SPANS OF COSPANS IN A TOPOS

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Abstract. For a topos $\mathbf{T}$, there is a bicategory $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ whose objects are those of $\mathbf{T}$, morphisms are cospans in $\mathbf{T}$, and 2-morphisms are isomorphism classes of monic spans of cospans in $\mathbf{T}$. Using a result of Shulman, we prove that $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ is symmetric monoidal, and moreover, that it is compact closed in the sense of Stay. We provide an application which illustrates how to encode double pushout rewrite rules as 2-morphisms inside a compact closed sub-bicategory of $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{Graph}))$.

1. Introduction

There has been extensive work done on bicategories involving spans or cospans in some way. Given a finitely complete category $\mathbf{D}$, Bénabou [5] was the first to construct the bicategory $\mathbf{Sp}(\mathbf{D})$ consisting of objects of $\mathbf{D}$, spans in $\mathbf{D}$, and maps of spans in $\mathbf{D}$, and this was in fact one of the first bicategories ever constructed. Later, Stay showed that $\mathbf{Sp}(\mathbf{D})$ is a compact closed symmetric monoidal bicategory [20]. Various other authors have considered bicategories and higher categories with maps of spans or spans of spans as 2-morphisms [13, 14, 16, 18, 20], as well as categories and bicategories in which the morphisms are cospans 'decorated' with extra structure [1, 2, 3, 4, 8, 10]. Here, however, we pursue a different line of thought and study spans of cospans.

A span of cospans is a commuting diagram with shape

$$\begin{tikzcd}
 & \bullet & \\
\bullet \arrow[ur] \arrow[r] \arrow[dr] & \bullet \arrow[u] \arrow[d] & \bullet \arrow[ul] \arrow[l] \arrow[dl] \\
 & \bullet &
\end{tikzcd}$$

These were found to satisfy a lax interchange law by Grandis and Paré in their paper on 'intercategories' [11]. Later on, the first listed author of the present work constructed a bicategory with isomorphism classes of spans of cospans as 2-morphisms [6]. However, for the interchange law to be invertible, the 2-morphisms were restricted to spans of cospans inside a topos $\mathbf{T}$ such that the span legs were monomorphisms.

We denote the bicategory of monic spans of cospans inside of a topos $\mathbf{T}$ by $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$. It has the objects of $\mathbf{T}$ for objects, cospans in $\mathbf{T}$ for morphisms, and isomorphism classes of monic spans of cospans in $\mathbf{T}$ for 2-morphisms. Horizontal

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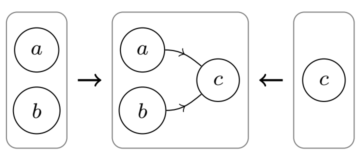
composition is given by pushouts and vertical composition is given by pullbacks. Thus we require that certain limits and colimits exist and, moreover, that pushouts preserve monomorphisms. This occurs in every topos.

Our first result is that the bicategory **MonicSp**(**Csp**(**T**)) is symmetric monoidal. The definition of symmetric monoidal bicategory is long [20], so checking every condition by hand is time consuming. Shulman [19] provides a less tedious process. The idea is to construct an 'isofibrant pseudo double category' that restricts, in a suitable sense, to the bicategory that we are interested in. If this double category is symmetric monoidal, then the 'restricted' bicategory is symmetric monoidal as well. The advantage of this method is its relative efficiency; it is much easier to check that a double category is symmetric monoidal than it is to check that a bicategory is symmetric monoidal.

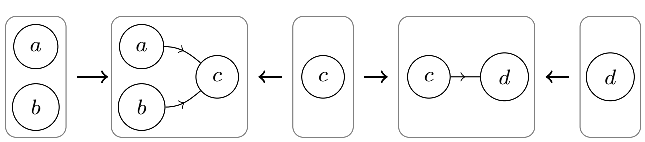
Our second result is that **MonicSp**(**Csp**(**T**)) is compact closed. Stay [20] has defined a compact closed bicategory to be a symmetric monoidal bicategory in which every object has a dual. The details seem more involved than in the ordinary category case due to the coherence laws. However, Pstrągowski [17] has found a way to avoid checking the worst of these laws: the 'swallowtail equations'.

The primary motivation for constructing **MonicSp**(**Csp**(**T**)) is the case when **T** is the topos **Graph** of directed graphs. We are also interested in certain labeled graphs obtained from various slice categories of **Graph**. This is a useful framework to study open graphs; that is, graphs with a subset of nodes serving as 'inputs' and 'outputs' and 'rewrite rules' of such [6]. Graphical calculi fit nicely into this picture as well [7].

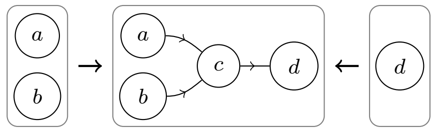
Let us illustrate how this works with open graphs. We begin with a compact closed sub-bicategory of **MonicSp**(**Csp**(**Graph**)) which we shall call **Rewrite**. The sub-bicategory **Rewrite** is 1-full and 2-full on edgeless graphs and the 1-morphisms model open graphs by specifying the inputs and outputs with the legs of the cospans. For example, consider the following cospan of graphs.



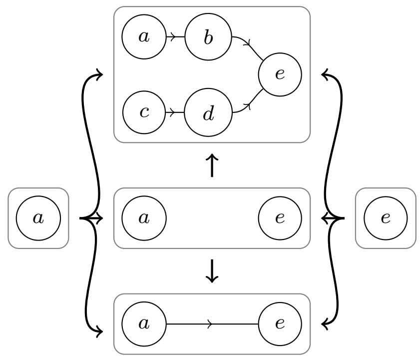
The node labels indicate the graph morphism behaviors. Here, the nodes a and b are inputs and c is an output. The use of the terms *inputs* and *outputs* is justified by the composition, which can be thought of as connecting the inputs of one graph to compatible outputs of another graph. This is made precise with pushouts. For instance, we can compose



to obtain



A 2-morphism in **Rewrite** is the rewriting of one graph into another in a way that preserves inputs and outputs. For instance,



(1)

The bicategory **Rewrite** is not of interest for its own sake. It serves as an ambient context in which to freely generate a compact closed bicategory from some collection of morphisms and 2-morphisms. Of course, the choice of collection depends on one's interests.

The structure of the paper is as follows. In Section 2, we introduce the bicategory $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$. In Section 3, we review how symmetric monoidal double categories can be used to show that certain bicategories have a symmetric monoidal structure, and moreover, when a symmetric monoidal bicategory is compact closed. Nothing in this section is new, but we use these results in Section 4 to show that $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ is compact closed. Finally, in Section 5, we discuss an application that illustrates the consequeces of the bicategory $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{Graph}))$ being compact closed in regard to rewriting open graphs.

2. The bicategory $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$

In this section, we recall the bicategory $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ and some important related concepts. Throughout this paper, $\mathbf{T}$ is a topos.

Spans of cospans were considered by Kissinger in his thesis [15] in the context of rewriting, and also by Grandis and Paré [11] who found a lax interchange law. Later, the first listed author of the present work showed that the interchange law is invertible when restricting attention to a topos $\mathbf{T}$ and monic spans [6], meaning that each morphism is a monomorphism. This gives a bicategory $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ with $\mathbf{T}$-objects for objects, cospans for morphisms, and isomorphism classes of monic spans of cospans for

2-morphisms. A monic span of cospans is a commuting diagram of the form

$$\begin{tikzcd}
& y & \\
x \arrow[ur] \arrow[r] \arrow[dr] & y' \arrow[u, tail] \arrow[d, tail] & z \arrow[ul] \arrow[l] \arrow[dl] \\
& y'' &
\end{tikzcd}$$

where the '$\rightarrowtail$' arrows denotes a monomorphism, and two of these are isomorphic if there is an isomorphism θ such that the following diagram commutes:

$$\begin{tikzcd}
& & & y & \\
& & y'_0 \arrow[ur, tail] \arrow[dd, "\theta"] \arrow[dddl, tail] & & \\
x \arrow[urr] \arrow[drr] \arrow[uurrr, bend left] \arrow[ddr, bend right] & & & & z \arrow[ull] \arrow[dll] \arrow[uul, bend right] \arrow[ddlll, bend left] \\
& & y'_1 \arrow[uuur, tail] \arrow[dl, tail] & & \\
& y'' & & &
\end{tikzcd}$$

As usual, composition of morphisms is by pushout. Given vertically composable 2-morphisms

$$\begin{tikzcd}
& \ell & \\
x \arrow[ur] \arrow[r] \arrow[dr] & y' \arrow[u, tail] \arrow[d, tail] & z \arrow[ul] \arrow[l] \arrow[dl] \\
& y &
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
& y & \\
x \arrow[ur] \arrow[r] \arrow[dr] & y'' \arrow[u, tail] \arrow[d, tail] & z \arrow[ul] \arrow[l] \arrow[dl] \\
& r &
\end{tikzcd}$$

their vertical composite is given by

$$\begin{tikzcd}
& \ell & \\
x \arrow[ur] \arrow[r] \arrow[dr] & y' \times_y y'' \arrow[u, tail] \arrow[d, tail] & z \arrow[ul] \arrow[l] \arrow[dl] \\
& r &
\end{tikzcd}$$

The legs of the inner span are monic because pullbacks preserve monomorphisms. Given two horizontally composable 2-morphisms

$$\begin{tikzcd}
& y & \\
x \arrow[ur] \arrow[r] \arrow[dr] & y' \arrow[u, tail] \arrow[d, tail] & z \arrow[ul] \arrow[l] \arrow[dl] \\
& y'' &
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
& w & \\
z \arrow[ur] \arrow[r] \arrow[dr] & w' \arrow[u, tail] \arrow[d, tail] & v \arrow[ul] \arrow[l] \arrow[dl] \\
& w'' &
\end{tikzcd}$$

their horizontal composite is given by

$$\begin{tikzcd} & y +_z w & \\ x \arrow[ur] \arrow[r] \arrow[dr] & y' +_z w' \arrow[u, tail] \arrow[d, tail] & v \arrow[ul] \arrow[l] \arrow[dl] \\ & y'' +_z w'' & \end{tikzcd}$$

where the legs of the inner span are monic by a previous result of the first author [6, Lem. 2.2]. The monoidal structure is given by coproducts, and this again preserves the legs of the inner spans being monic.

$$\begin{tikzcd} & y & \\ x \arrow[ur] \arrow[r] \arrow[dr] & y' \arrow[u, tail] \arrow[d, tail] & z \arrow[ul] \arrow[l] \arrow[dl] \\ & y'' & \end{tikzcd} + \begin{tikzcd} & w & \\ v \arrow[ur] \arrow[r] \arrow[dr] & w' \arrow[u, tail] \arrow[d, tail] & u \arrow[ul] \arrow[l] \arrow[dl] \\ & w'' & \end{tikzcd} = \begin{tikzcd} & y + w & \\ x + v \arrow[ur] \arrow[r] \arrow[dr] & y' + w' \arrow[u, tail] \arrow[d, tail] & z + u \arrow[ul] \arrow[l] \arrow[dl] \\ & y'' + w'' & \end{tikzcd}$$

This bicategory is explored further in Section 5.1 where we take **T** to be the topos of directed graphs.

3. Double categories and duality

Bicategories are nice, but symmetric monoidal bicategories are much nicer. Unfortunately, checking the coherence conditions for a symmetric monoidal bicategory is daunting due to the sheer number of them. However, using a result of Shulman [19], we can circumvent checking these conditions by promoting our bicategories to double categories and showing that the double categories are symmetric monoidal. This involves proving that a pair of categories are symmetric monoidal, which is a much more manageable task. Even better than symmetric monoidal bicategories are those that are compact closed. The notion of compact closedness we consider is given by Stay [20]. After summarizing Shulman's and Stay's work, we use their machinery to show that **MonicSp**(**Csp**(**T**)) is not only symmetric monoidal, but also compact closed.

3.1. Monoidal bicategories from monoidal double categories. Double categories, or pseudo double categories to be precise, have been studied by Fiore [9] and Paré and Grandis [12] among others. Before giving a formal definition, it is helpful to have the following picture in mind. A double category has 2-morphisms that look like this:

$$\begin{tikzcd} A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow a"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "\shortmid"{marking}] & D \end{tikzcd} \tag{2}$$

We call A, B, C and D **objects** or **0-cells**, f and g **vertical 1-morphisms**, M and N **horizontal 1-morphisms**, and a a **2-morphism**. Note that vertical 1-morphisms go between objects and 2-morphisms go between horizontal 1-morphisms. In our definitions, we denote a double category with a bold font '$\mathbb{D}$' either as a stand-alone letter or as the first letter in a longer name.

3.2. Definition. *A **pseudo double category** $\mathbb{D}$, or simply **double category**, consists of a category of objects $\mathbb{D}_0$ and a category of arrows $\mathbb{D}_1$ together with the following functors*

$$\begin{aligned} U\colon \mathbb{D}_0 &\to \mathbb{D}_1, \\ S,T\colon \mathbb{D}_1 &\rightrightarrows \mathbb{D}_0, \text{ and} \\ \odot\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 &\to \mathbb{D}_1 \end{aligned}$$

where the pullback $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ is taken over S and T. These functors satisfy the equations

$$\begin{aligned} S(U_A) = A &= T(U_A) \\ S(M \odot N) &= SN \\ T(M \odot N) &= TM. \end{aligned}$$

This also comes equipped with natural isomorphisms

$$\begin{aligned} \alpha&\colon (M \odot N) \odot P \to M \odot (N \odot P) \\ \lambda&\colon U_B \odot M \to M \\ \rho&\colon M \odot U_A \to M \end{aligned}$$

such that $S(\alpha)$, $S(\lambda)$, $S(\rho)$, $T(\alpha)$, $T(\lambda)$, and $T(\rho)$ are each identities and that the coherence axioms of a monoidal category are satisfied.

*To match this definition with the more intuitive terms used, we say **vertical 1-morphisms** for the $\mathbb{D}_0$-morphisms, **horizontal 1-morphisms**[1] for the $\mathbb{D}_1$-objects, and **2-morphisms** for the $\mathbb{D}_1$-morphisms. As for notation, we write vertical and horizontal morphisms with the arrows $\to$ and $\nrightarrow$, respectively, and 2-morphisms we draw as in (2).*

An equivalent perspective to this definition is that a *pseudo* double category is a category 'weakly internal' to **Cat**, whereas a category internal to **Cat** is an ordinary double category, meaning that the natural isomorphisms above are identities.

To bypass checking that a bicategory is monoidal, we instead need to check that a certain double category is monoidal. To define a monoidal double category, however, we need the notion of a **globular 2-morphism**. This is a 2-morphism whose source and target vertical 1-morphisms are identities.

[1]Sometimes the term **horizontal 1-cell** is used for these [19], and for good reason. A $(n \times 1)$-category consists of categories $\mathbf{D_i}$ for $0 \leq i \leq n$ where the objects of $\mathbf{D_i}$ are i-cells and the morphisms of $\mathbf{D_i}$ are vertical $i+1$-morphisms. A double category is then just a (1×1)-category. From this perspective, 'cells' are always objects with morphisms going between them.

3.3. Definition. *A **monoidal double category** is a double category $\mathbb{D}$ such that:*

(a) $\mathbb{D}_0$ *and* $\mathbb{D}_1$ *are both monoidal categories.*

(b) If I *is the monoidal unit of* $\mathbb{D}_0$*, then* U_I *is the monoidal unit of* $\mathbb{D}_1$*.*

(c) The functors S *and* T *are strict monoidal and preserve the associativity and unit constraints.*

(d) There are globular 2-isomorphisms

$$\mathfrak{x}\colon (M_1 \otimes N_1) \odot (M_2 \otimes N_2) \to (M_1 \odot M_2) \otimes (N_1 \odot N_2)$$

and

$$\mathfrak{u}\colon U_{A\otimes B} \to (U_A \otimes U_B)$$

such that the following diagrams commute:

(e) The following diagrams commute expressing the constraint data for the double functor $\otimes$*.*

$$\begin{tikzcd}
((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \odot (M_3 \otimes N_3) \arrow[d, "\alpha"'] & ((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \odot (M_3 \otimes N_3) \arrow[l, "\mathfrak{x} \odot 1"'] \arrow[d, "\mathfrak{x}"] \\
(M_1 \otimes N_1) \odot ((M_2 \otimes N_2) \odot (M_3 \otimes N_3)) \arrow[d, "1 \odot \mathfrak{x}"'] & ((M_1 \odot M_2) \odot M_3) \otimes ((N_1 \odot N_2) \odot N_3) \arrow[d, "\alpha \otimes \alpha"] \\
(M_1 \otimes N_1) \odot ((M_2 \odot M_3) \otimes (N_2 \odot N_3)) & (M_1 \odot (M_2 \odot M_3)) \otimes (N_1 \odot (N_2 \odot N_3)) \arrow[l, "\mathfrak{x}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(M \otimes N) \odot U_{C\otimes D} \arrow[d, "\rho"'] & (M \otimes N) \odot (U_C \otimes U_D) \arrow[l, "1 \odot \mathfrak{u}"'] \arrow[d, "\mathfrak{x}"] \\
M \otimes N & (M \odot U_C) \otimes (N \odot U_D) \arrow[l, "\rho \otimes \rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{A\otimes B} \odot (M \otimes N) \arrow[r, "\mathfrak{u} \odot 1"] \arrow[d, "\lambda"'] & (U_A \otimes U_B) \odot (M \otimes N) \arrow[d, "\mathfrak{x}"] \\
M \otimes N & (U_A \odot M) \otimes (U_B \odot N) \arrow[l, "\lambda \otimes \lambda"']
\end{tikzcd}$$

(f) The following diagrams commute expressing the associativity isomorphism for $\otimes$ *is a transformation of double categories.*

$$\begin{tikzcd}
((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2) \arrow[d, "\mathfrak{x}"'] & (M_1 \otimes (N_1 \otimes P_1)) \odot (M_2 \otimes (N_2 \otimes P_2)) \arrow[l, "a \odot a"'] \arrow[d, "\mathfrak{x}"] \\
((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \otimes (P_1 \odot P_2) \arrow[d, "\mathfrak{x} \otimes 1"'] & (M_1 \odot M_2) \otimes ((N_1 \otimes P_1) \odot (N_2 \otimes P_2)) \arrow[d, "1 \otimes \mathfrak{x}"] \\
((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \otimes (P_1 \odot P_2) & (M_1 \odot M_2) \otimes ((N_1 \odot N_2) \otimes (P_1 \odot P_2)) \arrow[l, "a"']
\end{tikzcd}$$

$$\begin{tikzcd}
U_{(A\otimes B)\otimes C} \arrow[r, "U_a"] \arrow[d, "\mathfrak{u}"'] & U_{A\otimes(B\otimes C)} \arrow[d, "\mathfrak{u}"] \\
U_{A\otimes B} \otimes U_C \arrow[d, "\mathfrak{u} \otimes 1"'] & U_A \otimes U_{B\otimes C} \arrow[d, "1 \otimes \mathfrak{u}"] \\
(U_A \otimes U_B) \otimes U_C \arrow[r, "a"] & U_A \otimes (U_B \otimes U_C)
\end{tikzcd}$$

(g) The following diagrams commute expressing that the unit isomorphisms for $\otimes$ are transformations of double categories.

$$
\begin{tikzcd}
(M \otimes U_I) \odot (N \otimes U_I) \arrow[r, "\mathfrak{x}"] \arrow[d, "r \odot r"'] & (M \odot N) \otimes (U_I \odot U_I) \arrow[d, "1 \otimes \rho"] \\
M \odot N & (M \odot N) \otimes U_I \arrow[l, "r"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U_A \otimes U_I \arrow[dd, "r"] \\
U_{A \otimes I} \arrow[ur, "\mathfrak{u}"] \arrow[dr, "U_r"'] & \\
 & U_A
\end{tikzcd}
$$

$$
\begin{tikzcd}
(U_I \otimes M) \odot (U_I \otimes N) \arrow[r, "\mathfrak{x}"] \arrow[d, "\ell \odot \ell"'] & (U_I \odot U_I) \otimes (M \odot N) \arrow[d, "\lambda \otimes 1"] \\
M \odot N & U_I \otimes (M \odot N) \arrow[l, "\ell"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U_I \otimes U_A \arrow[dd, "\ell"] \\
U_{I \otimes A} \arrow[ur, "\mathfrak{u}"] \arrow[dr, "U_\ell"'] & \\
 & U_A
\end{tikzcd}
$$

A ***braided monoidal double category*** *is a monoidal double category such that:*

(h) $\mathbb{D}_0$ and $\mathbb{D}_1$ are braided monoidal categories.

(i) The functors S and T are strict braided monoidal functors.

(j) The following diagrams commute expressing that the braiding is a transformation of double categories.

$$
\begin{tikzcd}
(M_1 \odot M_2) \otimes (N_1 \odot N_2) \arrow[r, "\mathfrak{s}"] \arrow[d, "\mathfrak{x}"'] & (N_1 \odot N_2) \otimes (M_1 \odot M_2) \arrow[d, "\mathfrak{x}"] \\
(M_1 \otimes N_1) \odot (M_2 \otimes N_2) \arrow[r, "\mathfrak{s} \odot \mathfrak{s}"] & (N_1 \otimes M_1) \odot (N_2 \otimes M_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
U_A \otimes U_B \arrow[d, "\mathfrak{s}"'] & U_{A \otimes B} \arrow[l, "\mathfrak{u}"'] \arrow[d, "U_{\mathfrak{s}}"] \\
U_B \otimes U_A & U_{B \otimes A} \arrow[l, "\mathfrak{u}"']
\end{tikzcd}
$$

Finally, a ***symmetric monoidal double category*** *is a braided monoidal double category $\mathbb{D}$ such that*

(k) $\mathbb{D}_0$ and $\mathbb{D}_1$ are symmetric monoidal.

3.4. Definition. *Let $\mathbb{D}$ be a double category and $f \colon A \to B$ a vertical 1-morphism. A* ***companion*** *of f is a horizontal 1-morphism $\widehat{f} \colon A \nrightarrow B$ together with 2-morphisms*

$$
\begin{tikzcd}
A \arrow[r, "\widehat{f}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "B"] \\
B \arrow[r, "U_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, "A"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "\widehat{f}"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

such that the following equations hold:

$$
\begin{tikzcd}
A \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, "A"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "\widehat{f}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "B"] \\
B \arrow[r, "U_B"', "\shortmid"{marking}] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & A \arrow[d, "f"] \\
B \arrow[r, "U_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, "A"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[r, "\widehat{f}", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "B"] \\
A \arrow[r, "\widehat{f}"', "\shortmid"{marking}] & B \arrow[r, "U_B"', "\shortmid"{marking}] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\widehat{f}", "\shortmid"{marking}] \arrow[d, "A"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{\widehat{f}}"] & B \arrow[d, "B"] \\
A \arrow[r, "\widehat{f}"', "\shortmid"{marking}] & B
\end{tikzcd}
\tag{3}
$$

A ***conjoint*** *of f, denoted $\check{f} \colon B \nrightarrow A$, is a companion of f in the double category $\mathbb{D}^{h \cdot \mathrm{op}}$ obtained by reversing the horizontal 1-morphisms, but* not *the vertical 1-morphisms.*

3.5. Definition. *We say that a double category is* ***fibrant*** *if every vertical 1-morphism has both a companion and a conjoint. If every* invertible *vertical 1-morphism has both a companion and a conjoint, then we say the double category is* ***isofibrant****.*

The final piece we need to present the main theorem of this section is the following. Given a double category $\mathbb{D}$, the **horizontal edge bicategory** $H(\mathbb{D})$ of $\mathbb{D}$ is the bicategory whose objects are those of $\mathbb{D}$, morphisms are horizontal 1-morphisms of $\mathbb{D}$, and 2-morphisms are the globular 2-morphisms.

3.6. Theorem. [Shulman [19, Theorem 5.1]] *Let $\mathbb{D}$ be an isofibrant symmetric monoidal double category. Then $H(\mathbb{D})$ is a symmetric monoidal bicategory.*

Thanks to Theorem 3.6, we can show that $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ is symmetric monoidal much more efficiently than if we were to drudge through all of the axioms. But before we do this, we recall the notion of compactness in a bicategory.

3.7. Duality in bicategories. In this section, we introduce various notions of duality in order to define 'compact closed bicategories' as conceived by Stay [20]. We write LR for the tensor product of objects L and R and fg for the tensor product of morphisms f and g. This lets us reserve the symbol '$\otimes$' for the horizontal composition functor of a bicategory.

3.8. Definition. *A* ***dual pair*** *in a monoidal category is a tuple (L, R, e, c) with objects L and R, called the* ***left*** *and* ***right*** *duals, and morphisms*

$$e\colon LR \to I \qquad c\colon I \to RL,$$

called the ***counit*** *and* ***unit****, respectively, such that the following diagams commute.*

$$\begin{tikzcd} L \arrow[dr, "Lc"] \arrow[dd, "L"'] & \\ & LRL \arrow[dl, "eL"] \\ L & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[dr, "cR"] \arrow[dd, "R"'] & \\ & RLR \arrow[dl, "Re"] \\ R & \end{tikzcd}$$

3.9. Definition. *Inside a monoidal bicategory, a* ***dual pair*** *is a tuple $(L, R, e, c, \alpha, \beta)$ with objects L and R, morphisms*

$$e\colon LR \to I \qquad c\colon I \to RL,$$

and invertible 2-morphisms

$$\begin{tikzcd} L \arrow[r, "L"] \arrow[d] & L \\ LI \arrow[d, "Lc"'] \arrow[r, phantom, "\Downarrow\alpha"] & IL \arrow[u] \\ L(RL) \arrow[r] & (LR)L \arrow[u, "eL"'] \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "R"] \arrow[d] & R \\ RI \arrow[d, "cR"'] \arrow[r, phantom, "\Downarrow\beta"] & RI \arrow[u] \\ (RL)R \arrow[r] & R(LR) \arrow[u, "Re"'] \end{tikzcd}$$

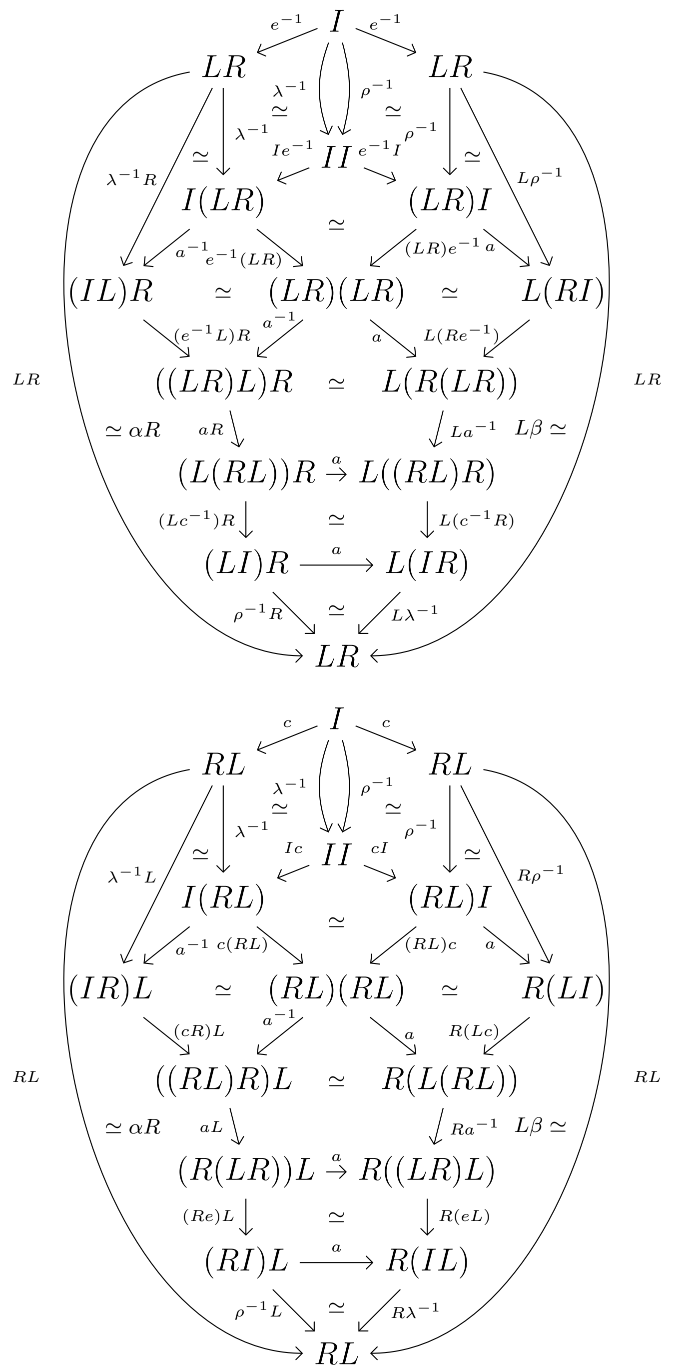


Figure 1: The swallowtail diagrams for the (co)unit.

called ***cusp isomorphisms****. If this data satisfies the swallowtail equations in the sense that the diagrams in Figure 1 are identities, then we call the dual pair* ***coherent****.*

Recall that a symmetric monoidal category is called **compact closed** if every object is part of a dual pair. We can generalize this idea to bicategories by introducing 2-morphisms and some coherence axioms. The following definition is due to Stay [20].

3.10. Definition. *A* ***compact closed*** *bicategory is a symmetric monoidal bicategory for which every object R is part of a coherent dual pair.*

The difference between showing compact closededness in categories versus bicategories might seem quite large because of the swallowtail equations. Looking at Figure 1, it is no surprise that these can be incredibly tedious to work with. Fortunately, Pstrągowski [17] proved a wonderful strictification theorem that effectively circumvents the need to consider the swallowtail equations.

3.11. Theorem. [17, p. 22] *Given a dual pair $(L, R, e, c, \alpha, \beta)$, we can find a cusp isomorphism β' such that $(L, R, e, c, \alpha, \beta')$ is a coherent dual pair.*

With the requisite background covered, we can move on to our main results.

4. Main results

4.1. Definition. *Let* $\mathbf{T}$ *be a topos. We will define a double category* $\mathbb{M}\mathbf{onicSp}(\mathbf{Csp}(\mathbf{T}))$ *whose objects are the objects of* $\mathbf{T}$*, vertical 1-morphisms are isomorphism classes of spans with invertible legs in* $\mathbf{T}$*, horizontal 1-morphisms are cospans in* $\mathbf{T}$*, and 2-morphisms are isomorphism classes of spans of cospans in* $\mathbf{T}$ *with monic legs. In other words, a* 2*-morphism is a commuting diagram in* $\mathbf{T}$ *of the form:*

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \\
\bullet \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & \bullet \arrow[u, tail] \arrow[d, tail] & \bullet \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

4.2. Lemma. $\mathbb{M}\mathbf{onicSp}(\mathbf{Csp}(\mathbf{T}))$ *is a double category.*

Proof. Denote $\mathbb{M}\mathbf{onicSp}(\mathbf{Csp}(\mathbf{T}))$ by $\mathbb{M}$. The object category $\mathbb{M}_0$ is given by objects of $\mathbf{T}$ and isomorphism classes of spans in $\mathbf{T}$ such that each leg is an isomorphism. The arrow category $\mathbb{M}_1$ has as objects the cospans in $\mathbf{T}$ and as morphisms the isomorphism classes of spans of cospans with monic legs as in the diagram above. We denote a span $x \leftarrow y \to z$ as $y \colon x \xrightarrow{\text{sp}} z$ and a cospan $x \to y \leftarrow z$ as $y \colon x \xrightarrow{\text{csp}} z$.

The functor $U \colon \mathbb{M}_0 \to \mathbb{M}_1$, introduced in 3.2, acts on objects by mapping x to the identity cospan on x and on morphisms by mapping $y \colon x \xrightarrow{\text{sp}} z$, whose legs are isomorphisms, to the square

$$\begin{tikzcd}
x \arrow[r] & x & x \arrow[l] \\
y \arrow[r] \arrow[u] \arrow[d] & y \arrow[u] \arrow[d] & y \arrow[l] \arrow[u] \arrow[d] \\
z \arrow[r] & z & z \arrow[l]
\end{tikzcd}$$

The functor $S \colon \mathbb{M}_1 \to \mathbb{M}_0$, also introduced in 3.2, acts on objects by sending $y \colon x \xrightarrow{\text{csp}} z$ to x and on morphisms by sending a square to the span occupying the square's left vertical side. The other functor T is defined similarly.

The horizontal composition functor $\odot \colon \mathbb{M}_1 \times_{\mathbb{M}_0} \mathbb{M}_1 \to \mathbb{M}_1$ acts on objects by composing cospans with pushouts in the usual way. It acts on morphisms by

$$\begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \arrow[r] & d & e \arrow[l] \\
a' \arrow[r] \arrow[u] \arrow[d] & b' \arrow[u] \arrow[d] & c' \arrow[l] \arrow[r] \arrow[u] \arrow[d] & d' \arrow[u] \arrow[d] & e' \arrow[l] \arrow[u] \arrow[d] \\
a'' \arrow[r] & b'' & c'' \arrow[l] \arrow[r] & d'' & e'' \arrow[l]
\end{tikzcd}
\quad \overset{\odot}{\mapsto} \quad
\begin{tikzcd}
a \arrow[r] & b +_c d & e \arrow[l] \\
a' \arrow[r] \arrow[u] \arrow[d] & b' +_{c'} d' \arrow[u] \arrow[d] & e' \arrow[l] \arrow[u] \arrow[d] \\
a'' \arrow[r] & b'' +_{c''} d'' & e'' \arrow[l]
\end{tikzcd}$$

This respects identities. We prove that $\odot$ preserves composition in Lemma 4.3 below. It is straightforward to check that the required equations are satisfied. The associator and unitors are given by natural isomorphisms that arise from universal properties. ■

4.3. Lemma. *The assignment $\odot$ from Lemma 4.2 preserves composition. In particular, $\odot$ is a functor.*

Proof. Let α, α', β, and β' be composable 2-morphisms given by

$$\alpha = \begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \\
a' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & b' \arrow[u, tail] \arrow[d, tail] & c' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
\ell \arrow[r] & m & n \arrow[l]
\end{tikzcd}
\qquad
\alpha' = \begin{tikzcd}
c \arrow[r] & d & e \arrow[l] \\
c' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & d' \arrow[u, tail] \arrow[d, tail] & e' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
n \arrow[r] & p & q \arrow[l]
\end{tikzcd}$$

$$\beta = \begin{tikzcd}
\ell \arrow[r] & m & n \arrow[l] \\
v' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & w' \arrow[u, tail] \arrow[d, tail] & x' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
v \arrow[r] & w & x \arrow[l]
\end{tikzcd}
\qquad
\beta' = \begin{tikzcd}
n \arrow[r] & p & q \arrow[l] \\
x' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & y' \arrow[u, tail] \arrow[d, tail] & z' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
x \arrow[r] & y & z \arrow[l]
\end{tikzcd}$$

Our goal is to show that

$$(\alpha \odot \alpha') \circ (\beta \odot \beta') = (\alpha \circ \beta) \odot (\alpha' \circ \beta'). \tag{4}$$

The left hand side of (4) corresponds to performing horizontal composition before vertical composition. The right hand side of (4) reverses that order.

First, we compute the left hand side of (4). Composing horizontally, $\alpha \odot \alpha'$ and $\beta \odot \beta'$ are, respectively,

$$
\begin{tikzcd}
a \arrow[r] & b +_c d & e \arrow[l] \\
a' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & b' +_{c'} d' \arrow[u, tail] \arrow[d, tail] & e' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
\ell \arrow[r] & m +_n p & q \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\ell \arrow[r] & m +_n p & q \arrow[l] \\
v' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & w' +_{x'} y' \arrow[u, tail] \arrow[d, tail] & z' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
v \arrow[r] & w +_x y & z \arrow[l]
\end{tikzcd}
$$

The preservation of monics under this operation follows from [6, Lem. 2.1]. Next, vertically composing $\alpha \odot \alpha'$ and $\beta \odot \beta'$, we get that $(\alpha \odot \alpha') \circ (\beta \odot \beta')$ is equal to

$$
\begin{tikzcd}
a \arrow[r] & b +_d d & e \arrow[l] \\
a' \times_\ell v' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & (b' +_{c'} d') \times_{(m +_n p)} (w' +_{x'} y') \arrow[u, tail] \arrow[d, tail] & e' \times_q z' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
v \arrow[r] & w +_x y & z \arrow[l]
\end{tikzcd}
\tag{5}
$$

Solving for the right hand side of (4), we first obtain that $\alpha \circ \beta$ and $\alpha' \circ \beta'$ are, respectively,

$$
\begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \\
a' \times_\ell v' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & b' \times_m w' \arrow[u, tail] \arrow[d, tail] & c' \times_n x' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
v \arrow[r] & w & x \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r] & d & e \arrow[l] \\
c' \times_n x' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & d' \times_p y' \arrow[u, tail] \arrow[d, tail] & e' \times_q z' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
x \arrow[r] & y & z \arrow[l]
\end{tikzcd}
$$

Composing these horizontally, we get that $(\alpha \circ \beta) \odot (\alpha' \circ \beta')$ equals

$$
\begin{tikzcd}
a \arrow[r] & b +_c d & e \arrow[l] \\
a' \times_\ell v' \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"'] & (b' \times_m w') +_{(c' \times_n x')} (d' \times_p y') \arrow[u, tail] \arrow[d, tail] & e' \times_q z' \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\cong"'] \\
v \arrow[r] & w +_x y & z \arrow[l]
\end{tikzcd}
\tag{6}
$$

Now, we need to show that (5) is equal to (6) as 2-morphisms. Note that the diagrams only differ in the middle. Thus, to complete the interchange law, it suffices to establish an isomorphism

$$(b' +_{c'} d') \times_{(m +_n p)} (w' +_{x'} y') \to (b' \times_m w') +_{(c' \times_n x')} (d' \times_p y')$$

But because the left and right vertical spans have isomorphisms for legs, the isomorphism we seek follows from [6, Lem. 2.5]. ∎

4.4. Lemma. $\mathbb{M}\mathbf{onicSp}(\mathbf{Csp}(\mathbf{T}))$ *is a symmetric monoidal double category.*

Proof. Let us first show that the category of objects $\mathbb{M}_0$ and the category of arrows $\mathbb{M}_1$ are symmetric monoidal categories. Note that $\mathbb{M}_0$ is the largest groupoid contained in $\mathbf{Sp}(\mathbf{T})$. We obtain the monoidal structure on $\mathbb{M}_0$ by lifting the cocartesian structure on $\mathbf{T}$ to the objects and by defining

$$(b\colon a \xrightarrow{\text{sp}} c) + (b'\colon a' \xrightarrow{\text{sp}} c') = (b + b'\colon a + a' \xrightarrow{\text{sp}} c + c')$$

on morphisms. Universal properties provide the associator and unitors as well as the coherence axioms. This monoidal structure is clearly symmetric.

Next, we have that $\mathbb{M}_1$ is the category whose objects are the cospans in $\mathbf{T}$ and morphisms are the isomorphism classes of monic spans of cospans in $\mathbf{T}$. We obtain a symmetric monoidal structure on the objects via

$$(b\colon a \xrightarrow{\text{csp}} c) + (b'\colon a' \xrightarrow{\text{csp}} c') = (b + b'\colon a + a' \xrightarrow{\text{csp}} c + c')$$

and on the morphisms by

$$
\begin{tikzcd}
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \\
\bullet \arrow[r] \arrow[u] \arrow[d] & \bullet \arrow[u, tail] \arrow[d, tail] & \bullet \arrow[l] \arrow[u] \arrow[d] \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}
\quad + \quad
\begin{tikzcd}
* \arrow[r] & * & * \arrow[l] \\
* \arrow[r] \arrow[u] \arrow[d] & * \arrow[u, tail] \arrow[d, tail] & * \arrow[l] \arrow[u] \arrow[d] \\
* \arrow[r] & * & * \arrow[l]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet + * \arrow[r] & \bullet + * & \bullet + * \arrow[l] \\
\bullet + * \arrow[r] \arrow[u] \arrow[d] & \bullet + * \arrow[u, tail] \arrow[d, tail] & \bullet + * \arrow[l] \arrow[u] \arrow[d] \\
\bullet + * \arrow[r] & \bullet + * & \bullet + * \arrow[l]
\end{tikzcd}
$$

Again, universal properties provide the associator, unitors, and coherence axioms. Hence both $\mathbb{M}_0$ and $\mathbb{M}_1$ are symmetric monoidal categories.

It remains to find globular isomorphisms $\mathfrak{x}$ and $\mathfrak{u}$ such that the required diagrams commute. To find $\mathfrak{x}$, fix horizontal 1-morphisms

$$
\begin{aligned}
b\colon a &\xrightarrow{\text{csp}} c, \qquad & w\colon v &\xrightarrow{\text{csp}} x, \\
d\colon c &\xrightarrow{\text{csp}} e, \qquad & y\colon x &\xrightarrow{\text{csp}} z'.
\end{aligned}
$$

The globular isomorphism $\mathfrak{x}$ is an invertible 2-morphism with domain

$$(a + v) \to (b + w) +_{(c+x)} (d + y) \leftarrow (e + z)$$

and codomain

$$(a + v) \to (b +_c d) + (w +_x y) \leftarrow (e + z).$$

This comes down to finding an isomorphism in $\mathbf{T}$ between the apices of the above cospans. Such an isomorphism exists, and is unique, because both apices are colimits of the non-connected diagram

$$
\begin{tikzcd}[column sep=small]
& b & & d & & & w & & y & \\
a \arrow[ur] & & c \arrow[ul] \arrow[ur] & & e \arrow[ul] & v \arrow[ur] & & x \arrow[ul] \arrow[ur] & & z \arrow[ul]
\end{tikzcd}
$$

Moreover, the resulting globular isomorphism is a monic span of cospans as the universal maps are isomorphisms. The globular isomorphism $\mathfrak{u}$ is similar.

Finally, we check that the coherence axioms, namely (a)-(k) of Definition 3.3, hold. These are straightforward, though tedious, to verify. For instance, if we have

$$M_1 = \begin{tikzcd}[column sep=small] & b & \\ a \arrow[ur] & & c \arrow[ul] \end{tikzcd} \quad M_2 = \begin{tikzcd}[column sep=small] & d & \\ c \arrow[ur] & & e \arrow[ul] \end{tikzcd} \quad M_3 = \begin{tikzcd}[column sep=small] & f & \\ e \arrow[ur] & & g \arrow[ul] \end{tikzcd}$$

$$N_1 = \begin{tikzcd}[column sep=small] & u & \\ t \arrow[ur] & & v \arrow[ul] \end{tikzcd} \quad N_2 = \begin{tikzcd}[column sep=small] & w & \\ v \arrow[ur] & & x \arrow[ul] \end{tikzcd} \quad N_3 = \begin{tikzcd}[column sep=small] & y & \\ x \arrow[ur] & & z \arrow[ul] \end{tikzcd}$$

then following diagram ((e)) around the top right gives the sequence of cospans

$$((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \odot (M_3 \otimes N_3) =$$

$$\begin{tikzcd}[column sep=small] & ((b+w) +_{c+v} (d+w)) +_{e+x} (f+y) & \\ a+t \arrow[ur] & & g+z \arrow[ul] \end{tikzcd}$$

$$((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \odot (M_3 \otimes N_3) =$$

$$\begin{tikzcd}[column sep=small] & ((b +_c d) + (u +_v w)) +_{e+x} (f+y) & \\ a+t \arrow[ur] & & g+z \arrow[ul] \end{tikzcd}$$

$$((M_1 \odot M_2) \odot M_3) \otimes ((N_1 \odot N_2) \odot N_3) =$$

$$\begin{tikzcd}[column sep=small] & ((b +_c d) +_e f) + ((u +_v w) +_x y) & \\ a+t \arrow[ur] & & g+z \arrow[ul] \end{tikzcd}$$

$$(M_1 \odot (M_2 \odot M_3)) \otimes (N_1 \odot (N_2 \odot N_3)) =$$

$$\begin{tikzcd}[column sep=small] & (b +_c (d +_e f)) + (u +_v (w +_x y)) & \\ a+t \arrow[ur] & & g+z \arrow[ul] \end{tikzcd}$$

Following the diagram ((e)) around the bottom left gives another sequence of cospans

$$((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \odot (M_3 \otimes N_3) =$$

$$\begin{tikzcd}[column sep=small] & ((b+w) +_{c+v} (d+w)) +_{e+x} (f+y) & \\ a+t \arrow[ur] & & g+z \arrow[ul] \end{tikzcd}$$

$$(M_1 \otimes N_1) \odot ((M_2 \otimes N_2) \odot (M_3 \otimes N_3)) =$$

$$\begin{tikzcd}[column sep=small] & (b+w) +_{c+v} ((d+w) +_{e+x} (f+y)) & \\ a+t \arrow[ur] & & g+z \arrow[ul] \end{tikzcd}$$

$$(M_1 \otimes N_1) \odot ((M_2 \odot M_3) \otimes (N_2 \odot N_3)) =$$

$$\begin{tikzcd}
 & (b+u) +_{c+v} ((d +_e f) + (w +_x y)) & \\
a+t \arrow[ur, bend left] & & g+z \arrow[ul, bend right]
\end{tikzcd}$$

$$(M_1 \odot (M_2 \odot M_3)) \otimes (N_1 \odot (N_2 \odot N_3)) =$$

$$\begin{tikzcd}
 & (b +_c (d +_e f)) + (u +_v (w +_x y)) & \\
a+t \arrow[ur, bend left] & & g+z \arrow[ul, bend right]
\end{tikzcd}$$

Putting these together gives the following commutative diagram.

$$\begin{tikzcd}
a+t \arrow[r] & (b +_c (d +_e f)) + (u +_v (w +_x y)) & g+z \arrow[l] \\
a+t \arrow[r] \arrow[u] & ((b +_c d) +_e f) + ((u +_v w) +_x y) \arrow[u] & g+z \arrow[l] \arrow[u] \\
a+t \arrow[r] \arrow[u] & ((b +_c d) + (u +_v w)) +_{e+x} (f+y) \arrow[u] & g+z \arrow[l] \arrow[u] \\
a+t \arrow[r] \arrow[u] \arrow[d] & ((b+w) +_{c+v} (d+w)) +_{e+x} (f+y) \arrow[u] \arrow[d] & g+z \arrow[l] \arrow[u] \arrow[d] \\
a+t \arrow[r] \arrow[d] & (b+w) +_{c+v} ((d+w) +_{e+x} (f+y)) \arrow[d] & g+z \arrow[l] \arrow[d] \\
a+t \arrow[r] \arrow[d] & (b+u) +_{c+v} ((d +_e f) + (w +_x y)) \arrow[d] & g+z \arrow[l] \arrow[d] \\
a+t \arrow[r] & (b +_c (d +_e f)) + (u +_v (w +_x y)) & g+z \arrow[l]
\end{tikzcd}$$

The vertical 1-morphisms on the left and right are the the respective identity spans on $a+t$ and $g+z$. The vertical 1-morphisms in the center are isomorphism classes of monic spans where each leg is given by a universal map between two colimits of the same diagram. The horizontal 1-morphisms are given by universal maps into coproducts and pushouts. The top cospan is the same as the bottom cospan, making a bracelet-like figure in which all faces commute. The other diagrams witnessing coherence are given in a similar fashion. ∎

4.5. Lemma. *The symmetric monoidal double category* $\mathbb{M}\mathbf{onicSp}(\mathbf{Csp}(\mathbf{T}))$ *is isofibrant.*

Proof. The companion of a vertical 1-morphism $f = (b\colon a \xrightarrow{\mathrm{sp}} c)$ is given by $\widehat{f} = (b\colon a \xrightarrow{\mathrm{csp}} c)$ whose legs are the inverses of the legs of f. The required 2-morphisms are given by

$$\begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \\
b \arrow[u] \arrow[d] \arrow[r] & c \arrow[u] \arrow[d] & c \arrow[l] \arrow[u] \arrow[d] \\
c \arrow[r] & c & c \arrow[l]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
a \arrow[r] & a & a \arrow[l] \\
a \arrow[u] \arrow[d] \arrow[r] & a \arrow[u] \arrow[d] & b \arrow[l] \arrow[u] \arrow[d] \\
a \arrow[r] & b & c \arrow[l]
\end{tikzcd}$$

The conjoint of f is given by $\check{f} = \widehat{f}^{\mathrm{op}}$. ∎

The benefit of laying down this groundwork is that the following theorem now follows from applying Theorem 3.6 to $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$.

4.6. Theorem. $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ *is a symmetric monoidal bicategory.*

Proof. We have shown that $\mathbb{M}\mathbf{onicSp}(\mathbf{Csp}(\mathbf{T}))$ is an isofibrant symmetric monoidal pseudo double category and so we obtain $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ as the horizontal edge bicategory of $\mathbb{M}\mathbf{onicSp}(\mathbf{Csp}(\mathbf{T}))$ by Shulman's result. ∎

It remains to show that this bicategory is compact closed. We start with the following lemma.

4.7. Lemma. *The diagram*

$$\begin{tikzcd}
X+X+X \arrow[r, "X+\nabla"] \arrow[d, "\nabla+X"'] & X+X \arrow[d, "\nabla"] \\
X+X \arrow[r, "\nabla"] & X
\end{tikzcd}$$

is a pushout square.

Proof. Suppose that we have maps $f, g\colon X+X \to Y$ forming a cocone over the span inside the above diagram. Let $\iota\colon X \to X+X+X$ include X into the middle copy. Observe that $\ell \coloneqq (\nabla + X)\circ\iota$ and $r \coloneqq (X+\nabla)\circ\iota$ are, respectively, the left and right inclusions $X \to X+X$. Then $f\circ\ell = g\circ r$ is a map $X \to Y$, which we claim is the unique map making the required diagram commute. Indeed, given $h\colon X \to Y$ such that $f = h\circ\nabla = g$, then $g\circ r = f\circ\ell = h\circ\nabla\circ\ell = h$. ∎

4.8. Theorem. *The symmetric monoidal bicategory* $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ *is compact closed.*

Proof. First we show that each object is its own dual. For an object X, define the counit $e\colon X+X \xrightarrow{\mathrm{csp}} 0$ and unit $c\colon 0 \xrightarrow{\mathrm{csp}} X+X$ to be the following cospans:

$$e = (X+X \xrightarrow{\nabla} X \leftarrow 0), \qquad c = (0 \to X \xleftarrow{\nabla} X+X).$$

Next we define the cusp isomorphisms, α and β. Note that α is a 2-morphism whose domain is the composite

$$X \xrightarrow{\ell} X+X \xleftarrow{X+\nabla} X+X+X \xrightarrow{\nabla+X} X+X \xleftarrow{r} X$$

and whose codomain is the identity cospan on X. From Lemma 4.7 we have the equations $\nabla + X = \ell \circ \nabla$ and $X + \nabla = r \circ \nabla$ from which it follows that the domain of α is the identity cospan on X, and the codomain of β is also the identity cospan on X obtained as the composite

$$X \xrightarrow{r} X+X \xleftarrow{\nabla+X} X+X+X \xrightarrow{X+\nabla} X+X \xleftarrow{\ell} X$$

Take α and β each to be the isomorphism class determined by the identity 2-morphism on X, which in particular is a monic span of cospans. Thus we have a dual pair $(X, X, e, c, \alpha, \beta)$. By Theorem 3.11, there exists a cusp isomorphism β' such that $(X, X, e, c, \alpha, \beta')$ is a coherent dual pair, and thus $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ is compact closed. ∎

5. An application

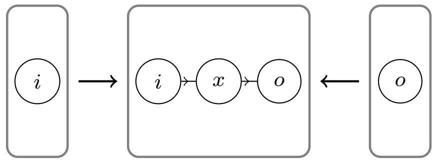
The primary motivation for constructing $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ is to provide a formalism in which to study networks that have inputs and outputs. Because of the symmetric monoidal and compact structure, we know that networks can be placed side by side by 'tensoring' and we can formally swap the inputs and outputs by compactness. For instance, the first author applied this formalism to the zx-calculus [7], which is a graphical language used for expressing operations on a pair of qubits.

In fact, this construction is a natural fit for other graphical calculi, too. The following example shows how to use our construction to rewrite a network. In particular, we replace a node with a more complex network and see how compactness affects this situation.

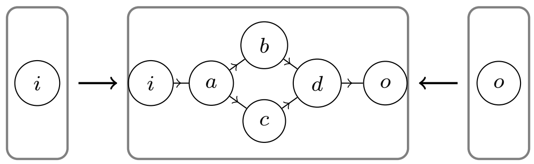
5.1. Replacing a node in a network. The following example was inspired by a comment of Michael Shulman [19] on the n-Cafe. The example involves replacing a particular node in some given network with another network, possibly more complex, whose inputs and outputs coincide with those of the node. For simplicity, we work with open graphs. First, we should place ourselves in the context of the compact closed sub-bicategory **Rewrite** of $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{Graph}))$ where **Rewrite** is the sub-bicategory that is 1-full and 2-full on the edgeless graphs as objects. The idea is that a 1-morphism is an **open graph**. That is, a graph with *input* and *output* nodes chosen by the legs of the cospan. The 2-morphisms contain all possible ways for an open graph to be rewritten into another open graph while preserving the inputs and outputs.

However, **Rewrite** is really only interesting as an ambient bicategory. We are particularly interested in sub-bicategories freely generated by various collections of 1-morphisms and 2-morphisms. The collection considered depends on our interests. The generators of the sub-bicategory **Rewrite** are chosen to provide a syntax for whatever types of networks we wish to study.

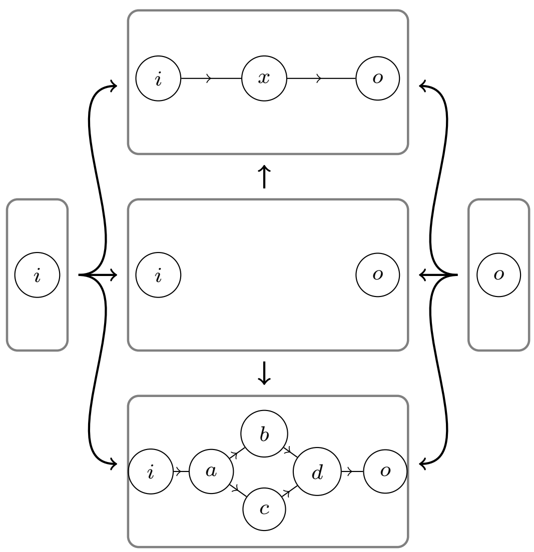
Suppose that we are working within a network where whenever we see



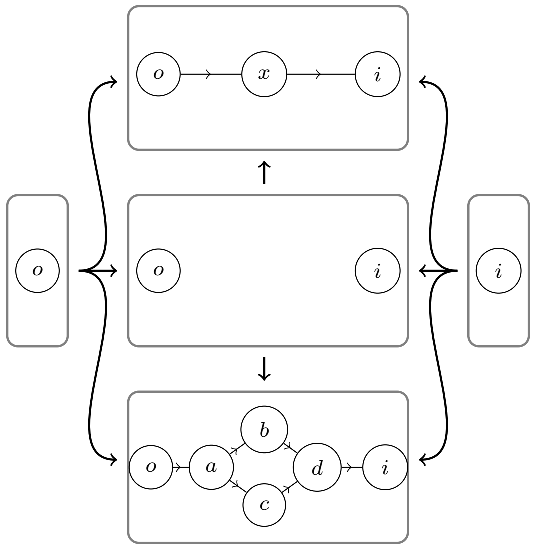
we want to replace it with the following:



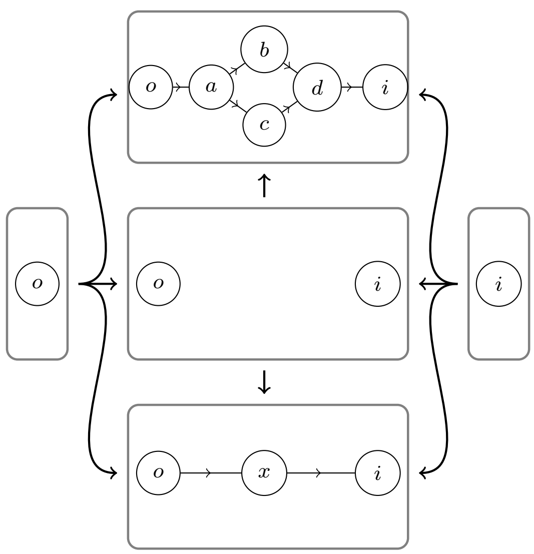
This corresponds to having the 2-morphism



where the rewritten network is obtained by taking a double pushout [6]. By compactness, once we have this 2-morphism, we can swap the roles of the inputs and outputs to obtain the following 2-morphism:



Hence there is no substantial difference between inputs and outputs. Also, we can flip this diagram vertically to obtain the following 2-morphism:



Thus the rewrite rules are also symmetric.

6. Conclusion

We have taken a closer look at spans of cospans, extending the results of the first author [6] by finding a symmetric monoidal and compact closed structure on the bicategory $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$. This structure is relevant when using $\mathbf{MonicSp}(\mathbf{Csp}(\mathbf{T}))$ as a framework in which to study various networks. Generally speaking, the symmetric monoidal structure allows us to consider disjoint networks as a single network by taking a coproduct of the two networks, and the compact closed structure allows us to turn open networks around, indicating that there is no substantial difference between inputs and outputs other than a shift in perspective. The primary advantage of having this structure to study networks is that in cases where we are able to find a sub-bicategory of **MonicSp**(**Csp**(**T**)) which gives the networks that are being considered a sense of inputs and outputs, we can freely generate compact closed bicategories with various families of 1-morphisms and 2-morphisms. This was illustrated in Section 5 by the compact closed sub-bicategory **Rewrite** of **MonicSp**(**Csp**(**Graph**)) which provided inputs and outputs to graphs.

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