Problem 1: Please prove that the sets $E = \{\text{even integers}\}$, $\mathbb{N}$, and $\mathbb{Z}$ are infinite sets and that they have the same cardinality.

Claim 1: The sets E, $\mathbb{N}$, and $\mathbb{Z}$ are infinite sets that have the same cardinality.

Proof 1: The claim is composed of several atomic statements: The first three claim that $E, \mathbb{N}$, and $\mathbb{Z}$ are infinite sets and the fourth claims that they have the same cardinality. These statements will be verified independently.

Subclaim 1-1a: The set E is an infinite set.

Subproof 1-1a: The set E is an infinite set if there exists a bijection from a proper subset of E to E. Let the set B be a subset of E such that $B = \{b \in E : 4\backslash b\}$. Additionally, let $f : E \to B$ such that $f(e) = 2 \cdot e \ \forall e \in E$. In this proof, we will show that the function f is a bijection between E and B, and thus that E is infinite.

For a function to be bijective, it must be both surjective and injective. For the function f to be surjective, $\forall b \in B, \exists e \in E \ni f(e) = b$. If $b \in B$, then we know that b is divisible by 4 such that $\exists c \in \mathbb{Z} \ni 4 \cdot c = b$ but 4 is also divisible by 2, implying that $2(2 \cdot c) = b$. Note that $2 \cdot c \in E$, which implies that $\exists e \in E \ \ni 2 \cdot e = b$, and hence that $\exists e \in E \ \ni f(e) = b$, making f surjective.

For a function to be injective, $f(a) = f(b)$ must imply that $a = b$. Let $e_1, e_2 \in E$. This implies that $f(e_1) = f(e_2) \Rightarrow 2e_1 = 2e_2$ and consequently that $2e_1 - 2e_2 = 0$ and $2(e_1 - e_2) = 0$. But then $-e_2$ must be the additive inverse of e_1, making $e_1 = e_2$. Therefore, $f(a) = f(b) \Rightarrow a = b$, making f injective. Because f is both injective and surjective, it is also bijective. Therefore, because B is a proper subset of E and f a bijection between B and E the cardinality of set E is infinite.

Subclaim 1-1b: The set $\mathbb{N}$ is an infinite set.

Subproof 1-1b: The set $\mathbb{N}$ is an infinite set if there exists a bijection from a proper subset of $\mathbb{N}$ to $\mathbb{N}$. Let the set B be a subset of $\mathbb{N}$ such that $B = \{b \in \mathbb{N} : 2\backslash b\}$. Additionally, let $f : \mathbb{N} \to B$ such that $f(n) = 2 \cdot n \ \forall n \in \mathbb{N}$. In this proof, we will show that the function f is a bijection between $\mathbb{N}$ and B, and thus that the cardinality of $\mathbb{N}$ is infinite.

For a function to be bijective, it must be both surjective and injective. For the function f to be surjective, $\forall b \in B, \exists n \in \mathbb{N} \ni f(n) = b$. If $b \in B$, then we know that b is divisible by 2 such that $\exists c \in \mathbb{Z} \ni 2 \cdot c = b$. Note that $c \in \mathbb{N}$, implying that $\exists n \in \mathbb{N} \ \ni 2 \cdot n = b$, and hence that $\exists n \in \mathbb{N} \ \ni f(n) = b$, making f surjective.

For a function to be injective, $f(a) = f(b)$ must imply that $a = b$. Let $n_1, n_2 \in \mathbb{N}$. This implies that $f(n_1) = f(n_2) \Rightarrow 2n_1 = 2n_2$ and consequently that $2n_1 - 2n_2 = 0$ and $2(n_1 - n_2) = 0$. But then $-n_2$ must be the additive inverse of n_1, making $n_1 = n_2$. Therefore, $f(a) = f(b) \Rightarrow a = b$, making f injective. Because f is both injective and surjective, it is also bijective. Therefore, because B is a proper subset of $\mathbb{N}$ and f is a bijection between B and $\mathbb{N}$, the cardinality of $\mathbb{N}$ is infinite.

Subclaim 1-1c: The set $\mathbb{Z}$ is an infinite set.

Subproof 1-1c: The set $\mathbb{Z}$ is an infinite set if there exists a bijection from a proper subset of $\mathbb{Z}$ to $\mathbb{Z}$. Let the set B be a subset of $\mathbb{Z}$ such that $B = \{b \in \mathbb{Z} : 2\backslash b\}$. Additionally, let $f : \mathbb{Z} \to B$ such that $f(z) = 2 \cdot z \ \forall z \in \mathbb{Z}$. In this proof, we will show that the function f is a bijection between $\mathbb{Z}$ and B, and thus that $\mathbb{Z}$ is infinite.

For a function to be bijective, it must be both surjective and injective. For the function f to be surjective, $\forall b \in B, \exists z \in \mathbb{Z} \ni f(z) = b$. If $b \in B$, then we know that b is divisible by 2 such that $\exists c \in \mathbb{Z} \ni 2 \cdot c = b$, implying that $\exists n \in \mathbb{N} \ \ni f(n) = b$, making f surjective.

For a function to be injective, $f(a) = f(b)$ must imply that $a = b$. Let $z_1, z_2 \in \mathbb{Z}$. This implies that $f(z_1) = f(z_2) \Rightarrow 2z_1 = 2z_2$ and consequently that $2z_1 - 2z_2 = 0$ and $2(z_1 - z_2) = 0$. But then $-z_2$ must be the additive inverse of z_1, making $z_1 = z_2$. Therefore, $f(a) = f(b) \Rightarrow a = b$, making f injective. Because f is both injective and surjective, it is also bijective. Therefore, because B is a subset of $\mathbb{Z}$, and f is a bijection between B and $\mathbb{Z}$, the cardinality of $\mathbb{Z}$ is infinite.

Lemma 1-2: Let $A, B, C \in \{\text{Sets}\}$. Before we can prove that the sets E, $\mathbb{N}$, and $\mathbb{Z}$ have the same cardinality, we must first prove that if the cardinality of set A is equal to the cardinality of B, and the cardinality of B is equal to the cardinality of C, then the cardinality of A is equal to the cardinality of C.

Because the cardinality of A is equal to the cardinality of B, there is a bijection between A and B such that

$f : A \xrightarrow[\text{onto}]{1:1} B$. By the same logic, there is also a bijection between B and C such that $g : B \xrightarrow[\text{onto}]{1:1} C$. Define a separate function $h : A \to C$ where $h(a) = g(f(a))$. We will show that h is a bijection from A to C and in doing so, that the cardinality of A and the cardinality of C are equal.

First, we show that h is surjective, or that $\forall a \in A,\ \exists c \in C \ni h(a) = c$. We know that $\forall b \in B,\ \exists a \in A \ni f(a) = b$ because f is a bijection between A and B. We also know that $\forall c \in C,\ \exists b \in B \ni g(b) = c$. Therefore, $\forall c \in C,\ \exists a \in A \ni g(f(a)) = c$, implying that $\forall c \in C,\ \exists a \in A \ni h(a) = c$ and making h surjective.

Next we show that h is injective, or that $h(a_1) = h(a_2) \Rightarrow a_1 = a_2 \forall a_1, a_2 \in A$ or, by its contrapositive, that $a_1 \neq a_2 \Rightarrow h(a_1) \neq h(a_2)$. Let $a_1 \neq a_2$, which implies that $f(a_1) \neq f(a_2)$ and finally that $g(f(a_1)) \neq g(f(a_2))$. The function h is defined as $h(a) = g(f(a))$, and therefore, $a_1 \ni a_2 \Rightarrow h(a_1) \neq h(a_2)$, making h injective. We have shown that h is both injective and surjective and therefore is bijective. The function h is a bijection between A and C which implies that the cardinality of A is the equal to the cardinality of C.

Subclaim 1-2: The sets E, $\mathbb{N}$, and $\mathbb{Z}$ have the same cardinality

Subproof 1-2: Two sets are said to have the same cardinality if there exists a bijection from one set to the other. We proved in Subproof 1-1c that there exists a bijection from E to $\mathbb{Z}$, which implies that the cardinality of E is equal to the cardinality of $\mathbb{Z}$. We will now prove that there exists a bijection from $\mathbb{N}$ to $\mathbb{Z}$ and in doing so show that the cardinality of $\mathbb{Z}$ is equal to the cardinality of $\mathbb{N}$.

Let the function $f : \mathbb{Z} \to \mathbb{N}$ be a piecewise function where

$$f(z) = \begin{cases} 2 \cdot z & z >= 0 \\ -2 \cdot z - 1 & z < 0 \end{cases}.$$

The function f maps the negative integers from $\mathbb{Z}$ to the odd integers in $\mathbb{N}$, and the non-negative integers from $\mathbb{Z}$ to the even integers in $\mathbb{N}$ although we will show that the function f is bijective more rigorously by showing that f is both surjective and injective.

For f to be surjective, we just show that $\forall n \in \mathbb{N},\ \exists z \in \mathbb{Z} \ni f(z) = n$. When $2 \backslash n$, then $f(z) = 2z$ and becuse $2 \backslash n$, then we know that there exists an element in $\mathbb{Z}$ such that $2z = n$. Furthermore, because the product of two positive values must be positive, then z must be greater than zero, satisfying the piecewise constraints in f. If n is odd, then $existsk \in \mathbb{N} \ \ni 2k - 1 = n$, implying that there also $\exists k_- \in \mathbb{Z} \ \ni k = -k_-$. Therefore, $\exists k \in \mathbb{N} \ \ni 2k - 1 = n \Rightarrow \exists k_- \in \mathbb{Z} \backslash \mathbb{N} \ \ni -2k_- = n$ and hence, that $f(k_-) = n$. Therefore, f is surjective.

To show that f is injective, we show that $f(a) = f(b) \Rightarrow a = b$. If $2 \backslash f(a)$, then $f(a) = 2 \cdot a$ and $f(b) = 2 \cdot b$ which implies that $2 \cdot a - 2 \cdot b = 0$ and consequently that $2(a - b) = 0$, which implies that $a = b$. If $2 \nmid f(a)$, then $f(a) = -2 \cdot a - 1$ and $f(b) = -2 \cdot b - 1$ which implies that $-2 \cdot a - 1 - (-2 \cdot b - 1) = 0$ and consequently that $-2 \cdot a - 1 + 2 \cdot b + 1 = 0$ which is equivalent to $-2(a - b) = 0$, and implies that $a = b$ which in turn implies that f is injective.

We have shown that the cardinality of $\mathbb{Z}$ is equal to the cardinality of E and that the cardinality of $\mathbb{Z}$ is also equal to the cardinality of $\mathbb{N}$. By lemma 1-2, we also know that the cardinality of E is equal to the cardinality of $\mathbb{N}$, implying that $\mathbb{Z}$, $\mathbb{N}$, and E all have the same cardinality.

Problem 2: Please prove that the set $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is infinite, but does not have the same cardinality as $\mathbb{N}$ (and hence not the same as E and $\mathbb{Z}$)

Claim 2: The set $(0, 1) = \{\mathbb{R} : 0 < x < 1\}$ is infinite but does not have the same cardinality as $\mathbb{N}$

Proof 2: The claim is composed of two atomic statements: The first claims that the set $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is infinite and the second claims that the same set does not have the same cardinality as $\mathbb{N}$. The two statements will be verified independently.

Subclaim 2-1: The set $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is infinite

Subproof 2-1: To show that the set $(0, 1)$ is infinite, we show that the function $f : (0, 1) \to (0, \frac{1}{2})$ is bijective where f is defined as $f(a) = \frac{a}{2}$. To show that f is bijective, we first show that f is surjective, that is $\forall b \in (0, \frac{1}{2}), \exists a \in (0, 1) \ni f(a) = b$. From the definition of f, we know that $f(a) = b \Rightarrow \frac{a}{2} = b$, which also implies that $2b = a$ and thus (dumbass reader) there does exist an a such that $f(a) = b$ and f is surjective.

We next show that f is injective, that is $f(a) = f(b) \Rightarrow a = b \ \forall a \in (0, 1),\ b \in (0, \frac{1}{2})$. From the definition of f, $f(a) = f(b) \Rightarrow \frac{a}{2} = \frac{b}{2}$, which implies that $\frac{a}{2} \cdot 2 = \frac{b}{2} \cdot 2$ and finally that $a = b$. Therefore, if $f(a) = f(b)$, then $a = b$ and f is injective.

Because f is both surjective and injective, f is also bijective, implying that there exists a bijection between $(0, \frac{1}{2}$ and $(0, 1)$. Furthermore, because $(0, \frac{1}{2})$ is a proper subset of $(0, 1)$, the set $(0, 1)$ is infinite.

Subclaim 2-2: The set $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ does not have the same cardinality as $\mathbb{N}$.

Subproof 2-2: To show that the set $(0, 1)$ does not have the same cardinality as $\mathbb{N}$, we must show that there cannot exist a bijection between the two sets. In doing so, it is sufficient to show that there cannot exist a surjection between the two sets. Suppose there exists a surjection $f : \mathbb{N} \to (0, 1)$ where the values in the domain and codomain are organized into a table such that

x (Domain)	$f(x)$ (codomain)
1	$f(1)$
2	$f(2)$
3	$f(3)$
4	$f(4)$
$\vdots$	$\vdots$

.

Because each element in the codomain of f is in the set $(0, 1)$, each value can be expressed in decimal form, where each decimal place contains a value from zero to nine such that $f(i) = 0.\, x_{i,1}\, x_{i,2}\, x_{i,3}\, x_{i,4} \cdots x_{i,n}$, where n represents the number of values in the decimal representation and can be any arbitrary value in the set of all positive integers. Let $x_{i,j}$ be the value of the j^{th} decimal place for $f(i)\ \forall i \in \mathbb{N}$. The table representing the domain and codomain of f can be rewritten where

x (Domain)	$f(x)$ (codomain)
1	$0.\, x_{1,1}\, x_{1,2}\, x_{1,3}\, x_{1,4} \cdots x_{1,n_1}$
2	$0.\, x_{2,1}\, x_{2,2}\, x_{2,3}\, x_{2,4} \cdots x_{2,n_2}$
3	$0.\, x_{3,1}\, x_{3,2}\, x_{3,3}\, x_{3,4} \cdots x_{3,n_3}$
4	$0.\, x_{4,1}\, x_{4,2}\, x_{4,3}\, x_{4,4} \cdots x_{4,n_4}$
$\vdots$	$\vdots$

.

Let $\Phi(i, j)$ map (i, j) to one if $x_{i,j} = 0$, and zero otherwise such that

$$\Phi(i, j) = \begin{cases} 1 & x_{i,j} = 0 \\ 0 & x_{i,j} \neq 1 \end{cases}.$$

Let $a \in (0, 1)$ which is constructed using the diagonal elements from the codomain of f as shown in Table 1 so that $a = 0.\Phi(1, 1)\Phi(2, 2)\Phi(3, 3) \cdots \Phi(n, n)$. The element a cannot be equal to $f(1)$. If the first entry of $f(1)$ is equal to

x (Domain)	$f(x)$ (codomain)
1	$0.\, x_{1,1}\, x_{1,2}\, x_{1,3}\, x_{1,4} \cdots x_{1,n_1}$
2	$0.\, x_{2,1}\, x_{2,2}\, x_{2,3}\, x_{2,4} \cdots x_{2,n_2}$
3	$0.\, x_{3,1}\, x_{3,2}\, x_{3,3}\, x_{3,4} \cdots x_{3,n_3}$
4	$0.\, x_{4,1}\, x_{4,2}\, x_{4,3}\, x_{4,4} \cdots x_{4,n_4}$
$\vdots$	$\vdots$

Table 1: Diagonal elements

0, then the first value of a is equal to 1. If the first value of $f(1)$ is not equal to 0, then the first decimal entry of a is equal zero, so that $f(1) \neq a$. The same logic applies for any $f(i),\ i \in \mathbb{N}$, implying that a has no preimage despite being an element in $(0, 1)$. Therefore, f is not a surjection, which contradicts the original assumption that f was surjective between $\mathbb{N}$ and $(0, 1)$. Therefore, there cannot exist any surjection (and by extension, any bijection) between $\mathbb{N}$ and $(0,1)$ which also means that the cardinality of $\mathbb{N}$ is not equal to the cardinality of $(0,1)$.

Problem 3: Recall $\implies, \vee, \wedge$, and ∇ are binary logical operations on $\mathcal{M}$ we've studied. Recall $\neg$ is a (the only) logical *operator* (or *unary operation*) on $\mathcal{M}$ we've studied. Please determine, with justification, the number of different binary logical operations on $\mathcal{M}$ there can be. Also, determine the number of logical operators there can be.

Claim 3-1: There are 16 possible binary operations on $\mathcal{M}$

Proof 3-1: A binary operation is essentially a mapping $f : (\{\text{True}, \text{False}\} \times \{\text{True}, \text{False}\}) \to \{\text{True}, \text{False}\}$, where the domain of f has four elements: $(\text{True}, \text{True}), (\text{True}, \text{False}), (\text{False}, \text{True})$ and $(\text{False}, \text{False})$. The image of each element in the domain can be either True, or False such that

P	Q	f(P,Q)
True	True	{True, False }
True	False	{True, False }
False	True	{True, False }
False	False	{True, False }

.

Let the set $\mathcal{S}$ include all possible combinations of the set $\{(P,Q) \times f(P,Q)\}$. Note that for each row in the table, the number of elements in $\mathcal{S}$ doubles, implying that there are $2^4 = 16$ modes of behavior for a binary operation in $\mathcal{M}$. Each mode of behavior can be represented by at most one binary operation and so, there can be at most 16 binary operations on $\mathcal{M}$.

Claim 3-2: There can exist up to four binary operators on $\mathcal{M}$

Proof 3-2: A binary operator on $\mathcal{M}$ is defined as a function $f : \mathcal{M} \to \mathcal{M}$. A truth table for such a function could be written as

P	f(P)
True	{True, False}
False	{True, False}

.

Note that there are two rows in the truth table for a binary operator on $\mathcal{M}$. By the same logic as Proof 3-1, this implies that the number of combinations of $(P, f(P))$ is equal to $2^2 = 4$ and because a binary operator on $\mathcal{M}$ can only represent one combination, there are four possible binary operators on $\mathcal{M}$.

Problem 4: Turn the logical system $(\mathcal{M}, \Phi, \wedge, \vee, \implies, \Longleftarrow, \iff, \neg)$ into a purely algebraic system over $(\mathbb{Z}_2, +, \cdot)$ with, for $x \in \mathcal{M}$, $\Phi(x) = 0$ if x is true and $\Phi(x) = 1$ if x is false. Use the logical system to show that any statement of the form $[\neg P \wedge ((\ negP \implies Q) \wedge \neg Q)] \implies P$ is a tautology.

The logical system $(\mathcal{M}, \Phi, \wedge, \vee, \implies, \Longleftarrow, \iff, \neg)$ can be expressed as an algebraic system over $(\mathbb{Z}_2, +, \cdot)$ by expressing $\Phi, \wedge, \vee, \implies, \Longleftarrow, \iff$, and $\neg$ in term of $+$ and $\cdot$ in $\mathbb{Z}_2$.

Claim 4-1: The operator $\neg$ can be expressed in terms of $+$ and $\cdot$ from $\mathbb{Z}_2$.

Proof 4-1: Recall how the multiplication table for $\mathbb{Z}_2$ is

+	0	1
0	0	1
1	1	0

.

and that the column corresponding to 1 can be written in a truth table format as

P	$P+1$
0	1
1	0

.

where 0 denotes 'True' and 1 denotes 'False'. Note how the truth table for P and $P+1$ is the same as the truth table for $\neg$, implying that $\neg$ can be expressed algebraically such that $\neg P \equiv P + 1$ in $\mathbb{Z}_2$.

Claim 4-2: The operation $\vee$ can be expressed in terms of $+$ and $\cdot$ from $\mathbb{Z}_2$.

Proof 4-2: The truth table for the operation $\vee$ is

P	$\Phi(P)$	Q	$\Phi(Q)$	$P \vee Q$	$\Phi(P \vee Q)$
True	0	True	0	True	0
False	1	True	0	True	0
True	0	False	1	True	0
False	1	False	1	False	1

.

Note how the table for multiplication in $\mathbb{Z}_2$ is

$+$	0	1
0	0	0
1	0	1

.

which can be expressed in terms of $\Phi(\cdot)$ and reorganized as a truth table such that

$\Phi(P)$	$\Phi(Q)$	$\Phi(P) \cdot \Phi(Q)$
0	0	0
1	0	0
0	1	0
1	1	1

,

which is equivalent to the truth table for $\vee$, therefore $\vee$ can be expressed algebraically in terms of $(\mathbb{Z}_2, +, \cdot)$.

Claim 4-3: The operation $\wedge$ can be expressed in terms of $+$ and $\cdot$ from $\mathbb{Z}_2$.

Proof 4-3: De Morgan's Theorem states that $\neg(P \wedge Q) \iff \neg P \vee \neg Q$, implying that $P \wedge Q \iff \neg(\neg P \vee \neg Q)$. Note that the expression $P \wedge Q$ is equivalent to another expression which contains only the $\neg$ operator and the $\vee$ operation which we have shown can be expressed algebraically in $(\mathbb{Z}_2, +, \cdot)$ which implies that $\wedge$ can also be expressed as such.

Claim 4-4: The operation $\implies$ can be expressed in terms of $+$ and $\cdot$ from $\mathbb{Z}_2$.

Proof 4-4: From homework 2, recall that $P \implies Q$ can be expressed in terms of the logical statement $P \vee \neg Q$. Because the operator $\vee$ and the operation $\neg$ can be expressed algebraically, then $P \vee \neg Q$ can also be expressed as such, implying that $P \implies Q$ can be written in terms of the algebraic system $(\mathbb{Z}_2, \cdot, +)$.

Claim 4-5: The operation $\impliedby$ can be expressed in terms of $+$ and $\cdot$ from $\mathbb{Z}_2$.

Proof 4-5: The truth table for the operator $P \impliedby Q$ is given as

P	Q	$P \impliedby Q$
True	True	True
True	False	False
False	True	True
False	False	True

and can be expressed as a logical statement $Q \vee \neg P$ as shown by their truth tables

P	Q	$P \impliedby Q$	$Q \vee \neg P$
True	True	True	True
True	False	False	False
False	True	True	True
False	False	True	True

.

The logical operator $\vee$ and the operation $\neg$ have already been shown to be expressable in the algebraic system $(\mathbb{Z}_2, +, \cdot)$ which implies that $\impliedby$ can as well because $\impliedby$ can be expressed in terms of $\vee$ and $\neg$.

Claim 4-6: The operation $\iff$ can be expressed in terms of $+$ and $\cdot$ from $\mathbb{Z}_2$.

Proof 4-6: The truth table for $\iff$ is given as

P	Q	$P \iff Q$
True	True	True
False	True	False
True	False	False
False	False	True

which implies that

$\Phi(P)$	$\Phi(Q)$	$\Phi(P \iff Q)$
0	0	0
1	0	1
0	1	1
1	1	0

and can be expressed logically as $(P \wedge Q) \vee (\neg P \wedge \neg Q)$. The logical expression for $\iff$ is built upon the operators/operations $\wedge, \vee$ and $\neg$ which can be expressed algebraically in $(\mathbb{Z}_2, +, \cdot)$. Therefore, $\iff$ can also be expressed algebraically in $(\mathbb{Z}_2, +, \cdot)$.

Claim 4-7: The statement $[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P$ is a tautology.

Proof 4-7: Begin by defining the following expressions

$$\neg P \equiv P + 1 \tag{1}$$
$$P \implies Q \equiv Q(P+1) \tag{2}$$
$$P \wedge Q \equiv (P+1)(Q+1)+1 \tag{3}$$
$$1 + 1 \equiv 0 \text{ in } \mathbb{Z}_2 \tag{4}$$
$$PP \equiv P \tag{5}$$

. Next, begin replacing all $\neg$ operations such that

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv [(P+1) \wedge (((P+1) \implies Q) \wedge (Q+1))] \implies P$$

Next, we replace the first use of the implies operation with it's mathematical definition from (2), yielding

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv [(P+1) \wedge (((P+(1+1))Q) \wedge (Q+1))] \implies P$$

But because $1 + 1 = 0$ in $\mathbb{Z}_2$, we can also say that

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv [(P+1) \wedge ((PQ) \wedge (Q+1))] \implies P.$$

The next step is to substitute the algebraic definition for $\wedge$ for the second $\wedge$ operation such that

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv [(P+1) \wedge ((PQ+1)(Q+1+1))] \implies P.$$

which reduces to

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv [(P+1) \wedge (PQQ+Q)] \implies P.$$

Note, that $QQ \equiv Q$ because when $Q = 1, QQ = 1$ and when $Q = 0$, $QQ = 0$, therefore the previous expression can be simplified again as

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv [(P+1) \wedge (PQ+Q)] \implies P.$$

Next, we substitute the last remaining $\wedge$ operation for its algebraic definition to obtain

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv [(P+1+1)(PQ+Q+1)] \implies P.$$

which simplifies to

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv [(PPQ+PQ+P)] \implies P$$

and alternatively as

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv [(PQ+PQ+P)] \implies P.$$

Note that when PQ = 1, PQ + PQ = 0, and when PQ = 0, PQ + PQ = 0, therefore, PQ + PQ = 0 can be applied to the previous expression as

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv P \implies P$$

and finally, the algebraic definition of $\implies$ is used to show that

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv P(P+1).$$

Because the expression P(P+1) = 0 when P = 1, and P(P+1) = 0 when P = 0, we can say that

$$[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P \equiv 0.$$

which implies that the statement $[\neg P \wedge ((\neg P \implies Q) \wedge \neg Q)] \implies P$ is a tautology as the statement is always true.