1 Rewriting

Definition 1.1. A category with pullbacks is **adhesive** if pushouts along monics exist and are *Van Kampen.*

Theorem 1.2. *Topoi are adhesive.*

Corollary 1.3. *Let $L \dashv R \colon \mathbf{A} \to \mathbf{X}$ be an adjunction between topoi. The category L is adhesive.*

Definition 1.4. For $\mathbf{C}$ an adhesive category, an **$\mathbf{C}$-rewrite rule** (often called a production) is a span $a \leftarrow b \rightarrow c$ inside $\mathbf{C}$. When both legs of the span are monic, we say the rewrite rule is **linear**.

Definition 1.5. Given composable arrows $a \to b \to y$ we say that an arrow $a \to x$ is a **pushout complement** if it fits into a pushout diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ x \arrow[r] & y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Definition 1.6. Given a $\mathbf{C}$-rewrite rule $a \leftarrow b \rightarrow c$ and a $\mathbf{C}$-arrow $a \to x$ such that $b \to a \to x$ has a pushout complement, a **derived (linear) rewrite rule** is the bottom row of the induced double pushout diagram

$$\begin{tikzcd} a \arrow[d] & b \arrow[l] \arrow[r] \arrow[d] & c \arrow[d] \\ x \arrow[ur, phantom, "\urcorner", very near start] & y \arrow[l] \arrow[r] & z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Definition 1.7. A **(linear) grammar** consists of an adhesive category $\mathbf{A}$ and a set of (linear) $\mathbf{A}$-rewrite rules. Observe that $\mathbf{A}$-rewrite rules are actually arrows in $\mathbf{Span}(\mathbf{A})$. Given a grammar Γ, the subcategory $\mathcal{L}(\Gamma)$ of $\mathbf{Span}(\mathbf{A})$ generated by the set of rewrites derived from Γ is called a **language**.

Lemma 1.8. *Let $(\mathbf{A}, \otimes_{\mathbf{A}}, I_{\mathbf{A}})$ be a symmetric monoidal category with pullbacks and $(\mathbf{X}, \otimes_{\mathbf{X}}, I_{\mathbf{X}})$ be a symmetric monoidal topos. Let $L \dashv R \colon \mathbf{A} \to \mathbf{X}$ be an adjunction where L preserves pullbacks.*

Fix a grammar Γ in the topos L. The generated language $\mathcal{L}(\Gamma)$ is a subbicategory of L.