

A NOTE ON FIBRATIONS AND ADJUNCTIONS

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ABSTRACT. Fibrations and Adjunctions rock.

1. PRELIMINARIES

In this section, we present the necessary background for this paper and take this opportunity to set our notation and conventions. We specifically cover two flavors of opfibrations: Grothendieck and Street. We also present, for each type of opfibration, a helpful lemma that relates colimits in the fibres to colimits in the total category.

Grothendieck Opfibrations. We recall some basic material from the theory of opfibrations; standard references include [4, 7, 2]. For our purposes, we center our presentation mostly around opfibrations.

Consider a functor $U: \mathbf{X} \rightarrow \mathbf{A}$. A morphism $\beta: x \rightarrow y$ in \mathbf{X} over a morphism $f = U(\beta): a \rightarrow b$ in \mathbf{A} is called **cocartesian** if and only if, for all $g: b \rightarrow b'$ in \mathbf{A} and $\gamma: x \rightarrow y'$ in \mathbf{X} with $U(\gamma) = g \circ f$, there exists a unique $\delta: y \rightarrow y'$ in \mathbf{X} such that $U(\delta) = g$ and $\gamma = \delta \circ \beta$:

$$(1) \quad \begin{array}{ccc} x & \xrightarrow{\beta} & y \\ & \searrow \gamma & \nearrow \exists! \delta \\ & & y' \\ & \searrow g \circ f = U\gamma & \nearrow \\ a & \xrightarrow{f=U\beta} & b \end{array} \quad \begin{array}{l} \text{in } \mathbf{X} \\ \\ \text{in } \mathbf{A} \end{array}$$

For any object $a \in \mathbf{A}$, we denote by \mathbf{X}_a the **fibre** of U over a , i.e. the subcategory of \mathbf{X} which consists of objects x above a , namely $U(x) = a$, and vertical morphisms β , namely $U(\beta) = 1_a$. The functor $U: \mathbf{X} \rightarrow \mathbf{A}$ is an **opfibration** if and only if, for all $f: a \rightarrow b$ in \mathbf{A} and $x \in \mathbf{X}_a$, there is a cocartesian morphism β with domain x above f ; it is called a **cocartesian lifting** of b along f . The category \mathbf{A} is called the **base** of the opfibration, and \mathbf{X} its **total category**. Of course, this is a dual notion to that of a **fibration**.

For any opfibration $U: \mathbf{X} \rightarrow \mathbf{A}$, assuming the axiom of choice we may select a cocartesian arrow over each $f: a \rightarrow b$ in \mathbf{A} and $x \in \mathbf{X}_a$, denoted by $\text{Cocart}(f, x): x \rightarrow f_!(x)$. Such a choice of cocartesian liftings is called a **cleavage** for U , which is then called a **cloven opfibration**; in this way, any opfibration can be assumed to be cloven. As a special case of the universal property, any arrow in the total category of an opfibration factorizes uniquely into a cocartesian morphism followed

by a vertical one:

$$(2) \quad \begin{array}{ccc} x & \xrightarrow{\gamma} & y \\ & \searrow \text{Cocart}(f,x) & \downarrow \delta \\ & & f_!x \\ \downarrow \gamma & & \downarrow \delta \\ a & \xrightarrow{f} & b \end{array} \quad \begin{array}{l} \text{in } \mathbf{X} \\ \\ \text{in } \mathbf{A}. \end{array}$$

The choice of cocartesian liftings in a cloven opfibration induces a so-called **reindexing functor** between the fibre categories

$$(3) \quad f_! : \mathbf{X}_a \rightarrow \mathbf{X}_b$$

for any $f : a \rightarrow b$ in the base category. It can be verified by the cocartesian lifting property that $(1_a)_! \cong 1_{\mathbf{X}_a}$ and $(f \circ g)_! \cong f_! \circ g_!$. If these isomorphisms are equalities, we have the notion of a **split** (op)fibration.

For what follows, it is necessary to clarify the relation between the existence of colimits in the total category of an opfibration to the existence of those colimits in the fibres in a coherent way. For more details, and a proof of the following result, see [5, Cor. 3.7].

Lemma 1.1. *Suppose \mathbf{J} is a small category and $U : \mathbf{X} \rightarrow \mathbf{A}$ is an opfibration. If the base \mathbf{A} has \mathbf{J} -colimits, the following are equivalent:*

- (a) *all fibres have \mathbf{J} -colimits, and the reindexing functors preserve them;*
- (b) *the total category \mathbf{X} has \mathbf{J} -colimits, and U (strictly) preserves them.*

The above formulation and its dual version relate opfibrations specifically with colimits and fibrations with limits. Notice that condition (a) is the usual formal definition of when an arbitrary opfibration has *opfibred \mathbf{J} -colimits*, which in principle does not require a \mathbf{J} -cocomplete base. The strict preservation of colimits is discussed in detail below 3.8.

Categories of opfibrations, depending on section 5. Now or then.

We'll have to decide on the 'right notion' of morphisms needed! Adjunction maps should remain fixed, start from there.

Street Opfibrations. While fibrations are well-studied and frequently used without hesitation, there is an “evilness” about them. The term “evil” is a tongue-in-cheek expression applied to any categorical concept that fails to transport across an equivalence of categories. The aspect of an opfibration $U : \mathbf{X} \rightarrow \mathbf{A}$ that fails occurs when, given any arrow $f : a \rightarrow b$ and cartesian lifting $\theta : x \rightarrow y$ where x is in the fiber of a , we require that Uy equals a . After rebasing \mathbf{X} with an equivalent category, it is entirely possible that the equality weakens to an isomorphism, meaning that the rebased version of U is no longer an fibration. In general, evil practices in category theory are morally dubious. Fibrations, however, frequently get a pass because in concrete examples, fibrations do satisfy this strict equality. However, there is a non-evil version of a fibration. I love the chatty style but also point of this paragraph. I'm wondering whether it should go to the intro though?

No strong opinion

Main references?

Definition 1.2. A functor $U: \mathbf{X} \rightarrow \mathbf{A}$ is a **Street opfibration** if, for any arrow $f: a \rightarrow b$ in \mathbf{A} and x above a , there exists a morphism $\theta: x \rightarrow y$ and an isomorphism $h: Ux \cong a$ such that $fh = U\theta$

Daniel, this is the 'easy' version, not the full definition. Please 'copy' grothendieck diagram from above and adjust here? the morphism up top can go anywhere not necessarily 'above' b

In the Street fibration, the evil equality is replaced by an isomorphism h . One can readily check that rebasing a *Street fibration* U with an equivalent category does, in fact, give another Street fibration.

Because we generalize our results in Section 2 from Grothendieck fibrations to Street fibrations, we need a generalized version of Lemma 1.1.

Lemma 1.3. Suppose \mathbf{J} is a small category and $U: \mathbf{X} \rightarrow \mathbf{A}$ is a Street (op)fibration. If \mathbf{A} has \mathbf{J} -(co)limits, the following are equivalent:

- (a) all fibres have \mathbf{J} -colimits and the reindexing functors preserve them;
- (b) \mathbf{X} has \mathbf{J} -(co)limits and U preserves them.

Proof. Every Street fibration can be factored as a Grothendieck fibration followed by an equivalence of categories. a reference is necessary here, and a diagram expressing that property could go here or earlier, discussing the relation of this new notion with the old. This lemma holds for Grothendieck fibrations plus categories having (co)limits and functors preserving them are both properties that satisfy the principle of equivalence. \square

Coreflections and laris. The main result of this paper is a correspondence between fibrations and adjoints, under certain assumptions. The adjunctions involved are a stricter version of the well-known *coreflections*, whose definition we recall.

Proposition 1.4. The following are equivalent for an adjunction $L \dashv U: \mathbf{X} \rightarrow \mathbf{A}$:

- (a) the left adjoint L is fully faithful;
- (b) the unit $\eta: 1_{\mathbf{A}} \Rightarrow UL$ is an isomorphism.

Under these conditions, $U\epsilon$ and ϵ_L are also isomorphisms.

The above clauses also imply that the induced comonad $(LU, L\eta_G, \epsilon)$ on \mathbf{X} is *idempotent*. Such an adjunction is called a **coreflection**, \mathbf{A} is a **coreflective** subcategory of \mathbf{U} and the right adjoint U is called the **coreflector**.

Connected to the issues of 'evilness' discussed already is a stricter version of a coreflector that actually appears in the desired correspondence between (Grothendieck) fibrations and adjoints. We use the acronym *lari*, originally introduced in [4], for the notion of a 'left adjoint, right inverse' functor. Explicitly, if $U: \mathbf{X} \rightarrow \mathbf{A}$ has a *lari* L , then $L \dashv U$ and $\eta: 1_{\mathbf{A}} = UL$. Equivalently, in this case we can say that U is a *rali*, namely it is a right adjoint, left inverse of some L — still the unit of the adjunction is the identity, with U now being a left inverse. Of course, under these conditions, $U\epsilon$ and ϵ_L are also identities $ULU = U$.

Analogously, if a functor has a *rari* or equivalently is a *lali*, then the counit of the adjunction is the identity; this comes as a special case of the dual notion of a *reflection* of categories.

Either here or in section 5

Recall that a *map of adjunctions* $(F, G): (L \dashv U) \rightarrow (L' \dashv U')$ is given by two functors F and G that make both squares below commute

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{F} & \mathbf{X}' \\ \begin{array}{c} \uparrow \\ L \end{array} \left(\dashv \right) \begin{array}{c} \downarrow \\ U \end{array} & & \begin{array}{c} \uparrow \\ L' \end{array} \left(\dashv \right) \begin{array}{c} \downarrow \\ U' \end{array} \\ \mathbf{A} & \xrightarrow{G} & \mathbf{A}' \end{array}$$

and moreover $G\eta = \eta'_G$ (or equivalently $\epsilon'_F = F\epsilon$). *Not sure if we should go 2-categorical after all. Haven't ever heard of 2-cells between maps of adjunctions. Leave out for now.*

We obtain a category \mathbf{Adj} with objects pairs *we need to decide if the adjoint is part of the object, probably yes* of adjoint functors (L, U) and morphisms maps of adjunctions between them. Moreover, there is a full subcategory \mathbf{Corefl} of coreflections, as well as a full subcategory \mathbf{Rali} of adjunctions whose right adjoint is left inverse (equivalently, whose left adjoint is right inverse). Notice that for the morphisms of the latter, the condition $G\eta = \eta'_G$ is automatically satisfied since the unit is the identity.

2. GROTHENDIECK FIBRATIONS AND RALIS

In this section, we investigate conditions under which opfibrations and adjunctions with identity unit relate to one another. The main result, Theorem 2.4, exhibits their bijective correspondence under the intersection of the respective conditions.

Initially we establish when an opfibration has a left adjoint; the following result is the dual of [4, Prop. 4.4].

Proposition 2.1. *Let $U: \mathbf{X} \rightarrow \mathbf{A}$ be an opfibration. Then U is a rali if (and only if) its fibers have initial objects which are preserved by the reindexing functors.*

Proof. Denote by \perp_a the initial object in each fiber \mathbf{X}_x above an object $a \in \mathbf{A}$ in the base category. Define a left adjoint L of U as follows:

$$\begin{array}{ccc} L: \mathbf{A} & \longrightarrow & \mathbf{X} \\ a & \longmapsto & \perp_a \\ f \downarrow & & \downarrow Lf \\ b & \longmapsto & \perp_b \end{array} \quad \begin{array}{c} \swarrow \text{Cocart}(f, \perp_a) \\ \searrow \\ f_!(\perp_a) \end{array} \quad \begin{array}{c} \longleftarrow \sim \\ \longleftarrow f_!(\perp_a) \end{array}$$

where the bottom isomorphism is the unique one induced by the fact that each $f_!$ preserves initial objects between the respective fibers. This assignment is strictly functorial due to the universal properties of cocartesian liftings. Notice that by definition, $La = \perp_a$ is in the fibre \mathbf{X}_a , which means $ULa = a$ for any object $a \in \mathbf{A}$.

To show this determines a left adjoint of U , it suffices to establish a natural bijection $\mathbf{X}(La, x) \cong \mathbf{A}(a, Ux)$ for any $a \in \mathbf{A}$, $x \in \mathbf{X}$. Indeed, each morphism in the

total category $k: La \rightarrow x$ above $f = Uk$, factorizes uniquely (2) as

$$\begin{array}{ccc}
 \perp_a & \xrightarrow{k} & x \\
 & \searrow \text{Cocart}(f, \perp_a) & \uparrow s \\
 & & f!(\perp_a) \\
 \downarrow \text{Y} & & \downarrow \text{Y} \\
 a & \xrightarrow{f} & Ux
 \end{array}
 \quad \begin{array}{l} \text{in } \mathbf{X} \\ \\ \text{in } \mathbf{A} \end{array}$$

but since $f!(\perp_a) \cong \perp_{Ux}$ is the initial object in the fibre above Ux , the morphism s is in fact unique. As a result, every k uniquely corresponds to some f in that way, and this isomorphism is natural.

Finally, the right adjoint U is indeed a left inverse of L , namely the unit of the adjunction $\eta: 1_{\mathbf{A}} \rightarrow UL$ is the identity natural transformation: $ULa = a$ and moreover $\mathbf{X}(La, La) \cong \mathbf{A}(a, ULa)$ ensures that the identity \perp_a corresponds to the identity on a . \square

The ‘only if’ direction of the statement is not needed for our purposes. Notice that in particular, since U is a right adjoint, it ends up preserving all limits that exist in \mathbf{X} ; this makes the above condition for its existence look slightly unintuitive.

The dual result states that a fibration is a lali, or equivalently has a rari namely the counit of the adjunction is the identity, if and only if the fibers have terminal objects that are preserved by the reindexing functors. Due to Lemma 1.1, we can express both these results as follows.

Corollary 2.2. *An (op)fibration is a (right) left adjoint left inverse if the base category has and the functor strictly preserves the (initial) terminal object.*

Next, we examine when a right adjoint left inverse has an opfibration structure. The relevant colimits in this case are pushouts, and the following result provides sufficient conditions in terms of those. Recall the discussion regarding strict preservation of colimits in Section 1.

Proposition 2.3. *Suppose that $U: \mathbf{X} \rightarrow \mathbf{A}$ is a rali, with left adjoint right inverse L . Then U is an opfibration if \mathbf{X} and \mathbf{A} have chosen pushouts such that U strictly preserves them.*

Proof. First of all, suppose that the rali $U: \mathbf{X} \rightarrow \mathbf{A}$ preserves pushouts, which are chosen such that pushouts of a morphism along an identity is the morphism itself. We will show it has the structure of a cloven Grothendieck opfibration. Indeed, take a morphism $f: a \rightarrow b$ in \mathbf{A} and an object $x \in \mathbf{X}$ above a , as in

$$\begin{array}{ccc}
 x & & \text{in } \mathbf{X} \\
 \downarrow U & & \\
 a & \xrightarrow{f} & b \\
 & & \text{in } \mathbf{A}.
 \end{array}$$

Define the cocartesian lifting of x along f to be the horizontal dashed arrow to the (chosen) pushout of the following diagram in \mathbf{X}

$$\begin{array}{ccc} LUx & \xrightarrow{Lf} & Lb \\ \varepsilon_a \downarrow & & \downarrow \\ x & \dashrightarrow & x +_{LUx} Lb \end{array}$$

where $\varepsilon: LU \Rightarrow 1_{\mathbf{X}}$ is the counit of the adjunction $L \dashv U$. To verify this is a cocartesian lifting, first of all it must be mapped to f via U : if we apply U to the above square, using the facts that $U\varepsilon_x = 1_a$, $ULf = f$ (by Proposition 1.4) and

$$U(x +_{LUx} Lb) = Ux +_{ULUx} ULb = a +_a b$$

since U strictly preserves pushouts, the resulting colimit diagram

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ 1_a \downarrow & & \downarrow \\ a & \dashrightarrow & a +_a b = b \end{array}$$

is the pushout of f along the identity, namely f due to our choice. Moreover, the universal property (1) of the proposed cocartesian lifting follows from universality of pushouts thus the one direction of the proof is complete. \square

Combining Propositions 2.1 and 2.3, we obtain the following result that establishes certain conditions under which a Grothendieck opfibration structure on a functor corresponds to a rali structure and vice versa.

Theorem 2.4. *Suppose that \mathbf{X} and \mathbf{A} have chosen pushouts and initial objects, and a functor $U: \mathbf{X} \rightarrow \mathbf{A}$ strictly preserves them. Then U is a right adjoint left inverse if and only if U is an opfibration.*

3. STREET FIBRATIONS AND COREFLECTIONS

In this section, we trod the same path as in Section 2 yet shed all strictness of the preservation of colimits and work with Street opfibrations instead of Grothendieck opfibrations.

Generalising Proposition 2.1, we obtain the following.

Proposition 3.1. *Let $U: \mathbf{X} \rightarrow \mathbf{A}$ be a Street opfibration. Then U is a coreflector if its fibers have initial objects which are preserved by the reindexing functors.*

Proof. We define L exactly as in Proposition 2.1.

The unit of the adjunction $L \dashv U$ is the identity $a \rightarrow ULa = U\perp_a = a$. The counit is the initial map $\perp_{Ux} = LUx \rightarrow x$. The triangle identities are satisfied as well. The first is the composite

$$La \xrightarrow{L\eta_a} LULa \xrightarrow{\epsilon_{La}} La$$

which is given by $\perp_a \rightarrow \perp_a \rightarrow \perp_a$. The second is the composite

$$Ux \xrightarrow{\eta_{Ux}} ULUx \xrightarrow{U\eta_x} Ux$$

which is given by $Ux \rightarrow Ux \rightarrow Ux$.

Because the unit of the adjunction is an isomorphism, L is full and faithful [3, Prop. 1.3] making U a coreflector. \square

Applying Lemma 1.3, we can transport the assumptions on the fibers to the base category of our Street fibration.

Corollary 3.2. *A Street (op)fibration is a coreflector if the base category has and the functor preserves the (initial) terminal object.*

Generalising Proposition 2.3, we obtain the following result.

Proposition 3.3. *Suppose that $U: \mathbf{X} \rightarrow \mathbf{A}$ is a coreflector. Then U is a Street opfibration if \mathbf{X} has pushouts and U preserves them.*

Proof. Fix an arrow $f: a \rightarrow b$ in \mathbf{A} and an object x in the fibre of a . We claim that \hat{f} , defined as the pushout of Lf along the counit ϵ_x

$$\begin{array}{ccc} LUx & \xrightarrow{Lf} & Lb \\ \downarrow \epsilon & & \downarrow \\ x & \xrightarrow{\hat{f}} & x +_{LUx} Lb \end{array}$$

is the cocartesian lift. There is a string of isomorphisms

$$U(x +_{LUx} Lb) \cong Ux +_{ULUx} Lb \cong Ux +_{Ux} b \cong b$$

whose composite we call h . Then

$$\begin{array}{ccc} & U(x +_{LUx} Lb) & \\ & \nearrow U\hat{f} & \downarrow h \\ a & \xrightarrow{f} & b \end{array}$$

commutes, and so \hat{f} is an appropriate lift. It remains to show that \hat{f} is cocartesian.

Consider a \mathbf{X} -arrow $g: x \rightarrow y$ with a \mathbf{A} -arrow $\theta: U(x +_{LUx} Lb) \rightarrow Uy$ so that $\theta U\hat{f} = Ug$. Can we uniquely lift θ ? Set up the diagram

$$\begin{array}{ccc} LUx & \xrightarrow{Lf} & Lb \\ \downarrow \epsilon_x & & \downarrow \\ x & \xrightarrow{\hat{f}} & x +_{LUx} Lb \end{array} \quad \begin{array}{c} \searrow g' \\ \downarrow \\ \searrow g \\ y \end{array}$$

where $g' := \epsilon_y L\theta Lh^{-1}$. To show the outer square commutes, it suffices to show that $g'Lf$ and $g\epsilon_x$ have the same image under the adjunction homset correspondence.

We have

$$g' \circ Lf = \epsilon_y \circ L\theta \circ Lh \circ Lf \mapsto U\epsilon_y \circ UL\theta \circ ULh \circ ULf \circ \eta_{ux} = \theta \circ h \circ f = \theta \circ U \circ \hat{f} = Ug$$

and

$$g\epsilon_x \mapsto Ug \circ \eta_{Ux} = Ug$$

□

We now intersect the hypothesis of Propositions Proposition 3.1 and Proposition 3.3 to provide the main result of this section.

Theorem 3.4. *Let $U: \mathbf{X} \rightarrow \mathbf{A}$ be a functor such that \mathbf{A} and \mathbf{X} have chosen initial objects and pushouts and U preserves them. Then U is a coreflector if and only if U is a Street opfibration.*

Choice of colimits. In what follows, and in particular for our main Theorem 2.4, we often require that certain colimits must be *strictly* preserved. Whereas such an uncommon condition sounds ‘evil’ at first, we remark that it is the necessary such in order to match to the inherent ‘evilness’ of Grothendieck fibrations. Moreover, in Section 3 we examine the non-strict context which then naturally matches to the notion of a Street fibration as discussed above.

In more detail, assuming the axiom of choice we can regard any category with (co)limits as having *chosen* ones, in the sense of choosing a specific adjoint (out of all isomorphic ones) to the constant diagram functor:

$$\begin{array}{ccc} & \text{colim}_J & \\ & \downarrow \perp & \\ \mathbf{C} & \xrightarrow{\Delta_c} & [\mathbf{J}, \mathbf{C}] \\ & \uparrow \perp & \\ & \text{lim}_J & \end{array}$$

Some categories, like **Set**, even have a canonical choice corresponding to well-known constructions of (co)limits of sets. In general, a functor $F: \mathbf{C} \rightarrow \mathbf{D}$ between categories with colimits, for example, preserves them when the following diagram

$$(4) \quad \begin{array}{ccc} [\mathbf{J}, \mathbf{C}] & \xrightarrow{\text{colim}} & \mathbf{C} \\ \downarrow [\mathbf{J}, F] & \cong & \downarrow F \\ [\mathbf{J}, \mathbf{D}] & \xrightarrow{\text{colim}} & \mathbf{D} \end{array}$$

which is the *mate* of the commutative square with Δ ’s, commutes up to natural isomorphism. The following two lemmas (due to Steve Lack) present two natural settings where functors between categories with chosen (co)limits *strictly* preserve them; evidently, such a thing is to be expected only when the limits in the categories have been both previously constructed from chosen limits in some fixed category.

Lemma 3.5. *Suppose \mathbf{C} is a category with chosen (co)limits (of any class). Then for any two categories \mathbf{A} and \mathbf{B} and any functor $F: \mathbf{A} \rightarrow \mathbf{B}$ between them, the pre-composition functor*

$$\begin{aligned} F^*: [\mathbf{B}, \mathbf{C}] &\longrightarrow [\mathbf{A}, \mathbf{C}] \\ \mathbf{B} \xrightarrow{H} \mathbf{C} &\longmapsto \mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{H} \mathbf{C} \end{aligned}$$

strictly preserves the chosen (co)limits.

Proof. This follows from the pointwise construction of colimits in functor categories. \square

As a particular case, the following result concerns our motivating example.

Corollary 3.6. *There is a canonical choice of (co)limits in \mathbf{Grph} inherited from those in \mathbf{Set} so that the domain and codomain functors $\mathbf{Grph} \rightarrow \mathbf{Set}$ strictly preserve all limits and colimits.*

Proof. Consider the two functors $1, 2: \mathbf{1} = \{\bullet\} \longrightarrow \{1 \rightrightarrows 2\} = \mathbf{2}$. Choosing the canonical (co)limits in \mathbf{Set} , by Lemma 3.5 we obtain two functors

$$\mathbf{Grph} = [\mathbf{2}, \mathbf{Set}] \begin{array}{c} \xrightarrow{\text{dom}} \\ \xleftarrow{\text{cod}} \end{array} [\mathbf{1}, \mathbf{Set}] = \mathbf{Set}$$

that strictly preserve them. \square

The following case again follows from a construction of chosen limits in common ground; colimits adhere to a dual result.

Lemma 3.7. *Suppose \mathbf{C} and \mathbf{D} have chosen limits and $F: \mathbf{C} \rightarrow \mathbf{D}$ is an arbitrary functor. Then the comma category $F \downarrow \mathbf{D}$ can be equipped with limits in such a way that both projections $\mathbf{C} \leftarrow F \downarrow \mathbf{D} \rightarrow \mathbf{D}$ strictly preserve them.*

Proof. This follows from the canonical construction of limits in comma categories, as found e.g. in [1, §2.16]. \square

Finally, another class of functors which is notable when it comes to strict preservation of (co)limits are (op)fibrations. In more detail, there is no loss of generality for a colimit-preserving opfibration if we assume that colimits are in fact preserved on-the-nose, whereas such a statement does definitely not apply to arbitrary functors. This is due to the following lemma, which is a special case of a more general fact: the embedding of \mathbf{OpFib} in the 2-category \mathbf{OpFib}_{\sim} where 2-cells are squares filled with isomorphisms, is locally fully faithful and essentially surjective (also for fibs...references...[6]?). We thank Claudio Hermida for these observations.

Lemma 3.8. *Suppose U and Q are opfibrations, and there is a natural isomorphism*

$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ U \downarrow & \phi \cong & \downarrow Q \\ A & \xrightarrow{F} & B \end{array}$$

Then ϕ factors as a commutative square composed by an isomorphism $\hat{\phi}$, as in

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{H} \\ \hat{\phi} \cong \\ \xrightarrow{\hat{H}} \end{array} & Y \\ U \downarrow & & \downarrow Q \\ A & \xrightarrow{F} & B \end{array}$$

As a result, if U is an opfibration that preserves J -colimits for some small J , we can factor the natural isomorphism (4), where the left leg is also a fibration, as

$$\begin{array}{ccc}
 [J, X] & \begin{array}{c} \xrightarrow{\text{colim}} \\ \cong \\ \xleftarrow{\text{colim}} \end{array} & X \\
 \downarrow [J, U] & & \downarrow U \\
 [J, A] & \xrightarrow{\text{colim}} & A
 \end{array}$$

essentially changing the choice of colimits in the total category and establishing that Q now strictly preserves them. In a dual way, we may assume that any fibration strictly preserves chosen limits, if it preserves limits in the ordinary sense.

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