COMPARING COMPOSITES OF LEFT AND RIGHT DERIVED FUNCTORS

MICHAEL SHULMAN

Abstract. We introduce a new categorical framework for studying derived functors, and in particular for comparing composites of left and right derived functors. Our central observation is that model categories are the objects of a double category whose vertical and horizontal arrows are left and right Quillen functors, respectively, and that passage to derived functors is functorial at the level of this double category. The theory of conjunctions and mates in double categories, which generalizes the theory of adjunctions and mates in 2-categories, then gives us canonical ways to compare composites of left and right derived functors. We give a number of sample applications, most of which are improvements of existing proofs in the literature.

Contents

This paper was published in the *New York Journal of Mathematics*, Vol. 17 (2011), p75–125. The published version is available at http://nyjm.albany.edu/j/2011/17-5.html.

Part 1. Theory

1. Introduction

Part of the general philosophy of category theory is that morphisms are often more important and subtler than objects. This applies also to categories and functors themselves, as well as to more complicated categorical structures, such as model categories and Quillen adjunctions. The passage from a model category to its homotopy category is well understood, but the passage from Quillen functors to derived functors seems more subtle and mysterious. In particular, the distinction between *left* and *right* derived functors is not well understood at a conceptual level.

For instance, it is well-known that taking derived functors of Quillen functors between model categories is pseudofunctorial—as long as all derived functors involved have the same "handedness." In other words, we have coherent isomorphisms such as $\mathbf{L}(G \circ F) \cong \mathbf{L}G \circ \mathbf{L}F$. However, not infrequently it happens that we want to compose a Quillen left adjoint with a Quillen right adjoint, and compare the result with another such composite. Standard model category theory has little to say about such questions, but such comparisons are often essential in applications.

For example, the authors of [MS06] construct, for every base space B, a model category $\mathbf{Ex}_B$ of ex-spaces over B, and for every continuous map $f\colon A \to B$, a Quillen adjunction $f_! \colon \mathbf{Ex}_A \rightleftarrows \mathbf{Ex}_B : f^*$. The derived functor $\mathbf{L}f_!$ is a parametrized version of homology, while the additional right adjoint f_* of $\mathbf{R}f^*$ (which is shown to exist using Brown representability) is a parametrized version of cohomology. One of their central lemmas about these adjunctions is that for any pullback square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

of base spaces, there are isomorphisms $\mathbf{L}f_! \circ \mathbf{R}h^* \cong \mathbf{R}k^* \circ \mathbf{L}g_!$ of derived functors, as long as either g or k is a fibration. (This sort of result is sometimes called a *Beck-Chevalley condition*.) Another analogous result is the proper base change theorem in sheaf theory. The aim of the current paper is to provide a general categorical framework in which to speak about such comparisons.

We should stress at the outset that we will not prove any general theorem about when two composites of left and right derived functors are isomorphic. Like the question of whether a given Quillen adjunction is a Quillen equivalence, the way to attack this question seems to depend a great deal on the particular situation. What we do give is a calculus describing the relationships between the natural transformations which compare such composites, which generalizes the calculus of "mates" in 2-categories (see §2). This gives a general framework in which to speak about such comparisons, and a way to make them more precise by identifying the particular natural transformation which is an isomorphism.

Many well-known theorems in homotopy theory and homological algebra, such as the proper base change theorem and the lemma from [MS06] mentioned above, can be restated in this language. We will work out a number of such examples in the second half of the paper. Usually, the existing proofs of such theorems already include all the real work, and only a little reformulation is required to place them in our abstract context. There are two main advantages accruing from the small amount of work involved in such a reformulation.

(i) When trying to prove a new commutation result between left and right derived functors, the abstract framework reduces the problem from *finding* an isomorphism to *proving* that a particular, canonically specified map is a weak equivalence.
(ii) The result obtained is strengthened from the mere *existence* of an isomorphism, which is usually all that the standard proofs provide, to the statement that a specific *canonically defined* map is an isomorphism. In some cases this does not matter, but in others (for instance, when coherence questions arise) it does.

In addition, we believe that our abstract framework sheds conceptual light on the distinction between left and right derived functors.

The central idea of this paper is to upgrade the category (or 2-category) of model categories and Quillen adjunctions to a more expressive structure called a *double category*. The standard 2-category of model categories and Quillen adjunctions is a somewhat uncomfortable thing, since to define it one must choose whether to consider a Quillen adjunction as pointing in the direction of the right adjoint or the left adjoint, and either choice is asymmetrical and aesthetically unsatisfactory. A double category, on the other hand, can include both the left and right Quillen functors as different types of morphism. Quillen adjunctions then appear as "conjunctions" in this double category. The central observation enabling us to compare left and right derived functors is that the passage from Quillen functors to derived functors is a functor of double categories.

Category theorists will be interested to see that there is also a formal analogy between *left* Quillen functors and *colax* monoidal functors (or colax morphisms for any 2-monad), and between *right* Quillen functors and *lax* monoidal functors. A functor which is both left and right Quillen corresponds to a strong monoidal functor, while a Quillen adjunction corresponds to a "doctrinal" or "lax/colax adjunction."

The plan of this paper is as follows. In the first part, comprising §§2–8, we develop the general theory. We begin in §2 by summarizing the theory of *mates* in 2-categories, which provides a general way to construct transformations comparing composites of adjoints; this is familiar to category theorists, but not as widely known as it should be. Then in §3 we summarize the standard theory of derived functors and note its deficiencies. The next three sections are devoted to setting up the double-categorical machinery we need. In §4 we recall the notion of double category and give our main examples, including the double category of model categories. In §5 we describe the theory of *companions* and *conjoints* in double categories, which generalizes the calculus of adjunctions and mates in 2-categories. And in §6 we define the relevant type of functor between double categories, which we call a *double pseudofunctor*; it differs from the most common notions of functor between double categories in that it only preserves the structure weakly in *both* directions.

The main result of the paper is proven in §7: passage to homotopy categories and derived functors is a double pseudofunctor on the double category of model categories. In §8 we extend this result to a more general context, involving categories equipped with "cofibrant" and "fibrant" approximations that admit more flexibility than those in a model structure; this generality turns out to be important in many examples. We call these *derivable categories* and the morphisms between them *derivable functors*; they are closely related to the *deformable functors* of [DHKS04].

In the second part of the paper, comprising §§9–13, we work out a number of example applications. Our goal is to show how the general theory can be applied in practice to compare composites of left and right derived functors, and to provide templates for future applications. In most of the examples we consider, the *existence* of an isomorphism is known; our contribution is to put all these facts in a general framework and show that the isomorphisms involved are actually the canonically defined maps which one would hope to be isomorphisms.

The procedure in all these examples is the following: use the general theory to identify a point-set-level representative of the canonically defined map in question, then invoke facts specific to the domain at hand to show that this map is (or, in some cases, is not) a weak equivalence. In general, the application of the general theory is easy, and the domain-specific facts are the same ones used in the standard proofs that an isomorphism exists (so that, in particular, the isomorphism constructed in the classical proof is in fact the canonical one). We do, however, include one example in §13 where the general theory does not apply so cleanly and a medium-sized diagram chase is still required. But even in this case, the general theory simplifies the problem significantly and provides a context in which to ask the right questions.

Our reference for model category theory is [Hov99]; in particular, we assume our model categories to be equipped with functorial factorizations. This is not strictly necessary, but it will make things easier. A good reference for the 2-category theory we will need is the first few sections of [KS74].

I would like to thank my thesis advisor, Peter May, for helpful conversations about derived functors, and the referee, for pointing out that more concrete examples were necessary.

2. Mates

Since left and right derived functors are, in particular, left and right adjoints, we begin by considering how to compare composites of left and right adjoints. The primary tool used for this purpose is the theory of *mates* in 2-categories. Though straightforward, this theory is not as well-known as it should be, and is thus frequently reinvented. Here we give a brief overview; a definitive treatment can be found in [KS74].

The most basic form of the mate correspondence says that if $f^*, g^* \colon \mathscr{A} \rightrightarrows \mathscr{B}$ are parallel functors with left adjoints $f_!$ and $g_!$, respectively, then there is a bijection between natural transformations $f_! \to g_!$ and natural transformations $g^* \to f^*$. A pair of natural transformations that correspond to each other under this bijection are called **mates** (or sometimes "conjugates" or "adjuncts").

More generally, for functors $f^* \colon \mathscr{A} \to \mathscr{B}$ and $g^* \colon \mathscr{C} \to \mathscr{D}$ with left adjoints $f_!$ and $g_!$, and any functors $h^* \colon \mathscr{D} \to \mathscr{B}$ and $k^* \colon \mathscr{C} \to \mathscr{A}$, there is a bijection between natural transformations $f_! h^* \to k^* g_!$ and natural transformations $h^* g^* \to f^* k^*$, i.e.

between transformations

$$\begin{tikzcd} \mathscr{D} \arrow[r, "h^*"] \arrow[d, "g_!"'] \arrow[dr, phantom, "\swarrow"] & \mathscr{B} \arrow[d, "f_!"] \\ \mathscr{C} \arrow[r, "k^*"'] & \mathscr{A} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{D} \arrow[r, "h^*"] \arrow[dr, phantom, "\searrow"] & \mathscr{B} \\ \mathscr{C} \arrow[r, "k^*"'] \arrow[u, "g^*"] & \mathscr{A}. \arrow[u, "f^*"'] \end{tikzcd}$$

Explicitly, the mate of $\alpha\colon f_! h^* \to k^* g_!$ is the composite

$$h^* g^* \xrightarrow{\eta h^* g^*} f^* f_! h^* g^* \xrightarrow{f^* \alpha g^*} f^* k^* g_! g^* \xrightarrow{f^* k^* \varepsilon} f^* h^*,$$

where η is the unit of the adjunction $f_! \dashv f^*$ and ε is the counit of the adjunction $g_! \dashv g^*$. This is also commonly described as the "pasting composite"

$$\begin{tikzcd} & \mathscr{D} \arrow[r, "h^*"] \arrow[d, "g_!"] \arrow[dr, phantom, "\swarrow"] & \mathscr{B} \arrow[r, equal] \arrow[d, "f_!"'] & \mathscr{B} \\ \mathscr{C} \arrow[ur, "g^*"] \arrow[r, equal] \arrow[r, phantom, shift left=4, "\Downarrow"] & \mathscr{C} \arrow[r, "k^*"'] & \mathscr{A} \arrow[ur, "f^*"'] \arrow[u, phantom, shift right=6, "\Downarrow"] & \end{tikzcd}$$

Dually, the mate of $\beta\colon h^* g^* \to f^* k^*$ is the composite

$$f_! h^* \xrightarrow{f_! h^* \eta} f_! h^* g^* g_! \xrightarrow{f_! \beta g_!} f_! f^* k^* g_! \xrightarrow{\varepsilon k^* g_!} k^* g_!.$$

The triangular diagrams for the adjunctions $f_! \dashv f^*$ and $g_! \dashv g^*$ are precisely what is required to make these into inverse bijections.

In general, the mate of an isomorphism need not be an isomorphism, but there are two important situations in which it is.

Lemma 2.1. *If h^* and k^* are identities, then a transformation $f_! \to g_!$ is an isomorphism if and only if its mate $g^* \to f^*$ is an isomorphism.*

Proof. The mate of the inverse of one supplies an inverse to the other. □

Note that this includes the standard fact that any two right adjoints of a given functor are canonically isomorphic.

Lemma 2.2. *If $f_! \dashv f^*$ and $g_! \dashv g^*$ are adjoint equivalences, then a transformation $f_! h^* \to k^* g_!$ is an isomorphism if and only if its mate $h^* g^* \to f^* k^*$ is an isomorphism.*

Proof. In this case the η and ε appearing in the definition of mates are isomorphisms, so composing with them preserves invertibility. □

However, in cases other than these, whether or not a given mate is an isomorphism can have substantial mathematical content. Here are two examples; we will see more in §§9–13.

Example 2.3. In the situation of [MS06] mentioned in the introduction, we have a category $\mathbf{Ex}_B$ associated to every space B and an adjunction $f_!\colon \mathbf{Ex}_A \rightleftarrows \mathbf{Ex}_B : f^*$ to every continuous map $f\colon A \to B$, and moreover for any commutative square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd} \tag{2.4}$$

of continuous maps, we have a natural isomorphism

$$h^*g^* \xrightarrow{\cong} f^*k^*. \tag{2.5}$$

This isomorphism has a mate

$$f_!h^* \longrightarrow k^*g_! \tag{2.6}$$

which may or may not be an isomorphism, depending on the square (2.4); in particular, it is an isomorphism whenever (2.4) is a pullback square. In category theory, the property of (2.6) being an isomorphism is often called the *Beck-Chevalley condition* for the square (2.4).

This example is paradigmatic of a very general situation: we have a category $\mathscr{S}$ (here $\mathscr{S} = \mathbf{Top}$) and a pseudofunctor $\mathscr{S}^{op} \to \mathcal{C}at$ (here this pseudofunctor sends B to $\mathbf{Ex}_B$), with the property that each morphism f of $\mathscr{S}$ is sent to a functor f^* having a left adjoint $f_!$. For any such pseudofunctor, we can ask whether a given commutative square in $\mathscr{S}$ satisfies the Beck-Chevalley condition; often this is the case for (some class of) pullback squares in $\mathscr{S}$.

Example 2.7. Let $\mathscr{C}$ and $\mathscr{D}$ be closed symmetric monoidal categories and let $f^*\colon \mathscr{D} \to \mathscr{C}$ be a lax monoidal functor; this means that we have natural transformations

$$f^*X \otimes f^*Y \longrightarrow f^*(X \otimes Y) \tag{2.8}$$

$$I \longrightarrow f^*I \tag{2.9}$$

satisfying certain axioms. Now we can also regard (2.8) as a transformation

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "f^*"] \arrow[d, "X\otimes -"'] \arrow[dr, phantom, "\swarrow"] & \mathscr{C} \arrow[d, "f^*X\otimes -"] \\
\mathscr{D} \arrow[r, "f^*"'] & \mathscr{C}
\end{tikzcd}$$

which therefore has a mate

$$f^* \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(f^*X, f^*Y). \tag{2.10}$$

We say that f^* is a *closed monoidal functor* if (2.10) is an isomorphism.

Now suppose that f^* has a left adjoint $f_!$. Then we also have composite adjunctions

$$X \otimes f_!(-) \;\dashv\; f^* \operatorname{Hom}(X, -) \qquad \text{and}$$
$$f_!(f^*X \otimes -) \;\dashv\; \operatorname{Hom}(f^*X, f^*-).$$

Under these adjunctions, (2.10) has a mate

$$f_!(f^*X \otimes A) \longrightarrow X \otimes f_!A. \tag{2.11}$$

By Lemma 2.1, (2.11) is an isomorphism if and only if (2.10) is. This alternate condition is sometimes easier to verify.

There are many other mates of this sort that compare various composites of adjoint functors between monoidal categories; see, for instance, [FHM03].

The thing to notice about both of these examples is that the given structure uniquely specifies a canonical transformation, and the important question is whether that transformation is an isomorphism. Thus, for instance, in the case of the Beck-Chevalley condition, it is important not merely that there *exists* an isomorphism $f_!h^* \cong k^*g_!$, but that *the particular transformation* $f_!h^* \to k^*g_!$ (the mate of the specified isomorphism $h^*g^* \cong f^*k^*$) is an isomorphism. The mere existence of an isomorphism may be sufficient for some applications, such as computing homology and cohomology groups up to isomorphism. However, for other purposes, such as proving the coherence axioms for the bicategory of parametrized spectra constructed in [MS06] (see [Shu08a] for details), it is essential to know *what* that isomorphism is.

We end this section with a useful observation about iterated mates.

Lemma 2.12. *Given a transformation*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "g^*"] \arrow[d, "h^*"'] \arrow[dr, phantom, "\swarrow"] & \mathscr{B} \arrow[d, "k^*"] \\ \mathscr{C} \arrow[r, "f^*"'] & \mathscr{D} \end{tikzcd}$$

where all the functors f^*, g^*, h^*, and k^* have left adjoints $f_!$, $g_!$, $h_!$, and $k_!$ respectively, if we first take its mate under the adjunctions $f_! \dashv f^*$ and $g_! \dashv g^*$ to obtain a transformation $f_!k^* \to h^*g_!$, and then take the mate of this under the adjunctions $h_! \dashv h^*$ and $k_! \dashv k^*$, the resulting transformation $h_!f_! \to g_!k_!$ is the same as the mate of the original transformation under the composite adjunctions $h_!f_! \dashv f^*h^*$ and $g_!k_! \dashv k^*g^*$.

Proof. By unraveling definitions. $\square$

3. Derived functors

A good deal of the power of model category theory, and of abstract homotopy theory more generally, comes from its ability to construct *derived* structure (that is, structure at the level of homotopy categories) from *point-set level* structure, in a tractable way. The most basic example, of course, is the construction of homotopy categories themselves; a few other examples include the constructions of

(i) derived functors from point-set level functors,
(ii) monoidal homotopy categories from monoidal model categories,
(iii) enriched homotopy categories from enriched model categories, and
(iv) triangulated homotopy categories from stable model categories.

Of course, structure-preserving passage from one world to another is a common phenomenon in mathematics; to describe it formally the term *functor* was invented. In the case of constructing derived structure, one general functoriality statement was proven in [Hov99]: passage to derived functors is a *pseudofunctor* from a 2-category $\mathcal{M}odel$ of model categories to the 2-category $\mathcal{C}at$ of categories, functors, and natural transformations.

In order to make such a statement precise, we need to specify what the morphisms are in $\mathcal{M}odel$. However, there are really two different types of morphism between model categories, so we end up with two different 2-categories. In $\mathcal{M}odel_L$, the morphisms are left Quillen functors (functors which preserve cofibrations and acyclic cofibrations and have a right adjoint), and in $\mathcal{M}odel_R$, the morphisms are right Quillen functors. In each case, we allow arbitrary natural transformations as the 2-cells.

For the reader's convenience, we now recall the usual definition of derived functors. If $f\colon \mathscr{C} \to \mathscr{D}$ is left Quillen, by Ken Brown's lemma it preserves weak equivalences between cofibrant objects, so the composite $f \circ Q\colon \mathscr{C} \to \mathscr{D}$ preserves all weak equivalences (where Q denotes a functorial cofibrant replacement). Thus, $f \circ Q$ induces a functor $\mathbf{L}f\colon \operatorname{Ho}(\mathscr{C}) \to \operatorname{Ho}(\mathscr{D})$ which we call the **left derived functor** of f. Dually, a right Quillen functor has a **right derived functor** $\mathbf{R}f$ induced by $f \circ R$ (where R denotes a functorial fibrant replacement).

Remark 3.1. One can show that such a left derived functor of f is, in particular, a right Kan extension of f along the localization $\mathscr{C} \to \operatorname{Ho}(\mathscr{C})$, and many authors take this as a definition of "derived functor". From our point of view it is fairly irrelevant, although it does imply that $\mathbf{L}f$ depends only on the weak equivalences in $\mathscr{C}$ rather than the model structure.

Recall that a *pseudofunctor* between 2-categories (also called a *weak 2-functor* or a *homomorphism of bicategories*) is a map which preserves composition not exactly, but only up to *constraint isomorphisms* $F(g) \circ F(f) \cong F(g \circ f)$ and $\operatorname{Id} \cong F(\operatorname{Id})$ (which are then required to satisfy standard *coherence axioms*).

Theorem 3.2 ([Hov99])**.** *There are pseudofunctors*

$$\mathbf{L}\colon \mathcal{M}odel_L \longrightarrow \mathcal{C}at$$
$$\mathbf{R}\colon \mathcal{M}odel_R \longrightarrow \mathcal{C}at$$

which take a model category $\mathscr{C}$ to $\operatorname{Ho}(\mathscr{C})$ and a left or right Quillen functor to its left or right derived functor, respectively.

Proof. Consider $\mathbf{L}$; of course $\mathbf{R}$ is dual. We have already defined the image of each model category and each left Quillen functor. If $f, g\colon \mathscr{C} \rightrightarrows \mathscr{D}$ are left Quillen and $\alpha\colon f \to g$ is a natural transformation, then the image of α under $\mathbf{L}$ is defined to be the natural transformation $\mathbf{L}f \to \mathbf{L}g$ whose components are represented by $\alpha_{QX}\colon fQX \to gQX$. We refer to this as the **derived natural transformation** of α. This operation clearly preserves composites of natural transformations. The pseudofunctor composition constraint $\mathbf{L}g \circ \mathbf{L}f \xrightarrow{\cong} \mathbf{L}(g \circ f)$ is represented by the natural transformation

$$gQfQ \xrightarrow{g\pi fQ} gfQ \tag{3.3}$$

where $\pi\colon Q \to \operatorname{Id}$ is a natural weak equivalence relating Q to the identity. Since f preserves cofibrant objects, πfQ is a weak equivalence between cofibrant objects, so (3.3) is also a weak equivalence and thus represents an isomorphism in the homotopy category. The unit isomorphism $\mathbf{L}(\operatorname{Id}) \xrightarrow{\cong} \operatorname{Id}$ is simply represented by π itself, and the axioms of a pseudofunctor follow by naturality of π. □

As usual, the existence of a functor implies the automatic preservation of any structure that can be defined in the relevant sort of category. In this case, that means any categorical structure that can be "internalized" to any 2-category.

Example 3.4. An *adjunction* $f \dashv g$ in a 2-category consists of morphisms $f\colon C \to D$ and $g\colon D \to C$ and 2-cells $\eta\colon \operatorname{id}_C \to gf$ and $\varepsilon\colon fg \to \operatorname{id}_D$ satisfying the usual triangle identities. An adjunction in $\mathcal{C}at$ is just an adjunction in the usual sense.

Since adjunctions are defined purely 2-categorically, they are preserved by any pseudofunctor. Thus, if $f \dashv g$ is an adjunction between model categories in which both f and g are left Quillen, then we also have $\mathbf{L}f \dashv \mathbf{L}g$.

Example 3.5. Of greatest interest to us is that *mates* can be defined internal to any 2-category. The definitions are the same as those given in §2: simply replace "functor" by "morphism" and "natural transformation" by "2-cell."

Since the definition is purely 2-categorical, such mates are also preserved by any pseudofunctor. Thus, in any of the examples given in §2, if all the categories are model categories and all the functors involved are (say) left Quillen, then *the mate of a derived natural transformation is the same as the derived natural transformation of a mate*. In particular, since pseudofunctors take 2-cell isomorphisms to 2-cell isomorphisms, if the mate of a given transformation is an isomorphism, then so is the mate of its derived transformation.

For example, given a square (2.4) which satisfies the Beck-Chevalley condition on the point-set level, and in which the functors f^*, g^*, h^*, k^*, $f_!$, and $g_!$ are all left Quillen, it follows that the square also satisfies the Beck-Chevalley condition at the derived level (i.e. the canonical transformation $\mathbf{L}f_! \circ \mathbf{L}h^* \to \mathbf{L}k^* \circ \mathbf{L}g_!$ is an isomorphism).

This is a very appealing formal setup—we have not just one but two functors—but unfortunately it is not all that useful in practice. It is certainly useful to know that passage to derived functors *of the same handedness* preserves composition (the pseudofunctor constraint $\mathbf{L}g \circ \mathbf{L}f \cong \mathbf{L}(g \circ f)$), but it turns out that one almost never encounters adjunctions in $\mathcal{M}odel_L$ or $\mathcal{M}odel_R$. Much more common are, of course, Quillen adjunctions, in which the *left* adjoint is *left* Quillen and (equivalently) the *right* adjoint is *right* Quillen. It is well-known that any Quillen adjunction $f \dashv g$ has a derived adjunction $\mathbf{L}f \dashv \mathbf{R}g$, but since f and g live in different 2-categories this does not follow from pseudofunctoriality as in Example 3.4. However, functoriality is such a useful type of framework that it is natural to ask whether there is some other type of "functor" which can serve to relate left and right derived functors.

In the rest of the paper we give an affirmative answer to this question. However, such an answer must move beyond 2-categories; it is impossible to have a 2-category $\mathcal{K}$ in which Quillen adjunctions are internal adjunctions and which admits a pseudofunctor $\mathcal{K} \to \mathcal{C}at$ combining $\mathbf{L}$ and $\mathbf{R}$. For if so, then as in Example 3.5, this would imply that any Beck-Chevalley condition that holds on the point-set level would remain true at the derived level. But this is known to be false; see, for instance, [MS06, Counterexample 0.0.1] (which we repeat below as Remark 9.7).

4. Double categories

Roughly speaking, the problem we encountered in the previous section is that the composite of a left Quillen functor and a right Quillen functor need not be any sort of Quillen functor. It turns out, however, that there is a well-known structure which precisely allows us to speak about 2-cells such as the η and ε in a Quillen adjunction, but without necessarily being able to actually "compose" f and g. This structure is called a **double category**.

Double categories are a fundamental categorical notion, like 2-categories (although historically, they have received less attention). As such, they can be seen from many different viewpoints and play many different roles in mathematics (which also leads to many variants of the definition). Double categories were originally introduced by Ehresmann [Ehr63]; a good reference with a point of view similar to ours is [KS74].

A double category $\underline{\mathsf{K}}$ consists of the following data. First of all, we have *two* categories with the same set of objects (or *0-cells*). We distinguish between the two types of morphisms (or *1-cells*) by calling one of them *vertical* and one of them *horizontal*, and usually drawing them accordingly. In addition, there are *squares* (or *2-cells*) which have the following shape:

(4.1)
$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\Swarrow\alpha"] & c \arrow[d, "k"] \\ b \arrow[r, "g"'] & d. \end{tikzcd}$$

Here a, b, c, and d are objects, f and g are horizontal morphisms, and h and k are vertical morphisms. We think of such an α as a morphism from "the composite kf" to "the composite gh," even though such composites do not actually exist (since the vertical and horizontal 1-cells live in different categories).

Finally, we require that the 2-cells can be composed both horizontally and vertically, forming the morphisms of a category in each direction, and that these two category structures respect each other and the given categories of horizontal and vertical 1-cells. We write $\alpha \boxbar \beta$ for the horizontal composite of 2-cells

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow\alpha"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow\beta"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

and $\beta \boxminus \alpha$ for the vertical composite

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow\alpha"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow\beta"] & \cdot \arrow[d] \\ \cdot \arrow[r] & . \end{tikzcd}$$

The compatibility requirement for composition is then that

$$(\alpha \boxbar \beta) \boxminus (\gamma \boxbar \delta) = (\alpha \boxminus \gamma) \boxbar (\beta \boxminus \delta).$$

Every object a has both a vertical identity 1_a and a horizontal identity 1^a, every vertical arrow $g\colon a \to b$ has an identity 2-cell

$$\begin{tikzcd} \cdot \arrow[r, "1^a"] \arrow[d, "g"'] \arrow[dr, phantom, "\Swarrow 1^g"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "1^b"'] & \cdot \end{tikzcd}$$

every horizontal arrow $f\colon a \to c$ has an identity 2-cell

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "1_a"'] \arrow[dr, phantom, "\Swarrow 1_f"] & \cdot \arrow[d, "1_c"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

and the compatibility requirements for units are that

$$1^{1_a} = 1_{1^a}, \qquad 1^g \boxminus 1^f = 1^{gf}, \quad \text{and} \quad 1_f \boxbar 1_g = 1_{gf}.$$

We will often write identity arrows simply as equalities.

Remark 4.2. A more concise definition of a double category is that it is an *internal* category in $\mathcal{C}at$ (as contrasted with a 2-category, which is a category *enriched* in $\mathcal{C}at$). This definition is often convenient for dealing with weak double categories (the double-category counterpart of bicategories, or weak 2-categories). However, for our purposes this approach merely muddies the water, since it breaks the symmetry between the horizontal and vertical directions.

The following examples are fundamental.

Example 4.3. There is a double category $\underline{\mathsf{Cat}}$ whose objects are categories, whose vertical and horizontal 1-cells are functors, and whose 2-cells of the form (4.1) are natural transformations $\alpha\colon kf \to gh$.

Example 4.4. A similar double category can be constructed with any 2-category $\mathcal{K}$ replacing $\mathcal{C}at$; we call this the double category $\underline{\mathsf{Sq}}(\mathcal{K})$ of **squares** in $\mathcal{K}$. Ehresmann, who first defined it, called it the double category of *quintets* in $\mathcal{K}$, since a 2-cell in $\underline{\mathsf{Sq}}(\mathcal{K})$ is defined by a quintet (f, g, h, k, α) where $\alpha\colon kf \to gh$ is a 2-cell in $\mathcal{K}$.

Example 4.5. Any ordinary category $\mathbf{C}$ can be regarded as a 2-category with only identity 2-cells, so we thereby obtain a double category $\underline{\mathsf{Sq}}(\mathbf{C})$ of commutative squares in $\mathbf{C}$.

In double categories of the form $\underline{\mathsf{Sq}}(\mathcal{K})$, the vertical and horizontal 1-cells are the same. Of course, the reason for considering double categories instead of 2-categories is that the two can also be different. The following example is the one in which we are most interested.

Example 4.6. There is a double category $\underline{\mathsf{Model}}$ whose objects are model categories, whose vertical arrows are left Quillen functors, whose horizontal arrows are right Quillen functors, and whose 2-cells of the form (4.1) are arbitrary natural transformations $\alpha\colon kf \to gh$. (The reason for these particular choices will become clear in §7.)

Any double category has two underlying 2-categories with the same objects, called its **horizontal 2-category** $\mathcal{H}(\underline{\mathsf{K}})$ and its **vertical 2-category** $\mathcal{V}(\underline{\mathsf{K}})$. The morphisms of $\mathcal{H}(\underline{\mathsf{K}})$ are the horizontal 1-cells of $\underline{\mathsf{K}}$, and its 2-cells are the squares in $\underline{\mathsf{K}}$ of the form

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swArrow"] & \cdot \arrow[d, equal] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

which we call **h-globular**. Dually, $\mathcal{V}(\underline{\mathsf{K}})$ is composed of the objects, vertical 1-cells, and **v-globular** squares in $\underline{\mathsf{K}}$.

Example 4.7. Of course, for any $\mathcal{K}$ we have $\mathcal{H}(\underline{\mathsf{Sq}}(\mathcal{K})) \cong \mathcal{K}$ and $\mathcal{V}(\underline{\mathsf{Sq}}(\mathcal{K})) \cong \mathcal{K}$. More interestingly, we have $\mathcal{H}(\underline{\mathsf{Model}}) \cong \mathcal{M}odel_R$ and $\mathcal{V}(\underline{\mathsf{Model}}) \cong \mathcal{M}odel_L$.

Any double category $\underline{\mathsf{K}}$ has three *opposites*, obtained by reversing it horizontally, vertically, or both. It also has a *transpose* $\underline{\mathsf{K}}^\top$ obtained by switching the vertical and horizontal arrows.

We end this section by mentioning one further class of examples. We will make no real use of these in this paper, but they are worth thinking about for purposes of comparison and intuition.

Example 4.8. There is a double category $\underline{\mathsf{MonCat}}$ whose objects are monoidal categories, whose horizontal arrows are *lax* monoidal functors, and whose vertical arrows are *colax* monoidal functors. A 2-cell

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\alpha"] & c \arrow[d, "k"] \\ b \arrow[r, "g"'] & d \end{tikzcd}$$

is a natural transformation $\alpha\colon kf \to gh$ such that the following diagrams commute:

$$\begin{tikzcd} & k(fx \otimes fy) \arrow[dl, "k_\otimes"'] \arrow[dr, "k(f_\otimes)"] & \\ kfx \otimes kfy \arrow[d, "\alpha\otimes\alpha"'] & & kf(x\otimes y) \arrow[d, "\alpha"] \\ ghx \otimes ghy \arrow[dr, "g_\otimes"'] & & gh(x\otimes y) \arrow[dl, "g(h_\otimes)"] \\ & g(hx \otimes hy) & \end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd} & k(I_c) \arrow[dl, "k_I"'] \arrow[dr, "k(f_I)"] & \\ I_d \arrow[d, equal] & & kf(I_a) \arrow[d, "\alpha"] \\ I_d \arrow[dr, "g_I"'] & & gh(I_a) \arrow[dl, "g(h_I)"] \\ & g(I_b) & \end{tikzcd}$$

The horizontal 2-category $\mathcal{H}(\underline{\mathsf{MonCat}})$ of $\underline{\mathsf{MonCat}}$ is the 2-category $\mathcal{M}onCat_\ell$ of monoidal categories, lax monoidal functors, and monoidal transformations, and dually for $\mathcal{V}(\underline{\mathsf{MonCat}})$. More generally, we have a double category T-$\underline{\mathsf{Alg}}$ of T-algebras, lax and colax T-morphisms, and generalized T-transformations for any 2-monad T. (A *2-monad* is a monad on a 2-category, for which we can define general notions of lax and colax morphisms of algebras; see [BKP89].) These double categories were apparently first considered in [GP04].

Remark 4.9. In $\underline{\mathsf{Model}}$ no axioms are imposed on the 2-cells, whereas in $\underline{\mathsf{MonCat}}$ there is a compatibility requirement with the structure of the 1-cells. Nevertheless, there is a connection between the two. One can "algebraicize" parts of the definition of model category so that "algebraicized" left and right Quillen functors become colax and lax morphisms for a 2-monad, respectively; see [GT06, Gar09, Rie09].

5. Companions and conjoints

Recall that our goal in introducing double categories was to find an abstract framework in which to express the adjointness between a left Quillen functor and a right Quillen functor. Inspecting the definition of $\underline{\mathsf{Model}}$, we immediately see how to write this down. The following terminology is due to [DPP10] (in [GP04] it was called an *orthogonal adjunction*).

Definition 5.1. A **conjunction** in a double category $\underline{\mathsf{K}}$ consists of a vertical 1-cell $f\colon a \to b$, a horizontal 1-cell $g\colon b \to a$, and 2-cells

$$\begin{tikzcd} a \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow\eta"] & a \arrow[d, equal] \\ b \arrow[r, "g"'] & a \end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd} b \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\varepsilon"] & a \arrow[d, "f"] \\ b \arrow[r, equal] & b \end{tikzcd}$$

(the **unit** and **counit**) such that $\varepsilon \boxbar \eta = 1_g$ and $\varepsilon \boxminus \eta = 1^f$. We say that f is the **left conjoint** and g is the **right conjoint**, and write $f \mathrel{-\!\!\prec} g$.

Example 5.2. A conjunction in $\underline{\mathsf{Cat}}$ is simply an ordinary adjunction. Likewise, a conjunction in $\underline{\mathsf{Sq}}(\mathcal{K})$ is simply an ordinary internal adjunction in $\mathcal{K}$.

Example 5.3. A conjunction in $\underline{\mathsf{Model}}$ is an adjunction in which the left adjoint is left Quillen and the right adjoint is right Quillen—in other words, a Quillen adjunction.

Example 5.4. A conjunction in T-$\underline{\mathsf{Alg}}$ (such as $\underline{\mathsf{MonCat}}$) is precisely a *doctrinal adjunction* as studied in [Kel74]. This is an adjunction between T-algebras in which the left adjoint is colax and the right adjoint is lax, and the colax and lax structure maps are mates under the adjunction.

Interpreting conjunctions in the horizontally-opposite double category (or, equivalently, the vertically-opposite one), we obtain a different useful notion.

Definition 5.5. A **companion pair** in a double category $\underline{\mathsf{K}}$ consists of a vertical 1-cell $f\colon a \to b$, a horizontal 1-cell $f'\colon a \to b$, and 2-cells

$$\begin{tikzcd} a \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow\varphi"] & b \arrow[d, equal] \\ b \arrow[r, equal] & a \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow\psi"] & a \arrow[d, "f"] \\ a \arrow[r, "f'"'] & b \end{tikzcd}$$

such that $\varphi \boxminus \psi = 1^f$ and $\psi \boxbar \varphi = 1_{f'}$. We say that f' is the **(horizontal) companion** of f and that f is the **(vertical) companion** of f', and write $f \approxeq f'$.

Example 5.6. In $\underline{\mathsf{Cat}}$ or $\underline{\mathsf{Sq}}(\mathcal{K})$, every 1-cell has a companion, namely itself; φ and ψ are both identities. More generally, a companion pair in $\underline{\mathsf{Sq}}(\mathcal{K})$ is precisely a natural isomorphism between two parallel morphisms $f, g\colon a \rightrightarrows b$ in $\mathcal{K}$.

Thus, from the double-categorical perspective, “adjunctions are dual to natural isomorphisms.”

Example 5.7. A 1-cell in $\underline{\mathsf{Model}}$ has a companion just when it both left and right Quillen.

Example 5.8. A 1-cell in $\underline{\mathsf{MonCat}}$ has a companion just when it is a *strong* monoidal functor. For the 2-cells φ and ψ show that f and f' are isomorphic as ordinary functors, and then the hexagon axioms in the definition of a 2-cell in $\underline{\mathsf{MonCat}}$ imply that the lax structure maps of f' are inverses to the colax structure maps of f, so that both are strong. An analogous statement is true in any T-$\underline{\mathsf{Alg}}$.

Motivated by these examples, we say that a 1-cell in a general double category is **strong** if it has a companion.

Companions and conjoints in a double category have most of the good properties of adjunctions in a 2-category. For instance, they are unique up to unique isomorphism when they exist, and are preserved under composition.

Proposition 5.9. *If f' and f'' are both horizontal companions of f, then there is a canonical isomorphism $f' \cong f''$ in $\mathcal{H}(\underline{\mathsf{K}})$, and similarly for vertical companions.*

Proof. An isomorphism is given by the following composite.

$$\begin{tikzcd}
{} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow"] & {} \arrow[r, "f'"] \arrow[d, "f"] \arrow[dr, phantom, "\Swarrow"] & {} \arrow[d, equal] \\
{} \arrow[r, "f''"'] & {} \arrow[r, equal] & {}
\end{tikzcd}$$

Its inverse is the obvious dual construction. ☐

Proposition 5.10. *If f and h have companions f' and h', then $h'f'$ is a companion of hf.*

Proof. It is straightforward to compose the 2-cells defining the companion pairs $f \approxeq f'$ and $h \approxeq h'$ to produce a companion pair $hf \approxeq h'f'$. ☐

By duality, we have the corresponding results for conjunctions.

Proposition 5.11. *If g and g' are both conjoints of f, then there is a canonical globular isomorphism $g \cong g'$.*

Proposition 5.12. *If f and h have conjoints g and k, respectively, then gh is a conjoint of hf.*

The most important property of companions and conjunctions for our purposes, however, is that they also have an associated mate correspondence. We begin with mates for companions.

Proposition 5.13. *If f and g have horizontal companions f' and g', then there is a canonical isomorphism*

$$\mathcal{V}(\underline{\mathsf{K}})(f,g) \cong \mathcal{H}(\underline{\mathsf{K}})(f',g'). \tag{5.14}$$

More generally, for any i, j, m, n there is a bijection between 2-cells of the following shapes:

$$\begin{tikzcd}
{} \arrow[r, "j"] \arrow[d, "fi"'] \arrow[dr, phantom, "\Swarrow\alpha"] & {} \arrow[d, "mg"] \\
{} \arrow[r, "n"'] & {}
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
{} \arrow[r, "g'j"] \arrow[d, "i"'] \arrow[dr, phantom, "\Swarrow\beta"] & {} \arrow[d, "m"] \\
{} \arrow[r, "nf'"'] & {}
\end{tikzcd}. \tag{5.15}$$

We say that a pair of 2-cells which correspond under (5.15) are ***mates****.*

Proof. The bijection is given by the following correspondences.

$$\begin{tikzcd}
{} \arrow[r, "j"] \arrow[d, "fi"'] \arrow[dr, phantom, "\Swarrow\alpha"] & {} \arrow[d, "mg"] \\
{} \arrow[r, "n"'] & {}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
{} \arrow[r, equal] \arrow[d, "i"'] \arrow[dr, phantom, "\Swarrow 1_i"] & {} \arrow[r, "j"] \arrow[d, "i"] \arrow[ddr, phantom, "\Swarrow\alpha"] & {} \arrow[r, "g'"] \arrow[d, "g"'] \arrow[dr, phantom, "\Swarrow\varphi"] & {} \arrow[d, equal] \\
{} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow\psi"] & {} \arrow[d, "f"] & {} \arrow[r, equal] \arrow[d, "m"'] \arrow[dr, phantom, "\Swarrow 1_m"] & {} \arrow[d, "m"] \\
{} \arrow[r, "f'"'] & {} \arrow[r, "n"'] & {} \arrow[r, equal] & {}
\end{tikzcd} \tag{5.16}$$

$$
\begin{tikzcd}
\bullet \arrow[r, "j"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_j"] & \bullet \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\psi"] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "j"] \arrow[d, "i"'] \arrow[drr, phantom, "\Downarrow\beta"] & \bullet \arrow[r, "g'"] & \bullet \arrow[d, "m"] \\
\bullet \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\varphi"] & \bullet \arrow[r, "n"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_n"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "n"'] & \bullet
\end{tikzcd}
\quad \longleftarrow\!\!\mapsfromchar \quad
\begin{tikzcd}
\bullet \arrow[r, "g'j"] \arrow[d, "i"'] \arrow[dr, phantom, "\Downarrow\beta"] & \bullet \arrow[d, "m"] \\
\bullet \arrow[r, "nf'"'] & \bullet
\end{tikzcd}. \tag{5.17}
$$

□

The correspondence (5.14) is preserved by composition, so we have a 2-category $\mathcal{S}tr(\underline{\mathsf{K}})$ whose 0-cells are the 0-cells of $\underline{\mathsf{K}}$, whose 1-cells are companion pairs in $\underline{\mathsf{K}}$, and whose 2-cells are mate-pairs of globular 2-cells. We also have canonical 2-functors

$$
\begin{aligned}
\mathcal{S}tr(\underline{\mathsf{K}}) &\longrightarrow \mathcal{V}(\underline{\mathsf{K}}) \\
\mathcal{S}tr(\underline{\mathsf{K}}) &\longrightarrow \mathcal{H}(\underline{\mathsf{K}})
\end{aligned}
$$

which are full and faithful on hom-categories.

Examples 5.18. $\mathcal{S}tr(\underline{\mathsf{Sq}}(\mathcal{K}))$ is not quite the same as $\mathcal{K}$; its morphisms are pairs of parallel morphisms in $\mathcal{K}$ with an isomorphism between them. However, it is "biequivalent" to $\mathcal{K}$ (this is the most general sort of equivalence between 2-categories).

Similarly, $\mathcal{S}tr(\underline{\mathsf{Model}})$ is biequivalent to the 2-category of model categories and functors which are both left and right Quillen, and $\mathcal{S}tr(\underline{\mathsf{MonCat}})$ is biequivalent to the 2-category of monoidal categories and strong monoidal functors.

Dualizing this correspondence, we immediately obtain the mate correspondence for conjunctions.

Proposition 5.19. *If $f \prec g$ and $h \prec k$ where $f, h\colon a \to b$, then we have a natural isomorphism*

$$
\mathcal{V}(\underline{\mathsf{K}})(f,h) \cong \mathcal{H}(\underline{\mathsf{K}})(k,g), \tag{5.20}
$$

under which isomorphisms $f \cong h$ correspond to isomorphisms $k \cong g$. More generally, for any i, j, m, n there is a bijection between 2-cells of the following shapes:

$$
\begin{tikzcd}
\bullet \arrow[r, "i"] \arrow[d, "mh"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, "fj"] \\
\bullet \arrow[r, "n"'] & \bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\bullet \arrow[r, "ik"] \arrow[d, "m"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, "j"] \\
\bullet \arrow[r, "gn"'] & \bullet
\end{tikzcd}. \tag{5.21}
$$

*We say that a pair of 2-cells which correspond under (5.21) are **mates**.*

As with companions, we obtain a 2-category $\mathcal{C}onj(\underline{\mathsf{K}})$ whose objects are those of $\underline{\mathsf{K}}$, whose 1-cells are the conjunctions in $\underline{\mathsf{K}}$, and whose 2-cells are the mate-pairs of globular 2-cells in $\underline{\mathsf{K}}$. Note, though, that to define $\mathcal{C}onj(\underline{\mathsf{K}})$ we must choose whether to consider a conjunction as pointing in the direction of the left conjoint or the right conjoint; it is precisely this arbitrariness which the double-categorical context avoids.

Examples 5.22. Of course, $\mathcal{C}onj(\underline{\mathsf{Cat}})$ is the usual 2-category of categories and adjunctions, while $\mathcal{C}onj(\underline{\mathsf{Model}})$ is the usual 2-category of model categories and Quillen adjunctions.

Remark 5.23. Mates for conjunctions are clearly analogous to mates for adjunctions in a 2-category. The mate correspondence for companion pairs also has an analogue in a 2-category, though it is too obvious to require comment (or a name): it simply says that if $f \cong f'$ and $g \cong g'$, then there is a bijection between 2-cells $f \to g$ and $f' \to g'$.

As an immediate application of mates, we show that companion pairs "mediate" between adjunctions and conjunctions.

Proposition 5.24. *Let $f\colon a \to b$ be a vertical 1-cell in $\underline{\mathsf{K}}$ and let $f'\colon a \to b$ and $g\colon b \to a$ be horizontal 1-cells. Then any two of the following statements imply the third.*

(i) *f' is a horizontal companion of f.*
(ii) *g is a right conjoint of f.*
(iii) *g is a right adjoint of f' in $\mathcal{H}(\underline{\mathsf{K}})$.*

More precisely, any companion pair $f \approxeq f'$ and conjunction $f \prec g$ determine a unique horizontal adjunction $f' \dashv g$, and similarly in the other cases.

Proof. Assuming (i), the correspondence of Proposition 5.13 transforms a unit and counit for a conjunction $f \prec g$ into a unit and counit for a horizontal adjunction $f' \dashv g$, and vice versa. The other cases are similar. □

Remark 5.25. In T-$\underline{\mathsf{Alg}}$, this implies part of one of the main results of [Kel74]: in a doctrinal adjunction, the left adjoint is a strong T-morphism precisely when the adjunction is an adjunction in the 2-category T-$\mathcal{A}lg_\ell$.

Remark 5.26. We believe that the notions of companion pair and conjunction are as central to the theory of double categories as the notions of equivalence and adjunction are to the theory of 2-categories. It is thus surprising that they seem only recently to have been isolated in the present form. The basic ideas, however, have been around a long time. For instance, a *folding* or *connection pair* on a double category, as considered in [BS76, BM99, Fio07], can be defined as a strictly functorial choice of a companion for each vertical 1-cell. Since companions are always pseudofunctorial (Proposition 5.10), an arbitrary choice of companions for each vertical 1-cell is the same as a *pseudo-folding* in the sense of [Fio07]. The *framed bicategories* of [Shu08a] are (pseudo) double categories in which every vertical 1-cell has both a horizontal companion and a right conjoint.

6. Double pseudofunctors

The way forward should now be clear: we aim to show that passage to homotopy categories and derived functors is a functor $\underline{\mathsf{Model}} \to \underline{\mathsf{Cat}}$. However, as in the 2-categorical case, it can only be expected to be a *pseudofunctor*, i.e. to preserve composition and identities up to coherent isomorphism. Strict functors of double categories are easy to define, and functors which are pseudo in one direction and strict in the other also appear in the literature under the name of *pseudo double functor* (see, for instance, [GP99]), but we require functors which are pseudo in both directions. We now define these precisely under the name of *double pseudofunctors*; the reader who is uninterested in the details may skim this section.

Definition 6.1. Let $\underline{\mathsf{K}}$ and $\underline{\mathsf{L}}$ be double categories. A **double pseudofunctor** $F \colon \underline{\mathsf{K}} \to \underline{\mathsf{L}}$ consists of the following structure and properties.

(i) Functions from the objects, vertical 1-cells, horizontal 1-cells, and 2-cells of $\underline{\mathsf{K}}$ to those of $\underline{\mathsf{L}}$, preserving sources, targets, and boundaries.

(ii) For each object a of $\underline{\mathsf{K}}$, 2-cells

$$\begin{tikzcd} Fa \arrow[r, "F(1^a)"] \arrow[d, equal, "1_{Fa}"'] \arrow[dr, phantom, "\Swarrow F_a"] & Fa \arrow[d, equal, "1_{Fa}"] \\ Fa \arrow[r, equal, "1^{Fa}"'] & Fa \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} Fa \arrow[r, equal, "1^{Fa}"] \arrow[d, equal, "1_{Fa}"'] \arrow[dr, phantom, "\Swarrow F^a"] & Fa \arrow[d, "F(1_a)"] \\ Fa \arrow[r, equal, "1^{Fa}"'] & Fa \end{tikzcd}$$

in $\underline{\mathsf{L}}$, of which the first is an h-globular isomorphism and the second a v-globular isomorphism.

(iii) For each composable pair $a \xrightarrow{f} b \xrightarrow{g} c$ of vertical 1-cells in $\underline{\mathsf{K}}$, a v-globular isomorphism

$$\begin{tikzcd} Fa \arrow[r, equal] \arrow[d, "Ff"'] & Fa \arrow[dd, "F(gf)"] \\ Fb \arrow[d, "Fg"'] \arrow[r, phantom, "\Swarrow F^{gf}"] & \\ Fc \arrow[r, equal] & Fc. \end{tikzcd}$$

(iv) For each composable pair $a \xrightarrow{h} b \xrightarrow{k} c$ of horizontal 1-cells in $\underline{\mathsf{K}}$, an h-globular isomorphism

$$\begin{tikzcd} Fa \arrow[rr, "F(kh)"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow F_{kh}"] & & Fc \arrow[d, equal] \\ Fa \arrow[r, "Fh"'] & Fb \arrow[r, "Fk"'] & Fc. \end{tikzcd}$$

(v) The following coherence axioms hold (the usual coherence axioms for a pseudofunctor in both directions).

$$F^{h(gf)} \boxbar \left(1^{Fh} \boxminus F^{gf}\right) = F^{(hg)f} \boxbar \left(F^{hg} \boxminus 1^{Ff}\right)$$
$$F^b \boxminus 1^{Ff} = F^{1_b f}$$
$$1^{Ff} \boxminus F^a = F^{f 1_a}$$
$$F_{h(gf)} \boxminus (F_{gf} \boxbar 1_{Fh}) = F_{(hg)f} \boxminus (1_{Ff} \boxbar F_{hg})$$
$$F_a \boxbar 1_{Ff} = F_{f 1_a}$$
$$1_{Ff} \boxbar F_b = F_{1_b f}.$$

(vi) The "double naturality" axioms displayed in figure 1 hold, as do their transposes involving F^{gf} and F^a.

Note that in general, a double pseudofunctor does not preserve globularity of 2-cells, since it does not preserve either vertical or horizontal identities strictly.

(6.2)
$$
\begin{tikzcd}
{} \arrow[rr, "F(gf)"] \arrow[d, equal] \arrow[drr, phantom, "\Swarrow F_{gf}"] & & {} \arrow[d, equal] \\
{} \arrow[r] \arrow[d, "Fu"'] \arrow[dr, phantom, "\Swarrow F\alpha"] & {} \arrow[r] \arrow[d, "Fv"] \arrow[dr, phantom, "\Swarrow F\beta"] & {} \arrow[d, "Fw"] \\
{} \arrow[r, "Fh"'] & {} \arrow[r, "Fk"'] & {}
\end{tikzcd}
=
\begin{tikzcd}
{} \arrow[rr, "F(gf)"] \arrow[d, "Fu"'] \arrow[drr, phantom, "\Swarrow F(\alpha \boxbar \beta)"] & & {} \arrow[d, "Fw"] \\
{} \arrow[rr] \arrow[d, equal] \arrow[drr, phantom, "\Swarrow F_{kh}"] & & {} \arrow[d, equal] \\
{} \arrow[r, "Fh"'] & {} \arrow[r, "Fk"'] & {}
\end{tikzcd}
$$

(6.3)
$$
\begin{tikzcd}
Fa \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow F_a^{-1}"] & Fa \arrow[d, equal] \\
Fa \arrow[r] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Swarrow F(1^f)"] & Fa \arrow[d, "Ff"] \\
Fa \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow F_a"] & Fa \arrow[d, equal] \\
Fa \arrow[r, equal] & Fa
\end{tikzcd}
=
\begin{tikzcd}
Fa \arrow[r, equal] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Swarrow 1^{Ff}"] & Fa \arrow[d, "Ff"] \\
Fa \arrow[r, equal] & Fa
\end{tikzcd}
$$

FIGURE 1. The horizontal double naturality axioms

However, any h-globular 2-cell

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow \alpha"] & b \arrow[d, equal] \\
a \arrow[r, "g"'] & b
\end{tikzcd}
$$

in $\underline{\mathsf{K}}$ gives rise to a canonical h-globular 2-cell

$$
\begin{tikzcd}
Fa \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow F^a"] & Fa \arrow[r, "Ff"] \arrow[d] \arrow[dr, phantom, "\Swarrow F(\alpha)"] & Fb \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\Swarrow (F^a)^{-1}"] & Fb \arrow[d, equal] \\
Fa \arrow[r, equal] & Fa \arrow[r, "Fg"'] & Fb \arrow[r, equal] & Fb
\end{tikzcd}
$$

in $\underline{\mathsf{L}}$, which we denote $\mathcal{H}F(\alpha)$. It is easy to check that this defines an ordinary pseudofunctor $\mathcal{H}F\colon \mathcal{H}\underline{\mathsf{K}} \to \mathcal{H}\underline{\mathsf{L}}$. Similarly, we have $\mathcal{V}F\colon \mathcal{V}\underline{\mathsf{K}} \to \mathcal{V}\underline{\mathsf{L}}$.

Examples 6.4. An ordinary pseudofunctor $F\colon \mathcal{K} \to \mathcal{L}$ gives rise to a double pseudofunctor $\underline{\mathsf{Sq}}(F)\colon \underline{\mathsf{Sq}}(\mathcal{K}) \to \underline{\mathsf{Sq}}(\mathcal{L})$ in a fairly straightforward way. The only wrinkle is that if $\alpha\colon kf \to gh$ is a 2-cell in $\underline{\mathsf{Sq}}(\mathcal{K})$, we must compose $F\alpha$ with the constraints of F on either side to obtain a 2-cell in $\underline{\mathsf{Sq}}(\mathcal{L})$.

In particular, if $F\colon \mathcal{C}at \to \mathcal{C}at$ is a pseudofunctor, we obtain a double pseudofunctor $\underline{\mathsf{Sq}}(F)\colon \underline{\mathsf{Cat}} \to \underline{\mathsf{Cat}}$, and some of the double pseudofunctors obtained in this way also give endofunctors of $\underline{\mathsf{Model}}$. For instance, there is a double pseudofunctor $\underline{\mathsf{Model}} \to \underline{\mathsf{Model}}$ which takes a model category $\mathscr{C}$ to its pointed variant $\mathscr{C}_*$; see [Hov99, 1.1.8, 1.3.5].

Example 6.5. Recall that $\underline{\mathsf{K}}^{\top}$ denotes the transpose of a double category, in which the vertical and horizontal arrows are interchanged. We then have a double pseudofunctor $\underline{\mathsf{Model}}^{\top} \to \underline{\mathsf{Model}}$ which takes $\mathscr{C}$ to $\mathscr{C}^{op}$.

Example 6.6. Recall that for any ordinary category $\mathbf{C}$ we have a double category $\underline{\mathsf{Sq}}(\mathbf{C})$ of commutative squares in $\mathbf{C}$. If we restrict the squares in $\underline{\mathsf{Sq}}(\mathbf{C})$ to a subclass $\mathcal{A}$ of commutative squares which are closed under composition and identities (such as all pullback squares), we obtain a smaller double category $\underline{\mathsf{Sq}}(\mathbf{C};\mathcal{A})$. Then for any 2-category $\mathcal{K}$, a double pseudofunctor $\underline{\mathsf{Sq}}(\mathbf{C};\mathcal{A}) \to \underline{\mathsf{Sq}}(\mathcal{K})$ is essentially the same as a *lower e-functor* relative to $\mathcal{A}$, in the sense of [Del, §4.1]. *Upper e-functors* and e^* *and* $e!$ *contradirectional functors* are defined by applying appropriate types of duality to $\underline{\mathsf{Sq}}(\mathcal{K})$. Finally, a *cross functor* is a double pseudofunctor $\underline{\mathsf{Sq}}(\mathbf{C};\mathcal{A}) \to \underline{\mathsf{Crs}}(\mathcal{K})$, where $\underline{\mathsf{Crs}}(\mathcal{K})$ is the double category defined as follows:

- Its objects are the objects of $\mathcal{K}$.
- Its horizontal 1-cells $A \to B$ are adjunctions $f^*\colon B \rightleftarrows A : f_*$ in $\mathcal{K}$, where f_* is the right adjoint.
- Its vertical 1-cells $A \to B$ are adjunctions $f_!\colon A \rightleftarrows B : f^!$ in $\mathcal{K}$, where $f_!$ is the left adjoint.
- Its 2-cells

$$\begin{tikzcd}
A \arrow[r, shift right, "f_*"'] \arrow[d, shift left, "h_!"] \arrow[dr, phantom, "\Swarrow"] & B \arrow[l, shift right, "f^*"'] \arrow[d, shift left, "k_!"] \\
C \arrow[u, shift left, "h^!"] \arrow[r, shift right, "g_*"'] & D \arrow[l, shift right, "g^*"'] \arrow[u, shift left, "k^!"]
\end{tikzcd}$$

  are isomorphisms $h_! f^* \cong g^* k_!$. Any such isomorphism has mates $f^* k^! \to h^! g^*$, $k^! g_* \cong f_* h^!$, and $k_! f_* \to g_* h_!$; thus a cross functor has underlying functors of all four sorts considered above.

The composite $G \circ F$ of two double pseudofunctors is defined in an obvious way, with one minor wrinkle: since G need not preserve the globularity of the constraints for F, we need to compose with the unit constraints of G when defining the constraints of GF. For example, the composition constraint of GF is given by the composite

$$\begin{tikzcd}
{} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow G^{Fa}"] & {} \arrow[r, "GF(gf)"] \arrow[d] \arrow[dr, phantom, "\Swarrow G(F_{gf})"] & {} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\Swarrow G^{Fc}"] & {} \arrow[d, equal] \\
{} \arrow[r, equal] \arrow[d, equal] \arrow[drrr, phantom, "\Swarrow G_{(Fg)(Ff)}"] & {} \arrow[r] & {} \arrow[r, equal] & {} \arrow[d, equal] \\
{} \arrow[rr, "GFf"'] & & {} \arrow[r, "GFg"'] & {}
\end{tikzcd}.$$

We thereby obtain a category $\mathfrak{Dbl}$ of double categories and double pseudofunctors. The operations $\mathcal{V}$ and $\mathcal{H}$ define functors from $\mathfrak{Dbl}$ to the category 2-$\mathfrak{Cat}$ of 2-categories and pseudofunctors. In the other direction, $\underline{\mathsf{Sq}}$ defines a functor from 2-$\mathfrak{Cat}$ to $\mathfrak{Dbl}$.

The most important observation about double pseudofunctors for our purposes is that they preserve companions, conjoints, and mates.

Proposition 6.7. *If f has a horizontal companion f' in $\underline{\mathsf{K}}$ and $F\colon \underline{\mathsf{K}} \to \underline{\mathsf{L}}$ is a double pseudofunctor, then $F(f)$ has a horizontal companion $F(f')$.*

Proof. We take the defining 2-cells to be

$$\begin{tikzcd}
\cdot \arrow[r, "Ff'"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow F\varphi"] & \cdot \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\Downarrow\cong"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\cong"] & \cdot \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "="] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, equal] & \cdot
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "="] & \cdot \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\cong"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\cong"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow F\psi"] & \cdot \arrow[d, "Ff"] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, "Ff'"'] & \cdot
\end{tikzcd}.$$

Verification of the equations defining a companion pair is straightforward using the double naturality axioms (Figure 1). ☐

Proposition 6.8. *If $f \prec g$ in $\underline{\mathsf{K}}$ and $F\colon \underline{\mathsf{K}} \to \underline{\mathsf{L}}$ is a double pseudofunctor, then $F(f) \prec F(g)$.*

Proof. By duality. ☐

Proposition 6.9. *The mate correspondences* (5.15) *and* (5.21) *are preserved by double pseudofunctors.*

Proof. Straightforward verification, again using the double naturality axioms and the horizontal and vertical pseudofunctor axioms. ☐

In particular, this means that we have two additional functors $\mathcal{S}tr\colon \mathfrak{Dbl} \to 2\text{-}\mathfrak{Cat}$ and $\mathcal{C}onj\colon \mathfrak{Dbl} \to 2\text{-}\mathfrak{Cat}$. It is of interest to note that $\mathcal{S}tr$ is right adjoint to $\underline{\mathsf{Sq}}$.

Remark 6.10. Double pseudofunctors, as we have defined them, do not seem to appear in the literature on double categories. They can, however, be shown to be equivalent to the morphisms of the tricategory $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$ defined in [Ver92, §1.4], after first identifying double categories with a certain strict subclass of the *double bicategories* which form the objects of $\underline{\underline{\underline{\mathcal{H}oriz}}}_{SH}$.

7. The double pseudofunctor Ho

We are now finally ready to construct the double pseudofunctor $\mathrm{Ho}\colon \underline{\mathsf{Model}} \to \underline{\mathsf{Cat}}$. We already know that it should take a model category to its homotopy category, a left Quillen functor to its left derived functor, and a right Quillen functor to its right derived functor, so it remains only to define its action on a 2-cell

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathscr{C} \arrow[d, "k"] \\
\mathscr{B} \arrow[r, "g"'] & \mathscr{D}
\end{tikzcd}$$

in $\underline{\mathsf{Model}}$. We define the derived transformation of such an α to be the transformation

$$\begin{tikzcd}
\mathrm{Ho}(\mathscr{A}) \arrow[r, "\mathbf{R}f"] \arrow[d, "\mathbf{L}h"'] \arrow[dr, phantom, "\Downarrow\mathrm{Ho}(\alpha)"] & \mathrm{Ho}(\mathscr{C}) \arrow[d, "\mathbf{L}k"] \\
\mathrm{Ho}(\mathscr{B}) \arrow[r, "\mathbf{R}g"'] & \mathrm{Ho}(\mathscr{D})
\end{tikzcd}$$

represented by the composite of the following zigzag:

$$kQfR \xleftarrow{\sim} kQfQR \longrightarrow kfQR \xrightarrow{\alpha} ghQR \longrightarrow gRhQR \xleftarrow{\sim} gRhQ \tag{7.1}$$

in $\mathrm{Ho}(\mathscr{D})$. Note that the backwards maps are weak equivalences, hence represent isomorphisms in $\mathrm{Ho}(\mathscr{D})$, so this makes sense. (Recall our convention that the functor Q preserves fibrant objects.)

Remark 7.2. We can express this more simply as follows. Assume that $X \in \mathscr{C}$ is both cofibrant and fibrant. Then we have a commutative diagram

$$\begin{tikzcd}
& kQfQRX \arrow[r] \arrow[dl, "\sim"'] & kfQRX \arrow[r, "\alpha"] & ghQRX \arrow[r] & gRhQRX & \\
kQfRX & kQfQX \arrow[u, "\sim"'] \arrow[d, "\sim"] \arrow[r] & kfQX \arrow[u, "\sim"'] \arrow[d, "\sim"] \arrow[r, "\alpha"] & ghQX \arrow[u, "\sim"] \arrow[d, "\sim"] \arrow[rr] & & gRhQX \arrow[ul, "\sim"'] \arrow[dl, "\circledast", "\sim"'] \\
& kQfX \arrow[ul, "\circledast"', "\sim"] \arrow[r] & kfX \arrow[r, "\alpha"] & ghX \arrow[r] & gRhX &
\end{tikzcd}$$

in which the zigzag along the top represents $\mathrm{Ho}(\alpha)$ as defined above. The two weak equivalences marked $\circledast$ represent the canonical isomorphisms $\mathbf{R}f(X) \cong fX$ and $\mathbf{L}h(X) \cong hX$ when X is fibrant and cofibrant, so this diagram shows that modulo these isomorphisms, $\mathrm{Ho}(\alpha)_X$ is represented by

$$kQfX \longrightarrow kfX \xrightarrow{\alpha} ghX \longrightarrow gRhX. \tag{7.3}$$

This suffices to determine $\mathrm{Ho}(\alpha)$, since every object is isomorphic in $\mathrm{Ho}(\mathscr{C})$ to a cofibrant and fibrant one.

Remark 7.4. We could equally well choose to represent $\mathrm{Ho}(\alpha)$ by the composite

$$kQfR \xleftarrow{\sim} kQfRQ \longrightarrow kfRQ \xrightarrow{\alpha} ghRQ \longrightarrow gRhRQ \xleftarrow{\sim} gRhQ, \tag{7.5}$$

where we use instead the assumption that R preserves cofibrant objects to conclude that the backwards maps are weak equivalences. A diagram chase shows that (7.1) and (7.5) represent the same morphism in $\mathrm{Ho}(\mathscr{D})$.

Theorem 7.6. *The above constructions define a double pseudofunctor*

$$\mathrm{Ho}\colon \underline{\mathsf{Model}} \longrightarrow \underline{\mathsf{Cat}}$$

Proof. We take the constraint 2-cells to be those of the pseudofunctors $\mathbf{L}$ and $\mathbf{R}$ defined in §3. The ordinary pseudofunctor coherence axioms (Definition 6.1(v)) follow from naturality of fibrant and cofibrant replacement, just as for the ordinary pseudofunctors $\mathbf{L}$ and $\mathbf{R}$.

Proving the double-naturality axioms is basically an exercise in filling up big diagrams with lots of naturality squares, though we do have to take care that enough "interior" arrows are weak equivalences that the diagram can be chased in the homotopy category. The diagrams for (6.2) and (6.3) are shown in Figures 2 and 3, respectively. In both cases the source and target of the zigzags in question are placed in boxes to be easily visible, and all the quadrilaterals are naturality squares. In Figure 2 the two zigzags go around the top-right and the bottom-left, and the marked arrows are weak equivalences. In Figure 3 the two zigzags go across the top and the bottom, and *all* the arrows are weak equivalences. The zigzag along the bottom of Figure 3 represents the identity since a backwards-pointing arrow represents the inverse of its forward-pointing version. $\square$

$$\begin{tikzcd}
wQgRfR & wQgQRfR \arrow[l, "\sim"'] \arrow[r] & wgQRfR \arrow[r, "\beta"] & kvQRfR \arrow[r] & kRvQRfR \\
\boxed{wQgfR} \arrow[u, "\sim"] & wQgQfR \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"'] & wgQfR \arrow[r, "\beta"] \arrow[u] & kvQfR \arrow[r] \arrow[u] & kRvQfR \arrow[u, "\sim"'] \\
 & wQgQfQR \arrow[u, "\sim"] \arrow[r] \arrow[dl, "\sim"] & wgQfQR \arrow[r, "\beta"] \arrow[u] \arrow[d] & kvQfQR \arrow[r] \arrow[u] \arrow[d] & kRvQfQR \arrow[u, "\sim"'] \arrow[d] \\
wQgfQR \arrow[uu, "\sim"] \arrow[rr] & & wgfQR \arrow[r, "\beta"] & kvfQR \arrow[r] \arrow[d, "\alpha"'] & kRvfQR \arrow[d, "\alpha"] \\
 & & & khuQR \arrow[r] \arrow[d] & kRhuQR \arrow[d] \\
 & & & khRuQR \arrow[r, "\sim"] & kRhRuQR \\
 & & & khRuQ \arrow[u, "\sim"] \arrow[r, "\sim"'] & \boxed{kRhRuQ} \arrow[u, "\sim"']
\end{tikzcd}$$

FIGURE 2. Proof of (6.2) for the homotopy double pseudofunctor

$$\begin{tikzcd}
 & fQQR \arrow[r, "f\pi QR"] \arrow[dl, "fQ\pi R"'] & fQR \arrow[r, "\rho fQR"] & RfQR & \\
fQR & & & & RfQ \arrow[ul, "RfQ\rho"'] \\
\boxed{fQ} \arrow[u, "fQ\rho"] & & fQQ \arrow[ul, "fQQ\rho"] \arrow[ll, "fQ\pi"] \arrow[rr, "f\pi Q"'] & & \boxed{fQ} \arrow[u, "\rho fQ"'] \arrow[uull, "fQ\rho"] \\
 & & fQQQ \arrow[u, "fQQ\pi"'] \arrow[dl, "fQ\pi Q"'] \arrow[dr, "f\pi QQ"] & & \\
 & fQQ \arrow[uul, "fQ\pi"] \arrow[dr, "f\pi Q"'] & & fQQ \arrow[uur, "fQ\pi"'] \arrow[dl, "f\pi Q"] & \\
 & & fQ & &
\end{tikzcd}$$

FIGURE 3. Proof of (6.3) for the homotopy double pseudofunctor

We can now fulfill our promise to exhibit the preservation of adjunctions as a functoriality statement.

Corollary 7.7. *If $f\colon \mathscr{C} \rightleftarrows \mathscr{D} : g$ is a Quillen adjunction, then we have a derived adjunction $\mathbf{L}f\colon \mathrm{Ho}(\mathscr{C}) \rightleftarrows \mathrm{Ho}(\mathscr{D}) : \mathbf{R}g$.*

Proof. This follows from Theorem 7.6 and Proposition 6.8. $\square$

In fact, applying the functor $\mathcal{C}onj$ to the morphism Ho in $\mathfrak{Dbl}$, we obtain the ordinary pseudofunctor defined in [Hov99, 1.4.3].

$$\mathcal{C}onj(\mathrm{Ho})\colon \mathcal{C}onj(\underline{\mathsf{Model}}) \longrightarrow \mathcal{C}onj(\underline{\mathsf{Cat}}).$$

Considering companion pairs instead, we obtain the following dual result, which is also occasionally useful:

Corollary 7.8. *If $f\colon \mathscr{C} \to \mathscr{D}$ is both left and right Quillen (with respect to the same model structures), then $\mathbf{L}f \cong \mathbf{R}f$.* □

The full power of Theorem 7.6, however, lies in the fact that double pseudofunctors also preserve mates (Proposition 6.9). In §§9–13 we will see how to apply this fact to compare composites of left and right derived functors.

Remark 7.9. Note that the problem mentioned at the end of §3 does not arise in the double-categorical context. It is perfectly possible to have a 2-cell α in $\underline{\mathsf{Model}}$ which is a natural isomorphism, but for which $\mathrm{Ho}(\alpha)$ is *not* an isomorphism. This is because the fact that α is an isomorphism in $\mathcal{C}at$ is not visible to the double category $\underline{\mathsf{Model}}$, and hence need not be preserved by the functor Ho.

Remark 7.10. Theorem 7.6 admits various generalizations. For instance, it is shown in [Hov99] that if $\mathscr{V}$ is a monoidal model category, then the homotopy category of any $\mathscr{V}$-model category is enriched over $\mathrm{Ho}(\mathscr{V})$; in this way we can construct a double pseudofunctor from $\mathscr{V}$-model categories to $\mathrm{Ho}(\mathscr{V})$-categories. We could also lift the codomain of Ho to the double category $\underline{\mathsf{QCat}} = \underline{\mathsf{Sq}}(\mathcal{QC}at)$ of quasicategories (see [Joy02, Joy, Lur09]), where $\mathcal{QC}at$ is the 2-category of quasicategories described in [Joy]. A third generalization is described in the next section.

8. Derivable categories

For many purposes, the powerful framework of model categories and Quillen adjunctions is adequate, but there are some examples in which it is too restrictive. This includes many examples where we want to compare left and right derived functors. The problem is roughly that when dealing with derived functors, we need more flexible notions of "fibrant" and "cofibrant" objects than are supplied by a model structure.

In this section we describe an extension of the double pseudofunctor Ho from $\underline{\mathsf{Model}}$ to a larger double category $\underline{\mathsf{Drv}}$ of *derivable categories*. This generalization also serves to clarify the essential properties necessary for the definition of derived functors and the double pseudofunctor Ho.

Definition 8.1. A **derivable structure** on a category $\mathscr{C}$ consists of:

(i) a class of "weak equivalences" satisfying the 2-out-of-3 property,
(ii) full subcategories $\mathscr{C}_Q$ and $\mathscr{C}_R$,
(iii) a functor $Q\colon \mathscr{C} \to \mathscr{C}$, whose image is contained in $\mathscr{C}_Q$, and a natural weak equivalence $\pi\colon Q \xrightarrow{\sim} \mathrm{Id}_{\mathscr{C}}$, and
(iv) a functor $R\colon \mathscr{C} \to \mathscr{C}$, whose image is contained in $\mathscr{C}_R$, and a natural weak equivalence $\rho\colon \mathrm{Id}_{\mathscr{C}} \xrightarrow{\sim} R$, such that
(v) either $Q(\mathscr{C}_R) \subset \mathscr{C}_R$ or $R(\mathscr{C}_Q) \subset \mathscr{C}_Q$.

A category equipped with a derivable structure is called a **derivable category**.

In any derivable category we write $\mathscr{C}_{QR} = \mathscr{C}_Q \cap \mathscr{C}_R$. The importance of condition (v) is visible in the following lemma.

Lemma 8.2. *If $\mathscr{C}$ is a derivable category, then every object is connected by a zigzag of weak equivalences to an object in $\mathscr{C}_{QR}$.*

Proof. If $Q(\mathscr{C}_R) \subset \mathscr{C}_R$, possible zigzags are

$$X \xrightarrow[\sim]{\rho} RX \xleftarrow[\sim]{\pi R} QRX \quad \text{and} \quad X \xleftarrow[\sim]{\pi} QX \xrightarrow[\sim]{Q\rho} QRX \ .$$

Similarly, if $R(\mathscr{C}_Q) \subset \mathscr{C}_Q$, possible zigzags are

$$X \xrightarrow[\sim]{\rho} RX \xrightarrow[\sim]{R\pi} RQX \quad \text{and} \quad X \xleftarrow[\sim]{\pi} QX \xrightarrow[\sim]{\rho Q} RQX \ .$$

□

Example 8.3. Of course, any model category $\mathscr{C}$ is a derivable category if we take $\mathscr{C}_Q$ and $\mathscr{C}_R$ to be its full subcategories of cofibrant and fibrant objects, respectively. In this case both disjuncts of (v) can be satisfied at once.

Example 8.4. If $\mathscr{C}$ is any category equipped with a class of weak equivalences satisfying the 2-out-of-3 property, we can make it into a derivable category with $\mathscr{C}_Q = \mathscr{C}_R = \mathscr{C}$ and $Q = R = \mathrm{Id}$.

Example 8.5. The product of two derivable categories is derivable, with a pointwise structure. Also, the opposite of any derivable category is also derivable (simply switch $\mathscr{C}_Q$ and $\mathscr{C}_R$).

We will see other examples of derivable categories in §§9–13.

Any derivable category $\mathscr{C}$ has a homotopy category $\mathrm{Ho}(\mathscr{C})$ obtained by formally inverting its weak equivalences (though $\mathrm{Ho}(\mathscr{C})$ may not have small hom-sets without additional assumptions on $\mathscr{C}$). An equivalent homotopy category is obtained by inverting the weak equivalences in $\mathscr{C}_Q$, $\mathscr{C}_R$, or (by Lemma 8.2) $\mathscr{C}_{QR}$.

Definition 8.6. If $\mathscr{C}$ and $\mathscr{D}$ are derivable categories, a functor $f\colon \mathscr{C} \to \mathscr{D}$ is **left derivable** if

(i) it preserves weak equivalences in $\mathscr{C}_Q$ and
(ii) $f(\mathscr{C}_Q) \subset \mathscr{D}_Q$.

The dual notion is **right derivable**.

Condition (i) ensures that any left derivable $f\colon \mathscr{C} \to \mathscr{D}$ has a left derived functor $\mathbf{L}f\colon \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$, defined to be induced by the composite $f \circ Q\colon \mathscr{C} \to \mathscr{D}$. Condition (ii) ensures that the composite of two left derivable functors is again left derivable.

Example 8.7. Any left Quillen functor between model categories is left derivable. Similarly, any right Quillen functor is right derivable.

Example 8.8. If $\mathscr{C}$ and $\mathscr{D}$ are derivable categories in which $Q = R = \mathrm{Id}$, as in Example 8.4, then a functor $\mathscr{C} \to \mathscr{D}$ is left or right derivable just when it preserves all weak equivalences.

Definition 8.9. We write $\underline{\mathsf{Drv}}$ for the double category whose objects are derivable categories, whose vertical arrows are left derivable functors, whose horizontal arrows are right derivable functors, and whose 2-cells are arbitrary natural transformations.

Since every model category is a derivable category and every Quillen functor is derivable, we have a forgetful functor $\underline{\mathsf{Model}} \to \underline{\mathsf{Drv}}$.

Theorem 8.10. *There is a double pseudofunctor*

$$\mathrm{Ho}\colon \underline{\mathsf{Drv}} \to \underline{\mathsf{Cat}}$$

sending each object $\mathscr{C}$ to $\mathrm{Ho}(\mathscr{C})$, each vertical 1-cell f to $\mathbf{L}f$, each horizontal 1-cell g to $\mathbf{R}g$, and each 2-cell to a derived transformation defined as in Theorem 7.6.

Proof. This is a slight generalization of the proof of Theorem 7.6. Pseudofunctoriality in each direction follows exactly as in that case. However, we are now forced to choose between the defining composites (7.1) and (7.5) based on whether $Q(\mathscr{D}_R) \subset \mathscr{D}_R$ or $R(\mathscr{D}_Q) \subset \mathscr{D}_Q$, since we have only required one or the other to hold. Note that in either case, Lemma 8.2 enables us to use the simpler version of Remark 7.2. Additional diagram chases, which differ inconsequentially from those in Figure 2, are required to verify the double naturality axioms for the composite of two 2-cells where one uses (7.1) and the other (7.5). □

This theorem is applied in the same way as Theorem 7.6. For example, we have the following immediate corollaries.

Corollary 8.11. *If $f\colon \mathscr{C} \to \mathscr{D}$ is a functor which is both left and right derivable (relative to the same derivable structures), then $\mathbf{L}f \cong \mathbf{R}f$.*

Proof. Such functors f are precisely the strong morphisms in $\underline{\mathsf{Drv}}$. □

Corollary 8.12. *If $f \dashv g$ is an adjunction between derivable categories in which f is left derivable and g is right derivable, then we have an adjunction $\mathbf{L}f \dashv \mathbf{R}g$.*

Proof. Such an adjunction is precisely a conjunction in $\underline{\mathsf{Drv}}$. □

We call a conjunction in $\underline{\mathsf{Drv}}$ a **derivable adjunction**.

Of central importance for us, of course, is that *mates* are additionally preserved. The rest of the paper will focus on example applications of this fact.

Example 8.13. The extra generality of derivable categories and functors can be useful even when simply comparing functors of the same handedness. For instance, if $\mathscr{C}$ is a *monoidal model category*, then although its tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ does satisfy a Quillen condition of sorts, it is not a left Quillen functor and not a morphism in $\underline{\mathsf{Model}}$. However, it does preserve cofibrant objects and weak equivalences between cofibrant objects, so it is left derivable.

In this way any such $\mathscr{C}$ becomes a *pseudomonoid* in the 2-category $\mathcal{V}(\underline{\mathsf{Drv}})$. (A pseudomonoid is the 2-categorical "internalization" of a monoidal category.) Since pseudomonoids are preserved by any product-preserving pseudofunctor, it follows immediately that $\mathrm{Ho}(\mathscr{C})$ is a monoidal category for any monoidal model category $\mathscr{C}$ (and more generally, for any "monoidal derivable category" $\mathscr{C}$).

Remark 8.14. In the terminology of [DHKS04], the subcategories $\mathscr{C}_Q$ and $\mathscr{C}_R$ of a derivable category $\mathscr{C}$ are a left and right "deformation retract" of $\mathscr{C}$, respectively, and our derivable functors are a special sort of "deformable functors." The difference in viewpoint is that we consider $\mathscr{C}_Q$ and $\mathscr{C}_R$ to be given structure on the category $\mathscr{C}$, whereas [DHKS04] allows deformation retracts to vary with the functors under consideration.

Of particular note is that if $f\colon \mathscr{C} \rightleftarrows \mathscr{D} :g$ is a *deformable adjunction* in the sense of [DHKS04], then it becomes a derivable adjunction in our sense if we choose $\mathscr{C}_Q$ to be a "left f-deformation retract," $\mathscr{D}_R$ to be a "right g-deformation retract," and

$\mathscr{C}_R = \mathscr{C}$ and $\mathscr{D}_Q = \mathscr{D}$. Therefore, the results of [DHKS04, §44] on derived adjunctions of deformable adjunctions follow from our Corollary 8.12, and [DHKS04, 44.3] is then a special case of the preservation of mates by the double pseudofunctor Ho.

Part 2. Applications

9. Base change for parametrized spaces

As we saw in §2, a number of important questions can be phrased in the form “is the mate of such-and-such a transformation an isomorphism or not?” The fact that the double pseudofunctor Ho preserves mates for conjunctions gives us a structured way to attack such questions at the level of homotopy categories, by giving an explicit formula for the mate of a derived transformation $\mathrm{Ho}(\alpha)$—namely, it is the derived transformation of the mate of α. In the remainder of the paper we present several worked examples of how to apply this technique, taken both from folklore and from recent work such as [MS06, Shu08b].

We begin with a simpler version of the situation of [MS06], where we deal with *unsectioned* spaces. Let **Top** denote the category of compactly generated topological spaces. Then for any space B we have a category $\mathbf{Top}/B$, and for any continuous $f\colon A \to B$ we have an adjunction

$$f_! \colon \mathbf{Top}/A \rightleftarrows \mathbf{Top}/B \,: f^*,$$

where $f_!$ is given by composition with f, and f^* is given by pullback along f. The categories $\mathbf{Top}/B$ and functors f^* assemble into a pseudofunctor $\mathbf{Top}^{op} \to \mathcal{C}at$.

Lemma 9.1. *This pseudofunctor satisfies the Beck-Chevalley condition for any pullback square*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D. \end{tikzcd}$$

That is, for such a square, the canonical transformation $f_!h^* \to k^*g_!$ is an isomorphism.

Proof. This follows from an elementary lemma about pullback squares (and thus remains true if **Top** is replaced by any category with pullbacks). $\square$

Now each category $\mathbf{Top}/B$ inherits a model structure from the “classical” one on **Top**. The weak equivalences are weak homotopy equivalences of total spaces, and the fibrations are Serre fibrations of total spaces (so in particular, the fibrant objects are Serre fibrations over B). With these model structures, each adjunction $f_! \dashv f^*$ is Quillen, so we have derived adjunctions $\mathbf{L}f_! \dashv \mathbf{R}f^*$, and it is natural to ask whether we still have isomorphisms $\mathbf{L}f_! \circ \mathbf{R}h^* \cong \mathbf{R}k^* \circ \mathbf{L}g_!$. This is no longer true for all pullback squares (see Remark 9.7, below), but the preservation of mates by the homotopy double pseudofunctor enables us to give a sufficient condition for it to hold.

Theorem 9.2. *The derived pseudofunctor $B \mapsto \mathrm{Ho}(\mathbf{Top}/B)$ satisfies the Beck-Chevalley condition for a pullback square*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D. \end{tikzcd}$$

as long as either g or k is a (Serre) fibration.

Proof. We have to show that the mate

$$\mathbf{L}f_! \circ \mathbf{R}h^* \longrightarrow \mathbf{R}k^* \circ \mathbf{L}g_! \tag{9.3}$$

of the isomorphism

$$\mathbf{R}h^* \circ \mathbf{R}g^* \cong \mathbf{R}f^* \circ \mathbf{R}k^* \tag{9.4}$$

is itself an isomorphism. Because the double pseudofunctor Ho preserves mates, and (9.4) is the derived transformation of the isomorphism $h^*g^* \cong f^*k^*$, it follows that (9.3) is the derived transformation of the mate $f_!h^* \to k^*g_!$ (which is an isomorphism by Lemma 9.1). Therefore, by Remark 7.2, (9.3) is represented by the composite

$$f_!Qh^*X \longrightarrow f_!h^*X \xrightarrow{\cong} k^*g_!X \longrightarrow k^*Rg_!X$$

where X is fibrant and cofibrant in $\mathbf{Top}/C$, i.e. X is a cofibrant space and $X \to C$ is a fibration. We want to show that this composite is a weak equivalence. But $f_!$ preserves all weak equivalences, since it is just given by composition, so $f_!Qh^*X \longrightarrow f_!h^*X$ is always a weak equivalence. Thus, it suffices to show that $k^*g_!X \longrightarrow k^*Rg_!X$ is also a weak equivalence; here is where we will use the hypothesis on g or k.

We know that $g_!X \to Rg_!X$ is a weak equivalence, so it suffices to show that this weak equivalence is preserved by the functor k^*. If k is a fibration, then this is clear, since pullback along a fibration preserves all weak equivalences (i.e. **Top** is right proper). On the other hand, if g is a fibration, then $g_!X \to D$, being the composite $X \to C \xrightarrow{g} D$, is also a fibration, and thus $g_!X$ is fibrant in $\mathbf{Top}/D$. Therefore, since weak equivalences between fibrant objects are preserved by right Quillen functors, k^* preserves the weak equivalence $g_!X \to Rg_!X$, as desired. $\square$

Remark 9.5. The same proof applies with any model category replacing **Top**, as long as it is either right proper or we assume that the objects C and D are fibrant.

Remark 9.6. Note that in the case when k is a fibration, and hence so is h, all the functors f^*, g^*, h^*, k^*, $k_!$, and $h_!$ lie in $\mathcal{H}(\underline{\mathsf{Drv}})$. Thus, this case of Theorem 9.2 could be deduced from the ordinary pseudofunctoriality of $\mathbf{R}\colon \mathcal{H}(\underline{\mathsf{Drv}}) \to \mathcal{C}at$. However, this is no longer the case when it is g that is a fibration.

Remark 9.7. The same techniques can also be used to show that a particular square *violates* the Beck-Chevalley condition. For instance, consider the situation of [MS06, Counterexample 0.0.1], where the pullback square is

$$\begin{tikzcd} \emptyset \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \star \arrow[d, "g=1"] \\ \star \arrow[r, "k=0"'] & {[0,1].} \end{tikzcd}$$

In this case the derived Beck-Chevalley transformation is represented by the composite

$$f_! Q h^* X \longrightarrow f_! h^* X \xrightarrow{\cong} k^* g_! X \longrightarrow k^* R g_! X$$

where X is a space fibrant and cofibrant over $\star$, i.e. just a cofibrant space. Since $\mathbf{Top}/\emptyset$ is trivial, $Qh^*X \to h^*X$ is an isomorphism $\emptyset \cong \emptyset$, and thus so is $f_!Qh^*X \to f_!h^*X$. However, $k^*g_!X$ is also empty, whereas $k^*Rg_!X$ is not (unless X is itself empty); thus the composite cannot be a weak equivalence.

The situation of greater interest in [MS06] is more complicated: instead of the category $\mathbf{Top}/B$ of spaces over B, we consider the category $\mathbf{Ex}_B$ of spaces over *and* under B. An object of $\mathbf{Ex}_B$, called an *ex-space* over B, is a space X equipped with a projection $p\colon X \to B$ and a section $s\colon B \to X$ such that $ps = \mathrm{id}_B$. Once again for any $f\colon A \to B$ we have an adjunction

$$f_! \colon \mathbf{Ex}_A \rightleftarrows \mathbf{Ex}_B \; : f^*$$

where f^* is given by pullback, except that now $f_!$ is given by pushout rather than mere composition.

Remark 9.8. To ensure good behavior of pushouts, in the sectioned case we allow X to be merely a k-space, but the base spaces must still be compactly generated; see [MS06, §1.3].

Each $\mathbf{Ex}_B$ again inherits a model structure from $\mathbf{Top}$, although in [MS06] a certain "qf-model structure" is constructed with better formal behavior. For the purposes of the Beck-Chevalley condition, however, it is most convenient to give $\mathbf{Ex}_B$ the following derivable structure: we take $(\mathbf{Ex}_B)_R$ to consist of ex-spaces whose projection is a Hurewicz fibration, and $(\mathbf{Ex}_B)_Q$ to consist of ex-spaces whose section is a fiberwise closed Hurewicz cofibration. In the terminology of [MS06], $(\mathbf{Ex}_B)_R$ consists of *h-fibrant* objects, $(\mathbf{Ex}_B)_Q$ of *well-sectioned* or $\bar{f}$*-cofibrant* objects, and $(\mathbf{Ex}_B)_{QR}$ of *ex-fibrations*. In [MS06, §8.3] it is shown that there are functors Q and R making $\mathbf{Ex}_B$ into a derivable category in this way.

By [MS06, 8.2.2], each functor $f_!$ preserves well-sectioned ex-spaces, and by [MS06, 7.3.4] it preserves weak equivalences between well-sectioned ex-spaces; thus it is left derivable. On the other hand, each functor f^* certainly preserves Hurewicz fibrations and weak equivalences between them; hence it is right derivable. It follows that each adjunction $f_! \dashv f^*$ is a derivable adjunction (i.e. a conjunction in $\underline{\mathsf{Drv}}$).

We now upgrade the proof of the Beck-Chevalley condition for ex-spaces given in [MS06, 9.4.6], making the use of Theorem 8.10 explicit and thus showing that the isomorphism constructed is, in fact, the canonical Beck-Chevalley transformation.

Theorem 9.9. *The derived pseudofunctor $B \mapsto \mathrm{Ho}(\mathbf{Ex}_B)$ satisfies the Beck-Chevalley condition for a pullback square*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D. \end{tikzcd}$$

as long as either g or k is a Serre fibration.

Proof. As in the proof of Theorem 9.2, we must show that the composite

$$f_! Q h^* X \longrightarrow f_! h^* X \xrightarrow{\cong} k^* g_! X \longrightarrow k^* R g_! X$$

is a weak equivalence, where X is an ex-fibration. Now by [MS06, 8.2.2], h^* preserves ex-fibrations, so in particular h^*X is well-sectioned. Thus $Qh^*X \to h^*X$ is a weak equivalence between well-sectioned ex-spaces, and so it is preserved by $f_!$. It remains to show that the weak equivalence $g_!X \to Rg_!X$ is preserved by k^*, and as before, this is evident if k is itself a fibration. If instead g is a fibration, we factor k as a homotopy equivalence followed by a Hurewicz fibration and consider the two cases separately. The second case we have already dealt with, whereas if k is a homotopy equivalence, then since g is a Serre fibration, h is also a homotopy equivalence. Hence by [MS06, 7.3.4], the adjunctions $\mathbf{L}h_! \dashv \mathbf{R}h^*$ and $\mathbf{L}k_! \dashv \mathbf{R}k^*$ are adjoint equivalences, and so in the composite

$$\mathbf{L}f_!\mathbf{R}h^* \xrightarrow{\eta \mathbf{L}f_!\mathbf{R}h^*} \mathbf{R}k^*\mathbf{L}k_!\mathbf{L}f_!\mathbf{R}h^* \cong \mathbf{R}k^*\mathbf{L}g_!\mathbf{L}h_!\mathbf{R}h^* \xrightarrow{\mathbf{R}k^*\mathbf{L}g_!\varepsilon} \mathbf{R}k^*\mathbf{L}g_! \tag{9.10}$$

both η and ε are isomorphisms. (Note that this composite is not the composite we have taken to *define* the Beck-Chevalley map; that would be

$$\mathbf{L}f_!\mathbf{R}h^* \xrightarrow{\mathbf{L}f_!\mathbf{R}h^*\eta} \mathbf{L}f_!\mathbf{R}h^*\mathbf{R}g^*\mathbf{L}g_! \cong \mathbf{L}f_!\mathbf{R}f^*\mathbf{R}k^*\mathbf{L}g_! \xrightarrow{\varepsilon\mathbf{R}k^*\mathbf{L}g_!} \mathbf{R}k^*\mathbf{L}g_!. \tag{9.11}$$

However, they are equal, because the isomorphism $\mathbf{L}k_!\mathbf{L}f_! \cong \mathbf{L}g_!\mathbf{L}h_!$ occurring in (9.10) is the mate of the isomorphism $\mathbf{R}h^*\mathbf{R}g^* \cong \mathbf{R}f^*\mathbf{R}k^*$ occurring in (9.11).) □

Analogous proofs apply to the study of the category $\mathbf{Sp}_B$ of *ex-spectra* over B; we leave the rephrasing of these to the interested reader. Since the derived versions of $f_!$ and f_* for ex-spectra are parametrized versions of homology and cohomology, these compatibility relations imply important calculational results.

10. Base change for sheaves

The examples in the previous section concerned *spaces over spaces* as one way to to do homotopy theory over a base space. Another widespread type of homotopy theory over a base space studies *sheaves* of various sorts. There are many different types of sheaves, of course, but almost all of them eventually require comparisons of left and right derived functors. For simplicity, we will consider only the category of sheaves of abelian groups on a topological space A, which we denote $\mathbf{Sh}(A)$. We leave it to the reader to apply the same language to sheaves on ringed spaces or topoi, quasicoherent sheaves, simplicial sheaves, sheaves of spectra, and so on.

The most noticeable difference between all sheaf-theoretic contexts and that of spaces over spaces is that for a map $f\colon A \to B$ of base spaces, the pullback functor

$f^*\colon \mathbf{Sh}(B) \to \mathbf{Sh}(A)$ of sheaves always has a well-behaved *right* adjoint f_*, rather than a left adjoint $f_!$. Furthermore, the Beck-Chevalley condition for these adjoints does not hold for all pullback squares even on the point-set level: given a pullback square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D. \end{tikzcd} \tag{10.1}$$

the Beck-Chevalley transformation $k^*g_* \to f_*h^*$ is only an isomorphism under additional hypotheses. Probably the most well-known and useful result along these lines is the following, which is a special case of the *Proper Base Change Theorem*.

Lemma 10.2 ([KS90, 2.5.11])**.** *If g (and hence also f) is a proper map in* (10.1) *and all spaces involved are locally compact Hausdorff, then the Beck-Chevalley transformation $k^*g_* \to f_*h^*$ for sheaves is an isomorphism.*

Since the derived version of f_* gives the sheaf-theoretic approach to cohomology, it is again of importance when and whether this isomorphism is preserved by passage to homotopy categories. This is, of course, a special case of the derived version of the Proper Base Change Theorem; a very classical argument can be found (for instance) in [KS90, 2.6.7]. Just as for spaces over spaces, the preservation of mates by passage to derived functors is an implicit ingredient in any proof of this result. Here we sketch one such proof, making this dependence explicit.

We write $\mathbf{Ch}^+(A)$ for the category of bounded below cochain complexes of sheaves of abelian groups on A. This category has a model structure in which the weak equivalences are the quasi-isomorphisms (homology isomorphisms), every object is cofibrant, and the fibrant objects are the complexes of injectives. Each continuous map $f\colon A \to B$ induces an adjunction $f^*\colon \mathbf{Ch}^+(B) \rightleftarrows \mathbf{Ch}^+(A) : f_*$ which is Quillen with respect to these model structures, so we have a derived adjunction $\mathbf{L}f^* \dashv \mathbf{R}f_*$. (In fact, f^* is exact and hence preserves all weak equivalences, so one usually writes simply f^* instead of $\mathbf{L}f^*$.)

Theorem 10.3. *If g in* (10.1) *is proper and all spaces involved are locally compact Hausdorff, then the derived Beck-Chevalley transformation*

$$\mathbf{L}k^* \circ \mathbf{R}g_* \longrightarrow \mathbf{R}f_* \circ \mathbf{L}h^*$$

is an isomorphism.

Proof. The model structures mentioned above are sufficient for defining the derived functors, but for this proof we need to give $\mathbf{Ch}^+(A)$ a different derivable structure. Recall that a sheaf X on a space A is *c-soft* if sections of X over compact subsets of A can be extended to all of A, i.e. if $\Gamma(A, X) \to \Gamma(K, X)$ is surjective for all compact $K \subset A$. Given a continuous map $f\colon A \to B$, a sheaf X on A is said to be *f-soft* if its restriction to every fiber of f is c-soft. We say that a bounded below complex of sheaves is c-soft or f-soft if it consists of c-soft or f-soft sheaves. Since every injective sheaf is c-soft and every c-soft sheaf is f-soft, every bounded below complex of sheaves has a c-soft or f-soft resolution.

In particular, $\mathbf{Ch}^+(B)$ is a derivable category with $\mathbf{Ch}^+(B)_Q = \mathbf{Ch}^+(B)$ and $\mathbf{Ch}^+(B)_R$ the full subcategory of g-soft complexes, and likewise $\mathbf{Ch}^+(A)$ is a derivable category with $\mathbf{Ch}^+(A)_Q = \mathbf{Ch}^+(A)$ and $\mathbf{Ch}^+(A)_R$ the full subcategory of f-soft complexes. We consider $\mathbf{Ch}^+(C)$ as a derivable category with $\mathbf{Ch}^+(C)_Q = \mathbf{Ch}^+(C)_R = \mathbf{Ch}^+(C)$, and likewise for $\mathbf{Ch}^+(D)$. In all cases, the weak equivalences are the quasi-isomorphisms.

Now g_* preserves weak equivalences between g-soft complexes, and likewise for f_* and f-soft complexes, so g_* and f_* are right derivable functors. Since the pullback functors f^*, g^*, h^*, k^* preserve all weak equivalences, they are left derivable, so the mate correspondence in question takes place in $\underline{\mathsf{Drv}}$. Therefore, the derived Beck-Chevalley transformation is the derived natural transformation of the point-set-level transformation $k^*g_* \to f_*h^*$, and thus is represented by the explicit composite

$$k^*Qg_*X \longrightarrow k^*g_*X \xrightarrow{\cong} f_*h^*X \longrightarrow f_*Rh^*X$$

where $X \in \mathbf{Ch}^+(C)_{QR}$, i.e. X is a g-soft complex of sheaves on C. However, Q is the identity functor, and the point-set transformation $k^*g_* \to f_*h^*$ is an isomorphism by Lemma 10.2, so it remains to show that $f_*h^*X \to f_*Rh^*X$ is a weak equivalence. This follows from the observation that h^* takes g-soft sheaves to f-soft ones, which is true since the fibers of f are the same as the fibers of g, and h^* doesn't change the restrictions of X to fibers. □

Remark 10.4. The category $\mathbf{Ch}(A)$ of *unbounded* chain complexes of sheaves also admits a model structure in which the weak equivalences are the quasi-isomorphisms, every object is cofibrant, and the fibrant objects are the "dg-injective" or "K-injective" complexes; see [Joy84, Spa88, Bek00, Gil07]. (This is true with any Grothendieck abelian category replacing $\mathbf{Sh}(A)$.) However, it seems that Theorem 10.3 fails for unbounded complexes; see [Lur09, 6.5.4.3], where it is claimed that the solution is to use a more restrictive notion of weak equivalence than quasi-isomorphism.

The full version of the Proper Base Change Theorem involves defining a new "direct image with proper supports" functor $f_!$ equipped with a map $f_! \to f_*$ (which is an isomorphism when f is proper), and showing that the restricted transformation $\mathbf{L}k^* \circ \mathbf{R}g_! \longrightarrow \mathbf{R}f_! \circ \mathbf{L}h^*$ is an isomorphism whether or not f is proper. (This $f_!$ is not a left adjoint of f^*, but it serves a similar function to the $f_!$ for parametrized spaces and spectra. Whereas the $f_!$ for parametrized spectra is a version of homology, this $f_!$ is a version of compactly supported cohomology, which plays a similar role in duality theory.) While this transformation is no longer itself defined directly as a mate, mate correspondences are still essential for comparing various different transformations relating these functors; see [FHM03].

11. Comparing homotopy theories

One of the most important uses of model category theory is that it provides a structured way to compare different "models" of the same or similar homotopy theories. The strongest such comparison is, of course, a Quillen equivalence. For example, there are many different model categories of spectra, all of which are connected by a web of Quillen equivalences (see [MMSS01, MM02]); thus one can work with whichever category of spectra is most appropriate for a particular application. However, in situations such as those considered in the previous two sections, we often have not just two homotopy theories, but two "families" of homotopy theories, each consisting of many model categories related by base change functors. Thus, we naturally want to compare not only the model categories themselves in the two theories, but the base change functors as well.

For instance, one might hope that each ordinary model category of spectra (or at least some of them) would have a parametrized version $\mathbf{Sp}_B$ over any base space, with base change adjunctions $f_! \dashv f^* \dashv f_*$ induced by any continuous map $f\colon A \to B$. Given some other family of categories of parametrized spectra—denoted $\mathbf{Sp}'_B$, say—one would like to know not only that there are Quillen equivalences $\mathbf{Sp}_B \rightleftarrows \mathbf{Sp}'_B$, but that these equivalences are compatible with the base change adjunctions, and thus in particular compute the same notions of homology and cohomology.

In fact, only parametrized orthogonal spectra are studied at any length in [MS06], and there are certain difficulties involved in extending other models of spectra to the parametrized context (see [MS06, Chapter 24]). To get across the ideas involved in such comparisons, therefore, we will consider a less hypothetical, and also much less complicated, example of the same phenomenon: comparing different model structures for parametrized spaces.

In §9 we considered the model structure on $\mathbf{Top}/B$ induced from the "Quillen" or q-model structure on $\mathbf{Top}$, which is constructed from weak homotopy equivalences, Serre fibrations, and relative cell complexes. However, there are also other model structures on $\mathbf{Top}$ which induce model structures on $\mathbf{Top}/B$. In [Str72] it is shown that there is a "Hurewicz" or h-model structure constructed from homotopy equivalences, Hurewicz fibrations, and Hurewicz cofibrations (see also [SV02], [MS06, Ch. 4], and [Col06a]). And in [Col06b] it is shown that there is also a "mixed" or m-model structure constructed from weak homotopy equivalences, Hurewicz fibrations, and maps of the homotopy type of relative cell complexes. We have a pair of Quillen adjunctions

$$\mathbf{Top}^q \underset{J_q^m}{\overset{I_q^m}{\rightleftarrows}} \mathbf{Top}^m \underset{J_m^h}{\overset{I_m^h}{\rightleftarrows}} \mathbf{Top}^h,$$

where all the functors are the identity functor. Each of these model structures lifts to $\mathbf{Top}/B$, and we have analogous Quillen adjunctions:

$$(\mathbf{Top}/B)^q \underset{J_q^m}{\overset{I_q^m}{\rightleftarrows}} (\mathbf{Top}/B)^m \underset{J_m^h}{\overset{I_m^h}{\rightleftarrows}} (\mathbf{Top}/B)^h. \tag{11.1}$$

In both cases the first adjunction, but not the second, is a Quillen equivalence. (The second is a *left Quillen embedding* in the sense of [Shu08b], i.e. the derived left adjoint $\mathrm{Ho}(\mathbf{Top}^m) \to \mathrm{Ho}(\mathbf{Top}^h)$ is full and faithful.)

Now the adjunctions $f_!\colon \mathbf{Top}/A \rightleftarrows \mathbf{Top}/B : f^*$ are Quillen relative to all three model structures, and the important questions regard the commutativity of squares such as

$$\begin{tikzcd}
(\mathbf{Top}/B)^m \arrow[r, "f^*"] \arrow[d, "J_q^m"'] \arrow[dr, phantom, "?"] & (\mathbf{Top}/A)^m \arrow[d, "J_q^m"] \\
(\mathbf{Top}/B)^q \arrow[r, "f^*"'] & (\mathbf{Top}/A)^q
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
(\mathbf{Top}/A)^m \arrow[r, "f_!"] \arrow[d, "J_q^m"'] \arrow[dr, phantom, "?"] & (\mathbf{Top}/B)^m \arrow[d, "J_q^m"] \\
(\mathbf{Top}/A)^q \arrow[r, "f_!"'] & (\mathbf{Top}/B)^q,
\end{tikzcd}$$

particularly after passage to homotopy categories. In general, the point-set level question may already be interesting, but in our toy example it is trivial: since all the functors in (11.1) are the identity functor, all such squares obviously commute on the point-set level.

Moreover, the commutativity of some of these squares at the homotopy category level is also easy: since f^*, J_q^m, and J_m^h are right Quillen, ordinary pseudofunctoriality of $\mathbf{R}\colon \mathcal{M}odel_R \to \mathcal{C}at$ gives us isomorphisms

$$\mathbf{R}^q f^* \circ \mathbf{R}J_q^m \cong \mathbf{R}J_q^m \circ \mathbf{R}^m f^* \qquad \text{and} \tag{11.2}$$

$$\mathbf{R}^m f^* \circ \mathbf{R}J_m^h \cong \mathbf{R}J_m^h \circ \mathbf{R}^h f^* \tag{11.3}$$

(where we decorate $\mathbf{R}$ to indicate which model structure we are taking the derived functor with respect to). Likewise, the left derived functors of $f_!$ commute with those of I_q^m and I_m^h:

$$\mathbf{L}^m f_! \circ \mathbf{L}I_q^m \cong \mathbf{L}I_q^m \circ \mathbf{L}^q f_! \qquad \text{and} \tag{11.4}$$

$$\mathbf{L}^h f_! \circ \mathbf{L}I_m^h \cong \mathbf{L}I_m^h \circ \mathbf{L}^m f_!. \tag{11.5}$$

Note that these isomorphisms are actually mates of the previous ones. We also have four additional mates relating composites of left and right derived functors:

$$\mathbf{L}^q f_! \circ \mathbf{R}J_q^m \longrightarrow \mathbf{R}J_q^m \circ \mathbf{L}^m f_! \tag{11.6}$$

$$\mathbf{L}^m f_! \circ \mathbf{R}J_m^h \longrightarrow \mathbf{R}J_m^h \circ \mathbf{L}^h f_! \tag{11.7}$$

$$\mathbf{L}I_q^m \circ \mathbf{R}^q f^* \longrightarrow \mathbf{R}^m f^* \circ \mathbf{L}I_q^m \qquad \text{and} \tag{11.8}$$

$$\mathbf{L}I_m^h \circ \mathbf{R}^m f^* \longrightarrow \mathbf{R}^h f^* \circ \mathbf{L}I_m^h. \tag{11.9}$$

(Each of these can actually be constructed in two different ways, which give the same result by Lemma 2.12.) Now since $I_q^m \dashv J_q^m$ is a Quillen *equivalence*, its derived adjunction $\mathbf{L}I_q^m \dashv \mathbf{R}J_q^m$ is an adjoint equivalence, so Lemma 2.2 implies that (11.6) and (11.8) are also isomorphisms. Thus, it remains to consider (11.7) and (11.9). (Of course, the h- and q-model structures can be compared by simply composing the other two comparisons; double pseudofunctoriality shows that this gives the same result as a direct treatment using $I_q^h = I_m^h \circ I_q^m$ and $J_q^h = J_q^m \circ J_m^h$.)

Remark 11.10. The situation with the q and m model structures is quite common, since the most desirable (and very frequently occurring) comparison between homotopy theories is a Quillen equivalence. Thus, Lemma 2.2 means that in most cases no fancy technology is required to compare functors. However, comparisons which are not equivalences do occur. The h versus m/q model structures is a fairly trivial example, but we will mention a more contentful one below.

Proposition 11.11. *The derived transformation* (11.7) *is an isomorphism.*

Proof. By double pseudofunctoriality, (11.7) is represented by the composite

$$f_!(Q^m X) \longrightarrow f_! X \longrightarrow R^h(f_! X)$$

where X is h-fibrant and h-cofibrant over A, i.e. X is an h-cofibrant space (a vacuous condition) and $X \to A$ is a Hurewicz fibration. Since $f_!$ preserves all weak equivalences of any sort, the first map in this composite is a weak homotopy equivalence (i.e. a weak equivalence in the mixed model structure). And the second map is by definition a homotopy equivalence, and therefore also a weak homotopy equivalence. $\square$

On the other hand, (11.9) is *not*, in general, an isomorphism. Double pseudo-functoriality tells us that it can be represented by the composite

$$Q^m(f^*X) \longrightarrow f^*X \longrightarrow f^*(R^hX)$$

where X is m-fibrant and m-cofibrant over B, i.e. X is m-cofibrant (of the homotopy type of a CW complex) and $X \to B$ is an m-fibration. Since m-fibrations are the same as Hurewicz fibrations, X is already h-fibrant; thus $X \to R^hX$ is an h-equivalence between h-fibrant objects and so is preserved by f^*. However, if f^*X is not m-cofibrant, then the *weak* homotopy equivalence $Q^m(f^*X) \to f^*X$ will not be an h-equivalence (otherwise, f^*X would be homotopy equivalent to the m-cofibrant $Q^m(f^*X)$, hence m-cofibrant itself). For example, we could take $B = X = \star$ to be the one-point space and $A = f^*X$ to be any non-m-cofibrant space. Thus, in such a case the above composite is not an h-equivalence, and so does not represent an isomorphism in $\mathrm{Ho}((\mathbf{Top}/A)^h)$.

Remark 11.12. The situation for ex-spaces is basically the same; we have the same set of Quillen adjunctions, and the same formal arguments apply to six out of the eight possible comparisons. The analogue of (11.7) is also an isomorphism, although the proof requires an invocation of the Gluing Lemma, while the analogue of (11.9) is not an isomorphism. We leave the details to the reader.

There are also interesting model structures on $\mathbf{Top}/B$ that are not inherited from model structures on $\mathbf{Top}$. One such is the "qf-model structure" of [MS06, Ch. 6]. However, since the identity functor is a Quillen equivalence between the qf-model structure and the q-model structure, all compatibility relations follow directly as before.

A more interesting question concerns the model structure on $\mathbf{Top}/B$ constructed in [IJ02], which in [Shu08b] we called the ij-model structure. In this model structure the cofibrations are built out of cells of the form $U \times D^n$, where U is any open subset of B. Thus, fibrations and weak equivalences are detected by "spaces of sections" over open sets; precise definitions can be found in [IJ02]. Since this model structure has a very sheaf-theoretic feel, it is unsurprising that it is Quillen equivalent to a model category of simplicial presheaves on B. Furthermore, the pullback functor $f^*\colon \mathbf{Top}/B \to \mathbf{Top}/A$ is *left* Quillen for the ij-model structures, just as it is for sheaves, and because we have a Quillen equivalence, the same arguments as before show that the derived adjunctions $\mathbf{L}f^* \dashv \mathbf{R}f_*$ agree for the ij-model structure and for simplicial presheaves.

The relationship of the ij-model structure to the other model structures on $\mathbf{Top}/B$ is more subtle. We showed in [Shu08b] that if B is a locally compact CW complex, then the identity adjunction of $\mathbf{Top}/B$ is a Quillen adjunction

$$\iota^\star\colon (\mathbf{Top}/B)^{ij} \rightleftarrows (\mathbf{Top}/B)^m \colon \iota_\star,$$

but not a Quillen equivalence (though it is a *right Quillen embedding*, i.e. the derived right adjoint is full and faithful). We showed moreover that for any map $f\colon A \to B$ between locally compact CW complexes, when the identity $f^* = f^*$ is considered as a square

$$\begin{tikzcd}
(\mathbf{Top}/B)^m \arrow[r, "\iota_\star"] \arrow[d, equal] \arrow[dr, phantom, "\not\Downarrow"] & (\mathbf{Top}/B)^{ij} \arrow[d, "f^*"] \\
(\mathbf{Top}/B)^m \arrow[r, "\iota_\star \circ f^*"'] & (\mathbf{Top}/A)^{ij}
\end{tikzcd}$$

in $\underline{\mathsf{Model}}$, its derived natural transformation is an isomorphism

$$\mathbf{L}^{ij} f^* \circ \mathbf{R}\iota_\star \cong \mathbf{R}\iota_\star \circ \mathbf{R}^m f^*.$$

The proof involves a careful analysis of the explicit formula for a derived natural transformation. We also gave an analysis of the mate of this isomorphism:

$$\mathbf{L}\iota^\star \circ \mathbf{L}^{ij} f^* \longrightarrow \mathbf{R}^m f^* \circ \mathbf{L}\iota^\star$$

which turns out to be an isomorphism when f is a q-fibration, but not much more generally. By Lemma 2.1, if $\mathbf{R}^m f^*$ happens to have a right adjoint $\mathbf{M}f_*$ (in [MS06] such a right adjoint is constructed for connected ex-spaces or spectra using Brown representability), the induced transformation

$$\mathbf{R}\iota_\star \circ \mathbf{M}f_* \longrightarrow \mathbf{R}^{ij} f_* \circ \mathbf{R}\iota_\star$$

is also an isomorphism whenever f is a q-fibration. This implies that parametrized generalized cohomology can be computed via passage to the ij-model structure, one of the main points of [Shu08b]. (In fact, this application was the original motivation for the theory of the present paper.)

12. The projection formula for sheaves

We now consider monoidal structures and monoidal functors, as in Example 2.7, beginning this time with sheaves. The usual "injective" model structure on chain complexes of sheaves is not a monoidal model structure, and there is no analogue for sheaves of the "projective" model structure for modules over a ring (which is monoidal), but the remedy for this is well-known: we consider *flat* resolutions rather than projective ones.

It is proven in [Gil06, Gil07] that the category $\mathbf{Ch}(A)$ of unbounded chain complexes of sheaves admits a *flat model structure* which is monoidal; its cofibrant objects are *dg-flat* complexes. (For our purposes, all we actually need is that all complexes admit dg-flat resolutions, which was proven by [Spa88].) Thus $\mathrm{Ho}(\mathbf{Ch}(A))$ is symmetric monoidal; its tensor product $X \otimes^{\mathbf{L}} Y$ is represented by $QX \otimes QY$, where Q denotes a dg-flat resolution. In fact, tensoring with a dg-flat complex preserves all weak equivalences, so $X \otimes^{\mathbf{L}} Y$ can equally be represented by $QX \otimes Y$ or $X \otimes QY$.

Now let $f : A \to B$ be a continuous map, which as always induces an adjunction $f^* : \mathbf{Ch}(B) \rightleftarrows \mathbf{Ch}(A) : f_*$. This adjunction is derivable for the flat model structures, and moreover f^* is strong monoidal. Thus, by pseudofunctoriality in $\mathcal{V}(\underline{\mathsf{Drv}})$, its left derived functor $\mathbf{L}f^*$ is again strong monoidal.

We now regard the isomorphism $f^*(X \otimes Y) \cong f^*X \otimes f^*Y$, for fixed X, as a transformation

$$\begin{tikzcd} \mathbf{Ch}(B) \arrow[r, "f^*"] \arrow[d, "X\otimes -"'] \arrow[dr, phantom, "\swArrow"] & \mathbf{Ch}(A) \arrow[d, "f^*X\otimes -"] \\ \mathbf{Ch}(B) \arrow[r, "f^*"'] & \mathbf{Ch}(A). \end{tikzcd} \tag{12.1}$$

Thus, under the adjunction $f^* \dashv f_*$, it has a mate

$$\begin{tikzcd} \mathbf{Ch}(A) \arrow[r, "f_*"] \arrow[d, "f^*X\otimes -"'] \arrow[dr, phantom, "\swArrow"] & \mathbf{Ch}(B) \arrow[d, "X\otimes -"] \\ \mathbf{Ch}(A) \arrow[r, "f_*"'] & \mathbf{Ch}(B). \end{tikzcd} \tag{12.2}$$

The following is part of the (point-set level) *projection formula*, and can be found in [KS90, 2.5.13].

Lemma 12.3. *If $f\colon A \to B$ is a proper map of locally compact Hausdorff spaces, then the component of (12.2) at any flat sheaf X is an isomorphism*

$$X \otimes f_*Y \cong f_*(f^*X \otimes Y).$$

Of course, generally of more interest is the derived projection formula. We follow our usual procedure of upgrading the standard proof (in this case, taken from [KS90, 2.6.6]) to use the derived mate correspondence and show that the isomorphism constructed is, in fact, the canonical comparison map.

Theorem 12.4. *If $f\colon A \to B$ is a proper map of locally compact Hausdorff spaces, then for any bounded below complexes X and Y of sheaves on A and B, respectively, the isomorphism*

$$\mathbf{L}f^*(X \otimes^{\mathbf{L}} Y) \cong \mathbf{L}f^*X \otimes^{\mathbf{L}} \mathbf{L}f^*Y \tag{12.5}$$

(which exhibits $\mathbf{L}f^*$ as strong monoidal) has a mate

$$X \otimes^{\mathbf{L}} \mathbf{R}f_*Y \longrightarrow \mathbf{R}f_*(\mathbf{L}f^*X \otimes^{\mathbf{L}} Y) \tag{12.6}$$

which is an isomorphism.

Proof. We may assume without loss of generality that X is dg-flat, and hence so is f^*X (which is isomorphic to $\mathbf{L}f^*X$, since f^* is exact). Since tensoring with a dg-flat complex preserves all weak equivalences, $(X \otimes -)$ and $(f^*X \otimes -)$ are left derivable when $\mathbf{Ch}(A)$ and $\mathbf{Ch}(B)$ are equipped with their *injective* model structures (in which everything is cofibrant and the fibrant objects are the dg-injective complexes). With these model structures, (12.2) becomes a 2-cell in $\underline{\mathsf{Drv}}$

as drawn, while (12.1) is a 2-cell in $\underline{\mathsf{Drv}}$ where all the arrows involved are vertical:

$$\begin{tikzcd}
\mathbf{Ch}(B) \arrow[r, equal] \arrow[d, "X\otimes -"'] & \mathbf{Ch}(B) \arrow[d, "f^*"] \\
\mathbf{Ch}(B) \arrow[d, "f^*"'] \arrow[r, phantom, "\swarrow"] & \mathbf{Ch}(A) \arrow[d, "f^*X\otimes -"] \\
\mathbf{Ch}(A) \arrow[r, equal] & \mathbf{Ch}(A).
\end{tikzcd}$$

Thus, since (12.5) is the derived transformation of (12.1), its mate (12.6) is in fact the derived natural transformation of (12.2), and is therefore represented by the composite

$$X \otimes Qf_*Y \longrightarrow X \otimes f_*Y \xrightarrow{\cong} f_*(f^*X \otimes Y) \longrightarrow f_*R(f^*X \otimes Y)$$

where Y is dg-injective. Since we have assumed that X and Y are bounded below, we may assume that in fact Y is a complex of injective sheaves. Since $(X \otimes -)$ preserves all weak equivalences, the first questionable map $X \otimes Qf_*Y \longrightarrow X \otimes f_*Y$ is always a weak equivalence. Now since f^*X is flat and Y is injective (hence c-soft), $(f^*X \otimes Y)$ is f-soft (this follows from [KS90, 2.5.12]). But as we saw in §10, f_* preserves weak equivalences between f-soft complexes, so the second questionable map $f_*(f^*X \otimes Y) \longrightarrow f_*R(f^*X \otimes Y)$ is also a weak equivalence. $\square$

As with the proper base change theorem, the full version of the projection formula applies to all maps f, but replaces f_* by the "direct image with proper supports" functor $f_!$.

13. The projection formula for parametrized spaces

Finally, we consider the version of the projection formula proven in [MS06] for parametrized spaces. In this case there is no known *monoidal* model structure on $\mathbf{Top}/B$ that is equivalent to the q-model structure, but we can still make do with a derivable tensor product functor. Recall that the cartesian monoidal structure on $\mathbf{Top}/B$ is given by pullback: $X \otimes Y = X \times_B Y$. Since pullbacks along fibrations preserve weak equivalences, the model category $(\mathbf{Top}/B)^q$ is a pseudomonoid in $\mathcal{H}(\underline{\mathsf{Drv}})$. (Recall from Example 8.13 that a monoidal model category is, in particular, a pseudomonoid in $\mathcal{V}(\underline{\mathsf{Drv}})$.) Therefore, $\mathrm{Ho}(\mathbf{Top}/B)$ (we drop the superscript q from now on) is a monoidal category; its monoidal structure is represented by $X \times_B^{\mathbf{R}} Y = RX \times_B RY$, where R denotes replacement by a (q-)fibration. Moreover, for any map $f\colon A \to B$, the functor f^* is strong monoidal and right derivable; thus $\mathbf{R}f^*$ is again strong monoidal by ordinary pseudofunctoriality.

Now the isomorphism $f^*X \times_A f^*Y \cong f^*(X \times_B Y)$ making f^* strong monoidal can be viewed as a transformation

$$\begin{tikzcd}
\mathbf{Top}/B \arrow[r, "X\times_B -"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\nearrow"] & \mathbf{Top}/B \arrow[d, "f^*"] \\
\mathbf{Top}/A \arrow[r, "f^*X\times_A -"'] & \mathbf{Top}/A
\end{tikzcd} \tag{13.1}$$

which therefore has a mate under the adjunction $f_! \dashv f^*$:

$$\begin{tikzcd} \mathbf{Top}/A \arrow[r, "f^*X\times_A -"] \arrow[d, "f_!"'] \arrow[dr, phantom, "\swarrow"] & \mathbf{Top}/A \arrow[d, "f_!"] \\ \mathbf{Top}/B \arrow[r, "X\times_B -"'] & \mathbf{Top}/B \end{tikzcd} \tag{13.2}$$

whose components are transformations

$$f_!(f^*X \times_A Y) \longrightarrow X \times_B f_!Y. \tag{13.3}$$

Lemma 13.4. *The transformation* (13.3) *is always an isomorphism.*

Proof. This follows again from elementary facts about pullback squares (and hence is true in any category with pullbacks). □

Once again, we are interested in the derived version.

Theorem 13.5. *The isomorphism*

$$\mathbf{R}f^*X \times_A^{\mathbf{R}} \mathbf{R}f^*Y \cong \mathbf{R}f^*(X \times_B^{\mathbf{R}} Y) \tag{13.6}$$

(which exhibits $\mathbf{R}f^*$ as strong monoidal) has a mate

$$\mathbf{L}f_!(\mathbf{R}f^*X \times_A^{\mathbf{R}} Y) \longrightarrow X \times_B^{\mathbf{R}} \mathbf{L}f_!Y \tag{13.7}$$

which is an isomorphism.

Proof. Fix a fibrant X, so that $X\times_B -$ and $f^*X\times_A -$ are right derivable. Then (13.2) is a 2-cell in $\underline{\mathsf{Drv}}$ as drawn, whereas (13.1) is a 2-cell in $\underline{\mathsf{Drv}}$ with all arrows horizontal:

$$\begin{tikzcd} \mathbf{Top}/B \arrow[r, "f^*"] \arrow[d, equal] & \mathbf{Top}/A \arrow[r, "f^*X\times_A -"] \arrow[d, phantom, "\Downarrow"] & \mathbf{Top}/A \arrow[d, equal] \\ \mathbf{Top}/B \arrow[r, "X\times_B -"'] & \mathbf{Top}/B \arrow[r, "f^*"'] & \mathbf{Top}/A \end{tikzcd}$$

Thus, since (13.6) is the derived transformation of (13.1), it follows that its mate (13.7) is the derived transformation of (13.2), and thus can be represented by the composite

$$f_!Q(f^*X \times_A Y) \longrightarrow f_!(f^*X \times_A Y) \xrightarrow{\cong} X \times_B f_!Y \longrightarrow X \times_B R(f_!Y)$$

where Y is fibrant and cofibrant over A. Now as in §9, $f_!$ preserves all weak equivalences, so the first map $f_!Q(f^*X\times_A Y) \to f_!(f^*X\times_A Y)$ is a weak equivalence. Likewise, since X is fibrant, $X \times_B -$ actually preserves all weak equivalences (that is, **Top** is right proper); thus the second map $X \times_B f_!Y \to X \times_B R(f_!Y)$ is also a weak equivalence. □

The projection formulas for sectioned spaces is more subtle, since in this case the monoidal structure is itself a composite of left and right derived functors. This is the only one of our examples in which double pseudofunctoriality does not solve all our problems, since we end up having to compose transformations in a way that is not possible in the double category $\underline{\mathsf{Drv}}$. Nevertheless, double pseudofunctoriality does simplify the problem from a composite of four transformations to a composite of two, and the rest of the argument is completed by a diagram chase. (This diagram chase is exactly the sort of argument that would have to be made explicit in all the other examples, if we didn't have double pseudofunctoriality to invoke. In other words, Figures 2 and 3 in the proof of Theorem 7.6 have the effect of chasing all such diagrams once and for all.)

Before defining the fiberwise smash product, we first define the **external smash product** of ex-spaces $X \in \mathbf{Ex}_A$ and $Y \in \mathbf{Ex}_B$ by the following pushout:

$$\begin{tikzcd}
(X \times B) \sqcup_{A \times B} (A \times Y) \arrow[r] \arrow[d] & X \times Y \arrow[d] \\
A \times B \arrow[r] & X \barwedge Y
\end{tikzcd}$$

This produces an ex-space $X \barwedge Y \in \mathbf{Ex}_{A \times B}$, and thereby defines a functor

$$\barwedge \colon \mathbf{Ex}_A \times \mathbf{Ex}_B \longrightarrow \mathbf{Ex}_{A \times B}.$$

Note that the fiber of $X \barwedge Y$ over $(a, b) \in A \times B$ is $X_a \wedge Y_b$. The external smash product is coherently associative, unital, and symmetric, and furthermore we have isomorphisms

$$f^* X \barwedge g^* Y \cong (f \times g)^* (X \barwedge Y) \tag{13.8}$$

satisfying their own coherence conditions. In the terminology of [Shu08a], this structure makes $\mathbf{Ex}_{(-)}$ into a *monoidal fibration* over **Top**.

We then define the **fiberwise smash product** of $X, Y \in \mathbf{Ex}_A$ to be

$$X \wedge_A Y = \Delta_A^* (X \barwedge Y),$$

where $\Delta_A \colon A \to A \times A$ is the diagonal map. The isomorphisms (13.8), together with the coherence isomorphisms of $\barwedge$, can be used to construct coherence isomorphisms making $\wedge_A$ a symmetric monoidal structure on $\mathbf{Ex}_A$. Furthermore, we have isomorphisms

$$\begin{aligned} f^*(X \wedge_B Y) &= f^* \Delta_B^* (X \barwedge Y) \\ &\cong \Delta_A^* (f \times f)^* (X \barwedge Y) \\ &\cong \Delta_A^* (f^* X \barwedge f^* Y) \\ &= f^* X \wedge_A f^* Y \end{aligned} \tag{13.9}$$

making each functor $f^* \colon \mathbf{Ex}_B \to \mathbf{Ex}_A$ strong symmetric monoidal.

Remark 13.10. In [MS06] this construction is performed in the other direction, starting from the fiberwise smash product and producing the external one. See [Shu08a] for a full equivalence of the two notions.

We will use the same derivable structure for ex-spaces as in §9, where $(\mathbf{Ex}_A)_{QR}$ consists of ex-fibrations. Recall that in this case each adjunction $f_! \dashv f^*$ is derivable, and in particular f^* is right derivable. On the other hand, it is not hard to see, using the "gluing lemma," that the functor $\barwedge$ is *left* derivable. Thus, $\wedge_A$ is a composite of a left and a right derivable functor, and so $\mathbf{Ex}_A$ is not even a pseudomonoid in $\mathcal{V}(\underline{\mathsf{Drv}})$ or $\mathcal{H}(\underline{\mathsf{Drv}})$. We can, however, produce a symmetric monoidal structure on $\mathrm{Ho}(\mathbf{Ex}_A)$ in the same way that we did on the point-set level, provided that the isomorphisms (13.8) descend to derived functors. This is precisely the argument used in [MS06, §9.4].

Lemma 13.11. *When* (13.8) *is regarded as a 2-cell*

$$\begin{tikzcd} \mathbf{Ex}_C \times \mathbf{Ex}_D \arrow[r, "f^* \times g^*"] \arrow[d, "\barwedge"'] & \mathbf{Ex}_A \times \mathbf{Ex}_B \arrow[d, "\barwedge"] \arrow[dl, phantom, "\swArrow"] \\ \mathbf{Ex}_{C\times D} \arrow[r, "(f\times g)^*"'] & \mathbf{Ex}_{A\times B} \end{tikzcd}$$

in $\underline{\mathsf{Drv}}$*, its derived natural transformation is an isomorphism.*

Proof. Its derived natural transformation is represented by the composite

$$Qf^*X \barwedge Qg^*Y \longrightarrow f^*X \barwedge g^*Y \xrightarrow{\cong} (f\times g)^*(X \barwedge Y) \longrightarrow (f\times g)^* R(X \barwedge Y).$$

where X and Y are ex-fibrations over A and B, respectively. But by [MS06, §8.2], the functors f^*, g^*, and $\barwedge$ all preserve ex-fibrations, so both questionable maps in this composite are weak equivalences. ☐

In particular, we have isomorphisms

$$(\mathbf{R}f^*X) \barwedge^{\mathbf{L}} (\mathbf{R}g^*Y) \cong \mathbf{R}(f\times g)^* \left(X \barwedge^{\mathbf{L}} Y\right). \tag{13.12}$$

The associativity, unit, and symmetry isomorphisms for $\barwedge$ descend to $\barwedge^{\mathbf{L}}$ (by ordinary pseudofunctoriality), as do the compatibility axioms between these and (13.8) (by double pseudofunctoriality). Therefore, we can define the "middle derived fiberwise smash product"

$$X \wedge_A^{\mathbf{M}} Y = \mathbf{R}\Delta_A^* \left(X \barwedge^{\mathbf{L}} Y\right)$$

and show:

Theorem 13.13. *Each* $\wedge^{\mathbf{M}}$ *is a symmetric monoidal structure on* $\mathrm{Ho}(\mathbf{Ex}_A)$*, and each functor* $\mathbf{R}f^*$ *is strong symmetric monoidal.*

Now we consider the projection formula. As in the unsectioned case, on the point-set level we have a canonical morphism

$$f_!(f^*X \wedge_A Y) \longrightarrow X \wedge_B f_!Y \tag{13.14}$$

defined as the mate of $f^*X \wedge_A f^*Y \cong f^*(X \wedge_B Y)$, and once again we have:

Lemma 13.15. *The transformation* (13.14) *is always an isomorphism.*

Proof. This is a straightforward computation with limits and colimits, using only the fact that colimits are preserved by pullback in our convenient category of spaces (see Remark 9.8). ☐

In order to pass to derived functors, we need to rephrase this in terms of the external smash product. The morphism (13.14) becomes

$$f_!\Delta_A^*(f^*X \barwedge Y) \longrightarrow \Delta_B^*(X \barwedge f_!Y). \tag{13.16}$$

The definition of (13.14) as a mate of the isomorphism (13.9) then becomes the following definition of (13.16) in terms of (13.8):

$$\begin{tikzcd}
\mathbf{Ex}_A \arrow[r, equal] \arrow[d, "f_!"'] \arrow[dr, phantom, "\Swarrow"] & \mathbf{Ex}_A \arrow[d, equal] & \\
\mathbf{Ex}_B \arrow[r, "f^*"] \arrow[d, "X\barwedge -"'] \arrow[dr, phantom, "\Swarrow"] & \mathbf{Ex}_A \arrow[d, "f^*X\barwedge -"] & \\
\mathbf{Ex}_{B\times B} \arrow[r, "(f\times f)^*"'] \arrow[d, equal] & \mathbf{Ex}_{A\times A} \arrow[r, "\Delta_A^*"'] \arrow[d, phantom, "\Swarrow\cong"] & \mathbf{Ex}_A \arrow[d, equal] \\
\mathbf{Ex}_{B\times B} \arrow[r, "\Delta_B^*"'] & \mathbf{Ex}_B \arrow[r, "f^*"] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow"] & \mathbf{Ex}_A \arrow[d, "f_!"] \\
& \mathbf{Ex}_B \arrow[r, equal] & \mathbf{Ex}_B
\end{tikzcd}$$

If we choose X to be an ex-fibration (in which case f^*X is also), then all the 2-cells in this diagram live in $\underline{\mathsf{Drv}}$ as drawn. However, this diagram cannot be *composed* in $\underline{\mathsf{Drv}}$, so a single application of functoriality is insufficient. Instead, we can break it into two composites, each of which can be composed in $\underline{\mathsf{Drv}}$. (In fact, each is the construction of a mate.)

(13.17)
$$\begin{tikzcd}
\mathbf{Ex}_A \arrow[r, equal] \arrow[d, "f_!"'] \arrow[dr, phantom, "\Swarrow"] & \mathbf{Ex}_A \arrow[d, equal] \\
\mathbf{Ex}_B \arrow[r, "f^*"] \arrow[d, "X\barwedge -"'] \arrow[dr, phantom, "\Swarrow"] & \mathbf{Ex}_A \arrow[d, "f^*X\barwedge -"] \\
\mathbf{Ex}_{B\times B} \arrow[r, "(f\times f)^*"'] & \mathbf{Ex}_{A\times A}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{Ex}_{B\times B} \arrow[r, "(f\times f)^*"] \arrow[d, equal] & \mathbf{Ex}_{A\times A} \arrow[r, "\Delta_A^*"] \arrow[d, phantom, "\Swarrow\cong"] & \mathbf{Ex}_A \arrow[d, equal] \\
\mathbf{Ex}_{B\times B} \arrow[r, "\Delta_B^*"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathbf{Ex}_B \arrow[r, "f^*"] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow"] & \mathbf{Ex}_A \arrow[d, "f_!"] \\
\mathbf{Ex}_{B\times B} \arrow[r, "\Delta_B"'] & \mathbf{Ex}_B \arrow[r, equal] & \mathbf{Ex}_B
\end{tikzcd}$$

Now consider the transformation

$$\mathbf{L}f_!\Big(\mathbf{R}\Delta_A^*\Big(\mathbf{R}f^*X \mathbin{\barwedge^{\mathbf{L}}} Y\Big)\Big) \longrightarrow \mathbf{R}\Delta_B^*\Big(X \mathbin{\barwedge^{\mathbf{L}}} \mathbf{L}f_!Y\Big), \tag{13.18}$$

defined from (13.12) in the same way that (13.16) is defined from (13.8). For the same reasons, it is the composite of the morphisms corresponding to (13.17) at the level of homotopy categories. But, by pseudofunctoriality, the derived version of each of these is, in fact, the derived natural transformation of the corresponding point-set-level transformation (although this doesn't apply to the entire composite, which can't be composed in $\underline{\mathsf{Drv}}$). Thus, (13.18) is equal to the composite

$$\begin{aligned}
\mathbf{L}f_!\left(\mathbf{R}\Delta_A^*\left(\mathbf{R}f^*X \mathbin{\barwedge^{\mathbf{L}}} Y\right)\right) &\longrightarrow \mathbf{L}f_!\left(\mathbf{R}\Delta_A^*\left(\mathbf{R}\,(f\times f)^*\left(X \mathbin{\barwedge^{\mathbf{L}}} \mathbf{L}f_!Y\right)\right)\right)\\
&\longrightarrow \mathbf{R}\Delta_B^*\left(X \mathbin{\barwedge^{\mathbf{L}}} \mathbf{L}f_!Y\right).
\end{aligned}$$

Here the first map is $\mathbf{L}f_! \circ \mathbf{R}\Delta_A^*$ applied to the derived transformation of

$$f^*X \barwedge Y \longrightarrow (f\times f)^*(X \barwedge f_!Y)$$

and the second is the component at $X \overline{\wedge}^{\mathbf{L}} f_!Y$ of the derived transformation of

$$f_!\Delta_A^*(f \times f)^* \longrightarrow \Delta_B^*.$$

Filling in the definition of derived natural transformations, along with a pseudo-functoriality constraint for the composite $\Delta_A^*(f \times f)^*$, we conclude that (13.18) is represented by the zigzag along the top-right of the diagram in Figure 4, where Y is an ex-fibration.

To show that this composite represents an isomorphism in $\mathrm{Ho}(\mathbf{Ex}_B)$, we begin by "following our nose," filling in naturality properties and definitions, arriving at the zigzag along the bottom-left of Figure 4. (The region marked $\circledast$ is the definition of (13.16); all others are naturality squares.)

$$\begin{tikzcd}
\boxed{f_!Q\Delta_A^*R(f^*X\overline{\wedge}Y)} \arrow[r] & f_!Q\Delta_A^*R(f\times f)^*(X\overline{\wedge}f_!Y) \arrow[r] & f_!Q\Delta_A^*R(f\times f)^*R(X\overline{\wedge}f_!Y) & \\
f_!Q\Delta_A^*(f^*X\overline{\wedge}Y) \arrow[u, "\sim"'] \arrow[d, "\sim"'] \arrow[r] & f_!Q\Delta_A^*(f\times f)^*(X\overline{\wedge}f_!Y) \arrow[u] \arrow[d] \arrow[r] & f_!Q\Delta_A^*(f\times f)^*R(X\overline{\wedge}f_!Y) \arrow[u, "\sim"] \arrow[d] \arrow[r, "\cong"] & f_!Qf^*\Delta_B^*R(X\overline{\wedge}f_!Y) \arrow[dd] \\
f_!\Delta_A^*(f^*X\overline{\wedge}Y) \arrow[dd, "\cong"'] \arrow[r] & f_!\Delta_A^*(f\times f)^*(X\overline{\wedge}f_!Y) \arrow[d, "\cong"] \arrow[r] \arrow[dl, phantom, "\circledast"] & f_!\Delta_A^*(f\times f)^*R(X\overline{\wedge}f_!Y) \arrow[dr, "\cong"] & \\
& f_!f^*\Delta_B^*(X\overline{\wedge}f_!Y) \arrow[rr] \arrow[dl] & & \Delta_B^*R(X\overline{\wedge}Qf_!Y) \arrow[d] \\
\Delta_B^*(X\overline{\wedge}f_!Y) \arrow[rrr] & & & \boxed{\Delta_B^*R(X\overline{\wedge}f_!Y)}
\end{tikzcd}$$

FIGURE 4. The diagram chase for the projection formula

Thus it suffices to check that the maps along the bottom-left are weak equivalences. For the first two, this is because $(f^*X\overline{\wedge}-)$ and Δ_B^* preserve ex-fibrations, and the third is the isomorphism (13.16). We deal with the fourth by replacing $f_!Y$ with an ex-fibration as follows:

$$\begin{tikzcd}
\Delta_B^*(X\overline{\wedge}f_!Y) \arrow[r] & \Delta_B^*R(X\overline{\wedge}f_!Y) \\
\Delta_B^*(X\overline{\wedge}Qf_!Y) \arrow[u, "\sim"] \arrow[d, "\sim"'] \arrow[r] & \Delta_B^*R(X\overline{\wedge}Qf_!Y) \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
\Delta_B^*(X\overline{\wedge}QRf_!Y) \arrow[r, "\sim"'] & \Delta_B^*R(X\overline{\wedge}QRf_!Y)
\end{tikzcd} \tag{13.19}$$

Here the two left-hand vertical maps are weak equivalences because when X is an ex-fibration, $(X \wedge_A -) = \Delta_A^*(X\overline{\wedge}-)$ preserves h-equivalences between well-sectioned ex-spaces (see [MS06, 8.2.6]). The two right-hand vertical maps are weak equivalences because $(X\overline{\wedge}-)$ is left derivable and $f_!Y$, $Qf_!Y$, and $QRf_!Y$ are well-sectioned. Finally, the bottom horizontal map is a weak equivalence because $QRf_!Y$ is an ex-fibration, hence so is $X\overline{\wedge}QRf_!Y$. Thus, we have the projection formula for ex-spaces:

Theorem 13.20. *For any map $f\colon A \to B$, the natural isomorphism*

$$\mathbf{R}f^*\Big(X \wedge_B^{\mathbf{M}} Y\Big) \cong \Big(\mathbf{R}f^*X \wedge_A^{\mathbf{M}} \mathbf{R}f^*Y\Big) \tag{13.21}$$

(which exhibits $\mathbf{R}f^*$ as strong monoidal) has a mate

$$\mathbf{L}f_!\Big(\mathbf{R}f^*X \wedge_A^{\mathbf{M}} Y\Big) \longrightarrow \Big(X \wedge_B^{\mathbf{M}} \mathbf{L}f_!Y\Big) \tag{13.22}$$

which is also an isomorphism. □

Remark 13.23. The isomorphism given in the proof of the projection formula in [MS06, 9.4.5] is, essentially, the composite of the two maps along the left of (13.19):

$$\Delta_B^*(X \,\overline{\wedge}\, f_!Y) \xleftarrow{\sim} \Delta_B^*(X \,\overline{\wedge}\, Qf_!Y) \xrightarrow{\sim} \Delta_B^*(X \,\overline{\wedge}\, QRf_!Y).$$

The other weak equivalences in (13.19) and Figure 4 are implicitly present in the identification of the source and target of this zigzag as representing the source and target of (13.22). As always, the contribution of our theory is to identify this with the canonical comparison map, i.e. the mate of (13.21).

Remark 13.24. In fact, $\mathbf{Ex}_B$ is actually a *closed* monoidal category, and it is shown in [MS06, §9.3] (using Brown representability) that the subcategory of $\mathrm{Ho}(\mathbf{Ex}_B)$ consisting of connected spaces is also closed monoidal. Now Lemma 2.12 implies that for a functor between closed monoidal categories, the projection formula morphism is the same as the map (2.11) constructed via the closed structure. As remarked in Example 2.7, it then follows from Lemma 2.1 and the projection formula that f^* and $\mathbf{R}f^*$ are *closed* monoidal functors (the latter only insofar as $\mathrm{Ho}(\mathbf{Ex}_B)$ is closed). The analogous facts for spectra are true without any connectivity hypothesis.

These results, which play an important role in [MS06], seem impossible to approach without the technology of mates, since the internal-homs in $\mathrm{Ho}(\mathbf{Ex}_B)$ are so inexplicit. Note in particular that for this argument to work, it is essential that the isomorphism in the projection formula is not just any isomorphism, but the particular map defined as a mate of the derived transformation (13.21).

References

[Bek00] Tibor Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.*, 129(3):447–475, 2000. 31

[BKP89] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59(1):1–41, 1989. 12

[BM99] Ronald Brown and Ghafar H. Mosa. Double categories, 2-categories, thin structures and connections. *Theory Appl. Categ.*, 5:No. 7, 163–175 (electronic), 1999. 16

[BS76] Ronald Brown and Christopher B. Spencer. Double groupoids and crossed modules. *Cahiers Topologie Géom. Différentielle*, 17(4):343–362, 1976. 16

[Col06a] Michael Cole. Many homotopy categories are homotopy categories. *Topology Appl.*, 153(7):1084–1099, 2006. 32

[Col06b] Michael Cole. Mixing model structures. *Topology Appl.*, 153(7):1016–1032, 2006. 32

[Del] P. Deligne. Voevodsky's lectures on cross functors, fall 2001. Accessed July 2009 at http://www.math.ias.edu/~vladimir/1999_2001_seminar/delnotes01.ps. 19

[DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. *Homotopy Limit Functors on Model Categories and Homotopical Categories*, volume 113 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2004. 4, 25, 26

[DPP10] R. Dawson, R. Pare, and D. A. Pronk. The span construction. *Theory Appl. Categ.*, 24:302–377 (electronic), 2010. 12

[Ehr63] Charles Ehresmann. Catégories structurées. *Ann. Sci. École Norm. Sup. (3)*, 80:349–426, 1963. 9

[FHM03] H. Fausk, P. Hu, and J. P. May. Isomorphisms between left and right adjoints. *Theory Appl. Categ.*, 11:No. 4, 107–131 (electronic), 2003. 6, 31

[Fio07] Thomas M. Fiore. Pseudo algebras and pseudo double categories. *J. Homotopy Relat. Struct.*, 2(2):119–170, 2007, arXiv:math.CT/0608760. 16

[Gar09] Richard Garner. Understanding the small object argument. *Appl. Categ. Structures*, 17(3):247–285, 2009. arXiv:0712.0724. 12

[Gil06] James Gillespie. The flat model structure on complexes of sheaves. *Trans. Amer. Math. Soc.*, 358(7):2855–2874 (electronic), 2006. 35

[Gil07] James Gillespie. Kaplansky classes and derived categories. *Math. Z.*, 257(4):811–843, 2007. 31, 35

[GP99] Marco Grandis and Robert Pare. Limits in double categories. *Cahiers Topologie Géom. Différentielle Catég.*, XL(3):162–220, 1999. 16

[GP04] Marco Grandis and Robert Pare. Adjoints for double categories. *Cah. Topol. Géom. Différ. Catég.*, 45(3):193–240, 2004. 12

[GT06] Marco Grandis and Walter Tholen. Natural weak factorization systems. *Arch. Math. (Brno)*, 42(4):397–408, 2006. 12

[Hov99] Mark Hovey. *Model Categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1999. 4, 7, 8, 18, 23

[IJ02] Michele Intermont and Mark W. Johnson. Model structures on the category of ex-spaces. *Topology Appl.*, 119(3):325–353, 2002. 34

[Joy] A. Joyal. The theory of quasi-categories. In preparation. 23

[Joy84] A. Joyal. Letter to A. Grothendieck. 1984. 31

[Joy02] A. Joyal. Quasi-categories and Kan complexes. *Journal of Pure and Applied Algebra*, 175:207–222, 2002. 23

[Kel74] G. M. Kelly. Doctrinal adjunction. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 257–280. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974. 13, 16

[KS74] G. M. Kelly and Ross Street. Review of the elements of 2-categories. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, volume 420 of *Lecture Notes in Math.*, pages 75–103. Springer, Berlin, 1974. 4, 9

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1990. With a chapter in French by Christian Houzel. 30, 36, 37

[Lur09] Jacob Lurie. *Higher topos theory*. Number 170 in Annals of Mathematics Studies. Princeton University Press, 2009, arXiv:math.CT/0608040. 23, 31

[MM02] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002. 31

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc. (3)*, 82(2):441–512, 2001. 31

[MS06] J. P. May and J. Sigurdsson. *Parametrized Homotopy Theory*, volume 132 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., 2006, arXiv:math.AT/0411656. 2, 5, 7, 9, 26, 28, 29, 32, 34, 35, 37, 39, 40, 42, 43

[Rie09] Emily Riehl. Natural weak factorization systems in model structures. October 2009, arXiv:0910.2733. 12

[Shu08a] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20(18):650–738 (electronic), 2008, arXiv:0706.1286. 7, 16, 39

[Shu08b] Michael A. Shulman. Parametrized spaces model locally constant homotopy sheaves. *Topology and its Applications*, 155(5):412–432, 2008, arXiv:0706.2874. 26, 32, 34, 35

[Spa88] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988. 31, 35

[Str72] Arne Strøm. The homotopy category is a homotopy category. *Arch. Math. (Basel)*, 23:435–441, 1972. 32

[SV02] R. Schwänzl and R. M. Vogt. Strong cofibrations and fibrations in enriched categories. *Arch. Math. (Basel)*, 79(6):449–462, 2002. 32

[Ver92] Dominic Verity. *Enriched categories, internal categories, and change of base*. PhD thesis, University of Cambridge, April 1992. Macquarie Mathematics Report No. 93-123. 20

E-mail address: `mshulman@ucsd.edu`