

Section A.1 – Summation formulas and properties

A.1-1 Find a simple formula for $\sum_{k=1}^n (2k - 1)$.

$$\begin{aligned}
 \sum_{k=1}^n (2k - 1) &= \sum_{k=1}^n 2k - \sum_{k=1}^n 1 \\
 &= 2 \sum_{k=1}^n k - n \\
 &= 2 \cdot \frac{1}{2} n(n+1) - n \\
 &= n^2 + n - n \\
 &= n^2.
 \end{aligned}$$

A.1-2 (★) Show that $\sum_{k=1}^n 1/(2k - 1) = \ln(\sqrt{n}) + O(1)$ by manipulating the harmonic series.

$$\begin{aligned}
 \sum_{k=1}^n 1/(2k - 1) &= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-3} + \frac{1}{2n-1} \\
 &= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} \right) - \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) \\
 &= \sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^n \frac{1}{k} \\
 &= \ln 2n + O(1) - \frac{1}{2} (\ln n + O(1)) \\
 &= \ln n + \ln 2 + O(1) - \frac{1}{2} \ln n - \frac{1}{2} O(1) \\
 &= \frac{1}{2} \ln n + O(1) \\
 &= \ln(\sqrt{n}) + O(1).
 \end{aligned}$$

A.1-3 Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ for $0 < |x| < 1$.

From Equation A.8, we have

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

differentiating both sides and multiplying by x , we have

$$\begin{aligned}
 \sum_{k=0}^{\infty} k^2 x^k &= x \cdot \frac{1 \cdot (1-x)^2 - (2 \cdot (1-x) \cdot (-1) \cdot x)}{(1-x)^4} \\
 &= x \cdot \frac{(1-x)(1-x) + (1-x) \cdot 2x}{(1-x)^4} \\
 &= x \cdot \frac{(1-x) + 2x}{(1-x)^3} \\
 &= \frac{x(1+x)}{(1-x)^3}.
 \end{aligned}$$

A.1-4 (★) Show that $\sum_{k=0}^{\infty} (k-1)/2^k = 0$.

$$\begin{aligned}
 \sum_{k=0}^{\infty} (k-1)/2^k &= \sum_{k=0}^{\infty} \left(\frac{k}{2^k} - \frac{1}{2^k} \right) \\
 &= \sum_{k=0}^{\infty} k \frac{1}{2^k} - \sum_{k=0}^{\infty} \frac{1}{2^k} \\
 &= \sum_{k=0}^{\infty} k \left(\frac{1}{2} \right)^k - \sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^k \\
 &= \frac{(1/2)}{(1 - (1/2))^2} - \frac{1}{1 - (1/2)} \\
 &= \frac{(1/2)}{1 - 1 - (1/4)} - 2 \\
 &= 4/2 - 2 \\
 &= 0.
 \end{aligned}$$

A.1-5 (★) Evaluate the sum $\sum_{k=1}^{\infty} (2k+1)x^{2k}$ for $|x| < 1$.

$$\begin{aligned}
 \sum_{k=1}^{\infty} (2k+1)x^{2k} &= \frac{d}{dx} \cdot \sum_{k=1}^{\infty} x^{2k+1} \\
 &= \frac{d}{dx} \cdot x \cdot \sum_{k=1}^{\infty} x^{2k} \\
 &= \frac{d}{dx} \cdot x \cdot \sum_{k=0}^{\infty} (x^2)^k - x \\
 &= \frac{d}{dx} \cdot x \cdot \frac{1}{1-x^2} - x \\
 &= \frac{d}{dx} \cdot \frac{x - x(1-x^2)}{1-x^2} \\
 &= \frac{d}{dx} \cdot \frac{x^3}{1-x^2} \\
 &= \frac{3x^2(1-x^2) - (-2x)x^3}{(1-x^2)^2} \\
 &= \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2} \\
 &= \frac{(3-x^2) \cdot x^2}{(1-x^2)^2}.
 \end{aligned}$$

A.1-6 Prove that $\sum_{k=1}^n O(f_k(i)) = O(\sum_{k=1}^n f_k(i))$ by using the linearity property of summations.

Skipped.

A.1-7 Evaluate the product $\prod_{k=1}^n 2 \cdot 4^k$.

We have

$$\prod_{k=1}^n (2 \cdot 4^k) = 2^{\lg(\prod_{k=1}^n (2 \cdot 4^k))},$$

and

$$\begin{aligned} \lg \left(\prod_{k=1}^n (2 \cdot 4^k) \right) &= \sum_{k=1}^n \lg(2 \cdot 2^{2k}) \\ &= \sum_{k=1}^n \lg 2^{2k+1} \\ &= \sum_{k=1}^n (2k+1) \\ &= 2 \sum_{k=1}^n k + \sum_{k=1}^n 1 \\ &= n(n+1) + n \\ &= n(n+2). \end{aligned}$$

Thus,

$$\prod_{k=1}^n (2 \cdot 4^k) = 2^{n(n+2)}.$$

A.1-8 (★) Evaluate the product $\prod_{k=2}^n (1 - 1/k^2)$.

We have

$$\prod_{k=2}^n \left(1 - \frac{1}{k^2} \right) = 2^{\lg(\sum_{k=2}^n \lg(1 - 1/k^2))},$$

and

$$\begin{aligned} \sum_{k=2}^n \lg \left(1 - \frac{1}{k^2} \right) &= \sum_{k=2}^n \lg \left(\frac{k^2 - 1}{k^2} \right) \\ &= \sum_{k=2}^n \lg \left(\frac{(k-1)}{k} \cdot \frac{(k+1)}{k} \right) \\ &= \sum_{k=2}^n \left(\lg \left(\frac{k-1}{k} \right) + \lg \left(\frac{k+1}{k} \right) \right) \\ &= \lg \frac{1}{2} + \lg \frac{3}{2} + \lg \frac{2}{3} + \lg \frac{4}{3} + \lg \frac{3}{4} + \lg \frac{5}{4} + \dots + \lg \frac{n-2}{n-1} + \lg \frac{n}{n-1} + \lg \frac{n-1}{n} + \lg \frac{n+1}{n} \\ &= \lg 1 - \lg 2 + \lg 3 - \lg 2 + \lg 2 - \lg 3 + \lg 4 - \lg 3 + \lg 3 - \lg 4 + \lg 5 - \lg 4 + \dots \\ &\quad + \lg(n-2) - \lg(n-1) + \lg n - \lg(n-1) + \lg(n-1) - \lg n + \lg(n+1) - \lg n \\ &= 0 - 1 + \lg(n+1) - \lg n \\ &= \lg(n+1) - \lg(n) - 1. \end{aligned}$$

Thus,

$$\prod_{k=2}^n \left(1 - \frac{1}{k^2} \right) = 2^{(\lg(n+1) - (\lg(n) + 1))} = \frac{2^{\lg(n+1)}}{2^{\lg(n) + 1}} = \frac{n+1}{2^{\lg n} \cdot 2} = \frac{n+1}{2n}.$$