REWRITING STRUCTURED COSPANS

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Abstract. To foster the study of networks on an abstract level, we introduce the formalism of *structured cospans*. A structured cospan is a diagram of the form $La \to x \leftarrow Lb$ built from a geometric morphism with left exact left adjoint $L \dashv R\colon \mathsf{X} \to \mathsf{A}$. We show that this construction is functorial and results in a topos with structured cospans as objects. Additionally, structured cospans themselves are compositional. Combining these two perspectives, we define a double category of structured cospans. We then leverage adhesive categories to create a theory of rewriting for structured cospans. A well-known result of graph rewriting is that a graph grammar induces the same rewrite relation as its underlying graph grammar. We generalize this result to topoi under the assumption that the subobject algebra on each context in the grammar is well-founded. This fact is used to provide a compositional framework for double pushout rewriting in a topos X that is the domain of a geometric morphism.

1. Introduction

This paper fits into a program of codifying network theory into the language of category theory [2, 3, 4, 5, 6]. An important aspect of this program falls within a theme common to many areas of mathematics. That is the desire to study global properties using local phenominon as illustrated by the advent of sheaf theory. A concept related to the local study of systems is *compositionality* which occurs when properties of a system are completely determined by properties of its sub-systems and the manner in which they are connected together. A neccesary requirement of compositionality is the existence of a mechanism by which to connect sub-systems together. A recent proposal for such a mechanism given by Baez and Courser [1] is called *structured cospans*.

The structured cospans serve as a syntax for systems. A desirable feature of syntax is the ability to reflect a particular sematics, specifically as it relates to distinct syntactical terms with the same semantic representation. One way to accomplish this is using rewriting, which can be thought of as a relation similar to, but more finely grained, than equality. The main goal of this paper is to introduce a theory of rewriting for structured cospans.

To define a structured cospan, we begin with a functor $L\colon \mathsf{A} \to \mathsf{X}$ where A and X have finite colimits and L preserves them. The objects of the category X are your systems of interest, for example graphs, Markov processes, Petri nets, etc. The objects of A are interfaces for the systems in X. Often, this is Set. Then L is the channel allowing the interfaces to interact with the systems. A more formal description with that a structured cospan is a cospans of the form $La \to x \leftarrow Lb$, where x is the system, La the inputs, and Lb the outputs. Another open system whose inputs are Lb can be connected to this via pushout, which is the familiar composition in cospan categories

$$
\begin{tikzcd}
 & x & & & y & & & & x +_{Lb} y & \\
La \arrow[ur] & & Lb \arrow[ul] & Lb \arrow[ur] & & Lc \arrow[ul] \arrow[rr, "\text{connection}"] & & La \arrow[ur] & & Lc \arrow[ul]
\end{tikzcd}
$$

open sub-systems composite system

In this program, *functorial semantics* is used to describe system behavior. That is, the semantics of such systems is captured by a functor from our syntax category to a semantics category such as Set, Rel, or Pos. The functoriality ensures composition preserves properties of the system. There are early examples of applying functorial semantics to possive linear networks [3], Markov processes [4], and chemical reaction networks [5].

In order to introduce rewriting into this structured cospan syntax, we appeal to the most general class of objects that admit a rewriting theory. By this, we mean *adhesive categories* whose axioms were gathered specifically to encode the behavior of graph rewriting. However, instead of working in the full generality of adhesive categories, we restrict ourselves to elementary topoi, which are examples of adhesive categories. To place ourseves into this context, we define structured cospans stronger hypotheses than Baez and Courser. Specifically, we begin with a geometric morphism, that is an adjunction $(L\colon \mathsf{A} \to \mathsf{X}) \dashv (R\colon \mathsf{X} \to \mathsf{A})$ between topoi such that L preserves finite limits. In the systems analogy, A, X, and L continue to serve as the interfaces, systems, and interface channel. The compositional structure remains and another perspective emerges where structured cospans are objects with arrows between them. The arrows are commuting diagrams

$$\begin{array}{ccccc} La & \longrightarrow & x & \longleftarrow & La' \\ \downarrow & & \downarrow & & \downarrow \\ Lb & \longrightarrow & y & \longleftarrow & Lb' \end{array} \tag{1}$$

Structured cospans and their arrows form a category that we denote by StrCsp_L. The reason the object perspective of structured cospans is so important is captured in the first of several main results in this paper: StrCsp_L is a topos. This is neccesary for us to introduce the theory of rewriting to structured cospans.

finish the story

The structure of the paper is as follows. Section 2 defines structured cospans, the main object of study. There are two perspectives on structured cospans and we realize this by building two separate categories. The first category StrCsp_L, mentioned above, has structured cospans as objects and commuting diagrams (1) as arrows. The first result, and the keystone of the paper, is that StrCsp_L is a topos. This grants us access to the theory of adhesive rewriting. The second category, and the one introduced by Baez and Courser [1] views structured cospans as arrows. We then combine these two perspectives into a double category.

In Section **??**, we sweep over the basics of graph rewriting and cover the inductive viewpoint. We also recall the theory of adhesive categories before applying it to structured cospans in a categorical framework. We start with *grammars*. There are two types of grammars we are interested in. The first involves pairing an adhesive category C with a set of productions P inside C. These form a category. There is a subcategory we use too, consisting of grammars (StrCsp_L, P) where P is a set of spans in StrCsp_L of the form

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] \arrow[r] & \bullet \\
\bullet \arrow[u, "\cong"] \arrow[d, "\cong"'] & \bullet \arrow[l] \arrow[r] \arrow[u, tail] \arrow[d, tail] & \bullet \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
\bullet & \bullet \arrow[l] \arrow[r] & \bullet
\end{tikzcd}$$

and fitting them into a category $\mathsf{StrCspGram}$. Given a grammar (C, P), we define the rewriting relation on the objects of C using double pushouts in a manner similar to that as was done for graphs. However, we provide a functorial characterization as well. Namely, the functor $D\colon \mathsf{Gram} \to \mathsf{Gram}$ that send a grammar (C, P) to the grammar (C, P') where a production $g \leftarrow d \to h$ belongs to P' if there is a double pushout diagram

$$\begin{tikzcd}
\ell \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & k \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & r \arrow[d] \\
g & d \arrow[l] \arrow[r] & h
\end{tikzcd}$$

such that the top row is a production in P. We then define a semantics functor $S\colon \mathsf{Gram} \to \mathsf{DblCat}$ valued in double categories. These semantics capture the compositionality of the structured cospans and also the rewriting. The *language* of a grammar is its image under the composite functor SD. Our main result is Theorem 3.4. Given a grammar (X, P) on a topos X fitting into a suitable geometric morphism $L \dashv R\colon \mathsf{X} \to \mathsf{A}$, then we construct a grammar (StrCsp_L, Q) such that g is related to h from the rewriting relation on the grammar (X, P) exactly when there is a square in the double category $SD(\mathsf{StrCsp}_L, Q)$ of the form

$$\begin{tikzcd}
LR0 \arrow[r] & g & LR0 \arrow[l] \\
LR0 \arrow[r] \arrow[u] \arrow[d] & d \arrow[u] \arrow[d] & LR0 \arrow[l] \arrow[u] \arrow[d] \\
LR0 \arrow[r] & h & LR0 \arrow[l]
\end{tikzcd}$$

Simply stated, if X contains a type of network, then the rewritings on networks of type X can be understood by rewriting on pieces of the network inside of (StrCsp, Q) then gluing them back together.

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2. Structured Cospans

Structured cospans were introduced by Baez and Courser [1] to provide syntax for compositional systems. Their work has two aims: maximize the generality of the structured cospan construction using double categories and also to compare structured cospans to Fong's decorated cospans [12], an alternative syntax. Because structured cospans are a syntax, we want to set up a framework that can reflect the semantics. This paper proposes such a framework, for which we use the notion of double pushout rewriting. Due to our motivation, we assume different (but not disjoint) hypothesis than Baez and Courser. The purpose of this section is to set our hypothesis and explore the nature of structured cospans in this context. Instead of embarking on a mission to fully lay out a theory of structured cospans, we restrain ourselves to just those aspects needed to introduce rewriting.

To be specific, in this section we make explicit competing perspectives. The first is looking at structured cospans as objects of a category with appropriate morphisms between them. The second takes structured cospans as morphisms between certain "interfaces". The latter perspective encode the compositional structure. We then complete this section by marrying the two perspectives using double categories.

Fix an arbitrary geometric morphism $L \dashv R \colon \mathsf{X} \to \mathsf{A}$. This is an adjunction

$$\begin{tikzcd} \mathsf{X} \arrow[r, phantom, "\perp"] & \mathsf{A} \arrow[l, bend right, "L"'] \arrow[l, leftarrow, bend left, "R"] \end{tikzcd}$$

between (elementary) topoi with L left exact. Because spans and cospans factor heavily into this work, we use the notation $(f,g)\colon y \to x \times z$ for a span

$$x \xleftarrow{f} y \xrightarrow{g} z$$

and $(f,g)\colon x + z \to y$ for a cospan

$$x \xrightarrow{f} y \xleftarrow{g} z.$$

Because all of the categories in this paper have products and coproducts, this notation is sensible.

2.1. **Structured cospans as objects.**

Definition 2.1. A **structured cospan** is a cospan of the form $La + Lb \to x$. When we want to emphasize L, we use the term L-structured cospans.

The motivating force behind inventing structured cospans is to describe open systems, and so we do not hesitate to draw on the the intuition of open systems to better understand structured cospans. For instance, one should view the topos X as consisting of closed systems and their morphisms. By a *closed system*, we mean a system that cannot interact with the outside world. The topos A should be thought to contain possible interfaces for the closed systems. Equipping a closed system with an interface provides the system a way to interact with compatible elements of the outside world. Such a system is no longer closed, and so we call it an *open system*. The left adjoint L sends these interfaces into X so that they might interact with the closed systems. The right adjoint R can be though of as returning all possible interface elements of a closed system.

Through this perspective, a structured cospan consists of a closed system x equipped with the interface described by the arrows from La and Lb. By ignoring questions of causality, we may safely consider La as the input to x and Lb as the output. As expected, a morphism of open system ought to respect these components.

Definition 2.2. A morphism from one L-structured cospan $La+Lb \to x$ to another $Lc + Ld \to y$ is a triple of arrows (f, g, h) that fit into the commuting diagram

$$\begin{tikzcd} La \arrow[r] \arrow[d, "Lf"'] & x \arrow[d, "g"'] & Lb \arrow[l] \arrow[d, "Lh"] \\ Lc \arrow[r] & y & Ld \arrow[l] \end{tikzcd}$$

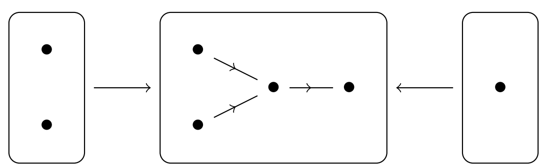
It is easy to check that L-structured cospans and their morphisms form a category, which we denote by StrCsp_L.

Example 2.3 (Open graphs)**.** Systems theory is intimately tied with graph theory. A natural example of a structured cospan is an *open graph*. While this notion is not new [10, 13], our infrastructure generalizes it.

Denote by RGraph the category of (directed reflexive multi-) graphs. There is an adjunction

$$\begin{tikzcd} \mathsf{RGraph} \arrow[r, shift left, "R"'] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, shift left, "L"'] \end{tikzcd}$$

where Rx is the node set of graph x and La is the edgeless graph with node set a. An **open graph** is a cospan $La + Lb \to x$ for sets a, b, and graph x. An illustrated example, with the reflexive loops suppressed, is



The boxed items are graphs and the arrows between boxes are graph morphisms defined as suggested by the illustration. In total, the three graphs and two graph morphisms make up a single open graph whose inputs and outputs are, respectively, the left and right-most graphs.

Having seen this example, it becomes more apparent about how open systems can "connect" together. Given another open graph whose inputs coincide with the outputs of the graph above, we can connect the inputs and outputs together to create a new open graph. By passing from graphs to open graphs, we are introducing *compositionality*. The category StrCsp_L does not encode the compositional structure, but we introduce a new category Cospan_L in Section 2.2 which does.

We now come to the first of our main results: that StrCsp_L is a topos. In terms of this paper, this result is critical because it allows for the introduction of rewriting onto structured cospans.

Theorem 2.4. *The category* StrCsp_L *is a topos.*

Proof. The category StrCsp_L constructed using the geometric morphism $L \dashv R \colon \mathsf{X} \to \mathsf{A}$ is equivalent to the category whose objects are cospans of form $a + b \to Rx$ and morphisms are triples (f, g, h) fitting into the commuting diagram

$$\begin{tikzcd} w \arrow[r] \arrow[d, "f"'] & Ra \arrow[d, "Rg"'] & x \arrow[l] \arrow[d, "h"'] \\ y \arrow[r] & Rb & z \arrow[l] \end{tikzcd}$$

This, in turn, is equivalent to the comma category $(\mathsf{A} \times \mathsf{A} \downarrow \Delta R)$, where $\Delta \colon \mathsf{A} \to \mathsf{A} \times \mathsf{A}$ is the diagonal functor. But the diagonal functor is right adjoint to taking binary coproducts. That means ΔR is also a right adjoint and, furthermore, that $(\mathsf{A} \times \mathsf{A} \downarrow \Delta R)$ is an instance of Artin gluing [18], hence a topos. $\square$

We now show that the construction of StrCsp_L is actually functorial.

Theorem 2.5. *There is a functor*

$$\mathsf{StrCsp}_{(-)} \colon [\bullet \to \bullet, \mathsf{Topos}] \to \mathsf{Topos}$$

defined by

$$\begin{tikzcd} \mathsf{X} \arrow[r, shift right, "R"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "G"'] \arrow[d, phantom, "\dashv"] & \mathsf{A} \arrow[l, shift right, "L"'] \arrow[d, shift left, "G'"] \arrow[d, phantom, "\vdash"] \\ \mathsf{X}' \arrow[r, shift left, "R'"] \arrow[r, phantom, "\top"] \arrow[u, shift right, "F"'] & \mathsf{A}' \arrow[l, shift left, "L'"] \arrow[u, shift left, "F'"] \end{tikzcd} \quad \xmapsto{\mathsf{StrCsp}_{(-)}} \quad \begin{tikzcd} \mathsf{StrCsp}_L \arrow[r, shift left, "\Theta"] \arrow[r, phantom, "\perp"] & \mathsf{StrCsp}_{L'} \arrow[l, shift left, "\Theta'"] \end{tikzcd}$$

which is in turn given by

$$\begin{tikzcd}
La \arrow[r, "m"] \arrow[d, "Lf"'] & x \arrow[d, "g"'] & Lb \arrow[l, "n"'] \arrow[d, "Lh"'] & {} \arrow[r, mapsto, "\Theta"] & {} & L'G'a \arrow[r, "Gm"] \arrow[d, "L'G'f"'] & Gx \arrow[d, "Gg"'] & L'G'b \arrow[l, "Gn"'] \arrow[d, "L'G'h"'] \\
Lc \arrow[r, "o"] & y & Ld \arrow[l, "p"'] & & & L'G'c \arrow[r, "Go"] & Gy & L'G'd \arrow[l, "Gp"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
L'a' \arrow[r, "m'"] \arrow[d, "L'f'"'] & x' \arrow[d, "g'"'] & L'b' \arrow[l, "n'"'] \arrow[d, "L'h'"'] & {} \arrow[r, mapsto, "\Theta'"] & {} & LF'a' \arrow[r, "Fm'"] \arrow[d, "LF'f'"'] & Fx' \arrow[d, "Fg'"'] & LF'b' \arrow[l, "Fn'"'] \arrow[d, "LF'h'"'] \\
L'c' \arrow[r, "o'"] & y' & L'd' \arrow[l, "p'"'] & & & LF'c' \arrow[r, "Fo'"] & Fy' & LF'd' \arrow[l, "Fp'"']
\end{tikzcd}$$

Proof. In light of Lemma 2.4, it suffices to show that $\Theta \dashv \Theta'$ gives a geometric morphism.

Denote the structured cospans

$$(m,n)\colon La + Lb \to x$$

in StrCsp_L by ℓ and

$$(m',n')\colon L'a' + L'b' \to x'$$

in $\mathsf{StrCsp}_{L'}$ by ℓ'. Also, denote the unit and counit for $F \dashv G$ by η, ε and for $F' \dashv G'$ by η', ε'. The assignments

$$((f,g,h)\colon \ell \to \Theta'\ell') \mapsto ((\varepsilon' \circ F'f, \varepsilon \circ Fg, \varepsilon' \circ F'h)\colon \Theta\ell \to \ell') \tag{2}$$

$$((f',g',h')\colon \Theta\ell \to \ell') \mapsto ((G'f' \circ \eta', Gg' \circ \eta, G'h' \circ \eta')\colon \ell \to \Theta'\ell') \tag{3}$$

give a bijection $\hom(\Theta\ell, \ell') \simeq \hom(\ell, \Theta'\ell')$. Moreover, it is natural in ℓ and ℓ'. This rests on the natural maps η, ε, η', and ε'. The left adjoint Θ' preserves finite limits because they are taken pointwise and L, F, and F' all preserve finite limits. $\square$

Even though StrCsp_L is a topos, and we are heavily dependent on the topos theory, we are not currently interested in developing the theory of stuctured cospans interal to Topos. The primary reason is that the sort of morphisms $\mathsf{StrCsp}_L \to \mathsf{StrCsp}_{L'}$ we are interested in are not geometric morphisms, but instead are the following.

Definition 2.6. A **structured cospan functor** is a pair of finitely continuous and cocontinuous functors $F \colon \mathsf{X} \to \mathsf{X}'$ and $G \colon \mathsf{A} \to \mathsf{A}'$ such that $FL = L'F$ and $GR = R'F$.

Structured cospan categories and their morphisms do form a category, but we leave it unnamed.

2.2. **Structured cospans as arrows.** We now turn to capturing the compositional structure that truly motivates the invention of structured cospans. To do this, we shift perspectives from structured cospans as objects in StrCsp_L to structured cospans as morphisms.

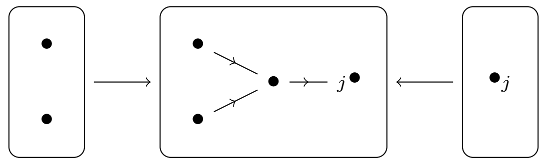
Definition 2.7. Denote by Cospan_L the category that has the same objects as A and structured cospans $La + Lb \to x$ as arrows of type $a \to b$.

Note that the composition of $La+Lb \to x$ with $Lb+Lc \to y$ is given by pushout:

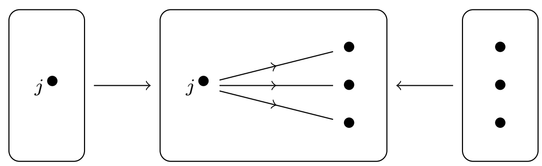
$$\begin{tikzcd} & x +_{Lb} y & \\ La \arrow[ur] & & Lc \arrow[ul] \end{tikzcd}$$

Pushouts, in a sense, are a way of gluing things together. Hence using pushouts as composition is a sensible way to model system connection. The composition above is like connecting along Lb. To illustrate this we return to the open graphs example.

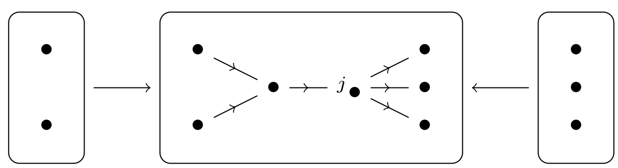
Example 2.8. The open graph



can be composed with the open graph



to obtain



This composition glued the two open graphs together along the node j.

2.3. **A double category of structured cospans.** Using double categories allows us to combine into a single instrument the competing perspectives of structured cospans as objects and as morphisms. For a precise definition of a symmetric monoidal double category, we point to Shulman [17], though for the sake of completeness, we list the key pieces. A (psuedo) double category $\mathbb{C}$ is a category weakly internal to Cat. Roughly, this is a pair of categories $(\mathsf{C}_0, \mathsf{C}_1)$ assembled together as follows.

- The $\mathbb{C}$-objects are exactly the C_0-objects.
- The vertical arrows $c \to d$ in $\mathbb{C}$ between C-objects are exactly the C_0-arrows.
- The horizontal arrows $c \nrightarrow d$ in $\mathbb{C}$ between $\mathbb{C}$-objects are C_1-objects.

- The squares of $\mathbb{C}$ are

$$\begin{tikzcd} c \arrow[r, "m", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \theta"] & d \arrow[d, "g"] \\ c' \arrow[r, "n"', "|" marking] & d' \end{tikzcd} \qquad \begin{gathered} c, c', d, d' \in \mathrm{ob}(\mathsf{C}_0) \\ f, g \in \mathrm{arr}(\mathsf{C}_0) \\ m, n \in \mathrm{ob}(\mathsf{C}_1) \\ \theta \in \mathrm{arr}(\mathsf{C}_1) \end{gathered}$$

  are the arrows of C_1.

In addition, there are structure maps ensuring the correct interplay between the elements of this data. The vertical arrows compose as they do in C_0 and there is a structure map for composing horizontal arrows. The squares can compose both horizontally and vertically.

Observe that the horizontal arrows play two roles: as objects in their origin category and arrows in the double category. This reflects the content of the categories StrCsp_L and Cospan_L. Here is a first example of a double category.

Definition 2.9. There is a double category $\mathbb{S}\mathsf{trCsp}_L \coloneqq (\mathsf{A}, \mathsf{StrCsp}_L)$:

- the objects are the A-objects
- the vertical arrows $a \to b$ the A-arrows,
- the horizontal arrows $a \nrightarrow b$ are the cospans $La + Lb \to x$, and
- the squares are the commuting diagrams

$$\begin{array}{ccccc} La & \longrightarrow & x & \longleftarrow & Lb \\ {\scriptstyle Lf}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Lh} \\ Lc & \longrightarrow & y & \longleftarrow & Ld \end{array}$$

Baez and Courser proved that this actually is a double category [1, Cor. 3.9]. Moreover, when A and X are cocartesian, their coproducts can be used to define a tensor product on $\mathbb{S}\mathsf{trCsp}_L$. This tensor encodes the idea that the disjoint union of considering the disjoint union of two systems as a single system. Because we have no need for this structure in this paper, we say no more about it.

Remark 2.10. Double categories are a nice way of capturing both the object-ness and arrow-ness of structured cospans. An alternative would be to use bicategories, but this doesn't reflect the nature of structured cospans as faithfully as does double categories.

3. Rewriting

We begin this final section by recalling the basics of double pushout rewriting within the context of topoi. We also present the second of our main results: a generalization, from rewriting graphs to rewriting topoi, about the expressiveness of certain graph grammars. We then apply this rewriting theory to structured cospans. In doing so, we introduce some new categorical bookkeeping devices that shows that the rewrite relation is obtained functorially. This section contains our main result which is a generalization of work by Gadducci and Heckle [13]. However, this result is not simply a mere generalization but justifies the study of systems using structured cospans. We end the section by exploring this justification.

Double pushout rewriting was introduced for graphs by Ehrig, et. al. [11]. It has since undergone extensive study and generalization. Currently, the most general setting to contain a rich theory of rewriting is adhesive categories, introduced by Lack and Sobociński [15]. Topoi, such as structured cospan categories, are examples of adhesive categories [16] so the theory we are developing in this paper admits rewriting.

Because topoi are the greatest level of generality we need, we only recall rewriting at this level. Of course, these concepts hold for adhesive categories in general, but restricting to topoi allows us to avoid an unessecary digression

3.1. **Rewriting topoi.** Fix a topos C. Since rewriting in topoi is abstracted from graph rewriting, the archetypal topos for us is RGraph.

Rewriting starts with the notion of a **rewrite rule**, or simply **rule**. This is a span

$$\ell \leftarrowtail k \rightarrowtail r$$

in C with monic legs. We continue to denote spans by $k \to \ell \times r$ and specifying it is a rule indicates that the legs are monic. The conciept of a rule is that ℓ is replaced by r with k a fixed subobject common to both. We can then apply this rule to suitable objects having ℓ as a subobject. Suitability for $m \colon \ell \rightarrowtail g$ means a **pushout complement** exists, that is an object d fitting into a pushout diagram

$$\begin{tikzcd} \ell \arrow[d, "m"'] \arrow[dr, phantom, "\text{p.o.}"] & k \arrow[l] \arrow[d] \\ g & d \arrow[l] \end{tikzcd}$$

A pushout complement need not exist, but if it does and the map $k \to \ell$ is monic, then it is unique up to isomorphism [15, Lem. 15]. Given a rule together with a suitable g, we obtain a **derived rule** on the bottom row of the *double pushout diagram*

$$\begin{tikzcd} \ell \arrow[d, tail] \arrow[dr, phantom, "\text{p.o.}"] & k \arrow[l, tail] \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & r \arrow[d] \\ g & d \arrow[l] \arrow[r] & h \end{tikzcd}$$

Indeed, the span $d \to g \times h$ is a rule because pushouts preserve monics in topoi [15, Lem. 12]. The intuition of this is that we are identifying an instance of ℓ in g and replacing it with r in a cohesive manner, thus resulting in a new object h.

A topos C together with a finite set P of rules $\{k_j \to \ell_j \times r_j\}$ in C is called a **grammar**. An arrow of grammars $(\mathsf{C}, P) \to (\mathsf{D}, Q)$ is a generic functor $F \colon \mathsf{C} \to \mathsf{D}$ such that $FP \subseteq Q$. Together these form a category Gram.

Every grammar (C, P) gives rise to a relation on the objects of C defined by $g \rightsquigarrow h$ whenever there exists a rule $d \to g \times h$ derived from a production in P. However, this relation is not sufficient. For one, it is not true in general that $x \rightsquigarrow x$ holds. Also, it doesn't capture multistep rewrites. That is, perhaps there are derived rules witnessing $g \rightsquigarrow g'$ and $g' \rightsquigarrow g''$ but not a derived rule witnessing $g \rightsquigarrow g''$. However, we want to be able to relate a pair of objects if one can be rewritten into another after a finite sequence of derived rules. Therefore, the relation we actually want is the reflexive and transitive closure of $\rightsquigarrow$, which we denote by $\rightsquigarrow^*$. This is called the **rewrite relation**. Every grammar gives rise to a unique rewrite relation. Moreover, this can be done functorially, though we content ourselves to restrict our attention to showing this in the context of structured cospan categories.

3.2. **Generalizing a result from graph rewriting.** In this section, we lift a well-known result [11, Prop. 3.3] from the theory of rewriting graphs into the theory of rewriting topoi.

The original result is as follows. Let $\flat \colon \mathsf{RGraph} \to \mathsf{RGraph}$ denote the underlying discrete graph comonad. Given a grammar (RGraph, P), define a new grammar $(\mathsf{RGraph}, P_\flat)$ where $P_\flat$ consists of rules $k_\flat \hookrightarrow k \to \ell \times r$ for each rule $k \to \ell \times r$ in P. Then a graph g is related to a graph h with respect to the rewrite relation induced by (RGraph, P) if and only if g is related to h with respect to the rewriting relation induced by $(\mathsf{RGraph}, P_\flat)$.

To generalize this result, we first need a few notions. Fix a geometric morphism $L \dashv R \colon \mathsf{X} \to \mathsf{A}$. Denote by $(\mathsf{X}, P_\flat)$ the **discrete grammar** underlying (X, P). This consists of all rules obtained by pulling back $k \to \ell \times r$ by the counit $LRk \to k$ for each rule in P.

Recall that a poset is **well-founded** if every non-empty subset has a minimal element. Whenever the axiom of choice is present, well-foundedness is equivalent to the lack of infinite descending chains. For a relevant example, as the axiom of choice holds in any presheaf category, the Heyting algebra $\mathrm{Sub}(x)$ for any finite-set valued presheaf x is well-founded.

Theorem 3.1. *Fix a geometric morphism $L \dashv R \colon \mathsf{X} \to \mathsf{A}$ with monic counit. Let (X, P) be a grammar such that for every X-object x in the apex of a production of P, the Heyting algebra $\mathrm{Sub}(x)$ is well-founded. The rewriting relation for a grammar (X, P) is equal to rewriting relation for the grammar $(\mathsf{X}, P_\flat)$*

Proof. For any derivation

$$\begin{tikzcd} \ell \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & k \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & r \arrow[d] \\ g & d \arrow[l] \arrow[r] & h \end{tikzcd}$$

arising from P, there is a derivation

$$\begin{tikzcd} \ell \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & k \arrow[l] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & LRk \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & k \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & r \arrow[d] \\ g & d \arrow[l] & w \arrow[l] \arrow[r] & d \arrow[r] & h \end{tikzcd}$$

where

$$w \coloneqq \bigwedge\{z \colon z \wedge k = x\} \vee LRk.$$

Note that $w \vee k = x$ and $w \wedge k = LRy$ which gives that the two inner squares of the lower diagram are pushouts. $\square$

3.3. **Rewriting structured cospans.** We now turn to rewriting structured cospans. The ability of structured cospans to give a nice theory of rewriting lies in the fact that they form a topos (Theorem 2.4). The first thing we do is appropriately restrict Gram to a subcategory $\mathsf{StrCspGram}$. The objects are (StrCsp_L, P) where P consists of productions of the form

$$\begin{tikzcd}
La \arrow[r] & x & La' \arrow[l] \\
Lb \arrow[u, "\cong"] \arrow[d, "\cong"'] \arrow[r] & y \arrow[u, tail] \arrow[d, tail] & Lb' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
Lc \arrow[r] & z & Lc' \arrow[l]
\end{tikzcd}$$

and the morphisms are the structured cospan functors (Definition 2.6) that are stable under the grammars.

Recall that to each grammar is associated a relation $\rightsquigarrow$ and its reflexive transitive closure, the rewrite relation $\rightsquigarrow^*$. We now show that this can be done functorially via a composite of two functors, $D\colon \mathsf{StrCspGram} \to \mathsf{StrCspGram}$ and $S\colon \mathsf{StrCspGram} \to \mathsf{DblCat}$, which we now define.

is composition preserved?

Lemma 3.2. *There is an idempotent functor $D\colon \mathsf{StrCspGram} \to \mathsf{StrCspGram}$. It is defined on objects by setting $D(\mathsf{StrCsp}_L, P)$ to be the grammar (StrCsp_L, P'), where P' consists of all rules $h \to g \times d$ witnessing the relation $g \rightsquigarrow h$ with respect to (StrCsp_L, P). On arrows, $DF\colon D(\mathsf{StrCsp}_K, P) \to D(\mathsf{StrCsp}_L, Q)$ is defined exactly as F. Moreover, the identity on $\mathsf{StrCspGram}$ is a subfunctor of D.*

Proof. That $D(\mathsf{StrCsp}, P)$ actually gives a grammar follows from the fact that pushouts respect monics in a topos [15, Lem. 12].

That D is idempotent is equivalent to saying that, for a set P of rules, $g \rightsquigarrow h$ with respect to $D(\mathsf{StrCsp}_L Gram, P)$ if and only if $g \rightsquigarrow h$ with respect to $DD(\mathsf{StrCspGram}_L, P)$. This follows from the fact that the outer box of the diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

is a pushout.

The identity is a subfunctor of D because $\ell \rightsquigarrow r$ for any production $k \to \ell \times r$ in (StrCsp_L, P) via a triple of identity arrows. Hence the identity functor on StrCsp_L turns (StrCsp_L, P) into a subobject of $D(\mathsf{StrCsp}_L, P)$. $\square$

what's the meaning of this lemma?

To define S, we reference the double category $\mathbb{M}\mathsf{onSpCsp}(\mathsf{C})$ for a topos C introduced in [9]. The objects are those in C, the vertical arrows are spans with invertible legs in C, the horizontal arrows are cospans in C, and the squares are diagrams in C with shape

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \\
\bullet \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & \bullet \arrow[u, tail] \arrow[d, tail] & \bullet \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

Given a structured cospan grammar (StrCsp_L, P), observe that the productions in P are admissible as squares in $\mathbb{M}\mathsf{onSpCsp}(\mathsf{X})$. Denote by $S(\mathsf{StrCsp}_L, P)$ the sub-double category of $\mathbb{M}\mathsf{onSpCsp}(\mathsf{X})$ that is full on objects, vertical and horizontal arrows, and generated by the productions in P. This assignment is functorial because

double check this

$$(F, G)\colon (\mathsf{StrCsp}_L, P) \to (\mathsf{StrCsp}_{L'}, P')$$

gives a mapping between the generators of $S(\mathsf{StrCsp}_L, P)$ and $S(\mathsf{StrCsp}_{L'}, P')$. Composition holds because F and G both preserve pullbacks and pushouts. This allows us to define the language functor $\mathrm{Lang} \coloneqq SD$.

rotate to main result

Here is a quick lemma that we use in the next theorm.

Lemma 3.3. *If $x \rightsquigarrow^* y$ and $x' \rightsquigarrow^* y'$, then $x + x' \rightsquigarrow^* y + y'$*

Proof. It the derivation $x \rightsquigarrow^* y$ comes from a string of double pushout diagrams

$$\begin{tikzcd}
\ell_1 \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & k_1 \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & r_1 \arrow[dr] & & \ell_2 \arrow[dl] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & k_2 \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & r_2 \arrow[d] & \cdots & \ell_n \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & k_n \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & r_n \arrow[d] \\
x & d_1 \arrow[l] \arrow[rr] & & w_1 & & d_2 \arrow[ll] \arrow[r] & w_2 & & w_{n-1} & d_n \arrow[l] \arrow[r] & y
\end{tikzcd}$$

and the derivation $x \rightsquigarrow^* y$ comes from a string of double pushout diagrams

$$\begin{tikzcd}
\ell'_1 \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & k'_1 \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & r'_1 \arrow[dr] & & \ell'_2 \arrow[dl] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & k'_2 \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & r'_2 \arrow[d] & \cdots & \ell'_n \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & k'_m \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\text{p.o.}"] & r'_m \arrow[d] \\
x' & d'_1 \arrow[l] \arrow[rr] & & w'_1 & & d'_2 \arrow[ll] \arrow[r] & w'_2 & & w'_{m-1} & d'_m \arrow[l] \arrow[r] & y'
\end{tikzcd}$$

then $x + x' \rightsquigarrow^* y + y'$ is realized by concatenating to the end of first string with x' summed with the bottom row the second string with y summed on the bottom row. $\square$

The desire behind the main result is the ability to study systems, as represented by objects in a topos X, locally. The mechanism (that is, structured cospans) by which we do this is to equip systems with interfaces that allow us to connect sub-systems together. Another way to view this is that given a system can be decomposed into sub-systems. These can be studied individually then reconnected along the interfaces this mechanism provides. The manner in which the main result can accomplish this is discussed below the theorem

We need the following definition. Associate to a grammar (X, P) the structured cospan grammar (StrCsp_L, P') where P' contains

$$\begin{tikzcd} L0 \arrow[r] & \ell & LRk \arrow[l] \\ L0 \arrow[u] \arrow[d] \arrow[r] & LRk \arrow[u] \arrow[d] & LRk \arrow[l] \arrow[u] \arrow[d] \\ L0 \arrow[r] & r & LRk \arrow[l] \end{tikzcd}$$

for every rule $LRk \to \ell \times r$ of $P_\flat$.

Theorem 3.4. *Fix a geometric morphism $L \dashv R \colon \mathsf{X} \to \mathsf{A}$ with monic counit. Let (X, P) be a grammar such that for every X-object x in the apex of a production of P, the Heyting algebra $\mathrm{Sub}(x)$ is well-founded. Given g, $h \in \mathsf{X}$, then $g \rightsquigarrow^* h$ in the rewriting relation for a grammar (X, P) if and only if there is a square*

$$\begin{tikzcd} LR0 \arrow[r] & g & LR0 \arrow[l] \\ LR0 \arrow[u] \arrow[d] \arrow[r] & d \arrow[u] \arrow[d] & LR0 \arrow[l] \arrow[u] \arrow[d] \\ LR0 \arrow[r] & h & LR0 \arrow[l] \end{tikzcd}$$

in the double category $\mathrm{Lang}(\mathsf{StrCsp}_L, P')$.

Proof. We show sufficiency by induction on the length of the derivation. If $g \rightsquigarrow h$

$$\begin{tikzcd} \ell \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & LRk \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & r \arrow[d] \\ g & d \arrow[l] \arrow[r] & h \end{tikzcd}$$

the desired square is the horizontal composition of

$$\begin{tikzcd}
L0 \arrow[r] & \ell & LRk \arrow[l] \arrow[r] & d & L0 \arrow[l] \\
L0 \arrow[r] \arrow[u] \arrow[d] & LRk \arrow[u] \arrow[d] & LRk \arrow[l] \arrow[r] \arrow[u] \arrow[d] & d \arrow[u] \arrow[d] & L0 \arrow[l] \arrow[u] \arrow[d] \\
L0 \arrow[r] & r & LRk \arrow[l] \arrow[r] & d & L0 \arrow[l]
\end{tikzcd}$$

The left square is a generator and the right square is the identity on the horizontal arrow $LRk + L \to d$. The square for a derivation $g \rightsquigarrow^* h \rightsquigarrow j$ is the vertical composition of

$$\begin{tikzcd}
L0 \arrow[r] & g & L0 \arrow[l] \\
L0 \arrow[r] \arrow[u] \arrow[d] & d \arrow[u] \arrow[d] & L0 \arrow[l] \arrow[u] \arrow[d] \\
L0 \arrow[r] & h & L0 \arrow[l] \\
L0 \arrow[r] \arrow[u] \arrow[d] & e \arrow[u] \arrow[d] & L0 \arrow[l] \arrow[u] \arrow[d] \\
L0 \arrow[r] & j & L0 \arrow[l]
\end{tikzcd}$$

The top square is from $g \rightsquigarrow^* h$ and the second from $h \rightsquigarrow j$.

Conversely, proceed by structural induction on the generating squares of $\mathrm{Lang}(\mathsf{StrCsp}_L, P')$. It suffices to show that the rewrite relation is preserved by vertical and composition by a generating square. Suppose we have a square

$$\begin{tikzcd}
L0 & w \arrow[l] \arrow[r] & L0 \\
L0 \arrow[u] \arrow[d] & x \arrow[l] \arrow[r] \arrow[u] \arrow[d] & L0 \arrow[u] \arrow[d] \\
L0 & y \arrow[l] \arrow[r] & L0
\end{tikzcd}$$

corresponding to a derivation $w \rightsquigarrow^* y$. Composing this vertically with a generating square, which must have form

$$\begin{tikzcd} L0 & y \arrow[l] \arrow[r] & L0 \\ L0 \arrow[u] \arrow[d] & L0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & L0 \arrow[u] \arrow[d] \\ L0 & z \arrow[l] \arrow[r] & L0 \end{tikzcd}$$

corresponding to a production $0 \to y + z$ gives

$$\begin{tikzcd} L0 & w \arrow[l] \arrow[r] & L0 \\ L0 \arrow[u] \arrow[d] & L0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & L0 \arrow[u] \arrow[d] \\ L0 & z \arrow[l] \arrow[r] & L0 \end{tikzcd}$$

which corresponds to a derivation $w \rightsquigarrow^* y \rightsquigarrow z$. Composing horizontally with a generating square

$$\begin{tikzcd} L0 & \ell \arrow[l] \arrow[r] & L0 \\ L0 \arrow[u] \arrow[d] & LRk \arrow[l] \arrow[r] \arrow[u] \arrow[d] & L0 \arrow[u] \arrow[d] \\ L0 & r \arrow[l] \arrow[r] & L0 \end{tikzcd}$$

corresponding with a production $LRk \to \ell + r$ results in the square

$$\begin{tikzcd} L0 & w + \ell \arrow[l] \arrow[r] & L0 \\ L0 \arrow[u] \arrow[d] \arrow[r] & x + LRk \arrow[u] \arrow[d] & L0 \arrow[l] \arrow[u] \arrow[d] \\ L0 & y + r \arrow[l] \arrow[r] & L0 \end{tikzcd}$$

But $w + \ell \rightsquigarrow^* y + r$ as seen in Lemma 3.3.

□

With this result, we have completely described the rewrite relation for a grammar (X, P) with those squares in $\mathrm{Lang}(\mathsf{StrCsp}_L, P')$ framed by the initial object of X.

These squares are rewrites of a closed system, which may be difficult to understand. We can instead begin with a closed system as represented by a horizontal arrow in $\mathrm{Lang}(\mathsf{StrCsp}_L, P')$ and decompose it into a composite of easier to understand sub-systems, rather a sequence of composable horizontal arrows. Rewriting can be performed on each of these sub-systems which, of course, is represented by squares. The composite of these squares gives are rewriting of the original system.

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