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ARTIN GLUEING

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If E and F are elementary topoi and $d : E \to F$ is a left exact functor, then the comma category (F, d) is also an elementary topos, and is said to be obtained by "glueing" E to F along d. A discussion of this in the context of Grothendieck topoi is to be found in [1]. Tierney also gave a proof for the context of elementary topoi in his lectures [4]. We present here a slightly simpler proof, which also admits a generalization to more complicated comma categories. The proof is based on the following theorem, proved in [3, ch. 2].

Theorem 1. *Let C be a left exact comonad on an elementary topos E. Then E_C, the category of C-coalgebras, is an elementary topos. There is a geometric morphism $E \to E_C$ whose inverse image part is the forgetful functor.*

In fact, any geometric morphism whose inverse image part reflects isomorphisms is, up to isomorphism, of the type described above.

We shall also need the following easily established result.

Theorem 2. *Let E_1, E_2 be elementary topoi. Then $E_1 \times E_2$ is an elementary topos.*

Unfortunately for the notation, $E_1 \times E_2$ is the coproduct of E_1 and E_2 in the category of elementary topoi and geometric morphisms.

The two projection functors form the inverse image parts of geometric morphisms

$$E_1 \to E_1 \times E_2 \leftarrow E_2$$

which are the canonical inclusions into the coproduct. These are open inclusions, amd we may make the identifications

$$E_1 \simeq (E_1 \times E_2)/(1, \phi), \qquad E_2 \simeq (E_1 \times E_2)/(\phi, 1).$$

Let us return to the data for Artin glueing, i.e., consider a left exact functor

$d : E \to F$ between elementary topoi. On $E \times F$ consider the endofunctor

$$C : E \times F \to E \times F : \qquad (A, B) \mapsto (A, d(A) \times B).$$

Because d is left exact, so is C. Furthermore, C has a comonad structure given by

$$\epsilon : C \to \mathrm{id} : \qquad (A, d(A) \times B) \mapsto (A, B),$$
$$\delta : C \to C^2 : \qquad (A, d(A) \times B) \mapsto (A, d(A) \times d(A) \times B),$$

given by the projection and the diagonal map, respectively. We conclude that $(E \times F)_C$ is an elementary topos; its objects are pairs $(A, B) \in E \times F$ together with a map

$$(A, B) \xrightarrow{(a,b)} (A, d(A) \times B)$$

satisfying the appropriate axioms to define a C-coalgebra. These tell us that $a = 1_A$ and that $b = \langle h, 1_B \rangle$ for an arbitrary map $B \xrightarrow{h} d(A)$. We conclude that $(E \times F)_C$ is equivalent to the comma category (F, d). We get an open inclusion $E \to (F, d)$ and a closed inclusion $F \to (F, d)$ from the composites of the canonical inclusions

$$E \to E \times F \leftarrow F$$

with $E \times F \to (E \times F)_C$.

It is helpful to see how this reflects what happens for topological spaces. Let X be a topological space, $X_1 \subseteq X$ an open subspace and $X_2 = X - X_1$ its closed complement. We have a continuous bijection $X_1 \amalg X_2 \to X$ which is not, of course, necessarily a homeomorphism. It is well known that $\mathrm{Top}(X)$, the topos of sheaves on X, may be obtained by glueing $\mathrm{Top}(X_1)$ to $\mathrm{Top}(X_2)$ along a functor d obtained by composing the direct image part of the inclusion $X_1 \subseteq X$ with the inverse image part of the inclusion $X_2 \subseteq X$. It is clear that $\mathrm{Top}(X_1) \times \mathrm{Top}(X_2)$ is just $\mathrm{Top}(X_1 \amalg X_2)$. Because the function $X_1 \amalg X_2 \to X$ is surjective, the inverse image functor it induces reflects isomorphisms, so that $\mathrm{Top}(X)$ is of the form $\mathrm{Top}(X_1 \amalg X_2)_C$ for a left exact comonad C. We claim that C is precisely the comonad constructed from d by the prescription shown above.

Now we consider how to generalize the construction. By a 2-diagram of elementary topoi and left exact functors we mean the following:

(1) We are given a finite category $\boldsymbol{D}$.

(2) For each object D of $\boldsymbol{D}$ we are given an elementary topos E_D.

(3) For each map $D' \xrightarrow{d} D$ of $\boldsymbol{D}$ we are given a left exact functor

$$E_{D'} \xrightarrow{f_d} E_D .$$

(4) For each object D of $\boldsymbol{D}$ we are given a natural map

$$a_D : f_{1_D} \to 1_{E_D}.$$

(5) For each composable pair of maps d, d' of $\boldsymbol{D}$ we are given a natural map

$$a_{d,d'} : f_{dd'} \to f_d\, f_{d'}\ .$$

This is to be subject to the obvious coherence conditions, namely:
For each map $D' \xrightarrow{d} D$ in $\boldsymbol{D}$,

$$1_{f_d} = (f_d\, a_{D'})\, a_{d,1_{D'}} = (a_D\, f_d)\, a_{1_D,d}$$

in $\boldsymbol{D}$, and for each composable triple d, d', d'',

$$(a_{d,d'}\, f_{d''})\, a_{dd',d''} = (f_d\, a_{d',d''})\, a_{d,d'd''}\,.$$

We denote such a 2-diagram by

$$L = (\boldsymbol{D}, E, f, a)\,.$$

To it we shall associate an elementary topos $\mathrm{Top}(L) = F_C$, where $F = \Pi_D E_D$ and C is the left exact comonad on F given by

$$C(\{A_D\}_{D \in \boldsymbol{D}}) = \left\{ \prod \{f_d(A_{D'}) \mid D' \xrightarrow{d} D\} \right\}_{D \in \boldsymbol{D}}$$

with counit $C \xrightarrow{\epsilon} \mathrm{id}$ and comultiplication $C \xrightarrow{\delta} C^2$ given by the diagrams

$$\begin{tikzcd}
\prod \{f_D(A_{D'}) \mid D' \xrightarrow{d} D\} \arrow[rr, "\epsilon"] \arrow[dr] & & A_D \\
& f_{1_D}(A_D) \arrow[ur, dashed, "a_d"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\prod \{f_{d_1}(A_{D_1}) \mid D_1 \xrightarrow{d_1} D\} \arrow[r, "\delta"] \arrow[d] & \prod \{f_d(f_{d'}(A_{D''})) \mid D'' \xrightarrow{d'} D' \xrightarrow{d} D\} \arrow[d] \\
f_{dd'}(A_{D''}) \arrow[r, "a_{d,d'}"] & f_d(f_{d'}(A_{D''}))
\end{tikzcd}$$

respectively, where unlabelled arrows denote projections. When one unravels the definition, one finds that an object of $\mathrm{Top}(L)$ is given by a family of objects $A_D \in E_D$ for each object D of $\boldsymbol{D}$, together with maps

$$A_D \xrightarrow{u_d} f_d(A_{D'})$$

for each map $D' \xrightarrow{d} D$ in $\boldsymbol{D}$ satisfying the obvious coherence conditions. Of course, the maps u_d simply define the coalgebra structure, and the coherence conditions correspond to the coalgebra axioms.

If we are to weaken the condition that D be a finite category, then we must correspondingly strengthen conditions on what limits exist in the categories E_D, and these must be preserved by the functors f_d. In the context of Grothendieck topoi, the above notions are fairly well known under the guise of fibred topoi. The most general elementary setting should require a base elementary topos S, that the categories E_D be S-topoi, that D be an internal category in S, and that the functors f_d should preserve S-limits. In particular, if the f_d's are direct image parts of morphisms of S-topoi, Top(L) will be the 2-colimit of L considered as a 2-diagram in the 2-category of S-topoi.

References

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