REWRITING OPEN OBJECTS

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ABSTRACT. open graphs, their rewrites, and an application

1. INTRODUCTION

Our goal is to model open networks and their rewrites. By an *open network*, we mean a network together with a boundary. To make this precise, we begin with a category of 'input and output types' **C** and another category of 'networks' **D**. To equip a network, an object of **D**, with a boundary, a pair of objects from **C**, we use an adjunction

$$\begin{tikzcd} C \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & D \arrow[l, shift left, "R"] \end{tikzcd}$$

With this setup, we focus on three categories.

The first category, denoted $L\text{-}\mathrm{Span}(\mathbf{D})$, has as objects, those from **C**, and as arrows, cospans of the form

$$Lc \to d \leftarrow Lc'$$

inside of **D**.

The second category, denoted $L\text{-}\mathrm{Open}$, has cospans

$$Lc \to d \leftarrow Lc'$$

in **D** for objects and triples of arrows (f, g, h) such that

$$\begin{tikzcd} Lc \arrow[r] \arrow[d, "Lf"'] & d \arrow[d, "g"'] & Lc' \arrow[l] \arrow[d, "Lh"'] \\ Lc'' \arrow[r] & d' & Lc''' \arrow[l] \end{tikzcd}$$

commutes. We show that, when **C** and **D** are topoi, then so is $L\text{-}\mathrm{Open}$.

The third category, denoted $L\text{-}\mathrm{Rewrite}$, again has cospans

$$Lc \to d \leftarrow Lc'$$

in **D** for objects and *cubical spans of cospans*, that is commuting diagrams

$$\begin{tikzcd} Lc \arrow[r] & d & Le \arrow[l] \\ Lc' \arrow[r] \arrow[u] \arrow[d] & d' \arrow[u] \arrow[d] & Le' \arrow[l] \arrow[u] \arrow[d] \\ Lc'' \arrow[r] & d'' & Le'' \arrow[l] \end{tikzcd}$$

for arrows.

How do these three categories help us to model open networks? To answer this, we first make the observation that cospans of the form

$$Lc \to d \leftarrow Lc'$$

have showed up in each of the above categories. We call such cospans L-*open objects*. The term "open" indicates that we are thinking of d as an object that can 'interact' with certain elements. More concretely, we say that d has inputs Lc and outputs Lc' which allow d to be glued together with any other L-open object with outputs Lc or inputs Lc'. This would give us a zig-zag which we turn into an L-open object via pushout. But this is exactly the composition in $L\text{-}\operatorname{Span}(\mathbf{D})$. Hence, through their 'openness' we can think of L-open objects as arrows. This is not the only perspective we take, however.

Through the categories $L\text{-}\operatorname{Open}$ and $L\text{-}\operatorname{Rewrite}$, we can think of L-open objects as, well, objects. Certainly, the arrows of $L\text{-}\operatorname{Open}$ are the best candidate for a morphism of L-open objects. We show that $L\text{-}\operatorname{Open}$ is actually a topos. Then, by work of Lack and Sobocinski, we know that L-open objects admit a nice (double pushout) rewriting theory. The sort of rewriting theory we are interested in, and that Lack and Sobocinski study, uses spans

$$\ell \to k \leftarrow r$$

to say that the object ℓ is rewritten to the object r, where k is some interface common to both ℓ and r. Translating this to the topos $L\text{-}\operatorname{Open}$, we consider spans of L-open objects which are exactly the arrows for $L\text{-}\operatorname{Rewrite}$. Therefore, we think of $L\text{-}\operatorname{Open}$ as the category of L-open objects with their morphisms and $L\text{-}\operatorname{Rewrite}$ as the category of L-open objects and their rewrite rules.

HERES A CHANGE