$$
\begin{tikzcd}
 & & L +_Y L' & & \\
 & L \arrow[ur, "\ell"] & & L' \arrow[ul, "\ell"'] & \\
 & T' \arrow[u, "t'_L"] \arrow[d, "t'_T"] & & S' \arrow[u, "s'_{L'}"] \arrow[d, "s'_S"] & \\
X \arrow[uur, "x_L"] \arrow[ur, "x_{T'}"] \arrow[r, "x_T"] \arrow[dr, "x_{T''}"'] \arrow[ddr, "x_R"'] & T & Y \arrow[uul, "y_L"'] \arrow[uur, "y_{L'}"] \arrow[ul, "y_{T'}"'] \arrow[ur, "y_{S'}"] \arrow[l, "y_T"'] \arrow[r, "y_S"] \arrow[dl, "y_{T''}"] \arrow[dr, "y_{S''}"'] \arrow[ddl, "y_R"] \arrow[ddr, "y_{R'}"'] & S & Z \arrow[uul, "z_{L'}"'] \arrow[ul, "z_{S'}"'] \arrow[l, "S"'] \arrow[dl, "S''"] \arrow[ddl, "R'"] \\
 & T'' \arrow[u, "t''_T"'] \arrow[d, "t''_R"] & & S'' \arrow[u, "s''_S"'] \arrow[d, "s''_{R'}"] & \\
 & R \arrow[dr, "r_+"'] & & R' \arrow[dl, "r'_+"] & \\
 & & R +_Y R' & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & B \arrow[dr, "\iota_b"] \arrow[ddd, "b_x"] & & \\
A \arrow[urr, "a_b"] \arrow[dr, "a_c"] \arrow[ddd, "\phi", "\cong"'] & & & B + C \arrow[r, "{[-]_A}"] \arrow[ddd, dashed, "i"] & B +_A C \arrow[ddd, dashed, "i'"] \\
 & C \arrow[urr, "\iota_c"] \arrow[ddd, "c_y"] & & & \\
 & & X \arrow[dr, "\iota_x"] & & \\
W \arrow[urr, "w_x"] \arrow[dr, "w_y"'] & & & X + Y \arrow[r, "{[-]_W}"] & X +_W Y \\
 & Y \arrow[urr, "\iota_y"] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & L +_Y L' & & \\
 & & & (T' +_Y S') \times_{T +_Y S} (T'' +_Y S'') \arrow[ul, "\pi'_X \left\langle t'_L \ell, s'_L \ell' \right\rangle"'] \arrow[dddl, "\pi''_X \left\langle t'_L \ell, s'_L \ell' \right\rangle"] & \\
X \arrow[uurr, bend left, "x_L \ell"] \arrow[urrr, "{[x_{T'} q_{T'}, x_{T''} q_{T''}]}"] \arrow[dr, "{[x_{T'}, x_{T''},] i_T}"] \arrow[ddrr, bend right, "x_R r"'] & & & & Z \arrow[uull, bend right, "z_L \ell'"'] \arrow[ul, "{[z_{S'} q_{S'}, z_{S''} q_{S''}]}"'] \arrow[dlll, "{[z_{S'}, z_{S''}] i_S}"'] \arrow[ddll, bend left, "z_R r'"] \\
 & (T' \times_T T'') +_Y (S' \times_S S'') \arrow[uuur, "\left\langle p_{T'} t'_L \ell, p_{S'} s'_{L'} \ell' \right\rangle"] \arrow[uurr, dashed, "\theta"] \arrow[dr, "\left\langle p_{T''} t''_R r, p_{S''} s''_{R'} r' \right\rangle"'] & & & \\
 & & R +_Y R' & &
\end{tikzcd}
$$

Theorem. *Let's assume that we're in a category where monic and epi implies iso, like a topos. There is an isomorphism* $A \cong B$, *where* $A := (T' +_Y S') \times_{T+_Y S} (T'' +_Y S'')$ *and* $B := (T' \times_T T'') +_Y (S' \times_S S'')$.

Proof. Write

$$
\begin{aligned}
&A := (T' \times_T T'') + (S' \times_S S'') \\
&A_y := (T' \times_T T'') +_Y (S' \times_S S'') \\
&B := (T' + S') \times_{T+S} (T'' + S'') \\
&B_y := (T' +_Y S') \times_{T+_Y S} (T'' +_Y S'')
\end{aligned}
$$

(I) Use lemma to get maps. The above lemma gives us monics c, c_y via the diagram

$$
\begin{tikzcd}
 & & T' \times_T T'' \arrow[dr] \arrow[ddd, tail] & & \\
Y \arrow[urr] \arrow[dr] \arrow[ddd, "\cong"] & & & A \arrow[r] \arrow[ddd, dashed, tail, "c"] & A_y \arrow[ddd, dashed, "c_y"] \\
 & S' \times_S S'' \arrow[urr] \arrow[ddd, tail] & & & \\
 & & T \arrow[dr] & & \\
Y \arrow[urr] \arrow[dr] & & & T + S \arrow[r, two heads] & T +_Y S \\
 & S \arrow[urr] & & &
\end{tikzcd}
$$

We get maps c', c'_y, c'', c''_y the same way.

(II) Find the desired map and show it's monic.

Consider the following diagram:

$$
\begin{tikzcd}
 & & & T'+S' \arrow[dr, tail] \arrow[ddd, two heads] & \\
A \arrow[r, tail, dashed, "d"'] \arrow[urrr, bend left, "c'"] \arrow[rrrr, bend left, "c"] \arrow[drr, bend right, "c''"] \arrow[ddd, two heads, "a"] & B \arrow[urr, tail] \arrow[dr, tail] \arrow[ddd, dashed, "b"] & & & T+S \arrow[ddd, two heads] \\
 & & T''+S'' \arrow[urr, tail] \arrow[ddd, two heads] & & \\
 & & & T'+_Y S' \arrow[dr, tail] & \\
A_Y \arrow[r, tail, dashed, "\theta"] \arrow[urrr, bend left, "c'_y"] \arrow[rrrr, bend right, "c_y"] \arrow[drr, bend right, "c''_y"] & B_Y \arrow[urr, tail] \arrow[dr, tail] & & & T+_Y S \\
 & & T''+_Y S'' \arrow[urr, tail] & &
\end{tikzcd}
$$

The map $d\colon A \to B$ is from the top square pull back and monic since c, c', c'' are. The map $b\colon B \to B_Y$ is from the bottom square pullback. The star of the show, $\theta\colon A_Y \to B_Y$ is from the bottom square pullback and is monic since c_y, c'_y, c''_y are.

Regardless of us assuming anything about the regularity of our category, the vertical maps $T+S \to T+_Y S$, $T'+S' \to T'+_Y S'$, $T''+S'' \to T''+_Y S''$ are regular. Recall that Y is a cone over the diagram in two different ways, via the T side and via the S side. This gives two different maps for each of $Y \to T+S$,$Y \to T'+S'$,$Y \to T''+S''$. It is easy to check that the vertical epis are co-equalizers of these maps.

The remainder of the proof is to show that θ is epic.

(III) Get the pushout of the left square. Let P be the pushout

$$
\begin{tikzcd}
A \arrow[r, tail, "d"] \arrow[d, two heads, "a"] & B \arrow[d, two heads, "p"] \\
A_Y \arrow[r, tail, "p'"] & P
\end{tikzcd}
$$

where p is epic and p' monic since pushouts preserve these properties.

(IV) The projection p is a co-equalizer. We have maps $t_a, s_a\colon Y \to A$ and $t_b, s_b\colon Y \to B$. Check that a is the co-equalizer of t_a, s_a. We will show that $p\colon B \to P$ is the co-equalizer of t_b, s_b.

Let's discuss the diagram

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, shift right, "t_a"'] \arrow[d, shift left, "s_a"] & Y \arrow[d, shift right, "t_b"'] \arrow[d, shift left, "s_b"] & \\
A \arrow[r, tail, "d"] \arrow[d, two heads, "a"] & B \arrow[d, two heads, "p"] \arrow[ddr, bend left, "x"] & \\
A_Y \arrow[r, tail, "p'"] \arrow[drr, bend right, "y"'] & P \arrow[dr, dashed, "z"] & \\
 & & Q
\end{tikzcd}$$

We start with the squares and consider an $x\colon B \to Q$ so that $t_b x = s_b x$. Then the upper square commutes so $t_a dx = s_a dx$. Since a co-equalizes t_a, s_a, this gives us $y \to A_Y \to Q$ so that $ay = dx$ which is universal. Then, since the bottom square is a pushout, we get a universal map z so the right triangle commutes. Hence, p is the co-equalizer of t_b, s_b.

(V) Show $A \to P$ is epic. This is the hole in the proof. We want to show that the composite $ap' = dp\colon A \to P$ is epic. Somehow use the regularity of a and p and the pushout square.

(VI) Make an isomorphism out of d, p'. From the composite $A \to P$ being epic, it follows that p' is epic. We want p' to be regular, which implies it's an isomorphism. By assumptions that pushouts of monics are pullbacks, the square

$$\begin{tikzcd}
A \arrow[r, tail, "d"] \arrow[d, two heads, "a"] & B \arrow[d, two heads, "p"] \\
A_Y \arrow[r, "p'", "\cong"'] & P
\end{tikzcd}$$

is a pullback, which respects regular epis (in a regular category) and so d is regular epic, hence an isomorphism.

(VII) Find a map $P \to B_Y$. We get the map $\psi\colon P \to B_Y$ from the pushout square

$$\begin{tikzcd}
A \arrow[r, "d", "\cong"'] \arrow[d, two heads, "a"] & B \arrow[d, two heads, "p"] \arrow[ddr, bend left, "b"] & \\
A_Y \arrow[r, "p'", "\cong"'] \arrow[drr, bend right, "\theta"'] & P \arrow[dr, dashed, "\psi"] & \\
 & & B_Y
\end{tikzcd}$$

(VIII) Show ψ is monic. *(I'm not certain this is needed. Maybe only ψ to be regular epic.)* Get a map $\eta\colon P \to T +_Y S$ from the pushout square

$$\begin{tikzcd}
A \arrow[r, "d", "\cong"'] \arrow[d, two heads, "a"] & B \arrow[d, two heads, "p"] \arrow[ddr, tail, bend left] & \\
A_Y \arrow[r, "p'", "\cong"'] \arrow[drr, bend right, "c_y"'] & P \arrow[dr, dashed, tail, "\eta"] & \\
 & & T +_Y S
\end{tikzcd}$$

Now we have the diagram

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "a"] & B \arrow[r, tail] \arrow[d, "p"] & T + S \arrow[d] \\
A_Y \arrow[r, "p'"] \arrow[rr, bend right, "b_y"'] & P \arrow[r, "\eta"] & T +_Y S
\end{tikzcd}$$

whose outer and left squares are pushouts. So the right square is a pushout. But pushouts respects monics, so η is monic. We have that

$$\begin{tikzcd}
A_Y \arrow[r, "p'"] \arrow[dr, tail, "\theta"] & P \arrow[r, tail, "\eta"] \arrow[d, "\psi"] & T + S \\
 & B_Y \arrow[ur, tail] &
\end{tikzcd}$$

which gives us that ψ is monic.

(IX) Show ψ is epic. Recall a map is epic if and only if its co-kernel pair coincides. That is we want to show that $\pi = \pi'$ in the pushout diagram

$$\begin{tikzcd}
P \arrow[r, "\psi"] \arrow[d, "\psi"] & B_Y \arrow[d, "\pi"] \arrow[ddr, tail, bend left] & \\
B_Y \arrow[r, "\pi'"] \arrow[drr, tail, bend right] & B_Y +_P B_Y \arrow[dr, dashed, "\rho"] & \\
 & & T +_Y S
\end{tikzcd}$$

This also gives us $\rho\colon B_Y +_P B_Y \to T +_Y S$ and that π, π' are monic. Note that we also have the pushout diagram

$$\begin{tikzcd}
P \arrow[r, tail, "\psi f"] \arrow[d, "\psi\pi"] & T +_Y S \arrow[d, equal] \arrow[ddr, bend left, "x"] & \\
B_Y +_P B_Y \arrow[r, "\rho"] \arrow[drr, bend right, "y"] & T +_Y S \arrow[dr, dashed, "x"] & \\
& & Q
\end{tikzcd}$$

where x, y are maps making the outer square commute and $f\colon B_Y \to T +_Y S$ is us finally naming this map. To show this really is a pushout, we need to show $\rho x = y$. This follows from $\psi\pi\rho x = \psi f x = \psi\pi y$ and the fact that $\psi\pi$ is monic. Because this is a pushout square, which respects monics, we know that ρ is a monic. Thus $\pi\rho = f = \pi'\rho$ so $\pi = \pi'$. Therefore ψ is epic.

(X) Wrap up. Because ψ is epic (regular?) it is an isomorphism. And so is p' and so because

$$\begin{tikzcd}
A_Y \arrow[r, "p'"] \arrow[dr, tail, "\theta"] & P \arrow[d, "\psi"] \\
& B_Y
\end{tikzcd}$$

we have that θ is an isomorphism.

So we have a diagram

$$
\begin{tikzcd}
 & & & T' \arrow[dr, tail] \arrow[ddd] & \\
 & T' \times_T T'' \arrow[urr, tail] \arrow[dr, tail] \arrow[dddl] \arrow[ddd, dashed] & & & T \arrow[ddd] \\
 & & T'' \arrow[urr, tail] \arrow[ddd] & & \\
 & & & T' +_Y S' \arrow[dr] & \\
B \arrow[r, dashed, "\theta"] \arrow[urrr] \arrow[drr] & A \arrow[urr, tail] \arrow[dr, tail] & & & T +_Y S \\
 & & T'' +_Y S'' \arrow[urr, tail] & & \\
 & & & S' \arrow[uuu] \arrow[dr, tail] & \\
 & S' \times_S S'' \arrow[urr, tail] \arrow[dr, tail] \arrow[uuul] \arrow[uuu, dashed] & & & S \arrow[uuu] \\
 & & S'' \arrow[uuu] \arrow[urr, tail] & &
\end{tikzcd}
$$

where the vertical dashed arrows to A are from the universal property of B as a pushout. This induces θ. The map $B \to T' +_Y S'$ is the same as the dashed arrow in

$$
\begin{tikzcd}
 & & T' \times_T T'' \arrow[dr] \arrow[ddd, tail] & & \\
Y \arrow[urr] \arrow[dr] \arrow[ddd, "\cong"] & & & (T' \times +_T T'') + (S' \times_S S'') \arrow[r] \arrow[ddd, tail] & B \arrow[ddd, dashed] \\
 & S' \times_S S'' \arrow[urr] \arrow[ddd, tail] & & & \\
 & & T' \arrow[dr] & & \\
Y \arrow[urr] \arrow[dr] & & & T' + S' \arrow[r] & T' +_Y S' \\
 & S' \arrow[urr] & & &
\end{tikzcd}
$$

By the above lemma, we know that the dashed arrow is monic. Thus we have

$$\begin{tikzcd}
B \arrow[r, "\theta"] \arrow[dr, tail] & A \arrow[d, tail] \\
& T' +_Y S'
\end{tikzcd}$$

which implies that θ is monic. $\square$