Continuity of the set of equilibria in problems with terms concentrating in the boundary.

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Abstract

We analyze the dynamics of a nonlinear parabolic problem when some terms are concentrated in a neighborhood of the boundary and this neighborhood shrinks to boundary, as a parameter ϵ goes to zero. Moreover, the upper boundary of this neighborhood presents a highly oscillatory behaviour. As a first step, we show in this paper the continuity of the set equilibria.

Keywords: Parabolic problem, concentrating terms, boundary reaction, oscillatory behaviour, equilibria, continuity.

1 Introduction

Let $\Omega = (0,1) \times (0,1)$, $\Gamma = \{(x_1, 0) \in \mathbb{R}^2 : 0 \leqslant x_1 \leqslant 1\}$ and $g \in L^\infty(\mathbb{R})$, T-periodic, with $0 < g_0 \leqslant g(x) \leqslant g_1$ almost every $x \in \mathbb{R}$. The mean value of g over $(0, T)$ is the real number $M(g)$ given by

$$M(g) = \frac{1}{T} \int_0^T g(s) ds.$$

We define

$$\omega_\epsilon = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \quad 0 \leqslant x_1 \leqslant 1 \quad \text{and} \quad 0 \leqslant x_2 < \epsilon g\left(\frac{x_1}{\epsilon}\right) \right\},$$

for sufficiently small ϵ, say $0 < \epsilon \leqslant \epsilon_0$, see Figure 1. We denote by $\mathcal{X}_{\omega_\epsilon}$ the characteristic function of the set ω_ϵ. We note that for small ϵ, the set ω_ϵ is a neighborhood of Γ, that collapses to the boundary when the parameter ϵ goes to zero. We observe that the upper boundary of set ω_ϵ presents a highly oscillatory behaviour and, moreover, the height of ω_ϵ, the amplitude and period of the oscillations are all of the same order, given by the small parameter ϵ.

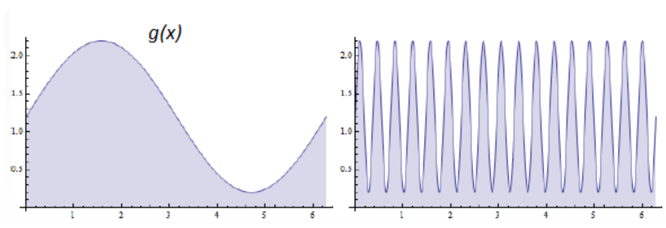

Figure 1: Function $g(\frac{x}{\epsilon})$.

We are interested in the behaviour, for small ϵ, of the solutions of the nonlinear elliptic problem

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = \dfrac{1}{\epsilon} \mathcal{X}_{\omega_\epsilon} f(u_\epsilon), & \Omega \\ \dfrac{\partial u_\epsilon}{\partial n} = 0, & \partial\Omega. \end{cases} \tag{1.1}$$

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We will prove that under certain conditions, the limit problem of (1.1) is the following elliptic problem with nonlinear boundary conditions

$$\begin{cases} -\Delta u_0 + u_0 = 0, & \Omega \\ \dfrac{\partial u_0}{\partial n} = M(g) f(u_0), & \Gamma \\ \dfrac{\partial u_0}{\partial n} = 0, & \partial\Omega \setminus \Gamma. \end{cases} \tag{1.2}$$

Moreover, we will show that if $\{u_\epsilon^*\}_{\epsilon\in(0,\epsilon_0]}$ is a family of solutions of (1.1), then we can get a subsequence, that we will still denote by $\{u_\epsilon^*\}_{\epsilon\in(0,\epsilon_0]}$, such that $u_\epsilon^* \to u_0^*$ in $H^1(\Omega)$, as $\epsilon \to 0$, with u_0^* a solution of the limit equation (1.2) and vice versa, if u_0^* is a hyperbolic solution of the limit equation (1.2), then there exists a family $\{u_\epsilon^*\}_{\epsilon\in(0,\epsilon_0]}$ of solutions of (1.1), with $\epsilon_0 > 0$ suficiently small, such that $u_\epsilon^* \to u_0^*$ in $H^1(\Omega)$, as $\epsilon \to 0$. In fact, we will show that there exists one and only one solution u_ϵ^* of (1.1) in a neighborhood of u_0^*.

Now, if we regard the solutions of (1.1) and (1.2), respectively, as the equilibrium solutions of the parabolic evolutionary equations

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t} = \Delta u^\epsilon - u^\epsilon + \dfrac{1}{\epsilon}\mathcal{X}_{\omega_\epsilon} f(u^\epsilon), & \Omega \times (0,\infty) \\ \dfrac{\partial u^\epsilon}{\partial n} = 0, & \partial\Omega \times (0,\infty) \\ u^\epsilon(0) = \phi^\epsilon \in H^1(\Omega) \end{cases} \tag{1.3}$$

$$\begin{cases} \dfrac{\partial u^0}{\partial t} = \Delta u^0 - u^0, & \Omega \times (0,\infty) \\ \dfrac{\partial u^0}{\partial n} = M(g) f(u^0), & \Gamma \times (0,\infty) \\ \dfrac{\partial u^0}{\partial n} = 0, & \partial\Omega \setminus \Gamma \times (0,\infty) \\ u^0(0) = \phi^0 \in H^1(\Omega) \end{cases} \tag{1.4}$$

then, in particular, we will get that the set of equilibria of (1.3) and (1.4) is continuous at $\epsilon = 0$. Since the equilibrium solutions are the simplest elements from the attractor, then the continuity of the set of equilibria is the first step to prove the continuity of attractors and to understood the dynamics behaviour of parabolic problems (1.3) and (1.4).

The behaviour of the solutions of elliptic problems with terms concentrated in a neighborhood of the boundary, initially, was studied in [6], when the neighborhood is a strip of width ϵ and base in the boundary, without oscillatory behaviour. Later, the asymptotic behaviour of the attractors of a parabolic problem was analyzed in [9], where the upper semicontinuity of attractors at $\epsilon = 0$ was proved. The same technique of [6] has been used in [2] and [3], where the results of [6] and [9] were extended to reaction-diffusion problems with delay.

Now, elliptic and parabolic problems in thin domains as

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : \quad 0 < x_1 < 1 \quad \text{and} \quad 0 < x_2 < \epsilon g\left(\frac{x_1}{\epsilon}\right)\right\},$$

with a highly oscillatory boundary, have been extensively studied, for example, in [4] and [7], where the homogenization theory is used to obtain the limit problem and analyze the convergence properties of the solutions and of the attractors. Here, although the domain Ω is fixed, we will use some ideas of [4] and [7].

Thus, our goal is to extend some results of [6] and [9] to elliptic and parabolic problems when the reaction term is concentrated in ω_ϵ, with the upper boundary of the ω_ϵ presented a highly oscillatory behaviour.

We will assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a $C^2(\mathbb{R})$-function and satisfies the dissipativeness assumption

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < 0. \tag{1.5}$$

It has been shown that the problems (1.3) and (1.4) is well posed in $W^{1,q}(\Omega)$, $q > 2$, without any restriction on the growth of f, see [5]. Moreover, under assumption (1.5), the problems (1.3) and (1.4) have a global attractor $\mathscr{A}_\epsilon$, which is essentially independent of q and that the attractors $\mathscr{A}_\epsilon$ are bounded in $L^\infty(\Omega)$, uniformly in ϵ. In particular, all solutions of (1.1) and (1.2) are bounded with a bound independent of ϵ. This enables us to cut the nonlinearity f in such a way that it becomes bounded with bounded derivatives up to second order without changing the attractors. After these considerations, we may assume, without loss of generality, that

(H) $f : \mathbb{R} \to \mathbb{R}$ is a $C^2(\mathbb{R})$-function satisfying (1.5) and

$$|f(u)| + |f'(u)| + |f''(u)| \leqslant K, \qquad \forall\, u \in \mathbb{R},$$

for some constant $K > 0$.

The fact that now the nonlinearity is globally Lipschitz allows us to study the problem in the space $H^1(\Omega)$. The attractors will lie in more regular spaces, but their continuity properties will be analyzed in the topology of the space $H^1(\Omega)$.

The paper will proceed as follows: in Section 2, we will give the notation that it will be used in this paper and we will see that the solutions of (1.1) and (1.2) will be obtained as fixed points of appropriate nonlinear maps defined in the space $H^1(\Omega)$. In Section 3, we will prove several important technical results that will be needed in the proof of continuity of the set equilibria. Afterwards, in Section 4, we will show the upper semicontinuity of the set of equilibria of (1.3) and (1.4) at $\epsilon = 0$. Finally, in Section 5, we will prove the lower semicontinuity of the set of equilibria at $\epsilon = 0$ and so the continuity, for this, we will also need to assume that the equilibrium points of (1.4) are stable under perturbation. This stability under perturbation can be given by the hyperbolicity of equilibrium point.

2 Solutions as fixed points

We consider the linear operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$, defined by

$$D(A) = \left\{ u \in H^2(\Omega) : \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \right\}$$

$$Au = -\Delta u + u, \qquad \forall\, u \in D(A).$$

The operator A is closed densely defined and sectorial in $L^2(\Omega)$, with compact resolvent set $\rho(A)$. Moreover, A is a positive operator.

Let us denote $X^0 = L^2(\Omega)$, $X^1 = D(A)$ and consider the scale of Hilbert spaces $\{X^\alpha, A_\alpha : \alpha \geqslant 0\}$ constructed by complex interpolation, see [1], which coincide, since we are in a Hilbert setting, with the standard fractional power spaces of operator A. This scale can also be extended to spaces of negative expoents by taking $X^{-\alpha} = (X^\alpha)'$ for $\alpha > 0$, and $X^{-\frac{1}{2}} = H^{-1}(\Omega)$. Let us denote the realization of operator A in the extrapolated spaces $X^{-\alpha}$, $0 < \alpha < 1$, by $A_{-\alpha}$, with domain $X^{-\alpha+1}$. The linear operator $A_{-\alpha} : X^{-\alpha+1} \subset X^{-\alpha} \longrightarrow X^{-\alpha}$ is closed densely defined, sectorial and positive, with compact resolvent set $\rho(A_{-\alpha})$ in $X^{-\alpha}$. Moreover, the operator $A_{-\frac{1}{2}} : X^{\frac{1}{2}} \subset X^{-\frac{1}{2}} \longrightarrow X^{-\frac{1}{2}}$ is given by

$$\left\langle A_{-\frac{1}{2}} u, \phi \right\rangle = \int_\Omega (\nabla u \nabla \phi + u\phi)\,,$$

for $\phi \in H^1(\Omega)$. With some abuse of notation we will identify all different realizations of this operator and we will write all of them as A.

For each $0 \leqslant \epsilon \leqslant \epsilon_0$, define $F_\epsilon : H^1(\Omega) \to H^{-\alpha}(\Omega)$, with $\frac{1}{2} < \alpha < 1$, by

$$\langle F_\epsilon(u), \phi \rangle := \frac{1}{\epsilon} \int_{\omega_\epsilon} f(u)\phi, \qquad 0 < \epsilon \leqslant \epsilon_0$$

$$\langle F_0(u), \phi \rangle := M(g) \int_\Gamma \gamma\left(f(u)\right) \gamma(\phi),$$

for $u \in H^1(\Omega)$ and $\phi \in H^\alpha(\Omega)$, where γ denotes the trace operator, $H^s(\Omega)$ denotes the usual Sobolev spaces for $s \in \mathbb{Z}$, the Slobodeckij spaces for $s > 0$ not integer and $H^s(\Omega) := (H^{-s}(\Omega))'$ for $s < 0$ (see [10] for trace Theorem in the Slobodeckij spaces in Lipschitz domain). Using the hypothesis (H), we can show that for each $0 \leqslant \epsilon \leqslant \epsilon_0$, F_ϵ is well defined.

With these considerations, we write (1.1) and (1.2) in an abstract form, respectively, as

$$Au_\epsilon = F_\epsilon(u_\epsilon), \qquad 0 < \epsilon \leqslant \epsilon_0 \tag{2.6}$$

$$Au_0 = F_0(u_0). \tag{2.7}$$

In particular, u_ϵ is a solution of (1.1) and (1.2) if and only if $u_\epsilon \in H^1(\Omega)$ satisfies

$$u_\epsilon = A^{-1} F_\epsilon(u_\epsilon),$$

that is, u_ϵ is a fixed point of the nonlinear map

$$A^{-1} F_\epsilon : H^1(\Omega) \to H^1(\Omega).$$

For each $\epsilon \in [0, \epsilon_0]$, we denote by $\mathscr{E}_\epsilon$ the set of solutions of (2.6) and (2.7), that is, the set of equilibrium points of (1.3) and (1.4),

$$\mathscr{E}_\epsilon = \left\{ u_\epsilon \in H^1(\Omega) : \quad Au_\epsilon - F_\epsilon(u_\epsilon) = 0 \right\}.$$

3 Some technical results

In this section we will prove several important technical results that will be needed in the proof of the main result.

Initially, we can analyse how concentrating integrals converge for certain families of functions which vary with ϵ and have some regularity properties. We note that since $0 < g_0 \leqslant g(x) \leqslant g_1$ almost every $x \in \mathbb{R}$, then our set ω_ϵ is contained in a strip of width ϵg_1 and base in Γ, without oscillatory behaviour, hence we can use the results of [6]. For this, we need of the following lemma:

Lemma 3.1. *Suppose that $v \in H^s(\Omega)$ with $\frac{1}{2} < s \leqslant 1$ and $s - 1 \geqslant -\frac{1}{q}$. Then, for sufficiently small ϵ_0, there exists a constant $C > 0$ independent of ϵ and v such that for any $0 < \epsilon \leqslant \epsilon_0$, we have*

$$\frac{1}{\epsilon} \int_{\omega_\epsilon} |v|^q \leqslant C \|v\|^q_{H^s(\Omega)}.$$

Proof. We note that

$$\frac{1}{\epsilon} \int_{\omega_\epsilon} |v|^q \leqslant \frac{1}{\epsilon} \int_{R_\epsilon} |v|^q,$$

where R_ϵ is the strip of width ϵg_1 and base in Γ, without oscillatory behaviour, given by

$$R_\epsilon = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \quad 0 \leqslant x_1 \leqslant 1 \quad \text{and} \quad 0 \leqslant x_2 < \epsilon g_1 \right\}.$$

Thus, the result follows using the Lemma 2.1 of [6]. $\square$

Now, we can prove how concentrating integrals converge to surface integrals, but for this we need of some regularity and of a result about weak limits of rapidly oscillating periodic functions, see Theorem 2.6 of [8]. More precisely, we have the following:

Lemma 3.2. *Suppose that h and φ are smooth functions, defined in $\bar{\Omega}$. Then,*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\omega_\epsilon} h\varphi = M(g)\int_\Gamma h\varphi. \tag{3.8}$$

Moreover, (3.8) also holds for $h,\varphi\in H^s(\Omega)$, with $\frac{1}{2}<s\leqslant 1$.

Proof. Let h and φ be smooth function. We note that

$$\left|\frac{1}{\epsilon}\int_{\omega_\epsilon} h\varphi - M(g)\int_\Gamma h\varphi\right| = \left|\frac{1}{\epsilon}\int_0^1\int_0^{\epsilon g\left(\frac{x_1}{\epsilon}\right)} h(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 - M(g)\int_0^1 h(x_1,0)\varphi(x_1,0)dx_1\right|.$$

Taking $x_2=\epsilon g\left(\frac{x_1}{\epsilon}\right)y$, we have

$$\frac{1}{\epsilon}\int_0^1\int_0^{\epsilon g\left(\frac{x_1}{\epsilon}\right)} h(x_1,x_2)\varphi(x_1,x_2)dx_2dx_1 = \int_0^1\int_0^1 h\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right)\varphi\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right)g\left(\frac{x_1}{\epsilon}\right)dydx_1.$$

Thus,

$$\begin{aligned}
&\left|\frac{1}{\epsilon}\int_{\omega_\epsilon} h\varphi - M(g)\int_\Gamma h\varphi\right|\\
\leqslant\ & \left|\int_0^1\int_0^1 h\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right)\varphi\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right)g\left(\frac{x_1}{\epsilon}\right)dydx_1 - \int_0^1 h(x_1,0)\varphi(x_1,0)g\left(\frac{x_1}{\epsilon}\right)dx_1\right|\\
+\ & \left|\int_0^1 h(x_1,0)\varphi(x_1,0)g\left(\frac{x_1}{\epsilon}\right)dx_1 - M(g)\int_0^1 h(x_1,0)\varphi(x_1,0)dx_1\right|\\
=\ & \left|\int_0^1\int_0^1 g\left(\frac{x_1}{\epsilon}\right)\left[h\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right)\varphi\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right) - h(x_1,0)\varphi(x_1,0)\right]dydx_1\right|\\
+\ & \left|\int_0^1 g\left(\frac{x_1}{\epsilon}\right)h(x_1,0)\varphi(x_1,0)dx_1 - M(g)\int_0^1 h(x_1,0)\varphi(x_1,0)dx_1\right|.
\end{aligned}$$

Using the Theorem 2.6 of [8], we get

$$\lim_{\epsilon\to 0}\int_0^1 g\left(\frac{x_1}{\epsilon}\right)h(x_1,0)\varphi(x_1,0)dx_1 = M(g)\int_0^1 h(x_1,0)\varphi(x_1,0)dx_1.$$

Now, since $\epsilon g\left(\frac{x_1}{\epsilon}\right)y\to 0$, as $\epsilon\to 0$, uniformly for $(x_1,y)\in[0,1]\times[0,1]$, then

$$g\left(\frac{x_1}{\epsilon}\right)\left[h\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right)\varphi\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right) - h(x_1,0)\varphi(x_1,0)\right]\to 0, \qquad \text{as } \epsilon\to 0,$$

uniformly for $(x_1,y)\in[0,1]\times[0,1]$. Thus,

$$\left|\int_0^1\int_0^1 g\left(\frac{x_1}{\epsilon}\right)\left[h\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right)\varphi\left(x_1,\epsilon g\left(\frac{x_1}{\epsilon}\right)y\right) - h(x_1,0)\varphi(x_1,0)\right]dydx_1\right|\to 0, \qquad \text{as } \epsilon\to 0.$$

Hence,

$$\left|\frac{1}{\epsilon}\int_{\omega_\epsilon} h\varphi - M(g)\int_\Gamma h\varphi\right|\to 0, \qquad \text{as } \epsilon\to 0.$$

Now, let $h \in H^s(\Omega)$, with $\frac{1}{2} < s \leqslant 1$, and let φ be smooth. We will prove that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} h\varphi = M(g) \int_\Gamma \gamma(h)\varphi, \tag{3.9}$$

where γ denotes the trace operator. In fact, since $C^\infty(\bar{\Omega})$ is dense in $H^s(\Omega)$ and $h \in H^s(\Omega)$, then there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in $C^\infty(\bar{\Omega})$ such that

$$h_n \to h \quad \text{in} \quad H^s(\Omega), \qquad \text{as } n \to \infty. \tag{3.10}$$

Since the trace operator $\gamma : H^s(\Omega) \longrightarrow L^2(\Gamma)$ is linear continuous, then

$$\gamma(h_n) \to \gamma(h) \quad \text{in} \quad L^2(\Gamma), \qquad \text{as } n \to \infty. \tag{3.11}$$

Moreover, for each $n \in \mathbb{N}$, from (3.8) we get

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi = M(g) \int_\Gamma h_n\varphi. \tag{3.12}$$

Now, we want to take the limit in (3.12), as $n \to \infty$, that is,

$$\lim_{n\to\infty} \left[\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi \right] = M(g) \lim_{n\to\infty} \int_\Gamma h_n\varphi. \tag{3.13}$$

Moreover, we want that

$$\lim_{n\to\infty} \left[\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi \right] = \lim_{\epsilon\to 0} \left[\lim_{n \to \infty} \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi \right].$$

But, for this we need of existence of the following limits

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi \qquad \text{and} \qquad \lim_{n \to \infty} \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi,$$

and that the second limit be uniformly in ϵ. From (3.12) we have that the first limit exists. Now, the second limit exists and uniformly in ϵ, since from Lemma 3.1 and (3.10), we have

$$\left| \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi - \frac{1}{\epsilon} \int_{\omega_\epsilon} h\varphi \right| \leqslant \left(\frac{1}{\epsilon} \int_{\omega_\epsilon} |h_n - h|^2 \right)^{\frac{1}{2}} \left(\frac{1}{\epsilon} \int_{\omega_\epsilon} |\varphi|^2 \right)^{\frac{1}{2}} \leqslant C \left\| h_n - h \right\|_{H^s(\Omega)} \left\| \varphi \right\|_{H^s(\Omega)} \to 0, \qquad \text{as } n \to \infty,$$

uniformly in ϵ. Therefore,

$$\lim_{n\to\infty} \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi = \frac{1}{\epsilon} \int_{\omega_\epsilon} h\varphi,$$

uniformly in ϵ. Hence,

$$\lim_{n\to\infty} \left[\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi \right] = \lim_{\epsilon\to 0} \left[\lim_{n \to \infty} \frac{1}{\epsilon} \int_{\omega_\epsilon} h_n\varphi \right] = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} h\varphi. \tag{3.14}$$

Moreover, using (3.11) we have

$$\left| \int_\Gamma h_n\varphi - \int_\Gamma \gamma(h)\varphi \right| = \left| \int_\Gamma \gamma(h_n)\varphi - \int_\Gamma \gamma(h)\varphi \right| \leqslant \left\| \gamma(h_n) - \gamma(h) \right\|_{L^2(\Gamma)} \left\| \varphi \right\|_{L^2(\Gamma)} \to 0, \qquad \text{as } n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} \int_\Gamma h_n\varphi = \int_\Gamma \gamma(h)\varphi. \tag{3.15}$$

Using (3.14) and (3.15) in (3.13), we get (3.9).

Similary, we prove that if $h, \varphi \in H^s(\Omega)$, with $\frac{1}{2} < s \leqslant 1$, then

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} h\varphi = M(g) \int_\Gamma \gamma(h)\gamma(\varphi). \quad \square$$

The following lemmas will be important to prove the continuity of the family of equilibria $\{\mathscr{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ of (1.3) and (1.4) at $\epsilon = 0$.

Lemma 3.3. *Suppose that (H) holds.*

1. *There exists $k>0$ independent of ϵ such that*

$$\|F_\epsilon(u)\|_{H^{-\alpha}(\Omega)} \leqslant k, \qquad \forall\ u \in H^1(\Omega) \quad \textit{and} \quad 0 \leqslant \epsilon \leqslant \epsilon_0.$$

2. *For each $0 \leqslant \epsilon \leqslant \epsilon_0$, the map $F_\epsilon : H^1(\Omega) \longrightarrow H^{-\alpha}(\Omega)$ is globally Lipschitz, uniformly in ϵ.*

3. *For each $u \in H^1(\Omega)$, we have*

$$\|F_\epsilon(u) - F_0(u)\|_{H^{-\alpha}(\Omega)} \to 0, \qquad \textit{as } \epsilon \to 0.$$

Furthermore, this limit is uniform for $u \in H^1(\Omega)$ such that $\|u\|_{H^1(\Omega)} \leqslant R$, for some $R>0$.

4. *If $u_\epsilon \to u$ in $H^1(\Omega)$, as $\epsilon \to 0$, then*

$$\|F_\epsilon(u_\epsilon) - F_0(u)\|_{H^{-\alpha}(\Omega)} \to 0, \qquad \textit{as } \epsilon \to 0.$$

Proof. 1. For each $u \in H^1(\Omega)$ and $0 \leqslant \epsilon \leqslant \epsilon_0$, we have

$$\|F_\epsilon(u)\|_{H^{-\alpha}(\Omega)} = \sup_{\substack{\phi \in H^\alpha(\Omega) \\ \|\phi\|_{H^\alpha(\Omega)} = 1}} |\langle F_\epsilon(u), \phi\rangle| .$$

Using that f is bounded and the Lemma 3.1, we have that for each $0 < \epsilon \leqslant \epsilon_0$ and $\phi \in H^\alpha(\Omega)$,

$$|\langle F_\epsilon(u), \phi\rangle| \leqslant \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f(u(x))|^2\, dx\right)^{\frac{1}{2}} \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |\phi(x)|^2\, dx\right)^{\frac{1}{2}} \leqslant CK(g_1)^{\frac{1}{2}} \|\phi\|_{H^\alpha(\Omega)} .$$

Hence, there exists a constant $k>0$ independent of ϵ such that

$$\|F_\epsilon(u)\|_{H^{-\alpha}(\Omega)} \leqslant k, \qquad \forall\ 0 < \epsilon \leqslant \epsilon_0.$$

Now, using again that f is bounded and the continuity of trace operator $\gamma : H^\alpha(\Omega) \longrightarrow L^2(\Gamma)$, we get

$$|\langle F_0(u), \phi\rangle| \leqslant M(g) \left(\int_\Gamma |\gamma(f(u(x)))|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Gamma |\gamma(\phi(x))|^2\, dx\right)^{\frac{1}{2}} \leqslant KM(g)\, \|\gamma(\phi)\|_{L^2(\Gamma)} \leqslant cKM(g)\, \|\phi\|_{H^\alpha(\Omega)} .$$

Hence, there exists $k>0$ such that

$$\|F_0(u)\|_{H^{-\alpha}(\Omega)} \leqslant k.$$

2. Let $u, v \in H^1(\Omega)$ and $0 \leqslant \epsilon \leqslant \epsilon_0$, we have

$$\|F_\epsilon(u) - F_\epsilon(v)\|_{H^{-\alpha}(\Omega)} = \sup_{\substack{\phi \in H^\alpha(\Omega) \\ \|\phi\|_{H^\alpha(\Omega)} = 1}} |\langle F_\epsilon(u) - F_\epsilon(v), \phi\rangle| .$$

For each $0 < \epsilon \leqslant \epsilon_0$ and $\phi \in H^\alpha(\Omega)$, from Lemma 3.1 we have

$$\begin{aligned} |\langle F_\epsilon(u) - F_\epsilon(v), \phi\rangle| &\leqslant \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f(u(x)) - f(v(x))|^2\, dx\right)^{\frac{1}{2}} \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |\phi(x)|^2\, dx\right)^{\frac{1}{2}} \\ &\leqslant C \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f(u(x)) - f(v(x))|^2\, dx\right)^{\frac{1}{2}} \|\phi\|_{H^\alpha(\Omega)} . \end{aligned}$$

Using (H) and the Lemma 3.1, we have

$$\begin{aligned}\|F_\epsilon(u)-F_\epsilon(v)\|_{H^{-\alpha}(\Omega)} &\leqslant C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}|f'(\theta(x)u(x)+(1-\theta(x))v(x))|^2\,|u(x)-v(x)|^2\,dx\right)^{\frac{1}{2}}\\ &\leqslant CK\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}|u(x)-v(x)|^2\,dx\right)^{\frac{1}{2}}\leqslant CK\,\|u-v\|_{H^1(\Omega)},\end{aligned}$$

for some $0\leqslant\theta(x)\leqslant 1$, $x\in\bar{\Omega}$. Hence, there exists $L>0$ independent of ϵ such that

$$\|F_\epsilon(u)-F_\epsilon(v)\|_{H^{-\alpha}(\Omega)}\leqslant L\,\|u-v\|_{H^1(\Omega)}.$$

Therefore, for each $0<\epsilon\leqslant\epsilon_0$, F_ϵ is globally Lipschitz, uniformly in ϵ. Similarly, F_0 is globally Lipschitz.

3. Initially, we take α_0 satisfying $\frac{1}{2}<\alpha_0<1$. For each $u\in H^1(\Omega)$ and $\phi\in H^{\alpha_0}(\Omega)$, we have

$$|\langle F_\epsilon(u),\phi\rangle-\langle F_0(u),\phi\rangle|=\left|\frac{1}{\epsilon}\int_{\omega_\epsilon}f(u(x))\phi(x)dx-M(g)\int_\Gamma\gamma\left(f(u(x))\right)\gamma\left(\phi(x)\right)dx\right|.$$

From Lemma 3.2, we get that for each $\phi\in H^{\alpha_0}(\Omega)$,

$$\langle F_\epsilon(u),\phi\rangle\to\langle F_0(u),\phi\rangle,\qquad\text{as }\epsilon\to 0.\tag{3.16}$$

Moreover, fixed $u\in H^1(\Omega)$ and using the item 1, we have that the set $\{F_\epsilon(u)\in H^{-\alpha_0}(\Omega):\epsilon\in(0,\epsilon_0]\}$ is equicontinuous. Thus, the limit (3.16) is uniform for ϕ in compact sets of $H^{\alpha_0}(\Omega)$. Hence, choosing α_0 such that $\frac{1}{2}<\alpha_0<\alpha<1$, we have that the embedding $H^\alpha(\Omega)\hookrightarrow H^{\alpha_0}(\Omega)$ is compact, and then, in particular,

$$\|F_\epsilon(u)-F_0(u)\|_{H^{-\alpha}(\Omega)}=\sup_{\substack{\phi\in H^\alpha(\Omega)\\ \|\phi\|_{H^\alpha(\Omega)}=1}}|\langle F_\epsilon(u)-F_0(u),\phi\rangle|\to 0,\qquad\text{as }\epsilon\to 0.\tag{3.17}$$

Now, we will show that the limit (3.17) is uniform for $u\in H^1(\Omega)$ such that $\|u\|_{H^1(\Omega)}\leqslant R$, for some $R>0$. Initially, we will show that, for each $0<\epsilon\leqslant\epsilon_0$, $F_\epsilon:H^1(\Omega)\longrightarrow H^{-\alpha}(\Omega)$ is continuous in $H^1(\Omega)$ with the weak topology.

Let $u_n\rightharpoonup u_0$ in $H^1(\Omega)$, as $n\to\infty$. Since $H^1(\Omega)\hookrightarrow H^s(\Omega)$ with compact embedding, for $s<1$, then we can assume that

$$u_n\to u_0\quad\text{in}\quad H^s(\Omega),\quad\text{as }n\to\infty.$$

For each $\phi\in H^\alpha(\Omega)$, from Lemma 3.1 we have

$$\begin{aligned}|\langle F_\epsilon(u_n)-F_\epsilon(u_0),\phi\rangle| &\leqslant \left(\frac{1}{\epsilon}\int_{\omega_\epsilon}|f(u_n(x))-f(u_0(x))|^2\,dx\right)^{\frac{1}{2}}\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}|\phi(x)|^2\,dx\right)^{\frac{1}{2}}\\ &\leqslant C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}|f(u_n(x))-f(u_0(x))|^2\,dx\right)^{\frac{1}{2}}\|\phi\|_{H^\alpha(\Omega)}.\end{aligned}$$

Using (H) and the Lemma 3.1, we have

$$\begin{aligned}\|F_\epsilon(u_n)-F_\epsilon(u_0)\|_{H^{-\alpha}(\Omega)} &\leqslant C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}|f'(\theta(x)u_n(x)+(1-\theta(x))u_0(x))|^2\,|u_n(x)-u_0(x)|^2\,dx\right)^{\frac{1}{2}}\\ &\leqslant CK\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}|u_n(x)-u_0(x)|^2\,dx\right)^{\frac{1}{2}}\leqslant CK\,\|u_n-u_0\|_{H^s(\Omega)}\to 0,\qquad\text{as }n\to\infty,\end{aligned}$$

for some $0 \leqslant \theta(x) \leqslant 1$, $x \in \bar{\Omega}$.

Therefore, for each $0 < \epsilon \leqslant \epsilon_0$, $F_\epsilon : H^1(\Omega) \longrightarrow H^{-\alpha}(\Omega)$ is continuous in $H^1(\Omega)$ with the weak topology. Hence, $F_\epsilon : H^1(\Omega) \longrightarrow H^{-\alpha}(\Omega)$ is uniformly continuous in compact sets of $H^1(\Omega)$ with the weak topology. We note that the closed ball $\bar{B}_R(0) = \{u \in H^1(\Omega) : \quad \|u\|_{H^1(\Omega)} \leqslant R\}$, with $R > 0$, is compact in $H^1(\Omega)$ with the weak topology. From this and (3.17), we get that the limit (3.17) is uniform in $\bar{B}_R(0)$.

4. Now, we take $u_\epsilon \to u$ in $H^1(\Omega)$, as $\epsilon \to 0$. We have,

$$\|F_\epsilon(u_\epsilon) - F_0(u)\|_{H^{-\alpha}(\Omega)} \leqslant \|F_\epsilon(u_\epsilon) - F_\epsilon(u)\|_{H^{-\alpha}(\Omega)} + \|F_\epsilon(u) - F_0(u)\|_{H^{-\alpha}(\Omega)}.$$

Using the item 2, we have that there exists $L > 0$ independent of ϵ such that

$$\|F_\epsilon(u_\epsilon) - F_\epsilon(u)\|_{H^{-\alpha}(\Omega)} \leqslant L \|u_\epsilon - u\|_{H^1(\Omega)}.$$

From this and item 3, we get

$$\|F_\epsilon(u_\epsilon) - F_0(u)\|_{H^{-\alpha}(\Omega)} \leqslant L \|u_\epsilon - u\|_{H^1(\Omega)} + \|F_\epsilon(u) - F_0(u)\|_{H^{-\alpha}(\Omega)} \to 0, \qquad \text{as } \epsilon \to 0. \quad \square$$

Using the hypothesis (H) we will show that, for each $0 \leqslant \epsilon \leqslant \epsilon_0$, $F_\epsilon : H^1(\Omega) \longrightarrow H^{-\alpha}(\Omega)$ is Fréchet differentiable, uniformly in ϵ, and your Fréchet differential is given by

$$\begin{aligned} F'_\epsilon : H^1(\Omega) &\longrightarrow \mathscr{L}\left(H^1(\Omega), H^{-\alpha}(\Omega)\right) \\ u^* &\longmapsto F'_\epsilon(u^*) : H^1(\Omega) \longrightarrow H^{-\alpha}(\Omega) \\ &\qquad\qquad\qquad\qquad w \longmapsto F'_\epsilon(u^*)w \end{aligned}$$

where $\mathscr{L}\left(H^1(\Omega), H^{-\alpha}(\Omega)\right)$ denotes the space of the continuous linear operators from $H^1(\Omega)$ in $H^{-\alpha}(\Omega)$, and

$$\begin{aligned} \langle F'_\epsilon(u^*)w, \phi\rangle &:= \frac{1}{\epsilon}\int_{\omega_\epsilon} f'\left(u^*(x)\right) w(x)\phi(x)dx, \qquad \forall\ \phi \in H^\alpha(\Omega) \quad \text{and} \quad 0 < \epsilon \leqslant \epsilon_0 \\ \langle F'_0(u^*)w, \phi\rangle &:= \int_\Gamma \gamma\left(f'\left(u^*(x)\right) w(x)\right)\gamma\left(\phi(x)\right)dx, \qquad \forall\ \phi \in H^\alpha(\Omega), \end{aligned} \tag{3.18}$$

where γ denotes the trace operator.

Lemma 3.4. *Suppose that (H) holds. Then, for each $0 \leqslant \epsilon \leqslant \epsilon_0$, $F_\epsilon : H^1(\Omega) \longrightarrow H^{-\alpha}(\Omega)$ is Fréchet differentiable, uniformly in ϵ, and your Fréchet differential is given by (3.18).*

Proof. From (H), in particular, we have that $f \in C^2(\mathbb{R})$, hence $f'(v) \in \mathscr{L}(\mathbb{R}, \mathbb{R})$, for each $v \in \mathbb{R}$. Using this and the linearity of integral and of trace operator, we get that for each $0 \leqslant \epsilon \leqslant \epsilon_0$, $F'_\epsilon(u^*) \in \mathscr{L}\left(H^1(\Omega), H^{-\alpha}(\Omega)\right)$, for each $u^* \in H^1(\Omega)$.

Now, we will show that given $\eta > 0$, there exists $\delta > 0$ independent of ϵ such that

$$\|F_\epsilon(u^* + w) - F_\epsilon(u^*) - F'_\epsilon(u^*)w\|_{H^{-\alpha}(\Omega)} \leqslant \eta \|w\|_{H^1(\Omega)}, \qquad \forall\ w \in H^1(\Omega) \quad \text{with} \quad \|w\|_{H^1(\Omega)} \leqslant \delta.$$

In fact, for each $0 < \epsilon \leqslant \epsilon_0$, $w \in H^1(\Omega)$ and $\phi \in H^\alpha(\Omega)$, from Lemma 3.1 we have

$$\begin{aligned} &|\langle F_\epsilon(u^* + w) - F_\epsilon(u^*) - F'_\epsilon(u^*)w, \phi\rangle| \\ \leqslant\ & \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f(u^*(x) + w(x)) - f(u^*(x)) - f'(u^*(x))w(x)|^2\, dx\right)^{\frac{1}{2}} \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |\phi(x)|^2\, dx\right)^{\frac{1}{2}} \\ \leqslant\ & C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f(u^*(x) + w(x)) - f(u^*(x)) - f'(u^*(x))w(x)|^2\, dx\right)^{\frac{1}{2}} \|\phi\|_{H^\alpha(\Omega)}. \end{aligned}$$

Using (H) and the Lemma 3.1, we have

$$\|F_\epsilon(u^*+w)-F_\epsilon(u^*)-F_\epsilon'(u^*)w\|_{H^{-\alpha}(\Omega)}$$

$$\begin{aligned}
&\leqslant C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}\left|f'(u^*(x)+\theta(x)w(x))-f'(u^*(x))\right|^2|w(x)|^2\,dx\right)^{\frac{1}{2}}\\
&\leqslant C\left[\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}\left|f'(u^*(x)+\theta(x)w(x))-f'(u^*(x))\right|^4 dx\right)^{\frac{1}{2}}\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}|w(x)|^4\,dx\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\\
&\leqslant C\left[\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}\left|f''\left[s(x)(u^*(x)+\theta(x)w(x))+(1-s(x))u^*(x)\right]\right|^4|\theta(x)w(x)|^4\,dx\right)^{\frac{1}{2}}\|w\|^2_{H^1(\Omega)}\right]^{\frac{1}{2}}\\
&\leqslant CK\left(\frac{1}{\epsilon}\int_{\omega_\epsilon}|w(x)|^4\,dx\right)^{\frac{1}{4}}\|w\|_{H^1(\Omega)}\leqslant CK\,\|w\|_{H^1(\Omega)}\,\|w\|_{H^1(\Omega)}\,,
\end{aligned}$$

for some $0\leqslant\theta(x)\leqslant 1$ and $0\leqslant s(x)\leqslant 1$, $x\in\bar{\Omega}$.

Therefore, given $\eta>0$, taking $\delta=\frac{\eta}{CK}>0$ we get that for $\|w\|_{H^1(\Omega)}\leqslant\delta$,

$$\|F_\epsilon(u^*+w)-F_\epsilon(u^*)-F_\epsilon'(u^*)w\|_{H^{-\alpha}(\Omega)}\leqslant\eta\,\|w\|_{H^1(\Omega)}\,.$$

We note that δ does not depend of ϵ. Hence, for each $0<\epsilon\leqslant\epsilon_0$, F_ϵ is Fréchet differentiable, uniformly in ϵ. Similarly, F_0 is also Fréchet differentiable. $\square$

Lemma 3.5. *Suppose that (H) holds.*

1. *There exist $k>0$ independent of ϵ such that*

$$\|F_\epsilon'(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))}\leqslant k,\qquad\forall\ u^*\in H^1(\Omega)\quad\text{and}\quad 0\leqslant\epsilon\leqslant\epsilon_0.$$

2. *For each $0\leqslant\epsilon\leqslant\epsilon_0$, the map $F_\epsilon':H^1(\Omega)\longrightarrow\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))$ is globally Lipschitz, uniformly in ϵ.*

3. *For each $u^*\in H^1(\Omega)$, we have*

$$\|F_\epsilon'(u^*)-F_0'(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))}\to 0,\qquad\text{as }\epsilon\to 0.$$

4. *If $u_\epsilon^*\to u^*$ in $H^1(\Omega)$, as $\epsilon\to 0$, then*

$$\|F_\epsilon'(u_\epsilon^*)-F_0'(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))}\to 0,\qquad\text{as }\epsilon\to 0.$$

5. *If $u_\epsilon^*\to u^*$ in $H^1(\Omega)$, as $\epsilon\to 0$, and $w_\epsilon\to w$ in $H^1(\Omega)$, as $\epsilon\to 0$, then*

$$\|F_\epsilon'(u_\epsilon^*)w_\epsilon-F_0'(u^*)w\|_{H^{-\alpha}(\Omega)}\to 0,\qquad\text{as }\epsilon\to 0.$$

Proof. 1. For each $u^*\in H^1(\Omega)$ and $0\leqslant\epsilon\leqslant\epsilon_0$, we have

$$\|F_\epsilon'(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))}=\sup_{\substack{w\in H^1(\Omega)\\ \|w\|_{H^1(\Omega)}=1}}\|F_\epsilon'(u^*)w\|_{H^{-\alpha}(\Omega)}\,.$$

Using that f' is bounded and the Lemma 3.1, we have that for each $0 < \epsilon \leqslant \epsilon_0$, $w \in H^1(\Omega)$ and $\phi \in H^\alpha(\Omega)$,

$$\begin{aligned} |\langle F'_\epsilon(u^*)w, \phi\rangle| &\leqslant \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f'(u^*(x))w(x)|^2\,dx\right)^{\frac12} \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |\phi(x)|^2\,dx\right)^{\frac12} \\ &\leqslant CK\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |w(x)|^2\,dx\right)^{\frac12} \|\phi\|_{H^\alpha(\Omega)} \leqslant CK\,\|w\|_{H^1(\Omega)}\,\|\phi\|_{H^\alpha(\Omega)}\,. \end{aligned}$$

Thus, there exists $k > 0$ independent of ϵ such that

$$\|F'_\epsilon(u^*)w\|_{H^{-\alpha}(\Omega)} \leqslant k\,\|w\|_{H^1(\Omega)}\,, \qquad \forall\, w \in H^1(\Omega).$$

Therefore,

$$\|F'_\epsilon(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \leqslant k, \qquad \forall\, 0 < \epsilon \leqslant \epsilon_0.$$

Similarly, there exists $k > 0$ such that

$$\|F'_0(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \leqslant k.$$

2. Let $u^*, v^* \in H^1(\Omega)$ and $0 \leqslant \epsilon \leqslant \epsilon_0$, we have

$$\|F'_\epsilon(u^*) - F'_\epsilon(v^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} = \sup_{\substack{w \in H^1(\Omega)\\ \|w\|_{H^1(\Omega)} = 1}} \|F'_\epsilon(u^*)w - F'_\epsilon(v^*)w\|_{H^{-\alpha}(\Omega)}\,.$$

For each $0 < \epsilon \leqslant \epsilon_0$, $w \in H^1(\Omega)$ and $\phi \in H^\alpha(\Omega)$, from Lemma 3.1 we have

$$\begin{aligned} |\langle F'_\epsilon(u^*)w - F'_\epsilon(v^*)w, \phi\rangle| &\leqslant \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f'(u^*(x))w(x) - f'(v^*(x))w(x)|^2\,dx\right)^{\frac12} \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |\phi(x)|^2\,dx\right)^{\frac12} \\ &\leqslant C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f'(u^*(x))w(x) - f'(v^*(x))w(x)|^2\,dx\right)^{\frac12} \|\phi\|_{H^\alpha(\Omega)}\,. \end{aligned}$$

Using (H) and the Lemma 3.1, we have

$$\begin{aligned} &\|F'_\epsilon(u^*)w - F'_\epsilon(v^*)w\|_{H^{-\alpha}(\Omega)} \\ &\leqslant C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f'(u^*(x)) - f'(v^*(x))|^2\,|w(x)|^2\,dx\right)^{\frac12} \\ &\leqslant C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f''(\theta(x)u^*(x) + (1-\theta(x))v^*(x))|^2\,|u^*(x) - v^*(x)|^2\,|w(x)|^2\,dx\right)^{\frac12} \\ &\leqslant CK\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |u^*(x) - v^*(x)|^2\,|w(x)|^2\,dx\right)^{\frac12} \\ &\leqslant CK\left[\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |u^*(x) - v^*(x)|^4\,dx\right)^{\frac12} \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |w(x)|^4\,dx\right)^{\frac12}\right]^{\frac12} \leqslant CK\,\|u^* - v^*\|_{H^1(\Omega)}\,\|w\|_{H^1(\Omega)}\,, \end{aligned}$$

for some $0 \leqslant \theta(x) \leqslant 1$, $x \in \bar{\Omega}$. Hence, there exists $L > 0$ independent of ϵ such that

$$\|F'_\epsilon(u^*) - F'_\epsilon(v^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \leqslant L\,\|u^* - v^*\|_{H^1(\Omega)}\,.$$

Therefore, for each $0 < \epsilon \leqslant \epsilon_0$, F'_ϵ is globally Lipschitz, uniformly in ϵ. Similarly, F'_0 is globally Lipschitz.

3. Initially, we take α_0 satisfying $\frac{1}{2} < \alpha_0 < 1$. For each $u^*, w \in H^1(\Omega)$ and $\phi \in H^{\alpha_0}(\Omega)$, we have

$$|\langle F'_\epsilon(u^*)w, \phi\rangle - \langle F'_0(u^*)w, \phi\rangle| = \left| \frac{1}{\epsilon} \int_{\omega_\epsilon} f'(u^*(x))w(x)\phi(x)dx - M(g) \int_\Gamma \gamma\left(f'(u^*(x))w(x)\right)\gamma\left(\phi(x)\right)dx \right|.$$

From Lemma 3.2, we get that for each $\phi \in H^{\alpha_0}(\Omega)$,

$$\langle F'_\epsilon(u^*)w, \phi\rangle \to \langle F_0(u^*)w, \phi\rangle, \qquad \text{as } \epsilon \to 0. \tag{3.19}$$

Moreover, fixed $u^*, w \in H^1(\Omega)$ and using the item 1, we have that the set $\{F'_\epsilon(u^*)w \in H^{-\alpha_0}(\Omega) : \epsilon \in (0, \epsilon_0]\}$ is equicontinuous. Thus, the limit (3.19) is uniform for ϕ in compact sets of $H^{\alpha_0}(\Omega)$. Hence, choosing α_0 such that $\frac{1}{2} < \alpha_0 < \alpha < 1$, we have that the embedding $H^\alpha(\Omega) \hookrightarrow H^{\alpha_0}(\Omega)$ is compact, and then, in particular,

$$\|F'_\epsilon(u^*)w - F'_0(u^*)w\|_{H^{-\alpha}(\Omega)} = \sup_{\substack{\phi \in H^\alpha(\Omega) \\ \|\phi\|_{H^\alpha(\Omega)} = 1}} |\langle F'_\epsilon(u^*)w - F'_0(u^*)w, \phi\rangle| \to 0, \qquad \text{as } \epsilon \to 0.$$

So, for each $w \in H^1(\Omega)$,

$$F'_\epsilon(u^*)w \to F'_0(u^*)w \quad \text{in} \quad H^{-\alpha}(\Omega), \qquad \text{as } \epsilon \to 0. \tag{3.20}$$

Now, fixed $u^* \in H^1(\Omega)$, we will show that, for each $0 < \epsilon \leqslant \epsilon_0$, $F'_\epsilon(u^*) : H^1(\Omega) \longrightarrow H^{-\alpha}(\Omega)$ is continuous in $H^1(\Omega)$ with the weak topology.

Let $w_n \rightharpoonup w_0$ in $H^1(\Omega)$, as $n \to \infty$. Since $H^1(\Omega) \hookrightarrow H^s(\Omega)$ with compact embedding, for $s < 1$, then we can assume that

$$w_n \to w_0 \quad \text{in} \quad H^s(\Omega), \quad \text{as } n \to \infty.$$

For each $\phi \in H^\alpha(\Omega)$, from Lemma 3.1 we have

$$\begin{aligned} |\langle F'_\epsilon(u^*)w_n - F_\epsilon(u^*)w_0, \phi\rangle| &\leqslant \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f'(u^*(x))(w_n(x) - w_0(x))|^2\,dx\right)^{\frac{1}{2}} \left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |\phi(x)|^2\,dx\right)^{\frac{1}{2}} \\ &\leqslant C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f'(u^*(x))(w_n(x) - w_0(x))|^2\,dx\right)^{\frac{1}{2}} \|\phi\|_{H^\alpha(\Omega)}. \end{aligned}$$

Using (H) and the Lemma 3.1, we have

$$\begin{aligned} \|F'_\epsilon(u^*)w_n - F'_\epsilon(u^*)w_0\|_{H^{-\alpha}(\Omega)} &\leqslant C\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |f'(u^*(x))|^2\,|w_n(x) - w_0(x)|^2\,dx\right)^{\frac{1}{2}} \\ &\leqslant CK\left(\frac{1}{\epsilon}\int_{\omega_\epsilon} |w_n(x) - w_0(x)|^2\,dx\right)^{\frac{1}{2}} \leqslant CK\,\|w_n - w_0\|_{H^s(\Omega)} \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, for each $0 < \epsilon \leqslant \epsilon_0$, $F'_\epsilon(u^*) : H^1(\Omega) \longrightarrow H^{-\alpha}(\Omega)$ is continuous in $H^1(\Omega)$ with the weak topology. Hence, $F'_\epsilon(u^*) : H^1(\Omega) \longrightarrow H^{-\alpha}(\Omega)$ is uniformly continuous in compact sets of $H^1(\Omega)$ with the weak topology. We note that the set $\{w \in H^1(\Omega) : \quad \|w\|_{H^1(\Omega)} = 1\}$ is compact in $H^1(\Omega)$ with the weak topology. From this and (3.20), we get

$$\|F'_\epsilon(u^*) - F'_0(u^*)\|_{\mathscr{L}(H^1, H^{-\alpha})} = \sup_{\substack{w \in H^1(\Omega) \\ \|w\|_{H^1(\Omega)} = 1}} \|F'_\epsilon(u^*)w - F'_0(u^*)w\|_{H^{-\alpha}(\Omega)} \to 0, \qquad \text{as } \epsilon \to 0.$$

4. We take $u_\epsilon^* \to u^*$ in $H^1(\Omega)$, as $\epsilon \to 0$. We have,

$$\|F'_\epsilon(u^*_\epsilon) - F'_0(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \leqslant \|F'_\epsilon(u^*_\epsilon) - F'_\epsilon(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} + \|F'_\epsilon(u^*) - F'_0(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))}.$$

Using the item 2, we have that there exists $L > 0$ independent of ϵ such that

$$\|F'_\epsilon(u^*_\epsilon) - F'_\epsilon(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \leqslant L \|u^*_\epsilon - u^*\|_{H^1(\Omega)}.$$

From this and item 3, we get

$$\|F'_\epsilon(u^*_\epsilon) - F'_0(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \leqslant L \|u^*_\epsilon - u^*\|_{H^1(\Omega)} + \|F'_\epsilon(u^*) - F'_0(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \to 0, \qquad \text{as } \epsilon \to 0.$$

5. We take $u_\epsilon^* \to u^*$ in $H^1(\Omega)$, as $\epsilon \to 0$, and $w_\epsilon \to w$ in $H^1(\Omega)$, as $\epsilon \to 0$. Using the items 1 and 4, we get

$$\begin{aligned}
&\|F'_\epsilon(u^*_\epsilon)w_\epsilon - F'_0(u^*)w\|_{H^{-\alpha}(\Omega)} \\
\leqslant\;& \|F'_\epsilon(u^*_\epsilon)w_\epsilon - F'_\epsilon(u^*_\epsilon)w\|_{H^{-\alpha}(\Omega)} + \|F'_\epsilon(u^*_\epsilon)w - F'_0(u^*)w\|_{H^{-\alpha}(\Omega)} \\
\leqslant\;& \|F'_\epsilon(u^*_\epsilon)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \|w_\epsilon - w\|_{H^1(\Omega)} + \|F'_\epsilon(u^*_\epsilon) - F'_0(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \|w\|_{H^1(\Omega)} \\
\leqslant\;& k \|w_\epsilon - w\|_{H^1(\Omega)} + \|F'_\epsilon(u^*_\epsilon) - F'_0(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \|w\|_{H^1(\Omega)} \to 0, \qquad \text{as } \epsilon \to 0. \quad \square
\end{aligned}$$

4 Upper semicontinuity of the set of equilibria

To prove the continuity of the family of equilibria $\{\mathscr{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ of (1.3) and (1.4) at $\epsilon = 0$, we need to prove the upper and lower semicontinuity of this family at $\epsilon = 0$. Initially, we have the following lemma:

Lemma 4.1. *Suppose that (H) holds. Then, for each $\epsilon \in [0, \epsilon_0]$ fixed, the operator $A^{-1}F_\epsilon : H^1(\Omega) \longrightarrow H^1(\Omega)$ is compact. For any bounded family $\{u_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ in $H^1(\Omega)$, the family $\{A^{-1}F_\epsilon(u_\epsilon)\}_{\epsilon\in(0,\epsilon_0]}$ is relatively compact in $H^1(\Omega)$. Moreover, if $u_\epsilon \to u$ in $H^1(\Omega)$, as $\epsilon \to 0$, then*

$$A^{-1}F_\epsilon(u_\epsilon) \to A^{-1}F_0(u) \quad \textit{in} \quad H^1(\Omega), \qquad \textit{as } \epsilon \to 0.$$

Proof. We note that the linear operator $A^{-1} : H^{-\alpha}(\Omega) \longrightarrow H^{2-\alpha}(\Omega)$ is continuous and using the compact embedding of $H^{2-\alpha}(\Omega)$ in $H^1(\Omega)$, with $2 - \alpha > 1$, we get that $A^{-1} : H^{-\alpha}(\Omega) \longrightarrow H^1(\Omega)$ is compact. Now, for each $\epsilon \in [0, \epsilon_0]$ fixed, using the item 1 of Lemma 3.3, we have that if B is bounded set in $H^1(\Omega)$ then $F_\epsilon(B)$ is bounded set in $H^{-\alpha}(\Omega)$. Hence, by compactness of $A^{-1} : H^{-\alpha}(\Omega) \longrightarrow H^1(\Omega)$, we get that $A^{-1}F_\epsilon : H^1(\Omega) \longrightarrow H^1(\Omega)$ is compact.

Let $\{u_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ be a bounded family in $H^1(\Omega)$. Again, from item 1 of Lemma 3.3, $\{F_\epsilon(u_\epsilon)\}_{\epsilon\in(0,\epsilon_0]}$ is a bounded family in $H^{-\alpha}(\Omega)$, uniformly in ϵ. By compactness of $A^{-1} : H^{-\alpha}(\Omega) \longrightarrow H^1(\Omega)$, we have that $\{A^{-1}F_\epsilon(u_\epsilon)\}_{\epsilon\in(0,\epsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$. Therefore, the family $\{A^{-1}F_\epsilon(u_\epsilon)\}_{\epsilon\in(0,\epsilon_0]}$ is relatively compact.

Now, we take $u_\epsilon \to u$ in $H^1(\Omega)$, as $\epsilon \to 0$. Using the item 4 of Lemma 3.3, we have

$$F_\epsilon(u_\epsilon) \to F_0(u) \quad \text{in} \quad H^{-\alpha}(\Omega), \qquad \text{as } \epsilon \to 0.$$

By continuity of operator $A^{-1} : H^{-\alpha}(\Omega) \longrightarrow H^1(\Omega)$, we get

$$A^{-1}F_\epsilon(u_\epsilon) \to A^{-1}F_0(u) \quad \text{in} \quad H^1(\Omega), \qquad \text{as } \epsilon \to 0. \quad \square$$

Before we prove the upper semicontinuity of the family of equilibria $\{\mathscr{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ at $\epsilon = 0$, we will show that each set of equilibria is not empty and it is compact.

Lemma 4.2. *Suppose that (H) holds. Then, for each $\epsilon \in [0, \epsilon_0]$ fixed, the set $\mathscr{E}_\epsilon$ of the solutions of (2.6) and (2.7) is not empty. Moreover, $\mathscr{E}_\epsilon$ is compact in $H^1(\Omega)$.*

Proof. Show that for each $\epsilon \in [0, \epsilon_0]$ fixed, the set $\mathscr{E}_\epsilon$ of the solutions of (2.6) and (2.7) is not empty, that is, that the equations (2.6) and (2.7) have at least one solution in $H^1(\Omega)$, it is equivalent to show that the compact operator $A^{-1}F_\epsilon : H^1(\Omega) \to H^1(\Omega)$ has at least one fixed point.

From item 1 of Lemma 3.3, we have that there exists $k > 0$ independent of ϵ such that

$$\|F_\epsilon(u)\|_{H^{-\alpha}(\Omega)} \leqslant k, \qquad \forall\, u \in H^1(\Omega) \quad \text{and} \quad 0 \leqslant \epsilon \leqslant \epsilon_0.$$

We consider the closed ball $\bar{B}_r(0)$ in $H^1(\Omega)$, where $r = k\|A^{-1}\|_{\mathscr{L}(H^{-\alpha}(\Omega), H^1(\Omega))}$. For each $u \in H^1(\Omega)$, we have

$$\|A^{-1}F_\epsilon(u)\|_{H^1(\Omega)} \leqslant \|A^{-1}\|_{\mathscr{L}(H^{-\alpha}(\Omega), H^1(\Omega))} \|F_\epsilon(u)\|_{H^{-\alpha}(\Omega)} \leqslant r. \tag{4.21}$$

Therefore, the compact operator $A^{-1}F_\epsilon : H^1(\Omega) \to H^1(\Omega)$ takes $H^1(\Omega)$ in the ball $\bar{B}_r(0)$, in particular, $A^{-1}F_\epsilon$ takes $\bar{B}_r(0)$ into itself. From Schauder fixed point Theorem, we obtain that $A^{-1}F_\epsilon$ has at least one fixed point in $H^1(\Omega)$. Thus, the equations (2.6) and (2.7) have at least one solution in $H^1(\Omega)$.

Now, for each $\epsilon \in [0, \epsilon_0]$ fixed, we will prove that $\mathscr{E}_\epsilon$ is compact in $H^1(\Omega)$. For each $\epsilon \in [0, \epsilon_0]$ fixed, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{E}_\epsilon$, then $u_n = A^{-1}F_\epsilon(u_n)$, $\forall\, n \in \mathbb{N}$. Similarly to (4.21) and using again the item 1 of Lemma 3.3, we get that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\Omega)$. By Lemma 4.1, $\{A^{-1}F_\epsilon(u_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence, that we will denote by $\{A^{-1}F_\epsilon(u_{n_k})\}_{k\in\mathbb{N}}$, with limit $u \in H^1(\Omega)$, that is,

$$A^{-1}F_\epsilon(u_{n_k}) \to u \quad \text{in} \quad H^1(\Omega), \qquad \text{as } k \to \infty.$$

Hence,

$$u_{n_k} \to u \quad \text{in} \quad H^1(\Omega), \qquad \text{as } k \to \infty.$$

By continuity of operator $A^{-1}F_\epsilon : H^1(\Omega) \longrightarrow H^1(\Omega)$, we get

$$A^{-1}F_\epsilon(u_{n_k}) \to A^{-1}F_\epsilon(u) \quad \text{in} \quad H^1(\Omega), \qquad \text{as } k \to \infty.$$

By the uniqueness of the limit, $u = A^{-1}F_\epsilon(u)$. Thus, $Au - F_\epsilon(u) = 0$ and $u \in \mathscr{E}_\epsilon$. Therefore, $\mathscr{E}_\epsilon$ is a compact set in $H^1(\Omega)$. $\square$

Theorem 4.3. *Suppose that (H) holds. Then, the family of equilibria $\{\mathscr{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ is upper semicontinuous at $\epsilon = 0$.*

Proof. Let $\{u^*_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ be a family of solutions of (2.6), then $u^*_\epsilon = A^{-1}F_\epsilon(u^*_\epsilon)$, for all $\epsilon \in (0, \epsilon_0]$. Now,

$$\|u^*_\epsilon\|_{H^1(\Omega)} = \|A^{-1}F_\epsilon(u^*_\epsilon)\|_{H^1(\Omega)} \leqslant \|A^{-1}\|_{\mathscr{L}(H^{-\alpha}(\Omega), H^1(\Omega))} \|F_\epsilon(u^*_\epsilon)\|_{H^{-\alpha}(\Omega)}.$$

Since from item 1 of Lemma 3.3, $\{F_\epsilon(u^*_\epsilon)\}_{\epsilon\in(0,\epsilon_0]}$ is a bounded family in $H^{-\alpha}(\Omega)$, uniformly in ϵ, then $\{u^*_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ is uniformly bounded in $H^1(\Omega)$. From Lemma 4.1 follows that the family $\{A^{-1}F_\epsilon(u^*_\epsilon)\}_{\epsilon\in(0,\epsilon_0]}$ is relatively compact in $H^1(\Omega)$. Then, there exist a subsequence, that we will still denote by $\{A^{-1}F_\epsilon(u^*_\epsilon)\}_{\epsilon\in(0,\epsilon_0]}$, and $u^*_0 \in H^1(\Omega)$ such that

$$A^{-1}F_\epsilon(u^*_\epsilon) \to u^*_0 \quad \text{in} \quad H^1(\Omega), \qquad \text{as } \epsilon \to 0.$$

Hence,

$$u^*_\epsilon \to u^*_0 \quad \text{in} \quad H^1(\Omega), \qquad \text{as } \epsilon \to 0.$$

Again by Lemma 4.1, we have

$$A^{-1}F_\epsilon(u^*_\epsilon) \to A^{-1}F_0(u^*_0) \quad \text{in} \quad H^1(\Omega), \qquad \text{as } \epsilon \to 0.$$

By the uniqueness of the limit, $u^*_0 = A^{-1}F_0(u^*_0)$ and u^*_0 is a solution of (2.7). Therefore, the family of equilibria $\{\mathscr{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ is upper semicontinuity at $\epsilon = 0$. $\square$

5 Lower semicontinuity of the set of equilibria

The proof of lower semicontinuity of the family of equilibria $\{\mathscr{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ at $\epsilon = 0$, requeres additional assumptions. We need to assume that the equilibrium points of (1.4) are stable under perturbation. This stability under perturbation can be given by the hyperbolicity.

Definition 5.1. *For each $\epsilon \in [0, \epsilon_0]$, we say that the solution u_ϵ^* of (2.6) and (2.7) is hyperbolic if the spectrum $\sigma(A - F_\epsilon'(u_\epsilon^*))$ is disjoint from the imaginary axis, that is, $\sigma(A - F_\epsilon'(u_\epsilon^*)) \cap \imath\mathbb{R} = \emptyset$.*

Theorem 5.2. *Suppose that (H) holds. If u_0^* is a solution of (2.7), that is, an equilibrium point of (1.4), which satisfies $0 \notin \sigma(A - F_0'(u_0^*))$, then u_0^* is isolated.*

Proof. Since $0 \notin \sigma(A - F_0'(u_0^*))$ then $0 \in \rho(A - F_0'(u_0^*))$. Thus, there exists $C > 0$ such that

$$\left\|(A - F_0'(u_0^*))^{-1}\right\|_{\mathscr{L}(H^{-\alpha}(\Omega), H^1(\Omega))} \leqslant C.$$

Now, we note that u is a solution of (2.7) if and only if

$$\begin{aligned} 0 &= Au - F_0'(u_0^*)u + F_0'(u_0^*)u - F_0(u) \\ &\Leftrightarrow \ (A - F_0'(u_0^*))\, u = F_0(u) - F_0'(u_0^*)u \\ &\Leftrightarrow \ u = (A - F_0'(u_0^*))^{-1} (F_0(u) - F_0'(u_0^*)u)\,. \end{aligned}$$

So, u is a solution of (2.7) if and only if u is a fixed point of the map

$$\begin{aligned} \Phi : H^1(\Omega) &\longrightarrow H^1(\Omega) \\ u &\longmapsto \Phi(u) = (A - F_0'(u_0^*))^{-1} (F_0(u) - F_0'(u_0^*)u)\,. \end{aligned}$$

We will show that there exists $r > 0$ such that $\Phi : \bar{B}_r(u_0^*) \longrightarrow \bar{B}_r(u_0^*)$ is a contraction, where $\bar{B}_r(u_0^*)$ is a closed ball in $H^1(\Omega)$ with center in u_0^* and ray r.

In fact, from Lemma 3.4 we have that there exists $\delta > 0$ such that

$$C\left\|F_0(u) - F_0(v) - F_0'(u_0^*)(u - v)\right\|_{H^{-\alpha}(\Omega)} \leqslant \frac{1}{2}\left\|u - v\right\|_{H^1(\Omega)}, \quad \text{for} \quad \left\|u - v\right\|_{H^1(\Omega)} \leqslant \delta.$$

Taking $r = \frac{\delta}{2}$ and $u,\, v \in \bar{B}_r(u_0^*)$, we have

$$\begin{aligned} \left\|\Phi(u) - \Phi(v)\right\|_{H^1(\Omega)} &= \left\|(A - F_0'(u_0^*))^{-1}\left[F_0(u) - F_0(v) - F_0'(u_0^*)(u - v)\right]\right\|_{H^1(\Omega)} \\ &\leqslant C\left\|F_0(u) - F_0(v) - F_0'(u_0^*)(u - v)\right\|_{H^{-\alpha}(\Omega)} \leqslant \frac{1}{2}\left\|u - v\right\|_{H^1(\Omega)}. \end{aligned}$$

Thus, Φ is a contraction on the $\bar{B}_r(u_0^*)$. Moreover, if $u \in \bar{B}_r(u_0^*)$ then

$$\left\|\Phi(u) - u_0^*\right\|_{H^1(\Omega)} = \left\|\Phi(u) - \Phi(u_0^*)\right\|_{H^1(\Omega)} \leqslant \frac{1}{2}\left\|u - u_0^*\right\|_{H^1(\Omega)} \leqslant \frac{1}{2}r < r.$$

Hence, $\Phi\left(\bar{B}_r(u_0^*)\right) \subset \bar{B}_r(u_0^*)$.

Therefore, from Contraction Theorem, Φ has an unique fixed point in $\bar{B}_r(u_0^*)$. Since u_0^* is a fixed point of Φ, then u_0^* is the unique fixed point of Φ in $\bar{B}_r(u_0^*)$. Thus, u_0^* is isolated. $\square$

Corollary 5.3. *Suppose that (H) holds. If u_0^* is a hyperbolic solution of (2.7), then u_0^* is an isolated equilibrium point.*

Proposition 5.4. *Suppose that (H) holds. If all points in $\mathscr{E}_0$ are isolated, then there is only a finite number of them. Moreover, if $0 \notin \sigma(A - F_0'(u_0^*))$ for each $u_0^* \in \mathscr{E}_0$, then $\mathscr{E}_0$ is a finite set.*

Proof. We suppose that the number of elements in $\mathscr{E}_0$ is infinite, hence there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathscr{E}_0$. From Lemma 4.2, $\mathscr{E}_0$ is compact, thus there exist a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ and $u^* \in \mathscr{E}_0$ such that

$$u_{n_k} \to u^* \quad \text{in} \quad H^1(\Omega), \qquad \text{as } k \to \infty.$$

Thus, for all $\delta > 0$, there exists $k_0 \in \mathbb{N}$ such that $u_{n_k} \in B_\delta(u^*)$, for all $k > k_0$, which is a contradiction with the fact that each fixed point in $\mathscr{E}_0$ is isolated and $u^* \in \mathscr{E}_0$.

Now, if $0 \notin \sigma(A - F_0'(u_0^*))$ for each $u_0^* \in \mathscr{E}_0$, then, by Theorem 5.2, u_0^* is isolated. Thus, $\mathscr{E}_0$ is a finite set. $\square$

To prove the lower semicontinuity of the family of equilibria $\{\mathscr{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ at $\epsilon = 0$, we will need of the following lemmas:

Lemma 5.5. *Suppose that (H) holds and let $u^* \in H^1(\Omega)$. Then, for each $\epsilon \in [0, \epsilon_0]$ fixed, the operator $A^{-1}F_\epsilon'(u^*) : H^1(\Omega) \longrightarrow H^1(\Omega)$ is compact. For any bounded family $\{w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ in $H^1(\Omega)$, the family $\{A^{-1}F_\epsilon'(u^*)w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ is relatively compact in $H^1(\Omega)$. Moreover, if $w_\epsilon \to w$ in $H^1(\Omega)$, as $\epsilon \to 0$, then*

$$A^{-1}F_\epsilon'(u^*)w_\epsilon \to A^{-1}F_0'(u^*)w \quad \textit{in} \quad H^1(\Omega), \qquad \textit{as } \epsilon \to 0.$$

Proof. For each $\epsilon \in [0, \epsilon_0]$ fixed, the compactness of linear operator $A^{-1}F_\epsilon'(u^*) : H^1(\Omega) \longrightarrow H^1(\Omega)$ follows from item 1 of Lemma 3.5 and of compactness of linear operator $A^{-1} : H^{-\alpha}(\Omega) \longrightarrow H^1(\Omega)$.

Let $\{w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ be a bounded family in $H^1(\Omega)$. Since

$$\|F_\epsilon'(u^*)w_\epsilon\|_{H^{-\alpha}(\Omega)} \leqslant \|F_\epsilon'(u^*)\|_{\mathscr{L}(H^1(\Omega),H^{-\alpha}(\Omega))} \|w_\epsilon\|_{H^1(\Omega)}, \qquad \forall\, \epsilon \in (0, \epsilon_0],$$

and from item 1 of Lemma 3.5, $\{F_\epsilon'(u^*)\}_{\epsilon\in(0,\epsilon_0]}$ is a bounded family in $\mathscr{L}(H^1(\Omega), H^{-\alpha}(\Omega))$, uniformly in ϵ, then $\{F_\epsilon'(u^*)w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ is a bounded family in $H^{-\alpha}(\Omega)$. By compactness of $A^{-1} : H^{-\alpha}(\Omega) \longrightarrow H^1(\Omega)$, we have that $\{A^{-1}F_\epsilon'(u^*)w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$. Therefore, the family $\{A^{-1}F_\epsilon'(u^*)w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ is relatively compact.

Now, let us take $w_\epsilon \to w$ in $H^1(\Omega)$, as $\epsilon \to 0$. Thus, from item 5 of Lemma 3.5,

$$F_\epsilon'(u^*)w_\epsilon \to F_0'(u^*)w \quad \text{in} \quad H^{-\alpha}(\Omega), \qquad \text{as } \epsilon \to 0.$$

By continuity of operator $A^{-1} : H^{-\alpha}(\Omega) \longrightarrow H^1(\Omega)$, we get

$$A^{-1}F_\epsilon'(u^*)w_\epsilon \to A^{-1}F_0'(u^*)w \quad \text{in} \quad H^1(\Omega), \qquad \text{as } \epsilon \to 0. \quad \square$$

Lemma 5.6. *Suppose that (H) holds and let $u^* \in H^1(\Omega)$ such that $0 \notin \sigma(A - F_0'(u^*))$. Then, there exist $\epsilon_0 > 0$ and $C > 0$ independent of ϵ such that $0 \notin \sigma(A - F_\epsilon'(u^*))$ and*

$$\left\|(A - F_\epsilon'(u^*))^{-1}\right\|_{\mathscr{L}(H^{-\alpha}(\Omega),H^1(\Omega))} \leqslant C, \qquad \forall\, \epsilon \in [0, \epsilon_0]. \tag{5.22}$$

Furthermore, for each $\epsilon \in [0, \epsilon_0]$ fixed, the operator $(A - F_\epsilon'(u^*))^{-1} : H^{-\alpha}(\Omega) \longrightarrow H^1(\Omega)$ is compact. For any bounded family $\{w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ in $H^{-\alpha}(\Omega)$, the family $\{(A - F_\epsilon'(u^*))^{-1} w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ is relatively compact in $H^1(\Omega)$. Moreover, if $w_\epsilon \to w$ in $H^{-\alpha}(\Omega)$, as $\epsilon \to 0$, then

$$(A - F_\epsilon'(u^*))^{-1} w_\epsilon \to (A - F_0'(u^*))^{-1} w \quad \textit{in} \quad H^1(\Omega), \qquad \textit{as } \epsilon \to 0.$$

Proof. Initially, we note that

$$(A - F_\epsilon'(u^*))^{-1} = \left[A(I - A^{-1}F_\epsilon'(u^*))\right]^{-1} = (I - A^{-1}F_\epsilon'(u^*))^{-1}A^{-1}, \qquad \epsilon \in [0, \epsilon_0].$$

Then, prove that $0 \notin \sigma(A - F'_\epsilon(u^*))$ it is equivalent to prove that $1 \in \rho(A^{-1}F'_\epsilon(u^*))$. Moreover, to prove that there exist $\epsilon_0 > 0$ and $C > 0$ independent of ϵ such that (5.22) holds, it is enough to prove that there exist $\epsilon_0 > 0$ and $M > 0$ independent of ϵ such that

$$\|(I - A^{-1}F'_\epsilon(u^*))^{-1}\|_{\mathscr{L}(H^1(\Omega))} \leqslant M, \qquad \forall\, \epsilon \in [0, \epsilon_0]. \tag{5.23}$$

In fact, we note that

$$\begin{aligned} \|(A - F'_\epsilon(u^*))^{-1}\|_{\mathscr{L}(H^{-\alpha}(\Omega), H^1(\Omega))} &\leqslant \|(I - A^{-1}F'_\epsilon(u^*))^{-1}\|_{\mathscr{L}(H^1(\Omega))}\|A^{-1}\|_{\mathscr{L}(H^{-\alpha}(\Omega), H^1(\Omega))} \\ &\leqslant M\|A^{-1}\|_{\mathscr{L}(H^{-\alpha}(\Omega), H^1(\Omega))} = C, \qquad \forall\, \epsilon \in [0, \epsilon_0], \end{aligned}$$

where $C > 0$ does not depend of ϵ.

Then we will show (5.23). Initially, from hypothesis $0 \notin \sigma(A - F'_0(u^*))$, then $1 \in \rho(A^{-1}F'_0(u^*))$. Hence, there exists the inverse $(I - A^{-1}F'_0(u^*))^{-1} : H^1(\Omega) \longrightarrow H^1(\Omega)$ and, in particular, the kernel $\mathcal{N}(I - A^{-1}F'_0(u^*)) = \{0\}$.

Now, let $B_\epsilon = A^{-1}F'_\epsilon(u^*)$, for all $\epsilon \in [0, \epsilon_0]$. From Lemma 5.5 we have that, for each $\epsilon \in [0, \epsilon_0]$ fixed, the operator $B_\epsilon : H^1(\Omega) \longrightarrow H^1(\Omega)$ is compact. Using the compactness of B_ϵ, we can show that estimate (5.23) is equivalent to say

$$\|(I - B_\epsilon)u_\epsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M}, \qquad \forall\, \epsilon \in [0, \epsilon_0] \quad \text{and} \quad \forall\, u_\epsilon \in H^1(\Omega) \quad \text{with} \quad \|u_\epsilon\|_{H^1(\Omega)} = 1. \tag{5.24}$$

In fact, suppose that (5.23) holds, then there exists the inverse $(I - B_\epsilon)^{-1} : H^1(\Omega) \longrightarrow H^1(\Omega)$ and it is continuous. Moreover,

$$\left\|(I - B_\epsilon)^{-1}v_\epsilon\right\|_{H^1(\Omega)} \leqslant M \|v_\epsilon\|_{H^1(\Omega)}, \qquad \forall\, \epsilon \in [0, \epsilon_0] \quad \text{and} \quad \forall\, v_\epsilon \in H^1(\Omega).$$

Let $u_\epsilon \in H^1(\Omega)$ such that $\|u_\epsilon\|_{H^1(\Omega)} = 1$ and taking $v_\epsilon = (I - B_\epsilon)u_\epsilon$, we have

$$\left\|(I - B_\epsilon)^{-1}(I - B_\epsilon)u_\epsilon\right\|_{H^1(\Omega)} \leqslant M \|(I - B_\epsilon)u_\epsilon\|_{H^1(\Omega)} \Rightarrow 1 = \|u_\epsilon\|_{H^1(\Omega)} \leqslant M \|(I - B_\epsilon)u_\epsilon\|_{H^1(\Omega)}.$$

$$\Rightarrow \|(I - B_\epsilon)u_\epsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M}.$$

Therefore, (5.24) holds. Reversely, suppose that (5.24) holds. We want to prove that there exists the inverse $(I - B_\epsilon)^{-1} : H^1(\Omega) \longrightarrow H^1(\Omega)$, it is continuous and satisfies (5.23). For this, we will prove the following estimative

$$\|(I - B_\epsilon)u_\epsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M} \|u_\epsilon\|_{H^1(\Omega)}, \qquad \forall\, \epsilon \in [0, \epsilon_0] \quad \text{and} \quad \forall\, u_\epsilon \in H^1(\Omega). \tag{5.25}$$

We note that (5.25) is immediate for $u_\epsilon = 0$. Let $u_\epsilon \in H^1(\Omega)$, $u_\epsilon \neq 0$, and we take $v_\epsilon = \dfrac{u_\epsilon}{\|u_\epsilon\|_{H^1(\Omega)}}$. Thus, $\|v_\epsilon\|_{H^1(\Omega)} = 1$ and using (5.24), we get

$$\|(I - B_\epsilon)v_\epsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M} \Rightarrow \left\|(I - B_\epsilon)\frac{u_\epsilon}{\|u_\epsilon\|_{H^1(\Omega)}}\right\|_{H^1(\Omega)} \geqslant \frac{1}{M} \Rightarrow \frac{1}{\|u_\epsilon\|_{H^1(\Omega)}} \|(I - B_\epsilon)u_\epsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M}.$$

$$\Rightarrow \|(I - B_\epsilon)u_\epsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M} \|u_\epsilon\|_{H^1(\Omega)}.$$

Now, let $u_\epsilon \in H^1(\Omega)$ such that $(I - B_\epsilon)u_\epsilon = 0$. From (5.25) follows $u_\epsilon = 0$. Thus, for each $\epsilon \in [0, \epsilon_0]$, $\mathcal{N}(I - B_\epsilon) = \{0\}$ and the operator $I - B_\epsilon$ is injective. So there exists the inverse $(I - B_\epsilon)^{-1} : \mathcal{R}(I - B_\epsilon) \longrightarrow H^1(\Omega)$, where $\mathcal{R}(I - B_\epsilon)$ denotes the image of the operator $I - B_\epsilon$.

Since B_ϵ is compact, for all $\epsilon \in [0, \epsilon_0]$, then by Fredholm Alternative Theorem, we have

$$\mathcal{N}(I - B_\epsilon) = \{0\} \Leftrightarrow \mathcal{R}(I - B_\epsilon) = H^1(\Omega).$$

Hence, $\mathcal{R}(I-B_\epsilon)=H^1(\Omega)$ and $I-B_\epsilon$ is bijective, thus there exists the inverse $(I-B_\epsilon)^{-1}:H^1(\Omega)\longrightarrow H^1(\Omega)$.

Now, taking $v_\epsilon\in H^1(\Omega)$ we have that there exists $u_\epsilon\in H^1(\Omega)$ such that $(I-B_\epsilon)u_\epsilon=v_\epsilon$ and $u_\epsilon=(I-B_\epsilon)^{-1}v_\epsilon$. From (5.25) we have

$$\left\|(I-B_\epsilon)^{-1}v_\epsilon\right\|_{H^1(\Omega)}=\|u_\epsilon\|_{H^1(\Omega)}\leqslant M\,\|(I-B_\epsilon)u_\epsilon\|_{H^1(\Omega)}=M\,\|v_\epsilon\|_{H^1(\Omega)},\qquad \forall\ \epsilon\in[0,\epsilon_0]\quad\text{and}\quad\forall\ v_\epsilon\in H^1(\Omega).$$

$$\Rightarrow\left\|(I-B_\epsilon)^{-1}\right\|_{\mathscr{L}(H^1(\Omega))}\leqslant M,\qquad\forall\ \epsilon\in[0,\epsilon_0].$$

Therefore, (5.23) holds.

Since (5.23) and (5.24) are equivalents, then it is enough to show (5.24). Suppose that (5.24) is not true, that is, there exist a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H^1(\Omega)$, with $\|u_n\|_{H^1(\Omega)}=1$ and $\epsilon_n\to0$, as $n\to\infty$, such that

$$\|(I-B_{\epsilon_n})u_n\|_{H^1(\Omega)}\to0,\qquad\text{as } n\to\infty.$$

From Lemma 5.5 we get that $\{B_{\epsilon_n}u_n\}_{n\in\mathbb{N}}$ is relatively compact. Thus, $\{B_{\epsilon_n}u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, which we again denote by $\{B_{\epsilon_n}u_n\}_{n\in\mathbb{N}}$, with limit $u\in H^1(\Omega)$, that is,

$$B_{\epsilon_n}u_n\to u\quad\text{in}\quad H^1(\Omega),\qquad\text{as } n\to\infty.$$

Since $u_n-B_{\epsilon_n}u_n\to0$ in $H^1(\Omega)$, as $n\to\infty$, then $u_n\to u$ in $H^1(\Omega)$, as $n\to\infty$. Hence, $\|u\|_{H^1(\Omega)}=1$. Moreover, since $u_n\to u$ in $H^1(\Omega)$, as $n\to\infty$, then using the Lemma 5.5, we have $B_{\epsilon_n}u_n\to B_0u$ in $H^1(\Omega)$, as $n\to\infty$. Thus,

$$u_n-B_{\epsilon_n}u_n\to u-B_0u\quad\text{in}\quad H^1(\Omega),\qquad\text{as } n\to\infty.$$

By the uniqueness of the limit, $u-B_0u=0$. This implies that $(I-B_0)u=0$, with $u\neq0$, contradicting our hypothesis. Therefore, (5.24) holds.

With this we conclude that there exist $\epsilon_0>0$ and $C>0$ independent of ϵ such that (5.22) holds.

Now, for each $\epsilon\in[0,\epsilon_0]$, the operator $(A-F'_\epsilon(u^*))^{-1}$ is compact and the prove of this compactness follows similarly to account below.

Let $\{w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ a bounded family in $H^{-\alpha}(\Omega)$. For each $\epsilon\in(0,\epsilon_0]$, let

$$v_\epsilon=(A-F'_\epsilon(u^*))^{-1}\,w_\epsilon.$$

From (5.22) we have

$$\|v_\epsilon\|_{H^1(\Omega)}\leqslant\left\|(A-F'_\epsilon(u^*))^{-1}w_\epsilon\right\|_{H^1(\Omega)}\leqslant\left\|(A-F'_\epsilon(u^*))^{-1}\right\|_{\mathscr{L}(H^{-\alpha}(\Omega),H^1(\Omega))}\|w_\epsilon\|_{H^{-\alpha}(\Omega)}\leqslant C\,\|w_\epsilon\|_{H^{-\alpha}(\Omega)}.$$

Hence, $\{v_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ is a bounded family in $H^1(\Omega)$. Moreover,

$$v_\epsilon=(A-F'_\epsilon(u^*))^{-1}\,w_\epsilon=(I-A^{-1}F'_\epsilon(u^*))^{-1}A^{-1}w_\epsilon\Leftrightarrow(I-A^{-1}F'_\epsilon(u^*))v_\epsilon=A^{-1}w_\epsilon\Leftrightarrow v_\epsilon=A^{-1}F'_\epsilon(u^*)v_\epsilon+A^{-1}w_\epsilon.$$

By compactness of $A^{-1}:H^{-\alpha}(\Omega)\longrightarrow H^1(\Omega)$, we get that $\{A^{-1}w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$. Moreover, using the Lemma 5.5, we have that $\{A^{-1}F'_\epsilon(u^*)v_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ is relatively compact in $H^1(\Omega)$, then $\{A^{-1}F'_\epsilon(u^*)v_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$. Therefore, $\{v_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$, that is, the family $\{(A-F'_\epsilon(u^*))^{-1}\,w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$, thus it is relatively compact in $H^1(\Omega)$.

Now, we take $w_\epsilon\to w$ in $H^{-\alpha}(\Omega)$, as $\epsilon\to0$. By continuity of operator $A^{-1}:H^{-\alpha}(\Omega)\longrightarrow H^1(\Omega)$, we have

$$A^{-1}w_\epsilon\to A^{-1}w\quad\text{in}\quad H^1(\Omega),\qquad\text{as }\epsilon\to0.$$

Moreover, $\{w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ is bounded in $H^{-\alpha}(\Omega)$, for some $\epsilon_0>0$ sufficiently small, and we have that from the above that $\{v_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$, with $\epsilon_0>0$ sufficiently small, has a convergent subsequence, which we again denote by $\{v_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$, with limit $v\in H^1(\Omega)$, that is,

$$v_\epsilon\to v\quad\text{in}\quad H^1(\Omega)\qquad\text{as }\epsilon\to0.$$

From Lemma 5.5 we get

$$A^{-1}F'_\epsilon(u^*)v_\epsilon \to A^{-1}F'_0(u^*)v \quad \text{in} \quad H^1(\Omega), \qquad \text{as } \epsilon \to 0.$$

Thus, v satisfies $v = A^{-1}F'_0(u^*)v + A^{-1}w$, and so

$$v = (A - F'_0(u^*))^{-1}w.$$

Therefore,

$$(A - F'_\epsilon(u^*))^{-1}w_\epsilon \to (A - F'_0(u^*))^{-1}w \quad \text{in} \quad H^1(\Omega), \qquad \text{as } \epsilon \to 0.$$

The limit above is independent of the subsequence, thus whole family $\{(A - F'_\epsilon(u^*))^{-1}w_\epsilon\}_{\epsilon\in(0,\epsilon_0]}$ converges to $(A - F'_0(u^*))^{-1}w$ in $H^1(\Omega)$, as $\epsilon \to 0$. $\square$

Theorem 5.7. *Suppose that (H) holds and that u_0^* is a solution of (2.7) which satisfies $0 \notin \sigma(A - F'_0(u_0^*))$. Then, there exist $\epsilon_0 > 0$ and $\delta > 0$ such that, for each $0 < \epsilon \leqslant \epsilon_0$, the equation (2.6) has exactly one solution, u_ϵ^*, in*

$$\{v_\epsilon \in H^1(\Omega) : \quad \|v_\epsilon - u_0^*\|_{H^1(\Omega)} \leqslant \delta\}.$$

Furthermore,

$$u_\epsilon^* \to u_0^* \quad \textit{in} \quad H^1(\Omega), \qquad \textit{as } \epsilon \to 0.$$

In particular, the family of equilibria $\{\mathscr{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ is lower semicontinuous at $\epsilon = 0$.

Proof. Initially, using the Lemma 5.6, we have that there exist $\epsilon_0 > 0$ and $C > 0$, independent of ϵ, such that $0 \notin \sigma(A - F'_\epsilon(u_0^*))$ and

$$\big\|(A - F'_\epsilon(u_0^*))^{-1}\big\|_{\mathscr{L}(H^{-\alpha}(\Omega),H^1(\Omega))} \leqslant C, \qquad \forall\, \epsilon \in (0, \epsilon_0]. \tag{5.26}$$

By Lemma 3.4, there exists $\tilde{\delta} = \tilde{\delta}(C) > 0$ independent of ϵ such that

$$C\,\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon) - F'_\epsilon(u_0^*)(u_\epsilon - v_\epsilon)\|_{H^{-\alpha}(\Omega)} \leqslant \frac{1}{2}\,\|u_\epsilon - v_\epsilon\|_{H^1(\Omega)}\,, \qquad \forall\, \epsilon \in (0, \epsilon_0], \tag{5.27}$$

for $\|u_\epsilon - v_\epsilon\| \leqslant \tilde{\delta}$.

We note that u_ϵ, $0 < \epsilon \leqslant \epsilon_0$, is a solution of (2.6) if and only if u_ϵ is a fixed point of the map

$$\begin{aligned} \Phi_\epsilon : H^1(\Omega) &\longrightarrow H^1(\Omega) \\ u_\epsilon &\longmapsto \Phi_\epsilon(u_\epsilon) = (A - F'_\epsilon(u_0^*))^{-1}\left(F_\epsilon(u_\epsilon) - F'_\epsilon(u_0^*)u_\epsilon\right). \end{aligned}$$

Initially, we affirm that

$$\Phi_\epsilon(u_0^*) \to u_0^* \quad \text{in} \quad H^1(\Omega), \qquad \text{as } \epsilon \to 0. \tag{5.28}$$

In fact, using (5.26), for $0 < \epsilon \leqslant \epsilon_0$, we have

$$\begin{aligned} &\|\Phi_\epsilon(u_0^*) - u_0^*\|_{H^1(\Omega)} \\ \leqslant\;& \big\|(A - F'_\epsilon(u_0^*))^{-1}\left[(F_\epsilon(u_0^*) - F'_\epsilon(u_0^*)u_0^*) - (F_0(u_0^*) - F'_0(u_0^*)u_0^*)\right]\big\|_{H^1(\Omega)} \\ +\;& \big\|\left[(A - F'_\epsilon(u_0^*))^{-1} - (A - F'_0(u_0^*))^{-1}\right](F_0(u_0^*) - F'_0(u_0^*)u_0^*)\big\|_{H^1(\Omega)} \\ \leqslant\;& C\left(\|F_\epsilon(u_0^*) - F_0(u_0^*)\|_{H^{-\alpha}(\Omega)} + \|F'_\epsilon(u_0^*)u_0^* - F'_0(u_0^*)u_0^*\|_{H^{-\alpha}(\Omega)}\right) \\ +\;& \big\|\left[(A - F'_\epsilon(u_0^*))^{-1} - (A - F'_0(u_0^*))^{-1}\right](F_0(u_0^*) - F'_0(u_0^*)u_0^*)\big\|_{H^1(\Omega)} \to 0, \qquad \text{as } \epsilon \to 0. \end{aligned}$$

This follows from item 3 of Lemma 3.3, item 5 of Lemma 3.5 and of Lemma 5.6.

Next, we show that, for $0 < \epsilon \leqslant \epsilon_0$, for some $\epsilon_0 > 0$ sufficiently small, Φ_ϵ is a contraction map from the closed ball $\bar{B}_\delta(u_0^*) = \{v_\epsilon \in H^1(\Omega) : \quad \|v_\epsilon - u_0^*\|_{H^1(\Omega)} \leqslant \delta\}$ into itself, where $\delta = \frac{\tilde{\delta}}{2}$. First, we show that Φ_ϵ is a contraction on the $\bar{B}_\delta(u_0^*)$ (uniformly in ϵ). Let u_ϵ, $v_\epsilon \in \bar{B}_\delta(u_0^*)$ and using (5.26) and (5.27), for $0 < \epsilon \leqslant \epsilon_0$, we have

$$\begin{aligned}
\|\Phi_\epsilon(u_\epsilon) - \Phi_\epsilon(v_\epsilon)\|_{H^1(\Omega)} &= \left\|(A - F_\epsilon'(u_0^*))^{-1}\left[F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon) - F_\epsilon'(u_0^*)(u_\epsilon - v_\epsilon)\right]\right\|_{H^1(\Omega)} \\
&\leqslant C\,\|F_\epsilon(u_\epsilon) - F_\epsilon(v_\epsilon) - F_\epsilon'(u_0^*)(u_\epsilon - v_\epsilon)\|_{H^{-\alpha}(\Omega)} \\
&\leqslant \frac{1}{2}\,\|u_\epsilon - v_\epsilon\|_{H^1(\Omega)}, \qquad \text{for } \epsilon \in (0, \epsilon_0].
\end{aligned}$$

To show that Φ_ϵ maps $\bar{B}_\delta(u_0^*)$ into itself, we observe that if $u_\epsilon \in \bar{B}_\delta(u_0^*)$, then

$$\begin{aligned}
\|\Phi_\epsilon(u_\epsilon) - u_0^*\|_{H^1(\Omega)} &\leqslant \|\Phi_\epsilon(u_\epsilon) - \Phi_\epsilon(u_0^*)\|_{H^1(\Omega)} + \|\Phi_\epsilon(u_0^*) - u_0^*\|_{H^1(\Omega)} \\
&\leqslant \frac{\delta}{2} + \|\Phi_\epsilon(u_0^*) - u_0^*\|_{H^1(\Omega)}, \qquad \text{for } \epsilon \in (0, \epsilon_0].
\end{aligned}$$

By convergence in (5.28), we have that there exists $\epsilon_0 > 0$ such that

$$\|\Phi_\epsilon(u_\epsilon) - u_0^*\|_{H^1(\Omega)} \leqslant \frac{\delta}{2} + \frac{\delta}{2} = \delta, \qquad \text{for } \epsilon \in (0, \epsilon_0].$$

Hence, $\Phi_\epsilon : \bar{B}_\delta(u_0^*) \longrightarrow \bar{B}_\delta(u_0^*)$ is a contraction for all $0 < \epsilon \leqslant \epsilon_0$. By Contraction Theorem follows that, for each $0 < \epsilon \leqslant \epsilon_0$, Φ_ϵ has an unique fixed point, u_ϵ^*, in the $\bar{B}_\delta(u_0^*)$.

To show that $u_\epsilon^* \to u_0^*$ in $H^1(\Omega)$, as $\epsilon \to 0$, we proceed in the following manner: since Φ_ϵ is a contraction map from $\bar{B}_\delta(u_0^*)$ into itself, then

$$\begin{aligned}
\|u_\epsilon^* - u_0^*\|_{H^1(\Omega)} &= \|\Phi_\epsilon(u_\epsilon^*) - u_0^*\|_{H^1(\Omega)} \leqslant \|\Phi_\epsilon(u_\epsilon^*) - \Phi_\epsilon(u_0^*)\|_{H^1(\Omega)} + \|\Phi_\epsilon(u_0^*) - u_0^*\|_{H^1(\Omega)} \\
&\leqslant \frac{1}{2}\,\|u_\epsilon^* - u_0^*\|_{H^1(\Omega)} + \|\Phi_\epsilon(u_0^*) - u_0^*\|_{H^1(\Omega)}.
\end{aligned}$$

Thus, using (5.28),

$$\|u_\epsilon^* - u_0^*\|_{H^1(\Omega)} \leqslant 2\,\|\Phi_\epsilon(u_0^*) - u_0^*\|_{H^1(\Omega)} \to 0, \qquad \text{as } \epsilon \to 0.$$

Hence and by compactness of $\mathscr{E}_0$ (Lemma 4.2), we have that the family of equilibria $\{\mathscr{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ is lower semicontinuity at $\epsilon = 0$. $\square$

The Theorems 4.3 and 5.7 show the continuity of the set of equilibria $\mathscr{E}_\epsilon$, $\epsilon \in [0, \epsilon_0]$, at $\epsilon = 0$, in the following sense: if $\{u_\epsilon^*\}_{\epsilon\in(0,\epsilon_0]}$ is a family of solutions of equation (2.6), then we can get a subsequence, that we will also denote by $\{u_\epsilon^*\}_{\epsilon\in(0,\epsilon_0]}$, such that $u_\epsilon^* \to u_0^*$ in $H^1(\Omega)$, as $\epsilon \to 0$, with u_0^* a solution of the limit equation (2.7) and vice versa, if u_0^* is a solution of the limit equation (2.7), which satisfies $0 \notin \sigma(A - F_0'(u_0^*))$, then there exists a family $\{u_\epsilon^*\}_{\epsilon\in(0,\epsilon_0]}$ of solutions of (2.6), with $\epsilon_0 > 0$ suficiently small, such that $u_\epsilon^* \to u_0^*$ in $H^1(\Omega)$, as $\epsilon \to 0$. Moreover, the Theorem 5.7 shows that if u_0^* is a solution of the equation (2.7), which satisfies $0 \notin \sigma(A - F_0'(u_0^*))$, then, for each $0 < \epsilon \leqslant \epsilon_0$, with ϵ_0 suficiently small, there exists an unique solution u_ϵ^* of (2.6) in a neighborhood of u_0^*.

Corollary 5.8. *Suppose that (H) holds and that u_0^* is a hyperbolic solution of (2.7). Then, there exist $\epsilon_0 > 0$ and $\delta > 0$ such that, for each $0 < \epsilon \leqslant \epsilon_0$, the equation (2.6) has exactly one solution, u_ϵ^*, in*

$$\{v_\epsilon \in H^1(\Omega) : \quad \|v_\epsilon - u_0^*\|_{H^1(\Omega)} \leqslant \delta\}.$$

Furthermore,

$$u_\epsilon^* \to u_0^* \quad \textit{in} \quad H^1(\Omega), \qquad \textit{as } \epsilon \to 0.$$

Theorem 5.9. *Suppose that (H) holds. If all solutions u_0^* of (2.7) satisfy $0 \notin \sigma(A - F_0'(u_0^*))$, then (2.7) has a finite number m of solutions, $u_{0,1}^*, ..., u_{0,m}^*$, and there exists $\epsilon_0 > 0$ such that, for each $0 < \epsilon \leqslant \epsilon_0$, the equation (2.6) has exactly m solutions, $u_{\epsilon,1}^*, ..., u_{\epsilon,m}^*$. Moreover, for all $i = 1, ..., m$,*

$$u_{\epsilon,i}^* \to u_{0,i}^* \quad in \quad H^1(\Omega), \qquad as\ \epsilon \to 0.$$

Proof. The proof follows of Proposition 5.4 and Theorem 5.7. $\square$

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