STRUCTURED COSPANS

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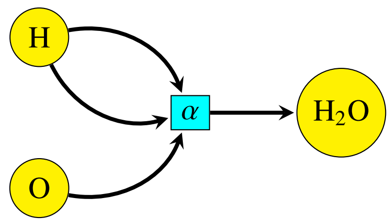
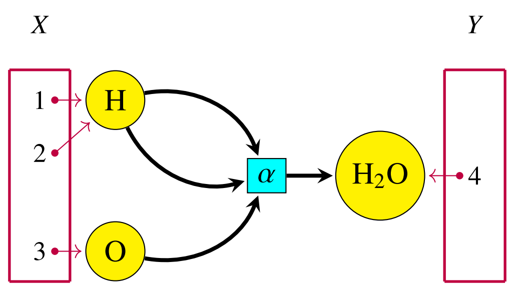
Abstract. Applied category theory has been instrumental in providing a framework in which to better understand networks. One kind of network is given by an 'open dynamical system' which is a 'dynamical system' equipped with the extra structure of prescribed 'inputs' and 'outputs'. These inputs and outputs then allow one open dynamical system to be composed with another whose inputs coincide with the outputs of the first, which naturally leads to open dynamical systems being morphisms in a category. Fong has developed a theory of 'decorated cospans' which is well-suited for describing isomorphism classes of open dynamical systems as morphisms in a symmetric monoidal category. Here we provide another compositional framework suitable for networks using 'double categories' which in some cases has advantages: one, that open dynamical systems no longer need be considered up to isomorphism class, and two, that we have greater flexibility in what we can allow a 2-morphism to be and thus what isomorphism classes of cospans with extra structure can consist of.

1. Introduction

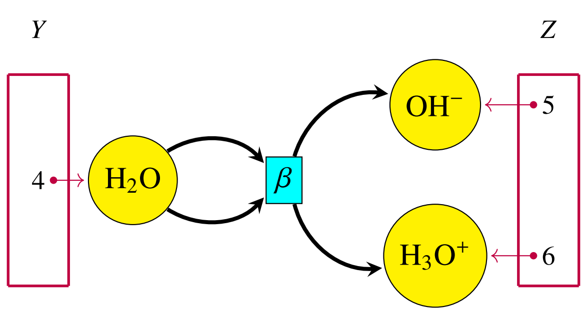
Category theory has played a prominent role in recent developments aimed at providing a suitable framework to better study networks. Networks arise all over various areas of science and come in many flavors and types. For example, the chemical reaction that takes two atoms of hydrogen and one atom of oxygen and produces a molecule of water can be represented as a 'Petri net':



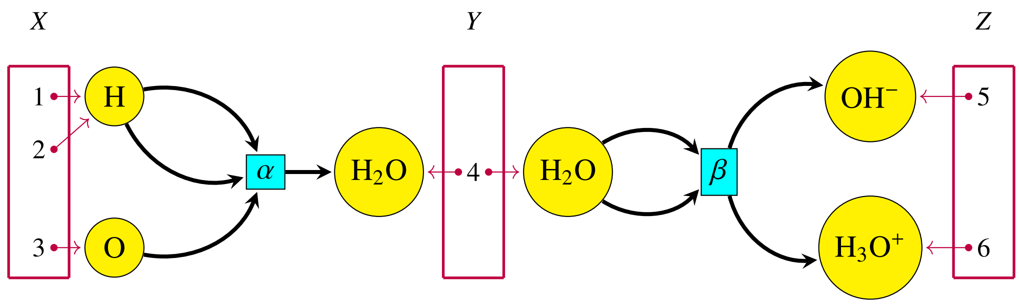
Here we have a set of 'species', the yellow nodes, and a set of 'transitions', the blue node. The disjoint union of these two sets then form the vertex set of a directed bipartite graph which is one characterization of a Petri net. A Petri net is one example of something more general known as a 'dynamical system', where a dynamical system can be viewed as a smooth vector field on a manifold. In many cases, networks, and hence dynamical systems, can be seen as pieces of even larger networks and systems linked together. This naturally leads to the idea of an 'open dynamical system', where here 'open' means that the dynamical system is equipped with 'inputs' and 'outputs'. We can view the above Petri net as an *open* Petri net by prescribing sets and functions that pick out these inputs and outputs. For example:



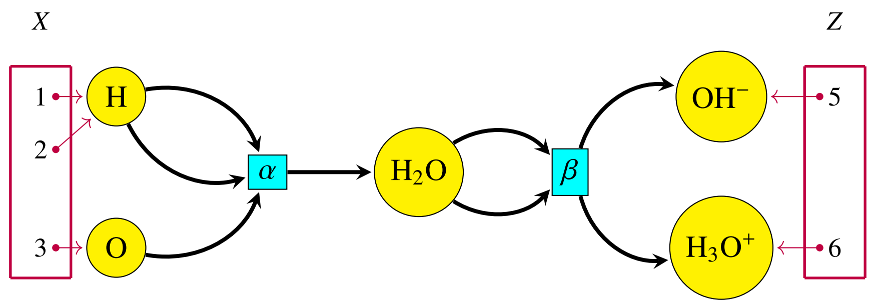
What these inputs and outputs allow us to do is compose two open dynamical systems such that the inputs of one coincide with the outputs of the other, which naturally leads to open dynamical systems being able to be thought of as morphisms in a category. For example, if we have another open Petri net that represents the chemical reaction of two molecules of water turning into a radical of hydronium and a radical of hydroxide:



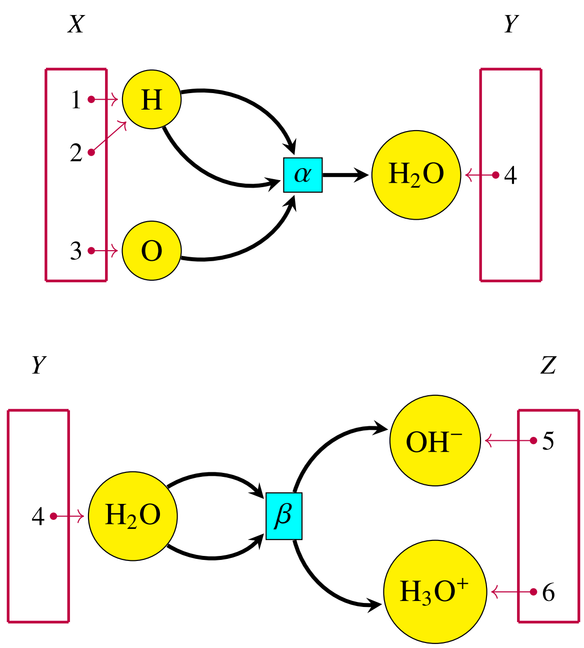
and the outputs of the first open Petri net coincide with the inputs of the second open Petri net, as they do here, we can compose them by identifying the outputs of the first with the inputs of the second:



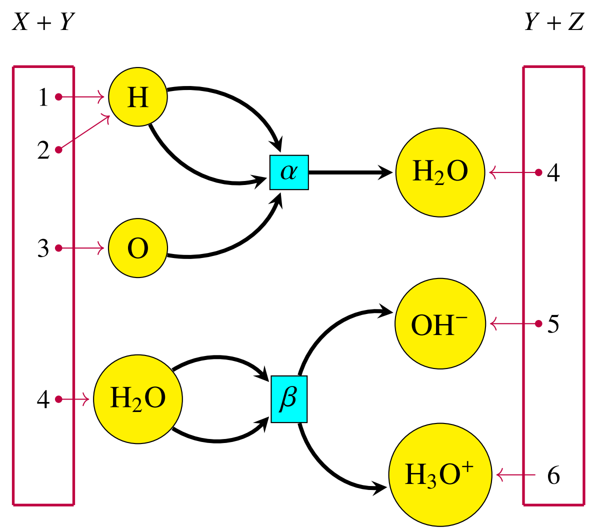
and this results in a new open Petri net whose inputs are that of the first and whose outputs are that of the second.



Similarly we can 'tensor' two open Petri nets by placing them side by side:



and this results in another open Petri net whose inputs are the disjoint union of the inputs of the two open Petri nets and likewise for the outputs.



Fong [3] has developed a theory of 'decorated cospans' in which isomorphism classes of open dynamical systems like the ones above can be viewed as morphisms in a symmetric monoidal category. A **cospan** in any category is a diagram of the form:

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

Here we call X and Y the **inputs** and **outputs**, respectively, and N the **apex**. We call the morphisms i and o the **legs** of the cospan. One of Fong's main results 'decorates' the apex of a cospan by adding extra structure or stuff to it - for example, the stucture of a Petri net. The morphisms i and o then pick out which nodes in N are to be inputs and outputs, respectively. These morphisms need not be injective in the case where objects are given by finite sets, as in the example above. Decorated cospans can then be composed by taking pushouts of the underlying cospans and carrying over the decorations of the two apices in a coherent way. This will be explained in detail in the next section. However, there are some subtleties to this approach in certain applications which become even more apparent when decorated cospan categories are promotoed to decorated cospan *bicategories* [10]. Namely, sometimes the isomorphism classes are too small in the sense that two decorated cospans which morally should be members of the same isomorphism class are not.

The alternative approach to adding extra structure onto cospans that we present in this paper uses 'double categories'. Double categories were first introduced by Ehresmann [11, 12], and have long been used in topology and other branches of pure mathematics [7, 8]. More recently they have been used to study open dynamical systems [19], open discrete-time Markov chains [21] and open continuous-time Markov processes [1]. While a mere *category* has only objects and morphisms, in a double category we have a few more types of entities. Similar to how bicategories have 2-morphisms which are morphisms between morphisms, double categories also have 2-morphisms which look like this:

$$\begin{tikzcd} A \arrow[r, "M", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow a"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "|" marking] & D \end{tikzcd}$$

In this diagam, we call A, B, C and D objects, f and g 'vertical 1-morphisms', M and N 'horizontal 1-cells', and a a '2-morphism'. The barred arrows for horizontal 1-cells help us distinguish them from vertical 1-morphisms even when we write the latter horizontally, e.g., $f : A \to C$. We can compose vertical 1-morphisms to get new vertical 1-morphisms, much as in a category. Similarly we can compose horizontal 1-cells. We can compose the 2-morphisms in two ways: horizontally by setting squares side by side, and vertically by setting one on top of the other. In an ordinary 'strict' double category all these forms of composition are associative. In a 'pseudo' double category, horizontal 1-cells compose in a weakly associative manner: that is, the associative law holds only up to an invertible 2-morphism, called the 'associator'. This is just a quick sketch of the idea, so for the full definitions one should turn elsewhere, for example the works of Grandis and Paré [14, 15].

The double categories that we utilize in this paper are these latter pseudo double categories. In many of the examples that we present in the final section, cospans appear as horizontal 1-cells. Cospans are composed by taking pushouts, which are unique only up to isomorphism, and so pseudo double categories are well-suited for capturing this weakly-associative composition. The existence of 2-morphisms in double categories also lets us consider individual cospans rather than cospans only up to isomorphism class. Throughout this paper, we will call both types of double categories, pseudo and strict, simply as double categories.

The outline of the paper is as follows: In the second section we review Fong's theory of decorated cospans and give an example in which isomorphism classes are defined by too fine of an equivalence. In the third section we discuss the main results of the paper which is another approach to adding extra structure to cospans, similar to Fong's. In the fourth section we discuss maps and maps between maps of the double categories containing cospans with extra structure. In the final section, we revisit the example in the second section using one of the main results of this paper and show that equivalences are now as one would expect, as well as revisit some applications of Fong, Pollard and the first author that utilize the decorated cospans approach.

2. Decorated cospans

The following theorem is due to Fong [13].

Theorem 2.1. *Let $(\mathbf{C}, +)$ be a category with finite colimits and let $(\mathbf{D}, \otimes)$ be a symmetric monoidal category. Let $F : \mathbf{C} \to \mathbf{D}$ be a symmetric lax monoidal functor. Then FCospan($\mathbf{C}$) is a symmetric monoidal category, where FCospan($\mathbf{C}$) is the category whose objects are that of $\mathbf{C}$ and whose morphisms are given by isomorphism classes of F-decorated cospans, where an F-decorated cospan is a pair*

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} \quad , \quad \begin{tikzcd} F(N) \\ I \arrow[u, "d"'] \end{tikzcd} \right)$$

and the composite of this F-decorated cospan with

$$\left(\begin{tikzcd} & N' & \\ Y \arrow[ur, "i'"] & & Z \arrow[ul, "o'"'] \end{tikzcd} , \quad \begin{tikzcd} F(N') \\ I \arrow[u, "d'"'] \end{tikzcd} \right)$$

is given by

$$\left(\begin{tikzcd} & N +_Y N' & \\ X \arrow[ur, "J_N \circ i"] & & Z \arrow[ul, "J_{N'} \circ o'"'] \end{tikzcd} , \quad \begin{tikzcd} F(N +_Y N') \\ I \arrow[u, "d''"'] \end{tikzcd} \right)$$

where d'' is the composite

$$I \xrightarrow{\lambda^{-1}} I \otimes I \xrightarrow{s \otimes s'} F(N) \otimes F(N') \xrightarrow{\phi_{N,N'}} F(N + N') \xrightarrow{F(J_{N,N'})} F(N +_Y N').$$

Here, $\phi_{N,N'}$ is the natural transformation of the lax monoidal functor F, $J_{N,N'} \colon N + N' \to N +_Y N'$ is the natural morphism from the coproduct to the pushout, and the maps J_N and $J_{N'}$ are the map $J_{N,N'}$ restricted to N and N', respectively.

A map of decorated cospans between two decorated cospans with the same inputs and outputs is then a morphism between the apices of the underlying cospans that make the following two diagrams commute [10]:

$$\left(\begin{tikzcd} & N \arrow[dd, "h"] & \\ X \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & N' & \end{tikzcd} , \quad \begin{tikzcd} & F(N) \arrow[dd, "F(h)"] \\ I \arrow[ur, "s_1"] \arrow[dr, "s_2"'] & \\ & F(N') \end{tikzcd} \right)$$

We say that two decorated cospans are in the same isomorphism class if $h \colon N \to N'$ is an isomorphism.

Much work has been done that builds off of Fong's theory of decorated cospans [3, 4, 5, 10]. As an example, let $F \colon \mathbf{FinSet} \to \mathbf{Set}$ be the symmetric lax monoidal functor that assigns to a finite set N the (large) set $F(N)$ of all graph structures whose underlying vertex set is the set N. Then a decorated cospan in this context is a graph (E, N, s, t) together with prescribed input and output vertices given by functions $i \colon X \to N$ and $o \colon Y \to N$. The set of edges E and source and target functions $s, t \colon E \to N$ are determined by the element of $F(N)$ given by the morphism $d \colon I \to F(N)$, where I is the monoidal unit of **Set** under cartesian products.

To illustrate, let $N = \{v_1, v_2\}$ and let $d \colon I \to F(N)$ be the graph structure on N given by:

$$v_1 \xrightarrow{\quad e \quad} v_2$$

We can view this graph as a decorated cospan in the symmetric monoidal category $F\mathrm{Cospan}(\mathbf{FinSet})$ by defining functions $i\colon X \to N$ and $o\colon Y \to N$ that specify inputs and outputs. For example:

$$
\begin{tikzcd}
 & v_1 \arrow[r, "e"] & v_2 & \\
X = \{\star\} \arrow[ur, dashed, "i"] & & & Y = \{\star\} \arrow[ul, dashed, "o"']
\end{tikzcd}
$$

Given another two element set $N' = \{w_1, w_2\}$ and a bijection $h\colon N \to N'$ defined by $h(v_i) = w_i$ for $i = 1, 2$, we can push forward the above decoration $d\colon I \to F(N)$ along $F(h)\colon F(N) \to F(N')$ to obtain a decoration $d' = F(h)d\colon I \to F(N')$ on N'.

$$
\begin{tikzcd}
 & v_1 \arrow[r, "e"] & v_2 & \\
X = \{\star\} \arrow[ur, dashed, "i"] \arrow[dr, dashed, "i'"'] & & & Y = \{\star\} \arrow[ul, dashed, "o"'] \arrow[dl, dashed, "o'"] \\
 & w_1 \arrow[r] & w_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(N) \arrow[dd, "F(h)"] \\
I \arrow[ur, "d"] \arrow[dr, "d'"'] & \\
 & F(N')
\end{tikzcd}
$$

In order for the triangle above on the right to commute, we must have that the bottom edge $w_1 \to w_2$ is labeled e - the function $F(h)$ takes a decoration on the finite set N, in this case an edge e with source and target v_1 and v_2, and turns this into a decoration on the finite set N'. This will be an edge with source and target $F(v_1) = w_1$ and $F(v_2) = w_2$ also labeled e, and herein lies the subtlety. The function $F(h)\colon F(N) \to F(N')$ is an isomorphism if and only if the edge at the bottom is also labeled e. If the bottom edge is not labeled e, then there is no function $F(h)\colon F(N) \to F(N')$ that will make the triangle on the right commute, because the underlying function $h\colon N \to N'$ nor the functor $F\colon \mathbf{FinSet} \to \mathbf{Set}$ know that the edge $v_1 \to v_2$ on top is labeled e. The labeling of the edge up top as e is extra *stuff* that h and F cannot detect. This results in more isomorphism classes than should otherwise be present. For example:

$$X = \{\star\} \overset{i}{\dashrightarrow} v_1 \xrightarrow{\quad e \quad} v_2 \overset{o}{\dashleftarrow} Y = \{\star\}$$

$$X = \{\star\} \overset{i'}{\dashrightarrow} w_1 \xrightarrow{\quad e' \quad} w_2 \overset{o'}{\dashleftarrow} Y = \{\star\}$$

Ethically speaking, there should be an isomorphism between these two decorated cospans, but here there is not, solely because the bottom edge is labeled e' and not e. In the next section we will present another method for applying extra structure to cospans that does not have this issue.

It is worth noting that this phenomenon does not occur in every situtation. For example, if we define a symmetric lax monoidal functor $F\colon \mathbf{FinSet} \to \mathbf{Set}$ that assigns to each finite set s the set of functions of the form

$$F(s) = \{\phi_s\colon s \times s \to \mathbb{R}\}$$

then isomorphism classes of decorated cospans look as they should [1, 27]. It is also worth noting that in this example, each finite set s is only being decorated with extra *structure* in the form of these functions, whereas before in the previous example, each finite set s was being decorated with extra *stuff* in the form of a graph.

3. A symmetric monoidal double category of structured cospans

In this section we present another approach for adding extra structure to cospans. The objects will reside in some category $\mathbf{C}$ and given a functor $L\colon \mathbf{C} \to \mathbf{D}$, we will transport objects of $\mathbf{C}$ to the feet of cospans in the category $\mathbf{D}$. The category $\mathbf{C}$ should be thought of as a category whose objects and morphisms are simple in which inputs and outputs reside, and the category $\mathbf{D}$ contains more complicated structures, analogous to what one would like a decorated object of $\mathbf{C}$ to look like in the case of decorated cospans. In many applications that we investigate in the last section, the functor L will be a left adjoint, such as in the example $L\colon \mathbf{FinSet} \to \mathbf{Graph}$ where L takes a finite set N to edgeless graph with vertex set N. We first prove a more general result for which this will end up being a particular case.

Theorem 3.1. *Given a double category $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ and a functor $L\colon \mathbf{C} \to \mathbb{D}_0$, there is a double category ${}_L\mathbb{D}$ for which:*

(i) objects are given by objects of $\mathbf{C}$,
(ii) vertical 1-morphisms are given by morphisms of $\mathbf{C}$,
(iii) a horizontal 1-cell with source c and target c' is a horizontal 1-cell $L(c) \xrightarrow{M} L(c')$ of $\mathbb{D}$, and

(iv) a 2-morphism is a 2-morphism in $\mathbb{D}$ of the form:

$$\begin{tikzcd} L(c) \arrow[r, "M"] \arrow[d, "L(f)"'] \arrow[dr, phantom, "\Downarrow \alpha"] & L(c') \arrow[d, "L(g)"] \\ L(c'') \arrow[r, "N"'] & L(c''') \end{tikzcd}$$

Proof. The objects and different types of morphisms are given as above. That the objects and vertical 1-morphisms constitute a category follows from $\mathbf{C}$ being a category and that the horizontal 1-cells and 2-morphisms form a category follows from $\mathbb{D}_1$ being a category. The unit structure functor $U \colon {}_L\mathbb{D}_0 \to {}_L\mathbb{D}_1$ takes an object c to the identity horizontal 1-cell given by $\hat{U}(L(c))$ where $\hat{U} \colon \mathbb{D}_0 \to \mathbb{D}_1$ is the unit structure functor for the double category $\mathbb{D}$:

$$L(c) \xrightarrow{1_{L(c)}} L(c).$$

Similarly for a morphism f in $\mathbf{C}$, we have that $U(f)$ is given by $\hat{U}(L(f))$:

$$\begin{tikzcd} L(c) \arrow[r, "1_{L(c)}"] \arrow[d, "L(f)"'] \arrow[dr, phantom, "\Downarrow U_{L(f)}"] & L(c) \arrow[d, "L(f)"] \\ L(c') \arrow[r, "1_{L(c')}"] & L(c'). \end{tikzcd}$$

We also have source and target structure functors

$$S \colon {}_L\mathbb{D}_1 \to {}_L\mathbb{D}_0$$

$$T \colon {}_L\mathbb{D}_1 \to {}_L\mathbb{D}_0$$

which do the obvious things: the source of a horizontal 1-cell $L(c) \xrightarrow{M} L(c')$ is given by c and the source of a 2-morphism

$$\begin{tikzcd} L(c) \arrow[r, "M"] \arrow[d, "L(f)"'] \arrow[dr, phantom, "\Downarrow \alpha"] & L(c') \arrow[d, "L(g)"] \\ L(c'') \arrow[r, "N"'] & L(c''') \end{tikzcd}$$

is given by f. The functor T acts similarly. We have a composition functor

$$\odot \colon {}_L\mathbb{D}_1 \times_{{}_L\mathbb{D}_0} {}_L\mathbb{D}_1 \to {}_L\mathbb{D}_1$$

where the pullback is taken over the source and target functors S and T. The natural isomorphisms α, ℓ and ρ are given by the corresponding natural isomorphisms in the double category $\mathbb{D}$. Lastly, the interchange law for 2-morphisms in ${}_L\mathbb{D}$ also holds since $\mathbb{D}$ is a double category to begin with. □

The following result is due to Shulman [24].

Theorem 3.2 (Shulman)**.** *Let $\mathbb{D}$ be an isofibrant symmetric monoidal double category. Then $H(\mathbb{D})$ is a symmetric monoidal bicategory, where $H(\mathbb{D})$ is the horizontal bicategory of $\mathbb{D}$ for which:*

(i) objects are given by objects of $\mathbb{D}$,

(ii) morphisms are given by horizontal 1-cells of $\mathbb{D}$, and
(iii) 2-morphisms are given by globular 2-morphisms of $\mathbb{D}$.

Corollary 3.3. *Given a double category $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ and a functor $L\colon \mathbf{C} \to \mathbb{D}_0$, there is a bicategory $H({}_L\mathbb{D})$ for which:*

(i) objects are given by objects of $\mathbf{C}$,
(ii) morphisms are given by horizontal 1-cells of $\mathbb{D}$, and
(iii) 2-morphisms are given by globular 2-morphisms of $\mathbb{D}$.

Theorem 3.4. *If $\mathbb{D}$ is a symmetric monoidal double category and $L\colon (C, \otimes, 1) \to \mathbb{D}_0$ is a symmetric monoidal functor, then ${}_L\mathbb{D}$ is a symmetric monoidal double category. Moreover, if $\mathbb{D}$ is isofibrant, then ${}_L\mathbb{D}$ is also isofibrant.*

Proof. First we note that the category of objects ${}_L\mathbb{D}_0$ and the category of arrows ${}_L\mathbb{D}_1$ are both symmetric monoidal double categories: ${}_L\mathbb{D}_0$ consists of objects and morphisms of the symmetric monoidal category $\mathbf{C}$. These objects and morphisms together with the associators, unitors and braidings of the symmetric monoidal category $\mathbf{C}$ make ${}_L\mathbb{D}_0$ into a symmetric monoidal category. The category of arrows ${}_L\mathbb{D}_1$ has horizontal 1-cells in $\mathbb{D}$ of the form

$$L(c) \xrightarrow{M} L(c')$$

as objects and 2-morphisms as above for morphisms. These objects and morphisms together with the associators, unitors and braidings of the symmetric monoidal category $\mathbb{D}_1$ make ${}_L\mathbb{D}_1$ into a symmetric monoidal category. Given two horizontal 1-cells

$$L(c) \xrightarrow{M} L(c')$$

$$L(c'') \xrightarrow{N} L(c''')$$

we can tensor them using the tensor product in $\mathbb{D}$ to obtain a horzontal 1-cell from $c \otimes c''$ to $c' \otimes c'''$

$$L(c \otimes c'') \cong L(c) \otimes L(c'') \xrightarrow{M \otimes N} L(c') \otimes L(c''') \cong L(c' \otimes c''')$$

using the natural isomorphisms $\mu_{c,c''}$ and $\mu_{c',c'''}$ of the symmetric monoidal functor $F\colon \mathbf{C} \to \mathbb{D}_0$. Note that this requires the symmetric monoidal functor F to be *strong*. We can also tensor 2-morphisms in an obvious way - given two 2-morphisms

$$\begin{tikzcd} L(c) \arrow[r, "M"] \arrow[d, "L(f)"'] & L(c') \arrow[d, "L(g)"] \\ L(c'') \arrow[r, "N"'] \arrow[u, phantom, shift left=6ex, "\Downarrow \alpha"] & L(c''') \end{tikzcd} \qquad \begin{tikzcd} L(d) \arrow[r, "M'"] \arrow[d, "L(f')"'] & L(d') \arrow[d, "L(g')"] \\ L(d'') \arrow[r, "N'"'] \arrow[u, phantom, shift left=6ex, "\Downarrow \beta"] & L(d''') \end{tikzcd}$$

we can tensor them to obtain another 2-morphism where we again use the natural ismorphisms that come from the functor $F\colon \mathbf{C} \to \mathbb{D}_0$ being symmetric monoidal.

$$\begin{tikzcd} L(c \otimes d) \arrow[r, "M \otimes M'"] \arrow[d, "L(f \otimes f')"'] & L(c' \otimes d') \arrow[d, "L(g \otimes g')"] \\ L(c'' \otimes d'') \arrow[r, "N \otimes N'"'] \arrow[u, phantom, shift left=8ex, "\Downarrow \alpha \otimes \beta"] & L(c''' \otimes d''') \end{tikzcd}$$

The monoidal unit for objects is given by the monoidal unit of the symmetric monoidal category $\mathbf{C}$, and tensoring of objects and morphisms is done as in $\mathbf{C}$. The monoidal unit for horizontal 1-cells is given by the monoidal unit for horizontal 1-cells of the symmetric monoidal category $\mathbb{D}$, which by the functoriality of the structure functor $\hat{U}\colon \mathbb{D}_0 \to \mathbb{D}_1$ and the symmetric monoidal functor $L\colon \mathbf{C} \to \mathbb{D}_0$, is given by

$$L(1) \xrightarrow{\hat{U}(L(1))} L(1)$$

where 1 is the monoidal unit for $\mathbf{C}$. The monoidal unit for 2-morphisms is then given by the 2-morphism

$$\begin{tikzcd}
L(1) \arrow[r, "\hat{U}(L(1))"] \arrow[d, "L(\mathrm{id}_1)"'] \arrow[dr, phantom, "\Downarrow \hat{U}(L(\mathrm{id}_1)) = \mathrm{id}_{\hat{U}(L(1))}"] & L(1) \arrow[d, "L(\mathrm{id}_1)"] \\
L(1) \arrow[r, "\hat{U}(L(1))"] & L(1).
\end{tikzcd}$$

Given four horizontal 1-cells M_1, M_2, N_1 and N_2 such that M_1 and M_2 are horizontally composable and likewise for N_1 and N_2, and two objects c and d, the globular isomorphisms

$$\chi\colon (M_1 \otimes N_1) \odot (M_2 \otimes N_2) \to (M_1 \odot M_2) \otimes (N_1 \odot N_2)$$

$$\mu\colon U_{c\otimes d} \to U_c \otimes U_d$$

come from the corresponding globular isomorphisms of the symmetric monoidal double category $\mathbb{D}$. All of the diagrams in the definition that are required to commute do so because $\mathbb{D}$ is symmetric monoidal to begin with.

Lastly, if $\mathbb{D}$ is isofibrant, then given a vertical 1-isomorphism $f\colon c \to c''$ in $\mathbf{C}$, $L(f)\colon L(c) \to L(c'')$ will be a vertical 1-isomorphism in $\mathbb{D}$ which by the isofibrancy of $\mathbb{D}$ we can lift to a companion horizontal 1-cell

$$L(c) \xrightarrow{L(\hat{f})} L(c'').$$

That this horizontal 1-cell has the two 2-morphisms that satisfy the necessary equations as well as that there exists a conjoint $\check{f}$ of f follows from the isofibrancy of $\mathbb{D}$. $\square$

Corollary 3.5. *If $\mathbb{D}$ is an isofibrant symmetric monoidal double category and $L\colon (C, \otimes, 1) \to \mathbb{D}_0$ is a symmetric monoidal functor, then $H({}_L\mathbb{D})$ is a symmetric monoidal bicategory for which:*

(i) objects are given by objects of $\mathbf{C}$,
(ii) morphisms are given by horizontal 1-cells of $\mathbb{D}$, and
(iii) 2-morphisms are given by globular 2-morphisms of $\mathbb{D}$.

Proof. This follows immediately from Shulman's Theorem 3.2 applied to 3.4. $\square$

Theorem 3.6. *Given a category $\mathbf{D}$ with pushouts, there is a double category $\mathbb{C}\mathrm{sp}(\mathbf{D})$ for which:*

(i) objects are given by objects of $\mathbf{D}$,
(ii) vertical 1-morphisms are given by morphisms of $\mathbf{D}$,
(iii) a horizontal 1-cell from c to c' is given by a cospan $c \to d \leftarrow c'$ in $\mathbf{D}$, and

(iv) a 2-morphism is a map of cospans in $\mathbf{D}$ *which is a commutative diagram of the form:*

$$\begin{tikzcd} c \arrow[r] \arrow[d, "f"'] & d \arrow[d, "h"] & c' \arrow[l] \arrow[d, "g"] \\ c'' \arrow[r] & d' & c''' \arrow[l] \end{tikzcd}$$

If $\mathbf{D}$ *has finite colimits, then this double category is symmetric monoidal.*

Proof. This is well known [10, 20, 25]. □

Corollary 3.7. *Let* $L\colon \mathbf{C} \to \mathbf{D}$ *be a functor where* $\mathbf{D}$ *is a category with pushouts. Then there exists a double category* ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ *for which:*

(i) objects are given by objects of $\mathbf{C}$*,*
(ii) vertical 1-morphisms are given by morphisms of $\mathbf{C}$*,*
(iii) a horizontal 1-cell is a horizontal 1-cell in $\mathbb{C}\mathrm{sp}(\mathbf{D})$ *of the form*

$$\begin{tikzcd} & d & \\ L(c) \arrow[ur] & & L(c') \arrow[ul] \end{tikzcd}$$

with composition given by taking pushouts, and
(iv) a 2-morphism is a commutative diagram of the form:

$$\begin{tikzcd} L(c) \arrow[r] \arrow[d, "L(f)"'] & d \arrow[d, "h"] & L(c') \arrow[l] \arrow[d, "L(g)"] \\ L(c'') \arrow[r] & d' & L(c''') \arrow[l] \end{tikzcd}$$

Proof. We have that $\mathbb{C}\mathrm{sp}(\mathbf{D})$ is a double category by Theorem 3.6 and so we get that ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ is a double category by Theorem 3.1. □

Corollary 3.8. *Let* $L\colon \mathbf{C} \to \mathbf{D}$ *be a functor where* $\mathbf{D}$ *is a category with pushouts. Then there exists a bicategory* $H({}_L\mathbb{C}\mathrm{sp}(\mathbf{D}))$ *for which:*

(i) objects are given by objects of $\mathbf{C}$*,*
(ii) morphisms are given by cospans of $\mathbb{D}$ *of the form*

$$\begin{tikzcd} & d & \\ L(c) \arrow[ur] & & L(c') \arrow[ul] \end{tikzcd}$$

and

(iii) 2-morphisms are given by maps of cospans which are commutative diagrams of the form:

$$\begin{tikzcd} & d \arrow[dd, "h"] & \\ L(c) \arrow[ur] \arrow[dr] & & L(c') \arrow[ul] \arrow[dl] \\ & d' & \end{tikzcd}$$

Theorem 3.9. *Let* $\mathbf{C}$ *be a symmetric monoidal category and let* $\mathbf{D}$ *be a category with finite colimits regarded as a symmetric monoidal category with coproduct as its tensor product. Given a symmetric monoidal functor* $L\colon \mathbf{C} \to \mathbf{D}$ *there exists a symmetric monoidal double category* ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ *for which:*

(i) objects are given by objects of $\mathbf{C}$*,*
(ii) vertical 1-morphisms are given by morphisms of $\mathbf{C}$*,*
(iii) a horizontal 1-cell is a horizontal 1-cell in $\mathbb{C}\mathrm{sp}(\mathbf{D})$ *of the form*

$$\begin{tikzcd} & d & \\ L(c) \arrow[ur] & & L(c') \arrow[ul] \end{tikzcd}$$

with composition given by taking pushouts, and
(iv) a 2-morphism is a commutative diagram of the form:

$$\begin{tikzcd} L(c) \arrow[r] \arrow[d, "L(f)"'] & d \arrow[d, "h"] & L(c') \arrow[l] \arrow[d, "L(g)"] \\ L(c'') \arrow[r] & d' & L(c''') \arrow[l] \end{tikzcd}$$

Tensoring of objects is given by the tensor product of $(\mathbf{C}, \otimes, 1)$ *and tensoring of two horizontal 1-cells is given by:*

$$\begin{tikzcd}[column sep=small] & d_1 & & & & d_2 & & & & d_1 \otimes d_2 & \\ L(c_1) \arrow[ur, "s"] & & L(c_1') \arrow[ul, "t"'] & \otimes & L(c_2) \arrow[ur, "s'"] & & L(c_2') \arrow[ul, "t'"'] & = & L(c_1 \otimes c_2) \arrow[ur, "s \otimes s'"] & & L(c_1' \otimes c_2') \arrow[ul, "t \otimes t'"'] \end{tikzcd}$$

Proof. This follows immediately by Theorem 3.4 and Theorem 3.6. □

Corollary 3.10. *Given the same hypothesis as in Theorem 3.9, there exists a symmetric monoidal bicategory* $H({}_L\mathbb{C}\mathrm{sp}(\mathbf{D}))$ *for which:*

(i) objects are given by objects of $\mathbf{C}$*,*

(ii) morphisms are given by cospans in $\mathbf{D}$ *of the form*

$$\begin{tikzcd} & d & \\ L(c) \arrow[ur] & & L(c') \arrow[ul] \end{tikzcd}$$

and

(iii) 2-morphisms are given by maps of cospans which are commutative diagrams of the form:

$$\begin{tikzcd} & d \arrow[dd, "h"'] & \\ L(c) \arrow[ur] \arrow[dr] & & L(c') \arrow[ul] \arrow[dl] \\ & d' & \end{tikzcd}$$

Tensoring of morphisms is given as in the previous Corollary and tensoring of 2-morphisms is given by:

$$\begin{tikzcd} & d_1 \arrow[dd, "f_1"'] & \\ L(c_1) \arrow[ur, "s_1"] \arrow[dr, "s_1'"'] & & L(c_1') \arrow[ul, "t_1"'] \arrow[dl, "t_1'"] \\ & d_1' & \end{tikzcd} \otimes \begin{tikzcd} & d_2 \arrow[dd, "f_2"'] & \\ L(c_2) \arrow[ur, "s_2"] \arrow[dr, "s_2'"'] & & L(c_2') \arrow[ul, "t_2"'] \arrow[dl, "t_2'"] \\ & d_2' & \end{tikzcd} = \begin{tikzcd} & d_1 \otimes d_2 \arrow[dd, "f_1 \otimes f_2"'] & \\ L(c_1 \otimes c_2) \arrow[ur, "s_1 \otimes s_2"] \arrow[dr, "s_1' \otimes s_2'"'] & & L(c_1' \otimes c_2') \arrow[ul, "t_1 \otimes t_2"'] \arrow[dl, "t_1' \otimes t_2'"] \\ & d_1' \otimes d_2' & \end{tikzcd}$$

4. Functors and transformations of structured cospan double categories

In this section we define what maps between structured cospan double categories are, and maps between these maps. As structured cospans naturally live in a double category, a map between two structured cospan double categories should somehow involve a double functor. A structured cospan double category ${}_L\mathbb{D}$ consists of a pair

$${}_L\mathbb{D} = (\mathbb{D}, L\colon \mathbf{C} \to \mathbb{D}_0)$$

where $\mathbb{D}$ is a double category and $L\colon \mathbf{C} \to \mathbb{D}_0$ is a functor that maps the category $\mathbf{C}$, which contains the objects and morphisms of the structured cospan double category ${}_L\mathbb{D}$, into the category of objects of the double category $\mathbb{D}$. Suppose that we have two structured cospan double categories:

$${}_L\mathbb{D} = (\mathbb{D}, L\colon \mathbf{C} \to \mathbb{D}_0)$$

and

$${}_{L'}\mathbb{D}' = (\mathbb{D}', L'\colon \mathbf{C}' \to \mathbb{D}'_0).$$

A map between these two will consist of a functor between the object categories $\mathbf{C}$ and $\mathbf{C}'$ together with a double functor $\mathbb{F}\colon \mathbb{D} \to \mathbb{D}'$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "L"] \arrow[d, "F"'] & \mathbb{D}_0 \arrow[d, "\mathbb{F}_0"] \\ \mathbf{C}' \arrow[r, "L'"] & \mathbb{D}'_0 \end{tikzcd} \qquad \mathbb{F} = (\mathbb{F}_0, \mathbb{F}_1)$$

In the case where ${}_L\mathbb{D}$ and ${}_{L'}\mathbb{D}'$ are symmetric monoidal we will then require that both the functor F and double functor $\mathbb{F}$ are symmetric monoidal.

Theorem 4.1. *Let ${}_L\mathbb{D} = (\mathbb{D}, L\colon \mathbf{C} \to \mathbb{D}_0)$ and ${}_{L'}\mathbb{D}' = (\mathbb{D}', L'\colon \mathbf{C}' \to \mathbb{D}'_0)$ be two structured cospan double categories. Then a functor from the first to the second consists of a functor $F\colon \mathbf{C} \to \mathbf{C}'$ and a double functor $\mathbb{F} = (\mathbb{F}_0, \mathbb{F}_1)\colon \mathbb{D} \to \mathbb{D}'$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathbf{C} \arrow[r, "L"] \arrow[d, "F"'] & \mathbb{D}_0 \arrow[d, "\mathbb{F}_0"] \\ \mathbf{C}' \arrow[r, "L'"] & \mathbb{D}'_0 \end{tikzcd}$$

This functor maps objects, vertical 1-morphisms, horizontal 1-cells and 2-morphisms as such.

(i) Objects:

$$c \mapsto F(c)$$

(ii) Vertical 1-morphisms:

$$\begin{tikzcd} c \arrow[d, "f"'] \\ c' \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} F(c) \arrow[d, "F(f)"] \\ F(c') \end{tikzcd}$$

(iii) Horizontal 1-cells:

$$L(c) \xrightarrow{M} L(d)$$

$$\mapsto$$

$$L'(F(c)) = \mathbb{F}_0(L(c)) \xrightarrow{\mathbb{F}_1(M)} \mathbb{F}_0(L(d)) = L'(F(d))$$

(iv) 2-morphisms:

$$\begin{tikzcd} L(c) \arrow[r, "M"] \arrow[d, "L(f)"'] & L(c') \arrow[d, "L(g)"] \\ L(c'') \arrow[r, "N"'] \arrow[ur, phantom, "\Downarrow \alpha"] & L(c''') \end{tikzcd} \mapsto \begin{tikzcd} \mathbb{F}_0(L(c)) \arrow[r, "\mathbb{F}_1(M)"] \arrow[d, "\mathbb{F}_0(L(f'))"'] & \mathbb{F}_0(L(c')) \arrow[d, "\mathbb{F}_0(L(g'))"] \\ \mathbb{F}_0(L(c'')) \arrow[r, "\mathbb{F}_1(N)"'] \arrow[ur, phantom, "\Downarrow \mathbb{F}_1(\alpha)"] & \mathbb{F}_0(L(c''')) \end{tikzcd}$$

Proof. We will show that the pair $(F, \mathbb{F})$ constitutes a double functor $(F, \mathbb{F})\colon {}_L\mathbb{D} \to_{L'} \mathbb{D}'$. This means that we must have

$$F\colon {}_L\mathbb{D}_0 \to_{L'} \mathbb{D}'_0$$

and

$$\mathbb{F}_1\colon {}_L\mathbb{D}_1 \to_{L'} \mathbb{D}'_1$$

such that the following diagarms commute:

$$\begin{tikzcd} {}_L\mathbb{D}_1 \arrow[r, "\mathbb{F}_1"] \arrow[d, "S"'] & {}_{L'}\mathbb{D}'_1 \arrow[d, "S'"] \\ {}_L\mathbb{D}_0 \arrow[r, "F"'] & {}_{L'}\mathbb{D}'_0 \end{tikzcd} \qquad \begin{tikzcd} {}_L\mathbb{D}_1 \arrow[r, "\mathbb{F}_1"] \arrow[d, "T"'] & {}_{L'}\mathbb{D}'_1 \arrow[d, "T'"] \\ {}_L\mathbb{D}_0 \arrow[r, "F"'] & {}_{L'}\mathbb{D}'_0 \end{tikzcd}$$

where S, T and S', T' are the source and target structure functors of the double categories ${}_L\mathbb{D}$ and ${}_{L'}\mathbb{D}'$, repsectively, together with natural transformations $\mathbb{F}_\odot\colon \mathbb{F}(M)\odot\mathbb{F}(N) \to \mathbb{F}(M\odot N)$ for every pair of composable horizontal 1-cells M and N and $\mathbb{F}_U\colon U'_{F(c)} \to \mathbb{F}(U_c)$ for every object c that satisfy the standard coherence axioms given by the associativity hexagon and unitality squares.

The functors F and $\mathbb{F}$ are defined as in the statement of the theorem. To see that the above squares commute, if we focus on the left one, starting up the upper left corner, for an object of ${}_L\mathbb{D}_1$ which is given by a horizontal 1-cell, we have going right that:

$$L(c) \xrightarrow{M} L(d)$$

$$\mapsto$$

$$\mathbb{F}_0(L(c)) \xrightarrow{\mathbb{F}_1(M)} \mathbb{F}_0(L(d))$$

and by the commutivity of

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "L"] \arrow[d, "F"'] & \mathbb{D}_0 \arrow[d, "\mathbb{F}_0"] \\
\mathbf{C}' \arrow[r, "L'"] & \mathbb{D}'_0
\end{tikzcd}$$

this horizontal 1-cell is the same as the horizontal 1-cell

$$L'(F(c)) \xrightarrow{\mathbb{F}_1(M)} L'(F(d))$$

which has source $F(c)$. If we go down and then right, we get that the source of the top horizontal 1-cell is the object c which then maps to $F(c)$ under the functor F. A morphism in ${}_L\mathbb{D}$ is given by a 2-morphism of the form

$$\begin{tikzcd}
L(c) \arrow[r, "M"] \arrow[d, "L(f)"'] \arrow[dr, phantom, "\Downarrow\alpha"] & L(d) \arrow[d, "L(g)"] \\
L(c') \arrow[r, "M'"] & L(d')
\end{tikzcd}$$

so, again focusing on the left square, going right gives

$$\begin{tikzcd}
\mathbb{F}_0(L(c)) \arrow[r, "\mathbb{F}_1(M)"] \arrow[d, "\mathbb{F}_0(L(f))"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathbb{F}_0(L(d)) \arrow[d, "\mathbb{F}_0(L(g))"] \\
\mathbb{F}_0(L(c')) \arrow[r, "\mathbb{F}_1(M')"] & \mathbb{F}_0(L(d'))
\end{tikzcd}$$

which by the commutivity of the above square is the same as

$$\begin{tikzcd}
L'(F(c)) \arrow[r, "\mathbb{F}_1(M)"] \arrow[d, "L'(F((f))"'] \arrow[dr, phantom, "\Downarrow\alpha"] & L'(F(d)) \arrow[d, "L'(F(g))"] \\
L'(F(c')) \arrow[r, "\mathbb{F}_1(M')"] & L'(F(d'))
\end{tikzcd}$$

which has source $F(f)$. On the other hand, going down we get that the source of the original 2-morphism is f which then maps to $F(f)$ under the functor F, and so the left square commutes. The right square is analogous.

That $(F,\mathbb{F})$ is functorial on morphisms is clear, as the pair $(F,\mathbb{F})$ acts as the functor $F\colon \mathbf{C} \to \mathbf{C}'$ on objects and morphisms. Given two vertically composable 2-morphisms in ${}_L\mathbb{D}$:

$$\begin{tikzcd}
L(c) \arrow[r, "M"] \arrow[d, "L(f)"'] \arrow[dr, phantom, "\Downarrow \alpha"] & L(d) \arrow[d, "L(g)"] \\
L(c') \arrow[r, "M'"'] & L(d')
\end{tikzcd}$$

$$\begin{tikzcd}
L(c') \arrow[r, "M'"] \arrow[d, "L(f')"'] \arrow[dr, phantom, "\Downarrow \beta"] & L(d') \arrow[d, "L(g')"] \\
L(c'') \arrow[r, "M''"'] & L(d'')
\end{tikzcd}$$

we wish to show that $(F,\mathbb{F})$ is functorial. If we first compose the above two 2-morphisms in ${}_L\mathbb{D}$, we get

$$\begin{tikzcd}
L(c) \arrow[r, "M"] \arrow[d, "L(f' \circ f)"'] \arrow[dr, phantom, "\Downarrow \beta\alpha"] & L(d) \arrow[d, "L(g' \circ g)"] \\
L(c'') \arrow[r, "M''"'] & L(d'')
\end{tikzcd}$$

and then $(F,\mathbb{F})$ acts as the double functor $\mathbb{F}$ on this 2-morphism, which yields

$$\begin{tikzcd}
\mathbb{F}_0(L(c)) \arrow[r, "\mathbb{F}_1(M)"] \arrow[d, "\mathbb{F}_0(L(f' \circ f))"'] \arrow[dr, phantom, "\Downarrow \mathbb{F}_1(\beta\alpha)"] & \mathbb{F}_0(L(d)) \arrow[d, "\mathbb{F}_0(L(g' \circ g))"] \\
\mathbb{F}_0(L(c'')) \arrow[r, "\mathbb{F}_1(M'')"'] & \mathbb{F}_0(L(d''))
\end{tikzcd}$$

On the other hand, if we first map over the two 2-morphisms, we get

$$
\begin{tikzcd}
\mathbb{F}_0(L(c)) \arrow[r, "\mathbb{F}_1(M)"] \arrow[d, "\mathbb{F}_0(L(f))"'] \arrow[dr, phantom, "\Downarrow \mathbb{F}_1(\alpha)"] & \mathbb{F}_0(L(d)) \arrow[d, "\mathbb{F}_0(L(g))"] \\
\mathbb{F}_0(L(c')) \arrow[r, "\mathbb{F}_1(M')"'] & \mathbb{F}_0(L(d'))
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{F}_0(L(c')) \arrow[r, "\mathbb{F}_1(M')"] \arrow[d, "\mathbb{F}_0(L(f'))"'] \arrow[dr, phantom, "\Downarrow \mathbb{F}_1(\beta)"] & \mathbb{F}_0(L(d')) \arrow[d, "\mathbb{F}_0(L(g'))"] \\
\mathbb{F}_0(L(c'')) \arrow[r, "\mathbb{F}_1(M'')"'] & \mathbb{F}_0(L(d''))
\end{tikzcd}
$$

and then composing these in ${}_{L'}\mathbb{D}'$ yields

$$
\begin{tikzcd}
\mathbb{F}_0(L(c)) \arrow[r, "\mathbb{F}_1(M)"] \arrow[d, "\mathbb{F}_0(L(f' \circ f))"'] \arrow[dr, phantom, "\Downarrow \mathbb{F}_1(\beta\alpha)"] & \mathbb{F}_0(L(d)) \arrow[d, "\mathbb{F}_0(L(g' \circ g))"] \\
\mathbb{F}_0(L(c'')) \arrow[r, "\mathbb{F}_1(M'')"'] & \mathbb{F}_0(L(d''))
\end{tikzcd}
$$

by the functoriality of $\mathbb{F}_0, \mathbb{F}_1$ and L.

Now let M and N be two composable horizontal 1-cells in ${}_L\mathbb{D}$ given by:

$$
L(c) \xrightarrow{M} L(d) \qquad\qquad L(d) \xrightarrow{N} L(e)
$$

Then for composable horizontal 1-cells M and N, we get a natural transformation $\mathbb{F}_{M,N} \colon \mathbb{F}(M) \odot \mathbb{F}(N) \to \mathbb{F}(M \odot N)$ given by

$$\begin{tikzcd}
\mathbb{F}_0(L(c)) \arrow[r, "\mathbb{F}_1(M)"] \arrow[dd, "1"'] & \mathbb{F}_0(L(d)) \arrow[r, "\mathbb{F}_1(N)"] \arrow[dd, phantom, "\Downarrow \mathbb{F}_\odot"] & \mathbb{F}_0(L(e)) \arrow[dd, "1"] \\
 & & \\
\mathbb{F}_0(L(c)) \arrow[rr, "\mathbb{F}_1(M \odot N)"] & & \mathbb{F}_0(L(e))
\end{tikzcd}$$

and for any object c, a natural transformation

$$\begin{tikzcd}
L'(F(c)) \arrow[r, "U'_{F(c)}"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow \mathbb{F}_U"] & L'(F(c)) \arrow[d, "1"] \\
\mathbb{F}_0(L(c)) \arrow[r, "\mathbb{F}_1(U_c)"] & \mathbb{F}_0(L(c))
\end{tikzcd}$$

both of which come from the fact that $\mathbb{F}$ is a double functor. The double functor $(F, \mathbb{F})$ is pseudo, lax or oplax depending on whether the double functor $\mathbb{F}$ is pseudo, lax or oplax, respectively. □

If both $F \colon \mathbf{C} \to \mathbf{C}'$ and $\mathbb{F} \colon \mathbb{D} \to \mathbb{D}'$ are symmetric monoidal, then $(F, \mathbb{F}) \colon {}_L\mathbb{D} \to_{L'} \mathbb{D}'$ is a symmetric monoidal double functor.

Theorem 4.2. *Let ${}_L\mathbb{D} = (\mathbb{D}, L \colon \mathbf{C} \to \mathbb{D}_0)$ and ${}_{L'}\mathbb{D}' = (\mathbb{D}', L' \colon \mathbf{C}' \to \mathbb{D}')$ be symmetric monoidal structured cospan double categories. If $(F, \mathbb{F}) \colon {}_L\mathbb{D} \to_{L'} \mathbb{D}'$ is a double functor of structured cospan double categories with F and $\mathbb{F}$ symmetric monoidal, then $(F, \mathbb{F})$ is a symmetric monoidal double functor of structured cospan double categories.*

Proof. Since the functor $F \colon \mathbf{C} \to \mathbf{C}'$ is symmetric monoidal, for every pair of objects a and b of $\mathbf{C}$, we have a natural transformation

$$\mu_{a,b} \colon F(a) \otimes F(b) \to F(a \otimes b)$$

together with a morphism

$$\epsilon \colon 1_{{}_{L'}\mathbb{D}'} \to F(1_{{}_L\mathbb{D}})$$

where the unit object of ${}_{L'}\mathbb{D}'$ is given by $1_{{}_{L'}\mathbb{D}'} = 1_{\mathbf{C}'} \cong F(1_{\mathbf{C}})$ and the unit object of ${}_L\mathbb{D}$ is given by $1_{{}_L\mathbb{D}} = 1_{\mathbf{C}}$. These together make the following diagrams commute for every triple of objects a, b, c of ${}_L\mathbb{D}$, which are objects of $\mathbf{C}$.

$$\begin{tikzcd}
(F(a) \otimes F(b)) \otimes F(c) \arrow[r, "\alpha'"] \arrow[d, "\mu_{a,b} \otimes 1"'] & F(a) \otimes (F(b) \otimes F(c)) \arrow[d, "1 \otimes \mu_{b,c}"] \\
F(a \otimes b) \otimes F(c) \arrow[d, "\mu_{a \otimes b, c}"'] & F(a) \otimes F(b \otimes c) \arrow[d, "\mu_{a, b \otimes c}"] \\
F((a \otimes b) \otimes c) \arrow[r, "F\alpha"] & F(a \otimes (b \otimes c))
\end{tikzcd}$$

$$\begin{tikzcd}
F(a) \otimes 1_{{}_{L'}\mathbb{D}'} \arrow[r, "r_{F(a)}"] \arrow[d, "1 \otimes \epsilon"'] & F(a) \\
F(a) \otimes F(1_{{}_L\mathbb{D}}) \arrow[r, "\mu_{a, 1_{{}_L\mathbb{D}}}"] & F(a \otimes 1_{{}_L\mathbb{D}}) \arrow[u, "F(r_a)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{{}_{L'}\mathbb{D}'} \otimes F(a) \arrow[r, "\ell_{F(a)}"] \arrow[d, "\epsilon \otimes 1"'] & F(a) \\
F(1_{{}_L\mathbb{D}}) \otimes F(a) \arrow[r, "\mu_{1_{{}_L\mathbb{D}}, a}"] & F(1_{{}_L\mathbb{D}} \otimes a) \arrow[u, "F(\ell_a)"']
\end{tikzcd}$$

Moreover, the following diagram commutes where by an abuse of notation, we denote the braidings in both categories $\mathbf{C}$ and $\mathbf{C}'$ as β.

$$\begin{tikzcd}
F(a) \otimes F(b) \arrow[r, "\beta_{F(a),F(b)}"] \arrow[d, "\mu_{a,b}"'] & F(b) \otimes F(a) \arrow[d, "\mu_{b,a}"] \\
F(a \otimes b) \arrow[r, "F(\beta_{a,b})"] & F(b \otimes a)
\end{tikzcd}$$

The double functor $\mathbb{F} \colon \mathbb{D} \to \mathbb{D}'$ is also symmetric monoidal, which means that for every pair of horizontal 1-cells M and N, we have a natural transformation

$$\nu_{M,N} \colon F(M) \otimes F(N) \to F(M \otimes N)$$

and a morphism

$$\delta \colon U_{1_{{}_{L'}\mathbb{D}'}} \to \mathbb{F}(U_{1_{{}_L\mathbb{D}}})$$

which together make the following diagrams commute for every triple of horizontal 1-cells M, N, P of $\mathbb{D}$.

$$\begin{tikzcd}
(F(M) \otimes F(N)) \otimes F(P) \arrow[r, "\alpha'"] \arrow[d, "\nu_{M,N} \otimes 1"'] & F(M) \otimes (F(N) \otimes F(P)) \arrow[d, "1 \otimes \nu_{N,P}"] \\
F(M \otimes N) \otimes F(P) \arrow[d, "\nu_{M \otimes N, P}"'] & F(M) \otimes F(N \otimes P) \arrow[d, "\nu_{M, N \otimes P}"] \\
F((M \otimes N) \otimes P) \arrow[r, "F\alpha"] & F(M \otimes (N \otimes P))
\end{tikzcd}$$

$$
\begin{tikzcd}
F(M) \otimes U_{1_{L'\mathbb{D}'}} \arrow[r, "r_{F(M)}"] \arrow[d, "1 \otimes \delta"'] & F(M) \\
F(M) \otimes F(U_{1_{L\mathbb{D}}}) \arrow[r, "\nu_{M,U_{1_{L\mathbb{D}}}}"] & F(M \otimes U_{1_{L\mathbb{D}}}) \arrow[u, "F(r_M)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{1_{L'\mathbb{D}'}} \otimes F(M) \arrow[r, "\ell_{F(M)}"] \arrow[d, "\delta \otimes 1"'] & F(M) \\
F(U_{1_{L\mathbb{D}}}) \otimes F(M) \arrow[r, "\nu_{U_{1_{L\mathbb{D}}},M}"] & F(U_{1_{L\mathbb{D}}} \otimes M) \arrow[u, "F(\ell_M)"']
\end{tikzcd}
$$

Lastly, by another abuse of notation, the following diagram commutes where we denote the braiding in both $\mathbb{D}_1$ and $\mathbb{D}'_1$ by β.

$$
\begin{tikzcd}
F(M) \otimes F(N) \arrow[r, "\beta_{F(M),F(N)}"] \arrow[d, "\nu_{M,N}"'] & F(N) \otimes F(M) \arrow[d, "\nu_{N,M}"] \\
F(M \otimes N) \arrow[r, "F(\beta_{M,N})"] & F(N \otimes M)
\end{tikzcd}
$$

That the comparison constraints of the double functor $(F, \mathbb{F})\colon {}_L\mathbb{D} \to_{L'} \mathbb{D}'$ are monoidal natural transformations follows from the fact that the comparison constraints of the symmetric monoidal double functor $\mathbb{F}\colon \mathbb{D} \to \mathbb{D}'$ are monoidal natural transformations. $\square$

We can also consider double transformations between these double functors and symmetric monoidal versions of such. By the previous Theorem, a map between two structured cospan double categories ${}_L\mathbb{D} = (\mathbb{D}, L\colon \mathbf{C} \to \mathbb{D}_0)$ and ${}_{L'}\mathbb{D}' = (\mathbb{D}', L'\colon \mathbf{C}' \to \mathbb{D}'_0)$ is a pair $(F, \mathbb{F})$ such that the following diagram commutes.

$$
\begin{tikzcd}
\mathbf{C} \arrow[r, "L"] \arrow[d, bend right, "F"'] & \mathbb{D}_0 \arrow[d, bend right, "\mathbb{F}_0"'] \\
\mathbf{C}' \arrow[r, "L'"] & \mathbb{D}'_0
\end{tikzcd}
$$

Then given another structured cospan double functor $(G', \mathbb{G}')\colon {}_L\mathbb{D} \to_{L'} \mathbb{D}'$, a double transformation from $(F, \mathbb{F})$ to $(G, \mathbb{G})$ consists of a pair (ϕ, Φ) where $\phi\colon F \Rightarrow G$ is a natural transformation and $\Phi\colon \mathbb{F} \Rightarrow \mathbb{G}$ is a double transformation such that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "L"] \arrow[d, bend right, "F"'] \arrow[d, bend left, "G"] \arrow[d, phantom, "\phi \Rightarrow"] & \mathbb{D}_0 \arrow[d, bend right, "\mathbb{F}_0"'] \arrow[d, bend left, "\mathbb{G}_0"] \arrow[d, phantom, "\Phi_0 \Rightarrow"] \\
\mathbf{C}' \arrow[r, "L'"] & \mathbb{D}'_0
\end{tikzcd}$$

As with functors of structured cospan double categories, if both the transformation $\phi\colon F \Rightarrow G$ and the double transformation $\Phi\colon \mathbb{F} \Rightarrow \mathbb{G}$ are symmetric monoidal, then $(\phi, \Phi)\colon (F, \mathbb{F}) \Rightarrow (G, \mathbb{G})$ is a symmetric monoidal double transformation of symmetric monoidal strucutured cospan double functors. We recall the relevant definitions.

Definition 4.3. Let $F\colon \mathbf{C} \to \mathbf{C}'$ and $G\colon \mathbf{C} \to \mathbf{C}'$ be monoidal functors. A natural transformation $\phi\colon F \Rightarrow G$ is **monoidal** if the following diagrams commute.

$$\begin{tikzcd}
F(a) \otimes F(b) \arrow[r, "\phi_a \otimes \alpha_b"] \arrow[d, "\mu_{a,b}"'] & G(a) \otimes G(b) \arrow[d, "\mu'_{a,b}"] \\
F(a \otimes b) \arrow[r, "\phi_{a \otimes b}"] & G(a \otimes b)
\end{tikzcd}$$

$$\begin{tikzcd}
1_{\mathbf{C}'} \arrow[rr, "\epsilon'"] \arrow[dr, "\epsilon"'] & & G(1_{\mathbf{C}}) \\
& F(1_{\mathbf{C}}) \arrow[ur, "\phi_{1_{\mathbf{C}}}"'] &
\end{tikzcd}$$

A natural transformation is **braided monoidal** or **symmetric monoidal** if the functors F and G are braided monoidal or symmetric monoidal, respectively.

Definition 4.4. A **(vertical) double transformation** $\Phi \colon \mathbb{F} \Rightarrow \mathbb{G}$ between two double functors $\mathbb{F} \colon \mathbb{D} \to \mathbb{D}'$ and $\mathbb{G} \colon \mathbb{D} \to \mathbb{D}'$ consists of two natural transformations $\Phi_0 \colon \mathbb{F}_0 \Rightarrow \mathbb{G}_0$ and $\Phi_1 \colon \mathbb{F}_1 \Rightarrow \mathbb{G}_1$ such that for all horizontal 1-cells M we have that $S(\Phi_{1M}) = \Phi_{0S(M)}$ and $T(\Phi_{1M}) = \Phi_{0T(M)}$ and for composable horizontal 1-cells M and N, we have that

$$
\begin{tikzcd}
\mathbb{F}(a) \arrow[r, "\mathbb{F}(M)"] \arrow[d, "1"'] & \mathbb{F}(b) \arrow[r, "\mathbb{F}(N)"] \arrow[d, phantom, "\Downarrow \mathbb{F}_{M,N}"] & \mathbb{F}(c) \arrow[d, "1"] \\
\mathbb{F}(a) \arrow[rr, "\mathbb{F}(M \odot N)"] \arrow[d, "\Phi_{0a}"'] & \arrow[d, phantom, "\Downarrow \Phi_{1M \odot N}"] & \mathbb{F}(c) \arrow[d, "\Phi_{0c}"] \\
\mathbb{G}(a) \arrow[rr, "\mathbb{G}(M \odot N)"] & & \mathbb{G}(c)
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{F}(a) \arrow[r, "\mathbb{F}(M)"] \arrow[d, "\Phi_{0a}"'] \arrow[dr, phantom, "\Downarrow \Phi_{1M}"] & \mathbb{F}(b) \arrow[r, "\mathbb{F}(N)"] \arrow[d, "\Phi_{0b}"'] \arrow[dr, phantom, "\Downarrow \Phi_{1N}"] & \mathbb{F}(c) \arrow[d, "\Phi_{0c}"] \\
\mathbb{G}(a) \arrow[r, "\mathbb{G}(M)"] \arrow[d, "1"'] & \mathbb{G}(b) \arrow[r, "\mathbb{G}(N)"] \arrow[d, phantom, "\Downarrow \mathbb{G}_{M,N}"] & \mathbb{G}(c) \arrow[d, "1"] \\
\mathbb{G}(a) \arrow[rr, "\mathbb{G}(M \odot N)"] & & \mathbb{G}(c)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{F}(a) \arrow[r, "U_{\mathbb{F}(a)}"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow \mathbb{F}_U"] & \mathbb{F}(a) \arrow[d, "1"] \\
\mathbb{F}(a) \arrow[r, "\mathbb{F}(U_a)"] \arrow[d, "\Phi_{0a}"'] \arrow[dr, phantom, "\Downarrow \Phi_{1U_a}"] & \mathbb{F}(a) \arrow[d, "\Phi_{0a}"] \\
\mathbb{G}(a) \arrow[r, "\mathbb{G}(U_a)"] & \mathbb{G}(a)
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{F}(a) \arrow[r, "U_{\mathbb{F}(a)}"] \arrow[d, "\Phi_{0a}"'] \arrow[dr, phantom, "\Downarrow U_{\Phi_{0a}}"] & \mathbb{F}(a) \arrow[d, "\Phi_{0a}"] \\
\mathbb{G}(a) \arrow[r, "U_{\mathbb{G}(a)}"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow \mathbb{G}_U"] & \mathbb{G}(a) \arrow[d, "1"] \\
\mathbb{G}(a) \arrow[r, "\mathbb{G}(U_a)"] & \mathbb{G}(a)
\end{tikzcd}
$$

We call Φ_0 and Φ_1 the object component and arrow component of the double transformation Φ, respectively.

Theorem 4.5. *Let $(\mathbb{F}, F) \colon {}_L\mathbb{D} \to_{L'} \mathbb{D}'$ and $(\mathbb{G}, G) \colon {}_L\mathbb{D} \to_{L'} \mathbb{D}'$ be double functors between two structured cospan double categories ${}_L\mathbb{D}$ and ${}_{L'}\mathbb{D}'$. Given a double transformation $\Phi \colon \mathbb{F} \Rightarrow \mathbb{G}$ and a transformation $\phi \colon F \to G$ such that the diagram on the previous page commutes, then $(\Phi, \phi) \colon (\mathbb{F}, F) \Rightarrow (\mathbb{G}, G)$ is a double transformation between structured cospan double functors.*

Proof. Because $\Phi \colon \mathbb{F} \Rightarrow \mathbb{G}$ is a double transformation and the diagram on the previous page commutes, we have that the following diagrams commute.

$$
\begin{tikzcd}
(\mathbb{F}, F)(a) \arrow[r, "(\mathbb{F}, F)(M)"] \arrow[d, "1"'] & (\mathbb{F}, F)(b) \arrow[r, "(\mathbb{F}, F)(N)"] \arrow[d, phantom, "\Downarrow (\mathbb{F}, F)_{\odot}"] & (\mathbb{F}, F)(c) \arrow[d, "1"] \\
(\mathbb{F}, F)(a) \arrow[rr, "(\mathbb{F}, F)(M \odot N)"] \arrow[d, "\phi_a"'] & \arrow[d, phantom, "\Downarrow \Phi_{1M \odot N}"] & (\mathbb{F}, F)(c) \arrow[d, "\phi_c"] \\
(\mathbb{G}, G)(a) \arrow[rr, "(\mathbb{G}, G)(M \odot N)"] & & (\mathbb{G}, G)(c)
\end{tikzcd}
=
\begin{tikzcd}
(\mathbb{F}, F)(a) \arrow[r, "(\mathbb{F}, F)(M)"] \arrow[d, "\phi_a"'] \arrow[dr, phantom, "\Downarrow \Phi_{1M}"] & (\mathbb{F}, F)(b) \arrow[r, "(\mathbb{F}, F)(N)"] \arrow[d, "\phi_b"'] \arrow[dr, phantom, "\Downarrow \Phi_{1N}"] & (\mathbb{F}, F)(c) \arrow[d, "\phi_c"] \\
(\mathbb{G}, G)(a) \arrow[r, "(\mathbb{G}, G)(M)"] \arrow[d, "1"'] & (\mathbb{G}, G)(b) \arrow[r, "(\mathbb{G}, G)(N)"] \arrow[d, phantom, "\Downarrow (\mathbb{G}, G)_{\odot}"] & (\mathbb{G}, G)(c) \arrow[d, "1"] \\
(\mathbb{G}, G)(a) \arrow[rr, "(\mathbb{G}, G)(M \odot N)"] & & (\mathbb{G}, G)(c)
\end{tikzcd}
$$

$$\begin{tikzcd}
(\mathbb{F},F)(a) \arrow[r, "U_{(\mathbb{F},F)(a)}"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow (\mathbb{F},\mathbb{F})_U"] & (\mathbb{F},F)(a) \arrow[d, "1"] & & (\mathbb{F},F)(a) \arrow[r, "U_{(\mathbb{F},F)(a)}"] \arrow[d, "\phi_a"'] \arrow[dr, phantom, "\Downarrow U_{\phi_a}"] & (\mathbb{F},F)(a) \arrow[d, "\phi_a"] \\
(\mathbb{F},F)(a) \arrow[r, "(\mathbb{F},F)(U_a)"] \arrow[d, "\phi_a"'] \arrow[dr, phantom, "\Downarrow \Phi_{1U_a}"] & (\mathbb{F},F)(a) \arrow[d, "\phi_a"] & = & (\mathbb{G},G)(a) \arrow[r, "U_{(\mathbb{G},G)(a)}"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow (\mathbb{G},\mathbb{G})_U"] & (\mathbb{G},G)(a) \arrow[d, "1"] \\
(\mathbb{G},G)(a) \arrow[r, "(\mathbb{G},G)(U_a)"] & (\mathbb{G},G)(a) & & (\mathbb{G},G)(a) \arrow[r, "(\mathbb{G},G)(U_a)"] & (\mathbb{G},G)(a)
\end{tikzcd}$$

Here, the commutativity of the diagram on the previous page allows the object component Φ_0 of the double transformation Φ to be replaced with ϕ. In detail, every object of ${}_L\mathbb{D}$ is of the form $L(c)$ for some c in $\mathbf{C}$. We thus have for every c in $\mathbf{C}$ that $\mathbb{F}(L(c)) = L'(F(c))$. The natural transformation $\phi\colon F \to G$ whose component at c then gives a map $\phi_c\colon F(c) \to G(c)$ and we can apply L' to this map which gives $L'(\phi_c)\colon L'(F(c)) \to L'(G(c))$. Then, because the diagram commutes, we have that $L'(G(c)) = \mathbb{G}(L(c))$, and thus $\phi_c\colon (\mathbb{F},F)(c) \to (\mathbb{G},G)(c)$. Moreover, the map ϕ_c for each appropriate object c will make the above diagrams commute because the maps $\Phi_{0L(c)}$ do since $\Phi\colon \mathbb{D} \to \mathbb{D}'$ is a double transformation.

Finally, because $\Phi\colon \mathbb{F} \Rightarrow \mathbb{G}$ is a double transformation and by the commutativity of the diagram on the previous page, for a horizontal 1-cell M in ${}_L\mathbb{D}$ we have that $S(\Phi_{1M}) = \phi_{S(M)}$ and $T(\Phi_{1M}) = \phi_{T(M)}$. ◻

Definition 4.6. A **monoidal double transformation** $\Phi\colon \mathbb{F} \Rightarrow \mathbb{G}$ between two monoidal double functors $\mathbb{F}\colon \mathbb{D} \to \mathbb{D}'$ and $\mathbb{G}\colon \mathbb{D} \to \mathbb{D}'$ is a double transformation such that the following diagrams commute for objects $a, b \in \mathbb{D}$ and horizontal 1-cells $M, N \in \mathbb{D}$.

$$\begin{tikzcd}
\mathbb{F}(a) \otimes \mathbb{F}(b) \arrow[r, "\Phi_{0a} \otimes \Phi_{0b}"] \arrow[d, "\mu_{a,b}"'] & \mathbb{G}(a) \otimes \mathbb{G}(b) \arrow[d, "\mu'_{a,b}"] & \mathbb{F}(M) \otimes \mathbb{F}(N) \arrow[r, "\Phi_{1M} \otimes \Phi_{1N}"] \arrow[d, "\mu_{M,N}"'] & \mathbb{G}(M) \otimes \mathbb{G}(N) \arrow[d, "\mu'_{M,N}"] \\
\mathbb{F}(a \otimes b) \arrow[r, "\Phi_{0a\otimes b}"] & \mathbb{G}(a \otimes b) & \mathbb{F}(M \otimes N) \arrow[r, "\Phi_{1M\otimes N}"] & \mathbb{G}(M \otimes N)
\end{tikzcd}$$

$$\begin{tikzcd}
1_{\mathbb{D}'} \arrow[rr, "\epsilon'"] \arrow[dr, "\epsilon"'] & & \mathbb{G}(1_{\mathbb{D}}) & U_{1_{\mathbb{D}'}} \arrow[rr, "\delta'"] \arrow[dr, "\delta"'] & & \mathbb{G}(U_{1_{\mathbb{D}}}) \\
& \mathbb{F}(1_{\mathbb{D}}) \arrow[ur, "\Phi_{01_{\mathbb{D}}}"'] & & & \mathbb{F}(U_{1_{\mathbb{D}}}) \arrow[ur, "\Phi_{1U_{1_{\mathbb{D}}}}"'] &
\end{tikzcd}$$

A double transformation is **braided monoidal** or **symmetric monoidal** if the double functors $\mathbb{F}$ and $\mathbb{G}$ are braided monoidal or symmetric monoidal, respectively.

Theorem 4.7. *Let $(\Phi, \phi)\colon (\mathbb{F}, F) \Rightarrow (\mathbb{G}, G)$ be a double transformation between two symmetric monoidal structured cospan double functors $(\mathbb{F}, F)\colon {}_L\mathbb{D} \to_{L'} \mathbb{D}'$ and $(\mathbb{G}, G)\colon {}_L\mathbb{D} \to_{L'} \mathbb{D}'$, where ${}_L\mathbb{D} = (\mathbb{D}, L\colon \mathbf{C} \to \mathbb{D}_0)$ and ${}_{L'}\mathbb{D}' = (\mathbb{D}', L'\colon \mathbf{C}' \to \mathbb{D}'_0)$. If $\phi\colon F \Rightarrow G$ is a* *symmetric monoidal transformation and* $\Phi\colon \mathbb{F} \Rightarrow \mathbb{G}$ *is a symmetric monoidal double transformation, then* $(\Phi, \phi)\colon (\mathbb{F}, F) \Rightarrow (\mathbb{G}, G)$ *is a symmetric monoidal double transformation of structured cospan double functors.*

Proof. The double transformation (Φ, ϕ) acts as ϕ on objects and vertical 1-morphisms. This means that the following diagrams commute.

$$\begin{tikzcd}
(\mathbb{F}, F)(a) \otimes (\mathbb{F}, F)(b) \arrow[r, "\phi_a \otimes \alpha_b"] \arrow[d, "\mu_{a,b}"'] & (\mathbb{G}, G)(a) \otimes (\mathbb{G}, G)(b) \arrow[d, "\mu'_{a,b}"] \\
(\mathbb{F}, F)(a \otimes b) \arrow[r, "\phi_{a\otimes b}"] & (\mathbb{G}, G)(a \otimes b)
\end{tikzcd}$$

$$\begin{tikzcd}
1_{L'\mathbb{D}'} \arrow[rr, "\epsilon'"] \arrow[dr, "\epsilon"'] & & (\mathbb{G}, G)(1_{L\mathbb{D}}) \\
& (\mathbb{F}, F)(1_{L\mathbb{D}}) \arrow[ur, "\phi_{1_{L\mathbb{D}}}"'] &
\end{tikzcd}$$

Similarly, the double transformation (Φ, ϕ) acts as Φ on horizontal 1-cells and 2-morphisms, which means that the following diagrams commute.

$$\begin{tikzcd}
(\mathbb{F}, F)(M) \otimes (\mathbb{F}, F)(N) \arrow[r, "\Phi_{1M} \otimes \Phi_{1N}"] \arrow[d, "\mu_{M,N}"'] & (\mathbb{G}, G)(M) \otimes (\mathbb{G}, G)(N) \arrow[d, "\mu'_{M,N}"] \\
(\mathbb{F}, F)(M \otimes N) \arrow[r, "\Phi_{1M\otimes N}"] & (\mathbb{G}, G)(M \otimes N)
\end{tikzcd}$$

$$\begin{tikzcd}
U_{1_{L'\mathbb{D}'}} \arrow[rr, "\delta'"] \arrow[dr, "\delta"'] & & (\mathbb{G}, G)(U_{1_{L\mathbb{D}}}) \\
& (\mathbb{F}, F)(U_{1_{L\mathbb{D}}}) \arrow[ur, "\Phi_{1U_{1_{L\mathbb{D}}}}"'] &
\end{tikzcd}$$

Thus $(\Phi, \phi)\colon (\mathbb{F}, F) \Rightarrow (\mathbb{G}, G)$ is a symmetric monoidal double transformation. ∎

5. Applications

As a first example, define a functor $L\colon \textbf{FinSet} \to \textbf{Graph}$ where given a finite set N, $L(N)$ is the discrete graph on N with no edges and given a function $f\colon N \to N'$, $L(f)\colon L(N) \to L(N')$ is the graph morphism that takes vertices of $L(N)$ to $L(N')$ as prescribed by the function f. This functor L preserves finite coproducts as it is left adjoint to the forgetful functor $U\colon \textbf{Graph} \to \textbf{FinSet}$ that takes a graph (E, N, s, t) to its underlying set of vertices N. The category **FinSet** is symmetric monoidal under coproducts and **Graph** is a topos and thus has pushouts.

Corollary 5.1. *Let $L\colon \textbf{FinSet} \to \textbf{Graph}$ be the symmetric monoidal functor defined above. Then there exists a symmetric monoidal double category ${}_L\mathbb{C}\mathrm{sp}(\textbf{Graph})$ consisting of:*

(i) finite sets as objects,
(ii) functions as vertical 1-morphisms,
(iii) cospans of graphs of the form

$$\begin{tikzcd} & d & \\ L(c) \arrow[ur] & & L(c') \arrow[ul] \end{tikzcd}$$

as horizontal 1-cells, and
(iv) maps of cospans of graphs as 2-morphisms, as in the following commutative diagram:

$$\begin{tikzcd} L(c) \arrow[r] \arrow[d, "L(f)"'] & d \arrow[d, "h"] & L(c') \arrow[l] \arrow[d, "L(g)"] \\ L(c'') \arrow[r] & d' & L(c''') \arrow[l] \end{tikzcd}$$

This symmetric monoidal double category is in fact isofibrant, and so we have the following due to Shulman [24].

Theorem 5.2. *There exists a symmetric monoidal bicategory $\textbf{GRAPH} = H({}_L\mathbb{C}\mathrm{sp}(\textbf{Graph}))$ which has:*

(i) finite sets as objects,
(ii) cospans of graphs of the form

$$\begin{tikzcd} & d & \\ L(c) \arrow[ur] & & L(c') \arrow[ul] \end{tikzcd}$$

as morphisms, and
(iii) maps of cospans of graphs as 2-morphisms, as in the following commutative diagram:

$$\begin{tikzcd} & d \arrow[dd, "h"] & \\ L(c) \arrow[ur] \arrow[dr] & & L(c') \arrow[ul] \arrow[dl] \\ & d' & \end{tikzcd}$$

We can then decategorify this symmetric monoidal bicategory **GRAPH** to obtain a symmetric monoidal category $D(\mathbf{GRAPH})$ consisting of:

(i) finite sets as objects, and
(ii) isomorphism classes of cospans of graphs of the form

$$\begin{tikzcd} & d & \\ L(c) \arrow[ur] & & L(c') \arrow[ul] \end{tikzcd}$$

as morphisms, where two cospans of graphs are in the same isomorphism class if the following diagram commutes:

$$\begin{tikzcd} & d \arrow[dd, "h"', "\sim"] & \\ L(c) \arrow[ur] \arrow[dr] & & L(c') \arrow[ul] \arrow[dl] \\ & d' & \end{tikzcd}$$

Here, the graph isomorphism $h\colon d \to d'$ is really a pair of bijections $f\colon N \to N'$ and $g\colon E \to E'$ between the vertex and edge sets of the graphs d and d' that make the following diagram commute:

$$\begin{tikzcd} d \arrow[d, "h"'] & E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "g"', "\sim"] & N \arrow[d, "\sim"', "f"] \\ d' & E' \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N' \end{tikzcd}$$

We can obtain a similar symmetric monoidal category using Fong's decorated cospan machinery. Define a lax symmetric monoidal functor $F\colon \mathbf{FinSet} \to \mathbf{Set}$ where for a finite set N, $F(N)$ is the large set of all possible graph structures on the finite set N, where a graph structure on a finite set N is given by a diagram in **Set** of the form:

$$\begin{tikzcd} E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N \end{tikzcd}$$

Denote this symmetric monoidal category as $F\mathrm{Cospan}(\mathbf{FinSet})$. Then we can define a functor $G\colon F\mathrm{Cospan}(\mathbf{FinSet}) \to D(\mathbf{GRAPH})$ that is the identity on objects and morphisms. This then solves the issue of two isomorphic graphs with isomorphic but not equal edges not being members of the same isomorphism class. Explicitly, given two graphs:

$$
\begin{tikzcd}
E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N
\end{tikzcd}
$$

$$
\begin{tikzcd}
E' \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N'
\end{tikzcd}
$$

together with two bijections $g\colon E \to E'$ and $f\colon N \to N'$

$$
\begin{tikzcd}
E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, dashed, "g"'] & N \arrow[d, dashed, "f"] \\
E' \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N'
\end{tikzcd}
$$

such that $g \circ s = s' \circ f$ and $g \circ t = t' \circ f$, the pair (f, g) is an isomorphism in $D(\mathbf{GRAPH})$ that does not exist in $F\mathrm{Cospan}(\mathbf{FinSet})$.

Next, we revisit several works of Fong, Pollard and the first author that utilize the decorated cospan approach in a more sophisticated manner and compare how things look from the structured cospans approach. One work of Fong and the first author studies 'passive linear networks' where a passive linear network Γ is given by a diagram in Set of the form:

$$
\begin{tikzcd}
\mathbb{F}^{+} & E \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N
\end{tikzcd}
$$

Here $\mathbb{F}$ is a field and the sets E and N denote the sets of edges and nodes, respectively, of the passive linear network Γ. An *open* passive linear network is then given by a cospan of finite sets:

$$
\begin{tikzcd}
& N & \\
X \arrow[ur, "i"] & & Y \arrow[ul, "o"']
\end{tikzcd}
$$

where the apex N is decorated with a passive linear network as above. Fong and the first author use the decorated cospan machinery of Fong to construct a symmetric monoidal category FCospan(FinSet) from a symmetric lax monoidal functor $F\colon \mathrm{FinSet} \to \mathrm{Set}$. This functor F is defined on objects by:

$$N \qquad \mapsto \qquad \{\mathbb{F}^+ \xleftarrow{r} E \underset{t}{\overset{s}{\rightrightarrows}} N\}$$

and on morphisms by

$$\begin{tikzcd} N \arrow[d, "f"'] \\ N' \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} \mathbb{F}^+ & E \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "\mathrm{id}_E"'] & N \arrow[d, "f"] \\ \mathbb{F}^+ & E \arrow[l, "r'"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N' \end{tikzcd}$$

Fong and the first author then define a 'black-box functor' $\blacksquare\colon F\mathrm{Cospan}(\mathrm{FinSet}) \to \mathrm{LagRel}_{\mathbb{F}}$ which is a symmetric monoidal functor that 'observes' the behavior at the inputs and outputs of these passive linear networks; these are the finite sets X and Y in the cospan above. Here, $\mathrm{LagRel}_{\mathbb{F}}$ is the symmetric monoidal category of symplectic $\mathbb{F}$-vector spaces and Lagrangian relations. The functor F then maps a passive linear network given by:

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

$$\mathbb{F}^+ \xleftarrow{r} E \underset{t}{\overset{s}{\rightrightarrows}} N$$

to a particular linear subspace of $\mathbb{R}^X \oplus \mathbb{R}^X \oplus \mathbb{R}^Y \oplus \mathbb{R}^Y$ and likewise a map between passive linear networks to a linear map between such linear subspaces; see the original paper [3] for details.

To fit the above construction into the framework of structured cospans, first we define a symmetric monoidal category $\mathbb{F}$Graph whose objects are given by $\mathbb{F}$-graphs:

$$\mathbb{F}^+ \xleftarrow{r} E \underset{t}{\overset{s}{\rightrightarrows}} N$$

and a morphism from this $\mathbb{F}$-graph to another:

$$\mathbb{F}^+ \xleftarrow{r'} E' \underset{t'}{\overset{s'}{\rightrightarrows}} N'$$

consists of a pair of functions $f\colon N \to N'$ and $g\colon E \to E'$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{F}^+ & E \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "g"'] & N \arrow[d, "f"] \\
\mathbb{F}^+ & E' \arrow[l, "r'"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N'
\end{tikzcd}$$

Next we define a left adjoint $L\colon \mathrm{FinSet} \to \mathbb{F}\mathrm{Graph}$ which is defined on sets by:

$$N \qquad \mapsto \qquad \begin{tikzcd}
\mathbb{F}^+ & \emptyset \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N
\end{tikzcd}$$

and on morphisms by:

$$\begin{tikzcd}
N \arrow[d, "f"'] \\
N'
\end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd}
\mathbb{F}^+ & \emptyset \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N \arrow[d, "f"] \\
\mathbb{F}^+ & \emptyset \arrow[l, "r'"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N'
\end{tikzcd}$$

Theorem 5.3. *The above functor $L\colon$ Finset $\to$ $\mathbb{F}$Graph is a left adjoint.*

Proof. The functor $L\colon \mathrm{FinSet} \to \mathbb{F}\mathrm{Graph}$ has a right adjoint given by the forgetful functor $R\colon \mathbb{F}\mathrm{Graph} \to \mathrm{FinSet}$ which maps an $\mathbb{F}$Graph

$$\begin{tikzcd}
\mathbb{F}^+ & E \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N
\end{tikzcd}$$

to its underlying vertex set N. We then have a natural isomorphism $\hom_{\mathbb{F}\mathrm{Graph}}(L(c), d) \cong \hom_{\mathrm{FinSet}}(c, R(d))$. $\square$

Theorem 5.4. *The symmetric monoidal category $\mathbb{F}$Graph has finite colimits.*

Proof. To show that $\mathbb{F}$Graph has finite colimits, it suffices to show that $\mathbb{F}$Graph has an initial object and pushouts. The initial object is given by:

$$\begin{tikzcd}
\mathbb{F}^+ & \emptyset \arrow[l, "!"'] \arrow[r, bend left, "!"] \arrow[r, bend right, "!"'] & \emptyset
\end{tikzcd}$$

Given a span in $\mathbb{F}$Graph, which would look like:

$$
\begin{tikzcd}
\mathbb{F}^+ & \mathbb{F}^+ & \mathbb{F}^+ \\
E_1 \arrow[u, "r'"] \arrow[d, bend right, "t'"'] \arrow[d, bend left, "s'"] & E \arrow[l, "g_1"'] \arrow[r, "g_2"] \arrow[u, "r"] \arrow[d, bend right, "t"'] \arrow[d, bend left, "s"] & E_2 \arrow[u, "r''"] \arrow[d, bend right, "t''"'] \arrow[d, bend left, "s''"] \\
N_1 & N \arrow[l, "f_1"'] \arrow[r, "f_2"] & N_2
\end{tikzcd}
$$

the pushout is given by the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{F}^+ & \mathbb{F}^+ & \mathbb{F}^+ \\
E_1 \arrow[u, "r'"] \arrow[d, bend right, "t'"'] \arrow[d, bend left, "s'"] \arrow[ddddr, bend right=60, "\psi_1"'] & E \arrow[l, "g_1"'] \arrow[r, "g_2"] \arrow[u, "r"] \arrow[d, bend right, "t"'] \arrow[d, bend left, "s"] & E_2 \arrow[u, "r''"] \arrow[d, bend right, "t''"'] \arrow[d, bend left, "s''"] \arrow[ddddl, bend left=60, "\psi_2"] \\
N_1 \arrow[dr, "\phi_1"] & N \arrow[l, "f_1"'] \arrow[r, "f_2"] & N_2 \arrow[dl, "\phi_2"'] \\
& N_1 +_N N_2 & \\
& & \\
& E_1 +_E E_2 \arrow[uu, bend left, "t' +_t t''"] \arrow[uu, bend right, "s' +_s s''"'] \arrow[d, "\textbf{something}"'] & \\
& \mathbb{F}^+ &
\end{tikzcd}
$$

where $\phi_i \colon N_i \to N_1 +_N N_2$ and $\psi_i \colon E_i \to E_1 +_E E_2$ are the canonical maps into the respective pushouts in FinSet. $\square$

Theorem 5.5. *Let* $L \colon \mathrm{FinSet} \to \mathbb{F}\mathrm{Graph}$ *be the left adjoint as described above. Then there exists a symmetric monoidal double category* ${}_L\mathrm{Csp}(\mathbb{F}\mathrm{Graph})$ *which has:*

(i) *finite sets as objects,*
(ii) *functions as vertical 1-morphisms,*
(iii) *cospans of finite sets where the apex is decorated with the stuff of an* $\mathbb{F}$*-graph*

$$
\begin{tikzcd}
& N & \\
L(X) \arrow[ur, "i"] & & L(Y) \arrow[ul, "o"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{F}^+ & E \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N
\end{tikzcd}
$$

as horizontal 1-cells, and

(iv) maps of cospans of finite sets where the apices of the cospans are decorated with the stuff of an $\mathbb{F}$*-graph*

$$\begin{tikzcd} L(X) \arrow[r, "i"] \arrow[d, "L(h_1)"'] & N \arrow[d, "f"] & L(Y) \arrow[l, "o"'] \arrow[d, "L(h_2)"] \\ L(X') \arrow[r, "i'"] & N' & L(Y') \arrow[l, "o'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F}^+ & E \arrow[l] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"] \arrow[d, "g"'] & N \arrow[d, "f"] \\ \mathbb{F}^+ & E' \arrow[l] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"] & N' \end{tikzcd}$$

as 2-morphisms.

Proof. As $\mathbb{F}$Graph has finite colimits, we get a double category Csp($\mathbb{F}$Graph) and hence a structured cospan double category ${}_L$Csp($\mathbb{F}$Graph). □

This symmetric monoidal double category is in fact isofibrant and so we can apply a result of Shulman [24] to obtain a symmetric monoidal bicategory:

Theorem 5.6. *There exists a symmetric monoidal bicategory* $\mathbb{F}$**Graph** *where:*

(i) Objects are given by finite sets,

(ii) morphisms are given by cospans of finite sets whose apices are decorated with the stuff of an $\mathbb{F}$*-graph, and*

(iii) 2-morphisms are given by maps of cospans such that the following diagrams commute.

$$\begin{tikzcd} & N \arrow[dd, "f"] & \\ L(X) \arrow[ur, "i"] \arrow[dr, "i'"'] & & L(Y) \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & N' & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F}^+ & E \arrow[l] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"] \arrow[d, "g"'] & N \arrow[d, "f"] \\ \mathbb{F}^+ & E' \arrow[l] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"] & N' \end{tikzcd}$$

Proof. We have that $\mathbb{F}$**Graph** $= H({}_L\mathrm{Csp}(\mathbb{F}\mathrm{Graph}))$. □

We can then decategorify this symmetric monoidal bicategory $\mathbb{F}$**Graph** to obtain a symmetric monoidal category $D(\mathbb{F}$**Graph**$)$ where:

(i) Objects are given by finite sets, and

(ii) morphisms are given by isomorphism classes of cospans of finite sets whose apices are equipped with the stuff of an $\mathbb{F}$-graph, where two morphisms are in the same isomorphism class if the following diagrams commute:

$$\begin{tikzcd} & N \arrow[dd, "f"', "\sim"] & \\ L(X) \arrow[ur, "i"] \arrow[dr, "i'"'] & & L(Y) \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & N' & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F}^+ & E \arrow[l] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"] \arrow[d, "g"', "\sim"] & N \arrow[d, "f"', "\sim"] \\ \mathbb{F}^+ & E' \arrow[l] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"] & N' \end{tikzcd}$$

We thus have the two symmetric monoidal categories FCospan(FinSet) and $D(\mathbb{F}\mathbf{Graph})$ each of which have the same objects. However, the second of these contains more isomorphisms. For example, consider the following two passive linear networks:

$$
\begin{tikzcd}
 & N & \\
X \arrow[ur, "i"] & & Y \arrow[ul, "o"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N & \\
X \arrow[ur, "i"] & & Y \arrow[ul, "o"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{F}^+ & E \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F}^+ & E' \arrow[l, "r'"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N
\end{tikzcd}
$$

where there is a bijection $\phi\colon E \to E'$ such that $s = s' \circ \phi$ and $t = t' \circ \phi$; this just says that the two networks look the same but have different edge labels. Then these two passive linear networks each constitutes distinct isomorphism classes in the symmetric monoidal category FCospan(FinSet), but are members of the same isomorphism class in the symmetric monoidal category $D(\mathbb{F}\mathbf{Graph})$.

We can then define a functor $G\colon F\text{Cospan(FinSet)} \to D(\mathbb{F}\mathbf{Graph})$ that is the identity on objects and morphisms and then consider the following diagram:

$$
\begin{tikzcd}
F\text{Cospan(FinSet)} \arrow[r, "\blacksquare"] \arrow[d, "G"'] & \mathrm{LagRel}_{\mathbb{F}} \\
D(\mathbb{F}\mathbf{Graph}) \arrow[ur, dashed, "\blacksquare"'] &
\end{tikzcd}
$$

Here the top functor $\blacksquare\colon F\text{Cospan(FinSet)} \to \mathrm{LagRel}_{\mathbb{F}}$ is the original black-boxing functor constructed by Fong and the first author and we are 'extending' the domain of this functor from FCospan(FinSet) to $D(\mathbb{F}\mathbf{Graph})$. These two categories have the same objects but the former has few isomorphism classes in the sense of the example given above.

We can mimic this remedy to the other two applications of decorated cospans. In a previous work of Fong, Pollard and the first author [4], a symmetric monoidal category Mark is created from a symmetric lax monoidal functor $F\colon \mathrm{FinSet} \to \mathrm{Set}$. This functor is defined similarly as the functor F from the previous example: for a finite set N, $F(N)$ is the large set of all Markov processes whose underlying set of state spaces is the set N, where a Markov process on N is given by a diagram in Set of the form:

$$
\begin{tikzcd}
(0,\infty) & E \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N
\end{tikzcd}
$$

As in the previous example, the labels of the edges coming from the edge set E play no significant role, and a Markov process can instead be viewed as a finite set equipped with the extra *structure* of an 'infinitesimal stochastic operator' [1, 5]. The symmetric monoidal category Mark has finite sets for objects and isomorphism classes of cospans whose apices are the underlying set of states of a Markov process for morphisms.

$$
\begin{tikzcd}
 & N & \\
X \arrow[ur, "i"] & & Y \arrow[ul, "o"']
\end{tikzcd}
$$

$$\begin{tikzcd} (0,\infty) & E \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N \end{tikzcd}$$

As before, two Markov processes in Mark in the decorated cospan framework can only be in the same isomorphism class if both Markov processes have E as their set of edges. By defining a left adjoint $L\colon \mathrm{FinSet} \to \mathrm{Mark}$ that maps a finite set N to the Markov process with state space N and no edges, also known as the *discrete* Markov process on N, and a function $f\colon N \to N'$ to the obvious map of discrete Markov processes, we get the following.

Theorem 5.7. *The functor* $L\colon \mathrm{FinSet} \to \mathrm{Mark}$ *defined on objects by:*

$$N \quad \mapsto \quad \{\begin{tikzcd} \mathbb{F}^+ & \emptyset \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N \end{tikzcd}\}$$

and on morphisms by:

$$\begin{tikzcd} N \arrow[d, "f"'] \\ N' \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathbb{F}^+ & \emptyset \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N \arrow[d, "f"] \\ \mathbb{F}^+ & \emptyset \arrow[l, "r'"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N' \end{tikzcd}$$

is left adjoint to the forgetful functor $R\colon \mathrm{Mark} \to \mathrm{FinSet}$ *that sends a Markov process to its underlying finite set of states.*

Proof. This is similar to the proof that $L\colon \mathrm{FinSet} \to \mathbb{F}\mathrm{Graph}$ is a left adjoint. □

Theorem 5.8. *The symmetric monoidal category Mark has finite colimits.*

Proof. This is also similar to why the symmetric monoidal category $\mathbb{F}$Graph has finite colimits - it can be shown that Mark has an initial object given by the Markov process with no edges and the empty set as its set of states as well as pushouts. □

Theorem 5.9. *Let* $L\colon \mathrm{FinSet} \to \mathrm{Mark}$ *be the left adjoint as described above. Then there exists a symmetric monoidal double category* ${}_L\mathrm{Csp}(\mathrm{Mark})$ *which has:*

(i) Finite sets as objects,
(ii) functions as vertical 1-morphisms,
(iii) cospans of finite sets where the apex is decorated with the stuff of a Markov process:

$$\begin{tikzcd} & N & \\ L(X) \arrow[ur, "i"] & & L(Y) \arrow[ul, "o"'] \end{tikzcd}$$

$$\begin{tikzcd} (0,\infty) & E \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N \end{tikzcd}$$

and

(iv) maps of cospans of finite sets where the apices of the cospans are decorated with the stuff of a Markov process:

$$\begin{tikzcd} L(X) \arrow[r, "i"] \arrow[d, "L(h_1)"'] & N \arrow[d, "f"'] & L(Y) \arrow[l, "o"'] \arrow[d, "L(h_2)"'] \\ L(X') \arrow[r, "i'"] & N' & L(Y') \arrow[l, "o'"'] \end{tikzcd} \qquad \begin{tikzcd} (0,\infty) & E \arrow[l] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "g"'] & N \arrow[d, "f"] \\ (0,\infty) & E' \arrow[l] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N' \end{tikzcd}$$

Proof. As Mark has pushouts, we get a double category Csp(Mark) and hence a structured cospan double category ${}_L$Csp(Mark). □

This double category is also isofibrant, and so by Shulman's result [24] we can again obtain a symmetric monoidal bicategory.

Theorem 5.10. *There exists a symmetric monoidal bicategory* **Mark** *where:*

(i) Objects are given by finite sets,
(ii) morphisms are given by cospans of finite sets whose apices are equipped with the stuff of a Markov process, and
(iii) 2-morphisms are given by maps of cospans whose apices are equipped with the stuff of a Markov process such that the following diagams commute.

$$\begin{tikzcd} & N \arrow[dd, "f"'] & \\ L(X) \arrow[ur, "i"] \arrow[dr, "i'"'] & & L(Y) \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & N' & \end{tikzcd} \qquad \begin{tikzcd} (0,\infty) & E \arrow[l] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "g"'] & N \arrow[d, "f"] \\ (0,\infty) & E' \arrow[l] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N' \end{tikzcd}$$

We can then decategory **Mark** to obtain a symmetric monoidal category D(**Mark**) whose objects are finite sets and whose morphisms are isomorphism classes as such, where two Markov processes are in the same isomorphism class if the following diagrams commute:

$$\begin{tikzcd} & N \arrow[dd, "f"', "\sim"] & \\ L(X) \arrow[ur, "i"] \arrow[dr, "i'"'] & & L(Y) \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & N' & \end{tikzcd} \qquad \begin{tikzcd} (0,\infty) & E \arrow[l] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "g"', "\sim"] & N \arrow[d, "f"', "\sim"] \\ (0,\infty) & E' \arrow[l] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & N' \end{tikzcd}$$

Finally, we can then extend the black-boxing functor ■: Mark → LinRel constructed by Fong, Pollard and the first author by defining a functor G: Mark → D(**Mark**) which is the identity on objects and morphisms. To do this, we define a new black-boxing functor

$\blacksquare\colon D(\mathbf{Mark}) \to \mathrm{LinRel}$ which makes the following diagram commute.

$$\begin{tikzcd} \mathrm{Mark} \arrow[r, "\blacksquare"] \arrow[d, "G"'] & \mathrm{LinRel} \\ D(\mathbf{Mark}) \arrow[ur, dashed, "\blacksquare"'] & \end{tikzcd}$$

For the last example, Pollard and the first author have constructed a black-boxing functor $\blacksquare\colon \mathrm{Dynam} \to \mathrm{SemiAlgRel}$. Here, Dynam is the symmetric monoidal category of 'open dynamical systems' and SemiAlgRel is the symmetric monoidal category of 'semialgebraic relations'. A particular kind of dynamical system is given by a Petri net as mentioned in the introduction.

Definition 5.11. A **Petri net** consists of a finite set S of *species*, a finite set T of *transitions* and functions $s, t\colon S \times T \to \mathbb{N}$. For a species $\sigma \in S$ and a transition $\tau \in T$, $s(\sigma, \tau)$ is the number of times the species σ appears as an input for the transition τ and $t(\sigma, \tau)$ is the number of times the species σ appears as an output for the transition τ. A **Petri net with rates** is a Petri net together with a function $r\colon T \to [0, \infty)$ where $r(\tau)$ is the rate of the transition τ.

We can also say that a Petri net with rates is a diagram of the form:

$$\begin{tikzcd} {[0,\infty)} & T \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathbb{N}[S] \end{tikzcd}$$

where $\mathbb{N}[S]$ is the free commutative monoid on the set S. As a special case of the symmetric monoidal category Dynam created by Pollard and the first author, there is a sub-symmetric monoidal category Petri whose objects are given by finite sets and whose morphisms are given by isomorphism classes of cospans whose apices are decorated with the stuff of a Petri net with rates.

$$\begin{tikzcd} & S & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} \qquad \begin{tikzcd} {[0,\infty)} & T \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathbb{N}[S] \end{tikzcd}$$

Two Petri nets with rates are in the same isomorphism class if the following diagrams commute:

$$\begin{tikzcd} & S \arrow[dd, "f"', "\sim"] & \\ X \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & S' & \end{tikzcd} \qquad \begin{tikzcd} {[0,\infty)} & T \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "\mathrm{id}_T"'] & \mathbb{N}[S] \arrow[d, "\mathbb{N}[f]", "\sim"'] \\ {[0,\infty)} & T \arrow[l, "r"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & \mathbb{N}[S'] \end{tikzcd}$$

We can obtain a similar category using structured cospan double categories: define a functor $L\colon \mathrm{FinSet} \to \mathrm{Petri}$ where for a finite set S, $L(S)$ is the *discrete* Petri net with S as its set of species and no transitions. In other words,

$$S \qquad \mapsto \qquad \begin{tikzcd} {[0,\infty)} & \emptyset \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathbb{N}[S] \end{tikzcd}$$

Theorem 5.12. *The functor* $L\colon \mathrm{FinSet} \to \mathrm{Petri}$ *defined above is left adjoint to the forgetful functor* $R\colon \mathrm{Petri} \to \mathrm{FinSet}$.

Proof. This is similar as to why the functors used in the previous two applications are also left adjoints. □

Theorem 5.13. *The symmetric monoidal category Petri has finite colimits.*

Proof. This is shown in the work of the Master and the first author [18] but can also be shown in a manner similar to the previous two applications. □

As the functor $L\colon \mathrm{FinSet} \to \mathrm{Petri}$ is a left adjoint and the category Petri has finite colimits [18], we have the following:

Theorem 5.14. *There exists a symmetric monoidal double category* ${}_L\mathrm{Csp}(\mathrm{Petri})$ *which has:*

(i) Finite sets as objects,
(ii) functions as vertical 1-morphisms,
(iii) cospans of finite sets whose apices are equipped with the stuff of a Petri net as horizontal 1-cells, and
(iv) maps of cospans as above as 2-morphisms, as in the following commutative diagrams.

$$\begin{tikzcd} L(X) \arrow[r, "i"] \arrow[d, "L(f_1)"'] & S \arrow[d, "f"] & L(Y) \arrow[l, "o"'] \arrow[d, "L(f_2)"] \\ L(X') \arrow[r, "i'"] & S' & L(Y') \arrow[l, "o'"'] \end{tikzcd} \qquad \begin{tikzcd} {[0,\infty)} & T \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "g"'] & \mathbb{N}[S] \arrow[d, "\mathbb{N}[f]"] \\ {[0,\infty)} & T' \arrow[l, "r'"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & \mathbb{N}[S'] \end{tikzcd}$$

This symmetric monoidal double category is in fact isofibrant, and so we can once again obtain a symmetric monoidal bicategory using Shulman's Theorem [24].

Theorem 5.15. *There exists a symmetric monoidal bicategory* **Petri** *which has:*

(i) Finite sets as objects,
(ii) cospans of finite sets whose apices are equipped with the stuff of a Petri net, and
(iii) maps of cospans whose apices are equipped with the stuff of a Petri net such that the following diagrams commute.

$$\begin{tikzcd} & S \arrow[dd, "f"] & \\ L(X) \arrow[ur, "i"] \arrow[dr, "i'"'] & & L(Y) \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & S' & \end{tikzcd} \qquad \begin{tikzcd} {[0,\infty)} & T \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "g"'] & \mathbb{N}[S] \arrow[d, "\mathbb{N}[f]"] \\ {[0,\infty)} & T' \arrow[l, "r"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & \mathbb{N}[S'] \end{tikzcd}$$

Proof. We have that **Petri** $= H({}_L\mathrm{Csp}(\mathrm{Petri}))$. □

Once again, we can then decategorify this bicategory **Petri** to obtain a symmetric monoidal category $D(\mathbf{Petri})$ which has:

(i) finite sets as objects, and
(ii) isomorphism classes of cospans of finite sets whose apices are equipped with the stuff of a Petri net as morphisms, where two morphisms are in the same isomorphism class if the following diagrams commute:

$$\begin{tikzcd}
& S \arrow[dd, "f"', "\sim"] & \\
L(X) \arrow[ur, "i"] \arrow[dr, "i'"'] & & L(Y) \arrow[ul, "o"'] \arrow[dl, "o'"] \\
& S' &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[0,\infty)} & T \arrow[l, "r"'] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[d, "g"', "\sim"] & \mathbb{N}[S] \arrow[d, "\mathbb{N}[f]", "\sim"'] \\
{[0,\infty)} & T' \arrow[l, "r"'] \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] & \mathbb{N}[S']
\end{tikzcd}$$

Finally, as a special case of the black-boxing functor $\blacksquare\colon \mathrm{Dynam} \to \mathrm{SemiAlgRel}$ constructed by Pollard and the first author, we can obtain a black-boxing functor $\blacksquare\colon \mathrm{Petri} \to \mathrm{SemiAlgRel}$ [5] and then extend this functor by defining a functor $G\colon \mathrm{Petri} \to D(\mathbf{Petri})$ that is the identity on objects and morphisms. We can then extend the domain of the functor $\blacksquare\colon \mathrm{Petri} \to \mathrm{SemiAlgRel}$ to obtain a functor $\blacksquare\colon D(\mathbf{Petri}) \to \mathrm{SemiAlgRel}$ where this second black-boxing functor is defined on objects and morphisms in the same way that the first one is.

$$\begin{tikzcd}
\mathrm{Petri} \arrow[r, "\blacksquare"] \arrow[d, "G"'] & \mathrm{SemiAlgRel} \\
D(\mathbf{Petri}) \arrow[ur, dashed, "\blacksquare"'] &
\end{tikzcd}$$

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