}}} \simeq \mathcal{F}$ *and* $p_{\mathcal{F}_p} \simeq p$.

Proof. (Sketch)
(i) Given a cloven fibration $p : \mathbf{E} \to \mathbf{B}$, we obtain an indexed category $\mathcal{F}_p : \mathbf{B}^{op} \to \mathcal{C}at$ as follows:

- For every $A \in |\mathbf{B}|$, $\mathcal{F}_p A = \mathbf{E}_A$.
- For every $u : A \to B$, a cleavage $\overline{(\_)}$ induces a reindexing functor $u^* : \mathbf{E}_B \to \mathbf{E}_A$ as described before Def. 2.8. The universal property of cartesian morphisms uniquely determines natural isomorphisms $\delta_{v,u} : v^* \circ u^* \dot{\to} (u \circ v)^*$ and $\gamma_A : 1_A \dot{\to} 1^*_{pA}$, which satisfy the coherence conditions in Def. 2.1.

(ii) Given an indexed category $\mathcal{F} : \mathbf{B}^{op} \to \mathcal{C}at$ we define the total category $\mathcal{G}\mathcal{F}$, consisting of:

Objects: $\langle A, a\rangle \in |\mathcal{G}\mathcal{F}|$ iff $A \in |\mathbf{B}|$ and $a \in |\mathcal{F}A|$. That is (using a hopefully self-explanatory dependent sum notation)

$$|\mathcal{G}\mathcal{F}| = \Sigma A : \mathrm{B}.\mathcal{F}A$$

Morphisms: $\langle f, g\rangle : \langle A, a\rangle \to \langle B, b\rangle$ iff $f : A \to B$ in $\mathbf{B}$ and $g : a \to f^*(b)$ in $\mathcal{F}A$. That is

$$\mathcal{G}\mathcal{F}(\langle A, a\rangle, \langle B, b\rangle) = \Sigma f : \mathbf{B}(A, B).\mathcal{F}A(a, f^* b)$$

Identity: $\langle 1_A, \gamma_A\rangle : \langle A, a\rangle \to \langle A, a\rangle$
Composition: Given $\langle f, g\rangle : \langle A, a\rangle \to \langle B, b\rangle$ and $\langle h, j\rangle : \langle B, b\rangle \to \langle C, c\rangle$, let

$$\langle h, j\rangle \circ \langle f, g\rangle = \langle h \circ f, \delta_{f,h}(c) \circ f^* j \circ g\rangle$$

Note that the coherence conditions of Def. 2.1 are required in order to show associativity of composition and the identity laws. The projection functor $p_{\mathcal{F}} : \mathcal{GF} \to \mathbf{B}$ which takes $\langle A, a\rangle$ to A (for objects and arrows) is then a fibration: for an arrow $u : A \to B$ in $\mathbf{B}$ and an object X in $\mathcal{F}B$, we can choose as cartesian lifting $\bar{u}(X) = \langle u, 1_{u^* X}\rangle$.

(iii) Simply observe that the fibres of $p_{\mathcal{F}}$ are $\mathcal{GF}_B = \mathcal{F}B$ and the action of the reindexing functors is the same in both fibrations and indexed categories (respectively). Note also that any pair of cleavages for a given fibration give rise to naturally isomorphic indexed categories.

□

2.11. Remarks.

- By duality, we get an analogous result relating cofibrations $p : \mathbf{E} \to \mathbf{B}$ and (pseudo-)functors $\mathcal{G} : \mathbf{B} \to \mathcal{C}at$ ('covariant indexed categories').
- The equivalence in the above proposition clearly restricts to one between split fibrations $\begin{array}{c}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{array}$ and functors $\mathcal{F} : \mathbf{B}^{op} \to \mathcal{C}at$ (strict indexed categories). This simplified version of indexed categories is the one considered in [BGT91], with similar assumptions about indexed functors (which then amount to natural transformations).

2.12. Remark. The 2-categorical aspect of the equivalence in Prop. 2.10.iii implies in particular a correspondence between indexed and fibred adjunctions. Thus, a fibred adjunction $F \dashv G : p \to q$ (between $\begin{array}{c}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{array}$ and $\begin{array}{c}\mathbf{D}\\ \downarrow q\\ \mathbf{B}\end{array}$) amounts to a family of adjunctions $\{F\,|_B \dashv G\,|_B : \mathbf{E}_B \to \mathbf{D}_B\}_{B\in|\mathbf{B}|}$ such that for every $u : B \to B'$, (u^{*p}, u^{*q}) is a (pseudo-)map of adjunctions from $F\,|_{B'} \dashv G\,|_{B'}$ to $F\,|_B \dashv G\,|_B$, *cf.* Remark 2.5.

2.13. Examples.

- The family fibration $f(\mathbf{C}) : \mathcal{F}am(\mathbf{C}) \to \mathcal{S}et$ results from applying the Grothendieck construction to the (strict) $\mathcal{S}et$-indexed category

$$\begin{array}{rcl} I & \mapsto & \mathcal{S}et^I \\ u : I \to J & \mapsto & \_ \circ u : \mathcal{S}et^J \to \mathcal{S}et^I \end{array}$$

- (Many-sorted algebraic signatures) Applying the Grothendieck construction to $\mathcal{AS} : \mathcal{S}et^{op} \to \mathcal{C}at$ from Ex. 2.3.ii we obtain the usual category of algebraic signatures **AlgSig**, whose objects are pairs $\langle S, \{\Sigma_r\}_{r\in S^+}\rangle$, where S is a set of sorts and each Σ_r is a set of operation symbols of rank r. A morphism $\langle f, g\rangle : \langle S, \{\Sigma_r\}_{r\in S^+}\rangle \to \langle S', \{\Sigma'_r\}_{r\in (S')^+}\rangle$ consists of a renaming of sorts $f : S \to S'$ and a family of operation-symbol renamings $g = \{g_r : \Sigma_r \to \Sigma'_{f^+(r)}\}_{r\in S^+}$ compatible with f.

For a functor $p : \mathbf{E} \to \mathbf{B}$, given a morphism $u : A \to B$ in $\mathbf{B}$, $X \in |\mathbf{E}_A|$ and $Y \in |\mathbf{E}_B|$, let

$$\mathbf{E}_u(X, Y) = \{f : X \to Y \text{ in } \mathbf{E} \mid pf = u\}$$

With this notation we have the following proposition:

2.14. **Proposition.** *Let $p : \mathbf{E} \to \mathbf{B}$ be a functor, $u : A \to B$ a morphism in $\mathbf{B}$, $X \in |\mathbf{E}_A|$ and $Y \in |\mathbf{E}_B|$.*

(i) *If p is a fibration then $\mathbf{E}_u(X,Y) \cong \mathbf{E}_A(X, u^*(Y))$ (naturally in X and Y).*

(ii) *If p is a cofibration then $\mathbf{E}_u(X,Y) \cong \mathbf{E}_B(u_!(X), Y)$ (naturally in X and Y).*

(iii) *If p is a fibration then*

$$p \text{ is a cofibration iff for every } u : A \to B \text{ in } \mathbf{B},\ u^* : \mathbf{E}_B \to \mathbf{E}_A \text{ has a left adjoint}$$

(The claims above assume a given (co)cleavage for the (co)fibration p.)

Proof. (i) and (ii) are straightforward consequences of the definition of cartesian and cocartesian morphisms respectively. For (iii),

$$\mathbf{E}_A(X, u^*(Y)) \cong \mathbf{E}_u(X,Y) \cong \mathbf{E}_B(u_!(X), Y)$$

which means that the 'coreindexings' are left adjoints to the corresponding reindexing functors, *i.e.* $u_! \dashv u^* : \mathbf{E}_B \to \mathbf{E}_A$, where $u_! : \mathbf{E}_A \to \mathbf{E}_B$ is determined dually to u^* in the proof of Prop. 2.10.i. □

Finally, we introduce the change-of-base construction which, given a functor $K : \mathbf{B} \to \mathbf{A}$ and a fibration with base $\mathbf{A}$, yields a fibration with base $\mathbf{B}$.

2.15. **Proposition.** *Given a fibration $q : \mathbf{D} \to \mathbf{A}$ and an arbitrary functor $K : \mathbf{B} \to \mathbf{A}$, consider a pullback diagram*

$$\begin{tikzcd} K^*(\mathbf{D}) \arrow[r, "q^*(K)"] \arrow[d, "K^*(q)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{D} \arrow[d, "q"] \\ \mathbf{B} \arrow[r, "K"'] & \mathbf{A} \end{tikzcd}$$

$K^*(q)$ is a fibration, with a morphism f in $K^*(\mathbf{D})$ being $K^*(q)$-cartesian iff $q^*(K)(f)$ is q-cartesian. The above diagram is therefore a morphism of fibrations (fibred 1-cell).

In the situation of the above proposition, we say that $K^*(q)$ is obtained by change of base from q along K. Note that for $B \in \mathbf{B}$, $K^*(\mathbf{D})_B \cong \mathbf{D}_{KB}$. Clearly, the same construction applies to cofibrations.

2.16. **Example.** Regarding the indexed categories $\mathcal{SS}$ and $\mathcal{AS}$ (Ex. 2.3) as fibrations, $p_{\mathcal{AS}}$ is obtained from $p_{\mathcal{SS}}$ by change of base along the functor $(\_)^+ : \mathcal{S}et \to \mathcal{S}et$. This example illustrates the fact that change-of-base for an indexed category $\mathcal{F} : \mathbf{B}^{op} \to \mathcal{C}at$ along a functor $K : \mathbf{A} \to \mathbf{B}$ is obtained by composition, $K^*(\mathcal{F}) \cong \mathcal{F} \circ (K)^{op}$.

This concludes the prerequisites required to deal with some notions of completeness for fibrations (resp. indexed categories) in terms of adjunctions in the (2-)categories $\mathcal{F}ib(\mathbf{B})$ (resp. $\mathcal{I}\mathcal{C}at(\mathbf{B})$), which we consider next.

2.3. Fibrewise completeness and cocompleteness

We present fibrewise notions of completeness and cocompleteness for fibrations (and thus for indexed categories) in terms of fibred adjunctions. First, let us recall the situation with categories. Let $\mathbf{I}$ be a small category. For a category $\mathbf{C}$, a diagram of type $\mathbf{I}$ in $\mathbf{C}$ amounts to a functor $D : \mathbf{I} \to \mathbf{C}$, which is an object in the functor category $\mathbf{C}^{\mathbf{I}}$. The diagonal functor $\Delta_{\mathbf{I}} : \mathbf{C} \to \mathbf{C}^{\mathbf{I}}$ takes an object $X \in |\mathbf{C}|$ to the constant functor $\Delta_{\mathbf{I}}(X) = (I \mapsto X)$. The category $\mathbf{C}$ has $\mathbf{I}$-limits/colimits iff $\Delta_{\mathbf{I}}$ has a right/left adjoint [Mac71, p.85]. Notice that the 'only if' direction requires the axiom of choice to give an assignment of limit/colimit to every diagram of type $\mathbf{I}$.

For the analogous situation with fibrations, we need the following auxiliary result.

2.17. Proposition. *Given a fibration $p : \mathbf{E} \to \mathbf{B}$ and a small category $\mathbf{I}$, $p^I : \mathbf{E}^I \to \mathbf{B}^I$ is a fibration.*

Proof. A natural transformation $\alpha : F \dot{\to} G : \mathbf{I} \to \mathbf{E}$ is p^I-cartesian iff every component is p-cartesian. □

This fibration of functor categories and the change-of-base construction are used in the following definition of fibred (co)limits, due to J. Bénabou.

2.18. Definition. For any small category $\mathbf{I}$, a fibration $p : \mathbf{E} \to \mathbf{B}$ has fibred $\mathbf{I}$-limits (resp. colimits) iff the fibred functor $\widehat{\Delta_I} : p \to \Delta_I^*(p^I)$, uniquely determined in the diagram below, has a fibred right (resp. left) adjoint $\widehat{\Delta_I} \dashv \widehat{\varprojlim}_I$

$$\begin{tikzcd}
\mathbf{E} \arrow[dr, "\widehat{\Delta_I}"] \arrow[drr, "\widetilde{\Delta_I}"] \arrow[ddr, "p"'] & & \\
& \Delta_I^*(\mathbf{E}) \arrow[r, "(p^I)^*(\Delta_I)"'] \arrow[d, "\Delta_I^*(p^I)"] & \mathbf{E}^{\mathbf{I}} \arrow[d, "p^I"] \\
& \mathbf{B} \arrow[r, "\Delta_I"] & \mathbf{B}^{\mathbf{I}}
\end{tikzcd}$$

where $\Delta_I : \mathbf{B} \to \mathbf{B}^I$ and $\widetilde{\Delta_I} : \mathbf{E} \to \mathbf{E}^I$ are the diagonal functors taking objects A to constant functors $(I \mapsto A)$. A dual definition accounts for cofibred $\mathbf{I}$-limits/colimits for a cofibration.

Then, considering the fibrewise formulation of fibred adjunctions in Remark 2.12, a fibration $\begin{array}{c}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{array}$ has fibred $\mathbf{I}$-limits/colimits if every fibre has $\mathbf{I}$-limits/colimits (in their usual sense in $\mathcal{C}at$) and the reindexing functors (of any cleavage) are $\mathbf{I}$-continuous/cocontinuous (i.e. preserve $\mathbf{I}$-limits/colimits). If we were considering only split fibrations (or equivalently, strict indexed categories), the reindexing functors would have to preserve the specified limits/colimits 'on the nose', *i.e.* up to equality rather than up to isomorphism as required in the above definition.

2.19. Example. The fibration resulting from applying the Grothendieck construction to the indexed category $\mathcal{SS} : \mathcal{S}et^{op} \to \mathcal{C}at$ of Ex. 2.3.i has fibred $\mathbf{I}$-limits and colimits (for every small $\mathbf{I}$): the fibres $\mathcal{SS}(I) = \mathcal{S}et^I$ have all small limits and colimits, given pointwise, which are therefore preserved by the reindexing functors $u^* = \_ \circ u$.

We just mention that change-of-base preserves fibred limits/colimits; see [Jac91, Her93].

3. Fibred adjunctions and completeness

In this section we address the completeness of the total category of a fibration in terms of the fibrewise completeness of the fibration and the completeness of its base category. This will be done using a characterization of adjunctions in $\mathcal{F}ib$ in terms of those in $\mathcal{F}ib(\mathbf{B})$ (for suitable $\mathbf{B}$) and those in $\mathcal{C}at$.

Let us make explicit the definition of adjunctions in $\mathcal{F}ib$ which, as already mentioned, is determined by the fact that $\mathcal{F}ib$ is a 2-category. We will call such adjunctions general fibred adjunctions.

3.1. Definition. Given $\begin{array}{c}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{array}$ and $\begin{array}{c}\mathbf{D}\\ \downarrow q\\ \mathbf{A}\end{array}$, a *general fibred adjunction* between them is given by pair of fibred 1-cells $(\tilde{F}, F) : p \to q$ and $(\tilde{G}, G) : q \to p$ together with a pair of fibred 2-cells $(\tilde{\eta}, \eta) : (1_{\mathbf{E}}, 1_{\mathbf{B}}) \Rightarrow (\tilde{G} \circ \tilde{F}, G \circ F)$ and $(\tilde{\epsilon}, \epsilon) : (\tilde{F} \circ \tilde{G}, F \circ G) \Rightarrow (1_{\mathbf{D}}, 1_{\mathbf{A}})$ such that

(i) $\tilde{F} \dashv \tilde{G} : \mathbf{D} \to \mathbf{E}$ via $\tilde{\eta}$, $\tilde{\epsilon}$ (in $\mathcal{C}at$)

(ii) $F \dashv G : \mathbf{A} \to \mathbf{B}$ via η,ϵ (in $\mathcal{C}at$)

(iii) p and q constitute a map of adjunctions between the two above, i.e. $p\tilde{\eta} = \eta p$ (or equivalently $q\tilde{\epsilon} = \epsilon q$)

Such a fibred adjunction will be displayed in the following way

$$\begin{tikzcd}
\mathbf{E} \arrow[r, shift left, "\tilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, "p"'] & \mathbf{D} \arrow[l, shift left, "\tilde{G}"] \arrow[d, "q"] \\
\mathbf{B} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{A} \arrow[l, shift left, "G"]
\end{tikzcd}$$

and written $(\tilde{F}, F) \dashv (\tilde{G}, G) : q \to p$. When the components of $\tilde{\eta}$ and $\tilde{\epsilon}$ are cartesian and the square $(\tilde{F}, F) : p \to q$ is a pullback, we shall call this situation a *cartesian fibred adjunction*. This terminology will be justified by Thm. 3.4.

In order to avoid going into heavy technical detail, we will simply state the main results about general fibred adjunctions we need. Full proofs may be found in [Her93], where the results are proved using mild 2-categorical reformulations of cartesian morphisms.

Firstly, change-of-base of a fibration along a left adjoint functor yields a cartesian fibred adjunction. More precisely, we have the following lemma:

3.2. **Lemma**. *Given* $\begin{array}{c}\mathbf{E}\\ \downarrow{\scriptstyle q}\\ \mathbf{B}\end{array}$ *and an adjunction* $F \dashv G : \mathbf{B} \to \mathbf{A}$ *via* η, ϵ, *change-of-base along* F *yields a cartesian fibred adjunction*

$$\begin{tikzcd}
F^*(\mathbf{E}) \arrow[r, shift left=1.5, "q^*(F)"] \arrow[r, phantom, "\perp"] \arrow[d, "F^*(q)"'] & \mathbf{E} \arrow[l, shift left=1.5, "\overline{G}"] \arrow[d, "q"] \\
\mathbf{A} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathbf{B} \arrow[l, shift left=1.5, "G"]
\end{tikzcd}$$

Proof. We only make explicit the cofree object. For $X \in |\mathbf{E}|$, $\overline{G}X$ is the unique object of $F^*(\mathbf{E})$ such that $q^*(F)(\overline{G}X) = \epsilon^*_{qX}(X)$ and $F^*(q)(\overline{G}X) = GqX$. Thus $\overline{G}X$ is determined by the reindexing functor ϵ^*_{qX}. □

3.3. **Remark**. Dually, change-of-base of a cofibration along a right adjoint yields cartesian cofibred adjunctions. This yields as a particular case Lemma 2 in [BGT91].

The above lemma proves one direction of the equivalence in the following theorem, which provides a factorization of general fibred adjunctions in terms of cartesian fibred adjunctions and fibred adjunctions. Cartesian fibred adjunctions are thus cartesian morphisms for a suitable fibration, with change-of-base providing a cleavage for it. Without further details, we state our main result on general fibred adjunctions, which we will apply later to obtain the purported characterisation of completeness for the total category of a fibration.

3.4. **Theorem**. *Given* $\begin{array}{c}\mathbf{E}\\ \downarrow{\scriptstyle p}\\ \mathbf{B}\end{array}$, $\begin{array}{c}\mathbf{D}\\ \downarrow{\scriptstyle q}\\ \mathbf{A}\end{array}$, $F \dashv G : \mathbf{A} \to \mathbf{B}$ *via* η, ϵ *and a fibred 1-cell* $(\widetilde{F}, F) : p \to q$ *as shown in the following diagram*

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "\widetilde{F}"] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\
\mathbf{B} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathbf{A} \arrow[l, shift left=1.5, "G"]
\end{tikzcd}$$

let $\widehat{F} : p \to F^*(q)$ *in* $Fib(\mathbf{B})$ *be the unique mediating functor in*

$$\begin{tikzcd}
\mathbf{E} \arrow[dr, "\widehat{F}"] \arrow[drr, bend left, "\widetilde{F}"] \arrow[ddr, bend right, "p"'] & & \\
& F^*(\mathbf{D}) \arrow[r, "q^*(F)"'] \arrow[d, "F^*(q)"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{D} \arrow[d, "q"] \\
& \mathbf{B} \arrow[r, "F"'] & \mathbf{A}
\end{tikzcd}$$

Then, the following statements are equivalent:

(i) *There exists* $\widetilde{G} : \mathbb{D} \to \mathbb{E}$ *such that* $\widetilde{F} \dashv \widetilde{G}$ *(in* $\mathcal{C}at$*) and* $(\widetilde{F}, F) \dashv (\widetilde{G}, G) : q \to p$ *(in* $\mathcal{F}ib$*).*

(ii) *There exists* $\widehat{G} : F^*(q) \to p$ *such that* $\widehat{F} \dashv \widehat{G}$ *(in* $\mathcal{F}ib(\mathbf{B})$*).*

Proof. We only show $(ii) \Rightarrow (i)$ since we will need the explicit description of the resulting $\widetilde{G}$ later on. By Lemma 3.2, there is a $\overline{G} : \mathbf{D} \to F^*(\mathbf{D})$. We obtain the desired right adjoint by composition of adjoints: $\widetilde{G} = \widehat{G} \circ \overline{G}$. □

By duality, we get the following statement about adjunctions between cofibrations.

3.5. Corollary. *Given cofibrations* $p : \mathbf{E} \to \mathbf{B}$ *and* $q : \mathbf{D} \to \mathbf{A}$*,* $F \dashv G : \mathbf{B} \to \mathbf{A}$ *via* η, ϵ *and a fibred 1-cell* $(\widetilde{G}, G) : p \to q$ *as shown in the following diagram*

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\widetilde{G}"] \arrow[d, "p"'] & \mathbb{D} \arrow[d, "q"] \\
\mathbf{B} \arrow[r, shift left, "G"] \arrow[r, phantom, "\top"] & \mathbf{A} \arrow[l, shift left, "F"]
\end{tikzcd}$$

let $\widehat{G} : p \to G^*(q)$ *in* $\mathcal{C}o\mathcal{F}ib(\mathbf{B})$ *be the unique mediating functor in*

$$\begin{tikzcd}
\mathbb{E} \arrow[dr, "\widehat{G}"] \arrow[drr, "\widetilde{G}"] \arrow[ddr, "p"'] & & \\
& G^*(\mathbf{D}) \arrow[r, "q^*(G)"] \arrow[d, "G^*(q)"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{D} \arrow[d, "q"] \\
& \mathbf{B} \arrow[r, "G"'] & \mathbf{A}
\end{tikzcd}$$

Then, the following statements are equivalent:

(i) *There exists* $\widetilde{F} : \mathbf{D} \to \mathbb{E}$ *such that* $\widetilde{F} \dashv \widetilde{G}$ *(in* $\mathcal{C}at$*) and* $(\widetilde{F}, F) \dashv (\widetilde{G}, G) : p \to q$ *(in* $\mathcal{C}o\mathcal{F}ib$*).*

(ii) *There exists* $\widehat{F} : G^*(q) \to p$ *such that* $\widehat{F} \dashv \widehat{G}$ *(in* $\mathcal{C}o\mathcal{F}ib(\mathbf{B})$*).*

Furthermore, the correspondence $\widehat{F} \leftrightarrow \widetilde{F}$ *is one-to-one.*

In order to apply the Thm. 3.4 to obtain completeness conditions for the total category of a fibration, we will need the following property of right adjoints in a slice category $\mathcal{C}at/\mathbf{B}$. The first part is essentially [Win90, Lemma 4.5] while the second part is a straightforward generalisation. $\mathcal{C}at^{\to}$ is the category of functors and commutative squares.

3.6. Lemma. (i) *Given* $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbf{B}\end{array}$, $\begin{array}{c}\mathbb{D}\\ \downarrow q\\ \mathbf{B}\end{array}$, *a 1-cell* $G : q \to p$ *in* $\mathcal{C}at/\mathbb{B}$*, if there is* $F : p \to q$ *such that* $F \dashv G$ *in* $\mathcal{C}at/\mathbb{B}$ *then* G *is a fibred 1-cell, i.e.* G *preserves cartesian morphisms.*

(ii) *Given* $\begin{smallmatrix}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{smallmatrix}$, $\begin{smallmatrix}\mathbf{D}\\ \downarrow q\\ \mathbf{B}\end{smallmatrix}$, *a 1-cell* $(\tilde{G}, G) : q \to p$ *in* $\mathcal{C}at^{\to}$, *if there is* $(\tilde{F}, F) : p \to q$ *such that* $(\tilde{F}, F) \dashv (\tilde{G}, G)$ *in* $\mathcal{C}at^{\to}$ *then* $(\tilde{G}, G)$ *is a fibred 2-cell, i.e.* $\tilde{G}$ *preserves cartesian morphisms.*

The above lemma means that 'right adjoints preserve cartesian morphisms'. Now we can state the following characterisation of limits in the total category of a fibration, as well as its dual for colimits and cofibrations.

3.7. Corollary. *Let* $\mathbf{I}$ *be a small category and* $\begin{smallmatrix}\mathbf{E}\\ \downarrow p\\ \mathbf{B}\end{smallmatrix}$ *be a fibration such that* $\mathbf{B}$ *has* $\mathbf{I}$*-limits. Then* p *has fibred* $\mathbf{I}$*-limits iff* $\mathbf{E}$ *has and* p *strictly preserves* $\mathbf{I}$*-limits.*

Proof. Apply Thm. 3.4 to the following data (recall $p^I : \mathbf{E}^I \to \mathbf{B}^I$ is a fibration by Prop. 2.17)

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "\tilde{\Delta}_I"] \arrow[d, "p"'] & \mathbf{E}^I \arrow[d, "p^I"] \\
\mathbf{B} \arrow[r, shift left, "\Delta_I"] \arrow[r, phantom, "\perp"] & \mathbf{B}^I \arrow[l, shift left, "\underleftarrow{\lim}"]
\end{tikzcd}$$

$\mathbf{E}$ has and p strictly preserves $\mathbf{I}$-limits means precisely that the above diagram can be completed to an adjunction $(\widetilde{\Delta_I}, \Delta_I) \dashv (\widetilde{\underleftarrow{\lim}}, \underleftarrow{\lim})$ in $\mathcal{C}at^{\to}$, which by Lemma 3.6.(ii) is an adjunction in $\mathcal{F}ib$. □

3.8. Corollary. *Let* $r : \mathbf{D} \to \mathbf{A}$ *be a cofibration such that* $\mathbf{A}$ *has* $\mathbf{I}$*-colimits. Then* r *has cofibred* $\mathbf{I}$*-colimits iff* $\mathbf{D}$ *has and* r *strictly preserves* $\mathbf{I}$*-colimits.*

Let us show how the construction of products in $\mathbf{E}$ (from Corollary 3.7) works. Let $\{X_i\}_{i\in I}$ be a set of objects in $\mathbf{E}$, such that $pX_i = A_i$. From Lemma 3.2, $\overline{\underleftarrow{\lim}}(\{X_i\}_{i\in I})$ is obtained by reindexing the family $\{X_i\}_{i\in I}$ along the corresponding instance of the counit of the adjunction $\Delta_I \dashv \underleftarrow{\lim}$, which is simply the limiting cone of the product in $\mathbf{B}$ of the A_i's. That is, we take every object X_i to the fibre over $\underleftarrow{\lim}(\{A_i\}_{i\in I})$. Then, following the construction in Thm. 3.4, we take the product of the reindexed objects within the fibre $\mathbf{E}_{\underleftarrow{\lim}(\{A_i\}_{i\in I})}$. So, for $X, Y \in |\mathbf{E}|$ (over A and B), their product is $\pi^*_{A,B}(X) \times (\pi'_{A,B})^*(Y)$ (over $A \times B$). This construction of limits and the analogous one for colimits agree with those of [BGT91], although we obtained them in a different way.

3.9. Remark.

(i) Corollary 3.7 yields a stronger version of [BGT91, Thm. 1]. More precisely, Thm. 1 asserts that if an indexed category $\mathcal{F} : \mathbf{B}^{op} \to \mathcal{C}at$ has fibred $\mathbf{I}$-limits and $\mathbf{B}$ has $\mathbf{I}$-limits, so does $\mathcal{GF}$. The corollary yields this implication plus (strict) preservation of $\mathbf{I}$-limits by p, and it also shows the converse, *i.e.* the hypotheses are necessary and not only sufficient.

(ii) Thm. 2 in *ibid.* asserts that if an indexed category $\mathcal{F} : \mathbf{B}^{op} \to \mathcal{C}at$ is such that every reindexing functor has a left adjoint has $\mathbf{I}$-colimits in every fibre and $\mathbf{B}$ has $\mathbf{I}$-colimits, so does $\mathcal{GF}$. Recall that a fibration is a bifibration (*i.e.* it is also a cofibration) iff every reindexing functor has a left adjoint (*cf.* Prop. 2.14.iii). Therefore the above mentioned theorem is a consequence of Corollary 3.8. As before, the corollary gives a converse to this theorem as well. Thus the two main results in *ibid.* can be shown as immediate consequences of our Thm. 3.4.

3.10. EXAMPLES.

(i) *Fam*($\mathbb{C}$) is complete whenever $\mathbb{C}$ is, by Corollary 3.7. In case $\mathbb{C}$ is cocomplete, $f(\mathbb{C})$ is also a cofibration and hence *Fam*($\mathbb{C}$) is cocomplete by Corollary 3.8. In any case, *Fam*($\mathbb{C}$) always has small sums, although they need not be preserved by $f(\mathbb{C})$. See [Jac93a] for further details on this fibration.

(ii) $\mathcal{GSS}$, the category of many-sorted sets (Ex. 2.3.i) is complete, since $p_{\mathcal{SS}}$ is fibred complete (Ex.2.19). Corollary 3.8 applies as well, essentially because *Set* is cocomplete – see [BGT91, Ex. 1] – and hence $\mathcal{GSS}$ is cocomplete.

(iii) The prototypical example of fibration in algebraic specifications is that of many-sorted algebras ([BGT91, Ex. 3]). Consider the functor $\mathcal{MSS} : \mathbf{AlgSig}^{op} \to \mathit{Cat}$ defined as follows:

$$\begin{aligned} \mathcal{MSS}(\Sigma) &= \mathbf{ALG}(\Sigma), \text{ the category of many-sorted } \Sigma\text{-algebras} \\ \mathcal{MSS}(\sigma : \Sigma \to \Sigma') &= \_|_{\sigma}, \text{ the usual } \sigma\text{-reduct functor} \end{aligned}$$

The total category $\mathcal{GMSS}$ is then that of many-sorted algebras (with explicitly given signatures) and homomorphisms between them. As explained in *ibid.* the completeness and cocompleteness of this category can be shown by applying the above corollaries. Of course, the proof that the hypotheses of the above corollaries hold in this case is not trivial, but it belongs to the theory of algebraic structure on categories (using either sketches or monads) – see [BW85].

Conclusion

The main aim of this paper was to show how results which are relevant in the area of algebraic specifications can be understood more deeply by considering them in the context of fibred category theory and analysing the situation at a fairly general level. In particular, the results in [BGT91] about completeness and cocompleteness of total categories of fibrations are obtained as direct consequences of Thm. 3.4, which expresses an essential property of adjunctions between fibrations.

A secondary aim, which due to lack of space we might not have achieved, was to introduce the reader to fibred categories and, through the application abovementioned, show their relevance in situations which arise in the theory of algebraic specifications, particularly in the area of institutions. We apologise to the reader for the paucity of examples and informal explanations. [Her93] contains a more thorough and detailed account of some of the concepts involved (although it cannot be regarded as a tutorial introduction to the area) as well as further applications of relevance to computer science (*e.g.* logical predicates).

We think the main application of fibrations in computer science should be the categorical understanding of logical systems, including logics of programs in particular. We hope that further developments in the area will make their significance more evident. [Win90] considers fibrations (and fibred adjunctions in particular) in this light, as guidance for the design of a proof system for labelled transition systems.

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References

[Bén85] J. Bénabou. Fibred categories and the foundation of naive category theory. *Journal of Symbolic Logic*, 50, 1985.

[BGT91] R. M. Burstall, J. A. Goguen, and A. Tarlecki. Some fundamental algebraic tools for the semantics of computation. part 3: Indexed categories. *Theoretical Computer Science*, 91:239–264, 1991.

[BW85] M. Barr and C. Wells. *Toposes, Triples and Theories.* Springer Verlag, 1985.

[GB90] J.A. Goguen and R.M. Burstall. Introducing institutions. Technical Report ECS-LFCS-90-106, Laboratory for Foundations of Computer Science, Dept. of Comp. Sci., Univ. of Edinburgh, 1990. to appear in JACM.

[Gra66] J. W. Gray. Fibred and cofibred categories. In S. Eilenberg, editor, *Proceedings of the Conference on Categorical Algebra.* Springer Verlag, 1966.

[Gro71] A. Grothendieck. Catégories fibrées et descente. In A. Grothendieck, editor, *Revêtements étales et groupe fondamental, (SGA 1), Exposé VI*, volume 224 of *Lecture Notes in Mathematics.* Springer Verlag, 1971.

[Her93] C. Hermida. *Fibrations, logical predicates and related topics.* PhD thesis, University of Edinburgh, 1993. forthcoming.

[Jac91] B. Jacobs. *Categorical Type Theory.* PhD thesis, Nijmegen, 1991.

[Jac93a] B. Jacobs. Comprehension categories and the semantics of type dependency. *Theoretical Computer Science*, to appear, 1993.

[Jac93b] B. Jacobs. Parameters and parameterization in specification. *Fundamenta Informaticae*, to appear, 1993.

[KS74] G.M. Kelly and R.H. Street. Review of the elements of 2-categories. In A. Dold and B. Eckmann, editors, *Category Seminar*, volume 420 of *Lecture Notes in Mathematics.* Springer Verlag, 1974.

[Mac71] S. MacLane. *Categories for the Working Mathematician.* Springer Verlag, 1971.

[Mit90] J. Mitchell. Type systems for programming languages. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics*. North Holland, 1990.

[MR91] Q. Ma and J. C. Reynolds. Types, abstraction and parametric polymorphism 2. In S Brookes, editor, *Math. Found. of Prog. Lang. Sem.*, volume 589 of *Lecture Notes in Computer Science*, pages 1–40. Springer Verlag, 1991.

[PS78] R. Pare and D. Schumacher. Abstract families and the adjoint functor theorems. In P.T. Johnstone and R. Pare, editors, *Indexed Categories and their Applications*, volume 661 of *Lecture Notes in Mathematics*. Springer Verlag, 1978.

[Win90] G. Winskel. A compositional proof system for a category of labelled transition systems. *Information and Computation*, 87:2 – 57, 1990.