Section A.1 – Summation formulas and properties

A.1-1 Find a simple formula for $\sum_{k=1}^{n}(2k-1)$.

$$\begin{aligned}\sum_{k=1}^{n}(2k-1) &= \sum_{k=1}^{n}2k - \sum_{k=1}^{n}1\\ &= 2\sum_{k=1}^{n}k - n\\ &= 2\cdot\frac{1}{2}n(n+2) - n\\ &= n^2+n-n\\ &= n^2.\end{aligned}$$

A.1-2 ($\star$) Show that $\sum_{k=1}^{n}1/(2k-1) = \ln(\sqrt{n}) + O(1)$ by manipulating the harmonic series.

$$\begin{aligned}\sum_{k=1}^{n}1/(2k-1) &= \frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-3}+\frac{1}{2n-1}\\ &= \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right) - \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\\ &= \sum_{k=1}^{2n}\frac{1}{k} - \frac{1}{2}\sum_{k-1}^{n}\frac{1}{k}\\ &= \ln 2n + O(1) - \frac{1}{2}(\ln n + O(1))\\ &= \ln n + \ln 2 + O(1) - \frac{1}{2}\ln n - \frac{1}{2}O(1)\\ &= \frac{1}{2}\ln n + O(1)\\ &= \ln(\sqrt{n}) + O(1).\end{aligned}$$

A.1-3 Show that $\sum_{k=0}^{\infty}k^2x^k = x(1+x)/(1-x)^3$ for $0 < |x| < 1$.

From Equation A.8, we have

$$\sum_{k=0}^{\infty}kx^k = \frac{x}{(1-x)^2}.$$

differentiating both sides and multiplying by x, we have

$$\begin{aligned}\sum_{k=0}^{\infty}k^2x^k &= x\cdot\frac{1\cdot(1-x)^2-(2\cdot(1-x)\cdot(-1)\cdot x)}{(1-x)^4}\\ &= x\cdot\frac{(1-x)(1-x)+(1-x)\cdot 2x}{(1-x)^4}\\ &= x\cdot\frac{(1-x)+2x}{(1-x)^3}\\ &= \frac{x(1+x)}{(1-x)^3}.\end{aligned}$$

A.1-4 ($\star$) Show that $\sum_{k=0}^{\infty}(k-1)/2^k = 0$.

$$\begin{aligned}
\sum_{k=0}^{\infty}(k-1)/2^k &= \sum_{k=0}^{\infty}\left(\frac{k}{2^k} - \frac{1}{2^k}\right) \\
&= \sum_{k=0}^{\infty} k\frac{1}{2^k} - \sum_{k=0}^{\infty}\frac{1}{2^k} \\
&= \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \\
&= \frac{(1/2)}{(1-(1/2))^2} - \frac{1}{1-(1/2)} \\
&= \frac{(1/2)}{1-1-(1/4)} - 2 \\
&= 4/2 - 2 \\
&= 0.
\end{aligned}$$

A.1-5 ($\star$) Evaluate the sum $\sum_{k=1}^{\infty}(2k+1)x^{2k}$ for $|x| < 1$.

$$\begin{aligned}
\sum_{k=1}^{\infty}(2k+1)x^{2k} &= \frac{d}{dx}\cdot\sum_{k=1}^{\infty}x^{2k+1} \\
&= \frac{d}{dx}\cdot x\cdot\sum_{k=1}^{\infty}x^{2k} \\
&= \frac{d}{dx}\cdot x\cdot\sum_{k=0}^{\infty}(x^2)^k - x \\
&= \frac{d}{dx}\cdot x\cdot\frac{1}{1-x^2} - x \\
&= \frac{d}{dx}\cdot\frac{x - x(1-x^2)}{1-x^2} \\
&= \frac{d}{dx}\cdot\frac{x^3}{1-x^2} \\
&= \frac{3x^2(1-x^2) - (-2x)x^3}{(1-x^2)^2} \\
&= \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2} \\
&= \frac{(3-x^2)\cdot x^2}{(1-x^2)^2}.
\end{aligned}$$

A.1-6 Prove that $\sum_{k=1}^{n} O(f_k(i)) = O(\sum_{k=1}^{n} f_k(i))$ by using the linearity property of summations.

Skipped.

A.1-7 Evaluate the product $\prod_{k=1}^{n} 2 \cdot 4^k$.

We have

$$\prod_{k=1}^{n}(2 \cdot 4^k) = 2^{\lg\left(\prod_{k=1}^{n}(2\cdot 4^k)\right)},$$

and

$$\begin{aligned}
\lg\left(\prod_{k=1}^{n}(2 \cdot 4^k)\right) &= \sum_{k=1}^{n}\lg(2 \cdot 2^{2k}) \\
&= \sum_{k=1}^{n}\lg 2^{2k+1} \\
&= \sum_{k=1}^{n}(2k+1) \\
&= 2\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 \\
&= n(n+1) + n \\
&= n(n+2).
\end{aligned}$$

Thus,

$$\prod_{k=1}^{n}(2 \cdot 4^k) = 2^{n(n+2)}.$$

A.1-8 ($\star$) Evalute the product $\prod_{k=2}^{n}(1 - 1/k^2)$.

We have

$$\prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right) = 2^{\lg\left(\sum_{k=2}^{n}\lg(1-1/k^2)\right)},$$

and

$$\begin{aligned}
\sum_{k=2}^{n}\lg\left(1 - \frac{1}{k^2}\right) &= \sum_{k=2}^{n}\lg\left(\frac{k^2-1}{k^2}\right) \\
&= \sum_{k=2}^{n}\lg\left(\frac{(k-1)}{k}\cdot\frac{(k+1)}{k}\right) \\
&= \sum_{k=2}^{n}\left(\lg\left(\frac{k-1}{k}\right) + \lg\left(\frac{k+1}{k}\right)\right) \\
&= \lg\frac{1}{2} + \lg\frac{3}{2} + \lg\frac{2}{3} + \lg\frac{4}{3} + \lg\frac{3}{4} + \lg\frac{5}{4} + \cdots + \lg\frac{n-2}{n-1} + \lg\frac{n}{n-1} + \lg\frac{n-1}{n} + \lg\frac{n+1}{n} \\
&= \lg 1 - \lg 2 + \cancel{\lg 3} - \cancel{\lg 2} + \cancel{\lg 2} - \cancel{\lg 3} + \cancel{\lg 4} - \cancel{\lg 3} + \cancel{\lg 3} - \cancel{\lg 4} + \cancel{\lg 5} - \cancel{\lg 4} + \ldots \\
&\quad + \cancel{\lg(n-2)} - \cancel{\lg(n-1)} + \cancel{\lg n} - \cancel{\lg(n-1)} + \cancel{\lg(n-1)} - \cancel{\lg n} + \lg(n+1) - \lg n \\
&= 0 - 1 + \lg(n+1) - \lg n \\
&= \lg(n+1) - \lg n - 1.
\end{aligned}$$

Thus,

$$\prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right) = 2^{(\lg(n+1) - \lg n - 1)}.$$