1. Structured cospans

Definition 1.1. Given a functor $L\colon \mathbf{A} \to \mathbf{X}$, a L-structured cospan is a diagram of form $La \to x \leftarrow Lb$.

Definition 1.2. Fix a category $\mathbf{X}$ and a functor $L\colon \mathbf{A} \to \mathbf{X}$. Denote by L-**Csp** the category with objects are those from $\mathbf{A}$ and with morphisms of type $a \to b$ are isomorphism classes of L-structured cospans $La \to x \leftarrow Lb$.

Definition 1.3. Denote by L-**StrCsp** the category whose objects are L-open objects and arrows are triples (f, g, h) that fit into a commuting diagram

Theorem 1.4. *Let $L \dashv R\colon \mathbf{A} \to \mathbf{X}$ be an adjunction between topoi. Then L-***StrCsp** *is a topos.*

2. Rewriting

Definition 2.1. A category with pullbacks is **adhesive** if pushouts along monics exist and are *Van Kampen*.

Theorem 2.2. *Topoi are adhesive.*

Corollary 2.3. *Let $L \dashv R\colon \mathbf{A} \to \mathbf{X}$ be an adjunction between topoi. The category L-***StrCsp** *is adhesive.*

Definition 2.4. For $\mathbf{C}$ an adhesive category, an $\mathbf{C}$**-rewrite rule** (often called a production) is a span $a \leftarrow b \to c$ inside $\mathbf{C}$. When both legs of the span are monic, we say the rewrite rule is **linear**.

Definition 2.5. Given composable arrows $a \to b \to y$ we say that an arrow $a \to x$ is a **pushout complement** if it fits into a pushout diagram

Definition 2.6. Given a $\mathbf{C}$-rewrite rule $a \leftarrow b \to c$ and a $\mathbf{C}$-arrow $a \to x$ such that $b \to a \to x$ has a pushout complement, a **derived (linear) rewrite rule** is the bottom row of the induced double pushout diagram

Definition 2.7. A **(linear) grammar** consists of an adhesive category $\mathbf{A}$ and a set of (linear) $\mathbf{A}$-rewrite rules. Observe that $\mathbf{A}$-rewrite rules are actually arrows in $\mathbf{Sp}(\mathbf{A})$. Given a grammar Γ, the subcategory $\mathcal{L}(\Gamma)$ of $\mathbf{Sp}(\mathbf{A})$ generated by the set of rewrites derived from Γ is called a **language**.

Lemma 2.8. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ be a symmetric monoidal category with pullbacks and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a symmetric monoidal topos. Let $L \dashv R\colon \mathbf{A} \to \mathbf{X}$ be an adjunction where L preserves pullbacks.*

*Fix a grammar Γ in the topos L-***StrCsp***. The generated language $\mathcal{L}(\Gamma)$ is a sub-bicategory of* $\mathbf{Sp}(L\text{-}\mathbf{StrCsp})$.

3. Non-linear rewriting of open objects

Definition 3.1. Let $\mathbf{A}$ be a category with pullbacks, $\mathbf{X}$ be a category with pullbacks and pushouts, and $L\colon \mathbf{A} \to \mathbf{X}$ be a functor preserving pullbacks. Denote by $\mathcal{P}(\mathbf{Sp}(L\text{-}\mathbf{StrCsp}))$ the preorder whose objects are L-open objects and arrows $(La \to x \leftarrow La') \leq (Lc \to x \leftarrow Lc')$ whenever there is a $\mathbf{Sp}(L\text{-}\mathbf{StrCsp})$-arrow with form

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] \\
Lb \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & y \arrow[u] \arrow[d] & Lb' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
Lc \arrow[r] & z & Lc' \arrow[l]
\end{tikzcd}$$

Definition 3.2. Let $\mathbf{A}$ be a category with pullbacks, $\mathbf{X}$ a category with pullbacks and pushouts, and $L\colon \mathbf{A} \to \mathbf{X}$ be a functor preserving pullbacks. Define the double category L-$\mathbb{R}\mathbf{ewrite}$ to have object category $\mathbf{core}(\mathbf{Sp}(\mathbf{A}))$ and have arrow category $\mathbf{C}$ as described in Definition 3.1.

Alternatively, L-$\mathbb{R}\mathbf{ewrite}$ is the double category with $\mathbf{A}$-objects as 0-cells, spans in $\mathbf{A}$ with isomorphic legs as vertical 1-cells, L-open objects as horizontal 1-cells, and a unique 2-cell if there exists a commuting diagram in $\mathbf{X}$ of form

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] \\
Lb \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & y \arrow[u] \arrow[d] & Lb' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
Lc \arrow[r] & z & Lc' \arrow[l]
\end{tikzcd}$$

Theorem 3.3. *The double category L-$\mathbb{R}\mathbf{ewrite}$ is isofibrant.*

Theorem 3.4. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories so that $\mathbf{A}$ has a pullbacks and $\mathbf{X}$ has pullbacks and pushouts. If $L\colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ preserves pullbacks, then $(L\text{-}\mathbb{R}\mathbf{ewrite}, \otimes, I)$ is a symmetric monoidal double category with $\otimes$ defined by*

$$(La \to x \leftarrow Lb) \otimes (Lc \to y \leftarrow Ld) := L(a \otimes_{\mathbf{A}} c) \to x \otimes_{\mathbf{X}} y \leftarrow L(b \otimes_{\mathbf{A}} d)$$

and I defined by

$$I := (LI_{\mathbf{A}} \to I_{\mathbf{X}} \leftarrow LI_{\mathbf{X}}).$$

Theorem 3.5. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories where $\mathbf{A}$ has pullbacks and $\mathbf{X}$ has pullbacks and pushouts. Let $L\colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a pullback preserving symmetric monoidal functor.*

*The horizontal edge bicategory L-**Rewrite** $:= \mathcal{H}(L\text{-}\mathbb{R}\mathbf{ewrite})$ in the sense of Shulman is symmetric monoidal. Moreover, if the monoidal products $\otimes_{\mathbf{A}}$ and $\otimes_{\mathbf{X}}$ are coproducts, then the symmetric monoidal bicategory L-**Rewrite** is compact closed.*

Theorem 3.6. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories so that $\mathbf{A}$ has a pullbacks and $\mathbf{X}$ has pullbacks and pushouts. Let $L\colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a pullback preserving functor.*

*The bicategory L-**Rewrite** is a bicategory of relations in the sense of Carboni and Walters.*

Theorem 3.7. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ be a symmetric monoidal category with pullbacks and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a symmetric monoidal topos. Let $L \dashv R\colon \mathbf{A} \to \mathbf{X}$ be an adjunction where L preserves pullbacks.*

*Suppose each element from a grammar Γ in L-**StrCsp** is of the form*

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] \\
Lb \arrow[u, "\cong"] \arrow[d, "\cong"'] \arrow[r] & y \arrow[u] \arrow[d] & Lb'' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
Lc \arrow[r] & z & Lc' \arrow[l]
\end{tikzcd}$$

*Then Γ generates a sub-double category of L-$\mathbb{R}$**ewrite** as follows:*

- *generate the sub-bicategory $\mathcal{L}(\Gamma) \subseteq \mathbf{Sp}(L\text{-}\mathbf{StrCsp})$ as in Lemma 2.8,*
- *with $\mathcal{L}(\Gamma)$, define the subcategory $\mathcal{P}(\mathcal{L}(\Gamma))$ as in Definition 3.1,*

Form the category as described in Definition **??** *from $\mathcal{L}(\Gamma)$. Pair this, as an arrow category, with the object category $\mathbf{core}(\mathbf{Sp}(\mathbf{A}))$.*

Theorem 3.8. *If Γ has only elements of the form*

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] \\
La \arrow[u, "="] \arrow[d, "="'] \arrow[r] & y \arrow[u] \arrow[d] & La' \arrow[l] \arrow[u, "="'] \arrow[d, "="] \\
La \arrow[r] & z & La' \arrow[l]
\end{tikzcd}$$

*then Γ generates a sub-bicategory of L-**Rewrite**. This sub-bicategory corresponds to the sub-bicategory of L-**Rewrite** obtained by passing the construction through L-$\mathbb{R}$**ewrite** first, then applying $\mathcal{H}(-)$.*

X shd b topos for intrchng

4. Linear rewriting of open objects

Definition 4.1. Let $\mathbf{A}$ be a category with pullbacks, $\mathbf{X}$ be a topos, and $L\colon \mathbf{A} \to \mathbf{X}$ be a functor preserving pullbacks. Denote by $\mathcal{C}(\mathbf{Sp}(L\text{-}\mathbf{StrCsp}))$ the category whose objects are L-open objects and arrows are isomorphism classes of 1-cells of form

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] \\
Lb \arrow[u, "\cong"] \arrow[d, "\cong"'] \arrow[r] & y \arrow[u, tail] \arrow[d, tail] & Lb' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
Lc \arrow[r] & z & Lc' \arrow[l]
\end{tikzcd}$$

where the arrows marked "$\rightarrowtail$" are monic.

Definition 4.2. Denote by L-$\mathbb{M}\mathbf{on}\mathbb{R}\mathbf{ewrite}$ the double category with $\mathbf{A}$-objects as 0-cells, spans in $\mathbf{A}$ whose legs are isomorphisms as vertical 1-cells, L-open objects as horizontal 1-cells, and commuting diagrams in $\mathbf{X}$ of form

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] \\
Lb \arrow[u, "\cong"] \arrow[d, "\cong"'] \arrow[r] & y \arrow[u, tail] \arrow[d, tail] & Lb' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
Lc \arrow[r] & z & Lc' \arrow[l]
\end{tikzcd}$$

Theorem 4.3. *The double category L-$\mathbb{M}\mathbf{on}\mathbb{R}\mathbf{ewrite}$ is isofibrant.*

Theorem 4.4. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories where $\mathbf{A}$ has pullbacks and $\mathbf{X}$ is a topos. Let $L\colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a pullback preserving functor. Then $(L\text{-}\mathbb{M}\mathbf{on}\mathbb{R}\mathbf{ewrite}, \otimes, I)$ is a symmetric monoidal double category with $\otimes$ defined by*

$$(La \to x \leftarrow Lb) \otimes (Lc \to y \leftarrow Ld) := L(a \otimes_{\mathbf{A}} c) \to x \otimes_{\mathbf{X}} y \leftarrow L(b \otimes_{\mathbf{A}} d)$$

and I by

$$I := (LI_{\mathbf{A}} \to I_{\mathbf{X}} \leftarrow LI_{\mathbf{X}}).$$

Theorem 4.5. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories where $\mathbf{A}$ has pullbacks and $\mathbf{X}$ is a topos. Let $L\colon (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \to (\mathbf{X}, \otimes_{\mathbf{X}})$ be an adjunction where L preserves pullbacks.*

The horizontal edge bicategory L-$\mathbf{MonRewrite} := \mathcal{H}(L\text{-}\mathbb{M}\mathbf{on}\mathbb{R}\mathbf{ewrite})$ in the sense of Shulman is symmetric monoidal. Moreover, if the monoidal products $\otimes_{\mathbf{A}}$

*and $\otimes_{\mathbf{X}}$ are coproducts, then the symmetric monoidal bicategory L-**MonRewrite** is compact closed.*

Theorem 4.6. *Suppose each element from a grammar Γ in L-**StrCsp** is of the form*

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] \\
Lb \arrow[u, "\cong"] \arrow[r] \arrow[d, "\cong"'] & y \arrow[u, tail] \arrow[d, tail] & Lb' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
Lc \arrow[r] & z & Lc' \arrow[l]
\end{tikzcd}$$

*then Γ generates a sub-double category $\langle\langle\Gamma\rangle\rangle$ of L-$\mathbb{M}$**on**$\mathbb{R}$**ewrite**. The recipe is get the language $\mathcal{L}(\Gamma) \subseteq \mathbf{Sp}(L\text{-}\mathbf{StrCsp})$. Form the category as described in Definition **??** from $\mathcal{L}(\Gamma)$. Pair this, as an arrow category, with the object category $\mathbf{core}(\mathbf{Sp}(\mathbf{A}))$.*

Theorem 4.7. *Same as above with monics thrown in.*