✠ Sub-Experience One: Some Tournament Problems

Part One: Curling. *No outside resources, please.*) Suppose 10 teams compete in a curling competition where each team plays every other team and no ties are allowed. Officials decide that there will be two elimination rounds the first of which will eliminate all teams except for those which beat at least 7 other teams. The second elimination round will eliminate all teams which do not beat a majority of the other teams. The final competition will be among the teams which remain after the second elimination round.

(A) Determine with proof the maximum number of teams which may remain after the first elimination round.

- **Response:** We know that there are ten initial teams, where each team will play each other team once. This implies that there are $\binom{10}{2} = 45$ games, where each game results in a “win”. We also know that a team must accumulate 7 wins to avoid elimination. If we distribute the wins so that the number of finalists is maximized, then each final team will only have 7 wins. This implies that the number of finalist teams is $\frac{45}{7} = 6.428$. However, as there are no partial teams, the the maximum number of finalist teams is $\lfloor \frac{45}{7} \rfloor = 6$.

(B) Determine with proof the maximum number of teams which may remain after the second elimination round.

- **Response:** The term “majority” is not defined in the problem statement, and so for this answer, I assume that majority means more than half. Consider the table given below:

# of Teams	# of Games	# Required Wins	$\lfloor \frac{\text{Game}}{\text{Wins}} \rfloor$
6	15	4	3
5	10	4	2
4	6	3	2
3	3	2	1
2	1	1	1

The first, second, third and fourth columns represent the number of potential finalist teams, the number of games to be played, the number of required wins, and the number of teams with exactly the right number of wins respectively, where the number of finalist teams that move on (the fourth column) is computed the same as in part A. We will observe that the largest number of teams that pass the second round is 3.

Part Two: Strong Tournaments. Recall that a directed D graph is **strong** if between any pair of vertices x and y, there is an x, y-path, and a y, x- path.

(A) Prove the following theorem: *In any strong tournament T on at least 5 vertices, there exist two distinct vertices x and y such that $T - x$ and $T - y$ are both strong.* Prove or disprove whether the above theorem with conclusion “*such that $T - x - y$ is strong*” is true (where x and y are the vertices found in the preceding part).

- **Response:** I understand this question to be asking me to prove that there exist two distinct vertices x and y such that if both are removed from T at the same time, then T will remain strong. First we consider the case where the first vertex, x is removed. When x is removed, the vertices against whom x won are at risk, meaning that paths to them may be disrupted. If there exists a vertex which has lost at least twice, than one of its winners (or parent vertices) may be removed without it becoming an ‘orphaned’ vertex. Any path that must have passed through this vertex will still be valid as any vertex can still reach its remaining parent, by the definition of strong. We must prove that for a tournament with four or move vertices, there must be at least one vertex with two or more loses.

  We know that the number of losses is equal to the number of games, that is $\binom{n}{2}$. Therefore, x can be removed for all values of n that satisfy
  $$\binom{n}{2} \geq n + 1$$
  because even if all losses are distributed, one vertex must still lose twice. By substituting definitions and

algebra-fooing in the usual sense, we show that

$$\begin{aligned}\binom{n}{2} \geq n+1 &\implies \frac{n!}{2!(n-2)!} \geq n+1 \\ &\implies \frac{n!}{2!(n-2)!} \geq n+1 \\ &\implies \frac{n(n-1)(n-2)!}{2(n-2)!} \geq n+1 \\ &\implies \frac{n(n-1)}{2} \geq n+1.\end{aligned}$$

The above constraint is satistied with equality for $n \approx 3.5616$ so that $\frac{n(n-1)}{2} \geq n + 1 \forall n \geq 4$. If $n = 5$, then we remove x, making $n = 4$. We can then select another vertex, y, as n is still greater than or equal to 4. Therefore, if $n \geq 5$, then there are at least two vertices x and y that can be removed such that $T - x - y$ is strong.

(B-1) Let T be a tournament on $n \geq 3$ vertices and let s be a vertex of T. Please prove the following statement: *T is strong if and only if for every vertex $t \in V(T) \setminus \{s\}$ there is an s,t-path and a t,s-path.* Is this statement true if T is a digraph (not necessarily a tournament).

- **Response:** This proof will consist of two parts. In the first, we show that if T is strong, then $\forall t \in V(T) \setminus s$ there exists an s,t path and a t,s path. In the second we will show that if $\forall t \in V(T) \setminus s$ there exists an s,t path and a t,s path then T is strong.

  Claim: If T is strong, then for every $t \in V(T)$, there exists a s,t path and a t,s path.

  Proof: By the definition of strong, we know that there exists a path from each vertex in T to every other vertex in T. Therefore, as both s and t are elements of $V(T)$, then there exists an s,t and t,s path.

  Claim: If for every $t \in V(T)$, there exists a s,t path and a t,s path, then T is strong.

  Proof: We desire to show that if for every $t \in V(T)$, there exists a s,t path and a t,s path, then for every $t_1, t_2 \in V(T)$, there exists a t_1, t_2 and a t_2, t_1 path.Let $t_1, t_2 \in V(T)$ and suppose that for every $t \in V(T)$, there exists a s,t path and a t,s path. This implies that there exists a t_1, s path and a s, t_2 path. These two paths can be concatenated to form a t_1, t_2 path. We can also form a t_2, t_1 path by the same logic making T strong.

(B-2) Suppose you have an algorithm $\mathcal{A}$ that determines whether there is a path from x to y, where x and y are vertices in a digraph. Suppose $\mathcal{A}$, when given a pair of vertices and a digraph D, performs this calculation with M operations in the worst case. How many operations, in the worst case, are needed to use $\mathcal{A}$ to determine whether D is strong using the standard definition of 'strong'? How many operation, in the worst case, are needed if the alternative definition of 'strong' proved in B-1?

- **Response:** If we use the traditional definition of strong, or $\forall\ t_1, t_2 \in V(T), \exists\ t_1, t_2 -$ path and $\exists\ t_2, t_1 -$ path then we must use the algoritihm $\binom{n}{2} = \frac{n(n-1)}{2}$ times so that the number of operations is $\frac{Mn(n-1)}{2}$. If we use the alternative definitiion of 'strong' as proved in B-1, we only need compute the paths from any vertex to s and vica-versa, leading to 2(n-1) iterations so that the number of computations is 2M(n-1).

Part Three: Queens. Let n and q be positive integers with $q \leq n$. Define an (n,q)-tournament to be a tournament on n vertices with exactly q queens. In this sub-experience, you will determine all possible values of n and q for which there exists an (n,q)-tournament. Specifically please prove the following theorem.
Theorem 1.3.1: *There exists an (n,q)-tournament for all positive integers n and q with $n \geq q \geq 1$, except for $q = 2$ and n arbitrary, and $n = q = 4$.*
My suggestion is that you use the Principle of Mathematical Induction and the following lemmas (which I ask that you prove as well).
Lemma 1.3.2: *There exists a $(1,1)$-tournament.*
Lemma 1.3.3: *There does not exist a $(4,4)$-tournament.*
Lemma 1.3.4: *There is a $(6,6)$-tournament.*
Lemma 1.3.5: *If there exists an (k,k)-tournament, then there exists an $(k+2, k+2)$-tournament, where k is a*

positive integer.
Lemma 1.3.6: *For every positive integer n except 2 and 4, there exists an (n, n)-tournament.*

✠ Sub-Experience Two: An Optimization Problem

Let H denote an arbitrary graph. Recall that the distance between vertices in H is the length of the shortest path

that has those vertices as its endpoints. Denote by $d_{\max}(H)$ the maximum distance among all pairs of vertices of H. Recall that $\Delta(H)$ denotes the maximum degree among all vertices of H.
Define the function $N(d, k)$ to be the maximum number of vertices among all graphs H with $d_{\max}(H) = d$ and $\Delta(H) = k$.

One. *Determine $N(n, 2)$.*

Two. *Determine $N(2, 3)$.*

Three. *Verify that the graph G drwan to the right has $d_{\max}(G) = 2$.*

Four. *Determine $N(2, 4)$.*

✠ Sub-Experience Three: Non-Standard Dice

(*No outside resources, please.*) A 6-sided die labeled with the integers $1, 2, 3, 4, 5, 6$ will be called a *standard die.* The goal for this part of the *Final* Experience is to determine all ways to label a pair of dice with positive integers so that the probabilities of rolling the usual sums $2, 3, \ldots, 12$ are the same, but the labels are non-standard.

Step 1. Let $p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$, and explain why $(p(x))^2$ is the generating function for the probabilities of outcomes in rolling a pair of standard dice.

Step 2. Let $A = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $B = (b_1, b_2, b_3, b_4, b_5, b_6)$ be two lists of positive integers. Put $p_A(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$ and $p_B(x) = x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}$. Explain why finding a_is and b_is such that $p_A(x)p_B(x) = (p(x))^2$ is relevant this part of the Experience.

Step 3. Factor $p(x)$ into irreducible polynomials and use this factorization to help solve for the a_is and b_is. Specifically, the factorization will force the form of $p_A(x)$ to be something like $p_1(x)^q p_2(x)^r p_3(x)^s p_4(x)^t$, where $0 \le q, r, s, t \le 2$ and $p_i(x)$, for $1 \le i \le 4$, is a factor of $p(x)$. In your solution to this step, you must motivate why you take this step.

Step 4. Begin to reduce the possibilities for q, r, s, and t by using information from $p_A(1)$ and $p_A(0)$. Note that, on one hand $p_A(1) = 1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6} = 6$ (since $a_i > 0$), and on the other hand we have $p_A(1) = p_1(1)^q p_2(1)^r p_3(1)^s p_4(1)^t$. Similarly, there are two ways to view $p_A(0)$.

Step 5. List all possible ways to label a pair of dice so that the probabilities of obtaining the sums $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ are $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$, respectively. One such way will be the standard way. In your solution for this step, explain why you have proved that the labels you have found are the only possible ones that give the desired probabilities for roll- outcomes.

✠ Sub-Experience Four: Space Station Security

(*No outside resources, please*) We have defined space stations, and developed notation in class, in particular see *Meeting Thirty-Three.* Please use that notation in your responses to these prompts. To remind: S_w denotes a (specific) w-walled station, $\oint S_w$ denotes its interior, ∂S_w denotes its boundary (its walls and corners). Let's use $V(S_w)$ to denote the corners of S_w and $W(S_w)$ to denote its walls.

1. **Watched Eyes.** The function $gg(w)$ is the maximum number of robot eyes (REs) required to protect any w-walled station which can be seen by at least one other RE. Please make a conjecture for the value of $gg(w)$. The points you earn from your response (your conjecture) will be proportional to the type and quantity of justification you give.

2. **Rectangular Stations.** Define the function $g_\perp(w)$ to be the maximum number of REs required to protect a w-walled station whose interior angles between walls are each either $90°$ or $270°$. Please find the value for $g_\perp(w)$ and prove the value you find is correct.

3. **Watched Eyes in Rectangular Stations.** Define the function $gg_\perp(w)$ to be the maximum number of REs required to protect a w-walled rectangular station such that each RE can be seen by at least one other RE. Please determine the value for $gg_\perp(w)$ and prove the value is correct.