Section 4.1 – The maximum-subarray problem

4.1-1 What does FIND-MAXIMUM-SUBARRAY return when all elements of A are negative?

> A subarray with only the largest negative element of A.

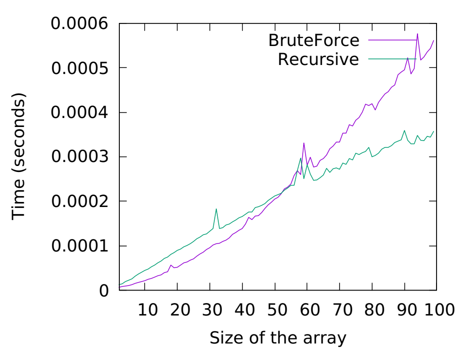
4.1-2 Write pseudocode for the brute-force method of solving the maximum-subarray problem. Your procedure should run in $\Theta(n^2)$ time.

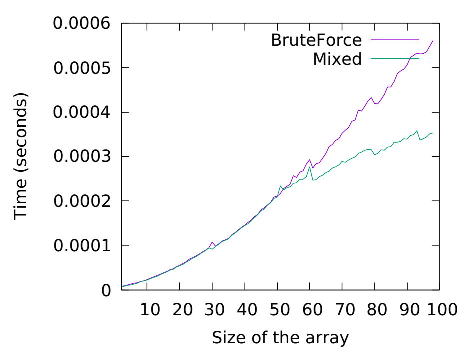
> The pseudocode is stated below.

```
   FindMaximumSubarray-BruteForce(A)
1      low = 0
2      high = 0
3      sum = −∞
4      for i = 1 to A.length do
5          cursum = 0
6          for j = i to A.length do
7              cursum = cursum + A[j]
8              if cursum > sum then
9                  sum = cursum
10                 low = i
11                 high = j
12     return low, high, sum
```

4.1-3 Implement both the brute-force and recursive algorithms for the maximum-subarray problem on your own computer. What problem size n_0 gives the crossover point at which the recursive algorithm beats the brute-force algorithm? Then, change the base case of the recursive algorithm to use the brute-force algorithm whenever the problem size is less than n_0. Does that change the crossover point?

> Figure below in the lhs ilustrates the crossover point between the BruteForce and Recursive solutions in my machine. In that comparison, $n_0 \approx 52$. Figure below in the rhs ilustrates the crossover point between the BruteForce and Mixed solutions in my machine. The crossover point does not change but the Mixed solution becomes as fast as the BruteForce solution when the problem size is lower than 52.





4.1-4 Suppose we change the definition of the maximum-subarray problem to allow the result to be an empty subarray, where the sum of the values of an empty subarray is 0. How would you change any of the algorithms that do not allow empty subarrays to permit an empty subarray to be the result?

> The BruteForce algorithm (stated above in Question 4.1-2) can be updated just by modifying line 3 to $sum = 0$, instead of $sum = -\infty$. In that case, if there is no subarray whose sum is greater than zero, the algorithm will return a invalid subarray ($low = 0, high = 0, sum = 0$), which will denote the empty subarray.
>
> The Recursive algorithm (stated in Section 4.1) can be updated as follows. In the FIND-MAX-CROSSING-SUBARRAY routine, update lines 1 and 8 to initialize *left-sum* and *right-sum* to 0, instead of $-\infty$. Also initialize *max-left* (after line 1) and *max-right* (after line 8) to 0. In the FIND-MAXIMUM-SUBARRAY routine, surround the return statement of line 2 with a conditional that verifies if $A[low]$ is greater than zero. If it is, return the values as it was before. If it is not, return a invalid subarray (denoted by $low = 0$ and $high = 0$) and the sum as zero.

4.1-5 Use the following ideas to develop a nonrecursive, linear-time algorithm for the maximum-subarray problem. Start at the left end of the array, and progress toward the right, keeping track of the maximum subarray seen so far. Knowing a maximum subarray of $A[1,\ldots,j]$, extend the answer to find a maximum subarray ending at index $j+1$ by using the following observation: a maximum subarray of $A[1,\ldots,j+1]$ is either a maximum subarray of $A[1,\ldots,j]$ or a subarray $A[i,\ldots,j+1]$, for some $1 \leq i \leq j+1$. Determine a maximum subarray of the form $A[i,\ldots,j+1]$ in constant time based on knowing a maximum subarray ending at index j.

The pseudocode is stated below.

```
FindMaximumSubarray-Linear(A)
1  low = 0
2  high = 0
3  sum = 0
4  current-low = 0
5  current-sum = 0
6  for i = 1 to A.length do
7      current-sum = max(A[i], current-sum + A[i])
8      if current-sum == A[i] then
9          current-low = i
10     if current-sum > sum then
11         low = current-low
12         high = i
13         sum = current-sum
14 return low, high, sum
```

We can make it a little faster (twice as fast on my machine) by avoiding executing lines 7, 8, and 10 when not necessary.

```
FindMaximumSubarray-Linear-Optimized(A)
low = 0
high = 0
sum = 0
current-low = 0
current-sum = 0
for i = 1 to A.length do
    if current-sum + A[i] ≤ 0 then
        current-sum = 0
    else
        current-sum = current-sum + A[i]
        if current-sum == A[i] then
            current-low = i
        if current-sum > sum then
            low = current-low
            high = i
            sum = current-sum
return low, high, sum
```

Section 4.2 – Strassen's algorithm for matrix multiplication

4.2-1 Use Strassen's algorithm to compute the matrix product

$$\begin{bmatrix} 1 & 3 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 4 & 2 \end{bmatrix}.$$

Show your work.

Let

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 5 \end{bmatrix}, B = \begin{bmatrix} 6 & 8 \\ 4 & 2 \end{bmatrix},$$

and $C = A \cdot B$. To compute C using Strassen's algorithm, we start by computing the S_i matrices:

$$\begin{aligned}
S_1 &= B_{12} - B_{22} = 8 - 2 = 6, \\
S_2 &= A_{11} + A_{12} = 1 + 3 = 4, \\
S_3 &= A_{21} + A_{22} = 7 + 5 = 12, \\
S_4 &= B_{21} - B_{11} = 4 - 6 = -2, \\
S_5 &= A_{11} + A_{22} = 1 + 5 = 6, \\
S_6 &= B_{11} + B_{22} = 6 + 2 = 8, \\
S_7 &= A_{12} + A_{22} = 3 - 5 = -2, \\
S_8 &= B_{21} + B_{22} = 4 + 2 = 6, \\
S_9 &= A_{11} - A_{21} = 1 - 7 = -6, \\
S_{10} &= B_{11} + B_{12} = 6 + 8 = 14.
\end{aligned}$$

Then we compute the P_i matrices:

$$\begin{aligned}
P_1 &= A_11 \cdot S_1 &&= 1 \cdot 6 &&= 6, \\
P_2 &= S_2 \cdot B_22 &&= 4 \cdot 2 &&= 8, \\
P_3 &= S_3 \cdot B_11 &&= 12 \cdot 6 &&= 72, \\
P_4 &= A_22 \cdot S_4 &&= 5 \cdot (-2) &&= -10, \\
P_5 &= S_5 \cdot S_6 &&= 6 \cdot 8 &&= 48, \\
P_6 &= S_7 \cdot S_8 &&= (-2) \cdot 6 &&= -12, \\
P_7 &= S_9 \cdot S_10 &&= (-6) \cdot 14 &&= -84.
\end{aligned}$$

Using matrices S_i and P_i, we compute C:

$$C = \begin{bmatrix} (P_5 + P_4 - P_2 + P_6) & (P_2 + P_2) \\ (P_3 + P_4) & (P_5 + P_1 - P_3 - P_7) \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ 62 & 66 \end{bmatrix}.$$

4.2-2 Write pseudocode for Strassen's algorithm.

The pseudocode is stated below.

```
Square-Matrix-Multiply-Strassen(A, B)
  n = A.rows
  let C be a new n × n matrix
  if n == 1 then
      c11 = a11 · b11
  else
      partition A, B, and C as into n/2 × n/2 submatrices
      let S1, S2, ..., S10 be new n/2 × n/2 matrices
      let P1, P2, ..., P7 be new n/2 × n/2 matrices
      S1 = B12 − B22
      S2 = A11 + A12
      S3 = A21 + A22
      S4 = B21 − B11
      S5 = A11 + A22
      S6 = B11 + B22
      S7 = A12 − A22
      S8 = B21 + B22
      S9 = A11 − A21
      S10 = B11 − B12
      P1 = Square-Matrix-Multiply-Strassen(A11, S1)
      P2 = Square-Matrix-Multiply-Strassen(S2, B22)
      P3 = Square-Matrix-Multiply-Strassen(S3, B11)
      P4 = Square-Matrix-Multiply-Strassen(A22, S4)
      P5 = Square-Matrix-Multiply-Strassen(S5, S6)
      P6 = Square-Matrix-Multiply-Strassen(S7, S8)
      P7 = Square-Matrix-Multiply-Strassen(S9, S10)
      C11 = P5 + P4 − P2 + P6
      C12 = P1 + P2
      C21 = P3 + P4
      C22 = P5 + P1 − P3 − P7
  return C
```

4.2-3 How would you modify Strassen's algorithm to multiply $n \times n$ matrices in which n is not an exact power of 2? Show that the resulting algorithm runs in time $\Theta(n^{\lg 7})$.

Pad each input $n \times n$ matrix (rows and columns) with $m - n$ zeros, resulting in an $m \times m$ matrix, where $m = 2^{\lceil \lg n \rceil}$. After computing the final matrix, cut down the last $m - n$ rows and $m - n$ columns (which will be zeros).
Padding the matrix with zeros is done once, in the root of the recursion tree, and takes $O(m^2)$. Since we now have an $m \times m$ matrix, the algorithm runs in $\Theta(m^{\lg 7}) + O(m^2) = \Theta(m^{\lg 7})$. We have that $n \leq m < 2^{(\lg n)+1} = 2^{\lg n} \cdot 2 = 2n$. Thus, the algorithm runs in $\Theta((2n)^{\lg 7}) = \Theta(n^{\lg 7})$.

4.2-4 What is the largest k such that if you can multiply 3×3 matrices using k multiplications (not assuming commutativity of multiplication), then you can multiply $n \times n$ matrices in time $o(n^{\lg 7})$? What would the running time of this algorithm be?

If we modify the Square-Matrix-Multiply-Recursive algorithm to partition the matrices into $n/3 \times n/3$ submatrices, we would have the following recurrence:

$$T(n) = \Theta(1) + 27T(n/3) + \Theta(n^2) = 27T(n/3) + \Theta(n^2).$$

Let's proceed to understand a little more about the above recurrence. Let A and B be the two input matrices in each node of the above recursion tree. Like in the original Square-Matrix-Multiply-Recursive algorithm, our modified version will take $\Theta(1)$ to partition A and B into $n/3 \times n/3$ submatrices. In each node of the tree, the product of A and B is recursively computed by the products of their submatrices. Since the number of recursive (submatrices) products to compute $A \cdot B$ in each node of the recurstion tree is 27 and each of these submatrices is 3 times smaller than A and B, the 27 recursive products takes $27T(n/3)$. Finally, the number of summations to compute the final matrix is $\Theta(3 \cdot 9 \cdot n^2/3) = \Theta(n^2)$.

If after partitioning A and B into $n/3 \times n/3$ submatrices we can compute their product with k multiplications (instead of 27), we would have the following recurrence:

$$T(n) = \Theta(1) + kT(n/3) + \Theta(n^2) = kT(n/3) + \Theta(n^2),$$

We can solve the above recurrence using the master method. We have $f(n) = n^2$ and $n^{log_b a} = n^{\log_3 k}$. Using the first case of the master method, we have

$$\forall\, k \mid \log_3 k > 2,\ n^2 = O(n^{(\log_3 k)-\epsilon}),\ 0 \leq \epsilon \leq \log_3 k - 2,$$

which implies

$$T(n) = \Theta(n^{\log_3 k}).$$

Since $\log_3 21 < \lg 7 < \log_3 22$, the largest value for k is 21. Its running time would be $n^{\log_3 21} \approx n^{2.7712}$.

4.2-5 V. Pan has discovered a way of multiplying 68×68 matrices using 132,464 multiplications, a way of multiplying 70×70 matrices using 143,640 multiplications, and a way of multiplying 72×72 matrices using 155,424 multiplications. Which method yields the best asymptotic running time when used in a divide-and-conquer matrix-multiplication algorithm? How does it compare to Strassen's algorithm?

The algorithms would take:

- $n^{\log_{68} 132{,}464} \approx n^{2.795128}$,
- $n^{\log_{70} 143{,}640} \approx n^{2.795122}$,
- $n^{\log_{72} 155{,}424} \approx n^{2.795147}$.

The fastest is the one that multiplies 70×70 matrices, but all of them are faster then the Strassen's algorithm.

4.2-6 How quickly can you multiply a $kn \times n$ matrix by an $n \times kn$ matrix, using Strassen's algorithm as a subroutine? Answer the same question with the order of the input matrices reversed.

Let A and B be $kn \times n$ and $n \times kn$ matrices, respectivelly. We can compute $A \cdot B$ as follows:

(a) Partition A and B into k submatrices $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$, each one of size $n \times n$.

(b) Compute the desired submatrices C_{ij} of the result matrix C by the product of $A_i \cdot B_j$. Use the Strassen's algorithm to compute each of those products.

Since each of the k^2 products takes $\Theta(n^{\lg 7})$, this algorithm runs in $\Theta(k^2 n^{\lg 7})$.
We can compute $B \cdot A$ as follows:

(a) Partition A and B into k submatrices $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$, each one of size $n \times n$.

(b) Compute the the result matrix $C = \sum_{i=1}^{k} A_i \cdot B_i$. Use the Strassen's algorithm to compute each of those products.

Since each of the k products takes $\Theta(n^{\lg 7})$ and the $k-1$ summations takes $\Theta((k-1)n^2/k) = O(n^2)$, this algorithm runs in $\Theta(kn^{\lg 7}) + O(n^2) = \Theta(kn^{\lg 7})$.

4.2-7 Show how to multiply the complex numbers $a + bi$ and $c + di$ using only three multiplications of real numbers. The algorithm should take a, b, c, and d as input and produce the real component $ac - bd$ and the imaginary component $ad + bc$ separately.

The pseudocode is stated below.

```
Complex-Product(a, b, c, d)
    x = a · c
    y = b · d
    real-part = x − y
    imaginary-part = (a + b) · (c + d) − x − y
    return real-part, imaginary-part
```

Section 4.3 – The substitution method for solving recurrences

4.3-1 Show that the solution of $T(n) = T(n-1) + n$ is $O(n^2)$.

Our guess is

$$T(n) \le cn^2 \;\; \forall\, n \ge n_0,$$

where c and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\le c(n-1)^2 + n \\ &= cn^2 - 2cn + c + n \quad (c = 1) \\ &= n^2 - 2n + n + 1 \\ &= n^2 - n + 1 \\ &\le n^2, \end{aligned}$$

where the last step holds as long as $n_0 \ge 1$.

4.3-2 Show that the solution of $T(n) = T(\lceil n/2 \rceil) + 1$ is $O(\lg n)$.

Our guess is

$$T(n) \le c \lg n - d \;\; \forall\, n \ge n_0,$$

where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\le c \lg(\lceil n/2 \rceil) - d + 1 \\ &\le c \lg n - d + 1 \\ &\le c \lg n, \end{aligned}$$

where the last step holds as long as $d \ge 1$.

4.3-3 We saw that the solution of $T(n) = 2T(\lfloor n/2 \rfloor) + n$ is $O(n \lg n)$. Show that the solution of this recurrence is also $\Omega(n \lg n)$. Conclude that the solution is $\Theta(n \lg n)$.

Our guess is

$$T(n) \ge cn \lg n \;\; \forall\, n \ge n_0,$$

where c and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\ge 2(c\lfloor n/2 \rfloor \lg\lfloor n/2 \rfloor) + n \\ &\ge 2c(n/4)\lg(n/4) + n \\ &= c(n/2)\lg n - c(n/2)\lg 4 + n \\ &= c(n/2)\lg n - cn + n \\ &\ge cn \lg n, \end{aligned}$$

where the last step holds as long as $c \le 1$.
Thus, we have

$$c_1 n \lg n \le T(n) \le c_2 n \lg n,$$

with $c_1 \le 1$ and $c_2 \ge 1$, which implies

$$T(n) = \Theta(n \lg n).$$

4.3-4 Show that by making a different inductive hypothesis, we can overcome the difficulty with the boundary condition $T(1) = 1$ for recurrence (4.19) without adjusting the boundary conditions for the inductive proof.

Our new guess is

$$T(n) \leq cn \lg n + n \;\; \forall\, n \geq n_0,$$

where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\leq 2(c\lfloor n/2 \rfloor \lg\lfloor n/2 \rfloor + \lfloor n/2 \rfloor) + n \\
&\leq cn \lg(n/2) + 2(n/2) + n \\
&= cn \lg n - cn \lg 2n + 2n \\
&= cn \lg n - cn + 2n \\
&\leq cn \lg n + n,
\end{aligned}$$

where the last step holds as long as $c \geq 1$.
Now on the boundary condition, we have

$$T(1) \leq c(n \lg n) + n = c1 \lg 1 + 1 = 0 + 1 = 1.$$

4.3-5 Show that $\Theta(n \lg n)$ is the solution to the "exact" recurrence (4.3) for merge sort.

First, we verify if (4.3) is $O(n \lg n)$. Our guess is

$$T(n) \leq c(n - d) \lg(n - d) \;\; \forall\, n \geq n_0,$$

where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\leq c(\lceil n/2 \rceil - d) \lg(\lceil n/2 \rceil - d) + c(\lfloor n/2 \rfloor - d) \lg(\lfloor n/2 \rfloor - d) + en \\
&\leq c(n/2 + 1 - d) \lg(n/2 + 1 - d) + c(n/2 - d) \lg(n/2 - d) + en \quad (d \geq 2) \\
&\leq c\left(\frac{n-d}{2}\right) \lg\left(\frac{n-d}{2}\right) + c\left(\frac{n-d}{2}\right) \lg\left(\frac{n-d}{2}\right) + en \\
&= c(n - d) \lg\left(\frac{n-d}{2}\right) + en \\
&= c(n - d) \lg(n - d) - c(n - d) + en \\
&= c(n - d) \lg(n - d) - cn + en + cd \\
&\leq c(n - d) \lg(n - d),
\end{aligned}$$

where the last step holds as long as $c > e$ and $n_0 \geq cd$.
Then we verify if (4.3) is $\Omega(n \lg n)$. Our guess is

$$T(n) \geq c(n + d) \lg(n + d) \;\; \forall\, n \geq n_0,$$

where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\geq c(\lceil n/2 \rceil + d) \lg(\lceil n/2 \rceil + d) + c(\lfloor n/2 \rfloor + d) \lg(\lfloor n/2 \rfloor + d) + en \\
&\geq c(n/2 + d) \lg(n/2 + d) + c(n/2 - 1 + d) \lg(n/2 - 1 + d) + en \quad (d \geq 2) \\
&\geq c\left(\frac{n+d}{2}\right) \lg\left(\frac{n+d}{2}\right) + c\left(\frac{n+d}{2}\right) \lg\left(\frac{n+d}{2}\right) + en \\
&= c(n + d) \lg\left(\frac{n+d}{2}\right) + en \\
&= c(n + d) \lg(n + d) - c(n + d) + en \\
&= c(n + d) \lg(n + d) - cn + en - cd \\
&\geq c(n + d) \lg(n + d),
\end{aligned}$$

where the last step holds as long as $e > c$ and $n_0 \geq cd$.

4.3-6 Show that the solution to $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$ is $O(n \lg n)$.

Our guess is

$$T(n) \leq c(n-d)\lg(n-d) \ \forall n \geq n_0,$$

where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\leq 2c(\lfloor n/2 \rfloor - d + 17)\lg(\lfloor n/2 \rfloor - d + 17) + n \\
&\leq 2c(n/2 - d + 17)\lg(n/2 - d + 17) + n \qquad (d \geq 34) \\
&\leq 2c\left(\frac{n-d}{2}\right)\lg\left(\frac{n-d}{2}\right) + n \\
&= c(n-d)\lg\left(\frac{n-d}{2}\right) + n \\
&= c(n-d)\lg(n-d) - c(n-d) + n \\
&= c(n-d)\lg(n-d) - cn + n + cd \\
&\leq c(n-d)\lg(n-d),
\end{aligned}$$

where the last step holds as long as $c \geq 2$ and $n_0 \geq cd$.

4.3-7 Using the master method in Section 4.5 you can show that the solution to the recurrence $T(n) = 4T(n/3) + n$ is $T(n) = \Theta(n^{\log_3 4})$. Show that a substitution proof with the assumption $T(n) \leq cn^{\log_3 4}$ fails. Then show how to subtract off a lower-order term to make a substitution proof work.

The initial guess is

$$T(n) \leq cn^{\log_3 4} \ \forall n \geq n_0,$$

where c, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\leq 4c\left(\frac{n}{3}\right)^{\log_3 4} + n \\
&= 4c\frac{n^{\log_3 4}}{4} + n \\
&= cn^{\log_3 4} + n
\end{aligned}$$

which does not imply $T(n) \leq cn^{\log_3 4}$ for any choice of c.
Our new guess is

$$T(n) \leq cn^{\log_3 4} - dn \ \forall n \geq n_0,$$

where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\leq 4\left(c\left(\frac{n}{3}\right)^{\log_3 4} - d\frac{n}{3}\right) + n \\
&= 4c\frac{n^{\log_3 4}}{4} - 4d\frac{n}{3} + n \\
&\leq cn^{\log_3 4},
\end{aligned}$$

where the last step holds as long as $d \geq 3/4$.

4.3-8 Using the master method in Section 4.5, you can show that the solution to the recurrence $T(n) = 4T(n/2) + n$ is $T(n) = \Theta(n^2)$. Show that a substitution proof with the assumption $T(n) \leq cn^2$ fails. Then show how to subtract off a lower-order term to make a substitution proof work.

The initial guess is

$$T(n) \leq cn^2 \ \forall n \geq n_0,$$

where c, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\leq 4c\left(\frac{n}{2}\right)^2 + n \\
&= cn^2 + n
\end{aligned}$$

which does not imply $T(n) \leq cn^2$ for any choice of c.

Our new guess is

$$T(n) \le cn^2 - dn \ \forall\, n \ge n_0,$$

where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\le 4\left(c\left(\frac{n}{2}\right)^2 - d\frac{n}{2}\right) + n \\ &= cn^2 - 2dn + n \\ &\le cn^2, \end{aligned}$$

where the last step holds as long as $d \ge 1/2$.

4.3-9 Solve the recurrence $T(n) = 3T(\sqrt{n}) + \log n$ by making a change of variables. Your solution should be asymptotically tight. Do not worry about whether values are integral.

Renaming $m = \log n$ yields

$$T(10^m) = 3T(10^{m/2}) + m.$$

Now renaming $S(m) = T(2^m)$ yields

$$S(m) = 3S(m/2) + m.$$

With the master method, we have $f(n) = m = \log n$ and $n^{\log_b a} = n^{\lg 3} \approx n^{1.585}$. Using the first case, we have

$$f(n) = \log n = O(n^{\lg 3 - \epsilon}),\ (\epsilon = 0.5)$$

which implies

$$S(m) = \Theta(m^{\lg 3}).$$

We can double-check if $S(m) = O(m^{\lg 3})$ using the substition method. Our guess is

$$S(m) \le cm^{\lg 3} - dm \ \forall\, m \ge m_0,$$

where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\le 3\left(c\left(\frac{m}{2}\right)^{\lg 3} - d\frac{m}{2}\right) + m \\ &= 3c\frac{m^{\lg 3}}{3} - 3d\frac{m}{2} + m \\ &\le cm^{\lg 3} + dm \end{aligned}$$

where the last step holds as long as $d \ge 2/3$.
Now verifying if $S(m) = \Omega(m^{\lg 3})$ with the substitution method. Our guess is

$$S(m) \ge cm^{\lg 3} \ \forall\, m \ge m_0,$$

where c, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\ge 3c\left(\frac{m}{2}\right)^{\lg 3} + m \\ &= 3c\frac{m^{\lg 3}}{3} + m \\ &\ge cm^{\lg 3}. \end{aligned}$$

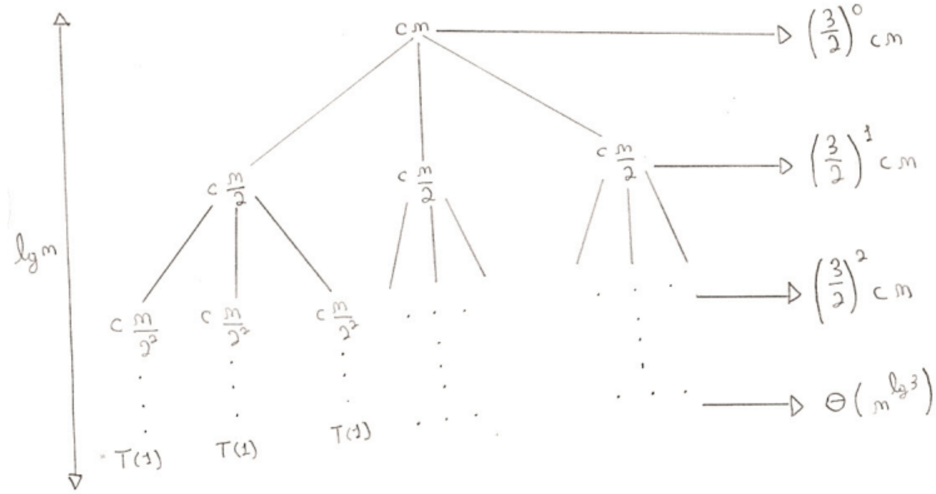
Finally, we have

$$T(n) = T(10^m) = S(m) = \Theta(m^{\lg 3}) = \Theta(\log^{\lg 3} n).$$

Section 4.4 – The recursion-tree method for solving recurrences

4.4-1 Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = 3T(\lfloor n/2 \rfloor) + n$. Use the substitution method to verify your answer.

Since floors/ceiling usually do not matter, we will draw a recursion tree for the recurrence $T(n) = 3T(n/2) + n$.



The number of nodes at depth i is 3^i. Since subproblem size reduce by a factor of 2, each node at depth i, for $i = 0, 1, 2, \ldots, \lg n - 1$, has a cost of $c(n/2^i)$. Thus, the total cost over all nodes at depth i, for $i = 0, 1, 2, \ldots, \lg n - 1$, is $(3/2)^i cn$. The bottom level, at deph $\lg n$, has $3^{\lg n} = n^{\lg 3}$ nodes, each contributing cost $T(1)$, for a total cost of $n^{\lg 3} T(1) = \Theta(n^{\lg 3})$. The cost of the entire tree is

$$
\begin{aligned}
T(n) &= cn + \frac{3}{2}cn + \left(\frac{3}{2}\right)^2 cn + \cdots + \left(\frac{3}{2}\right)^{\lg n - 1} cn + \Theta\left(n^{\lg 3}\right) \\
&= \sum_{i=0}^{\lg n - 1} \left(\frac{3}{2}\right)^i cn + \Theta(n^{\lg 3}) \\
&= cn \frac{\left(\frac{3}{2}\right)^{\lg n} - 1}{\frac{3}{2} - 1} + \Theta(n^{\lg 3}) \\
&= 2cn \left(\left(\frac{3}{2}\right)^{\lg n} - 1\right) + \Theta(n^{\lg 3}) \\
&= 2cn \frac{3^{\lg n}}{2^{\lg n}} - 2cn + \Theta(n^{\lg 3}) \\
&= 2cn \frac{n^{\lg 3}}{n} - 2cn + \Theta(n^{\lg 3}) \\
&= 2cn^{\lg 3} - 2cn + \Theta(n^{\lg 3}) \\
&= O(n^{\lg 3}).
\end{aligned}
$$

Our guess is

$$T(n) \le cn^{\lg 3} - dn \ \forall\, n \ge n_0,$$

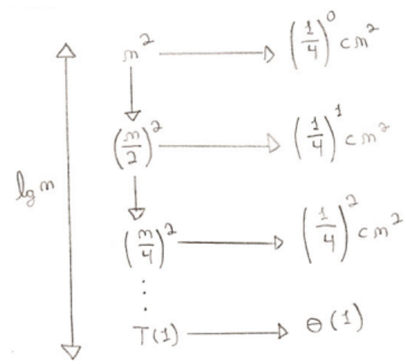
where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$
\begin{aligned}
T(n) &\le 3\left(c\left\lfloor \frac{n}{2} \right\rfloor^{\lg 3} - d\left\lfloor \frac{n}{2} \right\rfloor\right) + n \\
&\le \frac{3c}{3} n^{\lg 3} - \frac{3d}{2} n + n \\
&= cn^{\lg 3} - dn - \frac{d}{2} n + n \\
&\le cn^{\lg 3} - dn,
\end{aligned}
$$

where the last step holds as long as $d \ge 2$.

4.4-2 Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = T(n/2) + n^2$. Use the substitution method to verify your answer.

Figure below ilustrates the recursion tree $T(n) = T(n/2) + n^2$.



The tree has $\lg n$ levels and the cost at depth i is $c(n/2^i)^2 = (1/4)^i cn^2$.
The cost of the entire tree is

$$\begin{aligned} T(n) &= \sum_{i=0}^{\lg n} \left(\frac{1}{4}\right)^i cn^2 \\ &< \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i cn^2 \\ &= \frac{1}{1-(1/4)} cn^2 \\ &= \frac{4}{3} cn^2 \\ &= O(n^2). \end{aligned}$$

Our guess is

$$T(n) \le dn^2 \;\; \forall\, n \ge n_0,$$

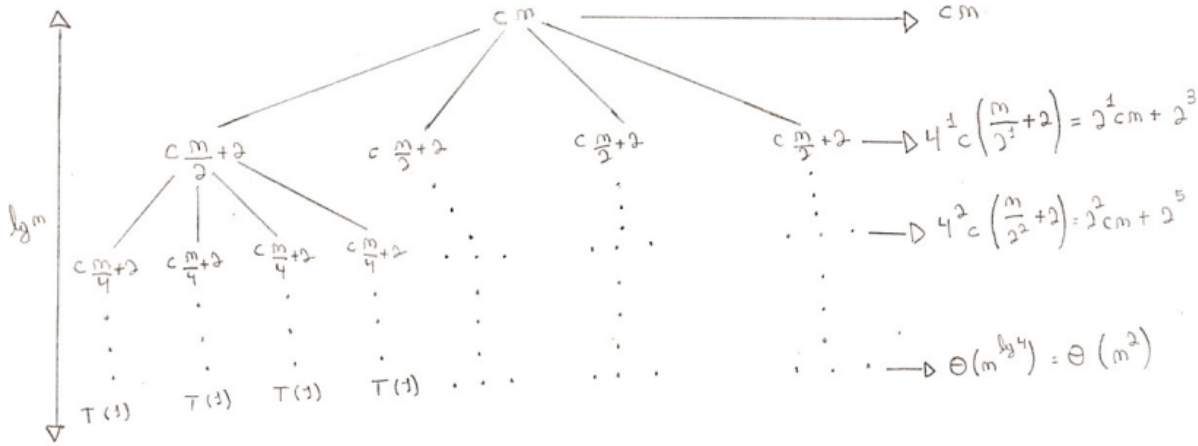
where d, and n_0 are positive constants. Substituting into the recurrence and using the same constant $c > 0$ as before yields

$$\begin{aligned} T(n) &\le d\left(\frac{n}{2}\right)^2 + cn^2 \\ &= \frac{1}{4} dn^2 + cn^2 \\ &\le dn^2, \end{aligned}$$

where the last step holds as long as $d \ge (4/3)c$.

4.4-3 Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = 4T(n/2+2)+n$. Use the substitution method to verify your answer.

Figure below ilustrates the recursion tree $T(n) = 4T(n/2 + 2) + n$.



The number of nodes at depth i is 4^i. Since subproblem size reduce by a factor of 2 and increment 2, each node at depth i, for $i = 0, 1, 2, \ldots, \lg n - 1$, has a cost of $c(n/2^i + 2)$. Thus, the total cost over all nodes at depth i, for $i = 0, 1, 2, \ldots, \lg n - 1$, is $4^i c(n/2^i + 2) = 2^i cn + 2^{2i+1}$. The bottom level, at deph $\lg n$, has $4^{\lg n} = n^{\lg 4}$ nodes, each contributing cost $T(1)$, for a total cost of $n^{\lg 4} T(1) = \Theta(n^{\lg 4})$.

The cost of the entire tree is

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\lg n - 1} \left(4^i c\left(\frac{n}{2^i} + 2\right)\right) + \Theta(n^2) \\
&= \sum_{i=0}^{\lg n - 1} \left(4^i c \cdot \frac{n}{2^i}\right) + \sum_{i=0}^{\lg n - 1} (4^i c \cdot 2) + \Theta(n^2) \\
&= cn \sum_{i=0}^{\lg n - 1} (2^i) + 2c \sum_{i=0}^{\lg n - 1} (4^i) + \Theta(n^2) \\
&= cn \frac{2^{\lg n} - 1}{2 - 1} + 2c \frac{4^{\lg n} - 1}{4 - 1} + \Theta(n^2) \\
&= cn(n-1) + \frac{2c}{3}(n^2 - 1) + \Theta(n^2) \\
&= cn^2 - cn + \frac{2cn^2}{3} - \frac{2c}{3} + \Theta(n^2) \\
&= O(n^2).
\end{aligned}
$$

Our guess is

$$T(n) \le cn^2 - dn \ \forall\, n \ge n_0,$$

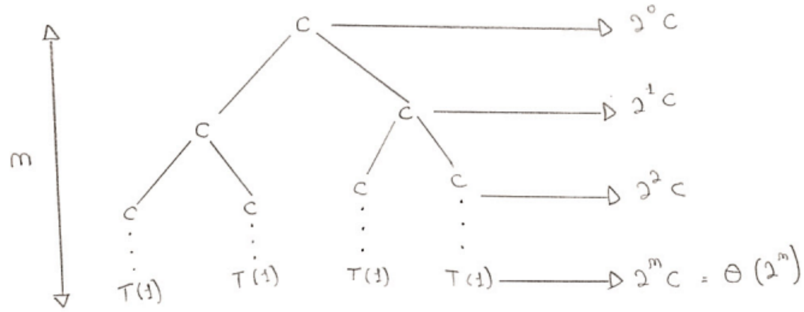
where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$
\begin{aligned}
T(n) &\le 4\left(c\left(\frac{n}{2} + 2\right)^2 - d\left(\frac{n}{2} + 2\right)\right) + n \\
&\le 4\left(c\frac{n^2}{4} + 2cn + 4c - \frac{dn}{2} - 2d\right) + n \\
&= cn^2 + 8cn + 16c - 2dn - 8d + n \\
&= cn^2 - dn - (d - 8c - 1)n - (d - 2c)8 \\
&\le cn^2 - dn,
\end{aligned}
$$

where the last step holds as long as $d - 8c - 1 \ge 0$.

4.4-4 Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = 2T(n-1) + 1$. Use the substitution method to verify your answer.

Figure below ilustrates the recursion tree $T(n) = 2T(n-1) + 1$.



The tree has n levels and 2^i nodes at each level. Since each node costs 1, the cost at depth i is 2^i. The bottom level, at deph n, has 2^n nodes, each contributing cost 1, for a total cost of $2^n = \Theta(2^n)$.
The cost of the entire tree is

$$\begin{aligned} T(n) &= \sum_{i=0}^{n-1} (2^i) + \Theta(2^n) \\ &= \frac{2^n - 1}{2 - 1} + \Theta(2^n) \\ &= 2^n - 1 + \Theta(2^n) \\ &= O(2^n). \end{aligned}$$

Our guess is

$$T(n) \leq c2^n - d \ \forall\, n \geq n_0,$$

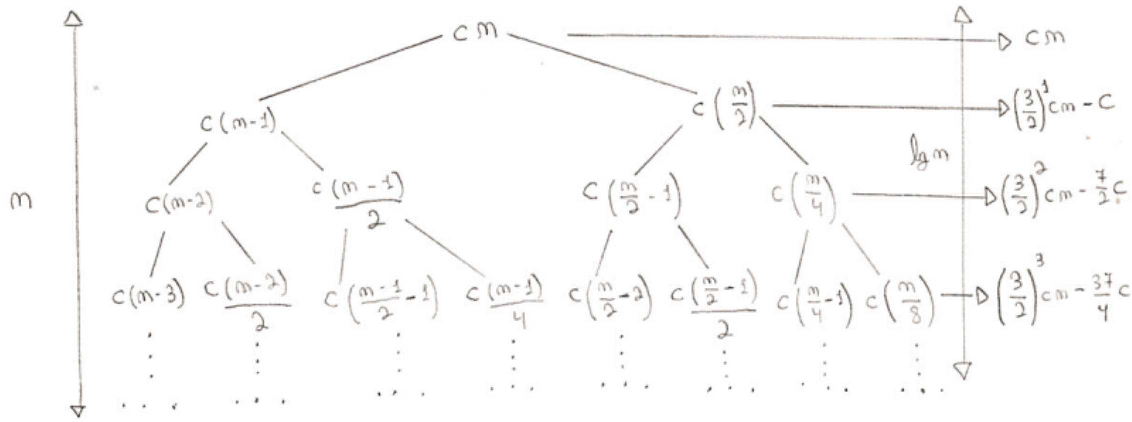
where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\leq 2(c2^{n-1} - d) + 1 \\ &= c2^n - 2d + 1 \\ &= c2^n - d - d + 1 \\ &\leq c2^n - d, \end{aligned}$$

where the last step holds as long as $d \geq 1$.

4.4-5 Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = T(n-1) + T(n/2) + n$. Use the substitution method to verify your answer.

Figure below ilustrates the recursion tree $T(n) = T(n-1) + T(n/2) + n$.



We start obtaining a lower bound. The cost of the initial levels (before level $\lg n$) of the tree are

$$cn \to (3/2)^1 cn - c \to (3/2)^2 cn - (7/2)c \to (3/2)^3 cn - (37/4)c.$$

Thus, the cost of the tree from the root to level $\lg n$ is at most

$$\sum_{i=0}^{\lg n} \left(\frac{3}{2}\right)^i cn = cn \frac{\left(\frac{3}{2}\right)^{\lg n+1} - 1}{\frac{3}{2} - 1} = 2cn\frac{3}{2}\left(\frac{3}{2}\right)^{\lg n} - 2cn = 3cn\frac{n^{\lg 3}}{n} - 2cn = 3cn^{\lg 3} - 2cn = O(n^{\lg 3}).$$

The cost of the longest simple path from the root to a leaf is

$$\sum_{i=0}^{n} c(n-i) = c\sum_{i=0}^{n} i = c\frac{n(n+1)}{2} = c\frac{n^2}{2} + \frac{c}{2} = O(n^2).$$

Thus, our guess for a lower bound for $T(n)$ is

$$T(n) \geq cn^2 \ \forall\, n \geq n_0,$$

where c, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\geq c(n-1)^2 + c\left(\frac{n}{2}\right)^2 + n \\
&= cn^2 - 2cn + 1 + \frac{cn^2}{4} + n \\
&= \frac{5}{4}cn^2 - 2cn + n + 1 \\
&\geq cn^2 - 2cn + n + 1 \\
&\geq cn^2,
\end{aligned}$$

where the last step holds as long as $c \geq 1$ and $n_0 \geq 1$. Thus, we have $T(n) = \Omega(n^2)$.
Consider now the recurrence

$$S(n) = 2T(n-1) + n,$$

which is more costly than $T(n)$. We can easily prove that $S(n) = O(2^n)$. Our guess for an upper bound of $S(n)$ is

$$S(n) \leq c2^n - dn \ \forall\, n \geq n_0,$$

where c, d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
S(n) &\leq 2(c2^{n-1} - d(n-1)) + n \\
&= c2^n - 2dn + 2d + n \\
&= c2^n - dn - n(d-1) + 2d \\
&\leq c2^n - dn,
\end{aligned}$$

where the last step holds as long as $d \geq 2$ and $n_0 \geq 3$. Thus, we have $T(n) = O(S(n)) = O(2^n)$.

We can obtain a more tight upper bound without using the recursion tree. Let $R(n) = T(n/2) + n$. We have

$$\begin{aligned} T(n) &= T(n-1) + R(n) \\ &= T(n-2) + R(n-1) + R(n) \\ &= R(1) + R(2) + \cdots + R(n-1) + R(n) \\ &\leq n \cdot R(n) \\ &= n \cdot T(n/2) + n^2, \end{aligned}$$

which can be solved using the master method. We have $f(n) = n^2$ and $n^{\log_b a} = n^{\lg n}$. Using the first case, we have

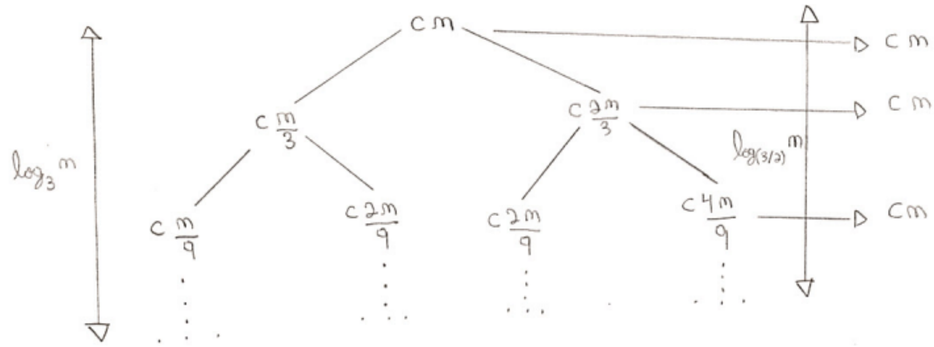
$$f(n) = n^2 = O(n^{\lg n - \epsilon}),\ (\epsilon = 1)$$

which implies

$$T(n) = O(n^{\lg n}).$$

4.4-6 Argue that the solution to the recurrence $T(n) = T(n/3) + T(2n/3) + cn$, where c is a constant, is $\Omega(n \lg n)$ by appealing to a recursion tree.

Figure below ilustrates the recursion tree $T(n) = T(n/3) + T(2n/3) + cn$.



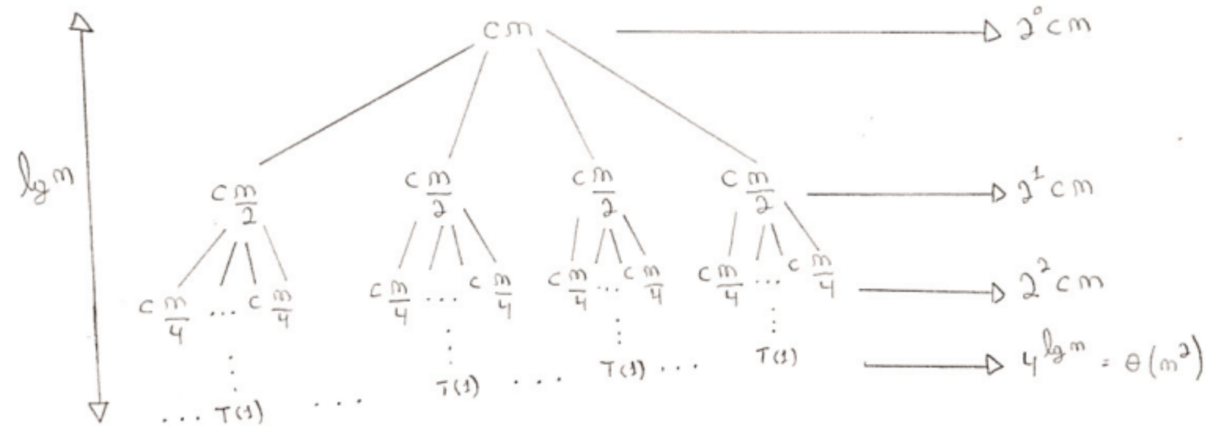
The tree is complete until level $\log_3 n$. The cost of the tree from the root to level $\log_3 n$ is

$$\sum_{i=0}^{\log_3 n} cn = cn \log_3 n,$$

which is $\Omega(n \lg n)$.

4.4-7 Draw the recursion tree for $T(n) = 4T(\lfloor n/2 \rfloor) + cn$, where c is a constant, and provide a tight asymptotic bound on its solution. Verify your bound by the substitution method.

Since floors/ceiling usually do not matter, we will draw a recursion tree for the recurrence $T(n) = 4T(n/2) + cn$.



The number of nodes at depth i is 4^i. Since subproblem size reduce by a factor of 2, each node at depth i, for $i = 0, 1, 2, \ldots, \lg n - 1$, has a cost of $c(n/2^i)$. Thus, the total cost over all nodes at depth i, for $i = 0, 1, 2, \ldots, \lg n - 1$, is $(4/2)^i cn = 2^i cn$. The bottom level has $4^{\lg n} = n^2$ nodes, each contributing cost $T(1)$, for a total cost of $n^2 T(1) = \Theta(n^2)$. The cost of the entire tree is

$$\sum_{i=0}^{\lg n - 1} (2^i cn) + \Theta(n^2) = cn\frac{2^{\lg n} - 1}{2 - 1} + \Theta(n^2) = cn(n-1) + \Theta(n^2) = cn^2 - cn + \Theta(n^2) = \Theta(n^2).$$

Lets verify with the substitution method. Our guess for an upper bound is

$$T(n) \le dn^2 - en \;\forall\, n \ge n_0,$$

where d, e, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\le 4\left(d\left\lfloor\frac{n}{2}\right\rfloor^2 - e\frac{n}{2}\right) + cn \\
&\le 4\left(d\left(\frac{n}{2}\right)^2 - e\frac{n}{2}\right) + cn \\
&= 4\left(d\frac{n^2}{4} - e\frac{n}{2}\right) + cn \\
&= dn^2 - 2en + cn \\
&= dn^2 - en - en + cn \\
&\le dn^2 - en,
\end{aligned}$$

where the last step holds as long as $e \ge c$.
Our guess for a lower bound is

$$T(n) \ge dn^2 \;\forall\, n \ge n_0,$$

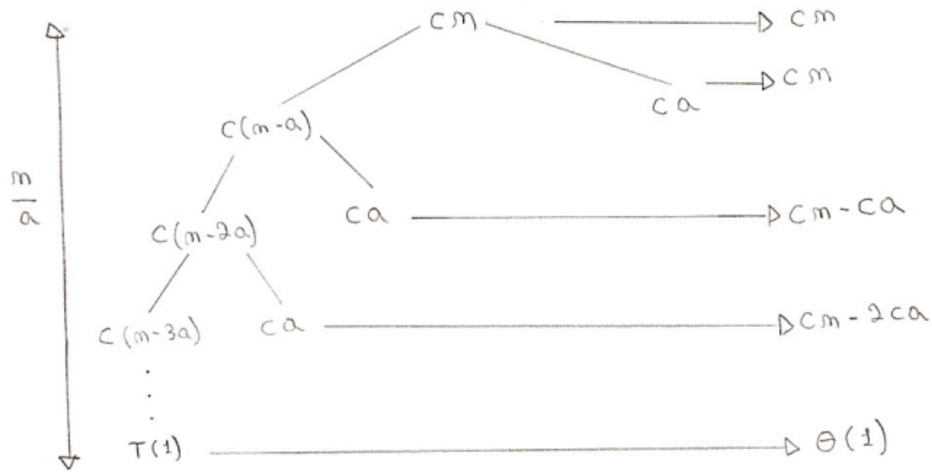
where d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\ge 4d\left\lfloor\frac{n}{2}\right\rfloor^2 + cn \\
&\ge 4d\left(\frac{n}{2} - 1\right)^2 + cn \\
&= 4d\left(\frac{n^2}{4} - n + 1\right) + cn \\
&= dn^2 - 4dn + 4d + cn \\
&= dn^2 - (4d - c)n + 4d
\end{aligned}$$

where the last step holds as long as $4d - c \ge 4$ and $n_0 \ge d$.

4.4-8 Use a recursion tree to give an asymptotically tight solution to the recurrence $T(n) = T(n-a) + T(a) + cn$, where $a \geq 1$ and $c > 0$ are constants.

Figure below ilustrates the recursion tree $T(n) = T(n-a) + T(a) + cn$.



The height of the tree is n/a. Each level i, for $i = 1, 2, \ldots, (n/a)$, has two nodes, one that costs $c(n - ia)$ and another that costs $T(a) = ca$. Thus, the cost over the nodes at depth i, for $i = 1, 2, \ldots, (n/a)$, is $c(n-a) + ca$. The root level, at deph 0, has a single node that costs cn.

The cost of the entire tree is

$$
\begin{aligned}
T(n) &= cn + \sum_{i=1}^{n/a}(c(n-ia) + ca) \\
&= cn + \sum_{i=1}^{n/a} cn - \sum_{i=1}^{n/a} cia + \sum_{i=1}^{n/a} ca \\
&= cn + c\frac{n^2}{a} - \frac{cn(a+n)}{2a} + cn \\
&= c\frac{n^2}{a} - c\frac{n^2}{2a} - c\frac{n}{2} + 2cn \\
&= c\frac{n^2}{2a} + \frac{3}{2}cn \\
&= \Theta(n^2).
\end{aligned}
$$

Lets verify with the substitution method. Our guess for an upper bound is

$$T(n) \leq cn^2 \ \forall n \geq n_0,$$

where c and n_0 are positive constants. Substituting into the recurrence yields

$$
\begin{aligned}
T(n) &\leq c(n^2 - 2an + a^2) + ca + cn \\
&= cn^2 - c(2an - a - n - a^2) \\
&\leq cn^2,
\end{aligned}
$$

where the last step holds as long as $n_0 \geq a$.

Our guess for a lower bound is

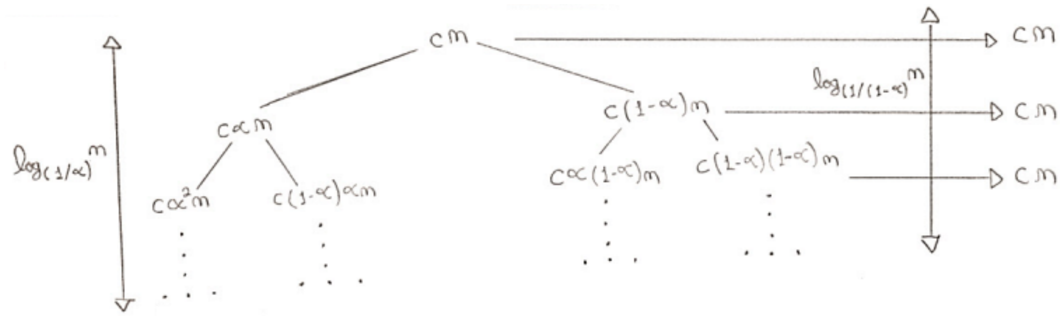
$$T(n) \geq \frac{c}{2a}n^2 \ \forall n \geq n_0,$$

where c, and n_0 are positive constants. Substituting into the recurrence yields

$$
\begin{aligned}
T(n) &\geq \frac{c}{2a}(n-a)^2 + ca + cn \\
&= \frac{c}{2a}(n^2 - 2an + a^2) + ca + cn \\
&= \frac{c}{2a}n^2 - cn + \frac{1}{2}ca + ca + cn \\
&= \frac{c}{2a}n^2 + \frac{3}{2}ca \\
&\geq \frac{c}{2a}n^2.
\end{aligned}
$$

4.4-9 Use a recursion tree to give an asymptotically tight solution to the recurrence $T(n) = T(\alpha n) + T((1-\alpha)n) + cn$, where α is a constant in the range $0 < \alpha < 1$ and $c > 0$ is also a constant.

Let $\alpha \geq 1 - \alpha$. Figure below ilustrates the recursion tree $T(n) = T(\alpha n) + T((1 - \alpha n)n) + cn$.



If it were a complete tree, all the $\log_{1-\alpha} n$ levels would cost cn and the entire tree $cn\log_{1-\alpha} n$. Thus, $T(n) = O(n\log_{1-\alpha} n) = O(n\lg n)$. The tree is complete until level $\log_{1/(1-\alpha)} n$. The cost of the tree from the root to level $\log_{1/(1-\alpha)} n$ is

$$\sum_{i=0}^{\log_{1/(1-\alpha)} n} cn = \left(\sum_{i=1}^{\log_{1/(1-\alpha)} n} cn\right) + cn = cn(\log_{1/(1-\alpha)} n) + cn,$$

which is $\Omega(n\log_{1/(1-\alpha)} n) = \Omega(n\lg n)$.
Lets verify with the substitution method. Our guess for an upper bound is

$$T(n) \leq dn\lg n \;\; \forall\, n \geq n_0,$$

where d and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\leq d\alpha n\lg(\alpha n) + d(1-\alpha)n\lg((1-\alpha)n) + dn \\
&= d\alpha n\lg\alpha + d\alpha n\lg n + d(1-\alpha)n\lg(1-\alpha) + d(1-\alpha)n\lg n + cn \\
&= d\alpha n\lg\alpha + d\alpha n\lg n + d(1-\alpha)n\lg(1-\alpha) + dn\lg n - d\alpha n\lg n + cn \\
&= dn\lg n + dn(\alpha\lg\alpha + (1-\alpha)\lg(1-\alpha)) + cn \\
&\leq dn\lg n,
\end{aligned}$$

where the last step holds as long as $d(\alpha\lg\alpha + (1-\alpha)\lg(1-\alpha)) + c \leq 0$.
Our guess for a lower bound is

$$T(n) \geq dn\lg n \;\; \forall\, n \geq n_0,$$

where d, and n_0 are positive constants. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\geq d\alpha n\lg(\alpha n) + d(1-\alpha)n\lg((1-\alpha)n) + dn \\
&= d\alpha n\lg\alpha + d\alpha n\lg n + d(1-\alpha)n\lg(1-\alpha) + d(1-\alpha)n\lg n + cn \\
&= d\alpha n\lg\alpha + d\alpha n\lg n + d(1-\alpha)n\lg(1-\alpha) + dn\lg n - d\alpha n\lg n + cn \\
&= dn\lg n + dn(\alpha\lg\alpha + (1-\alpha)\lg(1-\alpha)) + cn \\
&\geq dn\lg n,
\end{aligned}$$

where the last step holds as long as $d(\alpha\lg\alpha + (1-\alpha)\lg(1-\alpha)) + c \geq 0$.

Section 4.5 – The master method for solving recurrences

4.5-1 Use the master method to give tight asymptotic bounds for the following recurrences.

a. $T(n) = 2T(n/4) + 1$.

b. $T(n) = 2T(n/4) + \sqrt{n}$.

c. $T(n) = 2T(n/4) + n$.

d. $T(n) = 2T(n/4) + n^2$.

(a) Case 1 applies. $T(n) = \Theta(n^{\log_4 2}) = \Theta(\sqrt{n})$.

(b) Case 2 applies. $T(n) = \Theta(n^{\log_4 2} \lg n) = \Theta(\sqrt{n} \lg n)$.

(c) Case 3 applies. $T(n) = \Theta(n)$.

(d) Case 3 applies. $T(n) = \Theta(n^2)$.

4.5-2 Professor Caesar wishes to develop a matrix-multiplication algorithm that is asymptotically faster than Strassen's algorithm. His algorithm will use the divide-and-conquer method, dividing each matrix into pieces of size $n/4 \times n/4$, and the divide and combine steps together will take $\Theta(n^2)$ time. He needs to determine how many subproblems his algorithm has to create in order to beat Strassen's algorithm. If his algorithm creates a subproblems, then the recurrence for the running time $T(n)$ becomes $T(n) = aT(n/4) + \Theta(n^2)$. What is the largest integer value of a for which Professor Caesar's algorithm would be asymptotically faster than Strassen's algorithm?

Strassen's algorithm costs $\Theta(n^{\lg 7})$. The cost of $T(n)$ is stated below.

- If $a < 16$, Case 3 applies. $T(n) = \Theta(n^2) = o(n^{\lg 7})$.
- If $a = 16$, Case 2 applies. $T(n) = \Theta(n^2 \lg n) = o(n^{\lg 7})$.
- If $a > 16$, Case 1 applies. $T(n) = \Theta(n^{\log_4 a}) = o(n^{\lg 7})$ when $a < 49$.

Thus, the largest integer value of a is 48.

4.5-3 Use the master method to show that the solution to the binary-search recurrence $T(n) = T(n/2) + \Theta(1)$ is $T(n) = \Theta(\lg n)$. (See Exercise 2.3-5 for a description of binary search.)

We have

$$n^{\log_b a} = n^{\lg 1} = \Theta(1) = f(n).$$

Thus, Case 2 applies. $T(n) = \Theta(\lg n)$.

4.5-4 Can the master method be applied to the recurrence $T(n) = 4T(n/2) + n^2 \lg n$? Why or why not? Give an asymptotic upper bound for this recurrence.

We have
$$f(n) = n^2 \lg n,$$
and
$$n^{\log_b a} = n^{\log_2 4} = \Theta(n^2),$$
which is larger than $f(n)$, but not polynomially larger. Thus, we cannot use the master method to solve this recurrence.
We can use a recursion tree to guess the cost of $T(n)$ and verify with the substitution method. Figure below ilustrates the recursion tree of $T(n) = 4T(n/2) + n^2 \lg n$.

Figure here.

The tree has $\lg n$ levels and the number of nodes at depth i is 4^i. Each node at depth i has a cost $c((n/2^i)^2)\lg(n) = 1/4^i cn^2 \lg n$. Thus, the total cost at depth i is $4^i \times 1/4^i cn^2 \lg n = cn^2 \lg n$.
The cost of the entire tree is
$$\sum_{i=0}^{\lg n} cn^2 \lg n = O(n^2 \lg^2 n).$$
Lets verify with the substitution method. Our guess is
$$T(n) \le cn^2 \lg^2 n \ \forall n \ge n_0,$$
where c and n_0 are positive constants. Substituting into the recurrence yields
$$\begin{aligned} T(n) &\le 4c\left(\left(\frac{n}{2}\right)^2 \lg^2\left(\frac{n}{2}\right)\right) + n^2 \lg n \\ &= 4c\left(\frac{n^2}{4} \lg\left(\frac{n}{2}\right) \lg\left(\frac{n}{2}\right)\right) + n^2 \lg n \\ &= cn^2 \lg\left(\frac{n}{2}\right) \lg\left(\frac{n}{2}\right) + n^2 \lg n \\ &= cn^2 \lg\left(\frac{n}{2}\right) \lg n - cn^2 \lg\left(\frac{n}{2}\right) + n^2 \lg n \\ &= cn^2 \lg^2 n - cn^2 \lg n - cn^2 \lg n + cn^2 + n^2 \lg n \\ &\le cn^2 \lg^2 n, \end{aligned}$$
where the last step holds as long as $c \ge 1$.

4.5-5 Consider the regularity condition $af(n/b) \ge cf(n)$ for some constant $c < 1$, which is part of case 3 of the master theorem. Give an example of constants $a \ge 1$ and $b > 1$ and a function $f(n)$ that satisfies all the conditions in case 3 of the master theorem except the regularity condition.

Let $a = 1$, $b = 2$, and $f(n) = n \cos n$. We have
$$n^{\log_b a} = n^{\log_2 1} = \Theta(1),$$
which is polynomially smaller than $f(n)$ and satisfies the primary condition of Case 3. However, we have
$$af\left(\frac{n}{b}\right) \le cf(n) \to \frac{n}{2} \cos\left(\frac{n}{2}\right) \le c(n \cos n),$$
which is not valid for some constant $c < 1$ and all sufficiently large n since $\cos(\cdot)$ is not monotonic. Thus, it satisfies the primary condition of Case 3, but not the regularity condition