CONSTRUCTING SYMMETRIC MONOIDAL BICATEGORIES

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ABSTRACT. We present a method of constructing symmetric monoidal bicategories from symmetric monoidal double categories that satisfy a lifting condition. Such symmetric monoidal double categories frequently occur in nature, so the method is widely applicable, though not universally so.

1. Introduction

Symmetric monoidal bicategories are important in many contexts. However, the definition of even a monoidal bicategory (see [GPS95, Gur06]), let alone a symmetric monoidal one (see [KV94b, KV94a, BN96, DS97, Cra98, McC00, Gur]), is quite imposing, and time-consuming to verify in any example. In this paper we describe a method for constructing symmetric monoidal bicategories which is hardly more difficult than constructing a pair of ordinary symmetric monoidal categories. While not universally applicable, this method applies in many cases of interest. This idea has often been implicitly used in particular cases, such as bicategories of enriched profunctors, but to my knowledge the first general statement was claimed in [Shu08, Appendix B]. Our purpose here is to work out the details, independently of [Shu08].

Remark 1.1. Another approach to working out the details of this statement, from a different perspective, can be found in [GG09, §5]. The two approaches contain basically the same content and results, although the authors of [GG09] work with "locally-double bicategories" rather than monoidal double categories or 2x1-categories (see below). They also don't treat the symmetric case, but as we will see, that is a fairly easy extension once the theory is in place. Thus, this note really presents nothing very new, only a self-contained and (hopefully) convenient treatment of the particular case of interest.

The method relies on the fact that in many bicategories, the 1-cells are not the most fundamental notion of 'morphism' between the objects. For instance, in the bicategory *Mod* of rings, bimodules, and bimodule maps, the more fundamental notion of morphism between objects is a ring homomorphism. The addition of these extra morphisms promotes a bicategory to a *double category*, or a category internal to *Cat*. The extra morphisms are usually stricter than the 1-cells in the bicategory and easier to deal with for coherence questions; in many cases it is quite easy to show that we have a *symmetric monoidal double category*. The central observation is that in most cases (when the double category is 'fibrant') we can then 'lift' this symmetric monoidal structure to the original bicategory. That is, we prove the following theorem:

Theorem 1.2. If \mathbb{D} is a fibrant monoidal double category, then its underlying bicategory $\mathcal{H}(\mathbb{D})$ is a monoidal bicategory. If \mathbb{D} is braided or symmetric, so is $\mathcal{H}(\mathbb{D})$.

There is a good case to be made, however (see [Shu08]) that often the extra morphisms should *not* be discarded. From this point of view, in many cases symmetric monoidal bicategories are a red herring, and really we should be studying symmetric monoidal double

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categories. This is also true in higher dimensions; for instance, Chris Douglas [Dou09] has suggested that instead of tricategories we are usually interested in bicategories internal to *Cat* or categories internal to 2-*Cat*. In most such cases arising in practice, we can again 'lift' the coherence to give a tricategory after discarding the additional structure.

We propose the generic term $(n \times k)$ -category (pronounced "n-by-k-category") for an n-category internal to k-categories, a structure which has (n+1)(k+1) different types of cells or morphisms arranged in an (n+1) by (k+1) grid. Thus double categories may be called **1x1-categories**, while in place of tricategories we may consider 2x1-categories and 1x2-categories. Any $(n \times k)$ -category which satisfies a suitable lifting property should have an underlying (n+k)-category, but clearly as n and k grow an increasing amount of structure is discarded in this process.

However, even for those of the opinion that $(n \times k)$ -categories are fundamental (such as the author), sometimes it really is the underlying (n+k)-category that one cares about. This is particularly the case in the study of topological field theory, since the Baez-Dolan cobordism hypothesis asserts a universal property of the (n+1)-category of cobordisms which is not shared by the $(n \times 1)$ -category from which it is naturally constructed (see [Lur09]). Thus, regardless of one's philosophical bent, results such as Theorem 1.2 are of interest.

Proceeding to the contents of this paper, in §2 we review the definition of symmetric monoidal double categories, and in §3 we recall the notions of 'companion' and 'conjoint' whose presence supplies the necessary lifting property, which we call being *fibrant*. Then in §4 we describe a functor from fibrant double categories to bicategories, and in §5 we show that it preserves monoidal, braided, and symmetric structures.

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2. Symmetric monoidal double categories

In this section, we introduce basic notions of double categories. Double categories go back originally to Ehresmann in [Ehr63]; a brief introduction can be found in [KS74]. Other references include [BE74, GP99, GP04].

Definition 2.1. A (**pseudo**) **double category** \mathbb{D} consists of a 'category of objects' \mathbb{D}_0 and a 'category of arrows' \mathbb{D}_1 , with structure functors

$$U \colon \mathbb{D}_0 \to \mathbb{D}_1$$

$$S, T \colon \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$$

$$\odot \colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$$
(where the pullback is over $\mathbb{D}_1 \stackrel{T}{\longrightarrow} \mathbb{D}_0 \stackrel{S}{\longleftarrow} \mathbb{D}_1$) such that
$$S(U_A) = A$$

$$T(U_A) = A$$

$$S(M \odot N) = SN$$

$$T(M \odot N) = TM$$

equipped with natural isomorphisms

$$\mathfrak{a}: (M \odot N) \odot P \xrightarrow{\cong} M \odot (N \odot P)$$

$$\mathfrak{l}: U_B \odot M \xrightarrow{\cong} M$$

$$\mathfrak{r}: M \odot U_A \xrightarrow{\cong} M$$

such that $S(\mathfrak{a})$, $T(\mathfrak{a})$, $S(\mathfrak{l})$, $T(\mathfrak{l})$, $S(\mathfrak{r})$, and $T(\mathfrak{r})$ are all identities, and such that the standard coherence axioms for a monoidal category or bicategory (such as Mac Lane's pentagon; see [ML98]) are satisfied.

Just as a bicategory can be thought of as a category weakly *enriched* over *Cat*, a pseudo double category can be thought of as a category weakly *internal* to *Cat*. Since these are the kind of double category of most interest to us, we will usually drop the adjective "pseudo."

We call the objects of \mathbb{D}_0 objects or **0-cells**, and we call the morphisms of \mathbb{D}_0 (vertical) **1-morphisms** and write them as $f: A \to B$. We call the objects of \mathbb{D}_1 (horizontal) **1-cells**; if M is a 1-cell with S(M) = A and T(M) = B, we write $M: A \to B$. We call a morphism $\alpha: M \to N$ of \mathbb{D}_1 with $S(\alpha) = f$ and $T(\alpha) = g$ a **2-morphism** and draw it as follows:

$$(2.2) A \xrightarrow{M} B \cdot f \downarrow \qquad \downarrow g \\ C \xrightarrow{M} D$$

Note that we distinguish between *1-morphisms*, which we draw vertically, and *1-cells*, which we draw horizontally. In traditional double-category terminology these are both referred to with the same word (be it "cell" or "morphism" or "arrow"), the distinction being made only by the adjectives "vertical" and "horizontal." Our terminology is more concise, and allows for flexibility in the drawing of pictures without a corresponding change in names (some authors prefer to draw their double categories transposed from ours).

We write the composition of vertical 1-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ and the vertical composition of 2-morphisms $M \xrightarrow{\alpha} N \xrightarrow{\beta} P$ as $g \circ f$ and $\beta \circ \alpha$, or sometimes just gf and $\beta \circ \alpha$. We write the horizontal composition of 1-cells $A \xrightarrow{M} B \xrightarrow{N} C$ as $A \xrightarrow{N \circ M} C$ and that of 2-morphisms

 $\downarrow \alpha \qquad \downarrow \beta$ $\downarrow \beta \odot \alpha \qquad \downarrow$

as

The two different compositions of 2-morphisms obey an interchange law, by the functoriality of ⊙:

$$(M_1 \odot M_2) \circ (N_1 \odot N_2) = (M_1 \circ N_1) \odot (M_2 \circ N_2).$$

Every object A has a vertical identity 1_A and a horizontal unit U_A , every 1-cell M has an identity 2-morphism 1_M , every vertical 1-morphism f has a horizontal unit 2-morphism U_f , and we have $1_{U_A} = U_{1_A}$ (by the functoriality of U).

Note that the vertical composition ∘ is strictly associative and unital, while the horizontal one ⊙ is only weakly so. This is the case in most of the examples we have in mind. It is possible to define double categories that are weak in both directions (see, for instance, [Ver92]), but this introduces much more complication and is usually unnecessary.

Remark 2.3. In general, an $(n \times 1)$ -category consists of 1-categories \mathbb{D}_i for $0 \le i \le n$, together with source, target, unit, and composition functors and coherence isomorphisms. We refer to the objects of \mathbb{D}_i as *i*-cells and to the morphisms of \mathbb{D}_i as **morphisms of** *i*-cells or (vertical) (i + 1)-morphisms. A formal definition can be found in [Bat98] under the name *monoidal n-globular category*.

A 2-morphism (2.2) where f and g are identities (such as the constraint isomorphisms $\mathfrak{a},\mathfrak{l},\mathfrak{r}$) is called **globular**. Every double category $\mathbb D$ has a **horizontal bicategory** $\mathcal H(\mathbb D)$ consisting of the objects, 1-cells, and globular 2-morphisms. Conversely, many naturally occurring bicategories are actually the horizontal bicategory of some naturally occurring double category. Here are just a few examples.

Example 2.4. The double category Mod has as objects rings, as 1-morphisms ring homomorphisms, as 1-cells bimodules, and as 2-morphisms equivariant bimodule maps. Its horizontal bicategory $Mod = \mathcal{H}(Mod)$ is the usual bicategory of rings and bimodules.

Example 2.5. The double category $n\mathbb{C}$ ob has as objects closed n-manifolds, as 1-morphisms diffeomorphisms, as 1-cells cobordisms, and as 2-morphisms diffeomorphisms between cobordisms. Again $\mathcal{H}(n\mathbb{C}$ ob) is the usual bicategory of cobordisms.

Example 2.6. The double category \mathbb{P} rof has as objects categories, as 1-morphisms functors, as 1-cells *profunctors* (a profunctor $A \to B$ is a functor $B^{op} \times A \to \mathbf{Set}$), and as 2-morphisms natural transformations. Bicategories such as $\mathcal{H}(\mathbb{P}$ rof) are commonly encountered in category theory, especially the enriched versions.

As opposed to bicategories, which naturally form a tricategory, double categories naturally form a 2-category, a much simpler object.

Definition 2.7. Let \mathbb{D} and \mathbb{E} be double categories. A (**pseudo double**) functor $F : \mathbb{D} \to \mathbb{E}$ consists of the following.

- Functors $F_0: \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1: \mathbb{D}_1 \to \mathbb{E}_1$ such that $S \circ F_1 = F_0 \circ S$ and $T \circ F_1 = F_0 \circ T$.
- Natural transformations $F_{\odot} \colon F_1M \odot F_1N \to F_1(M \odot N)$ and $F_U \colon U_{F_0A} \to F_1(U_A)$, whose components are globular isomorphisms, and which satisfy the usual coherence axioms for a monoidal functor or pseudofunctor (see [ML98, §XI.2]).

Definition 2.8. A (**vertical**) **transformation** between two functors $\alpha: F \to G: \mathbb{D} \to \mathbb{E}$ consists of natural transformations $\alpha_0: F_0 \to G_0$ and $\alpha_1: F_1 \to G_1$ (both usually written as α), such that $S(\alpha_M) = \alpha_{SM}$ and $T(\alpha_M) = \alpha_{TM}$, and such that

for all 1-cells $M: A \rightarrow B$ and $N: B \rightarrow C$, and

$$FA \xrightarrow{U_{FA}} FA \qquad FA \xrightarrow{U_{FA}} FA$$

$$\parallel \qquad \Downarrow F_0 \qquad \parallel \qquad \qquad \alpha_A \qquad \qquad \Downarrow U_{\alpha_A} \qquad \downarrow \alpha_A$$

$$FA \xrightarrow{F(U_A)} FA = GA \xrightarrow{U_{GA}} GA$$

$$\alpha_A \qquad \qquad \downarrow \alpha_{U_A} \qquad \downarrow \alpha_A \qquad \parallel \qquad \Downarrow F_0 \qquad \parallel$$

$$GA \xrightarrow{G(U_A)} GA \qquad GA \xrightarrow{G(U_A)} GA.$$

for all objects A.

We write $\mathcal{D}bl$ for the 2-category of double categories, functors, and transformations, and **Dbl** for its underlying 1-category. Note that a 2-cell α in $\mathcal{D}bl$ is an isomorphism just when each α_A , and each α_M , is invertible.

The 2-category $\mathcal{D}bl$ gives us an easy way to define what we mean by a *symmetric monoidal double category*. In any 2-category with finite products there is a notion of a *pseudomonoid*, which generalizes the notion of monoidal category in Cat. Specializing this to $\mathcal{D}bl$, we obtain the following.

Definition 2.9. A **monoidal double category** is a double category equipped with functors $\otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ and $I : * \to \mathbb{D}$, and invertible transformations

$$\otimes \circ (\mathrm{Id} \times \otimes) \cong \otimes \circ (\otimes \times \mathrm{Id})$$

$$\otimes \circ (\mathrm{Id} \times I) \cong \mathrm{Id}$$

$$\otimes \circ (I \times \mathrm{Id}) \cong \mathrm{Id}$$

satisfying the usual axioms. If it additionally has a braiding isomorphism

$$\otimes \cong \otimes \circ \tau$$

(where $\tau \colon \mathbb{D} \times \mathbb{D} \cong \mathbb{D} \times \mathbb{D}$ is the twist) satisfying the usual axioms, then it is **braided** or **symmetric**, according to whether or not the braiding is self-inverse.

Unpacking this definition more explicitly, we see that a monoidal double category is a double category together with the following structure.

- (i) \mathbb{D}_0 and \mathbb{D}_1 are both monoidal categories.
- (ii) If I is the monoidal unit of \mathbb{D}_0 , then U_I is the monoidal unit of \mathbb{D}_1 .
- (iii) The functors S and T are strict monoidal, i.e. $S(M \otimes N) = SM \otimes SN$ and $T(M \otimes N) = TM \otimes TN$ and S and T also preserve the associativity and unit constraints.
- (iv) We have globular isomorphisms

$$\mathfrak{x}: (M_1 \otimes N_1) \odot (M_2 \otimes N_2) \stackrel{\cong}{\longrightarrow} (M_1 \odot M_2) \otimes (N_1 \odot N_2)$$

and

$$\mathfrak{u}\colon U_{A\otimes B}\stackrel{\cong}{\longrightarrow} (U_A\otimes U_B)$$

such that the following diagrams commute:

$$((M_{1} \otimes N_{1}) \odot (M_{2} \otimes N_{2})) \odot (M_{3} \otimes N_{3}) \longrightarrow ((M_{1} \odot M_{2}) \otimes (N_{1} \odot N_{2})) \odot (M_{3} \otimes N_{3})$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad$$

¹Actually, all the above definition requires is that U_I is coherently *isomorphic to* the monoidal unit of \mathbb{D}_1 , but we can always choose them to be equal without changing the rest of the structure.

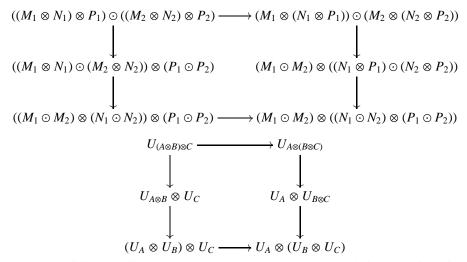
$$U_{A\otimes B}\odot (M\otimes N) \longrightarrow (U_{A}\otimes U_{B})\odot (M\otimes N)$$

$$\downarrow \qquad \qquad \downarrow$$

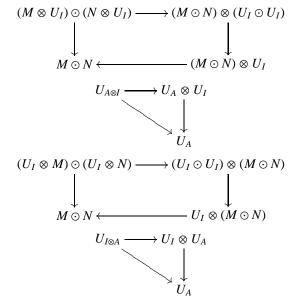
$$M\otimes N \longleftarrow (U_{A}\odot M)\otimes (U_{R}\odot N)$$

(these arise from the constraint data for the pseudo double functor \otimes).

(v) The following diagrams commute, expressing that the associativity isomorphism for \otimes is a transformation of double categories.



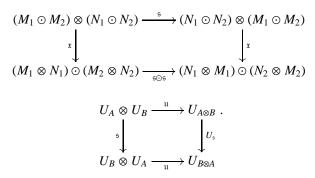
(vi) The following diagrams commute, expressing that the unit isomorphisms for \otimes are transformations of double categories.



Similarly, a braided monoidal double category is a monoidal double category with the following additional structure.

- (vii) \mathbb{D}_0 and \mathbb{D}_1 are braided monoidal categories.
- (viii) The functors S and T are strict braided monoidal (i.e. they preserve the braidings).

(ix) The following diagrams commute, expressing that the braiding is a transformation of double categories.



Finally, a symmetric monoidal double category is a braided one such that

(x) \mathbb{D}_0 and \mathbb{D}_1 are in fact symmetric monoidal.

While there are a fair number of coherence diagrams to verify, most of them are fairly small, and in any given case most or all of them are fairly obvious. Thus, verifying that a given double category is (braided or symmetric) monoidal is not a great deal of work.

Example 2.10. The examples Mod, nCob, and Prof are all easily seen to be symmetric monoidal under the tensor product of rings, disjoint union of manifolds, and cartesian product of categories, respectively.

Remark 2.11. In a 2-category with finite products there is additionally the notion of a cartesian object: one such that the diagonal $D \to D \times D$ and projection $D \to 1$ have right adjoints. Any cartesian object is a symmetric pseudomonoid in a canonical way, just as any category with finite products is a monoidal category with its cartesian product. Many of the "cartesian bicategories" considered in [CW87, CKWW08] are in fact the horizontal bicategory of some cartesian object in $\mathcal{D}bl$, and inherit their monoidal structure in this way.

Two further general methods for constructing symmetric monoidal double categories can be found in [Shu08].

Remark 2.12. The general yoga of internalization says that an X internal to Ys internal to Zs is equivalent to a Y internal to Xs internal to Zs, but this is only strictly true when the internalizations are all strict. We have defined a symmetric monoidal double category to be a (pseudo) symmetric monoid internal to (pseudo) categories internal to categories, but one could also consider a (pseudo) category internal to (pseudo) symmetric monoids internal to categories, i.e. a pseudo internal category in the 2-category SymMonCat of symmetric monoidal categories and strong symmetric monoidal functors. This would give almost the same definition, except that S and T would only be strong monoidal (preserving \otimes up to isomorphism) rather than strict monoidal. We prefer our definition, since S and T are strict monoidal in almost all examples, and keeping track of their constraints would be tedious.

Just as every bicategory is equivalent to a strict 2-category, it is proven in [GP99] that every pseudo double category is equivalent to a strict double category (one in which the associativity and unit constraints for \odot are identities). Thus, from now on we will usually omit to write these constraint isomorphisms (or equivalently, implicitly strictify our double categories). We *will* continue to write the constraint isomorphisms for the monoidal structure \otimes , since these are where the whole question lies.

3. Companions and conjoints

Suppose that $\mathbb D$ is a symmetric monoidal double category; when does $\mathcal H(\mathbb D)$ become a symmetric monoidal bicategory? It clearly has a unit object I, and the pseudo double functor $\otimes \colon \mathbb D \times \mathbb D \to \mathbb D$ clearly induces a functor $\otimes \colon \mathcal H(\mathbb D) \times \mathcal H(\mathbb D) \to \mathcal H(\mathbb D)$. However, the problem is that the constraint isomorphisms such as $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ are *vertical* 1-morphisms, which get discarded when we pass to $\mathcal H(\mathbb D)$. Thus, in order for $\mathcal H(\mathbb D)$ to inherit a symmetric monoidal structure, we must have a way to make vertical 1-morphisms into horizontal 1-cells. Thus is the purpose of the following definition.

Definition 3.1. Let \mathbb{D} be a double category and $f: A \to B$ a vertical 1-morphism. A **companion** of f is a horizontal 1-cell $\hat{f}: A \to B$ together with 2-morphisms

$$f \downarrow \xrightarrow{\hat{f}} \qquad \text{and} \qquad \left| \begin{array}{c} U_A \\ \downarrow \\ \downarrow \\ \hat{f} \end{array} \right| f$$

such that the following equations hold.

$$(3.2) \qquad \begin{array}{c} \stackrel{U_A}{ \downarrow \downarrow} \\ \stackrel{f}{ \downarrow \downarrow} \\ \stackrel{f}{ \downarrow \downarrow} \end{array} = f \begin{array}{c} \stackrel{U_A}{ \downarrow \downarrow} \\ \stackrel{\downarrow}{ \downarrow U_f} \\ \stackrel{f}{ \downarrow U_B} \end{array} = \begin{bmatrix} \stackrel{f}{ \downarrow \downarrow} \\ \stackrel{\downarrow}{ \downarrow U_f} \\ \stackrel{f}{ \downarrow U_B} \end{bmatrix} = \begin{bmatrix} \stackrel{f}{ \downarrow \downarrow} \\ \stackrel{\downarrow}{ \downarrow \downarrow} \\ \stackrel{f}{ \downarrow U_F} \\ \stackrel{f}{ \downarrow U_F} \\ \end{bmatrix}$$

A **conjoint** of f, denoted $\check{f}: B \to A$, is a companion of f in the double category $\mathbb{D}^{h \cdot \text{op}}$ obtained by reversing the horizontal 1-cells, but not the vertical 1-morphisms, of \mathbb{D} .

Remark 3.3. We momentarily suspend our convention of pretending that our double categories are strict to mention that the second equation in (3.2) actually requires an insertion of unit isomorphisms to make sense.

The form of this definition is due to [GP04, DPP], but the ideas date back to [BS76]; see also [BM99, Fio07]. In the terminology of these references, a *connection* on a double category is equivalent to a strictly functorial choice of a companion for each vertical arrow.

Definition 3.4. We say that a double category is **fibrant** if every vertical 1-morphism has both a companion and a conjoint.

Remark 3.5. In [Shu08] fibrant double categories were called *framed bicategories*. However, the present terminology seems to generalize better to $(n \times k)$ -categories, as well as avoiding a conflict with the *framed bordisms* in topological field theory.

Examples 3.6. \mathbb{M} od, $n\mathbb{C}$ ob, and \mathbb{P} rof are all fibrant. In \mathbb{M} od, the companion of a ring homomorphism $f: A \to B$ is B regarded as an A-B-bimodule via f on the left, and dually for its conjoint. In $n\mathbb{C}$ ob, companions and conjoints are obtained by regarding a diffeomorphism as a cobordism. And in \mathbb{P} rof, companions and conjoints are obtained by regarding a functor $f: A \to B$ as a 'representable' profunctor B(f-,-) or B(-,f-).

Remark 3.7. For an $(n \times 1)$ -category (recall Remark 2.3), the lifting condition we should require is simply that each double category $\mathbb{D}_{i+1} \rightrightarrows \mathbb{D}_i$, for $0 \le i < n$, is fibrant.

The existence of companions and conjoints gives us a way to 'lift' vertical 1-morphisms to horizontal 1-cells. What is even more crucial for our applications, however, is that

these liftings are unique up to isomorphism, and that these isomorphisms are canonical and coherent. This is the content of the following lemmas. We state most of them only for companions, but all have dual versions for conjoints.

Lemma 3.8. Let $\hat{f}: A \to B$ and $\hat{f}': A \to B$ be companions of f (that is, each comes equipped with 2-morphisms as in Definition 3.1). Then there is a unique globular isomorphism $\theta_{\hat{f},\hat{f}'}: \hat{f} \xrightarrow{\cong} \hat{f}'$ such that

$$(3.9) \qquad \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \downarrow \\ \end{array} \end{array} \end{array} f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \downarrow \\ \end{array} \end{array} f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_F \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \begin{array}{c} U_A \\ \downarrow \\ \end{array} \downarrow U_B \end{array} \downarrow f \\ \begin{array}{c} \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \end{array} \downarrow U_B \\ \begin{array}{c} \end{array} \downarrow U_B \\ \downarrow U_B \\ \end{array} \downarrow U_B \\ \end{split} \downarrow U_B \\ \downarrow U_B \\ \end{split} \end{split}$$

Proof. Composing (3.9) on the left with $\parallel \underbrace{\downarrow}_{f'}^{U_A} f$ and on the right with $f \downarrow \underbrace{\downarrow}_{U_B}^{\hat{f}} \parallel$, and

using the second equation (3.2), we see that if (3.9) is satisfied then $\theta_{\hat{f},\hat{f}'}$ must be the composite

$$\left\| \begin{array}{c} U_A \\ \downarrow \\ \downarrow \\ f \end{array} \right\| \begin{array}{c} \hat{f} \\ \downarrow \\ \hat{f}' \end{array} \right\|$$

Two applications of the first equation (3.2) shows that this indeed satisfies (3.9). As for its being an isomorphism, we have the dual composite $\theta_{\hat{f}',\hat{f}'}$:

$$\begin{vmatrix}
U_A & \hat{f}' \\
\downarrow f & \downarrow \\
\hat{f} & U_B
\end{vmatrix}$$

which we verify is an inverse using (3.2):

$$\begin{vmatrix} U_A \\ = \\ \downarrow \\ \downarrow \\ -U_A \\ -\hat{f}' \\ \end{vmatrix} - U_B \\ \end{vmatrix} \begin{vmatrix} \hat{f} \\ \downarrow \\ U_B \end{vmatrix} = \begin{vmatrix} U_A \\ \downarrow \\ \hat{f} \end{vmatrix} \begin{vmatrix} \hat{f} \\ \downarrow \\ \hat{f} \end{vmatrix} = \begin{vmatrix} \hat{f} \\ \downarrow \\ \hat{f} \end{vmatrix} \begin{vmatrix} \hat{f} \\ \downarrow \\ \hat{f} \end{vmatrix} = \begin{vmatrix} \hat{f} \\ \downarrow \\ \hat{f} \end{vmatrix}$$

(and dually).

Lemma 3.10. For any companion \hat{f} of f we have $\theta_{\hat{f},\hat{f}} = 1_{\hat{f}}$.

Proof. This is the second equation (3.2).

Lemma 3.11. Suppose that f has three companions \hat{f} , \hat{f}' , and \hat{f}'' . Then $\theta_{\hat{f},\hat{f}''} = \theta_{\hat{f}',\hat{f}''} \circ \theta_{\hat{f},\hat{f}'}$.

Proof. By definition, we have

$$\theta_{\hat{f}',\hat{f}''} \circ \theta_{\hat{f},\hat{f}'} = \begin{bmatrix} U_A & U_A & \hat{f} & \vdots & \vdots & \vdots \\ U_A & \vdots & \vdots & \vdots & \vdots & \vdots \\ U_A & \vdots & \vdots & \vdots & \vdots & \vdots \\ U_A & \vdots & \vdots & \vdots & \vdots & \vdots \\ U_A & \vdots & \vdots & \vdots & \vdots & \vdots \\ U_A & \vdots & \vdots & \vdots & \vdots & \vdots \\ U_A & \vdots & \vdots & \vdots & \vdots \\ U_A & \vdots & \vdots & \vdots & \vdots \\ U_A & \vdots & \vdots & \vdots \\ U$$

as desired.

Lemma 3.12. $U_A: A \rightarrow A$ is always a companion of $1_A: A \rightarrow A$ in a canonical way.

Proof. We take both defining 2-morphisms to be 1_{U_A} ; the truth of (3.2) is evident.

Lemma 3.13. Suppose that $f: A \to B$ has a companion \hat{f} and $g: B \to C$ has a companion \hat{g} . Then $\hat{g} \odot \hat{f}$ is a companion of gf.

Proof. We take the defining 2-morphisms to be the composites

It is easy to verify that these satisfy (3.2), using the interchange law for \odot and \circ in a double category.

Lemma 3.14. Suppose that $f: A \to B$ has companions \hat{f} and \hat{f}' , and that $g: B \to C$ has companions \hat{g} and \hat{g}' . Then $\theta_{\hat{g},\hat{g}'} \odot \theta_{\hat{f},\hat{f}'} = \theta_{\hat{g} \odot \hat{f},\hat{g}' \odot \hat{f}'}$.

Proof. Using the interchange law for ⊙ and ∘, we have:

as desired.

Lemma 3.15. If $f: A \to B$ has a companion \hat{f} , then $\theta_{\hat{f},\hat{f} \odot U_A}$ and $\theta_{\hat{f},U_B \odot \hat{f}}$ are equal to the unit constraints $\hat{f} \cong \hat{f} \odot U_A$ and $\hat{f} \cong U_B \odot \hat{f}$.

Proof. By definition, we have

$$\theta_{\hat{f},\hat{f}\odot U_A} = \begin{bmatrix} \begin{matrix} U_A \\ \downarrow 1_{U_A} \end{matrix} & \begin{matrix} U_A \\ \downarrow 1_{U_A} \end{matrix} & \begin{matrix} \vdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \end{matrix} & \begin{matrix} \vdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \end{matrix} & \begin{matrix}$$

which, bearing in mind our suppression of unit and associativity constraints, means that in actuality it is the unit constraint $\hat{f} \cong \hat{f} \odot U_A$. The other case is dual.

Lemma 3.16. Let $F: \mathbb{D} \to \mathbb{E}$ be a functor between double categories and let $f: A \to B$ have a companion \hat{f} in \mathbb{D} . Then $F(\hat{f})$ is a companion of F(f) in \mathbb{E} .

Proof. We take the defining 2-morphisms to be

The axioms (3.2) follow directly from those for \hat{f} .

Lemma 3.17. Suppose that \mathbb{D} is a monoidal double category and that $f: A \to B$ and $g: C \to D$ have companions \hat{f} and \hat{g} respectively. Then $\hat{f} \otimes \hat{g}$ is a companion of $f \otimes g$.

Proof. This follows from Lemma 3.16, since $\otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ is a functor, and a companion in $\mathbb{D} \times \mathbb{D}$ is simply a pair of companions in \mathbb{D} .

Lemma 3.18. Suppose that $f: \mathbb{D} \to \mathbb{E}$ is a functor and that $f: A \to B$ has companions \hat{f} and \hat{f}' in \mathbb{D} . Then $\theta_{F(\hat{f}),F(\hat{f}')} = F(\theta_{\hat{f},\hat{f}'})$.

Proof. Using the axioms of a pseudo double functor and the definition of the 2-morphisms in Lemma 3.16, we have

$$F(\theta_{\hat{f},\hat{f}'}) = \underbrace{ \begin{array}{c} F(\hat{f}) \\ F(U_A) \\ F(\hat{f}') \end{array} }_{F(\hat{f}')} = \underbrace{ \begin{array}{c} F(\hat{f}) \\ F(U_A) \\ F(\hat{f}') \end{array} }_{E(\hat{f}')} = \underbrace{ \begin{array}{c} U_{F(A)} \\ F(\hat{f}) \\ F(\hat{f}') \end{array} }_{E(U_A) \xrightarrow{F(\hat{f}')} F(\hat{f})} = \underbrace{ \begin{array}{c} U_{F(A)} \\ F(U_A) \\ F(\hat{f}') \end{array} }_{F(\hat{f}')} = \underbrace{ \begin{array}{c} U_{F(A)} \\ F(U_A) \\ F(\hat{f}') \end{array} }_{F(\hat{f}')} = \underbrace{ \begin{array}{c} U_{F(A)} \\ F(U_A) \\$$

as desired.

Lemma 3.19. Suppose that \mathbb{D} is a monoidal double category, that $f: A \to B$ has companions \hat{f} and \hat{f}' , and that $g: C \to D$ has companions \hat{g} and \hat{g}' . Then $\theta_{\hat{f},\hat{f}'} \otimes \theta_{\hat{g},\hat{g}'} = \theta_{\hat{f} \otimes \hat{g},\hat{f}' \otimes \hat{g}'}$.

Proof. This follows from Lemma 3.18 in the same way that Lemma 3.17 follows from Lemma 3.16. \Box

Lemma 3.20. If $f: A \to B$ is a vertical isomorphism with a companion \hat{f} , then \hat{f} is a conjoint of its inverse f^{-1} .

Proof. The composites

$$\begin{array}{c|c}
f & \downarrow & \downarrow \\
-U_B & \downarrow & \downarrow \\
f^{-1} \downarrow \downarrow U_{f^{-1}} \downarrow f^{-1}
\end{array}$$
 and
$$\begin{array}{c|c}
f^{-1} \downarrow \downarrow U_{f^{-1}} \downarrow f^{-1} \\
\downarrow U_A & \downarrow f
\end{array}$$

exhibit \hat{f} as a conjoint of f^{-1} .

Lemma 3.21. If $f: A \to B$ has both a companion \hat{f} and a conjoint \check{f} , then we have an adjunction $\hat{f} + \check{f}$ in \mathcal{HD} . If f is an isomorphism, then this is an adjoint equivalence.

Proof. The unit and counit of the adjunction $\hat{f} \dashv \check{f}$ are the composites

$$\left\| \begin{array}{c} U_A \\ \downarrow \\ \hat{f} \end{array} \right\| \quad \text{and} \quad \left\| \begin{array}{c} f \\ \downarrow \\ U_B \end{array} \right\| \quad \left\| \begin{array}{c} f \\ \downarrow \\ U_B \end{array} \right\|$$

The triangle identities follow from (3.2). If f is an isomorphism, then by the dual of Lemma 3.20, \check{f} is a companion of f^{-1} . But then by Lemma 3.13 $\check{f} \odot \hat{f}$ is a companion of $1_A = f^{-1} \circ f$ and $\hat{f} \odot \check{f}$ is a companion of $1_B = f \circ f^{-1}$, and hence \hat{f} and \check{f} are equivalences. We can then check that in this case the above unit and counit actually are the isomorphisms θ , or appeal to the general fact that any adjunction involving an equivalence is an adjoint equivalence.

Remark 3.22. Our intended applications actually only require our double categories to have companions and conjoints for vertical *isomorphisms*; we may call a double category with this property **isofibrant**. Note that by Lemma 3.20, having companions for all isomorphisms implies having conjoints for all isomorphisms. However, most examples we are interested in have all companions and conjoints, and these are useful for other purposes as well; see [Shu08]. Moreover, if we are given a double category in which only vertical isomorphisms have companions, we can still apply our theorems to it as written, simply by first discarding all noninvertible vertical 1-morphisms.

4. From double categories to bicategories

We are now equipped to lift structures on fibrant double categories to their horizontal bicategories. In this section we show that passage from fibrant double categories to bicategories is functorial; in the next section we show that it preserves monoidal structure.

As a point of notation, we write \odot for the composition of 1-cells in a bicategory, since our bicategories are generally of the form $\mathcal{H}(\mathbb{D})$. As advocated by Max Kelly, we say **functor** to mean a morphism between bicategories that preserves composition up to isomorphism; equivalent terms include *weak 2-functor*, *pseudofunctor*, and *homomorphism*.

Theorem 4.1. If \mathbb{D} is a double category, then $\mathcal{H}(\mathbb{D})$ is a bicategory, and any functor $F: \mathbb{D} \to \mathbb{E}$ induces a functor $\mathcal{H}(F): \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{E})$. In this way \mathcal{H} defines a functor of 1-categories **Dbl** \to **Bicat**.

Proof. The constraints of F are all globular, hence give constraints for $\mathcal{H}(F)$. Functoriality is evident.

The action of \mathcal{H} on transformations, however, is less obvious, and requires the presence of companions or conjoints. Recall that if $F, G: \mathcal{A} \to \mathcal{B}$ are functors between bicategories, then an **oplax transformation** $\alpha: F \to G$ consists of 1-cells $\alpha_A: FA \to GA$ and 2-cells

$$\alpha_A \downarrow \xrightarrow{Ff} \alpha_B \downarrow \alpha_B$$

such that for any 2-cell $A \underbrace{ \downarrow \downarrow x}_{g} B$ in \mathcal{A} ,

$$\alpha_{A} = \alpha_{A} \xrightarrow{Ff} \alpha_{B} = \alpha_{A} \xrightarrow{Ff} \alpha_{G} \xrightarrow{Gg} \alpha_{G}$$

$$\alpha_{A} \xrightarrow{Fg} \alpha_{B} = \alpha_{A} \xrightarrow{Gf} \alpha_{G} \xrightarrow{Gg} \alpha_{G}$$

and moreover for any A and any f, g in \mathcal{A} ,

It is a **lax transformation** if the 2-cells α_f go the other direction, and a **pseudo transformation** if they are isomorphisms.

By doctrinal adjunction [Kel74], given collections of 1-cells α_A : $FA \to GA$ and β_A : $GA \to FA$ and adjunctions $\alpha_A \dashv \beta_A$ in \mathcal{B} , there is a bijection between (i) collections of 2-cells α_f making α an oplax transformation and (ii) collections of 2-cells β_f making β a lax transformation. Two such transformations correspond under this bijection if and only if (4.4)

$$F(f) \xrightarrow{\eta \odot F(f)} \beta_B \odot \alpha_B \odot F(f) \qquad \alpha_B \odot F(f) \odot \beta_A \xrightarrow{\alpha_B \odot \beta_f} \alpha_B \odot \beta_B \odot G(f)$$

$$F(f) \odot \beta_A \odot \alpha_A \xrightarrow{\beta_f \odot \alpha_A} \beta_B \odot G(f) \odot \alpha_A \qquad G(f) \odot \alpha_A \odot \beta_A \xrightarrow{G(f) \odot \varepsilon} G(f)$$

commute. If we have a pointwise adjunction between an oplax and a lax transformation, whose 2-cell structures correspond under this bijection, we call it a **conjunctional transformation** $(\alpha \prec \beta)$: $F \rightarrow G$. (These are the conjoint pairs in a double category whose horizontal arrows are lax transformations and whose vertical arrows are oplax transformations.)

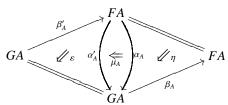
Of particular importance is the case when both α and β are pseudo natural and each adjunction $\alpha_A \dashv \beta_A$ is an adjoint equivalence. In this case we call $\alpha \prec \beta$ a **pseudo natural adjoint equivalence**. A pseudo natural adjoint equivalence can equivalently be defined as an internal equivalence in the bicategory $\mathcal{B}icat(\mathcal{A}, \mathcal{B})$ of functors, pseudo natural transformations, and modifications $\mathcal{A} \to \mathcal{B}$.

Recall also that if $\alpha, \alpha' \colon F \to G$ are oplax transformations, a **modification** $\mu \colon \alpha \to \alpha'$ consists of 2-cells $\mu_A \colon \alpha_A \to \alpha_A'$ such that

$$\alpha'_{A} \left(\underbrace{\rightleftharpoons}_{\mu_{A}} \right) \alpha_{A} \underset{\alpha_{f}}{\swarrow} \alpha_{B} = \alpha'_{A} \underbrace{\rightleftharpoons}_{\alpha_{f}} \alpha'_{B} \left(\underbrace{\rightleftharpoons}_{\mu_{B}} \right) \alpha_{B}$$

$$(4.5)$$

There is an evident notion of modification between lax transformations as well. Finally, given conjunctional transformations $\alpha \prec \beta$ and $\alpha' \prec \beta'$, there is a bijection between modifications $\alpha \to \alpha'$ and $\beta' \to \beta$, where $\mu \colon \alpha \to \alpha'$ corresponds to $\bar{\mu} \colon \beta' \to \beta$ with components $\bar{\mu}_A$ defined by:



The modifications $\bar{\mu}$ and μ are called **mates**, and are compatible with composition (see [KS74]). Thus, given \mathcal{A}, \mathcal{B} we can define a bicategory $Conj(\mathcal{A}, \mathcal{B})$, whose objects are functors $\mathcal{A} \to \mathcal{B}$, whose 1-cells are conjunctional transformations considered as pointing in the direction of their left adjoints, and whose 2-cells are mate-pairs of modifications.

Theorem 4.6. If \mathbb{D} is a double category and \mathbb{E} is a fibrant double category with chosen companions and conjoints, we have a functor

$$\mathcal{D}bl(\mathbb{D}, \mathbb{E}) \longrightarrow Conj(\mathcal{H}(\mathbb{D}), \mathcal{H}(\mathbb{E}))$$
$$F \mapsto \mathcal{H}(F)$$
$$\alpha \mapsto (\hat{\alpha} \prec \check{\alpha}).$$

Moreover, if α *is an isomorphism, then* $\hat{\alpha} \prec \check{\alpha}$ *is a pseudo natural adjoint equivalence.*

Note that we are here regarding the 1-category $\mathcal{D}bl(\mathbb{D},\mathbb{E})$ as a bicategory with only identity 2-cells.

Proof. We denote the chosen companion and conjoint of f in \mathbb{E} by \hat{f} and \check{f} , as usual. We define $\hat{\alpha}$ as follows: its 1-cell components are $\hat{\alpha}_A = \widehat{\alpha_A}$, and its 2-cell component $\hat{\alpha}_f$ is the composite

Equations (4.2) and (4.3) follow directly from Definition 2.8. The construction of $\check{\alpha}$ is dual, using conjoints, and Lemma 3.21 shows that $\hat{\alpha}_A + \check{\alpha}_A$. For the first equation in (4.4), we

have

$$\begin{vmatrix} U_{FA} & Ff & U_{FB} \\ = & = & \downarrow & \alpha_B & \downarrow \\ -U_{FA} & -Ff & -\hat{\alpha}_B & -\hat{\alpha}_B & -\hat{\alpha}_B & -\hat{\alpha}_B & -\hat{\alpha}_B & -\hat{\alpha}_A & -Ff & -\hat{\alpha}_B & -\hat{\alpha}_A & -Ff & -\hat{\alpha}_B & -\hat{\alpha}_A & -\hat{\alpha}_A & -Ff & -\hat{\alpha}_B & -\hat{\alpha}_A & -\hat$$

and the second is dual. Thus $(\hat{\alpha} \prec \check{\alpha})$ is a conjunctional transformation.

Now suppose given $\alpha \colon F \to G$ and $\beta \colon G \to H$. Then by Lemma 3.13, $\hat{\beta}_A \odot \hat{\alpha}_A$ is a companion of $\beta_A \circ \alpha_A$, so we have a canonical isomorphism

$$\theta_{\widehat{\beta\alpha}_A,\hat{\beta}_A\odot\hat{\alpha}_A}:\widehat{\beta\alpha}_A\stackrel{\cong}{\longrightarrow}\hat{\beta}_A\odot\hat{\alpha}_A.$$

Of course, we also have $\theta_{\widehat{1_A},U_A} \colon \widehat{1_A} \stackrel{\cong}{\longrightarrow} U_A$ by Lemma 3.12. These constraints are automatically natural, since $\mathcal{D}bl(\mathbb{D},\mathbb{E})$ has no nonidentity 2-cells. The axiom for the composition constraint says that two constructed isomorphisms

$$\widehat{\gamma\beta\alpha_A} \stackrel{\cong}{\longrightarrow} (\hat{\gamma}_A \odot \hat{\beta}_A) \odot \hat{\alpha}_A$$

are equal. However, both $\widehat{\gamma\beta\alpha_A}$ and $(\hat{\gamma}_A \odot \hat{\beta}_A) \odot \hat{\alpha}_A$ are companions of $\gamma_A\beta_A\alpha_A$, and both of these isomorphisms are constructed from composites (both \circ -composites and \odot -composites) of θ s; hence by Lemmas 3.11 and 3.14 they are both equal to

$$\theta_{\widehat{\gamma}\widehat{\beta}\alpha_A,\,(\hat{\gamma}_A\odot\hat{\beta}_A)\odot\hat{\alpha}_A}$$

and thus equal to each other. The same argument applies to the axioms for the unit constraint; thus we have a functor of bicategories.

Finally, if α is an isomorphism, then in particular each α_A is an isomorphism, so by Lemma 3.21 each $\hat{\alpha}_A \dashv \check{\alpha}_A$ is an adjoint equivalence. But α being an isomorphism also implies that each 2-cell

$$\begin{array}{c} \xrightarrow{Ff} \\ \downarrow \alpha_A \downarrow & \downarrow \alpha_B \\ \xrightarrow{Gf} \end{array}$$

is an isomorphism. From its inverse we form the composite

$$\begin{bmatrix} \hat{\alpha}_A & Gf & U_{GB} \\ \downarrow & \alpha_A^{-1} & \downarrow \alpha_f^{-1} & \alpha_B^{-1} & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ U_{FA} & Ff & \check{\alpha}_A \end{bmatrix}$$

which we can then verify to be an inverse of (4.7). Thus $\hat{\alpha}$, and dually $\check{\alpha}$, is pseudo natural, and hence $\hat{\alpha} \prec \check{\alpha}$ is a pseudo natural adjoint equivalence.

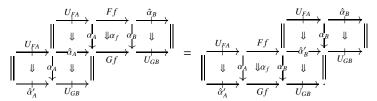
We can also promote Lemma 3.8 to a functorial uniqueness.

Lemma 4.8. Let \mathbb{D} be a double category and \mathbb{E} a fibrant double category with two different sets of choices \hat{f} , \check{f} and \hat{f}' , \check{f}' of companions and conjoints for each vertical 1-morphism f, giving rise to two different functors

$$\mathcal{H}, \mathcal{H}' \colon \mathcal{D}bl(\mathbb{D}, \mathbb{E}) \longrightarrow Conj(\mathcal{H}(\mathbb{D}), \mathcal{H}(\mathbb{E})).$$

Then the isomorphisms θ from Lemma 3.8 fit together into a pseudo natural adjoint equivalence $\mathcal{H} \simeq \mathcal{H}'$ which is the identity on objects.

Proof. We must first show that for a given transformation $\alpha \colon F \to G \colon \mathbb{D} \to \mathbb{E}$ in $\mathcal{D}bl$, the isomorphisms θ form an invertible modification $\hat{\alpha} \cong \hat{\alpha}'$. Substituting (4.7) and the definition of θ into (4.5), this becomes the assertion that



This follows from two applications of (3.2), one for $\hat{\alpha}_A$ and one for $\hat{\alpha}_B'$. (The mate of θ is, of course, uniquely determined.) Now, to show that these form a pseudo natural adjoint equivalence, it remains only to check that they do, in fact, form a pseudo natural transformation which is the identity on objects, i.e. that (4.2) and (4.3) are satisfied. But (4.2) is vacuous since $\mathcal{D}bl(\mathbb{D}, \mathbb{E})$ has no nonidentity 2-cells, and (4.3) follows from Lemmas 3.11 and 3.14 since all the constraints involved are also instances of θ .

It seems that we should have a functor from fibrant double categories to a tricategory of bicategories, functors, conjunctional transformations, and modifications, but there is no tricategory containing conjunctional transformations since the interchange law only holds laxly. However, we can say the following. Let $\mathcal{D}bl_g^f$ denote the sub-2-category of $\mathcal{D}bl$ containing the fibrant double categories, all functors between them, and only the transformations that are isomorphisms, and let $\mathcal{B}icat$ denote the tricategory of bicategories, functors, pseudo natural transformations, and modifications.

Theorem 4.9. There is a functor of tricategories $\mathcal{H}: \mathcal{D}bl_g^f \to \mathcal{B}icat$.

Proof. The definition of functors between tricategories can be found in [GPS95] or [Gur06]. In addition to Theorem 4.6, we require pseudo natural (adjoint) equivalences χ and ι relating composition and units in $\mathcal{D}bl_g^f$ and $\mathcal{B}icat$, and modifications relating composites of these, which satisfy various axioms. However, since composition of 1-cells in $\mathcal{D}bl_g^f$ and $\mathcal{B}icat$ is strictly associative and unital, \mathcal{H} strictly preserves this composition, and $\mathcal{D}bl_g^f$ has no nonidentity 3-cells, this merely amounts to the following.

Firstly, for every pair of transformations

$$\mathbb{C} \underbrace{ \frac{F}{\emptyset \alpha}}_{G} \mathbb{D} \underbrace{ \frac{H}{\emptyset \beta}}_{K} \mathbb{E}$$

between fibrant double categories, we require an invertible modification $\chi: \hat{\beta} * \hat{\alpha} \cong \widehat{\beta * \alpha}$ such that

$$1 \longrightarrow \hat{1} * \hat{1} \qquad \widehat{\gamma \alpha} * \widehat{\delta \beta} \longrightarrow (\hat{\gamma} * \hat{\delta})(\hat{\alpha} * \hat{\beta})$$

$$\downarrow \chi \qquad \text{and} \qquad \chi \downarrow \qquad \qquad \downarrow \chi \chi$$

$$\widehat{1 * 1} \qquad \widehat{\gamma \alpha} * \widehat{\delta \beta} \longrightarrow (\widehat{\gamma} * \widehat{\delta})(\widehat{\alpha} * \widehat{\beta})$$

commute. (Here we are writing * for the 'Godement product' of 2-cells in $\mathcal{D}bl$ and $\mathcal{B}icat$.) These are the 2-cell components of the composition constraint, its 1-cell components being identities. Now by Lemmas 3.13 and 3.16, $(\hat{\beta} * \hat{\alpha})_A = \hat{\beta}_{GA} \circ H(\hat{\alpha}_A)$ is a companion of $(\beta * \alpha)_A = \beta_{GA} \circ H(\alpha_A)$. Therefore, we take the component χ_A to be

$$\theta_{\widehat{\beta}_{GA} \circ H(\widehat{\alpha}_A), \widehat{\beta} * \widehat{\alpha}_A}$$
.

Equation (4.5), saying that these form a modification, becomes the equality of two large composites of 2-cells in \mathbb{D} , which as usual follows from (3.2).

Secondly, for every $F: \mathbb{D} \to \mathbb{E}$ we require an isomorphism $\iota: \widehat{1}_F \cong 1_{\mathcal{H}(F)}$ satisfying a couple of axioms which simply require it to be equal to the unit constraint of the local functor \mathcal{H} from Theorem 4.6; these are the 2-cell components of the unit constraint. Finally, the required modifications merely amount to the *assertions* that

$$\hat{\gamma} * \hat{\beta} * \hat{\alpha} \xrightarrow{\chi} \widehat{\gamma * \beta} * \hat{\alpha}
\chi \downarrow \qquad \qquad \downarrow \chi
\hat{\gamma} * \widehat{\beta} * \widehat{\alpha} \xrightarrow{\chi} \widehat{\gamma * \beta} * \alpha$$

$$\hat{\alpha} \xrightarrow{\iota} \widehat{1_F} * \hat{\alpha}
\downarrow \chi , \text{ and } \hat{\alpha} \xrightarrow{\iota} \widehat{\alpha} * \widehat{1_F}
\hat{\alpha} \xrightarrow{\chi} \widehat{\alpha} * \widehat{\alpha}$$

commute; again this follows from Lemma 3.11.

We end this section with one final lemma.

Lemma 4.10. Suppose $F,G: \mathbb{D} \to \mathbb{E}$ are functors, $\alpha: F \to G$ is a transformation, and that $f: A \to B$ has a companion \hat{f} in \mathbb{D} . Then the oplax comparison 2-cell for $\hat{\alpha}$:

$$\hat{\alpha}_{A} \downarrow \xrightarrow{F(\hat{f})} \hat{\alpha}_{B}$$

$$\xrightarrow{G(\hat{f})} \hat{\alpha}_{B}$$

is equal to $\theta_{\hat{\alpha}_B \odot F(\hat{f}), G(\hat{f}) \odot \hat{\alpha}_A}$ (and in particular is an isomorphism).

Proof. By definition $\hat{\alpha}_A$ and $\hat{\alpha}_B$ are companions of α_A and α_B , respectively, and by Lemma 3.16 $F(\hat{f})$ and $G(\hat{f})$ are companions of F(f) and G(f), respectively. Thus, by Lemma 3.13 the domain and codomain of $\hat{\alpha}_{\hat{f}}$ are both companions of $G(f) \circ \alpha_A = \alpha_B \circ F(f)$, so at least the asserted θ isomorphism exists. Now, by taking the definition (4.7) of $\hat{\alpha}_{\hat{f}}$ and substituting it for θ in (3.9), using the axioms for companions and the naturality of α on 2-morphisms, we see that $\hat{\alpha}_{\hat{f}}$ satisfies (3.9) and hence must be equal to θ .

5. Symmetric monoidal bicategories

We are now ready to lift monoidal structures from double categories to bicategories. If we had a theory of symmetric monoidal tricategories, we could do this by improving Theorem 4.9 to say that \mathcal{H} is a symmetric monoidal functor, and then conclude that it preserves pseudomonoids. However, in the absence of such a theory, we give a direct proof.

Theorem 5.1. *If* \mathbb{D} *is a fibrant monoidal double category, then* $\mathcal{H}(\mathbb{D})$ *is a monoidal bicategory. If* \mathbb{D} *is braided, so is* $\mathcal{H}(\mathbb{D})$ *, and if* \mathbb{D} *is symmetric, so is* $\mathcal{H}(\mathbb{D})$.

Remark 5.2. For monoidal bicategories, there is a notion in between braided and symmetric, called *sylleptic*, in which the braiding is self-inverse up to an isomorphism (the *syllepsis*) but this isomorphism is not maximally coherent. Since in our approach the syllepsis will be an isomorphism of the form $\theta_{\hat{f},\hat{f}'}$, it is *always* maximally coherent; thus our method cannot produce sylleptic monoidal bicategories that are not symmetric.

Proof of Theorem 5.1. A monoidal bicategory is defined to be a tricategory with one object. We use the definition of tricategory from [Gur06], which is the same as that of [GPS95] except that the associativity and unit constraints are pseudo natural adjoint equivalences, rather than merely pseudo transformations whose components are equivalences.

The functor \mathcal{H} evidently preserves products, so $\otimes \colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ induces a functor $\otimes \colon \mathcal{H}(\mathbb{D}) \times \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$, and of course I is still the unit. The associativity constraint of \mathbb{D} is a natural isomorphism

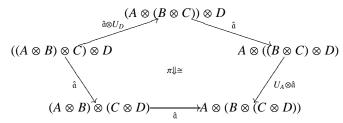
$$\mathbb{D} \times \mathbb{D} \times \mathbb{D} \xrightarrow{\otimes (\mathrm{Id} \times \otimes)} \mathbb{D}$$

so by Theorem 4.6 it gives rise to a pseudo natural adjoint equivalence

$$\mathcal{H}(\mathbb{D})\times\mathcal{H}(\mathbb{D})\times\mathcal{H}(\mathbb{D}) \xrightarrow{\otimes (\mathrm{Id}\times\otimes)} \mathcal{H}(\mathbb{D})$$

Likewise, the unit constraints of \mathbb{D} induce pseudo natural adjoint equivalences.

The final four pieces of data for a monoidal bicategory are invertible modifications relating various composites of the associativity and unit transformations. The first is a "pentagonator" which relates the two ways to go around the Mac Lane pentagon:



Now by Lemmas 3.13 and 3.17, both sides of this pentagon in $\mathcal{H}(\mathbb{D})$ are companions of the corresponding sides of the pentagon in \mathbb{D}_0 . Since the pentagon in \mathbb{D}_0 commutes, we have an isomorphism θ between the two sides of the pentagon in $\mathcal{H}(\mathbb{D})$, which we take to be π . That π is in fact a modification follows from Lemma 4.8. We construct the other invertible modifications μ , λ , ρ in the same way.

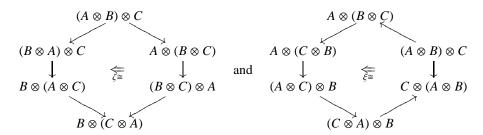
Finally, we must show that three equations between pasting composites of 2-cells hold, relating composites of π , μ , λ , ρ . However, in each of these equations, both the domain and the codomain of the 2-cells involved are companions of the same isomorphism in \mathbb{D}_0 . For the 5-associahedron, this isomorphism is the unique constraint

$$(((A \otimes B) \otimes C) \otimes D) \otimes E \xrightarrow{\cong} A \otimes (B \otimes (C \otimes (D \otimes E)));$$

for the other two it is simply the associator $(A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$. By Lemmas 3.15, 3.19, and 4.10, every 2-cell in these diagrams is a θ isomorphism relating two companions of the same vertical isomorphism. Therefore, Lemmas 3.11 and 3.14 imply that each pasting diagram is also a θ isomorphism between its domain and codomain. The uniqueness of θ then implies that the three equations hold.

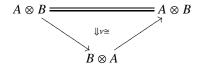
Now suppose that $\mathbb D$ is braided; to show that $\mathcal H(\mathbb D)$ is braided we seemingly must first have a definition of braided monoidal bicategory. The interested reader may follow the tortuous path of the definition of braided monoidal 2-categories and bicategories through the literature, starting from [KV94b, KV94a] and continuing, with occasional corrections, through [BN96, DS97, Cra98, McC00], and [Gur]. However, the details of the definition are essentially unimportant for us; since our constraints and coherence are produced in a universal way, any reasonable data can be produced and any reasonable axioms will be satisfied. For concreteness, we use the definition of [McC00].

The first piece of data we require to make $\mathcal{H}(\mathbb{D})$ braided is a pseudo natural adjoint equivalence $\otimes \stackrel{\simeq}{\longrightarrow} \otimes \circ \tau$, where τ is the switch isomorphism. This arises by Theorem 4.6 from the braiding of \mathbb{D} . We also require two invertible modifications filling the usual hexagons for a braiding:



As before, since the corresponding hexagons commute in \mathbb{D}_0 , and by Lemmas 3.13 and 3.17 each side of each hexagon in $\mathcal{H}(\mathbb{D})$ is a companion to the corresponding side in \mathbb{D}_0 , we have θ isomorphisms that we can take as ζ and ξ . Finally, we must verify that the four 2-cell diagrams in [McC00, p136–139] involving ζ and ξ commute. As with the axioms for a monoidal bicategory, both sides of these equalities are made up of θ s relating companions of a single morphism in \mathbb{D}_0 , and thus by uniqueness they must be equal.

Now suppose that \mathbb{D} is symmetric. To make $\mathcal{H}(\mathbb{D})$ symmetric, we require first a *syllepsis*, i.e. an invertible modification



Since the braiding in \mathbb{D}_0 is self-inverse, the top and bottom of this triangle are both companions of $1_{A\otimes B}$; thus we have a θ isomorphism between them which we take as ν . For $\mathcal{H}(\mathbb{D})$ to be sylleptic, the syllepsis must satisfy the two axioms on [McC00, p144–145]. As before, these diagrams of 2-cells are made up entirely of θ s relating companions of a single morphism in \mathbb{D}_0 , so they commute by uniqueness of θ .

Finally, for $\mathcal{H}(\mathbb{D})$ to be symmetric, the syllepsis must satisfy one additional axiom, given on [McC00, p91]. This follows automatically for the same reasons as before. \square

Combining the arguments of Theorems 4.9 and 5.1, we could show that passage from fibrant monoidal double categories to monoidal bicategories is a functor of tricategories, given a suitable definition of a tricategory of monoidal bicategories.

Remark 5.3. Essentially the same proof as that of Theorem 5.1 shows that any fibrant 2x1-category has an underlying tricategory. Note that unlike the construction of bicategories from 1x1-categories (i.e. double categories), this case requires fibrancy even in the absence of monoidal structure, since the associativity and unit constraints of a 2x1-category are not 1-cells but rather morphisms of 0-cells. There are many naturally occurring fibrant symmetric monoidal 2x1-categories, such as \mathbb{D}_0 = commutative rings, \mathbb{D}_1 = algebras, and \mathbb{D}_2 = modules, or the symmetric monoidal 2x1-category of *conformal nets* defined in [BDH09]. All of these have underlying tricategories, which will be symmetric monoidal for any reasonable definition of symmetric monoidal tricategory. More generally, as stated in §1, we expect any fibrant $(n \times k)$ -category to have an underlying (n + k)-category.

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