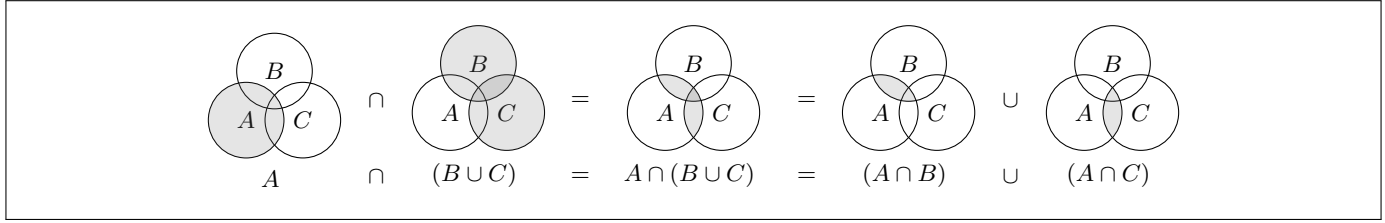


Section B.1 – Sets

B.1-1 Draw Venn diagrams that illustrate the first of the distributive laws (B.1).



B.1-2 Prove the generalization of DeMorgan's laws to any finite collection of sets:

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n},$$

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

The base case, which occurs when $n = 2$, is given (from the text book). Now, let's assume it holds for n and show that it also holds for $n + 1$.

For the first DeMorgan's law, we have

$$\begin{aligned}
 \overline{A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}} &= \overline{(A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1}} \\
 &= \overline{(A_1 \cap A_2 \cap \cdots \cap A_n)} \cup \overline{A_{n+1}} \\
 &= (\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}) \cup \overline{A_{n+1}} \\
 &= \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n} \cup \overline{A_{n+1}}.
 \end{aligned}$$

For the second DeMorgan's law, we have

$$\begin{aligned}
 \overline{A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}} &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}} \\
 &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n)} \cap \overline{A_{n+1}} \\
 &= (\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}) \cap \overline{A_{n+1}} \\
 &= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} \cap \overline{A_{n+1}}.
 \end{aligned}$$

B.1-3 (★) Prove the generalization of equation (B.3), which is called the *principle of inclusion and exclusion*:

$$\begin{aligned}
 |A_1 \cup A_2 \cup \cdots \cup A_n| &= \\
 &|A_1| + |A_2| + \cdots + |A_n| \\
 &- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots \quad (\text{all pairs}) \\
 &+ |A_1 \cap A_2 \cap A_3| + \cdots \quad (\text{all triples}) \\
 &\vdots \\
 &+ (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.
 \end{aligned}$$

Skipped.

B.1-4 Show that the set of odd natural numbers is countable.

Let \mathbb{O} denote the set of odd natural numbers.

The function $f(n) = 2n + 1$ is a 1-1 correspondence from \mathbb{N} to \mathbb{O} . Thus, \mathbb{O} is countable.

B.1-5 Show that for any finite set S , the power set 2^S has $2^{|S|}$ elements (that is, there are $2^{|S|}$ distinct subsets of S).

For the base case, consider a set with a single element x . We have

$$2^{\{x\}} = \{\emptyset, \{x\}\},$$

which shows that the power set of a set with a single element has cardinality $2^1 = 2$.

Let $C(\cdot)$ denote the cardinality of a power set. Let S be a set of size n . Let's assume that the power set of S has cardinality $C(S) = 2^{|S|} = 2^n$. Now, let S' be the set S with one additional element x , such that $|S'| = n + 1$. The power set of S' will consist of all sets in the power set of S plus all those same sets again, with the element x added. Thus, we have

$$C(S') = 2 \cdot C(S) = 2 \cdot 2^n = 2^{n+1}.$$

B.1-6 Give an inductive definition for an n -tuple by extending the set-theoretic definition for an ordered pair.

$$\begin{aligned} (a) &= \{a\} \\ (a, b) &= \{a, \{a, b\}\} \\ (a, b, c) &= \{a, \{a, b\}, \{a, b, c\}\} \\ (a_1, a_2, \dots, a_n) &= (a_1, a_2, \dots, a_{n-1}) \cup \{a_1, a_2, \dots, a_n\} \end{aligned}$$

Section B.2 – Relations

B.2-1 Prove that the subset relation “ \subseteq ” on all subsets of \mathbb{Z} is a partial order but not a total order.

Let \mathbb{S} denote all the subsets of \mathbb{Z} . Let $A = \{1\}$, $B = \{2\}$ be two subsets of \mathbb{Z} . We have $A \not\subseteq B$ and $B \not\subseteq A$. Thus, the subset relation “ \subseteq ” on $\mathbb{S} \times \mathbb{S}$ is not a total relation and therefore is not a total order.

For the relation \subseteq on \mathbb{S} to be a partial order, the following properties need to hold: (1) reflexivity, (2) antisymmetry, (3) transitivity. Since $A \subseteq A$, for all $A \in \mathbb{S}$, the relation “ \subseteq ” on $\mathbb{S} \times \mathbb{S}$ is reflexive. To be antisymmetric, we need to show that if $A \subseteq B$ and $B \subseteq A$, then $A = B$, for all $A, B \in \mathbb{S}$. Since $A \subseteq B$, for all $a \in A$ we have $a \in B$ and since $B \subseteq A$, for all $b \in B$ we have $b \in A$. Thus, $A = B$ and the relation “ \subseteq ” on $\mathbb{S} \times \mathbb{S}$ is antisymmetric. To be transitive, we need to show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$, for all $A, B, C \in \mathbb{S}$. So let $a \in A$. Since $A \subseteq B$, we have $a \in B$. Since $a \in B$ and $B \subseteq C$, we have $a \in C$. Thus, $A \subseteq C$ and the relation “ \subseteq ” on $\mathbb{S} \times \mathbb{S}$ is transitive.

B.2-2 Show that for any positive integer n , the relation “equivalent modulo n ” is an equivalence relation on the integers. (We say that $a \equiv b \pmod{n}$ if there exists an integer q such that $a - b = qn$.) Into what equivalence classes does this relation partition the integers?

To the relation “equivalent modulo n ” to be an equivalent relation on $\mathbb{Z} \times \mathbb{Z}$, the following needs to hold:

- (a) $a \equiv a \pmod{n}$, for all $a, n \in \mathbb{Z}$ (reflexivity)
- (b) $a \equiv b \pmod{n}$ implies $b \equiv a \pmod{n}$, for all $a, b, n \in \mathbb{Z}$ (symmetry)
- (c) $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ implies $a \equiv c \pmod{n}$, for all $a, b, c, n \in \mathbb{Z}$ (transitivity)

For the reflexivity property, we have that $a - a = 0n$ holds directly for $q = 0$.

For the symmetry property, we have that $a - b = pn$ implies $b - a = -pn$ holds directly for $q = -p$.

For the transitivity property, we have that $a - b = pn$ and $b - c = qn$ implies $a - c = rn$ holds for $r = p + q$, since

$$(a - b) + (b - c) = pn + qn \rightarrow a - c = (p + q)n.$$

B.2-3 Give examples of relations that are

- a. reflexive and symmetric but not transitive,
- b. reflexive and transitive but not symmetric,
- c. symmetric and transitive but not reflexive.

- (a) The relation “is neighbor of” is reflexive (one is neighbor of himself) and symmetric (a “is neighbor of” b imply b “is neighbor of” a), but not transitive (a “is neighbor of” b and b “is neighbor of” c does not imply a “is neighbor of” c).
- (b) The relation “ \leq ” is reflexive ($a \leq a$) and transitive ($a \leq b$ and $b \leq c$ imply $a \leq c$), but not symmetric ($a \leq b$ does not imply $b \leq a$).
- (c) Consider the relation “ $a + b > \infty$ ” on $\mathbb{Z} \times \mathbb{Z}$. This relation is empty. However, it is symmetric ($a R b$ imply $b R a$) and transitive ($a R b$ and $b R c$ imply $a R c$), but not reflexive since for no $a \in \mathbb{Z}$ is it the case that $a R a$.

B.2-4 Let S be a finite set, and let R be an equivalence relation on $S \times S$. Show that if in addition R is antisymmetric, then the equivalence classes of S with respect to R are singletons.

For every $a, b \in S$ such that $a R b$, by symmetry $b R a$, and by antisymmetry $a = b$. Thus, $[a] = \{b \in S : a R b\} = \{a\}$ for all $a \in S$, which implies that the equivalence classes are singletons.

B.2-5 Professor Narcissus claims that if a relation R is symmetric and transitive, then it is also reflexive. He offers the following proof. By symmetry, $a R b$ implies $b R a$. Transitivity, therefore, implies $a R a$. Is the professor correct?

No. This is only true for relations that for every a there is b such that $a R b$, by symmetry $b R a$, and by transitivity $a R a$. For instance, an empty relation (like the one from Question B.2-3, item (c)) are symmetric and transitive, but not reflexive.

Section B.3 – Functions

B.3-1 Let A and B be finite sets, and let $f : A \rightarrow B$ be a function. Show that

- a. if f is injective, then $|A| \leq |B|$;
- b. if f is surjective, then $|A| \geq |B|$.

(a) Since f is injective, we have that $|A| = |f(A)|$. Also, we have

$$\begin{cases} |B| = |f(A)|, & f \text{ is surjective,} \\ |B| > |f(A)|, & f \text{ is not surjective.} \end{cases}$$

Thus, $|B| \geq |f(A)| = |A| \rightarrow |A| \leq |B|$.

(b) Since f is surjective, we have $|f(A)| = |B|$. Also, we have

$$\begin{cases} |A| = |f(A)|, & f \text{ is injective,} \\ |A| > |f(A)|, & f \text{ is not injective.} \end{cases}$$

Thus, $|A| \geq |f(A)| = |B| \rightarrow |A| \geq |B|$.

B.3-2 Is the function $f(x) = x + 1$ bijective when the domain and the codomain are \mathbb{N} ? Is it bijective when the domain and the codomain are \mathbb{Z} ?

On the set of naturals, f is injective but not surjective, since there is no $a \in \mathbb{N}$ such that $0 = f(a)$, which makes $f(\mathbb{N}) \neq \mathbb{N}$.
On the set of integers, f is both injective and surjective, and therefore bijective.

B.3-3 Give a natural definition for the inverse of a binary relation such that if a relation is in fact a bijective function, its relational inverse is its functional inverse.

Let R be a binary relation on the sets A and B , such that $R \subseteq A \times B$. The general definition of the inverse of R is given by

$$R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}.$$

When R is a bijective function, we have: (1) each element of A has relation with precisely one element of B (injective) and (2) for all $b \in B$ there is a such that $a R b$ (surjective). Therefore, when R is bijective, each element of A is related to exactly one element of B and vice-versa, which implies

$$f(a) = b \iff f'(b) = a,$$

for all $a \in A$ and for all $b \in B$.

B.3-4 (★) Give a bijection from \mathbb{Z} to $\mathbb{Z} \times \mathbb{Z}$.

Skipped.