Goals

The goals of this is to turn the category $\texttt{F-Cospan}(\texttt{Sets})$ as discussed in Brendan's paper *Decorated Co-spans* into a bi-category using DPO Graph Rewriting.

Setup

As required in Brendan's construction, let $F\colon (\texttt{FinSet}, +) \to (\texttt{Set}, \times)$ be a monoidal functor

- sending a set n to the collection of graphs on nodes n
- sending a map $f\colon x \to y$ of finite sets to the map $Fx \to Fy$ which assigns to each graph $e \xrightarrow{s,t} x$ the graph $e \xrightarrow{s,t} x \xrightarrow{f} y$.

Then an *F-decorated cospan* is a pair

$$(x \to n \leftarrow y, \{\bullet\} \to Fn)$$

consisting of a co-span in $\texttt{Sets}$ and a graph with node set n. A morphism of f-decorated co-spans

$$(x \to n \leftarrow y, \{\bullet\} \xrightarrow{s} Fn) \to (x \to m \leftarrow y, \{\bullet\} \xrightarrow{t} Fm)$$

is a set map $f\colon n \to m$ such that the diagram

$$\begin{tikzcd} \{\bullet\} \arrow[d, "s"] \arrow[dr, "t"] & \\ Fn \arrow[r, "Ff"] & Fm \end{tikzcd}$$

commutes. We hope to show that there is a bi-category $\texttt{F-Cospan}(\texttt{Set})$ with

(a) 0-cells: finite sets,
(b) 1-cells: iso-classes of F-decorate co-spans between sets, and
(c) 2-cells: derivations $Fn_\bullet \Rightarrow Fm_\bullet$ via DPO graph transformations.

Brendan has already shown that $\texttt{F-Cospan}(\texttt{Set})$ is a (hyper-graph) category with respect to the 0-cells and 1 cells. Here, the composition of 1-cells

$$(x \to n \leftarrow y, \{\bullet\} \xrightarrow{s} Fn); (y \to m \leftarrow z, \{\bullet\} \xrightarrow{t} Fm)$$

is the pair $(x \to n +_y m \leftarrow y, \{\bullet\} \to F(n +_y m))$ where

- $n +_y m$ is a pushout obtained from the diagram

$$\begin{tikzcd} & & n +_y m & & \\ & n \arrow[ur] & & m \arrow[ul] & \\ x \arrow[ur] & & y \arrow[ul, "j"] \arrow[ur, "k"] & & z \arrow[ul] \end{tikzcd}$$

- the map $\{\bullet\} \to F(n +_y m)$ is the composite

$$\{\bullet\} \xrightarrow{\sigma^{-1}} \{\bullet\} \times \{\bullet\} \xrightarrow{(s,t)} Fn \times Fm \xrightarrow{\theta} F(n+m) \xrightarrow{F[j,k]} F(n +_y m)$$

where

- σ is the unit from the monoidal structure (here it's the diagonal)
- (s, t) is the product of the maps from the F-decorated cospans we are composing (here it picks out the corresponding graphs in Fn and Fm,
- θ comes from the monoidalness of the functor F (here is coproducts the chosen graphs in Fn and Fm,,
- $F[j, k]$ is the image of the co-pairing of j, k under F (here it amalgamates the graph under (y, j, k).

Bicategory Definition

Recall that a *bi-category* $\mathtt{C}$ consists of the data:

- collection of 0-cells,
- for each pair x, y of 0-cells, a category $\mathtt{C}(x, y)$ whose objects, we call 1-cells and morphisms we call 2-cells,
- a functor $\mathrm{id}_x \colon \mathtt{1} \to \mathtt{C}(x, x)$ (i.e. a 1-cell) that will represent an identity of for 1-cells and 2-cells which we will denote by $\mathtt{1}$ and use context to determine which identity we mean, and
- a functor $\circ_{x,y,z}$

$$\begin{aligned} \mathtt{C}(x, y) \times \mathtt{C}(y, z) &\to \mathtt{C}(x, y) \\ (f, g) &\mapsto f; g && (\text{on } 1 - \text{cells}) \\ (\alpha, \beta) &\mapsto \alpha * \beta && (\text{on } 1 - \text{cells}) \end{aligned}$$

which is described by the diagram

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & z \end{tikzcd}$$

- Natural Isomorphisms

$$\begin{tikzcd} \mathtt{C}(w, x) \times \mathtt{C}(x, y) \times \mathtt{C}(y, z) \arrow[r, "\circ_{w,x,y} \times \mathtt{C}(y, z)"] \arrow[d, "\mathtt{C}(w, x) \times \circ_{x,y,z}"'] \arrow[dr, phantom, "\overset{\alpha_{w,x,y,z}}{\Rightarrow}"] & \mathtt{C}(w, y) \times \mathtt{C}(y, z) \arrow[d, "\circ_{w,y,z}"] \\ \mathtt{C}(w, x) \times \mathtt{C}(x, z) \arrow[r, "\circ_{w,x,z}"'] & \mathtt{C}(w, z) \end{tikzcd}$$

$$\begin{tikzcd} \mathtt{C}(x, y) \times \mathtt{1} \arrow[d, "\mathtt{C}(x, y) \times id_y"'] \arrow[dr, "\cong"] \arrow[dr, phantom, "\overset{r_{x,y}}{\Rightarrow}"'] & \\ \mathtt{C}(x, y) \times \mathtt{C}(y, y) \arrow[r, "\circ_{x,y,y}"'] & \mathtt{C}(x, y) \end{tikzcd} \qquad \begin{tikzcd} \mathtt{1} \times \mathtt{C}(x, y) \arrow[d, "id_y \times \mathtt{C}(x, y)"'] \arrow[dr, "\cong"] \arrow[dr, phantom, "\overset{\ell_{x,y}}{\Rightarrow}"'] & \\ \mathtt{C}(x, x) \times \mathtt{C}(x, y) \arrow[r, "\circ_{x,x,y}"'] & \mathtt{C}(x, y) \end{tikzcd}$$

which are 2-cells

$$\begin{aligned} &\alpha_{f,g,h} \colon (f; g); h \xrightarrow{\cong} f; (g; h), \\ &r_f \colon \mathtt{1}; f \xrightarrow{\cong} f, \text{and} \\ &\ell_f \colon f; \mathtt{1} \xrightarrow{\cong} f \end{aligned}$$

- with the axioms that the following diagrams commute:

$$\begin{tikzcd}
f;(g;(h;k)) \arrow[rr, "1*\alpha"] \arrow[d, "\alpha"] & & f;((g;h);k) \arrow[d, "\alpha"] \\
(f;g)(h;k) \arrow[dr, "\alpha"] & & ((f;g);h);k \arrow[dl, "\alpha*1"] \\
& (f;(g;h));k &
\end{tikzcd}$$

$$\begin{tikzcd}
f;(\mathrm{id};g) \arrow[d, "\alpha"] \arrow[dr, "1*\ell"] & \\
(f;\mathrm{id});g \arrow[r, "r*1"] & f;g
\end{tikzcd}$$

DPO Graph Rewriting

Here we describe DPO graph rewriting. Start with a *production*, or a span $L \leftarrow K \to R$ of graphs where the left-hand morphism is an inclusion. Given such a production, a graph morphism $L \to G$ is called a *matching morphism*. Now, starting with a production and a compatible matching morphism, we can complete the diagram

$$\begin{tikzcd}
L \arrow[d, "g"] & K \arrow[l] \arrow[r] & R \\
G & &
\end{tikzcd} \tag{1}$$

to a double pushout

$$\begin{tikzcd}
L \arrow[d] & K \arrow[l] \arrow[r] \arrow[d] & R \arrow[d] \\
G & D \arrow[l] \arrow[r] & G'
\end{tikzcd} \tag{2}$$

with the following process:

(a) Start with a compatible production and matching morphism as in (1). Check
  - *(Dangling Condition)* that there is no edge in $G \smallsetminus g(L)$ incident to a node in $g(L) \smallsetminus g(K)$, and
  - *(Identification Condition)* that, given distinct nodes or edges $x, y \in L$, we have $g(x) = g(y)$ only if $x, y \in K$.

(b) Construct D by removing $g(L) \smallsetminus g(K)$ from G and let $K \to D$ be the restriction of g and $D \to G$ be the inclusion.

(c) Let G' be the pushout of $D \leftarrow K \to R$.

Thus we have obtained (2). If we denote the span $L \leftarrow K \to R$ by p, we say G' is a *direct derivation* of G, writen $G \Rightarrow_p G'$, if the inside squares in (2) are graph pushouts. Let P be a collection of productions. Then a graph H is a *derivation* of G is there are $p_1, \ldots, p_k$ in P and graphs $G_1, \ldots, G_{k-1}$ such that $G \Rightarrow_{p_1} G_1 \Rightarrow_{p_2} \cdots G_{k-1} \Rightarrow_{p_k} H$. Denote this by $G \Rightarrow_P H$.

We will consider the tradition approach to DPO graph transformations which requires that the right-hand morphism $K \to R$ is injective and the matching morphism can be arbitrary.

Central Idea

Given a pair of finite sets x, y, we want to form a category $\texttt{FinSet}(x, y)$ whose objects are F-decorated co-spans and morphisms come from DPO graph transformations. So, let's fix a set of productions $P = \{L \to K \leftarrow R\}$ and construct a graph whose nodes are the F-decorated co-spans (which can be unambiguously denoted by Fn_k) which is a graph in the set Fn of graphs with nodes n and we use number k to distinguish between different graphs on the same set n of nodes) and edges $Fn_0 \to Fm_0$ whenever $Fn_0 \Rightarrow_P Fm_0$. Define $\texttt{FinSet}(x, y)$ to be the free category on this graph.

The Dirty Details

It remains to show that the axioms for a bicategory are satisfied.