The final network having one less element is shown in Fig. 5 where A_6 is a star point, but not the reference point A_0 used in the development.

As an application of the alternate set of necessary and sufficient conditions mentioned earlier, let us consider the subnetwork of Fig. 4-a. Its terminal representation is given in Fig. 6. Since coefficient matrix of the terminal equations satisfy the conditions 1) and 2), existence of an equivalent star-network is ensured. The element values of the star network are determined as follows:

$$a_{11} = 1.6$$

$$\alpha_1 = 0.5$$

$$\alpha_2 = 2.5$$

$$\alpha_3 = 1$$

$$\sigma = 1 + 0.5 + 2.5 + 1 = 5.$$

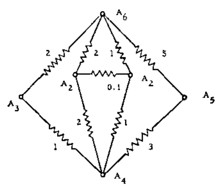
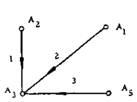


Fig. 5—6-Terminal R-network equivalent to that in Fig. 3 having one less element. (Element values in mhos).



$$\begin{bmatrix} i_1(t) \\ i_2(t) \\ i_3(t) \end{bmatrix} = \begin{bmatrix} 1.6 & -0.2 & -1 \\ -0.2 & 0.9 & -0.5 \\ -1 & -0.5 & 2.5 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix}$$

Fig. 6—Alternate terminal representation for subnetwork in Fig. 4-a.

Since

$$g_1 = \frac{a_{11}}{1 - \frac{1}{\sigma}} = \frac{1.6}{1 - \frac{1}{5}} = 2,$$

hence

$$g_2 = \alpha_1 g_1 = 1,$$

$$g_3 = \alpha_2 g_1 = 5 \quad \text{and} \quad g_4 = \alpha_3 g_1 = 2$$

These results check with the network derived from the first set of necessary and sufficient conditions.

CHUNG-LEI WANG
University of Illinois
Urbana, Ill.
YILMAZ TOKAD
Dept. of Elec. Engrg.
Michigan State University
East Lansing, Mich.

REFERENCES

[1] I. Cederbaum, "Applications of matrix algebra to network theory," IRE TRANS. ON CIRCUIT THEORY (special suppl.), vol. CT-6, pp. 127–137; May, 1959.

[2] E. A. Guillemin, "On the Analysis and Synthesis of Single-Element-Kind Networks," IRE TRANS. ON CIRCUIT THEORY, vol. CT-7; September, 1960.

[3] G. Biorci and P. P. Civalleri, "On the synthesis of resistive N-port networks," IRE TRANS. ON CIRCUIT THEORY, vol. CT-8, pp. 22–28; March, 1961.

[4] D. P. Brown and Y. Tokad, "On the synthesis of R-Networks," IRE TRANS. ON CIRCUIT THEORY, vol. CT-8, pp. 31–39; March, 1961.

[5] H. E. Koenig and W. A. Blackwell, "Electromechanical System Theory," McGraw-Hill Book Company, Inc., New York, N. Y.; 1961.

[6] P. Slepian and L. Weinberg, "Applications of paramount and dominant matrices," *Proc. Natl. Electronics Conf.*, vol. 14, pp. 611–630; October, 1958.

[7] J. Shekel, "Voltage reference node," *Wireless Engr.*, vol. 31, pp. 6–10; January, 1954.

[8] G. E. Sharpe and B. Spain, "On the solution of networks by means of the equicofactor matrix," IRE TRANS. ON CIRCUIT THEORY, vol. CT-7, pp. 230–239; September, 1960.

[9] A. Rosen, "A new network theorem," *J. IEE*, vol. 62, pp. 916–918; 1934.

[10] D. W. C. Shen, "Generalized star and mesh transformations," *Phil. Mag.*, ser. 7, vol. 38, pp. 267–275; April, 1947. In this article, this property is stated as Wheatstone's condition.

[11] G. Calabrese, "Notes on the equivalence of electrical networks," *GE Rev.*, vol. 42, pp. 323–325; July, 1939.

Converse of the Star-Mesh Transformation*

INTRODUCTION

Unlike the wye-delta transformation, the converse of the star-mesh transformation[1] does not always exist. A straight-forward technique for determining the necessary conditions for equivalence was given in 1939 by Calabrese.[2] In this paper we give his results a topological interpretation and in addition we indicate how to write the pertinent equations for the element values of the star graph in terms of the elements of the corresponding "complete graph". After completion of this work, it came to our attention that similar results had been obtained by Shen[3] using a different method. An important point here is that the existence of a general method for obtaining the converse of the star-mesh transformation has apparently gone unnoticed in texts on network theory. This convenient technique also provides a way of treating the "open question" regarding conditions for physical realizability of a symmetric matrix recently posed by Foster.[4]

TOPOLOGICAL PRELIMINARIES

To simplify discussion we shall use a regular polygon as a standardized representation for the "complete graph"[5] of N nodes which corresponds to the given "mesh" network. For ease in assigning branch designations and other reasons which will become apparent shortly, we also use the related $n \times n$ upper, off-diagonal, branch admittance matrix introduced by Nakagawa.[6] Note that n equals $N - 1$. The desired complete graph and its corresponding matrix were shown in an earlier paper[7] for a different purpose. Geometrically parallel branches in the standardized representation of the graph appear in the matrix either as elements along or parallel to the secondary diagonal (perpendicular to the main diagonal), or as elements $N/2$ steps apart along or parallel to the main diagonal itself. For example, consider the branch admittance matrix for $N = 7$:

$$\begin{array}{c} \text{(nodes)} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} A & C & F & J & \O & U \\ & B & E & I & N & T \\ & & D & H & M & S \\ & & & G & L & R \\ & & & & K & Q \\ & & & & & P \end{bmatrix} \end{array} = M.$$

Thus, branch A is parallel to branches L and S in the graph for $N = 7$ as shown in the previous paper.

NECESSARY AND SUFFICIENT CONDITIONS

Existence of an equivalent star for a given mesh network is contingent upon having the transfer admittances satisfy certain relationships.[2,3] A simple topological interpretation involving identification of quadrilaterals makes it possible to write them by inspection. The procedure automatically satisfies the formula given by Calabrese, namely

$$\text{number of additional conditions} = N(N-3)/2.$$

Now using the branch admittance matrix we can readily identify the "complete set" of quadrilaterals required for arbitrary N. This is shown in Fig. 1, where the standardized branch designations given in matrices of the type M are intentionally suppressed for emphasis. Only one quadrilateral is shown per matrix. The necessary relationship involves taking the product of the admittances of the two parallel branches and equating it with the product of the admittances of the two diagonal branches of the quadrilateral. Sufficiency requires that every one in the set satisfy the condition.

* Received by the PGCT, July 22, 1961.

[1] A. Rosen, "A new network theorem," *J. IEE*, vol. 62, pp. 916–918; 1924.

[2] G. O. Calabrese, "Notes on the equivalence of electrical networks," *Gen. Elec. Rev.*, vol. 42, pp. 323–325; July, 1939.

[3] D. W. C. Shen, "Generalized star and mesh transformations," *Phil. Mag.*, ser. 7, vol. 38, pp. 267–275; April, 1947.

[4] R. M. Foster, "An open question," IRE TRANS. ON CIRCUIT THEORY (correspondence), vol. CT-8, p. 175; June, 1961.

[5] S. Seshu, and M. B. Reed, "Linear Graphs and Electrical Networks," Addison-Wesley Publishing Co., Inc., Reading, Mass.; 1961.

[6] N. Nakagawa, "On evaluation of graph trees and the driving point admittance," IRE TRANS. ON CIRCUIT THEORY, vol. CT-5, pp. 122–127; June, 1958.

[7] S. D. Bedrosian, "Formulas for the Number of Trees in a Network," IRE TRANS. ON CIRCUIT THEORY, vol. CT-8, pp. 363–364; September, 1961.

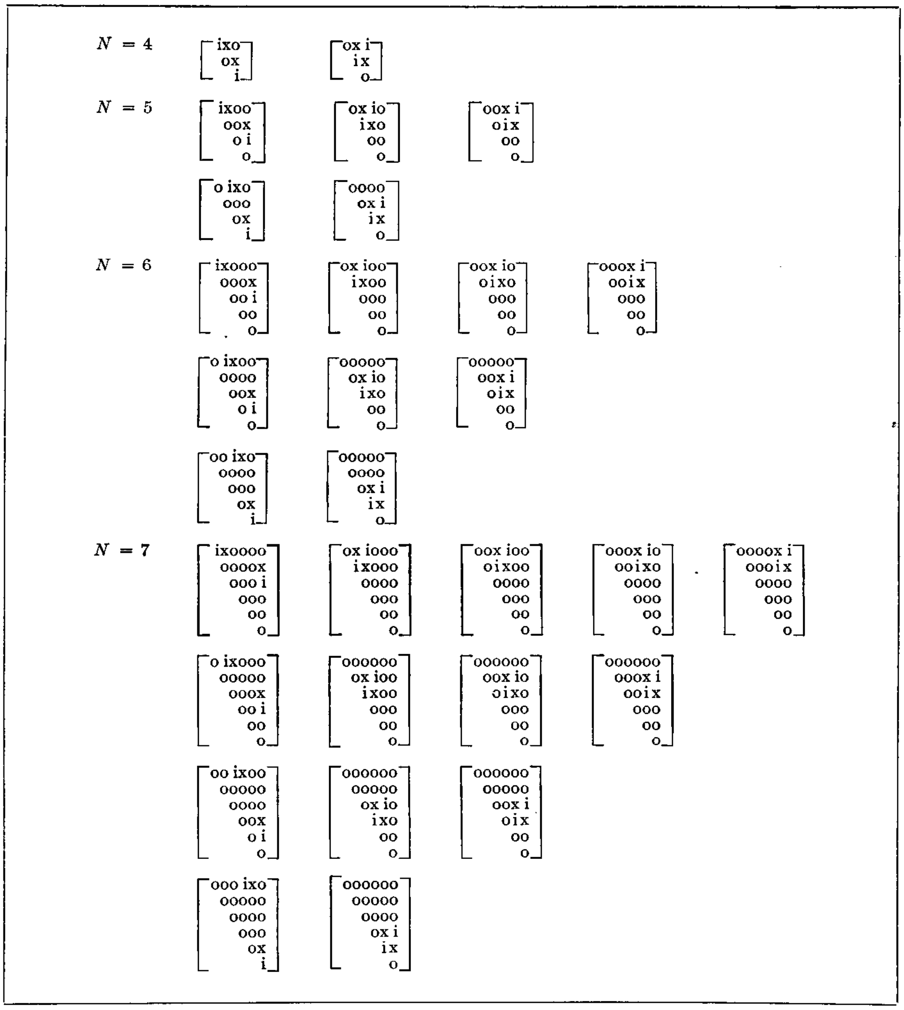


Fig. 1—Patterns for quadrilaterals in branch admittance matrix for a standardized complete graph. i = parallel branches, x = diagonal branches, and o = ignored branches.

To facilitate recognition observe a regular pattern by row and column for these quadrilaterals, and also that the number of matrices follows the pattern is indicated in Table I.

TABLE 1

N	Number of matrices in each column	Total $= N(N-3)/2$
3	0	0
4	1 1	2
5	2 2 1	5
6	3 3 2 1	9
7	4 4 3 2 1	14
8	5 5 4 3 2 1	20
9	6 6 5 4 3 2 1	27
10	7 7 6 5 4 3 2 1	35
⋮		⋮

Element Values

The final step is determination of the set of equations which give the element values of the star network in terms of the complete graph ("mesh" network). Once the conditions given above have been satisfied, one simply starts from Node 1 of the standardized graph. Then each equation consists of the sum of admittances of all the branches incident on the given node, plus the product of the two incident perimeter branches divided by the branch which forms a triangle with these two branches.

As a specific example, we give the pertinent equations for the branches of the star graph for $N = 7$. Taking the nodes consecutively,

$$g_{1k} = A + C + F + J + \varnothing + U + \frac{AU}{T} \tag{1}$$

$$g_{2k} = B + E + I + N + T + A + \frac{AB}{C} \tag{2}$$

$$g_{3k} = D + H + M + S + C + B + \frac{BD}{E} \tag{3}$$

$$g_{4k} = G + L + R + F + E + D + \frac{DG}{H} \tag{4}$$

$$g_{5k} = K + Q + J + I + H + G + \frac{GK}{L} \tag{5}$$

$$g_{6k} = P + \varnothing + N + M + L + K + \frac{KP}{Q} \tag{6}$$

$$g_{7k} = U + T + S + R + Q + P + \frac{PU}{\varnothing}. \tag{7}$$

Note that in g_{ik} the second subscript refers to the common node of the equivalent star network.

For $N = 7$, given the following conductance values in mhos:

$$A = \frac{2}{16} \quad D = \frac{3}{16} \quad G = \frac{2}{16} \quad J = \frac{2}{16} \quad M = \frac{9}{16} \quad P = \frac{12}{16} \quad S = \frac{12}{16}$$

$$B = \frac{6}{16} \quad E = \frac{2}{16} \quad H = \frac{6}{16} \quad K = \frac{6}{16} \quad N = \frac{6}{16} \quad Q = \frac{8}{16} \quad T = \frac{8}{16}$$

$$C = \frac{3}{16} \quad F = \frac{1}{16} \quad I = \frac{4}{16} \quad L = \frac{3}{16} \quad \varnothing = \frac{3}{16} \quad R = \frac{4}{16} \quad U = \frac{4}{16}.$$

Find the equivalent star network if it exists. Using Fig. 1, matrix M or the standardized graph one obtains the necessary and sufficient conditions which are satisfied by the given element values:

$$AS = CT \quad BF = CE \quad EJ = FI \quad \varnothing I = JN \quad NU = \varnothing T$$

$$CR = FS \quad DI = EH \quad HN = IM \quad MT = NS$$

$$FQ = JR \quad GM = HL \quad LS = MR$$

$$JP = \varnothing Q \quad KR = LQ.$$

Then using the set of equations given above the element values are

$$g_{1k} = 1 \quad g_{3k} = 3 \quad g_{5k} = 2 \quad g_{7k} = 4$$

$$g_{2k} = 2 \quad g_{4k} = 1 \quad g_{6k} = 3.$$

These results are easily verified by the star-mesh transformation.

Applications

An application of this transformation is in connection with the determination of the conditions for the equivalence of n-terminal planar graphs to nonplanar complete graphs. A particular case was considered at Foster's suggestion. It involves the equivalence, with respect to measurements made at the 5 terminals, of 10-branch 10-node planar graph designated P to the complete 5-node graph K (one of the basic Kuratowski nonplanar forms[5]). Graph P is a pentagon with a hinged branch at each of the nodes.

The solution depended on finding a sequence of elementary network transformations which accomplishes the transformation of the network whose graph is P to that whose graph is Q. Graph Q is a 5 star inside a pentagon. Then we have:

> A set of necessary and sufficient conditions for network P to be equivalent to a given nonplanar network K is that the network remaining after subtracting some positive conductance from each of the perimeter branches of network Q must satisfy the five conditions given in Fig. 1 under $N = 5$.

The object is to form a new graph K' having parallel branches about its perimeter. This provides for the existence, *i.e.*, elements with conductance > 0, of each of the perimeter branches in the equivalent network Q. This also ensures the existence of an equivalent network for the given nonplanar graph K.

In the light of these results, we can comment on Foster's "open question" regarding determination of conditions for physical realizability of a symmetric matrix as an admittance or impedance network. First convert the two given matrices to branch admittance matrices. Then it immediately becomes apparent that an equivalent star network exists for the second one (B) but not for the first (A). An attempt to adjust the values of the perimeter branches of the corresponding nonplanar graph to satisfy the necessary and sufficient conditions for a planar equivalent with positive elements, as given above, proved unsuccessful. (We can get an equivalent network by using one negative element however.) Thus we are unable to contradict Foster's contention that his matrix A cannot be realized as the open-circuit impedance matrix of a 5-port resistive network. Nevertheless we have, at least, shed some new light on necessary conditions by providing a means for categorizing the given matrices.

S. D. Bedrosian
Inst. for Cooperative Research
University of Pennsylvania
Philadelphia, Pa.

n-Port Resistive Networks and Communication Nets*

In recent years, there has been a rapid development in the theory of communication nets and in the theory of n-port resistive networks. However, very little has been done to relate the two fields. As a matter of fact, there is a great deal of similarity in the analysis techniques for the two systems and it is probable that many of the wealth of facts known about resistive networks have their counterparts in communication nets with new light being shed on both fields by the comparison. The similarity between the systems is so deep that we can actually show that the realizability conditions developed for one are applicable to the other. Specifically, we will show that the necessary and sufficient conditions for the realization of a terminal capacity matrix are a special case of a set of necessary and sufficient conditions for the realization of an n-port short-circuit admittance matrix.

Synthesis of a Terminal Capacity Matrix

Mayeda[1] has shown that the necessary and sufficient conditions for the realization of a matrix $\mathbf{T} = [t_{ij}]$, with positive real elements, as the terminal capacity matrix of an unoriented communication net N_c are the following:

a) $\mathbf{T}$ must be symmetric and it must be possible that $\mathbf{T}$ be partitioned after interchanging rows and columns, if necessary, as

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} \tag{1}$$

so that all elements of $\mathbf{T}_{12}$ and $\mathbf{T}_{21}$ are equal to the smallest elements in $\mathbf{T}$.

b) $\mathbf{T}_{11}$ and $\mathbf{T}_{22}$ are symmetric and each of them should be partitioned again by step a).

c) The process of step b) is repeated to the last element.

Synthesis of a Short-Circuit Admittance Matrix

Guillemin[2] has proved that the necessary and sufficient conditions for the realization of a non-semi-definite matrix $\mathbf{Y} = [y_{ij}]$, with positive real elements as the short-circuit admittance matrix of an n-port resistive network N_r with $(n + 1)$ nodes and a linear port structure are the following:

* Received by the PGCT, July 19, 1961. This work was supported by Natl. Sci. Foundation Grants G-6020 and G-10354.

[1] W. Mayeda, "Terminal and branch capacity matrices of a communication net," IRE Trans. on Circuit Theory, vol. CT-7, pp. 261–269; September, 1960.

[2] E. A. Guillemin, "On the analysis and synthesis of single element kind networks," IRE Trans. on Circuit Theory, vol. CT-7, pp. 303–312; September, 1960.