MATH 3310 – Meeting One, 8/29/2022

Objective: Share the Why, How, and What of this course and hope it will help you succeed. I'll also introduce and clarify some terms used in the lexicon of Math, as well as set some things up for the first homework assignment.

Key Ideas:

- The *Why, How,* and *What* of this course. Many Math students often know *what* they are studying and *what* their efforts are supposed to produce. Many Math students even know *how* they are to produce what they are supposed to. But I know many Math students who don't know *why* they study Math. (I was one such student.) And I'm discounting "because my major requires it".
- *Why*: To become better, more careful, more accurate, and more informed thinkers.

  Question: *When will I use this?*

  Answer: Never, if you don't know how.

  You cannot use a hammer if you don't have one; you cannot think mathematically if you don't know any Mathematics.
- *How*: Practice in a controlled system – Mathematics – exercising the innate faculties we have to think clearly and correctly and to communicate our thoughts unambiguously.
- *What*: Skill in Logic, counting, general problem solving, deduction, and technical writing.

Agenda items:

1. **Stupidity.** You cannot trust your brain. A lot of research focuses on why humans make some of the stupid decisions they do. Humans apparently don't maximize utility; they don't optimize efficiency; and they don't think much about their future. I read that, in economics theory, economists operate with one of the following assumptions: that the players of the economic games are either *econs* (entities that maximize utility), or that they are *humans* (entities that don't necessarily maximize utility). The economists that operate with the latter work to figure out what the 'humans' optimize, if they do, or what principles govern their behavior.

   If success in the World is measured by where you lie on the food chain, we humans are THE most successful. And this success is undeniably due to our very large energy-consuming brains. But our brains function the way they do for reasons based on variation, selection, and heredity (i.e., evolution). But nowadays it is clear that we must *train our brains* to think clearly and correctly; so that we may more easily identify "alternative facts", liars, inconsistencies, and also so that they can project and predict the effects of decisions in the here and now on the future.

   The *epistemology** of Mathematics is the IDEAL, and some like to argue that it is the method through which the universe operates. I think that argument is a waste of time. After experiencing the Method and Ethos† of Mathematics, I see that it provides the RIGHT way to live ... but it doesn't necessarily provide any answers.

2. **Lexicon of Math and Math's Distinction as an Academic Discipline.** There are a few terms academics (teachers and scientists alike) can't seem to get right, and it is a huge problem in my opinion. To facilitate the discussion, please consider the following statement known as **Goldbach's Conjecture**:

   Every even integer greater than 4 is the sum of two prime numbers.

   A **conjecture** is a statement (usually about objects indigenous to Mathematics) that has (1) no proof and (2) no piece of evidence contradicting the statement has been found. If such an example exists it is call a *counterexample* to the conjecture. Goldbach's Conjecture has been verified to be true for the first even

**Epistemology* is the theory of knowledge, especially with regard to its methods, validity, and scope. Epistemology is the investigation of what distinguishes justified belief from opinion.

†the characteristic spirit of a culture, era, or community as manifested in its beliefs and aspirations.

integers less than or equal to 4×10^{18}, but it is not regarded as a (Mathematical) *fact* because it has not been proved. A statement that *has* been proved is called a *theorem*.

Math is not Science. The preponderance of evidence supporting Goldbach's Conjecture together with the fact that mathematicians do not regard Goldbach's Conjecture as true shows this. Mathematicians I know *believe* the conjecture to be true, but belief has very little currency in Math. There are also major philosophical differences between Math and Science, one being that Science investigates causal relationships, while Math does not.

The most important distinction that Math has (with any other discipline) is in its philosophy and the role Truth plays in the practice of Math. Math is the only discipline in which we have access to Truth (Note that capital 't'). But there's a catch. What we talk about in Math and what we can deduce is true about the things we talk about says nothing about the 'real world' (whatever that is). Bertrand Russell[‡] said:

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

That statement is often taken out of context and used derisively, but I think it is very insightful, and a lot can be gained from understanding what this seemingly facetious statement is saying. Here's the original context in which the statement was written.

> Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true. Both these points would belong to applied mathematics. We start, in pure mathematics, from certain rules of inference, by which we can infer that if one proposition is true, then so is some other proposition. These rules of inference constitute the major part of the principles of formal logic. We then take any hypothesis that seems amusing, and deduce its consequences. If our hypothesis is about anything, and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. People who have been puzzled by the beginnings of mathematics will, I hope, find comfort in this definition, and will probably agree that it is accurate.
>
> — Bertrand Russell

Bertrand also said things like "Mathematics, rightly viewed, possesses not only truth, but supreme beauty — a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."

Anyway, these quotes deserve some discussion, which will hopefully happen in class. The point is that Math is concerned with pursuing Truth. I realize '*Truth*' is a relative term and the concept needs defining. Fortunately, it has a precise definition in Math, and I think you'll agree that the model for Truth in Math is the ideal one that no one would debate. Math's process for exhibiting Truth relies on the process of *deductions made from fundamental ideas*. These fundamental ideas are accepted as true, but not immune from questioning. The fundamental ideas are called *Axioms*.

I'll define 'axiom' and other terms from the lexicon of Math, but first I will editorialize a little. The terms I discuss now are used inconsistently, and hence sometimes incorrectly. *This is a* HUGE *problem.* I say so because I believe the pervasive improper use of these terms shows a misunderstanding of Math.

Theorem: A statement, probably expressing a containment relationship, that has been proved true. ... and therefore will remain true forever and ever and ever and ever, even after the sun grows cold and regardless of who's god wins. (Theorems are unique to Math.)

[‡] A philosopher/mathematician/political activist who died in 1970 and worked on the Foundations of Mathematics – he is also the main character in *Logicomix*.

Theory: A tentative statement, intended to provide a framework for thinking scientifically.

Mathematical Definition: A statement that creates an object, and typically gives the object a name, or stipulates how a symbol is to be interpreted.

Colloquial Definition: A description of how a symbol is to be interpreted. For example, the definition of the symbol 'asinine' in Meriam-Webster's dictionary tells us to interpret that symbol as "extremely or utterly foolish". Colloquial definitions are subject to change based on how the terms being defined are used. For example, in 1999, the term 'peruse' was defined as "to examine carefully"; but since it was used consistently in a way consistent with "casually examine", the definition changed sometime around 2005.

Axiom: First I need to say what an axiom is NOT: A axiom is not an intuitively obvious statement (something like this is what a CAPOTS[§] may believe). An axiom *is* a statement that is taken to be true, and usually defines a relationship between fundamental objects that are probably not defined and may not be defineable. But (and this is important to realize): Axioms are not arbitrarily chosen. Thinking axioms are arbitrarily chosen is to liken Math to a game, like football, whose rules of play *are* arbitrary. History shows that axioms have been reverse-engineered in attempts to describe precisely what relationships govern the Mathematics we use. Most of Mathematics was used extensively long before any of it was axiomatized. (Axioms are unique to Math. There are no axioms in Psychology, Biology, Physics, Sociology, Food Science, etc. But you could say that there are axioms in football – the axioms of football are the *rules* of the game.)

Postulate: This term should not be used as a noun. As a verb, as in "to *postulate*", it is perfectly acceptable and has a role in language. If you postulate $\mathcal{X}$, the way our language works make it appear to be stronger than conjecturing $\mathcal{X}$. But Goldbach's conjecture has been verified more than any human could ever verify what she has postulated.

Law: (Contrary to erroneous use in textbooks that is furthered by ill-advised teachers, there are no laws in Math; laws are akin to Science ... and the legal system, but I'm not talking about judicial laws here.) A statement based on repeated experimental observations that describes some aspects of the universe. A scientific law always applies under the same conditions, and implies that there is a causal relationship involving its elements. Laws are often quoted as fundamental controlling influences rather than a description of observed facts, e.g. "the laws of motion require that...".

Rule: Rules have no place in Mathematics. Outside of Mathematics they typically help us get along or prevent us from hurting ourselves ... or are simply annoying.

3. **Sloppy Language is Culturally Acceptable** (*but not Mathematically acceptable*) Communication is flawed at best. The cure for this is **Precision in Language.** Consider the following ridiculously vague statement (you may be familiar with it).

"All men are created equal."

This statement can only be true under some (unknown by me) interpretation of 'created' and of 'equal'. The 'men' part is strange and it possibly indicates that the statement (whatever it means) was obvious in the case of women, doesn't apply to women, or maybe women didn't exist when the statement was penned. But the way I see this statement used suggests it needs some serious revision; I suggest the following because it seems to be consistent with how the statement is used.

Each person is an individual who should not be prejudged by statistics related to particular groups to which they belong.

It is possible to be completely free of ambiguity in contexts outside Mathematics, but it is very difficult.

[§]Common *Attentive* Person On the Street.

4. **Example of Truth (in Math).** An example of a true statement: *The sum of two odd integers is an even integer.* This statement is *true* because it has a proof. It has a proof because it is about mathematical objects and concepts which have either been clearly defined or axiomatized.

   How is the statement "*The sum of two odd integers is an even integer*" proved? By displaying a string of correct deductions based on axioms, accepted definitions, and previously proved theorems. This "display" is the proof, and is best presented as prose.

   So where do we begin? We need to be clear on all the terms used such as *integer*, *even*, and *odd*, and possibly *sum*.

   If the definitions of terms used in a statement are not known, the statement cannot be proved.

   So ...

   An integer x is **even** if the equation $x = 2k$ is true for some integer k. An integer y is **odd** if the equation $y = 2k + 1$ is true for some integer k.

   I cannot define 'integer', and will have to take your knowledge and understanding of 'integer' for granted.

   We are now ready to prove that *The sum of two odd integers is an even integer.* I'll do this in class, possibly referencing the axioms below.

5. **The ~~Rules~~ Axioms For Stuff You've Been Doing for a Long Time.** In no particular order, below are the axioms you took for granted in calculus and whenever you've used arithmetic with real numbers. These axioms constitute a definition for the concept '*Field*' (not the kind with grasses or wheat or beans). A **field** is an amalgamation of three things: A set of 'numbers' I'll denote by F, and two operations defined on the numbers, say addition (which I'll denote by '$+$', and multiplication (which I'll denote by juxtaposition or '$\cdot$'. The operations must satisfy the properties -1 through 10 below.

   (-1) F is closed under addition; if $x, y \in F$, then $x + y \in F$.

   (0) F is closed under multiplication; if $x, y \in F$, then $x \cdot y \in F$.[¶]

   (1) Addition is commutative; If $x, y \in F$, then $x + y = y + x$.

   (2) Addition is associative; If $x, y, z \in F$, then $x + y + z$ makes sense as $(x + y) + z$ or $x + (y + z)$, and the two quantities are equal.

   (3) There exists an additive identity; There is some element, denote it by 0_F, for which $0_F + x = x + 0_F = x$ for every $x \in F$.

   (4) There is always an additive inverse; For every $x \in F$, there exists an object in F, we denote by $-x$, for which $x + (-x) = 0_F$.

   (5) Multiplication is commutative; If $x, y \in F$, then $x \cdot y = y \cdot x$.

   (6) Multiplication is associative; If $x, y, z \in F$, then $x \cdot y \cdot z$ makes sense and is equal to $(x \cdot y) \cdot z$ which is also equal to $x \cdot (y \cdot z)$.

   (7) There exists a multiplicative identity; There is an object in F, denote it by 1_F, for which $1_F \cdot x = x \cdot 1_F = x$ for every $x \in F$.

   (8) Every element of F has a multiplicative inverse, except for the additive identity; If $x \in F$ and $x \neq 0_F$, then there exists an object in F, usually denoted x^{-1}, such that $x \cdot x^{-1} = x^{-1} \cdot x = 1_F$.

   (9) Multiplication distributes over addition on the left and on the right[‖]; If $x, y, z \in F$, then $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

[¶]The axioms I numbered -1 and 0 are typically not included, because it is often stipulated that $+$ and $\cdot$ are *binary operations* on F, which implies the operations produce other objects of F when they operate on objects of F.

[‖]This is often called the "distributive law", and I loathe the fact that the term '*law*' is used. This detracts from the progress of humankind.

(10) The additive identity and multiplicative identity are distinct; $1_F \neq 0_F$.

Arcana: The first axiom is not necessary since it can be proved from the others, but it is traditionally listed even though doing so flies in the face of what mathematicians think about axioms. Ordinarily, *The list of axioms should be minimal.* A great deal of Mathematics was borne out of the plight to prove that Euclid's "*parallel postulate*" was not needed as an axiom.

Below I list some other number systems you might work with in another math class or need for some application.

Ring: If $+$ and $\cdot$ are defined is such a way that (-1), (0), (1), (2), (3), (4), (6), and (9) hold, then $(F, +, \cdot)$ is a **ring**.

Commutative Ring: If $(F, +, \cdot)$ is a ring and property (5) holds, then $(F, +, \cdot)$ is a **commutative ring**.

Ring with Identity: If $(F, +, \cdot)$ is a ring and property (7) holds, then $(F, +, \cdot)$ is a **ring with identity**.

Integral Domain: Suppose $(F, +, \cdot)$ is a commutative ring with identity with the property that whenever $x, y \in F$ and $x \cdot y = 0_F$, then $x = 0_F$ or $y = 0_F$; then $(F, +, \cdot)$ is an **integral domain**.

Homework #1: Due 9/9/2022 Please respond to as many of the following prompts as you can by writing readable and valid arguments that exhibit mathematical fluency to the extent that you can do this.

1. Let R be the set of positive real numbers and define addition, denoted by $\oplus$, and multiplication, denoted by $\otimes$, as follows. For every $a, b \in R$, $a \oplus b = ab$, and $a \otimes b = a^{\log b}$. Please prove or disprove $(R, \oplus, \otimes)$ is a field.

2. Denote the set $\{0, 1, 2, 3\}$ by $\mathbb{Z}_4$, and define addition, denoted $+$, and multiplication, denoted by $\cdot$ or juxtaposition, via the following tables:

$$\begin{array}{c|cccc} + & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 0 \\ 2 & 2 & 3 & 0 & 1 \\ 3 & 3 & 0 & 1 & 2 \end{array} \qquad \begin{array}{c|cccc} \cdot & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 0 & 2 & 0 & 2 \\ 3 & 0 & 3 & 2 & 1 \end{array}.$$

Please prove or disprove $(\mathbb{Z}_4, +, \cdot)$ is a field.

3. Denote the set $\{0, 1\}$ by $\mathbb{Z}_2$, and define addition, denoted $+$, and multiplication, denoted by $\cdot$ or juxtaposition, via the following tables:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}.$$

Please prove or disprove $(\mathbb{Z}_2, +, \cdot)$ is a field.

This lecture has been brought to you by ...
The numbers, zero and one, the letter $\mathbb{R}$, and Bertrand Russell: