How does $y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$ turn into $y = d_1 e^{ax} \cos bx + d_2 e^{ax} \sin bx$?

$\boxed{\text{ANS}}$

Use the following facts

$e^{ix} = \sin(x) i + \cos(x)$ & $e^{-ix} = \sin(-x)i + \cos(-x)$

$= -\sin(x) i + \cos(x)$

(use symmetries of sin & cos)

Then $y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$

$= c_1 e^{ax} e^{ibx} + c_2 e^{ax} e^{-ibx}$

$= e^{ax}(c_1 e^{ibx} + c_2 e^{-ibx})$

$\cancel{= e^{ax}(c_1 \sin(bx) i + c_1 \cos(bx) i}$

$= e^{ax}(c_1 \cos(bx) + c_1 i \sin(bx) + c_2 \cos(bx) - c_2 i \sin(bx))$

$= e^{ax}[(c_1 + c_2)\cos(bx) + (c_1 - c_2) i \sin(bx)]$

This gives new arbitrary constants

$d_1 = c_1 + c_2$ & $d_2 = (c_1 - c_2) i$

But we only want to work with the case that d_1 & d_2 are real, not complex. When does that happen? Only if c_1, c_2 are complex conjugates. So lets assume that's true, i.e. $c_1 = \alpha + \beta i$ & $c_2 = \alpha - \beta i$. Then $d_1 = c_1 + c_2 = (\alpha + \beta i) + (\alpha - \beta i) = 2\alpha$ & $d_2 = (c_1 - c_2) i = [(\alpha + \beta i) - (\alpha - \beta i)] i = 2\beta i^2 = -2\beta$. But 2α & -2β are arbitrary constants.