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Interpolation at critical points of an algebraic polynomial: A review of a result due to C. Davis

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Abstract. We consider an old problem on interpolation at critical points of an algebraic polynomial due to Chandler Davis. We essentially reproduce the proof given by this author in [C. Davis, 4714: Proposed solution: Problem 4653, Amer. Math. Monthly **64**(9) 729-730 (1957)]. We provide a detailed account of Davis' proof, adding some details that, in our opinion, make it even easier to follow. Some illustrative examples are also given.

Resumen. Consideramos un viejo problema de interpolación en puntos críticos de polinomios algebraicos propuesto por Chandler Davis. Básicamente, reproducimos la demostración dada por este autor en [C. Davis, 4714: Proposed solution: Problem 4653, Amer. Math. Monthly **64**(9) 729-730 (1957)]. Proporcionamos un reporte detallado de la demostración de Davis, agregando algunos detalles que, en nuestra opinión, hacen que la misma sea aún más fácil de seguir. También algunos ejemplos ilustrativos son dados.

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1 Introduction

Extremal properties for algebraic polynomials is an interesting subject in approximation theory and their applications permeate several fields in science and engineering [3, 8, 11, 12, 13, 18, 19, 21, 20]. In fact, new results continue to appear in some recent publications [2, 9, 16, 17].

In this note, we study in detail the following extremal problem:

Problem-CD: *For a real polynomial $P_n(x)$ of degree n, denote the zeros of $P_n'(x)$, multiplicity counted, by ξ_i, $i = 1, 2, \ldots, n-1$. Assume all ξ_i real, and $\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_{n-1}$. Now, to what extent are the numbers $P_n(\xi_i)$ arbitrary? More precisely, give necessary and sufficient conditions on an $(n-1)$-tuple of real numbers $\eta_1, \ldots, \eta_{n-1}$ in order that there exist P_n such that $P_n(\xi_i) = \eta_i$, $i = 1, 2, \ldots, n-1$.*

Davis* posed this extremal problem in 1956 and solved it (at least partially) by considering topological and differential properties of a mapping from a prescribed set of zeros for $P'(x)$ into the values of the corresponding $P(x)$ at its critical points (cf. [10] and the references therein).

We shall not provide an original proof or solution of **Problem-CD**. Instead of this, we are going to reproduce step by step the proof given by C. Davis in [6], providing as much details as possible about it. We believe that the proof proposed by C. Davis for solving **Problem-CD** is the first on the topic. Apart the interest for the solutions of extremal problems involving the zeros of the derivative of algebraic polynomials, the approach used in [6] seems a fine resolution by means of the implicit function theorem [14]. However, it is possible to use a differential equation approach or a method based on the theory of extremal problems in order to obtain the solution of **Problem-CD** (cf. e.g., [10, Theorem 1.1] and [22, Theorem 1.1], respectively).

Also, we shall add a few details which did not provide by Davis in his original presentation of the solution of **Problem-CD** and complete our study with some illustrative examples. We refer to [1, 7, 10, 15, 22] for some extensions of Davis' result.

2 On the solution for the Problem-CD proposed by Davis

Previously we have described **Problem-CD** in the words of Davis [5, 6]. It is not difficult to intuit that if we assume $\xi_1 < \xi_2 < \cdots < \xi_{n-1}$, then the extremal solution $P_n(x)$ satisfying

$$
\begin{aligned}
P'(\xi_i) &= 0, && (2.1)\\
P(\xi_i) &= \eta_i, \text{ for each } i = 1, \dots, n-1, && (2.2)
\end{aligned}
$$

should be a oscillating polynomial on some real interval [2, 9, 17].

Following Davis' method, in order to provide a characterization for an $(n-1)$-tuple $(\eta_1, \dots, \eta_{n-1}) \in \mathbb{R}^{n-1}$ subject to the constraints (2.1)-(2.2), we shall need some auxiliary results.

In what follows, we denote by $\mathbb{P}_n$ the linear space of polynomials with real coefficients and degree less than or equal to n.

Lemma 2.1. *Let $P \in \mathbb{P}_n$ be a polynomial of degree n satisfying the conditions of* **Problem-CD**. *Assume that $(\eta_1, \eta_2, \dots, \eta_{n-1}) \in \mathbb{R}^{n-1}$ is an $(n-1)$-tuple of real numbers subject to the constraint* (2.2). *Then the real numbers $(-1)^i(\eta_i - \eta_{i-1})$, $i = 2, \dots, n-1$ must be either all nonnegative or all nonpositive.*

*Professor Chandler Davis is Professor Emeritus in the Department of Mathematics at the University of Toronto. He joined the faculty in Toronto as Associate Professor in 1962 after holding a variety positions, among which included posts with the University of Michigan (instructor: 1950-54) and The Institute for Advanced Study at Princeton (1957-58). According to some biographical information about Professor Chandler Davis, he is a very unusual, interesting, and complex person [4]. In 2012 he became a fellow of the American Mathematical Society [23]. Since 2014 The Mathematical Intelligencer established *The Chandler Davis Prize for Expository Excellence* in order to encourage and reward excellent expository writing in mathematics. He is part of the 2019 class of fellows of the Association for Women in Mathematics [24]. This short sketch, however, is restricted to his early contributions to interpolation at critical points of an algebraic polynomial.

Proof. Assume that there exists at least a $k \in \{2, \dots, n-1\}$ such

$$(-1)^k(\eta_k - \eta_{k-1}) > 0.$$

If k is even, the above inequality implies that

$$\eta_k - \eta_{k-1} > 0, \tag{2.3}$$

i.e., $P(\xi_{k-1}) < P(\xi_k)$. Since P is a polynomial, necessarily $\xi_{k-1} < \xi_k$.

Now, we can consider the functions $f, g : [\xi_{k-1}, \xi_k] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{P(x)-P(\xi_{k-1})}{x-\xi_{k-1}}, \text{ si } x \neq \xi_{k-1}, \\ P'(\xi_{k-1}), \text{ si } x = \xi_{k-1}, \end{cases} \tag{2.4}$$

and

$$g(x) = \begin{cases} \frac{P(x)-P(\xi_k)}{x-\xi_k}, \text{ si } x \neq \xi_k, \\ P'(\xi_k), \text{ si } x = \xi_k. \end{cases} \tag{2.5}$$

It is clear that $f, g \in C[\xi_{k-1}, \xi_k]$. Furthermore, from (2.1) and our hypothesis we have

$$\begin{aligned} f(\xi_{k-1}) &= P'(\xi_{k-1}) = 0, \\ f(\xi_k) &= g(\xi_{k-1}) > 0, \\ g(\xi_k) &= P'(\xi_k) = 0. \end{aligned}$$

We define the function $h(x) = f(x) - g(x)$, for all $x \in [\xi_{k-1}, \xi_k]$. Then $h \in C[\xi_{k-1}, \xi_k]$ and

$$\begin{aligned} h(\xi_{k-1}) &= f(\xi_{k-1}) - g(\xi_{k-1}) = -g(\xi_{k-1}) < 0, \\ h(\xi_k) &= f(\xi_k) - g(\xi_k) = f(\xi_k) > 0. \end{aligned}$$

Consequently, there exists $c_k \in (\xi_{k-1}, \xi_k)$ such that $h(c_k) = 0$, i.e.,

$$\frac{P(c_k) - P(\xi_{k-1})}{c_k - \xi_{k-1}} = \frac{P(c_k) - P(\xi_k)}{c_k - \xi_k}. \tag{2.6}$$

Next, we consider the following cases:

Case 1: $f(c_k) = g(c_k) = 0$. Using (2.6) we have that

$$P(c_k) - P(\xi_{k-1}) = 0 = P(c_k) - P(\xi_k),$$

so, we deduce that $P(\xi_{k-1}) = P(\xi_k)$ and hence, $\eta_k - \eta_{k-1} = 0$, which contradicts (2.3).

Case 2: $f(c_k) = g(c_k) < 0$. From this condition we deduce that

$$\frac{P(c_k) - P(\xi_{k-1})}{c_k - \xi_{k-1}} < 0 \quad \text{and} \quad \frac{P(c_k) - P(\xi_k)}{c_k - \xi_k} < 0. \tag{2.7}$$

Then the first inequality in (2.7) leads to the inequality

$$P(c_k) - P(\xi_{k-1}) < 0, \tag{2.8}$$

and using the second inequality in (2.7) we deduce that

$$P(\xi_k) - P(c_k) < 0. \tag{2.9}$$

By adding both sides of inequalities (2.8) and (2.9), we obtain $\eta_k - \eta_{k-1} < 0$, which contradicts (2.3).

Case 3: Finally, we see that $f(c_k) = g(c_k) > 0$ cannot be satisfied. From our hypothesis (2.3), we deduce the following chain of implications

$$\begin{aligned} P(\xi_k) - P(\xi_{k-1}) > 0 \quad &\Rightarrow \quad [P(\xi_k) - P(c_k)] + [P(c_k) - P(\xi_{k-1})] > 0 \\ &\Rightarrow \quad -[P(c_k) - P(\xi_{k-1})] < -[P(c_k) - P(\xi_k)] \\ &\Rightarrow \quad P(c_k) - P(\xi_k) < P(c_k) - P(\xi_{k-1}) \\ &\Rightarrow \quad 1 < \frac{P(c_k) - P(\xi_{k-1})}{P(c_k) - P(\xi_k)}. \end{aligned}$$

Hence,

$$\frac{P(c_k) - P(\xi_{k-1})}{P(c_k) - P(\xi_k)} > 0. \tag{2.10}$$

Now, using (2.6) and the inequality $\frac{c_k - \xi_{k-1}}{c_k - \xi_k} < 0$, we derive

$$\frac{P(c_k) - P(\xi_{k-1})}{P(c_k) - P(\xi_k)} < 0. \tag{2.11}$$

Comparing (2.10) and (2.11) we obtain a contradiction. If we assume that k is odd, then a similar reasoning holds.

Finally, there is not exist $k \in \{2, \ldots, n-1\}$ such that $(-1)^k(\eta_k - \eta_{k-1}) > 0$. This finishes the proof of Lemma 2.1. $\square$

Lemma 2.2. *Let* $(\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ *such that*

$$\xi_0 \leqslant \xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_{n-1} \leqslant \xi_n \ \text{ with } \xi_0 = 0 \text{ and } \xi_n = 1.$$

Let $A > 0$ *and consider the polynomial* $Q \in \mathbb{P}_n$ *defined by*

$$Q(x) = A \int_0^x \prod_{j=1}^{n-1} (\xi_j - t)\, dt, \quad \text{with } Q(0) = 0, \tag{2.12}$$

the set

$$\mathbb{H}_n := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geqslant 0,\ 1 \leqslant i \leqslant n\},$$

and the functions $\phi_i : \mathbb{H}_n \to \mathbb{R}$ *defined by*

$$\phi_1(x_1, \ldots, x_n) = Q\left(\frac{x_1}{A}\right), \tag{2.13}$$

$$\phi_i(x_1, \ldots, x_n) = (-1)^{i-1} \left(Q\left(\frac{\sum_{k=1}^{i} x_k}{A}\right) - Q\left(\frac{\sum_{k=1}^{i-1} x_k}{A}\right) \right), \tag{2.14}$$

for $i = 2, \ldots, n$.

Assume $\phi_i \geqslant 0$ *for each* $i = 1, \ldots, n$. *Then* *(i)* $x_i = 0$, *for each* $i = 1, \ldots, n$, *if and only if* $\phi_i \equiv 0$.

(ii) *The function* $\Psi : \mathbb{H}_n \to \mathbb{H}_n$ *defined by*

$$\Psi(x_1, \ldots, x_n) = (\phi_1(x_1, x_2, \ldots, x_n), \ldots, \phi_n(x_1, x_2, \ldots, x_n)), \tag{2.15}$$

satisfies the following properties:

(a) Ψ *is continuous on* $\mathbb{H}_n$, *differentiable on* $\mathring{\mathbb{H}}_n$ *and* $\Psi(0, 0, \ldots, 0) = (0, 0, \ldots, 0)$.

(b) $\Psi(\partial \mathbb{H}_n) \subseteq \partial \mathbb{H}_n$.

(c) $\Psi\left(\mathring{\mathbb{H}}_n\right) \subseteq \mathring{\mathbb{H}}_n$.

(d) Ψ *maps every ray through the origin onto some ray through the origin.*

(iii) *For* $(x_1, x_2, \ldots, x_n) \in \mathring{\mathbb{H}}_n$, *let us consider the functions* $z_i : \mathring{\mathbb{H}}_n \to \mathbb{R}$ *given by*

$$\begin{aligned} z_1(x_1, x_2, \ldots, x_n) &= \frac{x_1}{A}, \\ z_i(x_1, x_2, \ldots, x_n) &= \frac{x_1 + \cdots + x_i}{A}, \quad i = 2, \ldots, n-1. \\ \rho(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} x_i. \end{aligned}$$

Then the function $\Phi : \mathring{\mathbb{H}}_n \to \mathring{\mathbb{H}}_n$ *defined by*

$$\Phi(x_1, \ldots, x_n) = (z_1(x_1, \ldots, x_n), \ldots, z_{n-1}(x_1, \ldots, x_n), \rho(x_1, \ldots, x_n)), \tag{2.16}$$

is differentiable on $\mathring{\mathbb{H}}_n$ *and*

$$\mathbf{J}\Phi(\xi_1, \ldots, \xi_n) = \frac{1}{A^{n-1}}, \tag{2.17}$$

whenever $0 < \xi_1 < \xi_2 < \cdots < \xi_n$.

Proof. (*i*). CASE I: $i = 1$. If $x_1 = 0$ then from (2.12) follows that

$$\phi_1(x) = \phi_1(0, x_2, \ldots, x_n) = Q\left(\frac{0}{A}\right) = Q(0) = 0.$$

Conversely, assume that $\phi_1(x_1, x_2, \ldots, x_n) = 0$, according to (2.13) we have $Q\left(\frac{x_1}{A}\right) = 0$, and using (2.12) we can deduce that

$$A \int_0^{\frac{x_1}{A}} \prod_{j=1}^{n-1} (\xi_j - t) dt = 0.$$

Since $A > 0$, the property of dilation/contraction of the interval of integration guarantees that

$$A \int_0^{\frac{x_1}{A}} \prod_{j=1}^{n-1} (\xi_j - t) dt = \int_0^{x_1} \prod_{j=1}^{n-1} (\xi_j - t) dt,$$

so,

$$\int_0^{x_1}\prod_{j=1}^{n-1}(\xi_j - t)dt = 0.$$

Now, we see that $x_1 = 0$ necessarily. Suppose that $x_1 \neq 0$, say, $x_1 > 0$. Since $q(t) = \prod_{j=1}^{n-1}(\xi_j - t)$ is a continuous function non constant, there exists $t_0 \in (0, x_1)$ such that $q(t_0) \neq 0$. Without loss of generality we can take $q(t_0) > 0$, and using the continuity of q again, there exists $\delta > 0$ such that

$$q(t) > 0, \text{ for any } t \in S := (t_0 - \delta, t_0 + \delta) \cap [0, x_1].$$

It then follows from the monotonicity of the integral that

$$\int_S q(t)dt > 0. \tag{2.18}$$

On the other hand,

$$\int_S q(t)dt \leqslant \int_0^{x_1} q(t)dt = 0,$$

that is,

$$\int_S q(t)dt \leqslant 0,$$

and this last inequality contradicts (2.18). Hence, x_1 cannot be a positive number.

Suppose $x_1 < 0$, then

$$\int_{x_1}^0 q(t)dt = -\int_0^{x_1} q(t)dt = 0,$$

and a similar reasoning to the above one leads to a contradiction. Therefore, $x_1 = 0$ is necessarily true.

Case II: $i = 2, \ldots, n$. If $x_i = 0$ by (2.14) we have

$$\begin{aligned}\phi_i(x_1, \ldots, x_n) &= (-1)^{i-1}\left(Q\left(\frac{\sum_{k=1}^{i} x_k}{A}\right) - Q\left(\frac{\sum_{k=1}^{i-1} x_k}{A}\right)\right)\\ &= (-1)^{i-1}\left(Q\left(\frac{\sum_{k=1}^{i-1} x_k}{A}\right) - Q\left(\frac{\sum_{k=1}^{i-1} x_k}{A}\right)\right) = 0.\end{aligned}$$

Conversely, assume $\phi_i(x_1, \ldots, x_n) = 0$, then

$$Q\left(\frac{\sum_{k=1}^{i} x_k}{A}\right) = Q\left(\frac{\sum_{k=1}^{i-1} x_k}{A}\right),$$

so, we can deduce that

$$\int_0^{a+x_i} q(t)dt = \int_0^a q(t)dt, \text{ with } a = \frac{\sum_{k=1}^{i-1} x_k}{A}. \tag{2.19}$$

We shall see that $x_i = 0$ is necessarily true. Suppose that $x_i \neq 0$, say, $x_i > 0$. Then

$$\int_0^{a+x_i} q(t)dt = \int_0^a q(t)dt + \int_a^{a+x_i} q(t)dt.$$

Substituting this on the left hand side of (2.19), we obtain

$$\int_0^a q(t)dt + \int_a^{a+x_i} q(t)dt = \int_0^a q(t)dt,$$

from this last equality we deduce that

$$\int_a^{a+x_i} q(t)dt = 0.$$

Using the same reasoning as in the CASE I, there exists $t_1 \in (a, a + x_i)$ such that $q(t_1) \neq 0$. Without loss of generality we can take $q(t_1) > 0$ and the continuity of q guarantees that there exists $\delta_1 > 0$ such that

$$q(t) > 0, \text{ for any } t \in S_1 := (t_1 - \delta_1, t_1 + \delta_1) \cap [a, a + x_i].$$

It then follows from the monotonicity of the integral that

$$\int_{S_1} q(t)dt > 0, \tag{2.20}$$

which contradicts that $\int_{S_1} q(t)dt \leqslant 0$. Therefore, x_i cannot be a positive number. An similar reasoning leads to x_i cannot be a negative number. Consequently, $x_i = 0$.

$(ii) - (a)$. The continuity of Ψ on $\mathbb{H}_n$ is a straightforward consequence of the continuity of its component functions ϕ_i. While Ψ is differentiable on $\mathring{\mathbb{H}}_n$ as a consequence of the so called conditions for differentiability (see for instance, [14, Theorem 4, Chap. 6, Sec. 6.4]). The equality $\Psi(0, 0, \ldots, 0) = (0, 0, \ldots, 0)$ follows of the condition $Q(0) = 0$ and (2.13)-(2.14).

$(ii)-(b)$. Let $y = (y_1, y_2, \ldots, y_n) \in \Psi(\partial\mathbb{H}_n)$, then there exists $x = (x_1, x_2, \ldots, x_n) \in \partial\mathbb{H}_n$ such that $y = \Psi(x)$. Since $x \in \partial\mathbb{H}_n$ there exists at least a component x_j of x, such that $x_j = 0$. Then,

$$\begin{aligned} y_j = \phi_j(x) &= \phi_j(x_1, x_2, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) \\ &= (-1)^{j-1}\left(Q\left(\frac{\sum_{k=1}^{j-1} x_k}{A}\right) - Q\left(\frac{\sum_{k=1}^{j-1} x_k}{A}\right)\right) = 0. \end{aligned}$$

Consequently,

$$\begin{aligned} y = \Psi(x) &= \Psi(x_1, x_2, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) \\ &= (\phi_1(x), \phi_2(x), \ldots, \phi_{j-1}(x), 0, \phi_{j+1}(x), \ldots, \phi_n(x)). \end{aligned}$$

So, y has at least a null component. Therefore, $y = \Psi(x) \in \partial\mathbb{H}_n$.

$(ii)-(c)$. Let $y=(y_1,y_2,\ldots,y_n)\in\Psi\left(\mathring{\mathbb{H}}_n\right)$, then there exists $x=(x_1,x_2,\ldots,x_n)$ in $\mathring{\mathbb{H}}_n$ such that $y=\Psi(x)$. Since $x\in\mathring{\mathbb{H}}_n$, $x_i>0$ for any $i=1,\ldots,n$. Suppose that some component of y is zero, for instance, suppose that $y_k=0$. This implies that

$$\phi_k(x)=y_k=0,$$

and using part (i) we have $x_k=0$. But, this last equality contradicts the fact $x\in\mathring{\mathbb{H}}_n$. Therefore, $y\in\mathring{\mathbb{H}}_n$, i.e., $\Psi\left(\mathring{\mathbb{H}}_n\right)\subseteq\mathring{\mathbb{H}}_n$.

$(ii)-(d)$. Let $x=(x_1,x_2,\ldots,x_n)\in\mathring{\mathbb{H}}_n$ and $y=\Psi(x)$. Consider the point $z=(z_1,z_2,\ldots,z_n)\in\mathring{\mathbb{H}}_n$ given by

$$\begin{aligned} z_1 &= \frac{x_1}{A}, \\ z_i &= \frac{x_1+\cdots+x_i}{A}, \quad i=2,\ldots,n. \end{aligned}$$

It is clear that

$$0<z_1<z_2<\cdots<z_{n-1}<z_n,$$

and

$$\begin{aligned} y &= \Psi(x)=(\phi_1(x),\phi_2(x),\ldots,\phi_n(x)) \\ &= (Q(z_1),-(Q(z_2)-Q(z_1)),\ldots,(-1)^{n-1}(Q(z_n)-Q(z_{n-1}))). \end{aligned}$$

Since $x\in\mathring{\mathbb{H}}_n$, then $\lambda x\in\mathring{\mathbb{H}}_n$ for any $\lambda>0$. Thus,

$$\Psi(\lambda x) = (Q(\lambda z_1),-(Q(\lambda z_2)-Q(\lambda z_1)),\ldots,(-1)^{n-1}(Q(\lambda z_n)-Q(\lambda z_{n-1}))).$$

Since $\lambda>0$, the property of dilation/contraction of the interval of integration guarantees that

$$\begin{aligned} Q(\lambda z_i) &= A\int_0^{\lambda z_i}\prod_{j=1}^{n-1}(\xi_j-t)dt \\ &= \lambda A\int_0^{z_i}\prod_{j=1}^{n-1}(\xi_j-t)dt=\lambda Q(z_i), \end{aligned} \tag{2.21}$$

for all $i=1,\ldots,n$.

Consequently,

$$\Psi(\lambda x)=\lambda\Psi(x), \tag{2.22}$$

for any $\lambda>0$. In other words, Ψ maps every ray through the origin onto some ray through the origin.

(iii). Let us consider $x=(x_1,\ldots,x_n)\in\mathring{\mathbb{H}}_n$ and $z_i:\mathring{\mathbb{H}}_n\to\mathbb{R}$ the functions given by

$$\begin{aligned} z_1(x_1,x_2,\ldots,x_n) &= \frac{x_1}{A}, \\ z_i(x_1,x_2,\ldots,x_n) &= \frac{x_1+\cdots+x_i}{A}, \quad i=2,\ldots,n-1. \\ \rho(x_1,x_2,\ldots,x_n) &= \sum_{i=1}^{n}x_i. \end{aligned}$$

Then, the function $\Phi : \mathring{\mathbb{H}}_n \to \mathring{\mathbb{H}}_n$ defined by

$$\Phi(x_1, \dots, x_n) = (z_1(x_1, \dots, x_n), \dots, z_{n-1}(x_1, \dots, x_n), \rho(x_1, \dots, x_n)),$$

is differentiable on $\mathring{\mathbb{H}}_n$ since each one of its component functions is differentiable on $\mathring{\mathbb{H}}_n$. Hence, there exists the linear map $\mathbf{D}\Phi(x_1, \dots, x_n)$ and its Jacobian matrix is:

$$\mathbf{D}\Phi(x_1,\dots,x_n) = \begin{pmatrix} \frac{\partial z_1}{\partial x_1}(x_1,\dots,x_n) & \frac{\partial z_1}{\partial x_2}(x_1,\dots,x_n) & \dots & \frac{\partial z_1}{\partial x_n}(x_1,\dots,x_n) \\ \frac{\partial z_2}{\partial x_1}(x_1,\dots,x_n) & \frac{\partial z_2}{\partial x_2}(x_1,\dots,x_n) & \dots & \frac{\partial z_2}{\partial x_n}(x_1,\dots,x_n) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial z_{n-1}}{\partial x_1}(x_1,\dots,x_n) & \frac{\partial z_{n-1}}{\partial x_2}(x_1,\dots,x_n) & \dots & \frac{\partial z_{n-1}}{\partial x_n}(x_1,\dots,x_n) \\ \frac{\partial \rho}{\partial x_1}(x_1,\dots,x_n) & \frac{\partial \rho}{\partial x_2}(x_1,\dots,x_n) & \dots & \frac{\partial \rho}{\partial x_n}(x_1,\dots,x_n) \end{pmatrix},$$

where

$$\begin{aligned} \frac{\partial z_k}{\partial x_j}(x_1,\dots,x_n) &= \frac{1}{A}, \text{ if } j \leqslant k, \\ \frac{\partial z_k}{\partial x_j}(x_1,\dots,x_n) &= 0, \text{ if } j > k, \\ \frac{\partial \rho}{\partial x_j}(x_1,\dots,x_n) &= 1, \end{aligned}$$

whenever $1 \leqslant k \leqslant n-1$, and $1 \leqslant j \leqslant n$.

Consequently,

$$\mathbf{D}\Phi(x_1,\dots,x_n) = \begin{pmatrix} \frac{1}{A} & 0 & \dots & 0 \\ \frac{1}{A} & \frac{1}{A} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{A} & \frac{1}{A} & \dots & 0 \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

By evaluating $\mathbf{D}\Phi(x_1, \dots, x_n)$ at the point $(\xi_1, \dots, \xi_{n-1}, A)$ we obtain

$$\mathbf{J}\Phi(\xi_1, \dots, \xi_{n-1}, A) = \frac{1}{A^{n-1}},$$

since $\mathbf{D}\Phi(\xi_1, \dots, \xi_{n-1}, A)$ is an upper triangular matrix. $\square$

Lemma 2.3. *Let $(z_1, z_2, \dots, z_{n-1}, \rho) \in \mathring{\mathbb{H}}_n$ as in Lemma 2.2. Consider the following functions*

$$\begin{aligned} \tilde{\phi}_1(z_1,\dots,z_{n-1},z_n) &= Q(z_1), \\ \tilde{\phi}_i(z_1,\dots,z_{n-1},z_n) &= (-1)^{i-1}(Q(z_i) - Q(z_{i-1})). \end{aligned} \tag{2.23}$$

where $z_n = \rho$ and $i = 2, \dots, n$. Then, the function $\tilde{\Psi} : \mathring{\mathbb{H}}_n \to \mathring{\mathbb{H}}_n$ defined by

$$\tilde{\Psi}(z_1,\dots,z_{n-1},\rho) = (\tilde{\phi}_1(z_1,\dots,z_{n-1},\rho),\dots,\tilde{\phi}_n(z_1,\dots,z_{n-1},\rho)) \tag{2.24}$$

is differentiable on $\mathring{\mathbb{H}}_n$ and the Jacobian determinant $\mathbf{J}\tilde{\Psi}(z_1, \dots, z_{n-1}, \rho)$ is nonzero.

Proof. Let us consider z_i, $i = 1, \dots, n-1$, and ρ, as in Lemma 2.2. Then

$$\begin{aligned} x_1 &= Az_1, \\ x_i &= Az_i - \sum_{j=1}^{i-1} x_j, \quad i = 2, \dots, n-1, \\ x_n &= \rho - Az_{n-1}, \end{aligned}$$

Thus, the functions ϕ_k given by (2.13) and (2.14) can be expressed as

$$\begin{aligned} \phi_1(x_1, \dots, x_n) &= Q(z_1), \\ \phi_k(x_1, \dots, x_n) &= (-1)^{k-1}(Q(z_i) - Q(z_{i-1})), \quad k = 2, \dots, n. \end{aligned}$$

In order to obtain (2.23) only should be noted that

$$\begin{aligned} \phi_1(x_1, \dots, x_n) &= \tilde{\phi}_1(z_1, \dots, z_{n-1}, z_n), \\ \phi_k(x_1, \dots, x_n) &= \tilde{\phi}_k(z_1, \dots, z_{n-1}, z_n), \quad k = 2, \dots, n. \end{aligned}$$

Assume in addition that $z_0(x) = 0$ and $z_n(x) = 1$ for any $x \in \mathring{\mathbb{H}}_n$. Then, the functions $\tilde{\phi}_i$, $i = 1, \dots, n$ are differentiable on $\mathring{\mathbb{H}}_n$, and consequently, $\tilde{\Psi}$ is differentiable on $\mathring{\mathbb{H}}_n$.

By part (*iii*) of Lemma 2.2 and the implicit function theorem, there exists an open neighborhood $U \subseteq \mathring{\mathbb{H}}_n$ of $(x_1, \dots, x_n)$ such that

$$(x_1, \dots, x_n) = (z_1, \dots, z_{n-1}, \rho).$$

On the other hand, by differentiation under the integral sign we have that

$$\frac{\partial \tilde{\phi}_i}{\partial z_j} = \tilde{q}(z_i) - \tilde{q}(z_{i-1}) + (-1)^{i-1} \int_{z_{i-1}}^{z_i} \frac{\partial \tilde{q}(t)}{\partial z_j}\, dt,$$

with $\tilde{q}(t) = \rho \prod_{s=1}^{n-1} (z_s - t)$, $i = 1, \dots, n$ and $j = 1, \dots, n-1$. By evaluating the partial derivatives at the point $(x_1, \dots, x_n)$ we obtain

$$\frac{\partial \tilde{\phi}_i}{\partial z_j}(x_1, \dots, x_n) = (-1)^{i-1} \int_{x_{i-1}}^{x_i} \frac{\hat{q}(t)}{(x_j - t)}\, dt. \tag{2.25}$$

where $\hat{q}(t) = x_n \prod_{s=1}^{n-1} (x_s - t)$.

Again, by differentiation under the integral sign the partial derivative of the function $\tilde{\phi}_i$ with respect ρ takes the form

$$\frac{\partial \tilde{\phi}_k}{\partial \rho} = \frac{\partial}{\partial \rho}\left[(-1)^{i-1} \int_{z_{i-1}}^{z_i} \tilde{q}(t)\, dt\right] = \frac{(-1)^{i-1}}{\rho} \int_{z_{i-1}}^{z_i} \tilde{q}(t)\, dt, \quad i = 1, \dots, n.$$

Next, by evaluating this partial derivative at the point $(x_1, \dots, x_n)$, we have

$$\frac{\partial \tilde{\phi}_k}{\partial \rho}(x_1, \dots, x_n) = \frac{(-1)^{i-1}}{x_n} \int_{x_{i-1}}^{x_i} \hat{q}(t)\, dt. \tag{2.26}$$

To prove that $\mathbf{J}\tilde{\Psi}(x_1,\ldots,x_n) \neq 0$ we argue by contradiction. Let $v \in \mathbb{R}^n \backslash \{0\}$, then the system of linear equations associated to the matrix equation

$$\mathbf{D}\tilde{\Psi}(x_1,\ldots,x_n)(v) = 0$$

has infinite solutions. So, for each row i of the matrix $\mathbf{D}\tilde{\Psi}(x_1,\ldots,x_n)$, there exists some linear combination of the form $\sum_{j=1}^{n} J_{ij} y_j = 0$, where J_{ij} is the (i,j)-th entry of the matrix $\mathbf{D}\tilde{\Psi}(x_1,\ldots,x_n)$.

Using the linearity property of the integral, we obtain

$$0 = \sum_{j=1}^{n} J_{ij} y_j = \sum_{j=1}^{n-1} \left((-1)^{i-1} \int_{x_{i-1}}^{x_i} \frac{\hat{q}(t)}{x_j - t}\, dt \right) y_j + \left(\frac{(-1)^{i-1}}{x_n} \int_{x_{i-1}}^{x_i} \hat{q}(t)\, dt \right) y_n,$$

which implies

$$0 = \sum_{j=1}^{n} J_{ij} y_j = \int_{x_{i-1}}^{x_i} (-1)^{i-1} \left(\sum_{j=1}^{n-1} \frac{\hat{q}(t)}{x_j - t} y_j + \frac{\hat{q}(t)}{x_n} y_n \right) dt.$$

Thus, the polynomial $F \in \mathbb{P}_{n-1}$, given by

$$F(x) = \sum_{j=1}^{n-1} \frac{\hat{q}(x)}{x_j - x} y_j + \frac{\hat{q}(x)}{x_n} y_n$$

is a polynomial of degree $n-1$ such that

$$\int_{x_{i-1}}^{x_i} F(x)\, dx = 0, \quad i = 1,\ldots,n.$$

Hence, the polynomial $G \in \mathbb{P}_n$, given by

$$G(x) = \int_0^x F(t)\, dt$$

is a polynomial of degree n, which is zero at the $n+1$ points $0, x_1, \ldots, x_n$, so that $F(x) = 0$.

Or equivalently,

$$\sum_{j=1}^{n-1} \frac{\hat{q}(x)}{x_j - x} y_j + \frac{\hat{q}(x)}{x_n} y_n = 0.$$

Since $x_1 < x_2 < \ldots < x_n$ then the set $\left\{ \frac{\hat{q}(x)}{x_1 - x}, \ldots, \frac{\hat{q}(x)}{x_{n-1} - x}, \frac{\hat{q}}{x_n}(x) \right\}$ is linearly independent, we have that the system of linear equations associated to the matrix equation $\mathbf{D}\tilde{\Psi}(x_1,\ldots,x_n)(v) = 0$ has a unique solution, which leads to a contradiction. Consequently, $\mathbf{J}\tilde{\Psi}(x_1,\ldots,x_n) \neq 0$. □

Finally, with Lemmas 2.1, 2.2 and 2.3 in mind we can present Davis solution for **Problem-CD** by means of following theorem.

Theorem 2.1. *Let* $(\xi_1, \ldots, \xi_{n-1}) \in \mathring{\mathbb{H}}_{n-1}$ *and* $A > 0$ *such that*

$$p(x) = A \prod_{i=1}^{n-1} (\xi_i - x),$$

Let $(\eta_1, \ldots, \eta_{n-1})$ *be a fixed point in* $\mathbb{R}^{n-1}$ *and* $z = (z_1, z_2, \ldots, z_{n-1}, \rho) \in \mathring{\mathbb{H}}_n$ *as in Lemma 2.2. Then there exists a unique polynomial* $P \in \mathbb{P}_n$ *of degree* n*, satisfying the conditions* (2.1)-(2.2) *if and only if the real numbers* $(-1)^i (\eta_i - \eta_{i-1})$, $i = 2, \ldots, n-1$ *are either all nonnegative or all nonpositive.*

Proof. Let us consider $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathring{\mathbb{H}}_{n-1}$ and $(\eta_1, \ldots, \eta_{n-1}) \in \mathbb{R}^{n-1}$. Without loss of generality we can assume that $P(0) = 0$, $\xi_i \in (0, 1)$, $\xi_0 = 0$, $\xi_n = 1$ and $\xi_i < \xi_{i+1}$, $i = 0, \ldots, n-1$.

Assume that there exists $P \in \mathbb{P}_n$ of degree n, satisfying the conditions (2.1)-(2.2). Then, by Lemma 2.1 the real numbers $(-1)^i (\eta_i - \eta_{i-1})$, $i = 2, \ldots, n-1$ are either all nonnegative or all nonpositive.

Conversely, suppose that for each $i = 2, \ldots, n-1$, the values $(-1)^i (\eta_i - \eta_{i-1})$ are nonnegative. Acoording to Lemma 2.2 and the implicit function theorem, the function Φ defined by (2.16), is locally invertible near $x = (\xi_1, \ldots, \xi_{n-1}, A)$.

Analogously, by Lemma 2.3 there exists an open neighborhood of the point $z = (z_1, z_2, \ldots, z_{n-1}, \rho) \in \mathring{\mathbb{H}}_n$ such that $z = x$. Furthermore, on this open neighborhood the function $\tilde{\Psi}$ defined by (2.24), is invertible.

Consequently, by the chain rule the function $\Psi = \tilde{\Psi} \circ \Phi$ is invertible on some neighborhood of x. Hence, it suffices to take $P(x) = Q(x)$ with $Q(x)$ defined by the equation (2.12).

Finally, the case $(-1)^i (\eta_i - \eta_{i-1})$, $i = 2, \ldots, n-1$, can be obtained by a similar reasoning. □

3 Some examples

As we have seen, the extremal solution P in Theorem 2.1 depends on the suitable use of the implicit function theorem, therefore one generally cannot give an explicit expression for P when n is arbitrarily large. In this section, three illustrative examples giving an explicit expression for the extremal solution P are presented. In order to do this, we implement Theorem 2.1 with the help of Mathematica 10, but MAPLE 18 also can be used.

The steps to implement Theorem 2.1 are the following.

Step 1. Consider the vectors $\eta \in \mathbb{R}^{n-1}$ and $\tilde{\eta} \in \mathbb{R}^{n+1}$ given by

$$\begin{aligned} \eta &= (\eta_1, \eta_2, \ldots, \eta_{n-1}), \\ \tilde{\eta} &= (\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_{n-1}, \tilde{\eta}_n), \end{aligned}$$

where the numbers $(-1)^i (\eta_i - \eta_{i-1})$, $i = 2, \ldots, n-1$ must be either all nonnegative or all nonpositive, the components of $\tilde{\eta}$ satisfy the following conditions

$$\begin{aligned} \tilde{\eta}_0 &= 0, \\ \tilde{\eta}_i &= \eta_i, \quad i = 1, \ldots, n-1, \end{aligned}$$

and $(-1)^i(\tilde{\eta}_i - \tilde{\eta}_{i-1})$, $i = 1, \ldots, n$ are either all nonnegative or all nonpositive. This step requires to choose a specific value for $\tilde{\eta}_n$.

Step 2. Consider as unknown variables the components of the vector

$$\xi = (\xi_1, \ldots, \xi_{n-1}, A) \in (0,1)^{n-1} \times (0, \infty).$$

Step 3. Define the polynomials

$$p(x) := A(\xi_1 - x)(\xi_2 - x)\cdots(\xi_{n-1} - x),$$

and

$$P(x) := A \int_0^x p(t)\, dt.$$

Step 4. Consider the vector $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ whose components are given by

$$\phi_k := (-1)^k (P\xi_k) - P(\xi_{k-1})), \quad k = 1, \ldots, n,$$

where $\xi_0 = 0$ and $\xi_n = 1$.

Step 5. Determine the n components of the vector ξ solving the following nonlinear system of n equations

$$\phi_k := (-1)^k (\tilde{\eta}_k - \tilde{\eta}_{k-1}), \quad k = 1, \ldots, n.$$

Example 3.1. Following the previous five steps with

$$\begin{aligned} \eta &= (\eta_1, \eta_2, \eta_3, \eta_4) = \left(\frac{1}{120}, \frac{-1}{120}, \frac{1}{120}, \frac{-1}{120} \right), \\ \tilde{\eta} &= (\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4, \tilde{\eta}_5) = \left(0, \frac{1}{120}, \frac{-1}{120}, \frac{1}{120}, \frac{-1}{120}, \frac{1}{120} \right), \end{aligned}$$

we derive

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4, A) = (0.0650677, 0.288935, 0.605482, 0.885001, 29.1399),$$

and the following is an explicit expression for the extremal polynomial of Theorem 2.1:

$$P(x) = 0.293561x - 3.172092x^2 + 10.512583x^3 - 13.437032x^4 + 5.827980x^5.$$

Figure 1 shows the plots for the polynomials P and p.

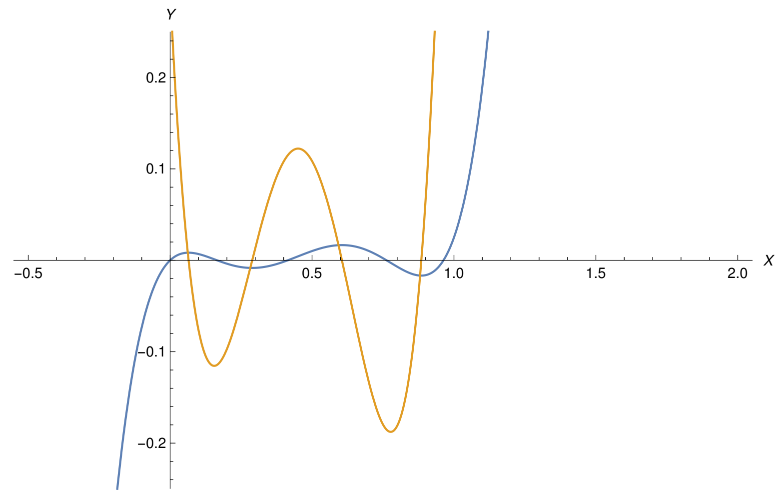

Figure 1: Extremal polynomial P (blue), derivative polynomial p (orange).

Example 3.2. Under a slight modification on the steps to implement Theorem 2.1 (involving affine transformations), we can solve the extremal problem on the interval $[1,2]$. For instance, we can consider

$$\begin{aligned}\eta &= (\eta_1,\eta_2) = \left(\frac{1}{2},-\frac{1}{4}\right),\\ \tilde{\eta} &= (\tilde{\eta}_0,\tilde{\eta}_1,\tilde{\eta}_2,\tilde{\eta}_3) = \left(0,\frac{1}{2},-\frac{1}{4},\frac{1}{6}\right).\end{aligned}$$

In this case, the unknown variables are the components of the vector $\xi = (\xi_1,\xi_2,A) \in (1,2)^2 \times (0,\infty)$, and the polynomials p and P are defined by

$$p(x) := A(\xi_1 - x)(\xi_2 - x),$$

and

$$P(x) := A\int_1^x p(t)\,dt.$$

So that, $P(1) = 0$ and $P(2)$ is preassigned as $\tilde{\eta}_3 = \frac{1}{6}$.

After the suitable modifications, we obtain that

$$\xi = (\xi_1,\xi_2,A) = (1.231538, 1.786792, 26.2867),$$

and

$$\begin{aligned}p(x) &= 26.2867(1.231538 - x)(1.786792 - x),\\ P(x) &= -26.93520812 + 57.84394239x - 39.67096761x^2 + 8.762233333x^3.\end{aligned}$$

Figure 2 shows the plots for the polynomials P and p.

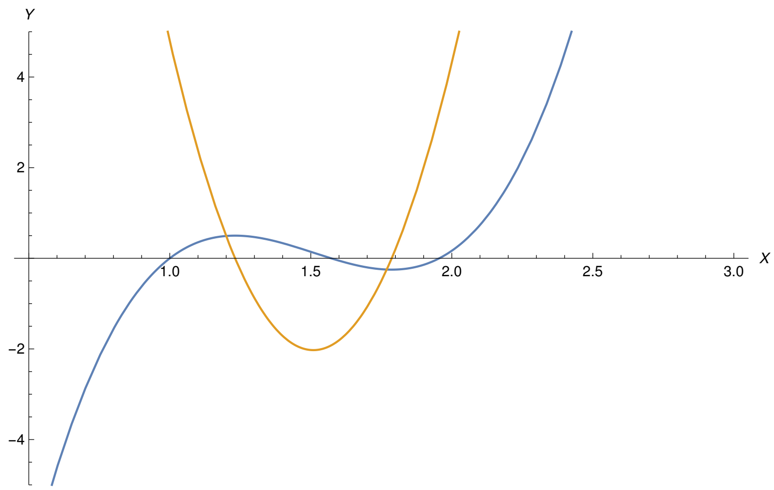

Figure 2: Extremal polynomial P (blue), derivative polynomial p (orange), on the interval $[1, 2]$.

Example 3.3. Analogously to the previous example, we can solve the extremal problem on the interval $[-1, 1]$. For instance, we can consider

$$\begin{aligned} \eta &= (\eta_1, \eta_2) = \left(\frac{1}{2}, -\frac{1}{4}\right), \\ \tilde{\eta} &= (\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3) = \left(-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{6}\right). \end{aligned}$$

In this case, the unknown variables are the components of the vector $\xi = (\xi_1, \xi_2, A) \in (-1, 1)^2 \times (0, \infty)$, and the polynomials p and P are defined by

$$p(x) := A(\xi_1 - x)(\xi_2 - x),$$

and

$$P(x) := -1 + A \int_{-1}^{x} p(t)\, dt.$$

Notice that $P(-1) = -1$ and $P(1)$ is prescribed as $\tilde{\eta}_3 = \frac{1}{6}$.

After the suitable modifications, we obtain that

$$\xi = (\xi_1, \xi_2, A) = (-0.342608, 0.627497, 4.92897),$$

and

$$\begin{aligned} p(x) &= 4.92897(-0.342608 - x)(0.627497 - x), \\ P(x) &= 0.285438 - 1.059657x + 0.702105x^2 + 1.642990x^3. \end{aligned}$$

Figure 3 shows the plots for the polynomials P and p.

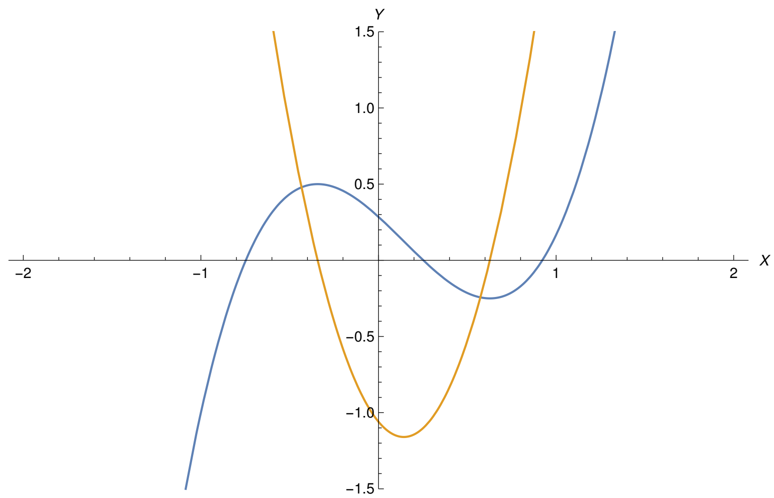

Figure 3: Extremal polynomial P (blue), derivative polynomial p (orange), on the interval $[-1,1]$.

4 Extensions of Problem CD: Chebyshev-Bojanov polynomials

Let $\nu_1,\dots,\nu_n$ be fixed natural numbers. For $N=\nu_1+\cdots+\nu_n-1$ and $f\in C^{N+1}[a,b]$ Newton interpolation formula establishes the following expression for f:

$$f(x)=L_N(f;x)+R_N(x),$$

where $L_N(f;\cdot)$ denotes the interpolating polynomial on the basis of the nodes $(x_k)_{k=1}^n$, $a\leqslant x_1<\cdots<x_n\leqslant b$, with multiplicities $(\nu_k)_{k=1}^n$, respectively, and R_N represents the remainder function.

The following estimates for the remainder function is well known

$$\|R_N\|_\infty\leqslant\left\|f^{(N+1)}\right\|_\infty\|(x-x_1)^{\nu_1}\cdots(x-x_n)^{\nu_n}\|_\infty .$$

In 1979, Borislav Bojanov solved [1] the following extremal problem

Problem-BB: *Determine*

$$\inf_{x_1<x_2<\cdots<x_n}\|(x-x_1)^{\nu_1}\cdots(x-x_n)^{\nu_n}\|_\infty . \tag{4.1}$$

Remark 4.1. When $\nu_1=\cdots=\nu_n=1$, the extremal solution of (4.1) is the Chebyshev polynomial of first kind $T_n(x)$. For this reason we call to the extremal solution of (4.1), $W_n(x)=(x-x_1^*)^{\nu_1}\cdots(x-x_n^*)^{\nu_n}$ Chebyshev-Bojanov polynomial.

Bojanov showed in [1] the existence and uniqueness of the basis of the nodes $\{x_k^*\}_{k=1}^n$ for any prescribed system of multiplicities $\{\nu_k\}_{k=1}^n$, that is, his result guaranties existence and uniqueness of Chebyshev-Bojanov polynomial W_n.

More precisely, using as main argument the implicit function theorem, Bojanov showed the following result:

Theorem 4.1. *[1, Theorem 1] Let $(\nu_k)_{k=1}^n$ be arbitrary fixed natural numbers and let $p(x)$ be a continuous function defined and $\geqslant 0$ on $[x_0, \infty)$, having a finite number of zeros in any finite subinterval $[x_0, x]$. Given positive numbers $(e_k)_{k=1}^n$, there exists a unique system of points $(x_k)_{k=1}^n$, $x_0 < x_1 < \cdots < x_n$, such that*

$$\left|\int_{x_{k-1}}^{x_k} p(x)\prod_{i=1}^{n}(x-x_i)^{\nu_i}\,dx\right| = e_k, \quad k = 1, \ldots, n.$$

As a consequence of Theorem 4.1, Bojanov obtained

Corollary 4.1. *[1, Corollary 1] Let $(\nu_k)_{k=1}^n$ be a fixed system of arbitrary natural numbers. Let the real numbers $(y_k)_{k=0}^{n+1}$ satisfy the requirements $y_k \neq y_{k-1}$, $k = 1, \ldots, n+1$, and*

$$\begin{aligned} |y_{k-1} - y_k| + |y_k - y_{k+1}| &= |y_{k-1} - y_{k+1}| \quad \text{if } \nu_k \text{ is even},\\ |y_{k-1} - y_k| + |y_k - y_{k+1}| &> |y_{k-1} - y_{k+1}| \quad \text{if } \nu_k \text{ is odd}, \end{aligned}$$

$k = 1, \ldots, n$. Given an interval $[a, b]$, there exists a unique polynomial $P \in \mathbb{P}_N$, $N = \nu_1 + \nu_2 + \ldots + \nu_n + 1$, and a unique system of points $(x_k)_{k=1}^n$, $a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b$, such that

$$P(x_k) = y_k, \quad k = 0, \ldots, n+1, \tag{4.2}$$

$$P^{(\lambda)}(x_k) = 0, \quad k = 1, \ldots, n, \quad \lambda = 1, \ldots, \nu_k. \tag{4.3}$$

Taking $\nu_1 = \cdots = \nu_n = 1$, the Chebyshev-Bojanov polynomial $P(x)$ in Corollary 4.1 coincides with the extremal polynomial in Theorem 2.1 when $A = 1$. Consequently, Corollary 4.1 is a generalization of Theorem 2.1.

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