Section 5.1 – The hiring problem

5.1-1 Show that the assumption that we are always able to determine which candidate is best, in line 4 of procedure Hire-Assistant, implies that we know a total order on the ranks of the candidates.

Let A be the set of candidates in random order and R the binary relation "is better than or equal" on the set A. R is a total order if

(a) R is **reflexive**. That is, $a\ R\ a\ \forall\, a \in A$;

(b) R is **antisymmetric**. That is, $a\ R\ b$ and $b\ R\ a$ imply $a = b$;

(c) R is **transitive**. That is, $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$;

(d) R is a **total relation**. That is, $a\ R\ b$ or $b\ R\ a\ \forall\, a, b \in A$.

The above properties are necessary because

(a) if two different candidates have the same qualification, it is necessary so that they can be compared;

(b) if both a is "better than or equal" than b and b is "better than or equal" than a and they qualifications are not equal, we would not be able to choose one of them and still be hiring "the best candidate we have seen so far";

(c) if we hire b because he is "better than or equal" than a and then we hire c because he is "better than or equal" than b and c is not "better than or equal" than a, we are not hiring "the best candidate we have seen so far";

(d) if the R is not a total relation, we would not be able to compare any two candidates.

5.1-2 ($\star$) Describe an implementation of the procedure Random(a, b) that only makes calls to Random(0, 1). What is the expected running time of your procedure, as a function of a and b?

The pseudocode is stated below.

```
RandomInterval(a, b)
  flips = ⌈lg(b − a)⌉
  count = ∞
  while count > b do
      count = 0
      for i = 1 to flips do
          count = count + (2^(i−1) · Random(0, 1))
  return count + a
```

The expected running time is

$$\underbrace{2^{\lceil \lg(b-a) \rceil}/(b-a)}_{\text{while loop}} \cdot \underbrace{\lceil \lg(b-a) \rceil}_{\text{for loop}} < 2 \cdot \lceil \lg(b-a) \rceil,$$

where the last inequality is valid since $1 \le 2^{\lceil \lg(b-a) \rceil}/(b-a) < 2$.

5.1-3 ($\star$) Suppose that you want to output 0 with probability 1/2 and 1 with probability 1/2. At your disposal is a procedure Biased-Random, that outputs either 0 or 1. It outputs 1 with some probability p and 0 with probability $1-p$, where $0 < p < 1$, but you do not know what p is. Give an algorithm that uses Biased-Random as a subroutine, and returns an unbiased answer, returning 0 with probability 1/2 and 1 with probability 1/2. What is the expected running time of your algorithm as a function of p?

The pseudocode is stated below.

```
Random()
  while 1 do
      r1 = Random(0, 1)
      r2 = Random(0, 1)
      if r1 ≠ r2 then
          return r1
```

The expected running time is

$$\frac{1}{\underbrace{(1-p)p}_{(r_1, r_2) = (0,1)} + \underbrace{p(1-p)}_{(r_1, r_2) = (1,0)}} \cdot 1 = \frac{1}{2p(1-p)}.$$