# SPANS OF COSPANS

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ABSTRACT. We study spans of cospans in a category **C** and explain how to horizontally and vertically compose these. When **C** is a topos and the legs of the spans are monic, these two forms of composition satisfy the interchange law. In this case there is a bicategory of objects, cospans, and 'monic-legged' spans of cospans in **C**. One motivation for this construction is an application to graph rewriting.

## 1. Introduction

There is currently interest in studying complex networks through the simpler networks of which they are comprised. This point of view is known as *compositionality*. Various flavors of graphs (directed, weighted, colored, etc.) play an important role in this program because they are particularly well suited to model networks [2, 3, 12, 13]. By adding a bit of structure to graphs, we can decide how to glue graphs together to make larger graphs.

The structure that we want to consider is given by choosing two subsets of nodes, named *inputs* and *outputs*. When the inputs of one graph equal the outputs of another, we can glue the graphs together. One method for adding this structure is to use a cospan of graphs $I \to G \leftarrow O$ where $I$ and $O$ are discrete. Then gluing graphs together becomes a matter of composing cospans, which Bénabou [4] described using pushouts in the context of a bicategory whose morphisms are cospans and 2-morphisms are maps of cospans. Rebro [11] extended this idea to give a bicategory whose 2-morphisms are cospans of cospans. See the figure below to see what this means.

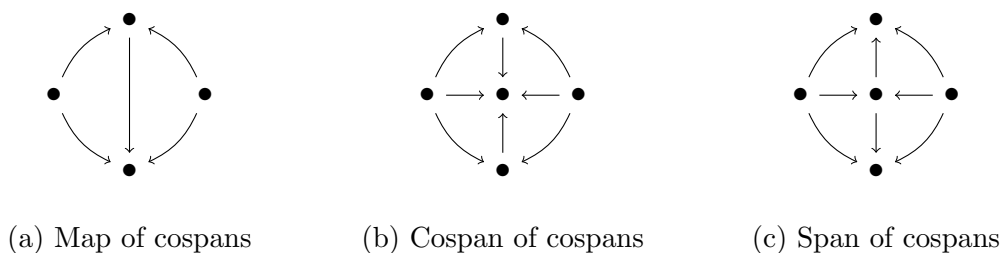(a) Map of cospans     (b) Cospan of cospans     (c) Span of cospans

Figure 1: Various morphisms of cospans

The goal of this paper is to explore another tool to study networks: spans of cospans (see Figure 1c). Grandis and Paré studied these in the context of intercategories and showed that lax interchange holds. We build a bicategory from certain spans of cospans. Both Bénabou and Rebro only required sufficient colimits to compose maps of cospans and cospans of cospans, respectively. For us, we require sufficient limits and colimits while also ensuring they play well together. For this reason, we start with a topos $\mathbf{C}$, then let our 0-cells be the objects of $\mathbf{C}$, 1-cells the cospans in $\mathbf{C}$, and 2-cells isomorphism classes of spans (with monic legs) of cospans. Our main theorem is that this construction, named $\mathbf{MonSp}(\mathbf{Csp}(\mathbf{C}))$, really gives a bicategory.

The specific motivation for this particular construction is to create a framework to house the rewriting of graphs (with inputs and outputs). Applying our construction to the topos **Graph** of presheaves on

$$\bullet \rightrightarrows \bullet$$

we consider **Rewrite**, the full sub-bicategory of $\mathbf{MonSp}(\mathbf{Csp}(\mathbf{Graph}))$ whose objects are the discrete graphs. The 2-cells of **Rewrite** represent all possible ways to rewrite one graph into another that respect the inputs and outputs. In this paper, graph rewrites are performed using the double pushout method. This is explained in Section 4, though more detailed accounts exist in the literature [5, 8]. Here is an example of what a 2-cell in **Rewrite** looks like:
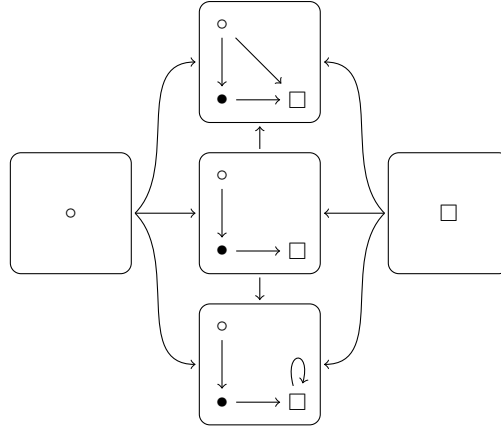


Inputs are depicted with '∘', outputs with '□', and nodes depicted with '•' are neither. The picture above illustrates the rewriting of the top graph to the bottom graph by deleting an edge and adding a loop.

The structure of this paper is as follows. We begin Section 2 by defining spans of cospans and two ways to compose them. When working in a topos and assuming that the legs of the spans of cospans are monic, we show that the compositions satisfy an interchange law. This section ends with a construction of a bicategory whose 2-cells are spans of cospans. Strictly speaking, there is no need for the full strength of a topos, so in Section 3, we discuss how we can weaken our assumptions so that interchange still holds. Then in Section 4 we give a brief introduction to graph rewriting and present our

motivating example: **Rewrite**, a bicategory that contains all possible ways to rewrite graphs.

## 2. Spans of cospans

We begin by introducing spans of cospans. Given that our motivation is to have these be 2-cells in some bicategory, we also show how to compose them. In fact, there are two ways to do so and these correspond to horizontal and vertical composition. Of course, we would like for these compositions to play nicely together and so we finish this section by showing that an interchange law holds under certain assumptions.

Before we begin, a few remarks on convention are in order. First, throughout this paper, tailed arrows "$\rightarrowtail$" refer to monics, and two headed arrows "$\twoheadrightarrow$" to quotient maps. Also, hooked arrows "$\hookrightarrow$" are canonical inclusions, which are labeled $\iota_x$ when its codomain is $x$. To declutter the diagrams, we name only those arrows that are explicitly referenced. Spans $A \leftarrow B \rightarrow C$ are denoted by $B \colon A \xrightarrow{sp} C$ and cospans $X \rightarrow Y \leftarrow Z$ by $Y \colon X \xrightarrow{csp} Z$. This notation is vague, but context should dispense with any confusion. Note that, when first defining spans of cospans below, the object names might seem to be oddly chosen. The intention is to develop a consistency that carries into the proof of the interchange law, at which point, the naming should seem more methodical.

2.1. SPANS OF COSPANS AND THEIR COMPOSITIONS. Suppose that we are working in a category **C**. Given a parallel pair of cospans $L, S \colon X \xrightarrow{csp} Y$, a *span of cospans* is a span $S' \colon L \xrightarrow{sp} S$ such that

$$
\begin{array}{ccc}
 & L & \\
 & \uparrow & \\
X & \rightarrow S' \leftarrow & Y \\
 & \downarrow & \\
 & S &
\end{array}
$$

commutes. Given spans of cospans $S'_1, S'_2 \colon L \xrightarrow{sp} S$, then a morphism of spans of cospans is a **C**-morphism $S'_1 \rightarrow S'_2$ such that the diagram

$$
\begin{array}{c}
\end{array}
\tag{1}
$$

commutes. This is an isomorphism of spans of cospans exactly when the **C**-morphism is.

There are two different ways to turn compatible pairs of spans of cospans into a single span of cospans. Actually, we are foreshadowing that spans of cospans are 2-cells in a

bicategory. Therefore, we immediately name these assignments for what they are: vertical and horizontal composition. For the compositions to be defined, $\mathbf{C}$ should have enough limits and colimits. Also, for the present moment, compositions are only be defined up to isomorphism. We also hold off on looking for the typical properties composition should satisfy until we introduce the bicategorical structure.

Take a pair of spans of cospans $S'\colon L \xrightarrow{sp} S$ and $S''\colon S \xrightarrow{sp} L'$. Define *vertical composition* $\circ_v$ by

$$S'' \circ_v S' := S' \times_S S''\colon L \xrightarrow{sp} L'. \tag{2}$$

Diagrammatically, this is



Now, let $L, S\colon X \xrightarrow{csp} Y$ and $R, T\colon Y \xrightarrow{csp} Z$ be cospans and let $S'\colon L \xrightarrow{sp} S$ and $T'\colon R \xrightarrow{sp} T$ be spans of cospans. Define the assignment *horizontal composition* $\circ_h$ by

$$T' \circ_h S' := S' +_Y T''\colon L +_Y R \xrightarrow{sp} S +_Y T, \tag{3}$$

which corresponds to



At this point, it is natural to ask whether the interchange law holds between vertical and horizontal composition. It does, but not without some further assumptions.

2.2. THE INTERCHANGE LAW. Let $\mathbf{C}$ be a topos with chosen pushouts and let both legs of each span of cospans be monic. To see examples of where the interchange law fails without these assumptions, see Section 3.

The first thing we want to do is to show that the vertical and horizontal compositions are well-defined, up to isomorphism. To this end, we give a lemma that will be put to work several times during the course of this section.

**2.3. LEMMA.** *Given a diagram*

$$
\begin{array}{ccc}
B & \leftarrow A \rightarrow & C \\
\updownarrow & \downarrow \cong & \updownarrow \\
B' & \leftarrow A \rightarrow & C'
\end{array}
\tag{4}
$$

*we get a pushout*

$$
\begin{array}{ccc}
B + C & \twoheadrightarrow & B +_A C \\
\downarrow \gamma & & \downarrow \gamma' \\
B' + C' & \twoheadrightarrow & B' +_A C'
\end{array}
\tag{5}
$$

*such that the canonical arrows $\gamma$ and $\gamma'$ are monic.*

PROOF. Using its universal property, we see that $\gamma$ factors through $B' + C$ as seen in diagram

$$
\begin{array}{ccc}
B & \lhook\joinrel\longrightarrow & B + C \\
\updownarrow & & \downarrow \\
B' & \lhook\joinrel\longrightarrow & B' + C & \longleftarrow & C \\
& & \downarrow & & \updownarrow \\
& & B' + C' & \longleftarrow & C'
\end{array}
$$

It is straightforward to check that the squares are both pushouts. By Lemma 4.1, we get that $\gamma$ must be monic and also that the monotonicity of $\gamma'$ follows from the diagram (5) being a pushout, which we now show.

One can check that the right hand square commutes by using the universal property of $B + C$. To see that this square is a pushout, set up a cocone $D$

$$
\begin{array}{ccc}
B + C & \xrightarrow{\ q\ } & B +_A C \\
\downarrow \gamma & \xrightarrow{\ q'\ } & \downarrow \gamma' \quad\searrow^{d} \\
B' + C' & \longrightarrow & B' +_{A'} C' \\
& \searrow_{d'} & \quad\downarrow \\
& & D
\end{array}
\tag{6}
$$

Then $d'\iota_{B'}$, $d'\iota_{C'}$, and $D$ form a cocone under the span $B' \leftarrow A' \rightarrow C'$ on the bottom face of diagram (4). This induces the canonical map $\gamma'' \colon B' +_A C' \rightarrow D$. It follows that $d'\iota_{B'} = \gamma''q'\iota_{B'}$ and $d'\iota_{C'} = \gamma''q'\iota_{C'}$. Therefore, $d' = \gamma''q'$ by the universal property of coproducts.

Furthermore, $dq\iota_B$, $dq\iota_C$, and $D$ form a cocone under the span $B \leftarrow A \rightarrow C$ on the top face of diagram (4). Then $dq\iota_B = d'\gamma\iota_B = \gamma''q'\gamma\iota_B = \gamma''\gamma'q\iota_B$ and $dq\iota_C = d'\gamma\iota_C = \gamma''q'\gamma\iota_C = \gamma''\gamma'q\iota_C$ meaning that both $d$ and $\gamma''\gamma'$ satisfy the canonical map $B +_A C \rightarrow D$. Hence $d = \gamma''\gamma'$.

The universality of $\gamma''$ with respect to diagram (6) follows from the universality of $\gamma''$ with respect to $B' +_A C'$. ∎

2.4. LEMMA. *Let $S'\colon L \xrightarrow{sp} S$ and $S''\colon S \xrightarrow{sp} L'$ be vertically composable spans of cospans, both of whose span legs are monic. The legs of the vertical composite span of cospans $S'' \circ_v S'\colon L \xrightarrow{sp} L'$ are monic. Similarly, let $L, S\colon X \xrightarrow{csp} Y$ and $R, T\colon Y \xrightarrow{csp} Z$ be cospans and let $S'\colon L \xrightarrow{sp} S$ and $T'\colon R \xrightarrow{sp} T$ be spans of cospans with monic legs. Then the legs of the horizontal composite span of cospans $T' \circ_h S'\colon L +_Y R \xrightarrow{sp} S +_Y T$ are monic.*

PROOF. The result for vertical composition follows from the fact that pullbacks respect monics. The result for horizontal composition follows from applying Lemma 2.3 to the diagrams

$$
\begin{array}{ccc}
S' \leftarrow Y \rightarrow T' & \quad & S' \leftarrow Y \rightarrow T' \\
\updownarrow \quad \downarrow = \quad \updownarrow & \quad & \updownarrow \quad \downarrow = \quad \updownarrow \\
S \leftarrow Y \rightarrow T & \quad & S \leftarrow Y \rightarrow T
\end{array}
$$

∎

The interchange law requires that, given composable spans of cospans



$$(7)$$

there is an isomorphism:

$$
(S' \circ_v S'') \circ_h (T' \circ_v T'') \cong (S' \circ_h T') \circ_v (S'' \circ_h T''). \tag{8}
$$

The left hand side corresponds to first applying vertical composition then horizontal composition. The right hand side swaps the order of composition. This isomorphism strengthens to an equality when isomorphism classes of spans of cospans are the 2-cells of a bicategory.

It is straightforward, using (2) and (3), to see that (8) reduces to finding an isomorphism

$$
(S' \times_S S'') +_Y (T' \times_T T'') \cong (S' +_Y T') \times_{S+_Y T} (S'' +_Y T'') \tag{9}
$$

of spans of cospans.

To simplify our notation, write:

$$
\begin{array}{ll}
A := (S' \times_S S'') + (T' \times_T T''), & A_Y := (S' \times_S S'') +_Y (T' \times_T T''), \\
B := (S' + T') \times_{S+T} (S'' + T''), & B_Y := (S' +_Y T') \times_{S+_Y T} (S'' +_Y T'').
\end{array}
$$

Now, apply Lemma 2.3 to the diagram

$$
\begin{array}{ccccc}
S' \times_S S'' & \longleftarrow & Y & \longrightarrow & T' \times_T T'' \\
\downarrow & & \downarrow = & & \downarrow \\
S & \longleftarrow & Y & \longrightarrow & T
\end{array}
$$

to get the pushout

$$
\begin{array}{ccc}
A & \longrightarrow & A_Y \\
\downarrow a & & \downarrow a_Y \\
S + T & \longrightarrow & S +_Y T
\end{array}
$$

Similarly, we get pushouts

$$
\begin{array}{ccc}
A & \longrightarrow & A_Y \\
\downarrow a' & & \downarrow a'_Y \\
S' + T' & \longrightarrow & S' +_Y T'
\end{array}
\qquad
\begin{array}{ccc}
A & \longrightarrow & A_Y \\
\downarrow a'' & & \downarrow a''_Y \\
S'' + T'' & \longrightarrow & S'' +_Y T''
\end{array}
$$

Now, $A$ forms a cone over the cospan $S + T\colon S' + T' \xrightarrow{csp} S'' + T''$ via the maps $a$, $a'$, and $a''$. And so, we get a canonical map $\theta\colon A \to B$.

2.5. LEMMA. *Given cospans $Y$, $W\colon X \xrightarrow{csp} Z$ where the legs of $W$ factor through a monic $Y \rightarrowtail W$, then there is a unique isomorphism $X \times_Y Z \cong X \times_W Z$.*

PROOF. Via the projection maps, $X \times_Y Z$ forms a cone over the cospan $W\colon X \xrightarrow{csp} Z$ and, also, $X \times_W Z$ forms a cone over the cospan $Y\colon X \xrightarrow{csp} Z$, though the latter requires the monic $Y \rightarrowtail W$ to do so. Universality implies that the induced maps are mutual inverses and they are the only such pair. ∎

2.6. LEMMA. *The map $\theta\colon A \to B$ is an isomorphism.*

PROOF. Because colimits are stable under pullback [10, Thm. 4.7.2], we get an isomorphism

$$
\gamma\colon (S' \times_{S+T} S'') + (S' \times_{S+T} T'') + (T' \times_{S+T} S'') + (T' \times_{S+T} T'') \to B.
$$

But $S' \times_{S+T} T''$ and $S'' \times_{S+T} T'$ are initial. To see this, recall that in a topos, all maps to the initial object are isomorphisms. Now, consider the diagram

$$
\begin{array}{ccc}
S' \times_{S+T} T'' & \longrightarrow & T'' \\
\downarrow & \searrow & \downarrow \\
& \emptyset \longrightarrow T & \\
\downarrow & \downarrow & \downarrow \\
S' & \longrightarrow S \longrightarrow S + T
\end{array}
$$

whose lower right square is a pullback because coproducts are disjoint in topoi. Similarly, $T' \times_{S+T} S''$ is initial. Hence we get a canonical isomorphism

$$\gamma' \colon (S' \times_{S+T} S'') + (T' \times_{S+T} T'') \to B \tag{10}$$

that factors through $\gamma$. But Lemma 2.5 gives unique isomorphisms $S' \times_S S'' \cong S' \times_{S+T} S''$ and $T' \times_T T'' \cong T' \times_{S+T} T''$. This produces a canonical isomorphism

$$\gamma'' \colon A \to (S' \times_{S+T} S'') + (T' \times_{S+T} T'').$$

One can show that $\theta = \gamma' \circ \gamma''$ using universal properties.                                   ∎

Now, let us consider the following diagram:



$$\tag{11}$$

where $\theta_Y$ and $\psi$ are the canonical maps. Observe that $\psi$ factors through $\theta_Y$ in the above diagram. This follows from the universal property of pullbacks. We also have that the top square is a pullback from the previous lemma.

2.7. LEMMA. *The map $\theta_Y \colon A_Y \to B_Y$ is an isomorphism.*

PROOF. Because we are working in a topos, it suffices to show that $\theta_Y$ is both monic and epic. It is monic because $a'_Y$ is monic.

To see that $\theta_Y$ is epic, it suffices to show that $\psi$ is epic. The front and rear right faces of (11) are pushouts by Lemma 2.3. Then because the top and bottom squares of (11) are pullbacks consisting of only monomorphisms, Lemma 4.2 implies that the front and rear left faces are pushouts. However, as pushouts over monos, Lemma 4.1 tells us they are pullbacks. But in a topos, regular epis are stable under pullback, and so $\psi$ is epic. ∎

It remains to show that $\theta_Y$ serves as an isomorphism between spans of cospans. This

amounts to showing that



$$(12)$$

commutes. Here $g$ and $k$ are induced from applying vertical composition before horizontal, $h$ from applying horizontal composition before vertical, $j$ is from composing in either order, $f$ is from (3), and $p$ is from (11). The top and bottom face commute by construction.

2.8. LEMMA. *The inner triangles of diagram* (12) *commute. That is, we have* $k = fp\theta_Y$ *and* $h = \theta_Y g$.

PROOF. To see that $k = fp\theta_Y$, consider the diagram



The bottom face is exactly the pushout diagram from which $f$ was obtained. Universality implies that $k = fa'_Y$ and, as seen in (11), $a'_Y = p\theta_Y$.

That $h = \theta_Y g$ follows from

$$fph = j = kg = fp\theta_Y g$$

and the fact that $fp$ is monic. ∎

Of course, we have only shown that two of the four inner triangles commute, but we can replicate our arguments to s how the remaining two commute as well. This lemma was the last step in proving the following interchange law.

2.9. THEOREM. *Given diagram* (7) *in a topos, there is a canonical isomorphism* $(S' \times_S S'') +_Y (T' \times_T T'') \cong (S' +_Y T') \times_{S+_Y T} (S'' +_Y T'')$.

Note that Section 3 contains a discussion on the assumptions taken in this theorem.

2.10. CONSTRUCTING THE BICATEGORY. Let $\mathbf{C}$ be any topos. We commence construction of a bicategory named $\mathbf{MonSp}(\mathbf{Csp}(\mathbf{C}))$, or $\widetilde{\mathbf{C}}$ for short. The 0-cells of $\widetilde{\mathbf{C}}$ are just the $\mathbf{C}$-objects. For 0-cells $X$ and $Y$, build a category $\widetilde{\mathbf{C}}(X, Y)$ whose objects are $\mathbf{C}$-cospans and morphisms are isomorphism classes of $\mathbf{C}$-spans of cospans whose legs are both monic. Composition in $\widetilde{\mathbf{C}}(X, Y)$ is the vertical composition $\circ_v$ introduced in (2). It is straightforward to check that associativity holds and that spans of cospans whose legs are identity serve as identities.

The composition functor is given by an assignment

$$\otimes \colon \widetilde{\mathbf{C}}(Y, Z) \times \widetilde{\mathbf{C}}(X, Y) \to \widetilde{\mathbf{C}}(X, Z)$$

that acts on 1-cells by $(T, S) \mapsto S \otimes_Y T$ and on 2-cells by horizontal composition $\circ_h$ from (3). It is straightforward to check that $\otimes$ preserves identities. Theorem 2.9 ensures that $\otimes$ preserves composition.

For every 0-cell $X$, the identity functor $\mathbf{1} \to \widetilde{\mathbf{C}}(X, X)$ picks out the 2-cell with all identity maps on $X$. The associator is made of 2-cells

$$R +_X S +_Y T \colon (R +_X S) +_Y T \xrightarrow{sp} R +_X (S +_Y T).$$

The right unitor is made of 2-cells $S \colon S +_Y Y \xrightarrow{sp} S$. Likewise, the left unitor has 2-cells $T \colon T \xrightarrow{sp} Y +_Y T$. The legs for each of the above are the obvious choices. The pentagon and triangle identities follow from the associativity, up to isomorphism, of pushouts.

Given all of the data just laid out, we have the main theorem of the paper.

2.11. THEOREM. *If $\mathbf{C}$ is a topos, then $\mathbf{MonSp}(\mathbf{Csp}(\mathbf{C}))$ is a bicategory.*

# 3. A discussion on the assumptions

Can we expand the domain on which this construction works? Apart from ensuring sufficiently many limits and colimits, the primary source of roadblocks is the interchange law. As we discuss below, it is not absolutely necessary to work strictly within a topos, but we choose to do so in order to be expeditious. To lay out, one by one, the requirements for our interchange law to hold would be exhausting and leave us little energy to work through its proof. To do this is even less reasonable given the ubiquity of topoi. However, listing these requirements is interesting enough to take a look at here.

Before digging deeper into the properties used, let's convince ourselves of the necessity for monic legs within our span of cospans.

3.1. EXAMPLE. Consider the category $\mathbf{Set}$ of sets and functions. Relax the assumption that the legs of the spans of cospans are both monic. Indeed, suppose that $S'$, $S''$, and $T'$ are two element sets and $S$, $Y$, $T$, and $T''$ are singletons. The functions can be any of the limited choices we have. After several routine calculations, we determine that $(S' \times_S S'') +_Y (T' \times_T T'')$ has cardinality 5 and $(S' +_Y T') \times_{S +_Y T} (S'' +_Y T'')$ has cardinality 6.

So we see that even in the archetypal topos, the legs of the spans must be monic for the interchange law to hold. But even if they are, this law may fail if our category $C$ is not a topos. The next example illustrates this.

3.2. EXAMPLE. Consider the Boolean algebra on a two element set. This is the category $0 \to 1$ with products given by meet and coproducts given by join. Note that this is not a topos. Indeed, the only non-identity morphism is both monic and epic but, as no inverse exists, it is not an isomorphism. Recalling the interchange equation (8), suppose that we have $Y = S'' = T' = 0$ and $S = S' = T = T'' = 1$. It is straightforward to check that $(S' \times_S S'') +_Y (T' \times_T T'') = 0$ and $(S' +_Y T') \times_{S+_Y T} (S'' +_Y T'') = 1$. That is, the interchange equation does not hold. Because this Boolean algebra can be embedded into any other, it follows that interchange does not hold for any Boolean algebra.

Now that we are convinced that we actually do need to assume something for interchange to work, we can list our requirements. The obvious place to start is by asking for our category to have enough limits and colimits.

In Lemma 2.3, we use that pushouts respect monics. This occurs in topoi. Indeed, this occurs in any adhesive category, which we discuss further in Section 4.

In Lemma 2.6, we use several facts. The first is that coproducts are disjoint, meaning that a pullback over coproduct inclusions is initial. We also use that colimits—particularly coproducts—are stable under pullback. The full force of a topos is not needed to satisfy this requirement. It suffices that our category is locally cartesian closed [10, Thm. 1.4.9].

In Lemma 2.7, we use that a monic epimorphism is an isomorphism in topoi. This is surely not true in general, but it is always true that a monic regular epimorphism is an isomorphism. Notice that the vertically aligned epimorphisms in Diagram (11) are all regular since they are all coequalizers. For instance $S + T \twoheadrightarrow S +_Y T$ is a coequalizer over $Y \rightrightarrows S + T$. So we can merely ask for pullbacks to preserve regular epimorphisms. Note that this does happen in a regular category, which is a larger class than topoi.

Also in Lemma 2.7, we make use of Lemmas 4.1 and 4.2. This holds in any adhesive category.

It is clear that the properties of adhesive categories (see Section 4) play an important role in ensuring that $\widetilde{\mathbf{C}}$ is a bicategory, as does being locally cartesian closed and regular. Topoi are a well-known, large family of categories that are contained in the intersection of all of these classes.

## 4. An informal introduction to graph rewriting

For those who are not familiar with rewriting systems, and graph rewriting in particular, we provide a brief introduction. For a more in-depth and rigorous viewpoint, see [1], [5], or [8].

There are many methods of rewriting found throughout mathematics, computer science, and linguistics. The general idea is that we begin with a collection of rules, a collection of terms, and a way to apply rules to certain, compatible terms. When applied

to a term, a rule replaces a sub-term with a new sub-term. A simple, informal example is found within the English language. Consider a set of rules

$$\{(\text{noun}) \mapsto x : x \text{ is an English noun}\}.$$

We can apply any one of these rules to

'The (noun) is far away.'

to obtain heaps of grammatically correct, if potentially ridiculous sentences.

The first rewriting methods were uni-dimensional, in that they are concerned with replacing a string of characters or letters. Many attempts to define multi-dimensional rewriting systems came up short in application and execution, but Ehrig, Pfender, and Scheider developed a categorical approach using graph morphisms and pushouts [5] that has since been studied extensively. This approach came to be known as *double pushout graph rewriting*. This is what we are interested in and consider in our bicategory **Rewrite** introduced below.

Here is how double pushout graph rewriting works. A *production* is a span

$$p \colon L \leftarrowtail K \rightarrowtail R$$

of graphs with monic legs. Some authors call this a 'linear production' to distinguish it from spans with potentially non-monic legs, but we do not adopt this convention here. Given a production $p$, and a graph morphism $L \to C$, called a *matching map*, such that there exists a diagram

$$\begin{array}{ccc}
L & \leftarrow K \to & R \\
\downarrow & \downarrow & \downarrow \\
C & \leftarrow E \to & D
\end{array} \tag{13}$$

consisting of two pushout squares, we say that that $D$ is a *derivation* of $C$ and write $C \rightsquigarrow_E D$ or just $C \rightsquigarrow D$. The arrow '$\rightsquigarrow$' is commonly decorated with more information: for instance, the name of the production or the matching map. But that is not necessary here. Observe that the objects $E$ and $D$ need not exist, but when they do, they are unique up to isomorphism [8, Lemma 4.5].

A *grammar* $(\mathcal{G}, \mathcal{P})$ is a set of graphs $\mathcal{G}$ paired with a set of productions $\mathcal{P}$. A *derivation of the grammar* is a string of derivations

$$G_0 \rightsquigarrow G_1 \rightsquigarrow \cdots \rightsquigarrow G_n$$

from productions in $\mathcal{P}$ and with $G_0 \in \mathcal{G}$. We say that $G_n$ is a *rewrite of* $G_0$. The *language* $\mathcal{L}(\mathcal{G}, \mathcal{P})$ generated by the grammar is the collection of all graphs $G$ such that there is a derivation $G_0 \rightsquigarrow^* G$ of the grammar. The idea is that one would study a language. Exactly what properties are interesting is beyond the scope of this discussion. Interested readers should consult the references mentioned at the beginning of this section. Instead of going deeper into the subject, we briefly zoom out.

Seeking to axiomatise graph rewriting, Lack and Sobocinski [8] introduced a class of categories they call *adhesive*. Roughly, a category is adhesive if it has pullbacks, pushouts along monomorphisms, and certain exactness conditions between pullbacks and pushouts hold. These exactness conditions, while subtle, imply all the properties one wants in double pushout rewriting.

Given that topoi are adhesive [9], there are many examples of adhesive categories. One advantage of the generalization is that adhesivity is preserved under coslicing. For instance, the category of pointed sets not a topos, but it is adhesive because it is equivalent to the coslice category $1 \downarrow \mathbf{Set}$.

Presently, we are interested in using the adhesivity of topoi, which allows us to use Lemmas 4.1 and 4.2, in our construction. While originally proved for adhesive categories, we restrict our assumptions to topoi as this suits our needs.

4.1. LEMMA. [8, Lemmas 4.2-3] *In a topos, monomorphisms are stable under pushout. Also, pushouts along monomorphisms are pullbacks.*

4.2. LEMMA. [8, Lemma 6.3] *In a topos, consider a cube*
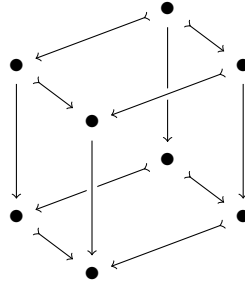


*whose top and bottom faces consist of only monomorphisms. If the top face is a pullback and the front faces are pushouts, then the bottom face is a pullback if and only if the back faces are pushouts.*

A good portion of the theory for double pushout graph rewriting has been extended to adhesive categories. So, while our focus is on double pushout graph rewriting, there may be variations of the bicatgory **Rewrite** (introduced below) that are of interest to computer scientists. For instance, the *Schanuel topos* was used to model the $\pi$-calculus [6] and adhesive categories allow us to extend rewriting to such settings. Our contribution is a bicategorical framework to house the rewriting as 2-cells, though only for double pushout graph rewriting.

4.3. **Rewrite**. Here, we introduce the bicategory **Rewrite** as promised. To prepare, we begin by introducing a slight generalization of the double pushout graph rewriting concepts discussed above. An *interface* $(I, O)$ is a pair of discrete graphs and a *production*

*with interface* $(I, O)$, or simply $(I, O)$-*production*, is a cospan of spans

$$
\begin{array}{ccc}
& L & \\
\nearrow \uparrow \nwarrow & & \\
I \rightarrow K \leftarrow O & & \\
\searrow \downarrow \swarrow & & \\
& R &
\end{array}
$$

Think of $I$ and $O$ as choosing inputs and outputs. Given a production with interface $(I, O)$, we say that a graph $G'$ is an $(I, O)$-*derivation* of $G$ if there is a diagram

$$
\begin{array}{ccccc}
& I & & O & \\
L & \longleftarrow & K & \longrightarrow & R \\
\downarrow & & \downarrow & & \downarrow \\
G & \longleftarrow & E & \longrightarrow & G'
\end{array}
$$

where the bottom squares are pushouts. We denote this as $G \rightsquigarrow G'$. An $(I, O)$-*grammar* $(\mathcal{G}, \mathcal{P})$ consists of a collection of graphs $\mathcal{G}$ and another of $(I, O)$-productions $\mathcal{P}$. Again, an $(I, O)$-grammar generates a language consisting of all graphs $G$ such that there is a chain of $(I, O)$-derivations $G_0 \rightsquigarrow G_1 \rightsquigarrow \cdots \rightsquigarrow G_n = G$ from $\mathcal{P}$ such that $G_0 \in \mathcal{G}$. However, this time, we require such a chain to respect the inputs and outputs in the sense that

$$
\begin{array}{ccccccccc}
& & I & \cdots & & \cdots & O & & \\
L_0 \leftarrow K_0 \rightarrow R_0 & & L_1 \leftarrow K_1 \rightarrow R_1 & & R_{n-1} & & L_n \leftarrow K_n \rightarrow R_n & & \\
\downarrow \quad \downarrow \quad \searrow \swarrow & & \downarrow \quad \downarrow \cdots \searrow \swarrow & & & & \downarrow \quad \downarrow & & \\
G_0 \leftarrow E_0 \longrightarrow G_1 \longleftarrow E_1 \rightarrow G_2 & & G_{n-1} \leftarrow E_{n-1} \rightarrow G_n & & &
\end{array}
\tag{14}
$$

Now we can put these productions with interfaces into our bicategorical framework.

Consider the full sub-bicategory of $\widetilde{\mathbf{C}}$ for $\mathbf{C} := \mathbf{Graph}$ with objects the finite, discrete graphs. Here, a 1-cell is a cospan $G \colon I \xrightarrow{csp} O$ where $G$ is a graph and $I$, $O$ are discrete graphs. A 2-cell between 1-cells $G'$ and $G''$ is a span of cospans $G \colon G' \xrightarrow{sp} G''$ which we think of as an $(I, O)$-derivation $G' \rightsquigarrow G''$. We call this sub-bicategory **Rewrite** because the 2-cells are exactly all the possible ways to rewrite one graph into another so that inputs and outputs are preserved. Given any 2-cell $G \colon G' \xrightarrow{sp} G''$ in **Rewrite**, then the diagram

$$
\begin{array}{ccccc}
& I & & O & \\
G' & \longleftarrow & G & \longrightarrow & G'' \\
\downarrow & & \downarrow & & \downarrow \\
G' & \longleftarrow & G & \longrightarrow & G''
\end{array}
$$

gives an $(I, O)$-derivation $G' \leadsto_G G''$. Conversely, any $(I, O)$-derivation (14) can be made into composable 2-cells $E_i \colon G_i \xrightarrow{sp} G_{i+1}$ where the maps from $I$ and $O$ are the evident composites. Then, the vertical composition of the resulting 2-cells gives us the desired span of cospans.

To better illustrate this, we provide a concrete example of the dictionary between **Rewrite** and double pushout graph rewriting. Suppose we were given an $(I, O)$-derivation with $I = \{*\} = O$, induced from the following double pushout graph rewriting diagram
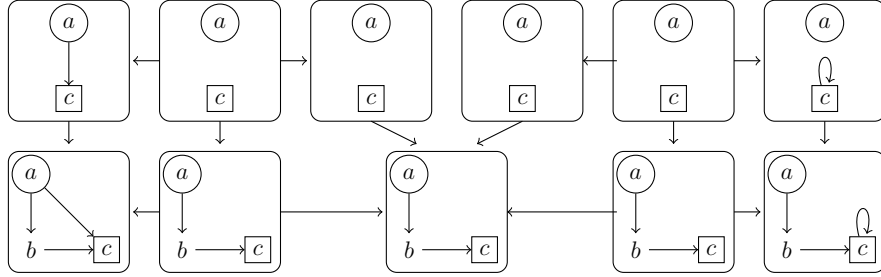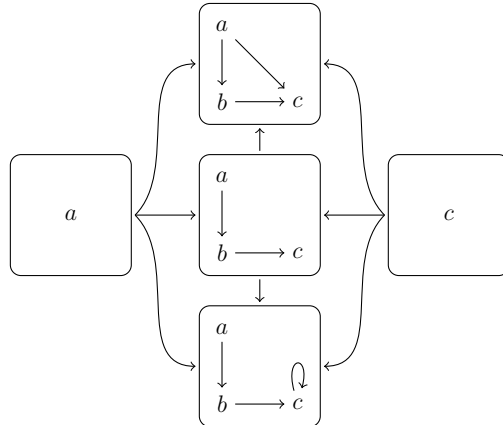


where the functions are described by the labeling, the inputs are circled, and the outputs are squared. In words, we have rewritten the graph on the lower left by removing an edge $a \to c$ and adding a loop $c \to c$. The pullback of the span
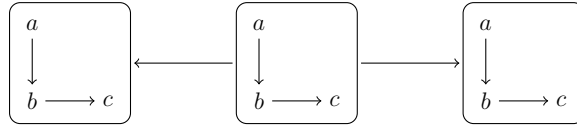


is the graph
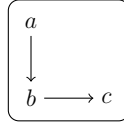


The corresponding 2-cell is the diagram

Here we witness the advantage that graph rewriting has over graph morphisms in the realm of expresivity. There is no way to replace the 2-cell above with a map of graph cospans.

There is nothing inherently special, from a mathematical point of view, about working with graphs and their morphisms. It is in applications where graphs gain importance. We can actually create categories analogous to **Rewrite** with any topos.

## 5. Conclusion and further work

Our primary motivation for constructing this bicategory is as a way to study the gluing of graphs together in a way compatible with chosen input and output nodes. To this end, we defined a bicategory **Rewrite**. This is a very large bicategory and in practice, one begins with a grammar and studies the resulting language. So, in an upcoming collaboration with Kenny Courser, we look at relating languages to sub-bicategories of **Rewrite** generated by a grammar. In this same paper, we study the structure of $\mathbf{MonSp}(\mathbf{Csp}(\mathbf{C}))$ alongside similar bicategories.

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