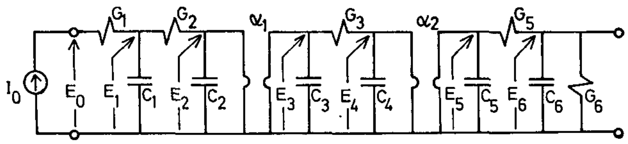

Fig. 3. Two-gyrator sixth-degree $RC\Gamma$ low-pass filter.

IV. An Example

Consider the network given in Fig. 3. This particular configuration is used to realize voltage transfer functions with passive components. The node-admittance matrix can be written as the following form [10], [11]:

$$\begin{bmatrix} I_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} G_1 & -G_1 & & & & & \\ -G_1 & C_1s + G_1 + G_2 & -G_2 & & & & \\ & -G_2 & C_2s + G_2 & \alpha_1 & & & \\ & & -\alpha_1 & C_3s + G_3 & -G_3 & & \\ & & & -G_3 & C_4s + G_3 & \alpha_2 & \\ & & & & -\alpha_2 & C_5s + G_5 & -G_5 \\ & & & & & -G_5 & C_6s + G_5 + G_6 \end{bmatrix} \begin{bmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix}. \tag{15}$$

Note that the determinant of the coefficient matrix in (15) is not a simple continuant. The driving-point impedance is

$$Z_d = \frac{K(2, 7)}{K(1, 7)}. \tag{16}$$

Suppose that the sensitivity of Z_d with respect to α_1 is desired. By observing (15) we have

$$b_3 = \alpha_1 \quad \text{and} \quad c_3 = -\alpha_1$$

$$\frac{\partial Z_d}{\partial \alpha_1} = \frac{\partial Z_d}{\partial b_3}\frac{\partial b_3}{\partial \alpha_1} + \frac{\partial Z_d}{\partial c_3}\frac{\partial c_3}{\partial \alpha_1}. \tag{17}$$

From (3a) and (4a)

$$\frac{\partial Z_d}{\partial \alpha_1} = 2\alpha_1 \frac{b_{1,2}c_{1,2}K(4, 7)K(5, 7)}{K^2(1, 7)}. \tag{18}$$

In order to compute (18), only $K(1, 7)$ has to be evaluated. The continuants $K(4, 7)$ and $K(5, 7)$ can be obtained in the intermediate steps of the algorithm.

V. Conclusion

It is shown that a well-known technique for the formulation of ladder networks can be extended to the sensitivity analysis of a wider class of networks with their network functions having the property of being expressible as the ratio of two continuants. An efficient use of the algorithm developed will not usually require the computation of an extra continuant other than the one which is required anyhow for the evaluation of the network function under consideration.

Acknowledgment

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Planarization by Transformation

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***Abstract*—A new approach to star–polygon transformations is introduced. For star-connected *R*-networks and for a certain class of polygon-connected *R*-networks the "terminal value" is defined. This concept allows for an elegant derivation of the necessary and sufficient conditions for the existence of a polygon-into-star transformation. The related formulas appear to be simple and easy to remember. More important, the terminal value turns out to be a useful tool in the search for planar equivalents and in the recognition of transformable polygon structures. Simplicity is the only reason for the restriction to resistance networks. The study of the general *RLC* case does not bring out any new aspects.**

I. Introduction

We start with establishing what we mean by a planar n-terminal network, since our definition is not the same as the graph-theoretical one. We call an n-terminal network planar if its graph is planar and remains planar if we add one extra vertex and connect every "terminal vertex" with it. Imagine, for example, a network of which the graph is complete with four vertices. This graph is planar from the graph-theoretical point of view. Yet we don't call this network planar, because the graph obtained by connecting every terminal with the new vertex is one of the basic nonplanar graphs of Kuratowski [7].

II. Star-Polygon Transformation

The best-known network transformation generating a planar network from a nonplanar one is the polygon-into-star transformation. By a polygon we mean a network that has a complete graph, while a star is a network consisting of n conductances connected to a common internal node, which is not available as a terminal. We number the n terminals going round the periphery of the network. The external behavior of an n-terminal network can be described by an $n \times n$ conductance matrix (which is always singular):

$$\begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ i_n \end{bmatrix} = \begin{bmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ k_{n,1} & k_{n,2} & & k_{n,n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \quad \text{or} \quad I = KV \tag{1}$$

where i_p is the current entering the network via terminal p and v_q is the voltage between terminal q and some arbitrary, but well-defined external reference.

Two networks are called equivalent if and only if their K-matrices are identical. From Kirchhoff's current and voltage laws, we know that

$$\sum_{i=1}^{n} k_{i,j} = 0 \quad \text{and} \quad \sum_{i=1}^{n} k_{j,i} = 0, \quad \text{for all } j. \tag{2}$$

In a star we denote the conductance between the inner node and terminal q by G_q and define the auxiliary entity G by

$$G = \sum_{i=1}^{n} G_i. \tag{3}$$

In a polygon we denote the conductance between terminal p and terminal q by $G_{p,q}$. Expressing the K-matrix entries of a star and a polygon in terms of their conductances, we get

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$$\left.\begin{aligned} k_{i,j} &= -\frac{G_iG_j}{G}, && \text{for } i \neq j \\ k_{i,j} &= G_i - \frac{G_i^2}{G}, && \text{for } i = j \end{aligned}\right\}\text{for a star} \tag{4}$$

and

$$\left.\begin{aligned} k_{i,j} &= -G_{i,j}, && \text{for } i \neq j \\ k_{i,j} &= \sum_{\substack{p=1 \\ p\neq i}}^{n} G_{i,p}, && \text{for } i = j \end{aligned}\right\}\text{for a polygon.} \tag{5}$$

It is easy to see in the case of the polygon that the correspondence between the network and its K-matrix is one to one.

Equating the entries of the K-matrices (4) and (5) we immediately find that the quantity $f(i)$, defined as

$$f(i) = \frac{G_{i,x}G_{i,y}}{G_{x,y}} \tag{6}$$

is independent of the choice of x and y (of course $x \neq i \neq y$ and $x \neq y$). This is a necessary condition for the transformability of a polygon. A polygon with this property will be referred to as a T-polygon. By writing $f(i)$ in terms of matrix entries we obtain

$$f(i) = -\frac{k_{i,x}k_{i,y}}{k_{x,y}} = \frac{(G_iG_x/G)\cdot(G_iG_y/G)}{G_xG_y/G} = \frac{G_i^2}{G}. \tag{7}$$

Note that for a T-polygon,

$$G_{i,j} = \frac{G_{j,x}}{G_{i,x}}f(i) = \frac{G_{i,x}}{G_{j,x}}f(j) \rightarrow G_{ij}^2 = f(i)f(j) = k_{i,j}^2 \tag{8}$$

and for a star

$$f(i)f(j) = \frac{G_i^2}{G}\cdot\frac{G_j^2}{G} = \left\{\frac{G_iG_j}{G}\right\}^2 = k_{i,j}^2. \tag{9}$$

We therefore define for each terminal of this network the terminal value as follows.

T-polygon:

$$\phi_i = \frac{\sqrt{G_{i,x}}\sqrt{G_{i,y}}}{\sqrt{G_{x,y}}}. \tag{10}$$

Star:

$$\phi_i = \frac{G_i}{\sqrt{G}}. \tag{11}$$

The matrix entries of *both* kinds of networks can be expressed in terms of terminal values:

$$\begin{aligned} k_{i,j} &= -\phi_i\phi_j, && \text{for } i \neq j \\ k_{i,j} &= \phi_i \sum_{\substack{p=1 \\ p\neq i}}^{n} \phi_p, && \text{for } i = j. \end{aligned} \tag{12}$$

From (10)–(12) we come to the conclusion that every star is transformable into a T-polygon, and conversely that every T-polygon is transformable into a star. It is easy to show now that the conditions (6) are also sufficient.

Suppose there is a polygon that doesn't satisfy (6), but has an equivalent star. Since every star can be tranformed into an equivalent T-polygon, it is possible to give a T-polygon equivalent to the original polygon which was not a T-polygon. This is contradictory to the one-to-one correspondence between polygons and their K-matrices.

We already mentioned that we permit negative conductance values (just as Klein and Bedrosian did before [1], [2]). However, some precautions have to be taken then. The square roots may become imaginary:

$$\sqrt{G_{i,j}} = i\sqrt{-G_{i,j}}. \tag{13}$$

However, in the formulas defining the elements of the new network, the power of i will always be even, thus yielding real entries.

Further, the case $G=0$ may occur. In this case the K-matrix will not be properly defined. Transformation is impossible in such a situation. The same degeneration for polygons occurs, if $\Sigma\phi = 0$. Thus in order for the polygon to be nondegenerate, the following condition must be satisfied:

$$\sum_{\substack{i=1 \\ i\neq x}}^{n} \frac{G_{i,p}}{G_{x,p}} \neq -1. \tag{14}$$

We now state the following (generalized) main theorem.

1) *Every star with $G \neq 0$ has an equivalent T-polygon with*

$$G_{i,j} = \phi_i\phi_j = \frac{G_iG_j}{G}. \tag{15}$$

2) *A polygon satisfying (14) has an equivalent star, if and only if it is a T-polygon. For this star we have*

$$G_i = \phi_i \sum_{p=1}^{n} \phi_p = \sum_{\substack{j=1 \\ j\neq i}}^{n} G_{i,j} + \phi_i^2. \tag{16}$$

III. Planar Equivalents

For some reasons it can be desirable to have a planar equivalent for a certain nonplanar network (e.g., construction of duals). For the complete 4-gon [1] and for the complete 5-gon [2] the problem is already solved. We have found planar networks for the complete 6-gon and 7-gon. The resulting structure for the 6-gon is canonical. This means that the number of elements is minimal [8]. The network equivalent to the 7-gon contains 28 conductances instead of the expected number of 21 [9]. A noncanonical solution for the 6-gon was known to us from private communication with S. Tirtoprodjo.

Our procedure for searching planar equivalents consists of repeatedly splitting networks into T-polygons and residual networks. Then the T-polygons obtained from the first step are transformed into

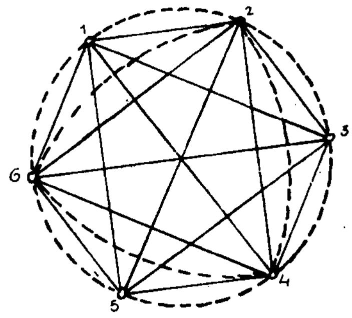

Fig. 1. Splitting of the 6-gon in a T-6-gon (full lines) and residual conductances (dotted lines).

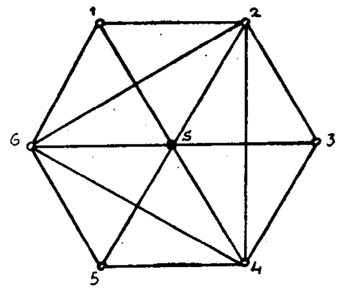

Fig. 2. The network after transformation of the T-6-gon.

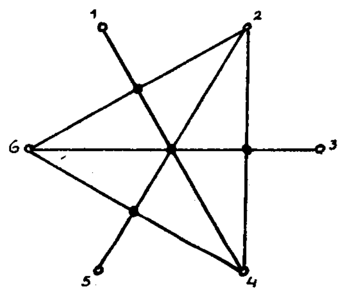

Fig. 3. The planar equivalent of a 6-gon.

stars. The conditions for degeneration must always be checked. As an example the case of the 6-gon is presented here.

We start by computing the "terminal values" of the T-polygon we want to split off. For this purpose we choose the following conductances: $G_{1,3}$, $G_{1,5}$, $G_{3,5}$, $G_{1,4}$, $G_{3,6}$, and $G_{2,5}$. Then we may write consecutively

$$\phi_1 = \frac{\sqrt{G_{1,3}}\sqrt{G_{1,5}}}{\sqrt{G_{3,5}}} \qquad \phi_i = \frac{G_{i,q}}{G_{p,q}}\phi_p, \qquad i = 3, 5, 4, 6, 2 \tag{17}$$

where p and q should be chosen in such a way that the conductances are in the set given above and that ϕ_p has already been calculated in another step. In order to obtain a network equivalent to the original 6-gon we construct a network with the configuration of Fig. 1. The full lines in this figure represent conductances $\phi_i\phi_j$, while the dashed lines represent conductances $G_{i,j}-\phi_i\phi_j$. The polygon given by the full lines now is a T-polygon, which can be transformed into a star (Fig. 2). Then the network consists of three 4-gons, which can be transformed according to [1], [8] into planar networks. The final result is depicted in Fig. 3.

IV. Conclusion

Introducing terminal values has permitted an easier proof of the main theorem as the existing methods [3]–[6]. At the same time practical formulas for the star–polygon transformation are obtained. Another advantage is in Section III, where the planarization of the 6-gon is demonstrated. This way of searching planar equivalents is not restricted to polygons. Both elementary Kuratowski graphs [7], for example, can be transformed into planar networks. Yet to conclude from the famous theorem is this paper [7] that every RLC network has a planar equivalent appears to be wrong after careful examination, but the restrictions are not very strong.

In applications, for example in energy distributions, it often occurs that the conductance values in a polygon are not all different. Let us assume an n-gon with only two different conductance values G_1 and G_2. With the concept of terminal values in mind we know that this polygon is transformable into a star if and only if all the $(n-1)$ conductances being incident to some node have value G_1 and all the others have value G_2. This structure is easy to recognize in a polygon network. It is easy to find other applications of the concept of terminal values that verify the usefulness of the notion.

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Noise Margins of Bandpass Filters

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***Abstract*—It is shown that if a bandpass filter is obtained from a prototype low-pass filter by the usual low-pass to bandpass transformation, the white-noise margin and the $(1/f)^2$ noise margin of the resulting filter will be identical to the white-noise margin of the prototype low-pass filter. A method for using this general result for evaluating some types of integrals that arise in noise calculations is indicated.**

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If an ideal low-pass square-response filter is defined as

$$\begin{aligned} \frac{1}{D_{\mathrm{LP}}(f)} &= 1, \qquad f \le f_0 \\ &= 0, \qquad f > f_0 \end{aligned} \tag{1}$$

the noise power at the output due to a unit magnitude white-noise power spectrum input will be given by

$$\int_0^\infty \frac{df}{D_{\mathrm{LP}}(f)} = f_0. \tag{2}$$

The white-noise margin for a practical low-pass filter intended to be equivalent to the ideal low-pass square-response filter is then defined as

$$\int_0^\infty \frac{d\left(\dfrac{f}{f_0}\right)}{D_{\mathrm{LPE}}(f)}. \tag{3}$$

As examples, Vandivere [1] has taken f_0 to be the 3-dB cutoff frequency of an equivalent Butterworth filter and the frequency at which the response reaches the bottom of the last ripple of an equivalent Chebyshev filter in his noise margin calculations. Hindin [2] has performed similar calculations for Legendre filters by taking f_0 to be the 3-dB cutoff frequency of an equivalent Legendre filter.

The purpose of this letter is to show that if a bandpass filter is obtained from a prototype low-pass filter by the usual low-pass to bandpass transformation

$$\bar{f} = f_m\left(\frac{f}{f_m} - \frac{f_m}{f}\right) \tag{4}$$

the white-noise margin and the $(1/f)^2$ noise margin of the resulting filter will be identical to the white-noise margin of the prototype low-pass filter. If an ideal bandpass square-response filter is defined as

$$\begin{aligned} \frac{1}{D_{\mathrm{BP}}(f)} &= 1, \qquad f_1 \le f \le f_2 \\ &= 0, \qquad f < f_1 \text{ and } f > f_2 \end{aligned} \tag{5}$$

where $(f_2-f_1)=f_0$, the white-noise margin of a practical bandpass filter intended to be equivalent to the ideal bandpass square-response filter will be given by

$$\int_0^\infty \frac{d\left(\dfrac{f}{f_0}\right)}{D_{\mathrm{BPE}}(f)}. \tag{6}$$

If $D_{\mathrm{BPE}}(f)$ is restricted to the class $D_{\mathrm{LPE}}(\bar{f})$, the integral of (6) can be evaluated by an integration from $f=f_m$ to $f=\infty$ and the use of the properties of the transformation of (4). Each frequency f_h above f_m has an image frequency f_l below f_m and

$$f_h f_l = f_m^2. \tag{7}$$

It therefore follows that

$$|df_l| = \left(\frac{f_m}{f_h}\right)^2 df_h. \tag{8}$$

Using (4), it is easy to show that

$$d\left(\frac{f}{f_0}\right) = \frac{d\left(\dfrac{\bar{f}}{f_0}\right)}{\left[1 + \left(\dfrac{f_m}{f}\right)^2\right]}. \tag{9}$$

Thus

$$\begin{aligned} \int_0^\infty \frac{d\left(\dfrac{f}{f_0}\right)}{D_{\mathrm{LPE}}(\bar{f})} &= \int_{(f_m/f_0)}^\infty \frac{\left[1 + \left(\dfrac{f_m}{f}\right)^2\right] d\left(\dfrac{f}{f_0}\right)}{D_{\mathrm{LPE}}(\bar{f})} \\ &= \int_0^\infty \frac{d\left(\dfrac{\bar{f}}{f_0}\right)}{D_{\mathrm{LPE}}(\bar{f})}. \end{aligned} \tag{10}$$

Since (f/f_0) and $(\bar{f}/f_0)$ in (3) and (10) are merely dummies of integration, the white-noise margin of a bandpass filter derived from a prototype low-pass filter by the transformation of (4) is identical to the white-noise margin of the prototype low-pass filter.