1 Non-linear rewriting

(Current goal : define double category non-linear rewriting. subgoals: define object // arrow categories)

Lemma 1.1. *Let $(\mathbf{A}, +, 0_{\mathbf{A}}, \tau_{\mathbf{A}})$ be a cocartesian category that is locally cartesian closed. There is a symmetric monoidal category $(\mathbf{core}(\mathbf{Span}(\mathbf{A})), \otimes, I, \tau)$ defined as follows:*

- *$\mathbf{core}(\mathbf{Span}(\mathbf{A}))$ is the subcategory of $\mathbf{Span}(\mathbf{A})$ consisting of all objects and whose arrows have invertible legs,*
- *$\otimes$ is the pointwise application of $+$,*
- *I is the span consisting of identities on $0_{\mathbf{A}}$,*
- *τ is the pointwise application of $\tau_{\mathbf{A}}$.*

Proof. The only non-trivial thing to check is that the interchange law holds between tensor and composition. That is, given two pairs of composable spans $a \leftarrow b \rightarrow c$, $c \leftarrow d \rightarrow e$ and $a' \leftarrow b' \rightarrow c'$, $c' \leftarrow d' \rightarrow e'$, we show that the span obtained by tensoring before composing

$$a + a' \leftarrow (b + b') \times_{c+c'} (d + d') \rightarrow e + e'$$

is equal to the span obtained by composing before tensoring

$$a + a' \leftarrow (b \times_c d) + (b' \times_{c'} d') \rightarrow e + e'.$$

In this context, equality means isomorphic as spans. But this follows from local cartesian closedness, because pullback functors are all left adjoints. $\square$

Lemma 1.2. *Let $(\mathbf{A}, +, 0_{\mathbf{A}}, \tau_{\mathbf{A}})$ and $(\mathbf{X}, +, 0_{\mathbf{X}}, \tau_{\mathbf{X}})$ be categories that are cocartesian and locally cartesian closed. Also, suppose that $\mathbf{X}$ has pushouts. Let $L \colon \mathbf{A} \rightarrow \mathbf{X}$ be a cocartesian functor that preserves pullbacks. There is a symmetric monoidal preorder $(\mathbf{P}, \otimes, I, \tau)$ defined as follows:*

- *$\mathbf{P}$ has L-structured cospans as objects and an arrow $(La \rightarrow x \leftarrow La') \leq (Lc \rightarrow x \leftarrow Lc')$ whenever there is commuting diagram with form*

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] \\
Lb \arrow[u, "Lf"] \arrow[r] \arrow[d, "Lg"'] & y \arrow[u] \arrow[d] & Lb' \arrow[l] \arrow[u, "Lf'"'] \arrow[d, "Lg'"] \\
Lc \arrow[r] & z & Lc' \arrow[l]
\end{tikzcd}$$

where f, f', g, and g' are isomorphisms in $\mathbf{A}$

- $\otimes$ *given by*

$$\begin{array}{ccccc} La & \longrightarrow & x & \longleftarrow & Lb \\ \uparrow & & \uparrow & & \uparrow \\ La' & \to & x' & \leftarrow & Lb' \\ \downarrow & & \downarrow & & \downarrow \\ La'' & \to & x'' & \leftarrow & Lb'' \end{array} \otimes \begin{array}{ccccc} Lc & \longrightarrow & y & \longleftarrow & Ld \\ \uparrow & & \uparrow & & \uparrow \\ Lc' & \to & y' & \leftarrow & Ld' \\ \downarrow & & \downarrow & & \downarrow \\ Lc'' & \to & y'' & \leftarrow & Ld'' \end{array} := \begin{array}{ccccc} L(a+c) & \longrightarrow & x+y & \longleftarrow & L(b+d) \\ \uparrow & & \uparrow & & \uparrow \\ L(a'+c') & \longrightarrow & x'+y' & \longleftarrow & L(b'+d') \\ \downarrow & & \downarrow & & \downarrow \\ L(a''+c'') & \to & x''+y'' & \leftarrow & L(b''+d'') \end{array}$$

- I *given by a pair of identities on* $L0_{\mathbf{A}}$
- τ *given by*

$$\begin{array}{ccccc} L(a+b) & \longrightarrow & x+y & \longleftarrow & L(c+d) \\ \uparrow & & \uparrow & & \uparrow \\ L(a'+b') & \longrightarrow & x'+y' & \longleftarrow & L(c'+d') \\ \downarrow & & \downarrow & & \downarrow \\ L(a''+b'') & \to & x''+y'' & \leftarrow & L(c''+d'') \end{array} \overset{\tau}{\mapsto} \begin{array}{ccccc} L(b+a) & \longrightarrow & (y+x) & \longleftarrow & L(d+c) \\ \uparrow & & \uparrow & & \uparrow \\ L(b'+a') & \longrightarrow & (y'+x') & \longleftarrow & L(d'+c') \\ \downarrow & & \downarrow & & \downarrow \\ L(b''+a'') & \to & (y''+x'') & \leftarrow & L(d''+c'') \end{array}$$

Proof. The only non-trivial thing to check is that the tensor and composition satisfy interchange. That is, given two pairs of composable arrows

$$\begin{array}{ccccc} La & \longrightarrow & v & \longleftarrow & La' \\ \uparrow & & \uparrow & & \uparrow \\ Lb & \longrightarrow & w & \longleftarrow & Lb' \\ \downarrow & & \downarrow & & \downarrow \\ Lc & \longrightarrow & x & \longleftarrow & Lc' \end{array} \qquad \begin{array}{ccccc} La'' & \to & v' & \leftarrow & La''' \\ \uparrow & & \uparrow & & \uparrow \\ Lb'' & \to & w' & \leftarrow & Lb''' \\ \downarrow & & \downarrow & & \downarrow \\ Lc'' & \to & x' & \leftarrow & Lc''' \end{array}$$

$$\begin{array}{ccccc} Lc & \longrightarrow & x & \longleftarrow & Lc' \\ \uparrow & & \uparrow & & \uparrow \\ Ld & \longrightarrow & y & \longleftarrow & Ld' \\ \downarrow & & \downarrow & & \downarrow \\ Le & \longrightarrow & z & \longleftarrow & Le' \end{array} \qquad \begin{array}{ccccc} Lc'' & \to & x' & \leftarrow & Lc''' \\ \uparrow & & \uparrow & & \uparrow \\ Ld'' & \to & y' & \leftarrow & Ld''' \\ \downarrow & & \downarrow & & \downarrow \\ Le'' & \to & z' & \leftarrow & Le'''' \end{array}$$

we want to show that the resulting arrow obtained by tensoring before composing

$$\begin{tikzcd}
L(a+a'') \arrow[dr] & & \\
& v+v' & \\
L((b+b'')\times_{(c+c'')}(d+d'')) \arrow[uu] \arrow[dd] \arrow[dr] & & L(a'+a'') \arrow[ul] \\
& (w+w')\times_{x+x'}(y+y') \arrow[uu] \arrow[dd] & \\
L(e+e'') \arrow[dr] & & L(b'+b''')\times_{(c'+c''')}(d'+d''') \arrow[uu] \arrow[dd] \arrow[ul] \\
& z+z' & \\
& & L(e'+e''') \arrow[ul]
\end{tikzcd}$$

is equal to the arrow obtained by composing before tensoring

$$\begin{tikzcd}
L(a+a'') \arrow[dr] & & \\
& v+v' & \\
L((b\times_c d)+(b''\times_{c''} d'')) \arrow[uu] \arrow[dd] \arrow[dr] & & L(a'+a'') \arrow[ul] \\
& (w\times_x y)+(w'\times_{x'} y') \arrow[uu] \arrow[dd] & \\
L(e+e'') \arrow[dr] & & L((b'\times_{c'} d')+(b'''\times_{c'''} d''')) \arrow[uu] \arrow[dd] \arrow[ul] \\
& z+z' & \\
& & L(e'+e''') \arrow[ul]
\end{tikzcd}$$

These arrows are parallel, hence equal by definition. □

Lemma 1.3. *The preorder* $\mathbf{P}$ *is symmetric.*

Proof. Any arrow $(La \to x \leftarrow La') \leq (Lc \to z \leftarrow Lc')$ in $\mathbf{P}$ gives an arrow $(Lc \to z \leftarrow Lc') \leq (La \to x \leftarrow La')$ by taking the dual span of L-structured cospans. □

Lemma 1.4. *Let* $(\mathbf{A}, +, 0_{\mathbf{A}}, \tau_{\mathbf{A}})$ *and* $(\mathbf{X}, +, 0_{\mathbf{X}}, \tau_{\mathbf{X}})$ *be categories that are cocartesian and locally cartesian closed. Also, suppose that* $\mathbf{X}$ *has pushouts. Let* $L\colon \mathbf{A} \to \mathbf{X}$ *be a cocartesian functor that preserves pullbacks. Then there is a symmetric monoidal double category* $(\mathbb{R}\mathbf{ewrite}_L, \otimes, I, \tau)$.

The double category $\mathbb{R}\mathbf{ewrite}_L$ *consists of the object category* $\mathbb{R}_0 := \mathbf{core}(\mathbf{Span}(\mathbf{A}))$*;*

arrow category $\mathbb{R}_1 \coloneqq \mathbf{P}$*; unit functor* $U\colon \mathbb{R}_0 \to \mathbb{R}_1$ *defined by*

$$\begin{tikzcd} a \\ b \arrow[u, "f"'] \arrow[d, "g"] \\ c \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} La \arrow[r, "\mathrm{id}"] & La & La \arrow[l, "\mathrm{id}"'] \\ Lb \arrow[u, "Lf"'] \arrow[d, "Lg"] \arrow[r, "\mathrm{id}"] & Lb \arrow[u, "Lf"'] \arrow[d, "Lg"] & Lb \arrow[l, "\mathrm{id}"'] \arrow[u, "Lf"'] \arrow[d, "Lg"] \\ Lc \arrow[r, "\mathrm{id}"] & Lc & Lc \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$

source and target functors $S, T\colon \mathbb{R}_1 \to \mathbb{R}_0$ *respectively defined by*

$$\begin{tikzcd} La \arrow[r] & x & La' \arrow[l] \\ Lb \arrow[u, "Lf"] \arrow[d, "Lg"'] \arrow[r] & y \arrow[u] \arrow[d] & Lb' \arrow[l] \arrow[u] \arrow[d] \\ Lc \arrow[r] & z & Lc' \arrow[l] \end{tikzcd} \mapsto \begin{tikzcd} a \\ b \arrow[u, "f"] \arrow[d, "g"'] \\ c \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} La \arrow[r] & x & La' \arrow[l] \\ Lb \arrow[u] \arrow[d] \arrow[r] & y \arrow[u] \arrow[d] & Lb' \arrow[l] \arrow[u, "Lf"] \arrow[d, "Lg"'] \\ Lc \arrow[r] & z & Lc' \arrow[l] \end{tikzcd} \mapsto \begin{tikzcd} a' \\ b' \arrow[u, "f"] \arrow[d, "g"'] \\ c' \end{tikzcd}$$

and composition functor $\odot\colon \mathbb{R}_1 \times_{\mathbb{R}_0} \mathbb{R}_1 \to \mathbb{R}_1$ *defined by*

$$\begin{tikzcd} La \arrow[r] & x & La' \arrow[l] \\ Lb \arrow[u] \arrow[d] \arrow[r] & y \arrow[u] \arrow[d] & Lb' \arrow[l] \arrow[u] \arrow[d] \\ Lc \arrow[r] & z & Lc' \arrow[l] \end{tikzcd} \odot \begin{tikzcd} La' \arrow[r] & x' & La'' \arrow[l] \\ Lb' \arrow[u] \arrow[d] \arrow[r] & y' \arrow[u] \arrow[d] & Lb'' \arrow[l] \arrow[u] \arrow[d] \\ Lc' \arrow[r] & z' & Lc'' \arrow[l] \end{tikzcd} \coloneqq \begin{tikzcd} La \arrow[r] & x +_{La'} x' & La'' \arrow[l] \\ Lb \arrow[u] \arrow[d] \arrow[r] & y +_{Lb'} y' \arrow[u] \arrow[d] & Lb'' \arrow[l] \arrow[u] \arrow[d] \\ Lc \arrow[r] & z +_{Lc'} z' & Lc'' \arrow[l] \end{tikzcd}$$

which uses pushouts in $\mathbf{X}$ *and their universal properties.*

The tensor $\otimes$ *is given by*

$$\begin{tikzcd} La \arrow[r] & x & Lb \arrow[l] \\ La' \arrow[u] \arrow[d] \arrow[r] & x' \arrow[u] \arrow[d] & Lb' \arrow[l] \arrow[u] \arrow[d] \\ La'' \arrow[r] & x'' & Lb'' \arrow[l] \end{tikzcd} \otimes \begin{tikzcd} Lc \arrow[r] & y & Ld \arrow[l] \\ Lc' \arrow[u] \arrow[d] \arrow[r] & y' \arrow[u] \arrow[d] & Ld' \arrow[l] \arrow[u] \arrow[d] \\ Lc'' \arrow[r] & y'' & Ld'' \arrow[l] \end{tikzcd} \coloneqq \begin{tikzcd} L(a+c) \arrow[r] & x+y & L(b+d) \arrow[l] \\ L(a'+c') \arrow[u] \arrow[d] \arrow[r] & x'+y' \arrow[u] \arrow[d] & L(b'+d') \arrow[l] \arrow[u] \arrow[d] \\ L(a''+c'') \arrow[r] & x''+y'' & L(b''+d'') \arrow[l] \end{tikzcd}$$

a monoidal unit I *defined by*

$$I \coloneqq (LI_{\mathbf{A}} \to LI_{\mathbf{A}} \leftarrow LI_{\mathbf{A}})$$

and braiding τ *defined by*

$$\begin{tikzcd} L(a+b) \arrow[r] & x+y & L(c+d) \arrow[l] \\ L(a'+b') \arrow[u] \arrow[d] \arrow[r] & x'+y' \arrow[u] \arrow[d] & L(c'+d') \arrow[l] \arrow[u] \arrow[d] \\ L(a''+b'') \arrow[r] & x''+y'' & L(c''+d'') \arrow[l] \end{tikzcd} \overset{\tau}{\mapsto} \begin{tikzcd} L(b+a) \arrow[r] & (y+x) & L(d+c) \arrow[l] \\ L(b'+a') \arrow[u] \arrow[d] \arrow[r] & (y'+x') \arrow[u] \arrow[d] & L(d'+c') \arrow[l] \arrow[u] \arrow[d] \\ L(b''+a'') \arrow[r] & (y''+x'') & L(d''+c'') \arrow[l] \end{tikzcd}$$

Proof. Composition is functorial because $\mathbb{R}_1$ is a preorder. It is straightforward to check that $S;U = \mathrm{id} = T;U$ as well as applying S and T to

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] & & La' \arrow[r] & x' & La'' \arrow[l] \\
Lb \arrow[r] \arrow[u] \arrow[d] & y \arrow[u] \arrow[d] & Lb' \arrow[l] \arrow[u] \arrow[d] & \odot & Lb' \arrow[r] \arrow[u] \arrow[d] & y' \arrow[u] \arrow[d] & Lb'' \arrow[l] \arrow[u] \arrow[d] \\
Lc \arrow[r] & z & Lc' \arrow[l] & & Lc' \arrow[r] & z' & Lc'' \arrow[l]
\end{tikzcd}$$

respectively returns

$$La \to Lb \leftarrow Lc \quad \text{and} \quad La'' \to Lb'' \leftarrow Lc''$$

The associator, plus left and right unitors are defined using universal properties. Therefore, $\mathbb{R}\mathbf{ewrite}_L$ is a double category.

We now show that it is symmetric monoidal. For this, we follow Shulman's unpacking of Definition (blah). Lemmas 1.1 and 1.2 show that our object and arrow categories are symmetric monoidal. We have that $U(0)$ is the pair of identities on $L0$ and that the source S and target T functors are strict monoidal by construction. citation

Next, given two pairs of composable vertical arrows

$$La \xrightarrow{f} w \xleftarrow{g} Lb \qquad Lb \xrightarrow{f'} x \xleftarrow{g'} Lc$$

$$La' \xrightarrow{h} y \xleftarrow{k} Lb' \qquad Lb' \xrightarrow{h'} z \xleftarrow{k'} Lc'$$

we construct an invertible 2-cell (denoted $\mathfrak{X}$ by Shulman) of form

$$\begin{tikzcd}
L(a+a') \arrow[r] & (w+y) +_{L(b+b')} (x+z) & L(c+c') \arrow[l] \\
L(a+a') \arrow[r] \arrow[u] \arrow[d] & (w+y) +_{L(b+b')} (x+z) \arrow[u] \arrow[d, "\theta"'] & L(c+c') \arrow[l] \arrow[u] \arrow[d] \\
L(a+a') \arrow[r] & (w +_{Lb} x) + (y +_{Le} z) & L(c+c') \arrow[l]
\end{tikzcd} \tag{1}$$

The cospans along the top and bottom of (1) follow from, respectively, tensoring before composition and composing before tensoring. The map θ is constructed below. Denote the monoidal structure map by s, a canonical inclusion by ι, and a canonical quotient by q. The cospan along the top of (1) has arrows from the diagram

$$
\begin{tikzcd}
L(a+a') \arrow[r] \arrow[d, "s"'] & (w+y)+_{L(b+b')}(x+z) & L(c+c') \arrow[l] \arrow[d, "s"] \\
La+La' \arrow[d, "f+h"'] & & Lc+Lc' \arrow[d, "g'+k'"'] \\
w+y \arrow[r, hook] & (w+y)+(x+z) \arrow[uu, two heads, "q"] & x+z \arrow[l, hook']
\end{tikzcd}
$$

and the cospan along the bottom has arrows from the diagram

$$
\begin{tikzcd}
L(a+a') \arrow[r] \arrow[d, "s"'] & (w+_{Lb}x)+(y+_{Lb'}z) & L(c+c') \arrow[l] \arrow[d, "s"] \\
La+La' \arrow[d, "f+h"'] & & Lc+Lc' \arrow[d, "g'+k'"'] \\
w+y \arrow[r, hook, "\iota+\iota"] & (w+x)+(y+z) \arrow[uu, two heads, "q+q"] & x+z \arrow[l, hook', "\iota+\iota"']
\end{tikzcd}
$$

The arrow θ in (1) exists because of the universal property of a pushout. The diagram

$$
\begin{tikzcd}
L(b+b') \arrow[rr, "s"] \arrow[d, "s"'] & & Lb+Lb' \arrow[r, "g+g'"] & w+y \arrow[d, "\iota+\iota"] \\
Lb+Lb' \arrow[d, "h+h'"'] & & & (w+x)+(y+z) \arrow[d, "q+q"] \\
x+z \arrow[r, "\iota+\iota"] & (w+x)+(y+z) \arrow[rr, "q+q"] & & (w+_{Lb}x)+(y+_{Lb'}z)
\end{tikzcd}
$$

commutes because the equations $g;\iota;q = h;\iota;q$ and $g';\iota;q = h';\iota;q$ hold. Indeed, these equations are exactly those from the pushout squares of $w+_{Lb}x$ and $y+_{Lb'}z$. It follows that θ fits into diagram (1). Because $\mathbb{R}_1$ is a symmetric preorder (Lemma 1.3), the 2-cell (1) is invertible as required.

Next, for objects a and b, we need an invertible 2-cell (denoted $\mathfrak{u}$ by Shulman) $U(a+b) \to Ua+Ub$. Again, to Lemma 1.3 ensures that all 2-cells are invertible.

Therefore, the 2-cell

$$\begin{tikzcd}
L(a+b) \arrow[r] & L(a+b) & L(a+b) \arrow[l] \\
L(a+b) \arrow[r] \arrow[u] \arrow[d] & L(a+b) \arrow[u] \arrow[d, "s"'] & L(a+b) \arrow[l] \arrow[u] \arrow[d] \\
L(a+b) \arrow[r, "s"] & La + Lb & L(a+b) \arrow[l, "s"']
\end{tikzcd}$$

provides $\mathfrak{u}$

It remains to check that various coherence diagrams commute. Each coherence diagram lives in the arrow category $\mathbb{R}_1$ which is a preorder, so commutes automatically. $\square$

Theorem 1.5. *The double category $\mathbb{R}\mathbf{ewrite}_L$ is fibrant.*

Proof. A companion for the vertical 1-cell $a \xrightarrow{f} b \xleftarrow{g} c$ consists of the horizontal 1-cell $La \xrightarrow{Lf^{-1}} Lb \xleftarrow{Lg^{-1}} Lc$ together with the 2-cells

$$\begin{tikzcd}
La \arrow[r, "Lf^{-1}"] & Lb & Lc \arrow[l, "Lg^{-1}"'] \\
Lb \arrow[u, "Lf"] \arrow[d, "Lg"'] \arrow[r, "\mathrm{id}"] & Lb \arrow[u, "\mathrm{id}"] \arrow[d, "Lg"'] & Lc \arrow[l, "Lg^{-1}"'] \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \\
Lc \arrow[r, "\mathrm{id}"] & Lc & Lc \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
La \arrow[r, "\mathrm{id}"] & La & La \arrow[l, "\mathrm{id}"'] \\
La \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[r, "Lf"] & Lb \arrow[u, "Lf"] \arrow[d, "\mathrm{id}"'] & Lb \arrow[l, "\mathrm{id}"'] \arrow[u, "Lf"] \arrow[d, "Lg"'] \\
La \arrow[r, "Lf"] & Lb & Lc \arrow[l, "Lg^{-1}"']
\end{tikzcd}$$

The equations hold because $\mathbb{R}\mathbf{ewrite}_L$ is locally posetal.

A conjoint for the vertical 1-cell $a \xrightarrow{f} b \xleftarrow{g} c$ consists of opposite horizontal 1-cell $Lc \xrightarrow{Lg^{-1}} Lb \xleftarrow{Lf^{-1}} La$ together with the same 2-cells as the companion. The equations hold because $\mathbb{R}\mathbf{ewrite}_L$ is locally posetal. $\square$

Corollary 1.6. *The horizontal edge bicategory* $\mathbf{Rewrite}_L \coloneqq \mathcal{H}\left(\mathbb{R}\mathbf{ewrite}_L\right)$ *in the sense of Shulman is symmetric monoidal.*

Proof. This follows from Theorem 5.1 in (cite Shulman) $\square$

Lemma 1.7. *Every 1-arrow of* **Rewrite** *is a left and right adjoint.*

Proof. It is straightforward to check that the left and right adjoint of a 1-arrow $La \to x \leftarrow Lb$ is obtained by turning the cospan around $Lb \to x \leftarrow La$. $\square$

Definition 1.8. (cite carb & walts)] Let $\mathbf{B}$ be a bicategory whose hom-categories are posets. A **Cartesian structure** on $\mathbf{B}$ consists of a tensor product $\otimes$ on $\mathbf{B}$ and a cocommutative comonoid structure $(\delta_x, \varepsilon_x, \sigma_x)$ on every object x in $\mathbf{B}$. In addition, this data satisfies two axioms. First, every 1-arrow $f \colon x \to y$ is a lax comonoid homomorphism, that is there are 2-arrows $\delta_y f \Rightarrow (f \otimes f)\delta_x$ and $\eta_y f \Rightarrow \eta_x$. Second, comultiplication and counit have right adjoints δ_x^*, ε_x^*. A Cartesian bicategory is said to be a **bicategory of relations** if every object is a Frobenius object.

Theorem 1.9. *Let $(\mathbf{A}, +, 0_{\mathbf{A}}, \tau_{\mathbf{A}})$ and $(\mathbf{X}, +, 0_{\mathbf{X}}, \tau_{\mathbf{X}})$ be categories that are cocartesian and locally cartesian closed. Also, suppose that $\mathbf{X}$ has pushouts. Let $L\colon \mathbf{A} \to \mathbf{X}$ be a cocartesian functor that preserves pullbacks.*

The bicategory $\mathbf{Rewrite}L$ is a bicategory of relations in the sense of Carboni and Walters.

Proof. We start by observing that $\mathbb{R}\mathbf{ewrite}$ is locally posetal because parallel 2-arrows are identified. The tensor product is provided in 1.6. We now show, in order, that each object has a cocommutative comonoid structure whose adjoints give a commutative monoid structure. These are compatible via the Frobenius equation. Finally, every 1-arrow is a lax comonoid homomorphism.

Given an object a in $\mathbb{R}\mathbf{ewrite}$, we use the folding map $\Delta a\colon a + a \to a$ in $\mathbf{A}$ to define comultiplication $\delta_a\colon a \to a + a$ as the cospan

$$\delta_a\colon La \to La \xleftarrow{L\delta_a} L(a+a)$$

and use the initial map to define the counit $\varepsilon_a\colon a \to 0_a$ as the cospan

$$La \to La \leftarrow L0_a.$$

The associativity and unity 2-arrows appear canonically, as does cocommutativity.

From that cocommutative comonoid structure, we obtain the commutative monoid structure by taking adjoints of all the 1-arrows (see 1.7).

The Frobenius equations are witnessed by the commuting diagram

$$\begin{tikzcd}
 & La & \\
L(a+a) \arrow[ur] \arrow[r] \arrow[dr] & La \arrow[u] \arrow[d] & L(a+a) \arrow[ul] \arrow[l] \arrow[dl] \\
 & La &
\end{tikzcd}$$

populated with arrows $L\delta_a$.

Finally, we need to check that any 1-arrow $La \xrightarrow{f} x \xleftarrow{g} Lb$ is a lax comonoid homomorphism. The lax comultiplication structure map comes from the commutating diagram

$$\begin{tikzcd}
 & x & \\
La \arrow[ur, "f"] \arrow[r] \arrow[dr, "\iota"'] & La + L(b+b) \arrow[u] \arrow[d] & L(b+b) \arrow[ul, "L\delta_b; g"'] \arrow[l] \arrow[dl, "s; g+g"] \\
 & La +_{L(a+a)} (x+x) &
\end{tikzcd}$$

made with f, g, the monoidal structure map $s\colon L(b+b) \to Lb+Lb$ and canonical arrows. The lax unit structure map comes from the commuting diagram

$$\begin{tikzcd} & x & \\ La \arrow[ur, "f"] \arrow[r] \arrow[dr] & La \arrow[u, "f"] \arrow[d] & L0 \arrow[l] \arrow[ul] \arrow[dl] \\ & La & \end{tikzcd}$$

□

Corollary 1.10. *Moreover, if the monoidal products $\otimes_{\mathbf{A}}$ and $+$ are coproducts, then the symmetric monoidal bicategory $\mathbf{Rewrite}_L$ is compact closed.*

Theorem 1.11. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ be a symmetric monoidal category with pullbacks and $(\mathbf{X}, +, I_{\mathbf{X}})$ be a symmetric monoidal topos. Let $L \dashv R\colon \mathbf{A} \to \mathbf{X}$ be an adjunction where L preserves pullbacks.*

Suppose each element from a grammar Γ in $\mathbf{Cospan}_L$ is of the form

$$\begin{tikzcd} La \arrow[r] & x & La' \arrow[l] \\ Lb \arrow[u, "\cong"] \arrow[r] \arrow[d, "\cong"'] & y \arrow[u] \arrow[d] & Lb'' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\ Lc \arrow[r] & z & Lc' \arrow[l] \end{tikzcd}$$

Then Γ generates a sub-double category of $\mathbb{R}\mathbf{ewrite}L$ as follows:

- *generate the sub-bicategory $\mathcal{L}(\Gamma) \subseteq \mathbf{Span}(\mathbf{Cospan}_L)$ as in Lemma* **??**,
- *with $\mathcal{L}(\Gamma)$, define the subcategory $\mathcal{P}(\mathcal{L}(\Gamma))$ as in Definition* **??**,

Form the category as described in Definition **??** *from $\mathcal{L}(\Gamma)$. Pair this, as an arrow category, with the object category* **coreSpanA**.

Theorem 1.12. *If Γ has only elements of the form*

$$\begin{tikzcd} La \arrow[r] & x & La' \arrow[l] \\ La \arrow[u, "="] \arrow[r] \arrow[d, "="'] & y \arrow[u] \arrow[d] & La' \arrow[l] \arrow[u, "="'] \arrow[d, "="] \\ La \arrow[r] & z & La' \arrow[l] \end{tikzcd}$$

then Γ *generates a sub-bicategory of* $\mathbf{Rewrite}L$*. This sub-bicategory corresponds to the sub-bicategory of* $\mathbf{Rewrite}L$ *obtained by passing the construction through* $\mathbb{R}\mathbf{ewrite}L$ *first, then applying* $\mathcal{H}(-)$*.*