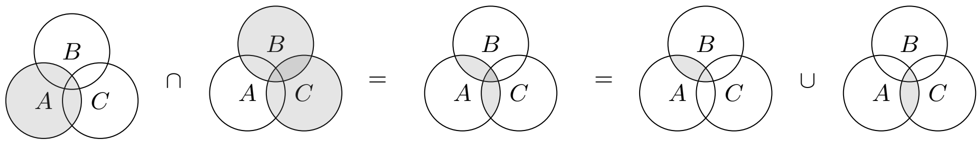
Section B.1 – Sets

B.1-1 Draw Venn diagrams that illustrate the first of the distributive laws (B.1).



$$A \quad \cap \quad (B \cup C) \quad = \quad A \cap (B \cup C) \quad = \quad (A \cap B) \quad \cup \quad (A \cap C)$$

B.1-2 Prove the generalization of DeMorgan's laws to any finite collection of sets:

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n},$$

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

The base case, which occurs when $n = 2$, is given (from the text book). Now, lets assume it holds for n and show that it also holds for $n + 1$.

For the first DeMongan's law, we have

$$\begin{aligned}
\overline{A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}} &= \overline{(A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1}} \\
&= \overline{(A_1 \cap A_2 \cap \cdots \cap A_n)} \cup \overline{A_{n+1}} \\
&= (\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}) \cup \overline{A_{n+1}} \\
&= \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} \cup \overline{A_{n+1}}.
\end{aligned}$$

For the second DeMongan's law, we have

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}} &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}} \\
&= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n)} \cap \overline{A_{n+1}} \\
&= (\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}) \cap \overline{A_{n+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} \cap \overline{A_{n+1}}.
\end{aligned}$$

B.1-3 ($\star$) Prove the generalization of equation (B.3), which is called the ***principle of inclusion and exclusion***:

$$\begin{aligned}
&|A_1 \cup A_2 \cup \cdots \cup A_n| = \\
&\quad |A_1| + |A_2| + \cdots + |A_n| \\
&\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots \qquad \text{(all pairs)} \\
&\quad + |A_1 \cap A_2 \cap A_3| + \cdots \qquad \text{(all triples)} \\
&\qquad \vdots \\
&\quad + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.
\end{aligned}$$

Skipped.

B.1-4 Show that the set of odd natural numbers is countable.

Let $\mathbb{O}$ denote the set of odd natural numbers.

The function $f(n) = 2n + 1$ is a 1-1 correspondence from $\mathbb{N}$ to $\mathbb{O}$. Thus, $\mathbb{O}$ is countable.

B.1-5 Show that for any finite set S, the power set 2^S has $2^{|S|}$ elements (that is, there are $2^{|S|}$ distinct subsets of S).

For the base case, consider a set with a single element x. We have

$$2^{\{x\}} = \{\emptyset, \{x\}\},$$

which shows that the power set of a set with a single element has cardinality $2^1 = 2$.
Let $C(\cdot)$ denote the cardinality of a power set. Let S be a set of size n. Lets assume that the power set of S has cardinality $C(S) = 2^{|S|} = 2^n$. Now, let S' be the set S with one additional element x, such that $|S'| = n + 1$. The power set of S' will consist of all sets in the power set of S plus all those same sets again, with the element x added. Thus, we have

$$C(S') = 2 \cdot C(S) = 2 \cdot 2^n = 2^{n+1}.$$

B.1-6 Give an inductive definition for an n-tuple by extending the set-theoretic definition for an ordered pair.

$$\begin{aligned}
(a) &= \{a\} \\
(a, b) &= \{a, \{a, b\}\} \\
(a, b, c) &= \{a, \{a, b\}, \{a, b, c\}\} \\
(a_1, a_2, \ldots, a_n) &= (a_1, a_2, \ldots, a_{n-1}) + \{a_1, a_2, \ldots, a_n\}
\end{aligned}$$