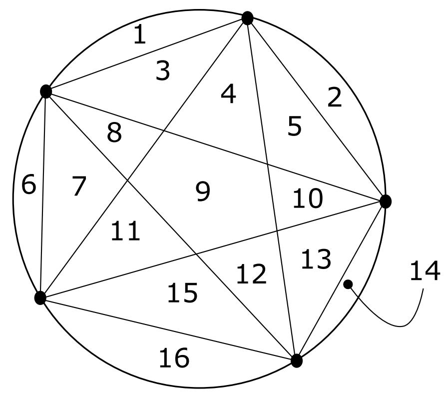
✠ **Sub-Experience One:** ***Pizza Cutting with Lasers***

No outside resources, please Given the difficulty of organizing and sequentially lining up cuts to get the maximum number of pieces, you invent a laser cutting system that allows you to make all cuts at once. You place n of your devices on the crust with spacing that will maximize the number of pieces made and, once activated (ideally remotely because they are dangerous), *laser sabers*® shoot between every pair of devices simultaneously cutting the pizza into $R(n)$ pieces[1]. Below is an illustration representing 5 of your devices cutting the pizza into 16 pieces.



Please determine a formula for $R(n)$.

A new piece is formed when a laser intersects with either another laser, or the edge of the pizza (Fun experiment... try this with Ghostbuster lasers). Each time a ‘laser’ is activated, the number of additional pieces it forms is therefore equal to the number of times it intersects an existing laser plus 1 (as each laser must intersect with the crust at some point) This leads to the expression

$$R(n) = \text{nLaser} + \text{nIntersection} + 1$$

where +1 is necessary because the entire pizza without lines or intersections is considered 1 piece. A laser can be described as a set of two points, making the the total number of lasers equal to the number of elements in the set of point pairs of which there are $\binom{n}{2}$ elements. Similarly, an intersection can be described as a set of two lasers, or four points, of which there are $\binom{n}{4}$ combinations. If we substitute these expressions for the number of lasers and the number of intersections into the existing equation, we get

$$\begin{aligned} R(n) &= \binom{n}{2} + \binom{n}{4} + 1 \\ &= \binom{n}{0} + \binom{n}{2} + \binom{n}{4} \\ &= \sum_{i=0}^{2} \binom{n}{2i} \end{aligned}$$

✠ **Sub-Experience Two:** ***Strange Walks***

After the pizza, you decide to take a walk.

Number of shortest paths using the *city block metric*. Starting at Center and Main, you decide to walk to X North and Y East. The distance is measured in blocks and so the distance from X_1 North and Y_1 East to X_2 North and Y_2 East is $|X_2 - X_1| + |Y_2 - Y_1|$.

SE 2.1. For yuks, determine π in the Logan metric. Remember that π is defined to be the ratio of the circumference to the diameter of a circle. And that a circle is the set of points equidistant from a given point (that distance is typically called the *radius* of the circle and the point is called its *center*.

In this proof, we define circumference as the number of steps it takes to travel to each point that is equidistant fromt the origin. Consider the case where the diameter is 2. The set of all points that are “on the circle” is:

[1] “R” for lase$\underline{\text{R}}$.

$\{(1,0),(0,1),(-1,0),(0,-1)\}$. If we were to walk from one point to the next, beginning at $(1,0)$, then we would walk 8 blocks, indicating that the circumference is 8. Because *pi* is defined as the ratio of circumference to diameter, then

$$\begin{aligned}\pi &= \frac{\text{Circumference}}{\text{Diameter}}\\ &= \frac{8}{2}\\ &= 4,\end{aligned}$$

indicating that π is equal to 4 under the city block metric.

SE 2.2. Count the number of shortest paths from Center and Main to X North and Y East. A more erudite way to pose this problem is as follows: count paths from $(0,0)$ to (X,Y) using steps of the form $R:(x,y)\mapsto(x+1,y)$ or $U:(x,y)\mapsto(x,y+1)$. So paths trace rectilinear “curves” in the first quadrant of the Cartesian coordinate system that visit only integer- valued coordinates.

Let m be the number of block to walk in the y direction and n be the number of blocks to walk in the x direction so that $m=|Y|$ and $n=|X|$. A path can be thought of as a linear board with $m+n$ tiles, where each tile is represented by either an x or y square. Note that each path is uniquely defined by only the locations of x or y tiles on the board because any non-x tile must be occupied by a y tile and vice versa. Therefore, the number of board combinations, and by extension the number of possible paths, is described as the number of ways the x tiles can be assigned to $m+n$ locations. Since there are n x tiles, the number of configurations is $\binom{m+n}{n}$.

SE 2.3. Walk in strange way. Start at Center and Main, thinking of it as the origin $(0,0)$ and take n steps, each of type R, L, or U, with R never followed by L and vice-versa. The steps are defined via $R:(x,y)\mapsto(x+1,y)$, $L:(x,y)\mapsto(x-1,y)$, and $U:(x,y+1)$. Determine the number of different paths with n steps.

Consider a path with n steps as an n element ordered sequence. The first element in the sequence has four options, one for north, south, east and west respectively. Because each element cannot double back on itself, once an element is placed, there are three remaining options so that the number of total options is equal to $4\cdot 3^{n-1}$.

✠ Sub-Experience Three: The Master Table of Distributions.

Recall the table of distribution functions $f:A\to B$, where A is a finite set with n elements and B is a finite set with x elements.

A	B	unrestricted	injective	onto
distinguishable	distinguishable	1.	2.	3.
indistinguishable	distinguishable	4.	5.	6.
distinguishable	indistinguishable	7.	8.	9.
indistinguishable	indistinguishable	10.	11.	12.

Please determine, with an argument for each, a formula for each entry in the table.

In this solution, we consider each entry in the table in reference to its number and give an appropriate response for each in the following list.

1. Because each element from both sets is distinguishable, any element from A can map to any element in B. Therefore, for a single element in A, there are two options for each element in B minus the option where that element does not map to any element in B so that each element in A has 2^x-1 options. Each element in A is also independent of the others, thereore the total number of combinations between A and B is $(2^x-1)^n$.

2. For a function to be injective, $f(a)=f(b) \implies a=b$, which essentially means that each element in A can only map to one element in B. If $|A|>|B|$, then the answer is zero because not all elements in the domain can map to a unique element in the codomain. The following proof is presented for the case where $|A|\leq|B|$. Consider an analogy where we place balls (representing elements from the domain) into boxes (representing elements in the codomain). Each matching is counted as a function and we therefore desire to find the number of ways in which to pair the balls from the domain with the boxes from the codomain. Consider the first element from the codomain. The first element can pair with any of the x elements in the codomain. The second element

in the domain can pair with any element in the codomain that is not paired with the first, yielding $x(x-1)$ combinations. Continuing as such until all elements in the domain have been paired with elements in the codomain yields

$$\text{nCombination} = x(x-1)(x-2)...(x-(n-1))$$

which is equivalent to

$$\text{nCombination} = \frac{x!}{(x-n)!}$$

3. A function is surjective if each box contains at least one ball, or more precisely $\forall b \in B,\ \exists a \in A \ni f(a) = b$. We will determine the number of surjective functiosn by subtractig the number that are not surjective from the total number of possible functions. A function is not surjective if at least one box does not contain a ball. Partition the set of non-surjective functions on the number of boxes that do not contain balls. First, we select the number of boxes that will not contain balls. Then we determine the number of permutations that may occur, and then we determine the number of possible functions per permutation. For i empty boxes, there are $\binom{x}{i}$ permutations. For each permutation, there is also n options per non-empty box so that each permutation yields n^{x-i} options. Therefore, the number of functions where i boxes do not contain any balls is described as $\binom{x}{i}n^{x-i}$. If we sum over all partitions, then the total number of empty boxes is

$$\sum_{i=1}^{x-1}\binom{x}{i}n^{x-i}$$

and the number of surjective functions is consequently $(2^x-1)^n - \sum_{i=1}^{x-1}\binom{x}{i}n^{x-i}$.

4. Because the elements of A are indistinguishable, each function would differ based on how many elements in A map to each element in B. For example, if we use the balls and boxes metaphore, then we would ask how many balls are in each box. Consider a scenario where there were two balls in box 1, three balls in box 2, and 1 ball in box 3. We describe this as an element in a multiset such that

$$f_i = \langle b_1^{(2)}, b_2^{(3)}, b_3^{(1)} \rangle.$$

In general, we refer to the set of all possible functions as the multiset of B size n, which has $\left(\binom{n}{x}\right)$ elements, where n is the number of elements from A, and x is the number of elements in B. Therefore, the number of elements in the multiset of B with size n is $\binom{x+n-1}{x}$.

5. A function is injective if $f(a) = f(b) \implies a = b$. If we adopt the balls and boxes metaphore, then this equates to if balls are in the same box then they are equal. Because the balls are indistinguishable, than all balls are equal. Hence, this is true for all functions which map A to B. Therefore, the number of injective functions is $\binom{n+x-1}{x}$.

6. Adopting again the ball and boxes metaphore, a surjective function is one where each box contains at least one ball. Because the balls are indistinct, we don't care which ball a box contains. Therefore, assume that the first x balls are placed in individual boxes. Then the number of possible functions is the multiset of B size $n-x$ and consequently, the number of functions is

$$\binom{n-1}{x}$$

7. Because the elements in B are indistinguishable from each other, the defining trait of a function is to how many elements in B each element in A maps. Because these functions are unconstrained, there are x possibilities for each element in A, one for each possible number of mappings to an element in B. Therefore, the number of functions is x^n.

8. Because a function is injective if $f(a) = f(b) \implies a = b$, and all elements in B are equal as they are indistinguishable, then no function mapping the domain to an element in B can be injective.

9. Recall that a function is surjective if $\forall b \in B,\ \exists a \in A \ni f(a) = b$. Suppose there are two element in B denoted b_1 and b_2, where b_1 and b_2 are indistinguishable so that $b_1 = b_2$. Then if $\exists a \in A \ni f(a) = b_1$, then there also exists an a in A such that $f(a) = b_2$. Hence, all functions that map from the domain to the codomain are surjective.

10. Assume the balls and boxes analogy we've used in the past. Because both elements contain sets that are indistinguishable, we characterize a function by how many balls are mapped to any one box. This leads to a unordered set of numbers of size x. Let f_i be defined as an element in a multiset

$$f_i = \langle 0^{()}, 1^{()}, 2^{()}, 3^{()}, \cdots n^{()} \rangle$$

of size x, where the value in each parenthesis for the i^{th} element gives the number of times i elements from A map to the same element in B. This yields $\left(\binom{n+1}{x}\right)$ functions which is expressed as $\binom{n+x}{x}$.

11. Because every element in A is equivalent and every element in B is equivalent, then $f(a) = f(b) \implies a = b$ is true for every element in A and B. Therefore, every function that maps A to B is injective.

12. The definition of surjective implies that $\forall b \in B,\ \exists a \in A \ni f(a) = b$. Since all element in A are equivalent and all element in B are also equivalent, than any mapping form A to B satisfies these constraints and therefore all functions that map from A to B are surjective.

✠ **Sub-Experience Four:** ***Game Time!***

SE 4.1: Modified Towers of Hanoi. The *Towers of Hanoi* are mythical diamond needles, three of them, with 64 gold discs impaled on one of the needles. God told the monks of Brahma to transfer all the discs to another needle with the constraints that no larger disc ever be placed atop a smaller disc, and only one disc at a time is to be moved. Once the monks finish moving the discs the World will end. Evidently the monks have not yet finished. Label the needles of the Towers of Hanoi L, M, and R, for the left, middle, and right, respectively. Consider the original Towers of Hanoi game, but with the additional constraint that a disk can only be moved to an adjacent peg; that is, a disk can only be moved to M from L or R, and can only be moved to L or R from M. Assume all discs are initially on L.

Determine the minimum number of moves (where a move *is defined to be the transfer of one disc from one needle to another) required to transfer n discs from L to R.*

Let F_n be the number of moves it takes to move a set of n discs to the left, or to the right by one. Suppose also that you know F_{n-1} and desire to compute F_n. We will demonstrate this proof for the case where we move the stack of tiles from the middle peg to the right peg, but the same logic can be applied for any movement.

Because we desire to move the bottom, or n^{th} tile to the right first, we move the $n-1$ pieces to the left, move the n^{th} peice to the right, and then move the $n-1$ pieces to the middle, and then right-most needle. In this transaction, we moved the $n-1$ stack 3 times, and the n^{th} piece once. Therefore, $F_n = 3F_{n-1} + 1$, which can be expressed as a sum where

$$F_n = \sum_{i=0}^{n-1} 3^i.$$

We desire to find a closed form expression for F_n. From the expression above, we know that

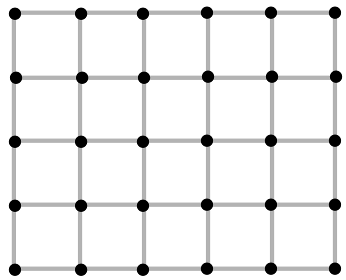
$$\begin{aligned} 3F_n - F_n &= 3\sum_{i=0}^{n-1} 3^i - \sum_{i=0}^{n-1} 3^i \\ &= \sum_{i=1}^{n} 3^i - \sum_{i=0}^{n-1} 3^i \\ &= 3^n + \left(\sum_{i=1}^{n-1} 3^i - \sum_{i=1}^{n-1} 3^i \right) - 1 \\ &= 3^n - 1 \end{aligned}$$

which also implies that $2F_n = 3^n - 1$. The expression for F_n gives the number of moves to transfer a stack of n pieces from one needle to an adjacent needle. Moving a stack of n pieces from L to R would require two repetitions and so the number of moves it would take transfer n pieces from the L needle to the R is equal to $3^n - 1$.

How Long? Assume the monks of Brahma were given this game with 64 discs, and can move a disc at a rate of one per second. How long in centuries will it take for the monks to complete their task?

The number of total moves to relocate a 64 piece stack is $3^{64} - 1 \approx 3.433683820292512 \times 10^{30}$ moves, which if done at a rate of 1 move per second implies that it would take approximately $1.088813996794937 \times 10^{21}$ centries to complete.

SE 4.2: A Two-Player Game with no Dire Consequences.
The Game
Two players, A and B, alternately select an *edge* (line segment connecting two dots) on the grid graph shown below and color it red. The loser of the game is the player who is forced to select an edge that creates a red C_4 — a red cycle on 4 vertices.



The Fun
Confirm or deny, with proof, whether player A (the first player) can always win if she employs a particular strategy for each move.

This proof shows that the first player can always win if they use the strategy given in this proof. Let each move be denoted $\overline{ab}$, where a and b represent the points to either side of the line. Furthermore, let the grid be numbered starting from (0,0) in the top left corner and incrementing down and to the right where the first element in the pair is the vertical index, and the second is the horizontal. The first move for player 1 must be $\overline{(2,2)(2,3)}$ and then aftward player 1 must mirror player 2's moves about the vertical and horizontal axis. By doing so, player 1 ensures that the board is symmetrical. If the board is symmetrical, than if player two can find a place to move, then symmetry dictates that player one can as well. By playing the first move directly in the center of the board, player 1 ensures that player two must be the first to select the first element in the symmetric pair. Therefore, the first player to make a choice on a board that has only losing options must be player two, making player one the winner.

✠ Sub-Experience Five – A Matter of Life and Death
No outside resources You are among a group of $n-1$ of your friends (pretend you have $n-1$ friends even if you don't, where n is possibly very large) and are captured by a horde of theater students who will force you and your $n-1$ friends to enact the play *Cats* with them over and over and over and ... . Your friends choose death over this fate and decide to form a circle and have every other person commit suicide (someone has a pistol and n bullets, which is quite reasonable to assume here at USU) until only one person survives, who will supposedly kill themselves. You want no part of this suicide madness since, you figure, the theater people will tire eventually and you can make your escape by taking Jennyanydots hostage with the pistol and remaining bullet and escape.
Number you and your friends 1 to n and assume you all form a vicious circle to facilitate the suicide.

Question: *In which position (call it $S(n)$) must you be in order to survive?*
Examples: $S(4) = 1$, $S(7) = 7$.

We will show that each value for n falls into one of three categories. The first is that $n = 1$, in which case $s(n) = 1$ is trivially satisfied. The second is where n is a power of two. If n is a power of two, then each time the gun traverses the circle, all evenly numbered members will kill themselves and the gun will safely pass by position 1 like an unholy peace pipe. Hence, if n is a power of 2, then $s(n) = 1$. The final case is where n is not 1, and n is also not a power of two (1 could be considered a power of two as well, but because 1 generally causes mayhem in proofs it made more sense to explicitly define the results). Finally, we show that if n is not a power of two, than it's safety position is

two higher than that of $n-1$.
Claim: If n is not a powerof two, than the safety position for n members is 2 more than that of $n-1$ members.
Proof: The movement of the gun is measured by how many positions it traverses. For example, if the gun moves from position 1 to position 3 (skipping the recently deceased person in position 2), than the gun has a movement of two. Define a *round* as all gun movement between instances where the gun crosses the 1 position. Furthermore, define the $n-1$ scenario as a scenario with n positions where the n^{th} persion is already dead. For example, consider a comparison between $n=4$ and $n=5$. Let the existing positions for the $n=4$ scenario be given as

$$\langle 1, 2, 3, 4, \textcolor{red}{5} \rangle$$

, where the fifth position is red to indicate that this person is already dead :) and the existing positions for the $n=5$ scenario be given as

$$\langle 1, 2, 3, 4, 5 \rangle.$$

The sequence the gun travels for the $n-1$ scenario is given as

move	Action	Number of Steps
$1 \to 2$	Kill	1
$2 \to 3$	Spare	1
$3 \to 4$	Kill	1
$4 \to 1$	Spare	2
$1 \to 3$	Kill	2

The gun sequence for the 5 persion scenario is given as

move	Action	Number of Steps
$1 \to 2$	Kill	1
$2 \to 3$	Spare	1
$3 \to 4$	Kill	1
$4 \to 5$	Spare	1
$5 \to 1$	Kill	1
$1 \to 3$	Spare	2
$3 \to 5$	Kill	2

Note how the 5 person scenario contained two more 1-step actions than the 4 person scenario. This will happen generally for all values of n and $n-1$ because the n-person scenario must traverse the extra location, and the $n-1$ scenario will skip over that location because that person is considered dead, yielding a two step move for the $n-1$-person scenario. These two additional one-step moves for the n-person scenario must include a spare and kill action, which equalizes the number of people between the two scenarios up to that point. Therefore, the number of moves afterward will be equal between the two scenarios and hence, the number of additional steps will also be equal. Therefore, the n-person scenario will always have two more steps than the $n-1$ person scenario, yielding a safety position that is two positions more than the $n-1$-person scenario.

To reiterate, if $n=1$, than $s(n)=1$. If n is a power of two, than $s(n)=1$, otherwise $s(n)=s(n-1)+2$. To form a final expression for $s(n)$, let $n=2^k+l$, where k and l are positive integers. The safety position for an n-person group is expressed as twice the distance from the nearest power of two, or $2l+1$. To write the expression explicitely, we have

$$s(n) = \left(n - 2^{\lfloor \log_2 n \rfloor}\right) + 1$$