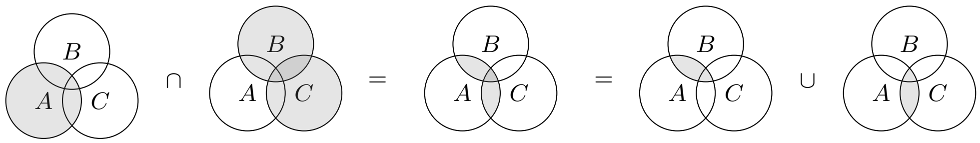
Section B.1 – Sets

B.1-1 Draw Venn diagrams that illustrate the first of the distributive laws (B.1).



$$A \quad \cap \quad (B \cup C) \quad = \quad A \cap (B \cup C) \quad = \quad (A \cap B) \quad \cup \quad (A \cap C)$$

B.1-2 Prove the generalization of DeMorgan's laws to any finite collection of sets:

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n},$$
$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

The base case, which occurs when $n = 2$, is given (from the text book). Now, lets assume it holds for n and show that it also holds for $n + 1$.

For the first DeMongan's law, we have

$$\begin{aligned}
\overline{A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}} &= \overline{(A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1}} \\
&= \overline{(A_1 \cap A_2 \cap \cdots \cap A_n)} \cup \overline{A_{n+1}} \\
&= (\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}) \cup \overline{A_{n+1}} \\
&= \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} \cup \overline{A_{n+1}}.
\end{aligned}$$

For the second DeMongan's law, we have

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}} &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}} \\
&= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n)} \cap \overline{A_{n+1}} \\
&= (\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}) \cap \overline{A_{n+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} \cap \overline{A_{n+1}}.
\end{aligned}$$

B.1-3 ($\star$) Prove the generalization of equation (B.3), which is called the ***principle of inclusion and exclusion***:

$$\begin{aligned}
&|A_1 \cup A_2 \cup \cdots \cup A_n| = \\
&\quad |A_1| + |A_2| + \cdots + |A_n| \\
&\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots \qquad \text{(all pairs)} \\
&\quad + |A_1 \cap A_2 \cap A_3| + \cdots \qquad \text{(all triples)} \\
&\quad \vdots \\
&\quad + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.
\end{aligned}$$

Skipped.

B.1-4 Show that the set of odd natural numbers is countable.

Let $\mathbb{O}$ denote the set of odd natural numbers.

The function $f(n) = 2n + 1$ is a 1-1 correspondence from $\mathbb{N}$ to $\mathbb{O}$. Thus, $\mathbb{O}$ is countable.

B.1-5 Show that for any finite set S, the power set 2^S has $2^{|S|}$ elements (that is, there are $2^{|S|}$ distinct subsets of S).

For the base case, consider a set with a single element x. We have

$$2^{\{x\}} = \{\emptyset, \{x\}\},$$

which shows that the power set of a set with a single element has cardinality $2^1 = 2$.
Let $C(\cdot)$ denote the cardinality of a power set. Let S be a set of size n. Lets assume that the power set of S has cardinality $C(S) = 2^{|S|} = 2^n$. Now, let S' be the set S with one additional element x, such that $|S'| = n + 1$. The power set of S' will consist of all sets in the power set of S plus all those same sets again, with the element x added. Thus, we have

$$C(S') = 2 \cdot C(S) = 2 \cdot 2^n = 2^{n+1}.$$

B.1-6 Give an inductive definition for an n-tuple by extending the set-theoretic definition for an ordered pair.

$$\begin{aligned}
(a) &= \{a\} \\
(a, b) &= \{a, \{a, b\}\} \\
(a, b, c) &= \{a, \{a, b\}, \{a, b, c\}\} \\
(a_1, a_2, \ldots, a_n) &= (a_1, a_2, \ldots, a_{n-1}) \cup \{a_1, a_2, \ldots, a_n\}
\end{aligned}$$

Section B.2 – Relations

B.2-1 Prove that the subset relation "$\subseteq$" on all subsets of $\mathbb{Z}$ is a partial order but not a total order.

Let $\mathbb{S}$ denote all the subsets of $\mathbb{Z}$. Let $A = \{1\}$, $B = \{2\}$ be two subsets of $\mathbb{Z}$. We have $A \nsubseteq B$ and $B \nsubseteq A$. Thus, the subset relation "$\subseteq$" on $\mathbb{S} \times \mathbb{S}$ is not a total relation and therefore is not a total order.

For the relation $\subseteq$ on $\mathbb{S}$ be a partial order, the following properties needs to hold: (1) reflexivity, (2) antisymmetry, (3) transitivity. Since $A \subseteq A$, for all $A \in \mathbb{S}$, the relation "$\subseteq$" on $\mathbb{S} \times \mathbb{S}$ is reflexive. To be antisymmetric, we need to show that if $A \subseteq B$ and $B \subseteq A$, then $A = B$, for all $A, B \in \mathbb{S}$. Since $A \subseteq B$, for all $a \in A$ we have $a \in B$ and since $B \subseteq A$, for all $b \in B$ we have $b \in A$. Thus, $A = B$ and the relation "$\subseteq$" on $\mathbb{S} \times \mathbb{S}$ is antisymmetric. To be transitive, we need to show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$, for all $A, B, C \in \mathbb{S}$. So let $a \in A$. Since $A \subseteq B$, we have $a \in B$. Since $a \in B$ and $B \subseteq C$, we have $a \in C$. Thus, $A \subseteq C$ and the relation "$\subseteq$" on $\mathbb{S} \times \mathbb{S}$ is transitive.

B.2-2 Show that for any positive integer n, the relation "equivalent modulo n" is an equivalence relation on the integers. (We say that $a \equiv b \pmod{n}$ if there exists an integer q such that $a - b = qn$.) Into what equivalence classes does this relation partition the integers?

To the relation "equivalent modulo n" be an equivalent relation on $\mathbb{Z} \times \mathbb{Z}$, the following needs to hold:

(a) $a \equiv a \pmod{n}$, for all $a, n \in \mathbb{Z}$ (reflexivity)

(b) $a \equiv b \pmod{n}$ implies $b \equiv a \pmod{n}$, for all $a, b, n \in \mathbb{Z}$ (symmetry)

(c) $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ implies $a \equiv c \pmod{n}$, for all $a, b, c, n \in \mathbb{Z}$ (transitivity)

For the reflexivity property, we have that $a - a = qn$ holds directly for $q = 0$.
For the symmetry property, we have that $a - b = pn$ implies $b - a = qn$ holds directly for $q = -p$.
For the transitivity property, we have that $a - b = pn$ and $b - c = qn$ implies $a - c = rn$ holds for $r = p + q$, since

$$(a - b) + (b - c) = pn + qn \to a - c = (p + q)n.$$

B.2-3 Give examples of relations that are

a. reflexive and symmetric but not transitive,

b. reflexive and transitive but not symmetric,

c. symmetric and transitive but not reflexive.

(a) The relation "is neighbor of" is reflexive (one is neighbor of himself) and symmetric (a "is neighbor of" b imply b "is neighbor of" a), but not transitive (a "is neighbor of" b and b "is neighbor of" c does not imply a "is neighbor of" c.

(b) The relation "$\leq$" is reflexive ($a \leq a$) and transitive ($a \leq b$ and $b \leq c$ imply $a \leq c$), but not symmetric ($a \leq b$ does not imply $b \leq a$).

(c) Consider the relation "$a + b > \infty$" on $\mathbb{Z} \times \mathbb{Z}$. This relation is empty. However, it is symmetric ($a\ R\ b$ imply $b\ R\ a$) and transitive ($a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$), but not reflexive since for no $a \in \mathbb{Z}$ is it the case that $a\ R\ a$.

B.2-4 Let S be a finite set, and let R be an equivalence relation on $S \times S$. Show that if in addition R is antisymmetric, then the equivalence classes of S with respect to R are singletons.

For every $a, b \in S$ such that $a\ R\ b$, by symmetry $b\ R\ a$, and by antisymmetry $a = a$. Thus, $[a] = \{b \in S : a\ R\ b\} = \{a\}$ for all $a \in S$, which implies that the equivalence classes are singletons.

B.2-5 Professor Narcissus claims that if a relation R is symmetric and transitive, then it is also reflexive. He offers the following proof. By symmetry, $a\ R\ b$ implies $b\ R\ a$. Transitivity, therefore, implies $a\ R\ a$. Is the professor correct?

No. This is only true for relations that for every a there is b such that $a\ R\ b$, by symmetry $b\ R\ a$, and by transitivity $a\ R\ a$. For instance, an empty relation (like the one from Question B.2-3, item (c)) are symmetric and transitive, but not reflexive.

Section B.3 – Functions

B.3-1 Let A and B be finite sets, and let $f : A \to B$ be a function. Show that

a. if f is injective, then $|A| \leq |B|$;

b. if f is surjective, then $|A| \geq |B|$.

(a) Since f is injective, we have that $A = f(A)$. Also, we have

$$\begin{cases} |B| = |f(A)|, & f \text{ is surjective,} \\ |B| > |f(A)|, & f \text{ is not surjective.} \end{cases}$$

Thus, $|B| \geq |f(A)| = |A| \to |A| \leq |B|$.

(b) Since f is surjective, we have $|f(A)| = |B|$. Also, we have

$$\begin{cases} |A| = |f(A)|, & f \text{ is injective,} \\ |A| > |f(A)|, & f \text{ is not injective.} \end{cases}$$

Thus, $|A| \geq |f(A)| = |B| \to |A| \geq |B|$.

B.3-2 Is the function $f(x) = x + 1$ bijective when the domain and the codomain are $\mathbb{N}$? Is it bijective when the domain and the codomain are $\mathbb{Z}$?

On the set of naturals, f is injective but not surjective, since there is no $a \in \mathbb{N}$ such that $0 = f(a)$, which makes $f(\mathbb{N}) \neq \mathbb{N}$. On the set of integers, f is both injective and surjective, and therefore bijective.

B.3-3 Give a natural definition for the inverse of a binary relation such that if a relation is in fact a bijective function, its relational inverse is its functional inverse.

Let R be a binary relation on the sets A and B, such that $R \subseteq A \times B$. The general definition of the inverse of R is given by

$$R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}.$$

When R is a bijective function, we have: (1) each element of A has relation with precisely one element of B (injective) and (2) for all $b \in B$ there is a such that $a \; R \; b$ (surjective). Therefore, when R is bijective, each element of A is related to exactly one element of B and vice-versa, which implies

$$f(a) = b \to f'(b) = a,$$

for all $a \in A$ and for all $b \in B$.

B.3-4 ($\star$) Give a bijection from $\mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$.

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