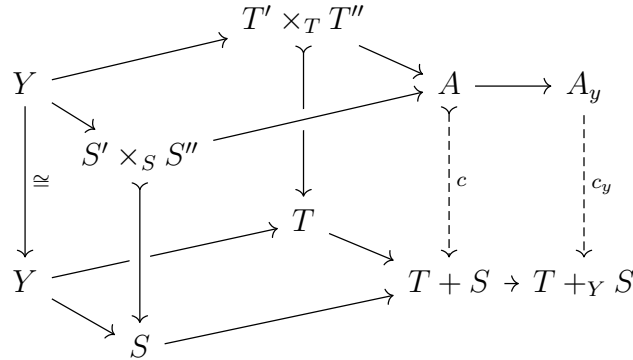


Theorem. *Let's assume that we're in a category where monic and epi implies iso, like a topos. There is an isomorphism $A \cong B$, where $A := (T' +_Y S') \times_{T+_Y S} (T'' +_Y S'')$ and $B := (T' \times_T T'') +_Y (S' \times_S S'')$.*

Proof. Write

$$\begin{aligned} A &:= (T' \times_T T'') + (S' \times_S S'') \\ A_y &:= (T' \times_T T'') +_Y (S' \times_S S'') \\ B &:= (T' + S') \times_{T+S} (T'' + S'') \\ B_y &:= (T' +_Y S') \times_{T+_Y S} (T'' +_Y S'') \end{aligned}$$

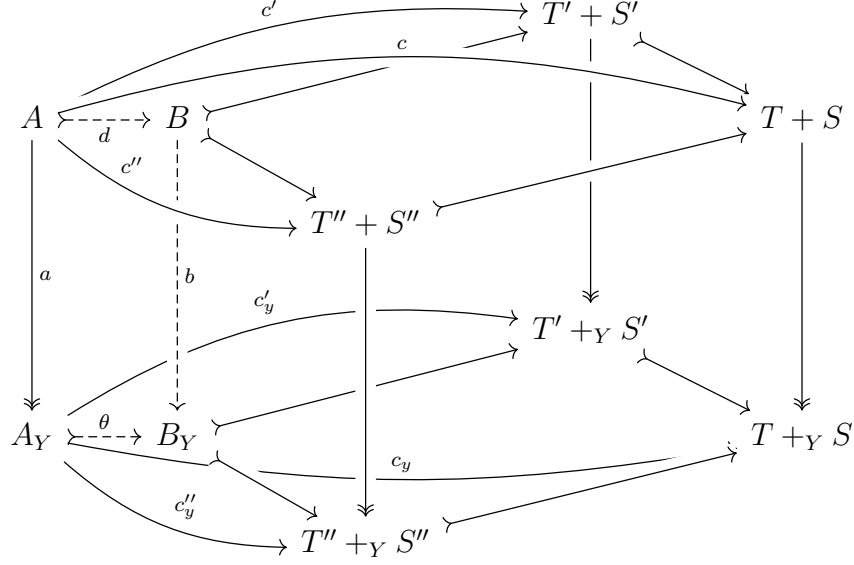
(I) Use lemma to get maps. The above lemma gives us monics c, c_y via the diagram



We get maps c', c'_y, c'', c''_y the same way.

(II) Find the desired map and show it's monic.

Consider the following diagram:



The map $d: A \rightarrow B$ is from the top square pull back and monic since c, c', c'' are. The map $b: B \rightarrow B_Y$ is from the bottom square pullback. The star of the show, $\theta: A_Y \rightarrow B_Y$ is from the bottom square pullback and is monic since c_y, c'_y, c''_y are.

Regardless of us assuming anything about the regularity of our category, the vertical maps $T + S \rightarrow T +_Y S$, $T' + S' \rightarrow T' +_Y S'$, $T'' + S'' \rightarrow T'' +_Y S''$ are regular. Recall that Y is a cone over the diagram in two different ways, via the T side and via the S side. This gives two different maps for each of $Y \rightarrow T + S, Y \rightarrow T' + S', Y \rightarrow T'' + S''$. It is easy to check that the vertical epis are co-equalizers of these maps.

The remainder of the proof is to show that θ is epic.

(III) Get the pushout of the left square. Let P be the pushout

$$\begin{array}{ccc} A & \xrightarrow{d} & B \\ \downarrow a & & \downarrow p \\ A_Y & \xrightarrow{p'} & P \end{array}$$

where p is epic and p' monic since pushouts preserve these properties.

(IV) The projection p is a co-equalizer. We have maps $t_a, s_a: Y \rightarrow A$ and $t_b, s_b: Y \rightarrow B$. Check that a is the co-equalizer of t_a, s_a . We will show that $p: B \rightarrow P$ is the co-equalizer of t_b, s_b .

Let's discuss the diagram

$$\begin{array}{ccc}
Y & \xlongequal{\quad} & Y \\
\downarrow t_a & & \downarrow t_b \\
\downarrow s_a & & \downarrow s_b \\
A & \xrightarrow{d} & B \\
\downarrow a & & \downarrow p \\
A_Y & \xrightarrow{p'} & P
\end{array}
\begin{array}{c}
\searrow x \\
\downarrow z \\
\searrow y
\end{array}
\begin{array}{c}
\\
\\
Q
\end{array}$$

We start with the squares and consider an $x: B \rightarrow Q$ so that $t_b x = s_b x$. Then the upper square commutes so $t_a dx = s_a dx$. Since a co-equalizes t_a, s_a , this gives us $y: A_Y \rightarrow Q$ so that $ay = dx$ which is universal. Then, since the bottom square is a pushout, we get a universal map z so the right triangle commutes. Hence, p is the co-equalizer of t_b, s_b .

(V) Show $A \rightarrow P$ is epic. This is the hole in the proof. We want to show that the composite $ap' = dp: A \rightarrow P$ is epic. Somehow use the regularity of a and p and the pushout square.

(VI) Make an isomorphism out of d, p' . From the composite $A \rightarrow P$ being epic, it follows that p' is epic. We want p' to be regular, which implies it's an isomorphism. By assumptions that pushouts of monics are pullbacks, the square

$$\begin{array}{ccc}
A & \xrightarrow{d} & B \\
\downarrow a & & \downarrow p \\
A_Y & \xrightarrow[p']{} & P
\end{array}$$

is a pullback, which respects regular epis (in a regular category) and so d is regular epic, hence an isomorphism.

(VII) Find a map $P \rightarrow B_Y$. We get the map $\psi: P \rightarrow B_Y$ from the pushout square

$$\begin{array}{ccc}
A & \xrightarrow[p']{} & B \\
\downarrow a & & \downarrow p \\
A_Y & \xrightarrow[p']{} & P
\end{array}
\begin{array}{c}
\searrow b \\
\downarrow \psi \\
\searrow \theta
\end{array}
\begin{array}{c}
\\
\\
B_Y
\end{array}$$

(VIII) Show ψ is monic. (*I'm not certain this is needed. Maybe only ψ to be regular epic.*) Get a map $\eta: P \rightarrow T +_Y S$ from the pushout square

$$\begin{array}{ccc}
 A & \xrightarrow[\cong]{d} & B \\
 \downarrow a & & \downarrow p \\
 A_Y & \xrightarrow[\cong]{p'} & P
 \end{array}
 \begin{array}{c}
 \nearrow \\
 \text{---} \eta \text{---} \\
 \searrow
 \end{array}
 \begin{array}{c}
 \\
 \\
 T +_Y S
 \end{array}$$

c_y

Now we have the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{d} & B & \twoheadrightarrow & T + S \\
 \downarrow a & & \downarrow p & & \downarrow \\
 A_Y & \xrightarrow{p'} & P & \xrightarrow{\eta} & T +_Y S \\
 & \searrow & \nearrow & & \\
 & & b_y & &
 \end{array}$$

whose outer and left squares are pushouts. So the right square is a pushout. But pushouts respects monics, so η is monic. We have that

$$\begin{array}{ccccc}
 A_Y & \xrightarrow{p'} & P & \twoheadrightarrow & T + S \\
 \searrow \theta & & \downarrow \psi & & \nearrow \\
 & & B_Y & &
 \end{array}$$

which gives us that ψ is monic.

(IX) Show ψ is epic. Recall a map is epic if and only if its co-kernel pair coincides. That is we want to show that $\pi = \pi'$ in the pushout diagram

$$\begin{array}{ccc}
 P & \xrightarrow{\psi} & B_Y \\
 \downarrow \psi & & \downarrow \pi \\
 B_Y & \xrightarrow{\pi'} & B_Y +_P B_Y
 \end{array}
 \begin{array}{c}
 \nearrow \\
 \text{---} \rho \text{---} \\
 \searrow
 \end{array}
 \begin{array}{c}
 \\
 \\
 T +_Y S
 \end{array}$$

This also gives us $\rho: B_Y +_P B_Y \rightarrow T +_Y S$ and that π, π' are monic. Note that we also have the pushout diagram

$$\begin{array}{ccc}
 P & \xrightarrow{\psi f} & T +_Y S \\
 \downarrow \psi \pi & & \parallel \\
 B_Y +_P B_Y & \xrightarrow{\rho} & T +_Y S
 \end{array}
 \begin{array}{c}
 \nearrow x \\
 \searrow x \\
 \searrow y
 \end{array}
 Q$$

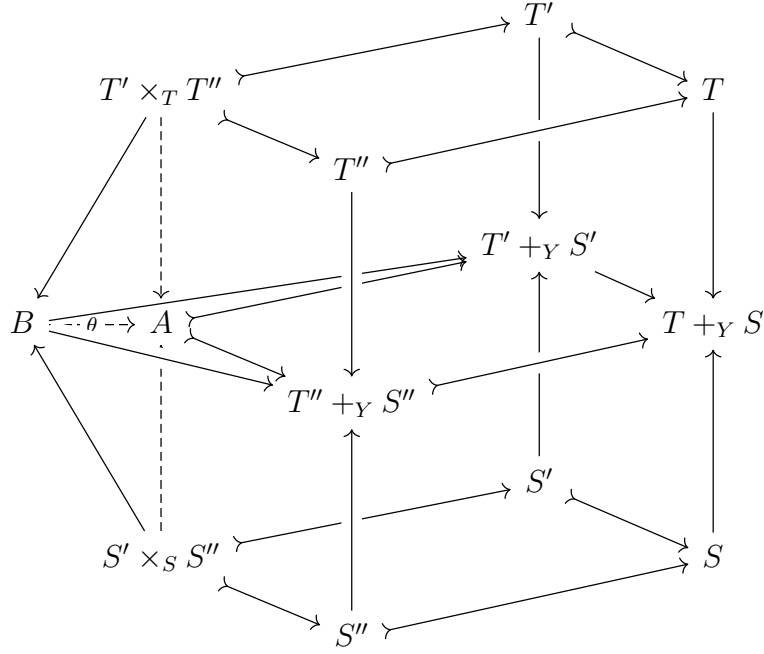
where x, y are maps making the outer square commute and $f: B_Y \rightarrow T +_Y S$ is us finally naming this map. To show this really is a pushout, we need to show $\rho x = y$. This follows from $\psi \pi \rho x = \psi f x = \psi \pi y$ and the fact that $\psi \pi$ is monic. Because this is a pushout square, which respects monics, we know that ρ is a monic. Thus $\pi \rho = f = \pi' \rho$ so $\pi = \pi'$. Therefore ψ is epic.

(X) Wrap up. Because ψ is epic (regular?) it is an isomorphism. And so is p' and so because

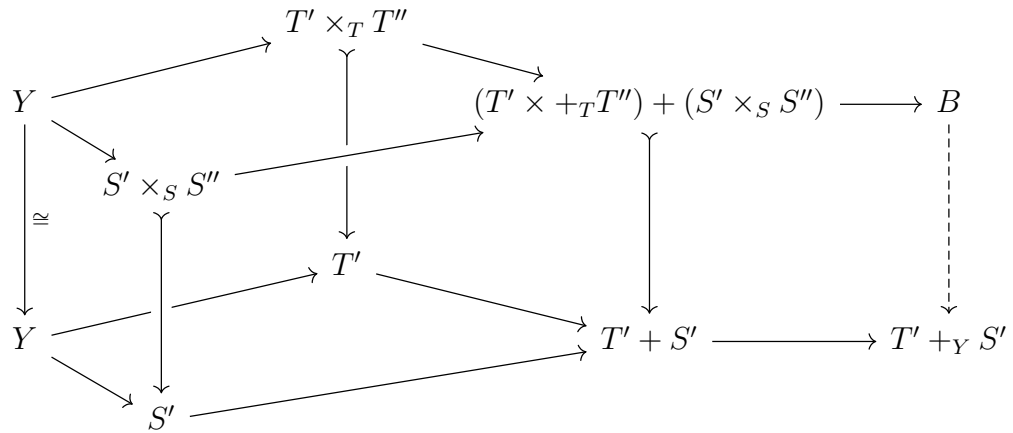
$$\begin{array}{ccc}
 A_Y & \xrightarrow{p'} & P \\
 \searrow \theta & & \downarrow \psi \\
 & & B_Y
 \end{array}$$

we have that θ is an isomorphism.

So we have a diagram



where the vertical dashed arrows to A are from the universal property of B as a pushout. This induces θ . The map $B \rightarrow T' +_Y S'$ is the same as the dashed arrow in



By the above lemma, we know that the dashed arrow is monic. Thus we

have

$$\begin{array}{ccc}
 B & \xrightarrow{\theta} & A \\
 & \searrow & \downarrow \\
 & & T' +_Y S'
 \end{array}$$

which implies that θ is monic.

□