Goals

The goals of this is to turn the category F-Cospan(Sets) as discussed in Brendan's paper Decorated Co-spans into a bi-category using DPO Graph Rewriting.

Setup

As required in Brendan's construction, let $F: (FinSet, +) \rightarrow (Set, \times)$ be a monoidal functor

- \bullet sending a set n to the collection of graphs on nodes n
- sending a map $f: x \to y$ of finite sets to the map $Fx \to Fy$ which assigns to each graph $e \xrightarrow{s,t} x$ the graph $e \xrightarrow{s,t} x \xrightarrow{f} y$. Then an F-decorated cospan is a pair

$$(x \to n \leftarrow y, \{\bullet\} \to Fn)$$

consisting of a co-span in Sets and a graph with node set n. A morphism of f-decorated co-spans

$$(x \to n \leftarrow y, \{\bullet\} \xrightarrow{s} Fn) \to (x \to m \leftarrow y, \{\bullet\} \xrightarrow{t} Fm)$$

is a set map $f: n \to m$ such that the diagram



commutes. We hope to show that there is a bi-category F-Cospan(Set) with

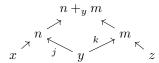
- (a) 0-cells: finite sets,
- (b) 1-cells: iso-classes of F-decorate co-spans between sets, and
- (c) 2-cells: derivations $Fn_{\bullet} \Rightarrow Fm_{\bullet}$ via DPO graph transformations.

Brendan has already shown that F-Cospan(Set) is a (hyper-graph) category with respect to the 0-cells and 1 cells. Here, the composition of 1-cells

$$(x \to n \leftarrow y, \{\bullet\} \xrightarrow{s} Fn); (y \to m \leftarrow z, \{\bullet\} \xrightarrow{t} Fm)$$

is the pair $(x \to n +_y m \leftarrow y, \{\bullet\} \to F(n +_y m))$ where

• $n +_y m$ is a pushout obtained from the diagram



• the map $\{\bullet\} \to F(n +_y m)$ is the composite

$$\{\bullet\} \xrightarrow{\sigma^{-1}} \{\bullet\} \times \{\bullet\} \xrightarrow{(s,t)} Fn \times Fm \xrightarrow{\theta} F(n+m) \xrightarrow{F[j,k]} F(n+_y m)$$

where

- $-\sigma$ is the unit from the monoidal structure (here it's the diagonal)
- (s,t) is the product of the maps from the F-decorated cospans we are composing (here it picks out the corresponding graphs in Fn and Fm.
- $-\theta$ comes from the monoidalness of the functor F (here is coproducts the chosen graphs in Fn and Fm,
- -F[j,k] is the image of the co-pairing of j,k under F (here it amalgamates the graph under (y,j,k).

Bicategory Definition

Recall that a bi-category C consists of the data:

- collection of 0-cells,
- for each pair x, y of 0-cells, a category C(x, y) whose objects, we call 1-cells and morphisms we call 2-cells,
- a functor $id_x : 1 \to C(x,x)$ (i.e. a 1-cell) that will represent an identity of for 1-cells and 2-cells which we will denote by 1 and use context to determine which identity we mean, and
- a functor $\circ_{x,y,z}$

$$\begin{split} \mathtt{C}(x,y) \times \mathtt{C}(y,z) &\to \mathtt{C}(x,y) \\ (f,g) &\mapsto f; g & \text{(on $1-$cells)} \\ (\alpha,\beta) &\mapsto \alpha * \beta & \text{(on $1-$cells)} \end{split}$$

which is described by the diagram

• Natural Isomorphisms

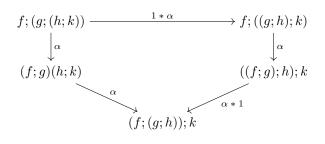
$$\begin{array}{c} \mathtt{C}(w,x) \times \mathtt{C}(x,y) \times \mathtt{C}(y,z) \xrightarrow{\circ_{w,x,y} \times \mathtt{C}(y,z)} \mathtt{C}(w,y) \times \mathtt{C}(y,z) \\ \\ \mathtt{C}(w,x) \times \circ_{x,y,z} \downarrow \qquad \qquad \qquad \downarrow \circ_{w,y,z} \\ \\ \mathtt{C}(w,x) \times \mathtt{C}(x,z) \xrightarrow{\circ_{w,x,z}} \mathtt{C}(w,z) \end{array}$$

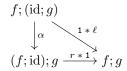
$$\begin{array}{c|c} \mathbf{C}(x,y)\times \mathbf{1} & \mathbf{1}\times \mathbf{C}(x,y) \\ \hline \mathbf{C}(x,y)\times id_y & & \cong & id_y\times \mathbf{C}(x,y) & & \cong \\ \mathbf{C}(x,y)\times \mathbf{C}(y,y) & & & \mathbf{C}(x,y) & & \mathbf{C}(x,y) \xrightarrow{\bigcirc_{x,x,y}} \mathbf{C}(x,y) \end{array}$$

which are 2-cells

$$\alpha_{f,g,h} \colon (f;g); h \xrightarrow{\cong} f; (g;h),$$
 $r_f \colon \mathbf{1}; f \xrightarrow{\cong} f, \text{ and}$
 $\ell_f \colon f; \mathbf{1} \xrightarrow{\cong} f$

• with the axioms that the following diagrams commute:





DPO Graph Rewriting

Here we describe DPO graph rewriting. Start with a production, or a span $L \leftarrow K \rightarrow R$ of graphs where the left-hand morphism is an inclusion. Given such a production, a graph morphism $L \rightarrow G$ is called a matching morphism. Now, starting with a production and a compatible matching morphism, we can complete the diagram

$$\begin{array}{ccc}
L & \longleftarrow & K & \longrightarrow & R \\
\downarrow g & & & & \\
G & & & & & \\
\end{array} \tag{1}$$

to a double pushout

$$\begin{array}{ccc}
L & \longleftarrow & K & \longrightarrow & R \\
\downarrow & & \downarrow & & \downarrow \\
G & \longleftarrow & D & \longrightarrow & G'
\end{array} \tag{2}$$

with the following process:

- (a) Start with a compatible production and matching morphism as in (1). Check
 - (Dangling Condition) that there is no edge in $G \setminus g(L)$ incident to a node in $g(L) \setminus g(K)$, and
 - (Identification Condition) that, given distinct nodes or edges $x, y \in L$, we have g(x) = g(y) only if $x, y \in K$.
- (b) Construct D by removing $g(L) \smallsetminus g(K)$ from G and let $K \to D$ be the restriction of g and $D \to G$ be the inclusion.
- (c) Let G' be the pushout of $D \leftarrow K \rightarrow R$.

Thus we have obtained (2). If we denote the span $L \leftarrow K \rightarrow R$ by p, we say G' is a direct derivation of G, writen $G \Rightarrow_p G'$, if the inside squares in (2) are graph pushouts. Let P be a collection of productions. Then a graph H is a derivation of G is there are p_1, \ldots, p_k in P and graphs G_1, \ldots, G_{k-1} such that $G \Rightarrow_{p_1} G_1 \Rightarrow_{p_2} \cdots G_{k-1} \Rightarrow_{p_k} H$. Denote this by $G \Rightarrow_P H$.

We will consider the tradition approach to DPO graph transformations which requires that the right-hand morphism $K \to R$ is injective and the matching morphism can be arbitrary.

Central Idea

Given a pair of finite sets x, y, we want to form a category $\mathtt{FinSet}(x, y)$ whose objects are F-decorated co-spans and morphisms come from DPO graph transformations. So, let's fix a set of productions $P = \{L \to K \leftarrow R\}$ and construct a graph whose nodes are the F-decorated co-spans (which can be unambiguously denoted by Fn_k) which is a graph in the set Fn of graphs with nodes n and we use number k to distinguish between different graphs on the same set n of nodes) and edges $Fn_0 \to Fm_0$ whenever $Fn_0 \Rightarrow_P Fm_0$. Define FinSet(x,y) to be the free category on this graph.

The Dirty Details

It remains to show that the axioms for a bicategory are satisfied.