Section 9.1 – Minimum and maximum

9.1-1 Show that the second smallest of n elements can be found with $n + \lceil \lg n \rceil - 2$ comparisons in the worst case. (*Hint:* Also find the smallest element.)

Lets find first the smallest element. Compare the elements in pairs and discard the largest element of each pair. The number of elements is now $\lceil n/2 \rceil$. Repeat this operation recursively to the remaining elements until the smallest element is found. Since we discard one element in each comparison, the number of comparisons is the number of elements that is not the smaller. Thus, $n-1$ comparisons. Note that the second smallest element can only be greater than the smallest element. Thus, the second smallest element is among these $\lceil \lg n \rceil$ elements that were discarded when compared to the smallest element. Use the same recursive approach on these $\lceil \lg n \rceil$ elements to find the second smallest with $\lceil \lg n \rceil - 1$ comparisons. The total number of comparisons in the worst-case is then $n - 1 + \lceil \lg n \rceil - 1 = n + \lceil \lg n \rceil - 2$.

9.1-2 ($\star$) Prove the lower bound of $\lceil 3n/2 \rceil - 2$ comparisons in the worst case to find both the maximum and minimum of n numbers. (*Hint:* Consider how many numbers are potentially either the maximum or minimum, and investigate how a comparison affects these counts.)

At the start, any of the n the elements can be both the minimum and the maximum. After the first comparison, we can discard the largest as not being the minimum and the smallest as not being the maximum. From now on we have two options: compare two different elements or compare one of the elements previously compared with a different element. The first option will decrease by one both the number of potential minimums and potential maximums, while the second option will only decrease one these totals. Thus, the best way to start is to group the elements in pairs and compare them, which requires $\lfloor n/2 \rfloor$ comparisons. After comparing all the pairs, we will have $\lceil n/2 \rceil$ potential maximums and $\lceil n/2 \rceil$ potential minimums. In the worst-case, those sets are disjoint and must be treated independently. We know from the previous question that the minimum number of comparisons needed to find the minimum or the maximum among $\lceil n/2 \rceil$ elements is $\lceil n/2 \rceil - 1$. Thus, the lower bound to find both the maximum and the minimum of n numbers is

$$\left\lfloor \frac{n}{2} \right\rfloor + 2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right).$$

If n is even, we have

$$\left\lfloor \frac{n}{2} \right\rfloor + 2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) = \frac{n}{2} + n - 2 = \frac{3n}{2} - 2 = \left\lceil \frac{3n}{2} \right\rceil - 2.$$

If n is odd, we have

$$\left\lfloor \frac{n}{2} \right\rfloor + 2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) = \frac{n-1}{2} + (n+1) - 2 = \frac{3n-3}{2} = \left\lceil \frac{3n}{2} \right\rceil - 2.$$