Article

A Scalable Approach to Minimize Charging Cost for Electric Bus Fleets

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Abstract: incorporating battery electric buses into bus fleets faces three primary challenges: a BEB's extended refuel time, the cost of charging, both by the consumer and the power provider, and large compute demands for planning methods. When BEBs charge, the additional demands on the grid may exceed hardware limitations and so power providers divide a consumer's energy needs into separate meters even though doing so is expensive for both power providers and consumers. Prior work has developed a number of strategies for computing charge schedules for bus fleets, however prior work has not worked to reduce cost by aggregating meters. Additionally, because many works use mixed integer linear programs, their compute needs make planning for commercial sized bus fleets intractable. This work presents a multi-program approach to computing charge plans for electric bus fleets. Rather than posing a single large MILP that incorporates every aspect of the charging problem, we solve a series of small subproblems in which the solution to the charging problem becomes successively more refined and moves closer to the optimal schedule. Our results show that intermediate subproblems can be solved with a dramatic reduction in runtimes allowing our method to be applied to significantly larger bus fleets. In fact, we will show that not only do the runtimes scale linearly with the number of buses, easily planning for fleets of 100+ buses, but the monthly cost does as well.

Keywords: Battery Electric Bus, Charge Schedule, Mixed Integer Linear Program, Bus Fleet, Grid Management

1. Introduction

Battery electric buses (BEBs) are replacing diesel and natural gas buses in public transportation because they offer many benefits [1] including reduced maintenance [2], zero emissions [3], and access to renewable energy [4]. The challenge of prolonged charging times has been addressed in prior research including distributed charging networks [5], bus availability, environmental impact [6], route scheduling [7], battery health [8], the cost of electricity [9], and the cost of charging infrastructure [10].

High power chargers can be used to avoid lengthy charing times, but fast charging puts high power demands on electrical infrastructure [11] so that power networks become unreliable [12] and may require expensive upgrades [13]. An effective charge plan must balance the need to charge quickly with the desire to maintain a low power profile [14].

Methods for developing charge plans range from heuristic approaches [15], to network flow on a graph [16], to reinforcement learning[17], to mixed integer linear programs (MILP) [18]. Generally, each method minimizes cost by either decreasing the instantaneous power needs for the fleet, or optimizing around time-of-use tarrifs [19].

Scaling these methods to large bus fleets (>100 BEBs) and numerous chargers is a challenge due to the size of the optimization problem that must be solved. For small fleets (<50 BEBs) and less than 10 chargers, the optimization problems in [16,18,19] have over 10^5 variables (including binary and integer variables) and over 10^5 constraints. Scaling to

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larger fleets and more chargers stresses computational resources and requires lengthy solve times.

This paper continues the theme of prior work which is to develop charging schedules for electric buses that minimize the monthly electricity bill (energy consumption plus power demand) while satisfying route constraints that demand buses be in specific locations at specific times. One novelty is that our formulation considers the agregated effects of loading across multiple meters. While meter agregation is not widespread today, distribution networks must be built to supply worst case loads to each metered circuit. Therefore, our approach begins to explore how optimization of loads across multilple meters can reduce the overall impact of BEB charging on the grid. In this work, meter agregation is modeled through the inclusion of uncontrolled (i.e. non-BEB charging) loads. Specifically, we incorporate historical load data from an electric train (UTA TRAX) that visits a central intermodal hub transit site in Salt Lake City, Utah which is also a charging stop for BEBs.

The main contribution of the present paper is addressing the matter of scale. Rather than posing a single large MILP that incorporates every aspect of the charging problem, we solve a series of small subproblems in which the solution to the charging problem becomes successively more refined and moves closer to the optimal schedule. Our results show that intermediate subproblems can be solved with a dramatic reduction in runtimes allowing our method to be applied to significantly larger bus fleets. In a sense, this work explores what is gained in runtime by sacrificing some optimality in the schedule. The subproblems fall into three groups as shown in Fig. 1. Each sub-problem is solved using a linear, quadratic, or integer program and when used together the series of programs provides a near optimal charge plan. Each sub-problem addresses elements from one of three areas: energy allocation and bus grouping, session length and bus-to-charger assignments, and second-by-second optimization.

1.1. Energy Allocation and Group Assignment

The first set of problems answers two primary questions: (1) at what time should energy be delivered to each bus, and (2) which buses are most able to share a charger. These questions are addressed through three sub-problems: unconstrained charge schedule, smooth charge schedule, and group separation.

The unconstrained schedule problem (denoted P_1), which is described in Section 2, computes an optimal charge schedule which minimises the monthly cost of power in the presence of uncontrolled loads under the assumption that each bus has a dedicated charger.

The smooth schedule problem (denoted P_2), which is described in Section 3, has the same form as P_1 except for two differences. First, the monthly cost is required to match the optimal cost from the solution to the unconstrained scheduling problem P_1. Second, the objective for the smooth schedule problem P_2 penalizes change in the scheduled charge rates.

The group assignment problem (denoted P_3), which is described in Section 4, uses the charge schedules from P_2 to separate buses into groups such that the bus schedules within a group overlap as little as possible. Separating the scheduling problem into groups helps to manage the number of computations in succeeding optimization problems by reducing the size of these problems.

1.2. Session Time and Charger Assignment

The problems in the session time and charger assignment section, which are computed on a per-group basis to reduce the number of computations, address two questions: (1) when should charge sessions start and stop, and (2) which charger should be used for each session. These questions are answered through three sub-problems: defragmentation, charger assignment, and session refinement.

The defragmentation problem (denoted P_4), which is described in Section 5, attempts to consolidate charge sessions with small amounts of energy to reduce the number of charge sessions and serves to both decrease the computational complexity of the charger

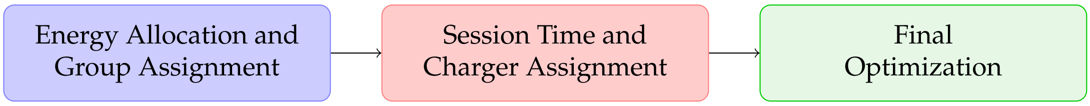

Figure 1. Overall Processing Chain

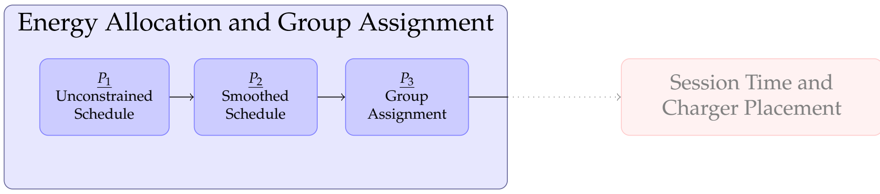

Figure 2. Processing chain for the energy allocation and group assignment problems

assignment problem by reducing the number of charge sessions and simplify the charge schedule to make it more operationally feasible.

After consolidation, each charge session is defined by a minimum/maximum start/stop time as given by the bus's arrival and departure times and an energy requirement in kW. The charger assignment problem (denoted P_5) is described in Section 6 uses the availability and energy constraints to assign chargers to charge sessions.

Once charge sessions are assigned to chargers, the final step is to ensure each session makes the most of each charger's availability. Many times the charge schedules do not use all available time on a charger. The charger refinement problem (denoted P_6) expands each charge session to fill unused time and prioritizes sessions with higher energy demands for adjacent sessions.

1.3. Final Optimization

Solutions to the previous problems provide a set of charge sessions, energy requirements, and time schedules for specific chargers. The final question to be answered is how should the energy for each session will be delivered. The two sub-problems in the final optimization section mirror problems P_1 and P_2 from the energy allocation and group assignment sections. The first problem (denoted P_7), uses the energy and time constraints from previos solutions to compute an optimal charge schedule in Section 8 and is analagous to the unconstrained charge problem P_1. The second problem (denoted P_8) computes a smoothed charge schedule with the same cost as the constrained schedule solution in Section 9 and is analagous to the smooth charge schedule problem P_2 in Section 3. The table given in Fig. 5 lists each sub-problem and which features each problem incorporates.

2. P_1: Unconstrained Schedule

This section describes a program that finds a charge schedule where buses are allowed to charge without regard to the number of available chargers. This solution is considered "optimal" and will be used in later sections to formulate a feasible solution that accounts for the actual number of chargers available.

2.1. Formulation

The cost objective we minimize is based on the rate schedule from [20], which contains two primary elements: the cost of energy and power demand. Energy is billed per kWh using different rates for on-peak and off-peak hours. Demand is divided into two

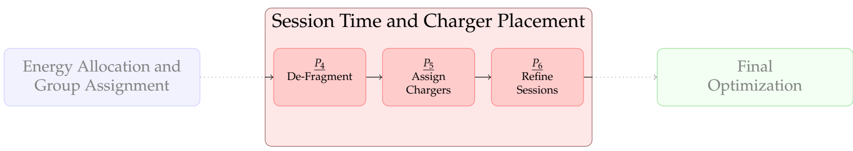

Figure 3. Processing chain for each group

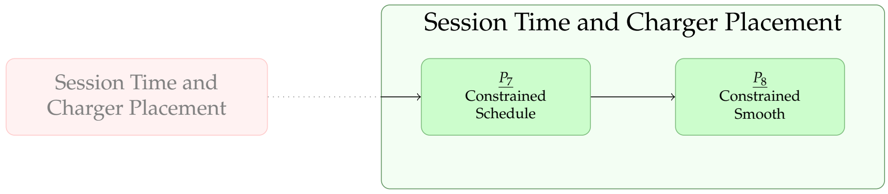

Figure 4. Processing chain for the Final Optimization set

Feature	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
Battery State of Charge	x	x		x			x	x
Minimize Cost	x	x		x			x	x
Charger Capacity	x	x	x	x	x		x	x
Energy Placement	x	x		x			x	x
Smooth Charge Plan		x						x
Computationally Scalable			x				x	x
Small Number of Charge Sessions				x			x	x
Number of Chargers					x		x	x
Efficient Charger Use						x	x	x
Precise Charge Plan							x	x

Figure 5. Descriptions of in which problems features are addressed

Table 1. Description of the billing structure

	On-Peak	Off-Peak	Facilities (Both)
Energy Rate	\$ 0.058282 /kWh	\$ 0.029624 /kWh	None
Energy Rate Symbol	$\mu_{\text{e-on}}$	$\mu_{\text{e-off}}$	None
Power Rate	\$ 15.73 /kW	None	\$ 4.81 /kW
Power Rate Symbol	$\mu_{\text{p-on}}$	None	$\mu_{\text{p-all}}$

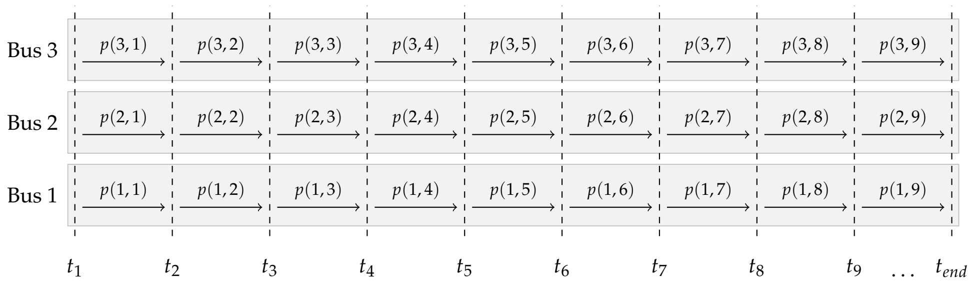

Figure 6. Demonstrates how bus power use is conceptualized

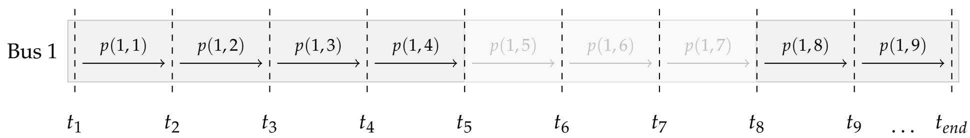

Figure 7. Bus schedule with availability

components. The first is a facilities charge which is billed per kW for the highest 15-minute average power use over the course of the month. The second is a demand charge, which is also billed per kW, but is only billed for the highest 15-minute average power usesd during on-peak hours. The rates for each component are given in Table 1.

Before computing the total monthly cost of electricity, we must define expressions for the average power and energy over time. Let each day be divided into time intervals of length ΔT where the average power consumed by bus i during time j is denoted $p(i,j)$ as shown in Fig. 6. Note that ΔT may be chosen to be on the order of a second or minute, and expressions for 15-minute averages will be derived later. The solution will yield the average power consumed by each bus during each time interval.

The time windows when each bus is available for charging must be accounted for as constraints. The maximum average power is set to zero when a bus is away from the station. For example, if bus 1 were out on route for times t_5, t_6, and t_7, then the average power for those periods would be equal to zero as shown in Fig. 7. Let $\boldsymbol{b}_{p(i,j)}$ be the average power used by bus i at time index j, and $\boldsymbol{b}$ be a vector which contains $b_{p(i,j)}$ for each bus and time index. Also let $\mathcal{A} \subset i \times j$ be the set of all indices where bus i is in the station during time t_j and let $\tilde{\mathcal{A}}$ be its complement. Also let $p_{\max}$ be the maximum power that a charger can deliver.

The set of constraints that buses do not use power when not in the station are given by

$$b_{p(i,j)} = 0 \; \forall i,j \in \tilde{\mathcal{A}} \tag{1}$$

$$0 \le b_{p(i,j)} \le p_{\max} \; \forall i,j \in \mathcal{A} \tag{2}$$

2.2. Battery

Each bus must also maintain its state of charge above a minimum acceptable levels throughout the day. When buses leave the station, their batteries discharges energy as it traverses its route. Let $\delta(i,j)$ be the amount of charge lost by bus i at time j and let $h(i,j)$ be the state of charge of bus i at time j. The state of charge for each bus can be defined as

$$\begin{aligned} &h(i,j) = h(i,j-1) + b_p(i,j-1) \cdot \Delta T - \delta(i,j) \; \forall i, j > 1 \\ &h(i,1) = \eta_i \; \forall i \end{aligned} \tag{3}$$

where η_i is the initial state of charge for bus i and ΔT is the difference in time between $t_{i,j}$ and $t_{i,j+1}$. Now that each value for the state of charge is defined, each value for h must be constrained so that it is greater than a minimum acceptable value, $h_{\min}$ but does not exceed the maximum battery capacity $h_{\max}$. This yields

$$h_{\min} \le h(i,j) \le h_{\max} \; \forall i,j \tag{4}$$

The final battery related constraint has to do with how we are planning for the bus. The expenses that come from power are computed monthly, but we desire to simulate the movements of the bus for only a day, and use this to extrapolate what the monthly cost may be. Therefore, the state of charge for a bus at the end of the day must reflect its starting value. This yields the following constraint:

$$h(i, \text{end}) = h(i,1) \; \forall i. \tag{5}$$

2.3. Cumulative Load Management

While this formulation does not directly account for the number of available chargers, we do account for the cumulative load capacities of all chargers. Let the number of chargers be denoted n_{charger}. We desire to maintain the average cumulative power for each time step at a level that is serviceable given n_{charger}. We define a slack variable $p_c(j)$ which represents the total average power consumed by all buses at time j. The variable $p_c(j)$ is computed as the sum of average bus powers so that

$$p_c(j) = \sum_i b_{p(i,j)}. \tag{6}$$

2.4. Objective

Now that the relevent constraints have been addressed, we turn attention to the objective function. We start by computing the total average power for the complete system. This total power is comprised of power used by the buses, and power used by external sources such as lights, ice melt, electric trains, etc which we refer to as "uncontrolled loads", where the average power for the uncontrolled loads at time step j is denoted $u(j)$. We compute the total power as the sum of power used by the buses, $p_c(j)$ and the power consumed by uncontrolled loads $u(j)$ so that the total power, denoted $p_t(j)$ is computed as

$$p_t(j) = p_c(j) + u(j). \tag{7}$$

The next step is to compute the fifteen minute average power use for each time step, denoted p_{15}. We do this by letting

$$p_{15}(j) = \frac{1}{n} \sum_{l \in \{j_{15}\}} p_t(l) \tag{8}$$

where $\{j_{15}\}$ is the set of all indices 15 minutes prior to time t_j and n is the cardinality of $\{j_{15}\}$. Next, note that the rate schedule requires both the maximum overall average power, denoted $p_{\text{facilities}}$, and the maximum average power during on-peak hours, or p_{demand}. Let $\mathcal{S}_{\text{on}}$ be the set of time indices belonging to on-peak hours, and recall that the max over all average power values is greater than or equal to $p_{15}(j)$ for all j. We can express this constraint as

$$p_{\text{facilities}} \geq p_{15}(j) \; \forall j. \tag{9}$$

Because $p_{\text{facilities}}$ will be used in the objective function, the value for $p_{\text{facilities}}$ will be minimised until it is equal to the largest value in p_{15}. Following a similar logic, we also define a set of constraints for the maximum average on-peak power, p_{demand} so that

$$p_{15}(i) \leq p_{\text{demand}} \; \forall i \in \mathcal{S}_{\text{on}}. \tag{10}$$

The next step in computing the objective function is to compute the total *energy* consumed during on and off-peak hours respectively. Let e_{on} be the total energy consumed during on-peak hours and e_{off} be the energy consumed during off-peak hours. We can compute energy as the product of average power and time. In our case, we compute this as

$$\begin{aligned} e_{\text{on}} &= \Delta T \cdot \sum_{i \in \mathcal{S}_{\text{on}}} p_t(i) \\ e_{\text{off}} &= \Delta T \cdot \sum_{i \notin \mathcal{S}_{\text{on}}} p_t(i). \end{aligned} \tag{11}$$

We can now compute the total monthly cost in dollars as

$$J_{\text{cost}} = \begin{bmatrix} e_{\text{on}} \\ e_{\text{off}} \\ p_{\text{facilities}} \\ p_{\text{demand}} \end{bmatrix}^T \begin{bmatrix} \mu_{\text{e-on}} \\ \mu_{\text{e-off}} \\ \mu_{\text{p-all}} \\ \mu_{\text{p-on}} \end{bmatrix} \tag{12}$$

The final optimization problem that computes a charge schedule without constraints on the number of chargers is descibed below.

Summary for P_1

$$\underset{\mathbf{y}}{\text{Min}}\ (12)\ \text{subject to}\ (1) - (11).$$

We have observed that charge commands in solutions to P_1 tend to switch frequently between 0 and p_{max}, which is difficult to implement in practice and imparts stress on charging hardware. Before additional steps can be taken, a smoother set of charge commands is computed, and this is the subject of the next section.

3. P_2: Unconstrained Smooth Schedule

This section implements a smoothing criteria so that the frequent "on-off" switching patterns from P_1 are reduced. This is done by modifying P_1 in two ways. The first is that the demand, facilities, on-peak energy, and off-peak energy are removed from the objective and constrained to equal their values obtained in the solution to P_1 so that

$$\begin{aligned} e_{\text{on}} &= \tilde{e}_{\text{on}} \\ e_{\text{off}} &= \tilde{e}_{\text{off}} \\ p_{\text{facilities}} &= \tilde{p}_{\text{facilities}} \\ p_{\text{demand}} &= \tilde{p}_{\text{demand}}, \end{aligned} \tag{13}$$

where values on the right-hand side are constants extracted from the solution to P_1. Next, we define an alternative objective that incentivizes continuity of charging between time steps. This objective is defined as

$$J_{\text{switch}} = \frac{1}{n} \sum_{i,j,\in\mathcal{K}} \| b(i,j) - b(i,j-1) \|_2^2, \tag{14}$$

where $\mathcal{K}$ is the set of all i, j where bus i may charge during time j and $j-1$. The final optimization problem that produces smooth charging schedules is given below.

Summary for P_2

$$\underset{\mathbf{y}}{\text{Min}}\ (14)\ \text{Subject to}\ (1) - (11),\ (13)$$

The solution to P_2 smooths charge schedules without increasing costs, but it presents the undesireable feature that the charge sessions tend to be fragmented into many short sessions. Additionally, the schedule does not account for the number of chargers or bus contention for charger use. Unfortunatly, addressing these problems requires the use of binary variables and optimization with binary variables becomes untractable for large numbers of buses and chargers. Before the fragmentation and charger assignment problems can be addressed, we first segment the buses into groups. Successive processing can be done separately in groups which helps to manage the computational complexity for later problems that incorporate binary variables.

4. P_3: Group Assignment

This section addresses the matter of problem size. The complexity of the problem is strongly influenced by contention, which arises as multiple buses must share limited charging resources. The number of binary variables in the optimization problem increases as n^2 where n is the number of charge sessions. Before we can formulate a solution to the bus problem that scales linearly with n, we propose a method to separate buses into groups to reduce the coupling between charge sessions.

The group assignment problem separates buses into n_{group} groups, where group m is allocated n^m_{charger} chargers and n^m_{bus} buses. Each group must have sufficient chargers to fill its needs and prefer buses with dissimilar schedules to better avoid contention. We know that the number of cross-terms in future problems will be reduced when each group has the same number of buses. Therefore, let n^m_{bus} be described as

$$\begin{aligned} n^m_{\text{bus}} &\geq \left\lfloor \frac{n_{\text{bus}}}{n_{\text{group}}} \right\rfloor \\ n^m_{\text{bus}} &\leq \left\lceil \frac{n_{\text{bus}}}{n_{\text{group}}} \right\rceil, \end{aligned} \tag{15}$$

where the values n_{bus} and n_{group} are user parameters.

The number of chargers assigned to each group must be exactly equal to the number of available chargers so that

$$n_{\text{charger}} = \sum_m n^m_{\text{charger}}. \tag{16}$$

The next set of constraints ensures that each bus is is part of a group exactly once. Let $\beta(i, m)$ be a binary variable which is one when bus i is in group m. Each bus is constrained to be a member of exactly one group by letting

$$\sum_m \beta(i, m) = 1 \; \forall i. \tag{17}$$

We must also ensure that buses are assigned to groups where the power delivered to each bus can be achieved with the number of chargers assigned to that group. Define a slack variable that gives the total power used in group m at time step j as $p(m, j)$. Recall, we also know the expected power use for each bus as this is a result of P_1 as $b_{p(i,j)}$, which allows us to describe the total power for any one group as

$$p(m, j) = \sum_i \beta(i, m) b_{p(i,j)}. \tag{18}$$

Next, we know that the total load of each group must be less than or equal to the collective capability of that group's chargers, which can be expressed as

$$n^m_{\text{charger}} \cdot p_{\max} \geq p(m, j) \; \forall m, j \tag{19}$$

so that the number of chargers is sufficient to charge the collective load of the group.

We also desire to group together buses whos routes have the least overlap. If two buses contain no overlap, they will be easiest to schedule on the same charger. The overlap is measured using the inner product of their schedules from P_1. If completely non-overlapping, the inner product will be equal to zero. Let

$$\phi(i, i') = \mathbf{b}(i, :)^T \mathbf{b}(j, :),$$

where $\mathbf{b}(i, :)$ is the charge schedule for bus i as computed in the P_1. We desire to minimize the total cross terms $\phi(i, i')$ for all buses in the same group. Define a slack variable $v(i, i', m)$ which is equal to $\phi(i, i')$ if buses i and i' are both in group m and zero otherwise so that

$$\begin{cases} v(i, i', m) = \phi(i, i') & \beta(i, m) = 1, \beta(i', m) = 1 \\ v(i, i', m) = 0 & \text{otherwise} \end{cases}$$

which can also be expressed by letting

$$\begin{aligned} & v(i, i', m) \leq \phi(i, i') \\ & v(i, i', m) \geq \phi(i, i') - M\big(2 - \beta(i, m) - \beta(i', m)\big) \\ & v(i, i', m) \leq 0 + M\beta(i, m) \\ & v(i, i', m) \leq 0 + M\beta(i', m) \\ & v(i, i', m) \geq 0. \end{aligned} \tag{20}$$

The final objective can then be expressed as

$$J_{\text{select}} = \sum_{i,i',m} v(i,i',m). \tag{21}$$

The final optimization problem may be expressed as shown below.

Summary for P_3

$$\underset{\mathbf{y}}{\text{Min}}\ (21)\ \text{subject to}\ (15) - (20)$$

Problems P_1 through P_3 have produced preliminary estimates for charge schedules as well as groups into which the buses can be subdivided but have not addressed the problem of fragmentation, where each bus's schedule contains many short charge sessions, whereas fewer charge sessions is desirable. Before we can address where buses should charge, we must first finalize each bus's charge schedule by decreasing the number of charge events.

5. P_4: Defragmentation

A minimum charge session length is another operational constraint that must be considered. We also consider constraints on minimum energy delivered per session. The intent of these constraints is to avoid charging for short durations or for small amounts of energy so that charge sessions are consolidated for convenience.

To solve these problems, assume there exists a "smoothed" solution from P_2 which has been appropriately placed in a group from P_3. Next, let the preliminary solution be subdivided into charge sessions, each with a specific amount of energy, a minimum start time, and a maximum stop time. If the energy for any charge session is less than the allowed, then this session is marked as "fragmented". The remaining sessions are either marked as "used" or "unused", where a used session delivers more power than specified by a "fragmentation-threshold", and an unused session delivers zero power.

We now propose a new optimization problem in which charge schedule will be defragmented so that each session exceeds a minimum charge threshold. The sessions in question are the "fragmented" sessions. Let $\theta(i,r)$ be a binary variable which indicates if session r from bus i will be active. Because the only sessions in question are fragmented, we only need to define $\theta(i,r)$ for fragmented sessions. Limiting the binary variables in this fashion significantly reduces the computational complexity of this step. The charge problem will be resolved using the same constraints and objective as in P_1, but with the following change. The first change constrains the minimum power delivery for each "active" charge session to be *at least* as large as the original power delivery from P_1. Let $\rho(i,r)$ be a vector which is ΔT, in hours, during the times bus i charges during session r and zero otherwise so that

$$\mathbf{b}(i,:)\rho(i,r) \geq \psi(i,j) \tag{22}$$

where $\psi(i,j)$ is the minimum energy for session i,r and session i,r is considered "active". For inactive sessions, the energy is constrained so that it is equal to zero. Finally, for fragmented sessions, the session energy must be greater than the minimum threshold, ω when active and zero otherwise which can be expressed as

$$\begin{aligned} \mathbf{b}(i,:)\rho(i,r) &\geq \omega - \omega(1-\theta(i,r)) \\ \mathbf{b}(i,:)\rho(i,r) &\leq 0 + \theta(i,r)e_{\max} \end{aligned} \tag{23}$$

where $e_{\max}$ is the maximum energy delivered in a session. The final optimization problem is given below.

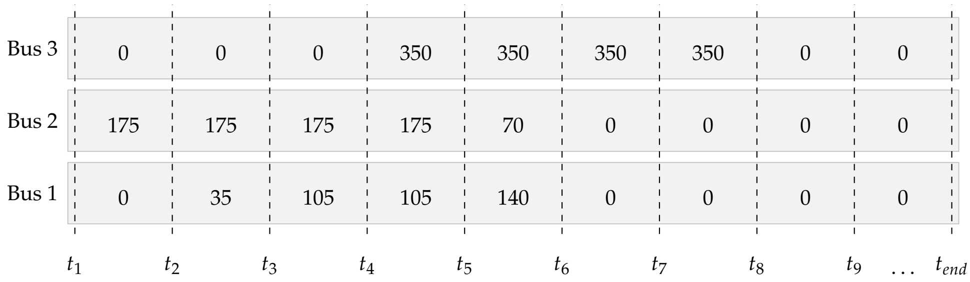

Figure 8. An example solution to a 3-bus, 2-charger scenario from p_4

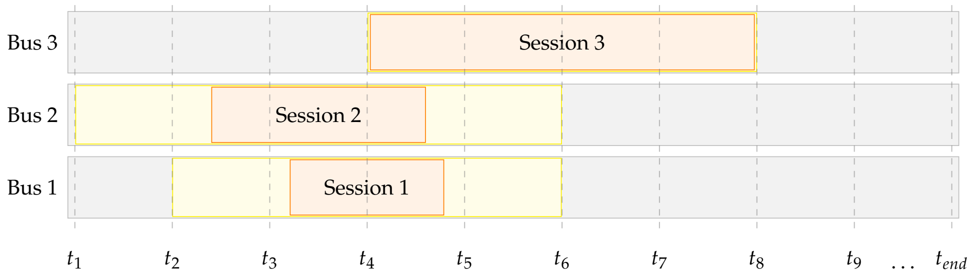

Figure 9. Demonstrates how results from p_4 can be reexpressed in terms of continuous variables

Summary for P_4

$$\min_{\mathbf{y}} \ (12) \text{ subject to } (1) - (11),\ (22),\ (23).$$

The solution to the defragmentation problem, P_4 provides a charge plan that optimizes the cost of power while requiring that each charge session meet a minimum energy criteria. Up to this point however, we still have not addressed constraints related to the number of chargers which is the focuse of the next section.

6. P_5: Charger Assignment

The results from P_4 give a general estimate of how much and when buses should charge, however we must still address two important issues. The first is defining concrete start and stop times for each charge session. The second is limiting the charge sessions to a finite number of chargers.

Consider a solution to a three bus, two charger scenario given in Fig. 8. Note that there appears to be three buses charging at the same time from t_5 to t_6 even though there are only two chargers. We can reformulate this solution in terms of continuous start and stop variables and a variable charge rate so that the *duration* of each charge session may be relaxed. The objective is to transfer the required energy to the corresponding bus within the optimized charge interval.

Note how few of the charge sessions utilize the chargers to full capacity. This implies that there exists a smaller charge window in which equivalent power can be delivered. This allow us to use the charge durations from the solution from Fig. 8 as bounds on *allowable* charge windows instead of enforcing equality.

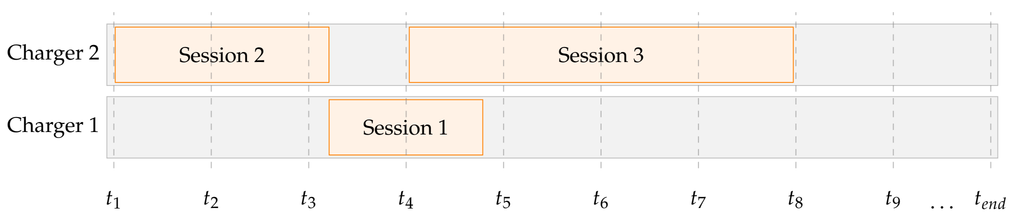

Figure 10. Demonstrates the solution to p_5

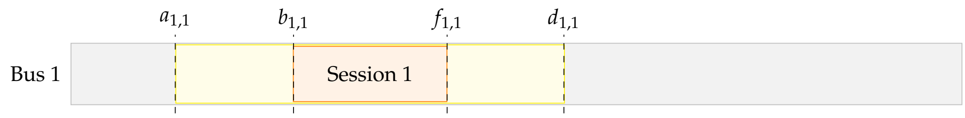

Figure 11. Gives variables of optimization for p_5

An example of how Fig. 8 may be reformulated is given in Fig. 9. Note how the actual charge sessions do not necessarily need to take up all the time they were initially allocated in the first solution and that these times can fluctuate if the average charge rate is less than the maximum charger capacity. In this example, we assume a maximum charge capacity of 350kW.

Note how the third charge session does have to be exactly where it was scheduled because the average is equal to the maximum charge rate. If we examin just the schedule for Bus 1, we note that there are four essential variables for the corresponding charge session: $a(i,r)$, $b(i,r)$, $f(i,r)$ and $d(i,r)$ which represent the minimum start time, actual start time, actual end time, and maximum end time, respectively.

The problem we must now solve is one of arranging these intervals such that each one is larger than its minimum width (or charge time). We must also account for the number of chargers. It can be helpful to view the problem as a bin packing problem, where each session must fit within the "swim lane" of a charger. For example, taking the charge sessions given in Fig. 9 and arranging them so that there is no overlap between sessions will yield a valid solution as shown in Fig. 10. From Fig. 11, we know that $a(i,r), b(i,r), f(i,r)$ and $d(i,r)$ must be such that

$$a(i,r) \leq b(i,r) \leq f(i,r) \leq d(i,r), \tag{24}$$

where $a(i,r)$ and $d(i,r)$ are known from P_4, and $b(i,r)$ and $f(i,r)$ are optimization variables.

To differentiate between different chargers, define $\sigma(i,r,k)$ as a binary selector variable which is one if charger k services bus i for session r and zero otherwise. Because only one charger can charge each bus at a time and each charge session *must* be serviced, we have

$$\sum_k \sigma(i,r,k) = 1 \ \forall i,r. \tag{25}$$

Next, we also know that during each session a certain amount of energy must be transfered from the charger to the battery. The amount of energy that must be transfered to bus i during session r is given in the solution to P_4 and are denoted $e(i,r)$. We can compute a minimum time window from these values by letting

$$w(i,r)_{\min} = \frac{e(i,r)}{p_{\max}}. \tag{26}$$

If we include constraints for a minimum time per session, then the previous expression becomes

$$w(i,r)_{\min} = \max\left(w_{\min}, \frac{e(i,r)}{p_{\max}}\right)$$

Because this is the minimum time window, we must ensure that the difference between the start and stop times is at least this large so that

$$f(i,r) - b(i,r) \geq w(i,r) \ \forall i,r. \tag{27}$$

The final set of constraints deals with contention so that no charger can be scheduled for two sessions that overlap. Let $\mathcal{L} = \{(i,r) \times (i',r')\}$ where charge sessions i,r and i',r' have the potential to overlap. Before we can prevent overlap, we must define a binary variable $l(i,r,i',r')$ which is equal to one when session i,r is scheduled before session i',r' and zero otherwise so that

$$\begin{cases} f(i,r) \leq b(i',r') & l(i,r,i',r') = 1 \\ f(i',r') \leq b(i',r') & l(i,r,i',r') = 0 \end{cases} \tag{28}$$

To simplify these constraints, let M have a large value such as the number of seconds in a day. We know what the top constraint must be trivially satisfied when $l(i,r,i',r') = 0$ and the bottom must also when $l(i,r,i',r') = 1$. This leads to a reformulation so that

$$\begin{aligned} f(i,r) - b(i',r') &\leq M(1 - l(i,r,i',r') \\ f(i',r') - b(i,r) &\leq l(i,r,i',r')M \end{aligned}$$

However, this constraint *only* needs to hold when sessions i,r and i',r' are scheduled to charge on the same charger or that $\sigma(i,r,k) = \sigma(i',r',k) = 1$. We can reformulate the above constraint to satisfy this condition by letting

$$\begin{aligned} f(i,r) - b(i',r') &\leq M(3 - \sigma(i,r,k) - \sigma(i',r',k) - l(i,r,i',r')) \\ f(i',r') - b(i,r) &\leq M(2 - \sigma(i,r,k) - \sigma(i',r',k) + l(i,r,i',r')) \end{aligned} \tag{29}$$

Finally, we desire the schedule to closly match the charge plan from P_4, which occures when each charge session matches the durations given in P_4 and so we formulated an objective function which minimizes the differences in the given plan and the results from P_4 by letting the objective be

$$\min_{f,b} \sum_{i,r} \|b(i,r) - a(i,r)\|_2^2 + \|f(i,r) - d(i,r)\|_2^2 \tag{30}$$

which has the effect of driving each variable to the desired value and more heavily penalizing values that are further from their optimal. The final optimization problem is given below.

Summary for P_5

$\underset{\mathbf{y}}{\text{Min}}$ (30) subject to (24) – (29).

Ideally, when P_5 is solved to optimality, the chargers are fully utilized. However, optimality for P_5 is computationally demanding and scalable solutions may require relaxations in the optimality gap of the solver. However, increasing the gap leads to a solution in which chargers are not fully utilized. The next section uses the session ordering from P_5, but recomputes session start/stop times to better utilize the charger availability even when sub-optimal gaps are given for P_5.

7. P_6: Optimizing Charge Schedules

Many times it is not feasible to compute the optimal set of charge schedules given in the previous sections. As the number of buses and charge sessions becomes large, computing a small-gap solution becomes computationally untractable. Allowing solutions with larger optimality gaps decreases the number of computations, but results in sub-optimal charge-time windows. In this section, a more optimal set of charge windows is computed using the results from P_5 to infer charger assignment and ordering for each charge session. We also know that the optimal solution will expand the charge windows to use any available time where a charger is unused, implying that the "stop" time for each session will either be equal to its buse's departure time, or the start time of the next window which can be expressed as

$$\begin{cases} c(s,i,r+1) = c(f,i,r) & c(d,i,r) > c(a,i,r+1) \\ c(s,i,r+1) = c(a,i,r+1) & \\ \quad c(f,i,r) = c(d,i,r) & c(d,i,r) <= c(a,i,r+1) \end{cases} \tag{31}$$

where $c(s,i,r)$ is the start time for charger i's r^{th} charge session, $c(f,i,r)$ is the stop time for charger i's r^{th} charge session, $c(d,i,r)$ is the departure time for the bus scheduled for charger i's r^{th} charge session, and $c(a,i,r)$ is the arrival time for the bus scheduled for charger i's r^{th} charge session. The minimum charge length must also be used so that energy can be properly delivered, so that

$$c(f,i,r) - c(s,i,r) \geq w(i,r) \tag{32}$$

where $w(i,r)$ is the minimum charge time for the corresponding session.

The final step to optimizing the charge windows is to give preference to windows with larger power deliveries. Let the objective for the optimization program be

$$J_{\text{window}} = \frac{1}{n} \sum_{i,r} \left\| \frac{c(f,i,r) - c(s,i,r)}{e(i,r)} \right\|_2^2. \tag{33}$$

When the function J contains windows with equal amounts of energy, the minimum will be found where each charge interval is the same width. As the amount of energy increases, the objective penalizes less for larger window sizes and thus gives preference to high energy sessions. The final optimization problem is given below.

Summary for P_6

$$\min_{\mathbf{y}} \ (33) \text{ subject to } (31),\ (32)$$

After solving P_6 each charge session is assigned to a charger so that contention for limited chargers has been managed for each group. Furthermore, each session also specifies target energy requirements which manage the risks of depleting batteries but does not give instructions on how the energy is to be delivered. The energy delivery problem is addressed in the next section and combined results for all groups so that the charge schedule begins to approach a globally optimal solution.

8. P_7 : Constrained Schedule

Up to this point, we have computed charge chedules which assume that any bus can charge regardless of the number of chargers. We then separated buses into groups to reduce the scale of the problem and treat each sub-problem separately while we defragment sessions and assign each charge session to specific chargers before determining the final start and stop times for each bus's charge session.

The final step in this process is to determine how the energy will be delivered so that cost is minimised. We begin with constraints for bus power, energy, and cost from Section 2 that are expressed as equations (1), (5), (7), (8), (9), (10) and (11). Next, include constraints for energy so that the energy for each charge session is properly delivered using a modified version of (22) so that

$$\mathbf{b}(i,:)\rho(i,r) = \psi(i,r), \tag{34}$$

where $\psi(i,r)$ is the required energy for bus i during rest period r as computed from the solution of the defragmentation problem. The resulting optimization problem is given below.

Summary for P_7

$$\underset{\mathbf{y}}{\text{Min}}\ (12)\ \text{subject to}\ (1),\ (5),\ (7)-(11),\ (34)$$

9. P_8 : Constrained Smooth Schedule

The charge schedule from P_7 will contain the same on-off switching defects as the solution to P_1 which can be managed as before by executing P_7 once again with the same modifications that lead to P_2: (1) constrain the cost terms in the objective to equal their values from P_7, and (2) reduce the difference of sequential charge rates with the smoothing objective from 14. The resulting optimization problem is given below.

Summary for P_8

$$\underset{\mathbf{y}}{\text{Min}}\ (14)\ \text{subject to}\ (1),\ (5),\ (7)-(11),\ (13),\ (34)$$

10. Results

The results given in this section aim to demonstrate how the proposed method can be used to find a scalable solution to the bus charge problem. Because the proposed solution contains various sub-problems, optimization parameters for each sub-problem may be tuned to best meet the demands of a given scenario, allowing for a degree of flexability that is not present in prior works which formulate solutions to the bus charge problem as a single optimization problem.

10.1. Overall Performance

In this section, we compare the proposed method with a baseline algorithm and a method developed by [21]. The baseline method models how bus drivers charge their electric vehicles at the Utah Transit Authority (UTA) in Salt Lake City, Utah. At UTA, when bus drivers arrive at the station, they recharge their batteries whenever a charger is available so that the number of charge sessions is maximized. The method from [21] works somewhat differently by minimizing the cost of energy with respect to time of use tarrifs μ_{e-on} and μ_{e-off}.

The comparison we observe is given for a 10-bus, 10-charger scenario and a single group. Each method was used to compute a charage schedule and the costs from demand, facilities, and energy charges are given in Fig. 12. Note how the baseline algorithm suffers significantly from the demand charges associated with on-peak power, and [21] incurres additional cost from the facilities charges, indicating that an emphasis on energy charges and habitual charging patterns can be improved.

We observe where the differences in cost originate in Fig. 14. Observe how the baseline charge profile achieved the largest 15-minute average power between 19:12 and 21:36, which is during on-peak hours and consequently yielded the large on-peak power charges given in Fig. 12. Additionally, note how the proposed method maintains a relatively flat power profile so that the load is balanced throughout the day, which we investigate in Fig. 13.

In Fig. 13, note how the proposed method produces a bus load that mirrors the uncontrolled load, yielding the flat load profile from Fig. 14 which is especially prevalent from 7:12 to 14:24. The results show that the proposed method works well, outperforming both historical patterns at UTA as well and improves upon prior methods.

10.2. Optimality Gap and Contention Trade-Off

In the previous section, we compared performance of three methods where each method was produced using a small gap in the numerical solver. In general, the most

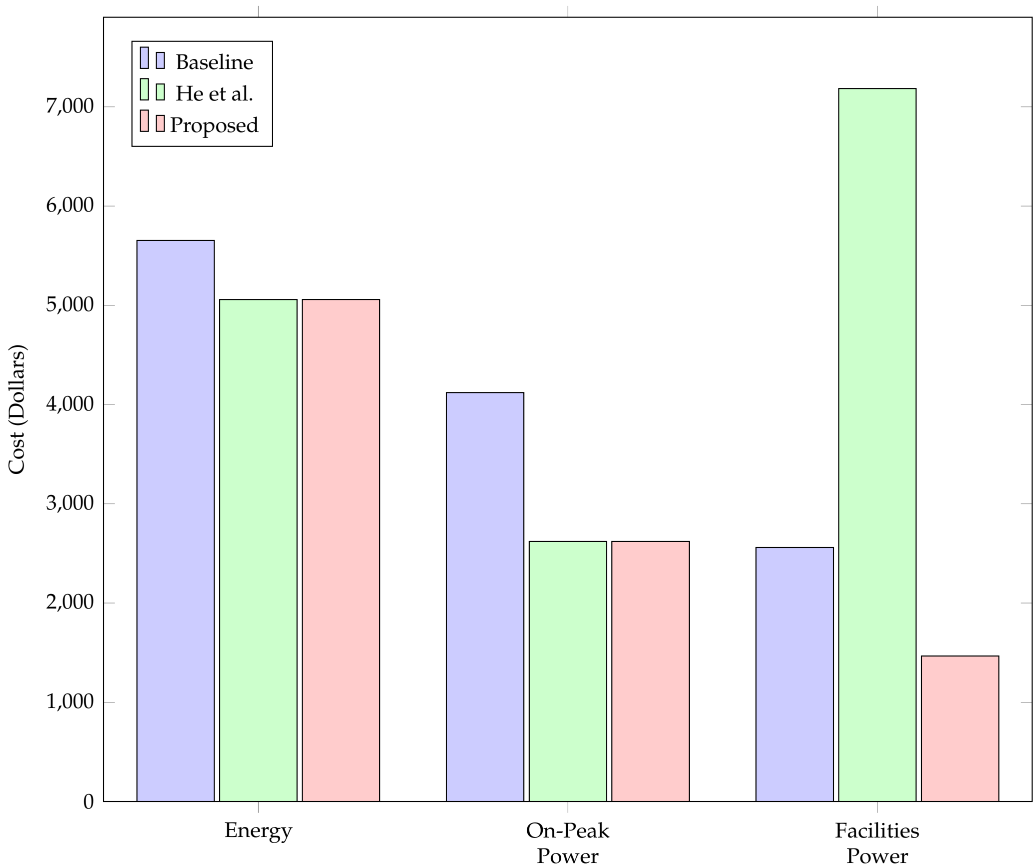

Figure 12. Cost comparison with prior work

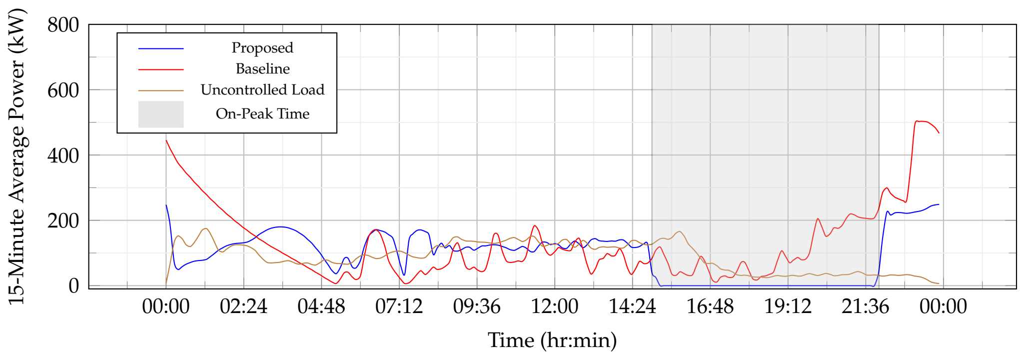

Figure 13. Comparison between uncontrolled and bus loads

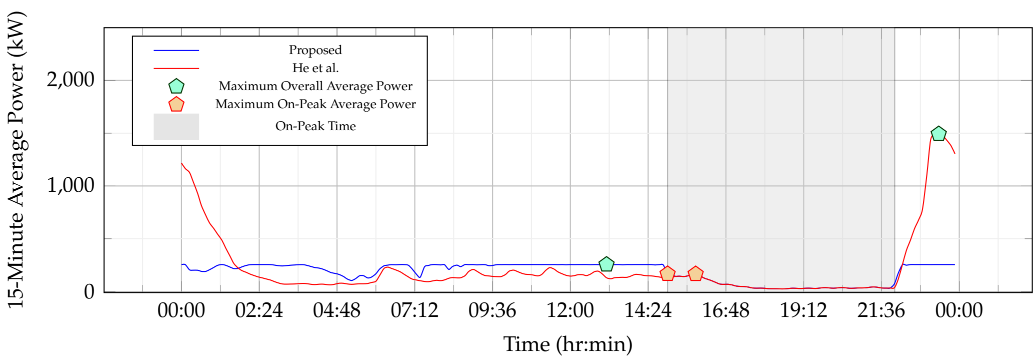

Figure 14. 15-Minute average power for one day

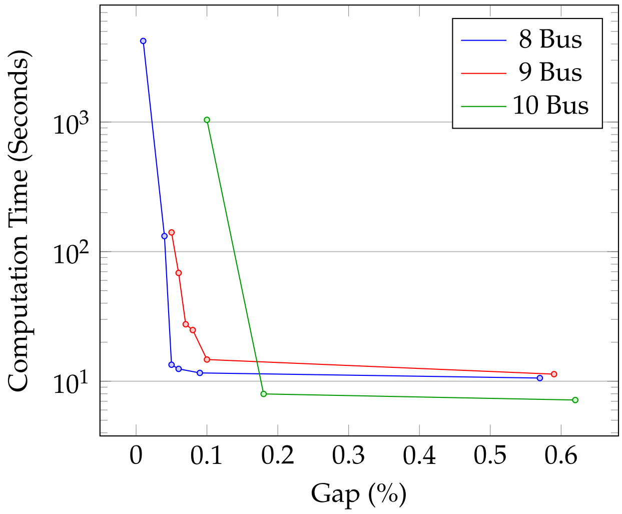

Figure 15. Comparison of Runtime for a 7-Charger Scenario

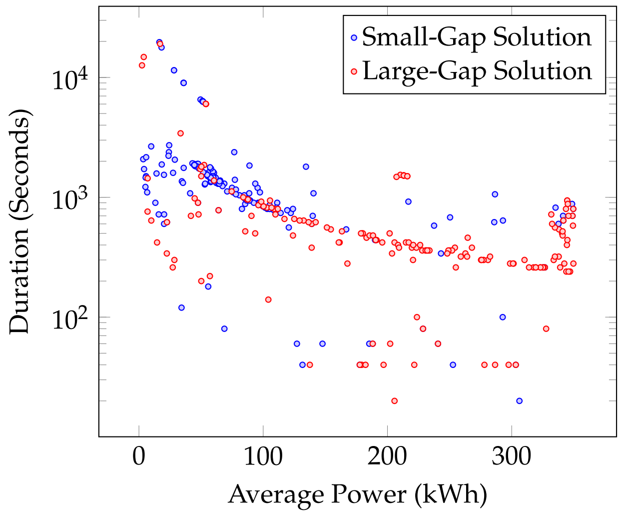

Figure 16. Comparison of charge session duration vs average charge rate

computationally demanding solution addressed bus-to-charger placement and can requre a very small gap to yield good solutions. 431 432

This work also seeks to address how to compute a solution in a scalable manner and so this section reviews the relationship between computation time and the number of buses by considering a 7-charger scenario with runtime comparisons for 8, 9, and 10 buses. 433 434 435

Fig. 15 plots the computation time as a function of optimality for a 7-charger 8-bus scenario in blue, a 7-charger 9-bus scenario in red, and a 7-charger 10-bus scenario in green. In each scenario, note how there exists a gap after which computation time dramatically increases for small improvements in the optimality gap. Solving P_5 to an optimality gap after this point becomes computationally expensive and should be avoided as the number of buses grows. 436 437 438 439 440 441

Additionally, note how the high solve-time region (near zero gap) for the 8-bus scenario begins at a smaller gap than that of the 9 or 10-bus scenarios. This demonstrates a relationship between contention and computation time as contention increases with the number of buses if the number of chargers is fixed. We can conclude therefore, that saving computation time as the number of buses increase can be accomplished by slackening (increasing) the optimality gap given to the numerical solver. 442 443 444 445 446 447

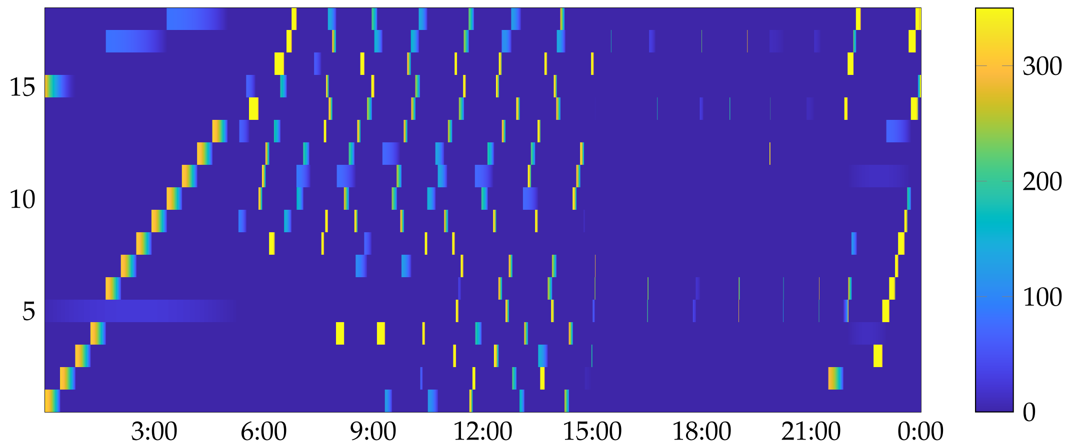

Figure 17. Routes with a large gap in the route placement problem

10.3. Contention: Sub-Optimal Schedules

In the previous section we observed that the proposed method cannot scale with contention if the optimality gap is too small. This section considers an experiment to motivate the division into groups from P_3. The focus of this experiment is to compare the the duration and charge rates of a small-gap and large-gap scenarios. Solutions to P_5 are preferred if session lengths are longer and require smaller charge rates because long charge sessions are more practical in the real world and small charge rates are easier to implement on charging hardware.

Fig. 16 displays the charge session durations as a function of charge rate for two solutions to an 18 bus 6 charger scenario. The first solution, shown in blue, was computed with a small optimality gap and the second, shown in red, was computed for a large optimality gap. Note how the charge sessions from the small-gap solution tend to have larger session durations and lower charge rates than the solution for the large-gap sessions, indicating the value of smaller gaps.

We further illustrate the difference in optimality gap by directly comparing the charge schedules for each scenario in Fig. 17 and Fig. 18. In each figure, the color at the i, j location represents the charge rate for bus i at time j. Observe how the first sessions for buses $1-4$ and $6-13$ are assigned to a single charger in Fig. **??** so that each charge session is compressed to accomodate the large number of buses. The remaining chargers appear to have at most one session which implies that charge sessions were poorly managed in the large-gap scenario. In comparison, the small-gap solution in Fig. 18 yields a more evenly distributed session load for each charger so that each session is lengthened, and contains smaller charge rates.

It is also interesting to note that the monthly costs of each solution may or may not be equivalent even though a small-gap solution is clearly superior. Therefore, a small gap is required to consistently achieve optimal session placement. We also know from Fig. 15 that small optimality gaps may increase the number of computations so that the charger assignment problem becomes untractable for large numbers of buses, making it necessary to reduce the problem scope by dividing buses into groups.

10.4. The Importance of Groups

One contribution this work provides is a way to compute cost-oriented charge schedules that scales well as the number of buses increases. We know from the previous section that the charger assignment problem will not scale for small optimality gaps. This section describes how the computational complexity of the charger assignment problem can be managed by separating the buses into groups so that the charger assignment problem can be solved for each group independently.

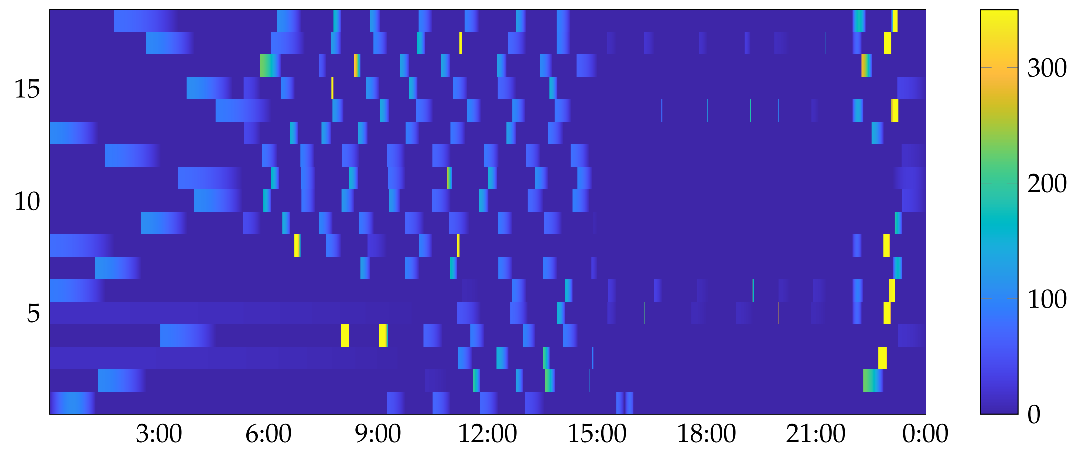

Figure 18. Routes with a small gap in the route placement problem

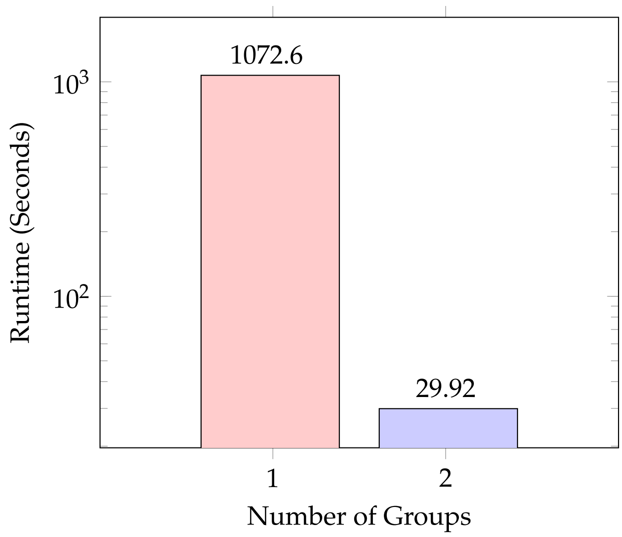

Figure 19. Runtimes for a 18 bus 12 charger scenario at a 0.13% gap

In this section, we consider a 18 bus, 12 charger scenario with a 0.13% gap in the charger assignment problem. Fig. 19, shows the respective runtimes for a one and two group scenario in P_5. Note how the runtime for the two group scenario is several orders of magnitude less then the runtime for the single group case which demonstrates how a small number of groups can manage the runtime for optimal charger assignment solutions.

10.5. Effects of De-Fragmentation

This paper also addresses the operational preference to consolidate charge sessions when possible, an operation we have called defragmentation. This section demonstrates the effectiveness of the defragmentation method given in P_4 and how consolidation affects the monthly cost. In section 5, the threshold for defragmentation is given by the minimum allowable energy per charge session. In this section we compare two 40 bus, 7 charger scenarios where the first contains results without defragmentation and the second consolidates charge sessions so that each session delivers at least 30 kWh. The results for each session are presented in Figures 20 and 21, where the color of i, j element of a figure represents the charge rate for bus i during time j. Note how Fig. 20 contains *many* short inconsequential charge sessions and requires each bus to charge each time it enters the station. In comparison, Fig. 21 contains only a handful of charge sessions so that each bus only needs to charge 4 to 5 times throughout the day.

Furthermore, Fig. 22, which plots monthly cost as a function of the minimum charging threshold, demonstrates that despite the additional constraints associated with consoli-

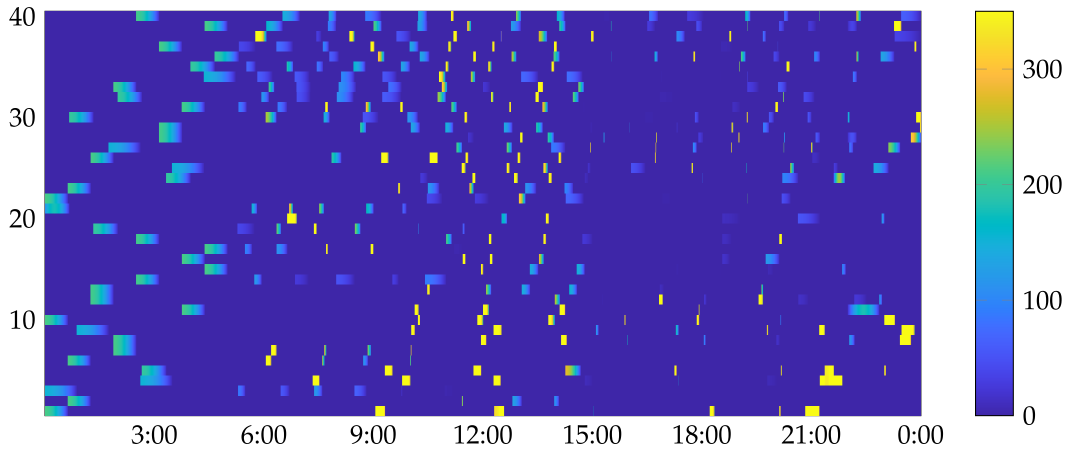

Figure 20. Routes without De-Fragmentation

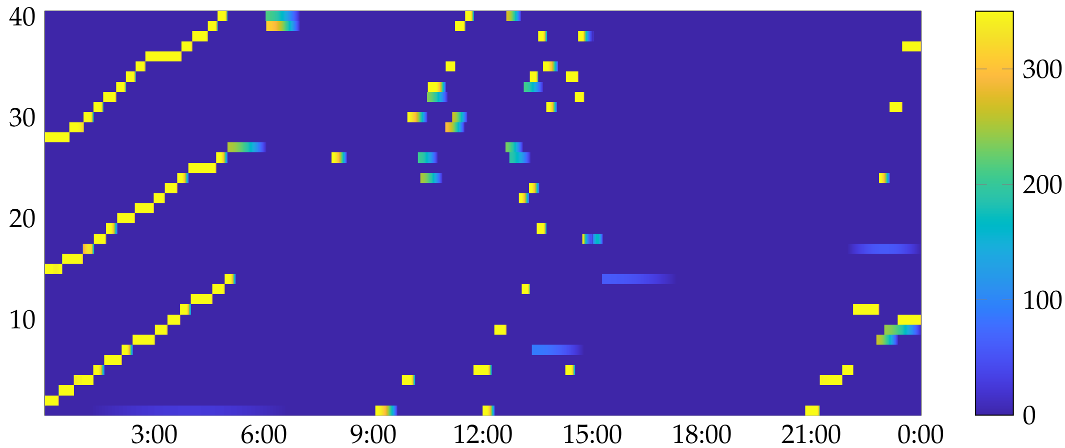

Figure 21. Routes with De-Fragmentation

dation, the monthly cost remains consistent over a large window of thresholds. As the minimum allowable energy per session increases, the number of binary variables in the de-fragmentation problem increases, resulting in significant runtimes for the defragmentation problem as shown in Fig. 23, which plots runtime as a function of the minimum charge threshold. However, because buses are divided into groups prior to defragmentation, the smaller groups decrease the computational complexity for defragmentation so that larger consolidation thresholds can be applied in a scalable manner.

10.6. Scalability

In this section, we consolidate what we have learned in the previous sections to demonstrate how the proposed framework can be used to compute a cost effective solution for large numbers of buses. This section focuses on a scenario with a minimum energy per session of 20 kWh, a large gap for the charger assignment solution, and a single group.

The results are given in Fig. 24, which plots the runtime as a function of the number of buses and shows how that runtime generally increases by one second per bus from 10 to 110 buses. One would expect the runtime to increase at least on the order of $O(n^2)$ for a globally optimal solution because of the coupling between bus variables. The fact that the proposed method is practical on the given range indicates a solution that scales well as the number of buses increases and can easily handle over 100 buses.

Generally, one would also expect such savings to come with significant increases to the monthly cost. However, the results in Fig. 25 demonstrate that the proposed solution yields a quasi-linear increase of approximately $404.10 dollars per bus per month.

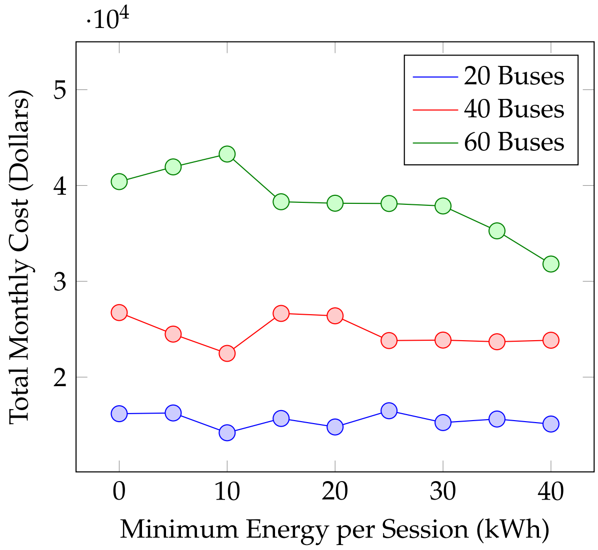

Figure 22. Cost comparison of different degragmentation thresholds in a pro-time optimization scheme.

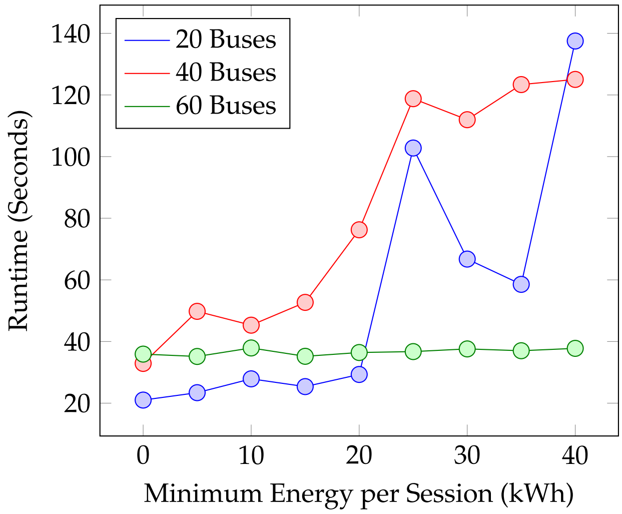

Figure 23. Comparison of runtime for the uncontested and contested scenarios over different defragmentation criteria

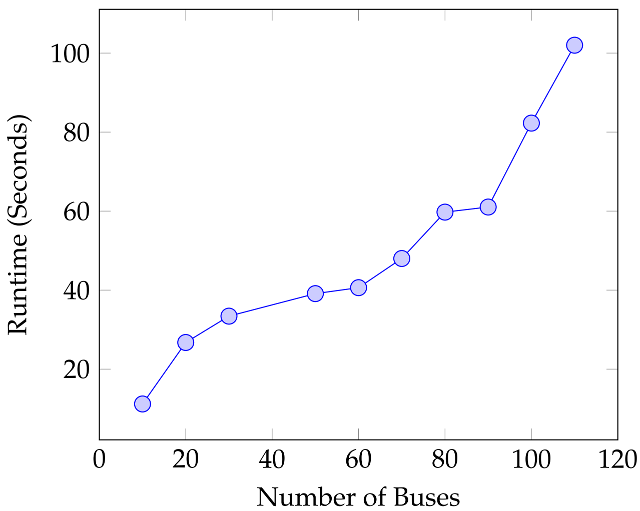

Figure 24. Runtime comparison for different numbers of buses

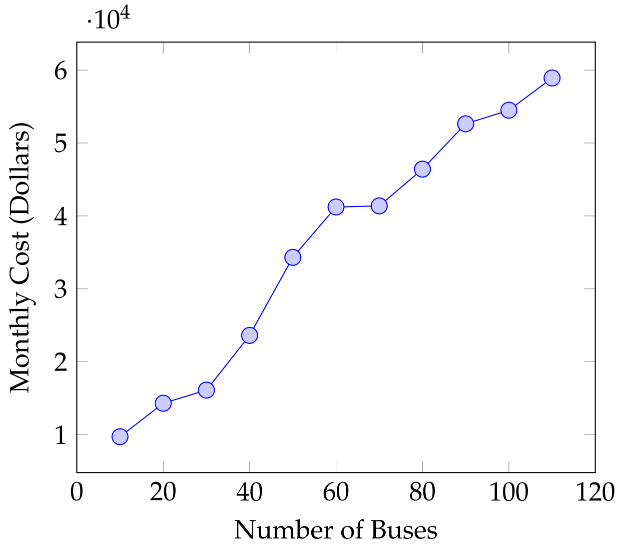

Figure 25. Cost comparison for different numbers of buses

11. Conclusions 525

In summary, this paper proposes a method to compute cost-oriented charge schedules 526 for large numbers of battery electric buses by dividing the charge problem into several 527 sub-problems which focus on energy placement and group separation, charge session 528 length and assignment, and cost optimization. The proposed method produces charge 529 schedules whose monthly cost and runtimes scale roughly linearly with the number of 530 buses and can easily handle over 100 buses. Furthermore, because the proposed method 531 contains a number of sub-problems, setting the optimization criteria for each sub-problem 532 gives the user flexability so that the proposed method can be adapated to solve a variety of 533 scenarios and optimization preferences. 534

Variable	**Description**	**Range**	**Variable**	**Description**	**Range**
Indices					
i	Bus index	$\mathbb{N}$	j	Time Index	$\mathbb{N}$
k	Charger index	$\mathbb{N}$	r	Route Index	
m	group index	$\mathbb{N}$			
Optimal Solution | Formulation					
n_{bus}	The number of buses in the optimization framework.	$\mathbb{Z}$	n_{time}	The number of time indices in a day.	$\mathbb{Z}^{+}$
$b_{p(i,j)}$	The average power consumed by bus i during time period j.	$\mathbb{R}$	t_j	The time at time index j. This paper also refers to the period of time from t_j to t_{j+1} as "period t_j".	$\mathbb{R}$
$\boldsymbol{b}$	A vector containing each value for $b_{p(i,j)}$.	$\mathbb{R}^{n_{\text{bus}} \cdot n_{\text{time}}}$	$\tilde{\mathcal{A}}$	The complement of $\mathcal{A}$.	$i \times j$
$\mathcal{A}$	The set of all $i \times j$ elements where bus i can charge at time index j	$i \times j$	$p_{\max}$	The maximum power a bus charger can deliver to a bus in kW. This paper assumes a value of 350 for most examples and results.	$\mathbb{R}^{+}$
Optimal Solution | Battery					
$h_{\min}$	The minimum allowable state of charge	$(0, h_{\max})$	$h_{\max}$	The maximum state of charge	$\mathbb{R}^{+}$
η_i	The beginning state of charge for bus i	$(h_{\min}, h_{\max})$	$h(ij)$	The state of charge for bus i at time t_j.	$(h_{\min}, h_{\max})$
ΔT	The change in time from t_j to t_{j+1}	$\mathbb{R}^{+}$	$\boldsymbol{h}$	A vector containing all state of charge values.	$\mathbb{R}_{+}^{n_{\text{bus}} \cdot n_{\text{time}}}$
$\delta(ij)$	The battery discharge for bus i during time period j.	$\mathbb{R}_{+}$	$h(i, \text{end})$	Bus i's final state of charge.	$(h_{\min}, h_{\max})$
Optimal Solution | Cumulative Load Management					

n_{charger}	The time index for the start of bus i's j^{th} stop	$\mathbb{Z}_+$	$p_c(j)$	The average power consumed by all buses during time period j.	$\mathbb{R}$
$\boldsymbol{p_c}$	A vector containing all values of $p_c(j)$.	$\mathbb{R}_+^{n_{\text{time}}}$	J_{thrash}	A secondary objective function which penalizes multiple plug-in instances per charge session.	$\mathbb{R}_+$
$g(i,j)$	A slack variable used to compute the absolute value of $\lvert b_{p(i,j)} - b_{p(i,j-1)} \rvert$	$\mathbb{R}_+$			
Optimal Solution | Objective					
$\mu_{\text{e-on}}$	On-Peak Energy Rate	$\mathbb{R}_+$	$\mu_{\text{e-off}}$	Off-Peak Energy Rate	$\mathbb{R}_+$
$\mu_{\text{p-on}}$	On-Peak Demand Power Rate	$\mathbb{R}_+$	$\mu_{\text{p-all}}$	Facilities Power Rate	$\mathbb{R}_+$
$\mathcal{S}_{\text{on}}$	The set of on-peak time indices	$\{1,...,n_{\text{time}}\}$	p_{demand}	Maximum average power during on-peak periods	$\mathbb{R}$
$p_{\text{facilities}}$	Maximum average power over all time instances.	$\mathbb{R}_+$	$p_t(j)$	The total average power co nsumed by both the bus chargers and the uncontrolled loads.	$\mathbb{R}_+^{n_{\text{time}}}$
$u(j)$	The average power over time j consumed by the uncontrolled loads	$\mathbb{R}_+^{n_{\text{time}}}$	$\boldsymbol{p_t}$	a vector containing $p_t(i)$ for all i.	$\mathbb{R}_+^{n_{\text{time}}}$
e_{on}	The total amount of energy consumed by the bus chargers and uncontrolled loads during off-peak hours.	$\mathbb{R}_+$	e_{off}	The total energy consumed by the bus chargers and uncontrolled loads during on-peak hours.	$\mathbb{R}_+$
J_{cost}	The section of the objective function pertaining to the fiscal expense of charging buses.	$\mathbb{R}$	J_{all}	The expression for the complete objective function.	$\mathbb{R}$
Scalability					
n_{group}	The number of groups in which to divide the buses and available chargers in preparation for the p_4, p_5, and p_6.	$\mathbb{Z}_+$	n^m_{charger}	The number of chargers assigned to group m.	$\mathbb{Z}_+$

n^m_{bus}	The number of buses in group m.	$\mathbb{Z}_+$	$p(j,m)$	The total power used during time index j by all buses in group m.	$\mathbb{R}_+$
$\beta(i,m)$	A binary selector variable which is one when bus i is in gropu m and zero otherwise.	$\{0,1\}$	n^m_{charger}	The number of chargers assigned to group m	$\mathbb{Z}_+$
$\phi(i,i')$	The inner product of the optimal charge schedules for buses i and i' respectively.	$\mathbb{R}_+$	$v(i,i',g)$	A variable that is $w(i,i')$ when buses i and i' are in group g and zero otherwise.	$\mathbb{Z}_+$
M_s	The maximum value for $\phi(i,i')$.	$\mathbb{R}_+$	J_{select}	The objective function for the group-selection problem	$\mathbb{R}_+$
De-Fragmentation					
$\theta(i,r)$	A binary variable which is one when charge session r from bus i will be used in a defragmented solution.	$\{0,1\}$	$\rho(i,r)$	A vector whose elements are equal to ΔT during time indices when bus i is charging during charge session r and zero otherwise.	$\mathbb{R}^{n_{\text{time}}}$
$\psi(i,j)$	The minimum allowable energy delivered to bus i during charge session r where the session in question is considered "active".	$\mathbb{R}$	ω	The minimum allowable energy for any charge session.	$\mathbb{R}$ 537
$e_{\max}$	The maximum allowable energy delivered in any session.	$\mathbb{R}$			
Charge Schedules					
$a(i,r)$	The beginning of the allowable charge interval for bus i's r^{th} charge session.	$\mathbb{R}_+$	$b(i,r)$	The commanded start time for bus i's r^{th} charge session	$\mathbb{N}$
$f(i,r)$	The commanded end time for bus i's r^{th} charge session.	$\mathbb{R}_+$	$d(i,r)$	The end time of the allowable charge interval for bus i's r^{th} charge session.	$\mathbb{R}_+$
$\sigma(i,r,k)$	A selector variable which is one when bus i charges at charger k for session r.	$\{0,1\}$	M	The number of seconds in a day	$\mathbb{Z}_+$
$\iota(i,r,i',r')$	A selector variable which is one when bus i charges before bus i' during the r and r' sessions respectively.	$\{0,1\}$			

Optimizing Charge Schedules					
$c(s,i,r)$	The start time for bus i's r^{th} charge session.	$\mathbb{R}$	$c(f,i,r)$	The stop time for bus i's r^{th} charge session.	$\mathbb{R}$
$c(a,i,r)$	The arrival time of bus i for charge session r.	$\mathbb{R}$	$c(d,i,r)$	The departure time for bus i after having completed the r^{th} charge session	$\mathbb{R}$
J_{window}	The loss function which drives charge windows to the desired length.	$\mathbb{R}$			
Multi-Rate Charging					
$x(i,j)$	The final charge schedule for bus i at time j, yielding the power at which bus i will charge.	$\mathbb{R}_+$	$z(j)$	The total power used by all buses at time j. $\mathbb{R}_+$	
$\gamma(i,d)$	A binary vector which is one at all time steps where bus i charges during charge session d.	$\{0,1\}^{n_{\text{time}}}$	$e(i,r)$	The amount of energy to be delivered to bus i during charge session r.	$\mathbb{R}_+$
$J_{\text{multi-rate}}$	The objective function over which we minimize to solve the multi-rate section of the bus charge problem.	R_+			

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