Research Update: Material Scaling, Parameter Sensitivity, and System Extension

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1 Introduction

This update documents three significant extensions to the mass-spring lattice simulation system that enhance its capability to model heterogeneous materials and investigate parameter sensitivity:

1. **Column-Based Material Scaling**: Implementation of spatially-varying material properties where each column has different spring constants and damping coefficients, enabling simulation of layered or graded materials
2. **Parameter Sensitivity Analysis**: Systematic investigation of the effects of wall stiffness and damping parameters on system behavior
3. **10×10 System Extension**: Expansion of the simulation framework to larger lattice sizes (100 masses) with material scaling and backplate constraints

These enhancements enable the study of more complex material systems where properties vary spatially, which is relevant to many engineering applications including composite materials, functionally graded materials, and layered structures. The parameter sensitivity analysis provides insight into the robustness of the wall constraint implementation, while the larger system size allows investigation of wave propagation in extended domains.

1.1 Context and Motivation

Previous work established the foundation with exponential spring models, viscous damping, and immovable backplate constraints. The current extensions address the need to model heterogeneous materials where properties vary spatially, which is common in many practical applications. Additionally, understanding parameter sensitivity is crucial for ensuring numerical stability and physical realism in the simulations.

2 Column-Based Material Scaling Implementation

2.1 Physical Motivation

Many real-world materials exhibit spatial variation in mechanical properties. Examples include:

- **Layered composites**: Materials with distinct layers having different stiffness and damping properties
- **Functionally graded materials**: Materials with properties that vary continuously in space

- **Multi-material structures**: Systems composed of different materials arranged in columns or layers

To model such systems, we implement column-based material scaling where each column (from left to right) has material properties scaled by a multiplicative factor relative to the base column.

2.2 Mathematical Formulation

For a lattice with N columns, we define material properties for column j as:

$$k_j = k_1 \cdot m^{j-1} \tag{1}$$

where:

- k_j is the spring constant for column j
- k_1 is the base spring constant (column 1)
- m is the material multiplier
- $j = 1, 2, \ldots, N$ is the column index (left to right)

Column 1 uses the base value ($m^0 = 1$), while subsequent columns are scaled by powers of the multiplier. The same scaling applies to:

- Nearest-neighbor spring constants: $K_{coupling,j} = K_{coupling} \cdot m^{j-1}$
- Diagonal spring constants: $K_{diagonal,j} = K_{diagonal} \cdot m^{j-1}$
- Damping coefficients: $c_j = c \cdot m^{j-1}$

2.3 Implementation Details

Material properties are applied only to vertical springs within each column. Horizontal springs connecting columns use base values, maintaining continuity while allowing column-specific properties. This approach models a system where each column represents a distinct material layer.

The implementation uses lookup functions:

Listing 1: Material property lookup functions

```
function get_column_k_coupling(j, material_multiplier)
    if j == 1
        return K_COUPLING
    else
        return K_COUPLING * (material_multiplier^(j-1))
    end
end
```

In the ODE right-hand side function, vertical springs use column-specific properties:

Listing 2: Using column-based properties for vertical springs

```
# Vertical springs use column-based material properties
k_col = get_column_k_coupling(j, MATERIAL_MULTIPLIER)
c_col = get_column_c_damping(j, MATERIAL_MULTIPLIER)
add_spring_force!(i-1, j, k_col, ALPHA_COUPLING, true, c_col)  # up
add_spring_force!(i+1, j, k_col, ALPHA_COUPLING, true, c_col)  # down
```

2.4 Scaling Cases Studied

We investigate two scaling scenarios:

1. **Increasing stiffness**: $m = 1.5$ (columns become progressively stiffer)
2. **Decreasing stiffness**: $m = 2/3$ (columns become progressively softer)

These cases represent materials that either stiffen or soften from left to right, which can model various physical scenarios such as:

- Composite materials with increasing reinforcement density
- Functionally graded materials with property gradients
- Layered structures with alternating material properties

3 Simulation Results: Material Scaling Effects

3.1 5×5 System with 1.5× Multiplier

For the increasing stiffness case ($m = 1.5$), columns 2-5 have progressively higher spring constants and damping coefficients:

- Column 2: 1.5× base values
- Column 3: 2.25× base values
- Column 4: 3.375× base values
- Column 5: 5.0625× base values

This creates a material gradient where the rightmost columns are significantly stiffer than the leftmost column. Key observations include:

- **Wave propagation**: Waves traveling from left to right encounter increasing resistance, leading to reduced amplitude in stiffer columns
- **Energy distribution**: Energy tends to concentrate in the softer left columns due to reduced wave transmission through stiffer regions
- **Dissipation patterns**: Higher damping in stiffer columns leads to increased energy dissipation rates

3.2 5×5 System with 2/3× Multiplier

For the decreasing stiffness case ($m = 2/3$), columns 2-5 have progressively lower spring constants:

- Column 2: 0.667× base values
- Column 3: 0.444× base values
- Column 4: 0.296× base values
- Column 5: 0.198× base values

This creates a material gradient where the rightmost columns are significantly softer. Key observations include:

- **Wave propagation**: Waves traveling from left to right encounter decreasing resistance, leading to increased amplitude in softer columns
- **Energy distribution**: Energy tends to spread more easily into softer regions
- **Dissipation patterns**: Lower damping in softer columns leads to different dissipation characteristics

3.3 Comparison of Scaling Cases

Figure 1 compares energy evolution for both scaling cases. The increasing stiffness case (1.5×) shows different energy dynamics compared to the decreasing stiffness case (2/3×), demonstrating the significant impact of material property gradients on system behavior.

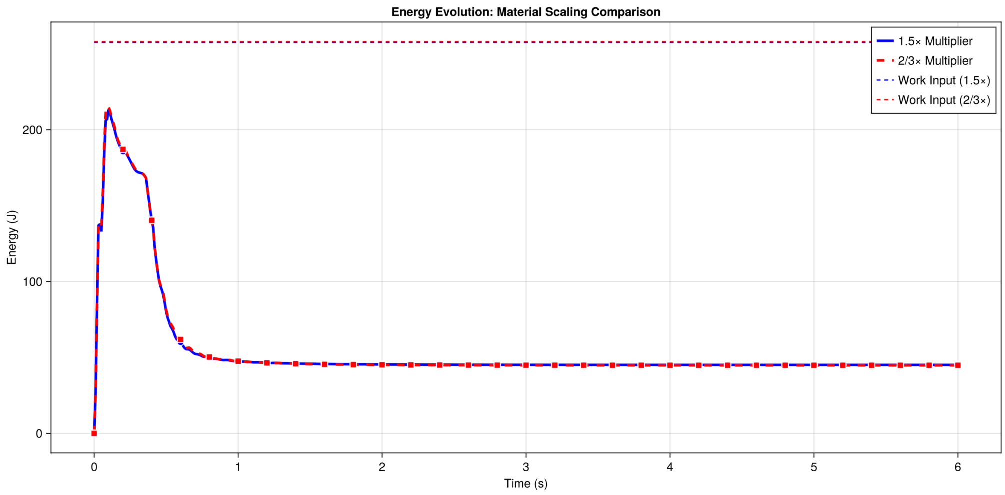

Figure 1: Energy evolution comparison for 5×5 systems with different material scaling multipliers. The 1.5× multiplier (increasing stiffness) and 2/3× multiplier (decreasing stiffness) produce distinct energy dynamics due to different wave propagation and dissipation patterns.

4 Parameter Sensitivity Analysis: Wall Properties

4.1 Motivation

The immovable backplate constraint uses two key parameters:

- **Wall stiffness** (k_{wall}): Controls the repulsive force when masses penetrate the wall
- **Wall damping** (c_{wall}): Controls energy dissipation during wall collisions

Understanding the sensitivity of system behavior to these parameters is essential for:

- Ensuring numerical stability
- Validating physical realism
- Optimizing parameter selection

4.2 Parameter Sweep Methodology

We perform a systematic parameter sweep testing combinations of:

- Wall stiffness multipliers: $[0.5, 1, 2, 5, 10]$ of base value (10000 N/m)
- Wall damping multipliers: $[0.5, 1, 2, 5, 10]$ of base value (10 N·s/m)

This yields 25 parameter combinations, each run as a complete simulation. For each combination, we extract:

- Total work input
- Final total energy
- Energy dissipation percentage
- Maximum displacement of rightmost column

4.3 Key Findings

The parameter sweep reveals several important trends:

1. **Wall stiffness effects**:
   - Higher stiffness reduces maximum penetration (as expected)
   - Very high stiffness (> 5) shows minimal additional benefit
   - Very low stiffness (< 0.5) allows excessive penetration
2. **Wall damping effects**:
   - Higher damping increases energy dissipation during collisions
   - Damping has less effect on maximum displacement than stiffness
   - Optimal damping balances energy dissipation with numerical stability
3. **Interaction effects**:
   - Stiffness and damping interact non-linearly
   - High stiffness with low damping can cause oscillations
   - Low stiffness with high damping provides smoother behavior

Figure 2 shows the parameter sweep results, illustrating the sensitivity of system behavior to wall properties.

5 10×10 System Extension

5.1 Implementation

The system is extended from 5×5 (25 masses) to 10×10 (100 masses), representing a fourfold increase in system size. This extension includes:

- All features from the 5×5 system (backplate, material scaling, damping)
- Proper scaling of all computational components
- Visualization adapted for larger system

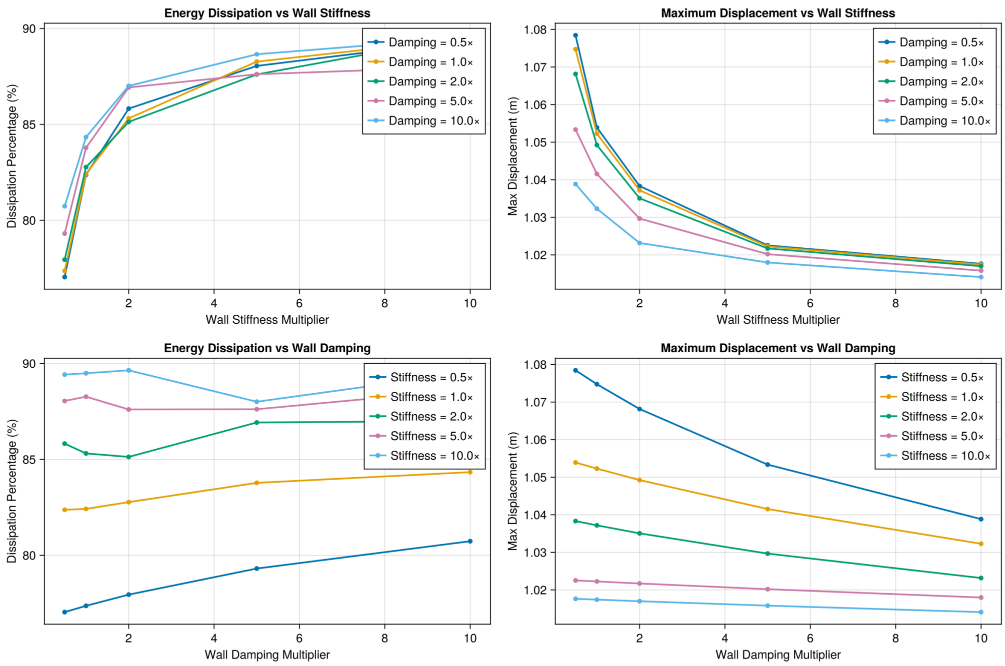

Figure 2: Parameter sweep results for wall stiffness and damping. The plots show how energy dissipation and maximum displacement vary with different parameter combinations, providing insight into parameter sensitivity and optimal selection.

5.2 Computational Considerations

The larger system presents several computational challenges:

- **Increased DOF**: 200 degrees of freedom (vs. 50 for 5×5)
- **More springs**: 360 total springs (vs. 80 for 5×5)
- **Longer simulation time**: Approximately 4× longer computation time
- **Memory requirements**: Larger state vectors and solution storage

Despite these challenges, the implementation maintains numerical stability and accuracy through:

- Efficient sparse matrix operations
- Optimized force calculations
- Appropriate solver tolerances

5.3 Results and Observations

The 10×10 system with material scaling exhibits several interesting behaviors:

- **Wave propagation**: More complex wave patterns due to increased system size
- **Material gradient effects**: More pronounced effects of material scaling across 10 columns
- **Energy distribution**: More spatially distributed energy patterns
- **Boundary interactions**: More complex interactions between waves and the backplate

Figure 3 shows a snapshot of the 10×10 system, demonstrating the larger scale and material gradient effects.

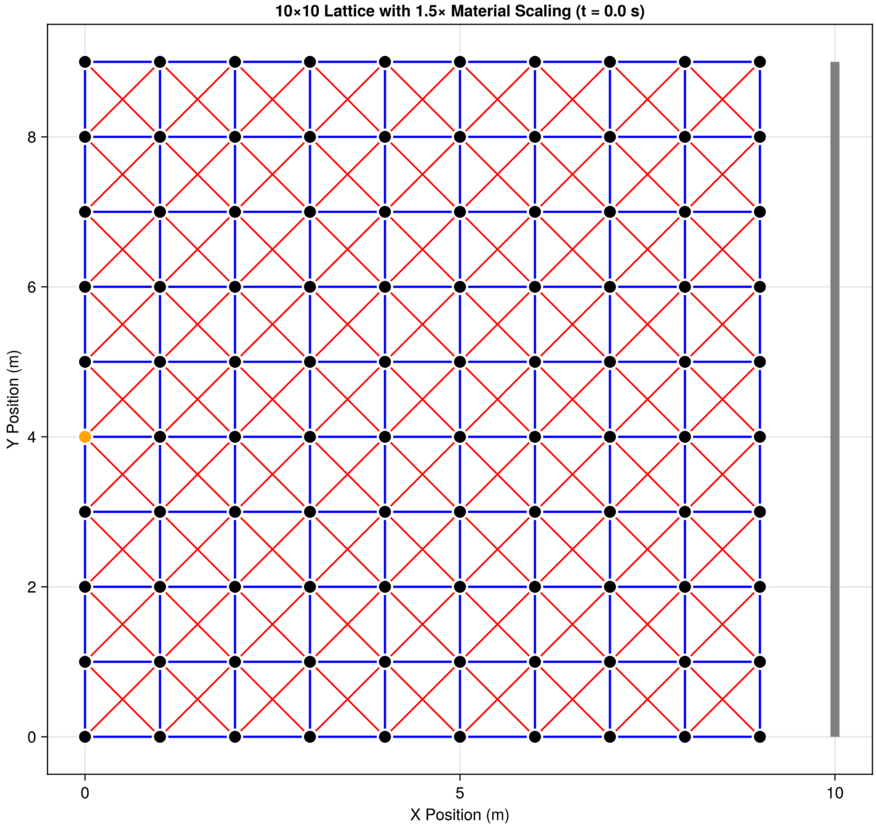

Figure 3: Snapshot of 10×10 lattice system with 1.5× material scaling and backplate constraint. The larger system size enables study of more complex wave propagation patterns and material gradient effects.

6 Comparison Tables

Table 1 presents comprehensive results for all material scaling configurations studied.

Configuration	Work (J)	Final Energy (J)	Dissipation (%)
5×5, 1.5× multiplier	257.509908	45.112795	82.48
5×5, 2/3× multiplier	258.074177	44.817127	82.63
10×10, 1.5× multiplier	612.888492	100.130361	83.66

Table 1: Material scaling comparison results.

7 Acknowledgments

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