Graphing a rational function.

Let's graph the function

$$f(x) = \frac{x^2 + 5x + 4}{x^2 - 4}$$

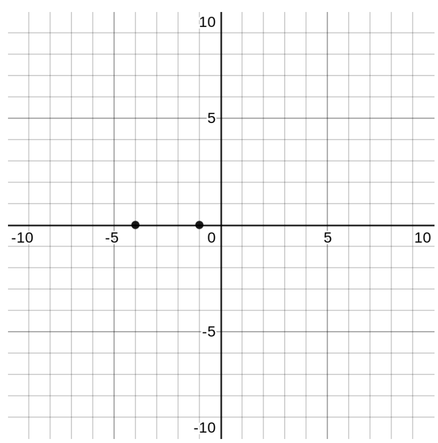
The overall strategy will go like this.

(1) Find zeros
(2) Find vertical and horizontal asymptotes
(3) Find function behavior near vertical asymptotes
(4) Graph the function

Step 1. Let's first find the zeros of f. The function is zero only when the numerator is zero. In other words, $f(x) = 0$ for any inputs x that make the equation $x^2 + 5x + 4 = 0$ true. So let's solve that:

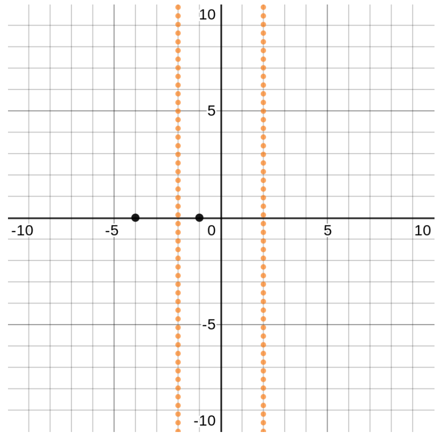
$$\begin{aligned} 0 &= x^2 + 5x + 4 \\ &= (x + 4)(x + 1) \end{aligned}$$

which gives that $x = -1$ and $x = -4$. These are our zeros, so let's plot them.



Step 2. Let's find the vertical and horizontal asymptotes: vertical first.

A vertical asymptote is like the pole that a function climbs as it explodes up to positive infinity or down to negative infinite. These "poles" occur wherever the denominator of a rational function is zero (you can't divide by zero!). Mathematically, this means that we need to find any input x such that the denominator of $f(x)$ is zero, which equationally is given by $x^2 - 4 = 0$. You can easily see that this happens at $x = \pm 2$. Graph these.



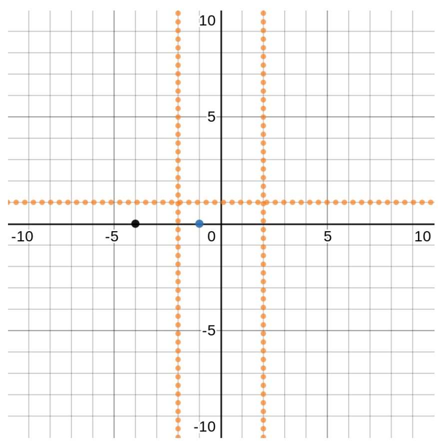
Next, find the horizontal aymptotes. Recall, to do this, we take our function

$$\frac{x^2 + 5x + 4}{x^2 - 4}$$

and ignore all of the terms except for the leading terms in both the numerator, giving

$$\frac{x^2}{x^2}$$

. But this is simply 1 because the x^2's cancel each other out. Thus, the horizontal asymptote is the line $y = 1$. Let's add that to our graph:



Step 3. Finally, it's time to graph the actual function. The vertical asymptotes separate the graph of f into three "stripes". Usually the outside stripes are easier to graph than the middle stripe, so let's start there.

The left stripe is split into an upper half and a lower half by the horizontal aymptote. The graph will only live in either the upper half or lower half of the graph. We can find out which by plugging in an x-value slightly to the left of the vertical asymptote and seeing if the output lands in the upper or lower half. But this is already done because of the root of f that we graphed earlier. That lives in the lower half, so our curve must live there also.

The right stripe is similar, so let's pick an x-value just to the right of the vertical asymptote and plug it into f. Let's try $x = 3$. We get that

$$f(3) = \frac{(3)^2 + 5(3) + 4}{(3)^2 - 4} = 28/5.$$

This is above the vertical asymptote, so the curve will live in the upper half in the right half.

The middle stripe requires us to plug in points that are just a shade inside the verticle asymptotes. Let's pick a point that's just to the right of the left vertical asymptotes $x = -2$ first. We already saw that $x = -1$ is a root, so that doesn't give us any information about whether the function runs up to positive infinity or down to negative infinity. We need to choose a closer point, like $x = -3/2$. Plugging this in, we get

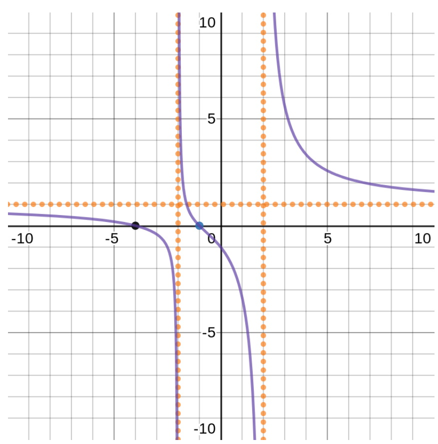
$$f(-3/2) = \frac{(-3/2)^2 + 5(-3/2) + 4}{(-3/2)^2 - 4}$$

which I'll let you compute, but I will say that it's positive. So the graph will run up to positive infinite along the right side of the left asymptote.

We need to repeat this for the left side of the right vertical asymptote. Let's plug in $x = 1$ and see if it's positive or negative:

$$f(1) = \frac{(1)^2 + 5(1) + 4}{(1)^2 - 4} = 10/-3$$

so it's negative, meaning that f runs down to negative infinity. We can now graph it, giving



And we're done.