1. Structured cospans

Definition 1.1. Given a functor $L: \mathbf{A} \to \mathbf{X}$, a L-structured cospan is a diagram of form $La \to x \leftarrow Lb$.

Definition 1.2. Fix a category **X** and a functor $L: \mathbf{A} \to \mathbf{X}$. Denote by L-**Csp** the category with objects are those from **A** and with morphisms of type $a \to b$ are isomorphism classes of L-structured cospans $La \to x \leftarrow Lb$.

Definition 1.3. Denote by L-**StrCsp** the category whose objects are L-open objects and arrows are triples (f, g, h) that fit into a commuting diagram

Theorem 1.4. Let $L \dashv R \colon \mathbf{A} \to \mathbf{X}$ be an adjunction between topoi. Then $L\text{-}\mathbf{StrCsp}$ is a topos.

2. Rewriting

Definition 2.1. A category with pullbacks is **adhesive** if pushouts along monics exist and are *Van Kampen*.

Theorem 2.2. Topoi are adhesive.

Corollary 2.3. Let $L \dashv R : \mathbf{A} \to \mathbf{X}$ be an adjunction between topoi. The category L-StrCsp is adhesive.

Definition 2.4. For **C** an adhesive category, an **C-rewrite rule** (often called a production) is a span $a \leftarrow b \rightarrow c$ inside **C**. When both legs of the span are monic, we say the rewrite rule is **linear**.

Definition 2.5. Given composable arrows $a \to b \to y$ we say that an arrow $a \to x$ is a **pushout complement** if it fits into a pushout diagram

Definition 2.6. Given a **C**-rewrite rule $a \leftarrow b \rightarrow c$ and a **C**-arrow $a \rightarrow x$ such that $b \rightarrow a \rightarrow x$ has a pushout complement, a **derived (linear) rewrite rule** is the bottom row of the induced double pushout diagram

Definition 2.7. A (linear) grammar consists of an adhesive category **A** and a set of (linear) **A**-rewrite rules. Observe that **A**-rewrite rules are actually arrows in $\mathbf{Sp}(\mathbf{A})$. Given a grammar Γ , the subcategory $\mathcal{L}(\Gamma)$ of $\mathbf{Sp}(\mathbf{A})$ generated by the set of rewrites derived from Γ is called a **language**.

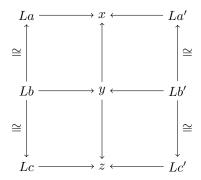
Lemma 2.8. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ be a symmetric monoidal category with pullbacks and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a symmetric monoidal topos. Let $L \dashv R \colon \mathbf{A} \to \mathbf{X}$ be an adjunction where L preserves pullbacks.

Fix a grammar Γ in the topos L-StrCsp. The generated language $\mathcal{L}(\Gamma)$ is a sub-bicategory of Sp(L-StrCsp).

3. Non-linear rewriting of open objects

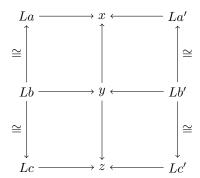
Definition 3.1. Let **A** be a category with pullbacks, **X** be a category with pullbacks and pushouts, and $L: \mathbf{A} \to \mathbf{X}$ be a functor preserving pullbacks. Denote by $\mathcal{P}(\mathbf{Sp}(L\mathbf{-StrCsp}))$ the preorder whose objects are $L\mathbf{-open}$ objects and arrows $(La \to x \leftarrow La') \leq (Lc \to x \leftarrow Lc')$ whenever there is a $\mathbf{Sp}(L\mathbf{-StrCsp})\mathbf{-arrow}$ with

form



Definition 3.2. Let **A** be a category with pullbacks, **X** a category with pullbacks and pushouts, and $L: \mathbf{A} \to \mathbf{X}$ be a functor preserving pullbacks. Define the double category L- \mathbb{R} ewrite to have object category $\mathbf{core}(\mathbf{Sp}(\mathbf{A}))$ and have arrow category \mathbf{C} as described in Definition 3.1.

Alternatively, L- \mathbb{R} ewrite is the double category with \mathbf{A} -objects as 0-cells, spans in \mathbf{A} with isomorphic legs as vertical 1-cells, L-open objects as horizontal 1-cells, and a unique 2-cell if there exists a commuting diagram in \mathbf{X} of form



Theorem 3.3. The double category L-Rewrite is isofibrant.

Theorem 3.4. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories so that \mathbf{A} has a pullbacks and \mathbf{X} has pullbacks and pushouts. If $L \colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ preserves pullbacks, then $(L\text{-}\mathbb{R}\mathbf{ewrite}, \otimes, I)$ is a symmetric monoidal double category with \otimes defined by

$$(La \to x \leftarrow Lb) \otimes (Lc \to y \leftarrow Ld) \coloneqq L(a \otimes_{\mathbf{A}} c) \to x \otimes_{\mathbf{X}} y \leftarrow L(b \otimes_{\mathbf{A}} d)$$
 and I defined by

$$I := (LI_{\mathbf{A}} \to I_{\mathbf{X}} \leftarrow LI_{\mathbf{X}}).$$

Theorem 3.5. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories where \mathbf{A} has pullbacks and \mathbf{X} has pullbacks and pushouts. Let $L \colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a pullback preserving symmetric monoidal functor.

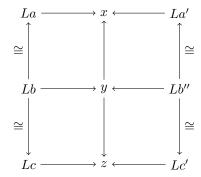
The horizontal edge bicategory L-Rewrite := $\mathcal{H}(L\text{-Rewrite})$ in the sense of Shulman is symmetric monoidal. Moreover, if the monoidal products $\otimes_{\mathbf{A}}$ and $\otimes_{\mathbf{X}}$ are coproducts, then the symmetric monoidal bicategory L-Rewrite is compact closed.

Theorem 3.6. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories so that \mathbf{A} has a pullbacks and \mathbf{X} has pullbacks and pushouts. Let $L \colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a pullback preserving functor.

The bicategory L-Rewrite is a bicategory of relations in the sense of Carboni and Walters.

Theorem 3.7. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ be a symmetric monoidal category with pullbacks and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a symmetric monoidal topos. Let $L \dashv R \colon \mathbf{A} \to \mathbf{X}$ be an adjunction where L preserves pullbacks.

Suppose each element from a grammar Γ in L-StrCsp is of the form

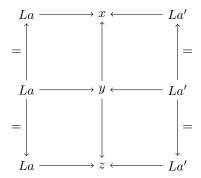


Then Γ generates a sub-double category of L-Rewrite as follows:

- generate the sub-bicategory $\mathcal{L}(\Gamma) \subseteq \mathbf{Sp}(L\text{-}\mathbf{StrCsp})$ as in Lemma 2.8,
- with $\mathcal{L}(\Gamma)$, define the subcategory $\mathcal{P}(\mathcal{L}(\Gamma))$ as in Definition 3.1,

Form the category as described in Definition ?? from $\mathcal{L}(\Gamma)$. Pair this, as an arrow category, with the object category $\mathbf{core}(\mathbf{Sp}(\mathbf{A}))$.

Theorem 3.8. If Γ has only elements of the form



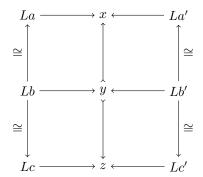
then Γ generates a sub-bicategory of L-Rewrite. This sub-bicategory corresponds to the sub-bicategory of L-Rewrite obtained by passing the construction through L-Rewrite first, then applying $\mathcal{H}(-)$.

4. Linear rewriting of open objects

X shd b topos for intrchng

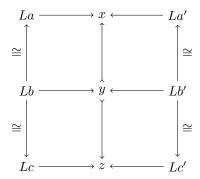
Definition 4.1. Let **A** be a category with pullbacks, **X** be a topos, and $L: \mathbf{A} \to \mathbf{X}$ be a functor preserving pullbacks. Denote by $\mathcal{C}(\mathbf{Sp}(L\text{-}\mathbf{Str}\mathbf{Csp}))$ the category whose

objects are L-open objects and arrows are isomorphism classes of 1-cells of form



where the arrows marked " \rightarrow " are monic.

Definition 4.2. Denote by L-MonRewrite the double category with A-objects as 0-cells, spans in A whose legs are isomorphisms as vertical 1-cells, L-open objects as horizontal 1-cells, and commuting diagrams in X of form



Theorem 4.3. The double category L-MonRewrite is isofibrant.

Theorem 4.4. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories where \mathbf{A} has pullbacks and \mathbf{X} is a topos. Let $L \colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a pullback preserving functor. Then $(L\text{-}Mon\mathbb{R}ewrite, \otimes, I)$ is a symmetric monoidal double category with \otimes defined by

$$(La \to x \leftarrow Lb) \otimes (Lc \to y \leftarrow Ld) \coloneqq L(a \otimes_{\mathbf{A}} c) \to x \otimes_{\mathbf{X}} y \leftarrow L(b \otimes_{\mathbf{A}} d)$$

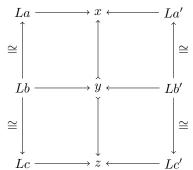
and I by

$$I := (LI_{\mathbf{A}} \to I_{\mathbf{X}} \leftarrow LI_{\mathbf{X}}).$$

Theorem 4.5. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories where \mathbf{A} has pullbacks and \mathbf{X} is a topos. Let $L \colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}})$ be an adjunction where L preserves pullbacks.

The horizontal edge bicategory L-MonRewrite := $\mathcal{H}(L\text{-}MonRewrite})$ in the sense of Shulman is symmetric monoidal. Moreover, if the monoidal products $\otimes_{\mathbf{A}}$ and $\otimes_{\mathbf{X}}$ are coproducts, then the symmetric monoidal bicategory L-MonRewrite is compact closed.

Theorem 4.6. Suppose each element from a grammar Γ in L-StrCsp is of the form



then Γ generates a sub-double category $\langle \langle \Gamma \rangle \rangle$ of L-MonRewrite. The recipe is get the language $\mathcal{L}(\Gamma) \subseteq \mathbf{Sp}(L\text{-}\mathbf{StrCsp})$. Form the category as described in Definition ?? from $\mathcal{L}(\Gamma)$. Pair this, as an arrow category, with the object category $\mathbf{core}(\mathbf{Sp}(\mathbf{A}))$.

Theorem 4.7. Same as above with monics thrown in.