Section A.1 – Summation formulas and properties

A.1-1 Find a simple formula for $\sum_{k=1}^{n}(2k-1)$.

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1) &= \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 \\
&= 2\sum_{k=1}^{n} k - n \\
&= 2 \cdot \frac{1}{2} n(n+2) - n \\
&= n^2 + n - n \\
&= n^2.
\end{aligned}$$

A.1-2 ($\star$) Show that $\sum_{k=1}^{n} 1/(2k-1) = \ln(\sqrt{n}) + O(1)$ by manipulating the harmonic series.

$$\begin{aligned}
\sum_{k=1}^{n} 1/(2k-1) &= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-3} + \frac{1}{2n-1} \\
&= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\
&= \sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2}\sum_{k-1}^{n} \frac{1}{k} \\
&= \ln 2n + O(1) - \frac{1}{2}(\ln n + O(1)) \\
&= \ln n + \ln 2 + O(1) - \frac{1}{2}\ln n - \frac{1}{2}O(1) \\
&= \frac{1}{2}\ln n + O(1) \\
&= \ln(\sqrt{n}) + O(1).
\end{aligned}$$

A.1-3 Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ for $0 < |x| < 1$.

From Equation A.8, we have

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

differentiating both sides and multiplying by x, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} k^2 x^k &= x \cdot \frac{1 \cdot (1-x)^2 - (2 \cdot (1-x) \cdot (-1) \cdot x)}{(1-x)^4} \\
&= x \cdot \frac{(1-x)(1-x) + (1-x) \cdot 2x}{(1-x)^4} \\
&= x \cdot \frac{(1-x) + 2x}{(1-x)^3} \\
&= \frac{x(1+x)}{(1-x)^3}.
\end{aligned}$$

A.1-4 ($\star$) Show that $\sum_{k=0}^{\infty}(k-1)/2^k = 0$.

$$\begin{aligned}
\sum_{k=0}^{\infty}(k-1)/2^k &= \sum_{k=0}^{\infty}\left(\frac{k}{2^k} - \frac{1}{2^k}\right) \\
&= \sum_{k=0}^{\infty} k\frac{1}{2^k} - \sum_{k=0}^{\infty}\frac{1}{2^k} \\
&= \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \\
&= \frac{(1/2)}{(1-(1/2))^2} - \frac{1}{1-(1/2)} \\
&= \frac{(1/2)}{1-1-(1/4)} - 2 \\
&= 4/2 - 2 \\
&= 0.
\end{aligned}$$

A.1-5 ($\star$) Evaluate the sum $\sum_{k=1}^{\infty}(2k+1)x^{2k}$ for $|x| < 1$.

$$\begin{aligned}
\sum_{k=1}^{\infty}(2k+1)x^{2k} &= \frac{d}{dx}\cdot\sum_{k=1}^{\infty}x^{2k+1} \\
&= \frac{d}{dx}\cdot x\cdot\sum_{k=1}^{\infty}x^{2k} \\
&= \frac{d}{dx}\cdot x\cdot\sum_{k=0}^{\infty}(x^2)^k - x \\
&= \frac{d}{dx}\cdot x\cdot\frac{1}{1-x^2} - x \\
&= \frac{d}{dx}\cdot\frac{x - x(1-x^2)}{1-x^2} \\
&= \frac{d}{dx}\cdot\frac{x^3}{1-x^2} \\
&= \frac{3x^2(1-x^2) - (-2x)x^3}{(1-x^2)^2} \\
&= \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2} \\
&= \frac{(3-x^2)\cdot x^2}{(1-x^2)^2}.
\end{aligned}$$

A.1-6 Prove that $\sum_{k=1}^{n} O(f_k(i)) = O(\sum_{k=1}^{n} f_k(i))$ by using the linearity property of summations.

Skipped.

A.1-7 Evaluate the product $\prod_{k=1}^{n} 2 \cdot 4^k$.

We have

$$\prod_{k=1}^{n}(2 \cdot 4^k) = 2^{\lg\left(\prod_{k=1}^{n}(2\cdot 4^k)\right)},$$

and

$$\begin{aligned}
\lg\left(\prod_{k=1}^{n}(2 \cdot 4^k)\right) &= \sum_{k=1}^{n} \lg(2 \cdot 2^{2k}) \\
&= \sum_{k=1}^{n} \lg 2^{2k+1} \\
&= \sum_{k=1}^{n} (2k+1) \\
&= 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= n(n+1) + n \\
&= n(n+2).
\end{aligned}$$

Thus,

$$\prod_{k=1}^{n}(2 \cdot 4^k) = 2^{n(n+2)}.$$

A.1-8 ($\star$) Evalute the product $\prod_{k=2}^{n}(1 - 1/k^2)$.

We have

$$\prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right) = 2^{\lg\left(\sum_{k=2}^{n} \lg(1-1/k^2)\right)},$$

and

$$\begin{aligned}
\sum_{k=2}^{n} \lg\left(1 - \frac{1}{k^2}\right) &= \sum_{k=2}^{n} \lg\left(\frac{k^2-1}{k^2}\right) \\
&= \sum_{k=2}^{n} \lg\left(\frac{(k-1)}{k} \cdot \frac{(k+1)}{k}\right) \\
&= \sum_{k=2}^{n} \left(\lg\left(\frac{k-1}{k}\right) + \lg\left(\frac{k+1}{k}\right)\right) \\
&= \lg\frac{1}{2} + \lg\frac{3}{2} + \lg\frac{2}{3} + \lg\frac{4}{3} + \lg\frac{3}{4} + \lg\frac{5}{4} + \cdots + \lg\frac{n-2}{n-1} + \lg\frac{n}{n-1} + \lg\frac{n-1}{n} + \lg\frac{n+1}{n} \\
&= \lg 1 - \lg 2 + \cancel{\lg 3} - \cancel{\lg 2} + \cancel{\lg 2} - \cancel{\lg 3} + \cancel{\lg 4} - \cancel{\lg 3} + \cancel{\lg 3} - \cancel{\lg 4} + \cancel{\lg 5} - \cancel{\lg 4} + \ldots \\
&\quad + \cancel{\lg(n-2)} - \cancel{\lg(n-1)} + \cancel{\lg n} - \cancel{\lg(n-1)} + \cancel{\lg(n-1)} - \cancel{\lg n} + \lg(n+1) - \lg n \\
&= 0 - 1 + \lg(n+1) - \lg n \\
&= \lg(n+1) - \lg(n) - 1.
\end{aligned}$$

Thus,

$$\prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right) = 2^{(\lg(n+1)-(\lg(n)+1))} = \frac{2^{\lg(n+1)}}{2^{\lg(n)+1}} = \frac{n+1}{2^{\lg n} \cdot 2} = \frac{n+1}{2n}.$$

Section A.2 – Bounding summations

A.2-1 Show that $\sum_{k=1}^{n} 1/k^2$ is bounded above by a constant.

Answer

A.2-2 Find an asymptotic upper bound on the summation

$$\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil .$$

Answer

A.2-3 Show that the nth harmonic number is $\Omega(\lg n)$ by splitting the summation.

Answer

A.2-4 Approximate $\sum_{k=1}^{n} k^3$ with an integral.

Answer

A.2-5 Why didn't we use the integral approximation (A.12) directly on $\sum k = 1^n 1/k$ to obtain an upper bound on the nth harmonic number?

Answer

Problems

A-1 Question

Answer.