Section A.1 – Summation formulas and properties

A.1-1 Find a simple formula for $\sum_{k=1}^{n}(2k-1)$.

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1) &= \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 \\
&= 2\sum_{k=1}^{n} k - n \\
&= 2 \cdot \frac{1}{2} n(n+2) - n \\
&= n^2 + n - n \\
&= n^2.
\end{aligned}$$

A.1-2 ($\star$) Show that $\sum_{k=1}^{n} 1/(2k-1) = \ln(\sqrt{n}) + O(1)$ by manipulating the harmonic series.

$$\begin{aligned}
\sum_{k=1}^{n} 1/(2k-1) &= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-3} + \frac{1}{2n-1} \\
&= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\
&= \sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2}\sum_{k-1}^{n} \frac{1}{k} \\
&= \ln 2n + O(1) - \frac{1}{2}(\ln n + O(1)) \\
&= \ln n + \ln 2 + O(1) - \frac{1}{2}\ln n - \frac{1}{2}O(1) \\
&= \frac{1}{2}\ln n + O(1) \\
&= \ln(\sqrt{n}) + O(1).
\end{aligned}$$

A.1-3 Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ for $0 < |x| < 1$.

From Equation A.8, we have

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

differentiating both sides and multiplying by x, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} k^2 x^k &= x \cdot \frac{1 \cdot (1-x)^2 - (2 \cdot (1-x) \cdot (-1) \cdot x)}{(1-x)^4} \\
&= x \cdot \frac{(1-x)(1-x) + (1-x) \cdot 2x}{(1-x)^4} \\
&= x \cdot \frac{(1-x) + 2x}{(1-x)^3} \\
&= \frac{x(1+x)}{(1-x)^3}.
\end{aligned}$$

A.1-4 ($\star$) Show that $\sum_{k=0}^{\infty}(k-1)/2^k = 0$.

$$\begin{aligned}
\sum_{k=0}^{\infty}(k-1)/2^k &= \sum_{k=0}^{\infty}\left(\frac{k}{2^k} - \frac{1}{2^k}\right) \\
&= \sum_{k=0}^{\infty} k\frac{1}{2^k} - \sum_{k=0}^{\infty}\frac{1}{2^k} \\
&= \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \\
&= \frac{(1/2)}{(1-(1/2))^2} - \frac{1}{1-(1/2)} \\
&= \frac{(1/2)}{1-1-(1/4)} - 2 \\
&= 4/2 - 2 \\
&= 0.
\end{aligned}$$

A.1-5 ($\star$) Evaluate the sum $\sum_{k=1}^{\infty}(2k+1)x^{2k}$ for $|x| < 1$.

$$\begin{aligned}
\sum_{k=1}^{\infty}(2k+1)x^{2k} &= \frac{d}{dx}\cdot\sum_{k=1}^{\infty}x^{2k+1} \\
&= \frac{d}{dx}\cdot x\cdot\sum_{k=1}^{\infty}x^{2k} \\
&= \frac{d}{dx}\cdot x\cdot\sum_{k=0}^{\infty}(x^2)^k - x \\
&= \frac{d}{dx}\cdot x\cdot\frac{1}{1-x^2} - x \\
&= \frac{d}{dx}\cdot\frac{x - x(1-x^2)}{1-x^2} \\
&= \frac{d}{dx}\cdot\frac{x^3}{1-x^2} \\
&= \frac{3x^2(1-x^2) - (-2x)x^3}{(1-x^2)^2} \\
&= \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2} \\
&= \frac{(3-x^2)\cdot x^2}{(1-x^2)^2}.
\end{aligned}$$

A.1-6 Prove that $\sum_{k=1}^{n} O(f_k(i)) = O(\sum_{k=1}^{n} f_k(i))$ by using the linearity property of summations.

Skipped.

A.1-7 Evaluate the product $\prod_{k=1}^{n} 2 \cdot 4^k$.

We have

$$\prod_{k=1}^{n}(2 \cdot 4^k) = 2^{\lg\left(\prod_{k=1}^{n}(2\cdot 4^k)\right)},$$

and

$$\begin{aligned}
\lg\left(\prod_{k=1}^{n}(2 \cdot 4^k)\right) &= \sum_{k=1}^{n} \lg(2 \cdot 2^{2k}) \\
&= \sum_{k=1}^{n} \lg 2^{2k+1} \\
&= \sum_{k=1}^{n}(2k+1) \\
&= 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= n(n+1) + n \\
&= n(n+2).
\end{aligned}$$

Thus,

$$\prod_{k=1}^{n}(2 \cdot 4^k) = 2^{n(n+2)}.$$

A.1-8 ($\star$) Evalute the product $\prod_{k=2}^{n}(1 - 1/k^2)$.

We have

$$\prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right) = 2^{\lg\left(\sum_{k=2}^{n}\lg(1-1/k^2)\right)},$$

and

$$\begin{aligned}
\sum_{k=2}^{n}\lg\left(1 - \frac{1}{k^2}\right) &= \sum_{k=2}^{n}\lg\left(\frac{k^2-1}{k^2}\right) \\
&= \sum_{k=2}^{n}\lg\left(\frac{(k-1)}{k}\cdot\frac{(k+1)}{k}\right) \\
&= \sum_{k=2}^{n}\left(\lg\left(\frac{k-1}{k}\right) + \lg\left(\frac{k+1}{k}\right)\right) \\
&= \lg\frac{1}{2} + \lg\frac{3}{2} + \lg\frac{2}{3} + \lg\frac{4}{3} + \lg\frac{3}{4} + \lg\frac{5}{4} + \cdots + \lg\frac{n-2}{n-1} + \lg\frac{n}{n-1} + \lg\frac{n-1}{n} + \lg\frac{n+1}{n} \\
&= \lg 1 - \lg 2 + \cancel{\lg 3} - \cancel{\lg 2} + \cancel{\lg 2} - \cancel{\lg 3} + \cancel{\lg 4} - \cancel{\lg 3} + \cancel{\lg 3} - \cancel{\lg 4} + \cancel{\lg 5} - \cancel{\lg 4} + \ldots \\
&\quad + \cancel{\lg(n-2)} - \cancel{\lg(n-1)} + \cancel{\lg n} - \cancel{\lg(n-1)} + \cancel{\lg(n-1)} - \cancel{\lg n} + \lg(n+1) - \lg n \\
&= 0 - 1 + \lg(n+1) - \lg n \\
&= \lg(n+1) - \lg(n) - 1.
\end{aligned}$$

Thus,

$$\prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right) = 2^{(\lg(n+1)-(\lg(n)+1))} = \frac{2^{\lg(n+1)}}{2^{\lg(n)+1}} = \frac{n+1}{2^{\lg n}\cdot 2} = \frac{n+1}{2n}.$$

Section A.2 – Bounding summations

A.2-1 Show that $\sum_{k=1}^{n} 1/k^2$ is bounded above by a constant.

$$\begin{aligned}
\sum_{k=1}^{n} &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} \\
&\leq 1 + \int_1^n \frac{dx}{x^2} \\
&= 1 + \left(-\frac{1}{x}\Big|_1^n \right) \\
&= 1 + \left(-\frac{1}{n} - \left(-\frac{1}{1} \right) \right) \\
&= 2 - \frac{1}{n} \\
&\leq 2.
\end{aligned}$$

A.2-2 Find an asymptotic upper bound on the summation

$$\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil.$$

$$\begin{aligned}
\sum_{k=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^k} \right\rceil &= n \cdot \sum_{k=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{1}{2^k} \right\rceil \\
&\leq n \cdot \sum_{k=0}^{\lg n} \left(\frac{1}{2^k} + 1 \right) \\
&= n \cdot \sum_{k=0}^{\lg n} \left(\frac{1}{2^k} \right) + \sum_{k=0}^{\lg n} 1 \\
&= n \cdot \frac{1}{1 - (1/2)} + \lg n + 1 \\
&= 2n + \lg n + 1 \\
&= O(n).
\end{aligned}$$

A.2-3 Show that the nth harmonic number is $\Omega(\lg n)$ by splitting the summation.

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &\geq \sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^i + j} \\
&\geq \sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^{i+1}} \\
&= \sum_{i=0}^{\lfloor \lg n \rfloor - 1} \frac{1}{2} \cdot \sum_{j=0}^{2^i - 1} \frac{1}{2^i} \\
&= \sum_{i=0}^{\lfloor \lg n \rfloor - 1} \frac{1}{2} \\
&\geq \sum_{i=0}^{\lg n - 2} \frac{1}{2} \\
&= \frac{1}{2}(\lg(n) - 1) \\
&= \Omega(\lg n).
\end{aligned}$$

A.2-4 Approximate $\sum_{k=1}^{n} k^3$ with an integral.

We have
$$\int_0^n x^3 dx \le \sum_{k=1}^{n} k^3 \le \int_1^{n+1} x^3 dx.$$
For a lower bound, we obtain
$$\sum_{k=1}^{n} k^3 \ge \int_0^n x^3 dx = \left.\frac{x^4}{4}\right|_0^n = \frac{n^4}{4} = \Omega(n^4).$$
For the upper bound, we obtain
$$\sum_{k=1}^{n} k^3 \le \int_1^{n+1} x^3 dx = \left.\frac{x^4}{4}\right|_1^{n+1} = \frac{(n+1)^4 - 1}{4} = O(n^4).$$
Thus,
$$\sum_{k=1}^{n} k^3 = \Theta(n^4).$$

A.2-5 Why didn't we use the integral approximation (A.12) directly on $\sum_{k=1}^{n} 1/k$ to obtain an upper bound on the nth harmonic number?

Applying (A.12) directly, we obtain
$$\sum_{k=1}^{n} \frac{1}{k} \le \int_0^n \frac{1}{x} dx,$$
but the function $1/x$ is undefined for $x = 0$ (because of the division by zero).

Problems

A-1 ***Bounding summations***
Give asymptotically tight bounds on the following summations. Assume that $r \geq 0$ and $s \geq 0$ are constants.

a. $\sum_{k=1}^{n} k^r$.
b. $\sum_{k=1}^{n} \lg^s k$.
c. $\sum_{k=1}^{n} k^r \lg^s k$.

(a) For a lower bound, we have

$$\begin{aligned}\sum_{k=1}^{n} k^r &\geq \int_0^n x^r dx \\ &= \left.\frac{x^{(r+1)}}{r+1}\right|_0^n \\ &= \frac{n^{(r+1)}}{r+1} - \frac{0^{(r+1)}}{r+1} \\ &\geq n^{(r+1)} \\ &= \Omega(n^{(r+1)}),\end{aligned}$$

and for the upper bound, we have

$$\sum_{k=1}^{n} k^r \leq \sum_{k=1}^{n} n^r = n^{(r+1)} = O(n^{(r+1)}).$$

Thus,

$$\sum_{k=1}^{n} = \Theta(n^{(r+1)}).$$

(b) For a lower bound, we have

$$\begin{aligned}\sum_{k=1}^{n} \lg^s k &= \sum_{k=1}^{n/2} \lg^s k + \sum_{k=n/2+1}^{n} \lg^s k \\ &\geq \sum_{k=1}^{n/2} 0 + \sum_{k=n/2+1}^{n} \lg^s \left(\frac{n}{2}\right) \\ &= \frac{n}{2} \lg^s \left(\frac{n}{2}\right) \\ &= \frac{n}{2} \lg^s n - \frac{n}{2} \lg^s 2 \\ &\geq \frac{1}{2} n \lg^s n - \frac{1}{2} n \\ &= \Omega(n \lg^s n),\end{aligned}$$

and for the upper bound, we have

$$\sum_{k=1}^{n} \lg^s k \leq \sum_{k=1}^{n} \lg^s n = n \lg^s n = O(n \lg^s n).$$

Thus,

$$\sum_{k=1}^{n} \lg^s k = \Theta(n \lg^s n).$$

(c) It is easy to see that this summation is greater than the one from item (a). Thus, it is $\Omega(n^{(r+1)})$. Also, we have

$$\sum_{k=1}^{n} k^r \lg^s k \leq \sum_{k=1}^{n} n^r \lg^s n = O(n^{(r+1)} \lg^s n).$$

Thus, I guess it is $\Theta(n^{(r+1)} \lg^s n)$.