

1. STRUCTURED COSPANS

Definition 1.1. Given a functor $L: \mathbf{A} \rightarrow \mathbf{X}$, a L -structured cospan is a diagram of form $La \rightarrow x \leftarrow Lb$.

Definition 1.2. Fix a category \mathbf{X} and a functor $L: \mathbf{A} \rightarrow \mathbf{X}$. Denote by $L\text{-Csp}$ the category with objects are those from \mathbf{A} and with morphisms of type $a \rightarrow b$ are isomorphism classes of L -structured cospans $La \rightarrow x \leftarrow Lb$.

Definition 1.3. Denote by $L\text{-StrCsp}$ the category whose objects are L -open objects and arrows are triples (f, g, h) that fit into a commuting diagram

Theorem 1.4. Let $L \dashv R: \mathbf{A} \rightarrow \mathbf{X}$ be an adjunction between topoi. Then $L\text{-StrCsp}$ is a topos.

2. REWRITING

Definition 2.1. A category with pullbacks is **adhesive** if pushouts along monics exist and are *Van Kampen*.

Theorem 2.2. Topoi are adhesive.

Corollary 2.3. Let $L \dashv R: \mathbf{A} \rightarrow \mathbf{X}$ be an adjunction between topoi. The category $L\text{-StrCsp}$ is adhesive.

Definition 2.4. For \mathbf{C} an adhesive category, an **C-rewrite rule** (often called a production) is a span $a \leftarrow b \rightarrow c$ inside \mathbf{C} . When both legs of the span are monic, we say the rewrite rule is **linear**.

Definition 2.5. Given composable arrows $a \rightarrow b \rightarrow y$ we say that an arrow $a \rightarrow x$ is a **pushout complement** if it fits into a pushout diagram

Definition 2.6. Given a \mathbf{C} -rewrite rule $a \leftarrow b \rightarrow c$ and a \mathbf{C} -arrow $a \rightarrow x$ such that $b \rightarrow a \rightarrow x$ has a pushout complement, a **derived (linear) rewrite rule** is the bottom row of the induced double pushout diagram

Definition 2.7. A **(linear) grammar** consists of an adhesive category \mathbf{A} and a set of (linear) \mathbf{A} -rewrite rules. Observe that \mathbf{A} -rewrite rules are actually arrows in $\mathbf{Sp}(\mathbf{A})$. Given a grammar Γ , the subcategory $\mathcal{L}(\Gamma)$ of $\mathbf{Sp}(\mathbf{A})$ generated by the set of rewrites derived from Γ is called a **language**.

Lemma 2.8. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ be a symmetric monoidal category with pullbacks and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a symmetric monoidal topos. Let $L \dashv R: \mathbf{A} \rightarrow \mathbf{X}$ be an adjunction where L preserves pullbacks.

Fix a grammar Γ in the topos $L\text{-StrCsp}$. The generated language $\mathcal{L}(\Gamma)$ is a sub-bicategory of $\mathbf{Sp}(L\text{-StrCsp})$.

3. NON-LINEAR REWRITING OF OPEN OBJECTS

Definition 3.1. Let \mathbf{A} be a category with pullbacks, \mathbf{X} be a category with pullbacks and pushouts, and $L: \mathbf{A} \rightarrow \mathbf{X}$ be a functor preserving pullbacks. Denote by $\mathcal{P}(\mathbf{Sp}(L\text{-StrCsp}))$ the preorder whose objects are L -open objects and arrows $(La \rightarrow x \leftarrow La') \leq (Lc \rightarrow x \leftarrow Lc')$ whenever there is a $\mathbf{Sp}(L\text{-StrCsp})$ -arrow with

form

$$\begin{array}{ccccc}
 La & \longrightarrow & x & \longleftarrow & La' \\
 \uparrow \cong & & \uparrow & & \uparrow \cong \\
 Lb & \longrightarrow & y & \longleftarrow & Lb' \\
 \downarrow \cong & & \downarrow & & \downarrow \cong \\
 Lc & \longrightarrow & z & \longleftarrow & Lc'
 \end{array}$$

Definition 3.2. Let \mathbf{A} be a category with pullbacks, \mathbf{X} a category with pullbacks and pushouts, and $L: \mathbf{A} \rightarrow \mathbf{X}$ be a functor preserving pullbacks. Define the double category $L\text{-}\mathbf{R}\mathbf{e}\mathbf{w}\mathbf{r}\mathbf{i}\mathbf{t}\mathbf{e}$ to have object category $\mathbf{core}(\mathbf{Sp}(\mathbf{A}))$ and have arrow category \mathbf{C} as described in Definition 3.1.

Alternatively, $L\text{-}\mathbf{R}\mathbf{e}\mathbf{w}\mathbf{r}\mathbf{i}\mathbf{t}\mathbf{e}$ is the double category with \mathbf{A} -objects as 0-cells, spans in \mathbf{A} with isomorphic legs as vertical 1-cells, L -open objects as horizontal 1-cells, and a unique 2-cell if there exists a commuting diagram in \mathbf{X} of form

$$\begin{array}{ccccc}
 La & \longrightarrow & x & \longleftarrow & La' \\
 \uparrow \cong & & \uparrow & & \uparrow \cong \\
 Lb & \longrightarrow & y & \longleftarrow & Lb' \\
 \downarrow \cong & & \downarrow & & \downarrow \cong \\
 Lc & \longrightarrow & z & \longleftarrow & Lc'
 \end{array}$$

Theorem 3.3. *The double category $L\text{-}\mathbf{R}\mathbf{e}\mathbf{w}\mathbf{r}\mathbf{i}\mathbf{t}\mathbf{e}$ is isofibrant.*

Theorem 3.4. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories so that \mathbf{A} has pullbacks and \mathbf{X} has pullbacks and pushouts. If $L: (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \rightarrow (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ preserves pullbacks, then $(L\text{-}\mathbf{R}\mathbf{e}\mathbf{w}\mathbf{r}\mathbf{i}\mathbf{t}\mathbf{e}, \otimes, I)$ is a symmetric monoidal double category with \otimes defined by*

$$(La \rightarrow x \leftarrow Lb) \otimes (Lc \rightarrow y \leftarrow Ld) := L(a \otimes_{\mathbf{A}} c) \rightarrow x \otimes_{\mathbf{X}} y \leftarrow L(b \otimes_{\mathbf{A}} d)$$

and I defined by

$$I := (LI_{\mathbf{A}} \rightarrow I_{\mathbf{X}} \leftarrow LI_{\mathbf{X}}).$$

Theorem 3.5. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories where \mathbf{A} has pullbacks and \mathbf{X} has pullbacks and pushouts. Let $L: (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \rightarrow (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a pullback preserving symmetric monoidal functor.*

The horizontal edge bicategory $L\text{-}\mathbf{R}\mathbf{e}\mathbf{w}\mathbf{r}\mathbf{i}\mathbf{t}\mathbf{e} := \mathcal{H}(L\text{-}\mathbf{R}\mathbf{e}\mathbf{w}\mathbf{r}\mathbf{i}\mathbf{t}\mathbf{e})$ in the sense of Shulman is symmetric monoidal. Moreover, if the monoidal products $\otimes_{\mathbf{A}}$ and $\otimes_{\mathbf{X}}$ are coproducts, then the symmetric monoidal bicategory $L\text{-}\mathbf{R}\mathbf{e}\mathbf{w}\mathbf{r}\mathbf{i}\mathbf{t}\mathbf{e}$ is compact closed.

Theorem 3.6. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories so that \mathbf{A} has a pullbacks and \mathbf{X} has pullbacks and pushouts. Let $L: (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \rightarrow (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a pullback preserving functor.

The bicategory $L\text{-}\mathbf{Rewrite}$ is a bicategory of relations in the sense of Carboni and Walters.

Theorem 3.7. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ be a symmetric monoidal category with pullbacks and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a symmetric monoidal topos. Let $L \dashv R: \mathbf{A} \rightarrow \mathbf{X}$ be an adjunction where L preserves pullbacks.

Suppose each element from a grammar Γ in $L\text{-}\mathbf{StrCsp}$ is of the form

$$\begin{array}{ccccc} La & \longrightarrow & x & \longleftarrow & La' \\ \cong \uparrow & & \uparrow & & \uparrow \cong \\ Lb & \longrightarrow & y & \longleftarrow & Lb'' \\ \cong \downarrow & & \downarrow & & \downarrow \cong \\ Lc & \longrightarrow & z & \longleftarrow & Lc' \end{array}$$

Then Γ generates a sub-double category of $L\text{-}\mathbf{Rewrite}$ as follows:

- generate the sub-bicategory $\mathcal{L}(\Gamma) \subseteq \mathbf{Sp}(L\text{-}\mathbf{StrCsp})$ as in Lemma 2.8,
- with $\mathcal{L}(\Gamma)$, define the subcategory $\mathcal{P}(\mathcal{L}(\Gamma))$ as in Definition 3.1,

Form the category as described in Definition ?? from $\mathcal{L}(\Gamma)$. Pair this, as an arrow category, with the object category $\mathbf{core}(\mathbf{Sp}(\mathbf{A}))$.

Theorem 3.8. If Γ has only elements of the form

$$\begin{array}{ccccc} La & \longrightarrow & x & \longleftarrow & La' \\ = \uparrow & & \uparrow & & \uparrow = \\ La & \longrightarrow & y & \longleftarrow & La' \\ = \downarrow & & \downarrow & & \downarrow = \\ La & \longrightarrow & z & \longleftarrow & La' \end{array}$$

then Γ generates a sub-bicategory of $L\text{-}\mathbf{Rewrite}$. This sub-bicategory corresponds to the sub-bicategory of $L\text{-}\mathbf{Rewrite}$ obtained by passing the construction through $L\text{-}\mathbf{Rewrite}$ first, then applying $\mathcal{H}(-)$.

4. LINEAR REWRITING OF OPEN OBJECTS

Definition 4.1. Let \mathbf{A} be a category with pullbacks, \mathbf{X} be a topos, and $L: \mathbf{A} \rightarrow \mathbf{X}$ be a functor preserving pullbacks. Denote by $\mathcal{C}(\mathbf{Sp}(L\text{-}\mathbf{StrCsp}))$ the category whose

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objects are L -open objects and arrows are isomorphism classes of 1-cells of form

$$\begin{array}{ccccc}
 La & \longrightarrow & x & \longleftarrow & La' \\
 \uparrow \cong & & \uparrow & & \uparrow \cong \\
 Lb & \longrightarrow & y & \longleftarrow & Lb' \\
 \downarrow \cong & & \downarrow & & \downarrow \cong \\
 Lc & \longrightarrow & z & \longleftarrow & Lc'
 \end{array}$$

where the arrows marked “ \rightarrow ” are monic.

Definition 4.2. Denote by $L\text{-}\mathbf{MonRewrite}$ the double category with \mathbf{A} -objects as 0-cells, spans in \mathbf{A} whose legs are isomorphisms as vertical 1-cells, L -open objects as horizontal 1-cells, and commuting diagrams in \mathbf{X} of form

$$\begin{array}{ccccc}
 La & \longrightarrow & x & \longleftarrow & La' \\
 \uparrow \cong & & \uparrow & & \uparrow \cong \\
 Lb & \longrightarrow & y & \longleftarrow & Lb' \\
 \downarrow \cong & & \downarrow & & \downarrow \cong \\
 Lc & \longrightarrow & z & \longleftarrow & Lc'
 \end{array}$$

Theorem 4.3. The double category $L\text{-}\mathbf{MonRewrite}$ is isofibrant.

Theorem 4.4. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories where \mathbf{A} has pullbacks and \mathbf{X} is a topos. Let $L: (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \rightarrow (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a pullback preserving functor. Then $(L\text{-}\mathbf{MonRewrite}, \otimes, I)$ is a symmetric monoidal double category with \otimes defined by

$$(La \rightarrow x \leftarrow Lb) \otimes (Lc \rightarrow y \leftarrow Ld) := L(a \otimes_{\mathbf{A}} c) \rightarrow x \otimes_{\mathbf{X}} y \leftarrow L(b \otimes_{\mathbf{A}} d)$$

and I by

$$I := (LI_{\mathbf{A}} \rightarrow I_{\mathbf{X}} \leftarrow LI_{\mathbf{X}}).$$

Theorem 4.5. Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be symmetric monoidal categories where \mathbf{A} has pullbacks and \mathbf{X} is a topos. Let $L: (\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}}) \rightarrow (\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be an adjunction where L preserves pullbacks.

The horizontal edge bicategory $L\text{-}\mathbf{MonRewrite} := \mathcal{H}(L\text{-}\mathbf{MonRewrite})$ in the sense of Shulman is symmetric monoidal. Moreover, if the monoidal products $\otimes_{\mathbf{A}}$ and $\otimes_{\mathbf{X}}$ are coproducts, then the symmetric monoidal bicategory $L\text{-}\mathbf{MonRewrite}$ is compact closed.

Theorem 4.6. *Suppose each element from a grammar Γ in $L\text{-}\mathbf{StrCsp}$ is of the form*

$$\begin{array}{ccccc}
 La & \longrightarrow & x & \longleftarrow & La' \\
 \cong \uparrow & & \uparrow & & \uparrow \cong \\
 Lb & \longrightarrow & y & \longleftarrow & Lb' \\
 \cong \downarrow & & \downarrow & & \downarrow \cong \\
 Lc & \longrightarrow & z & \longleftarrow & Lc'
 \end{array}$$

then Γ generates a sub-double category $\langle\langle\Gamma\rangle\rangle$ of $L\text{-}\mathbf{MonRewrite}$. The recipe is get the language $\mathcal{L}(\Gamma) \subseteq \mathbf{Sp}(L\text{-}\mathbf{StrCsp})$. Form the category as described in Definition ?? from $\mathcal{L}(\Gamma)$. Pair this, as an arrow category, with the object category $\mathbf{core}(\mathbf{Sp}(\mathbf{A}))$.

Theorem 4.7. *Same as above with monics thrown in.*