

## STAT 433: HOMEWORK 5

DANIEL PFEFFER

**Problem 1.** Let  $X$  and  $Y$  be two independent Poisson random variables with  $X \sim \text{Poisson}(\lambda)$  and  $Y \sim \text{Poisson}(\mu)$ . Use probability generating functions to find the distribution of  $X + Y$ .

*Solution.* The PGF of  $X$  is

$$G_X(s) = \sum_{k=0}^{\infty} s^k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} = e^{-\lambda} e^{s\lambda} = e^{\lambda(s-1)},$$

and by the same logic, the PGF of  $Y$  is  $G_Y(s) = e^{\mu(s-1)}$ . By independence of  $X$  and  $Y$ ,

$$G_{X+Y}(s) = G_X(s)G_Y(s) = e^{\lambda(s-1)}e^{\mu(s-1)} = e^{(\lambda+\mu)(s-1)}.$$

which, by uniqueness of the PGFs,  $X + Y \sim \text{Poisson}(\lambda + \mu)$ .

**Problem 2.** Let  $X_1, X_2, \dots$  be a sequence of i.i.d. Bernoulli random variables with parameter  $p$ . Let  $N$  be a Poisson random variables with  $N \sim \text{Poisson}(\lambda)$ , which is independent of the  $X_i$ 's.

- (1) Find the probability generating function of  $Z = \sum_{i=1}^N X_i$ .
- (2) Use (1) to identify the probability distribution of  $Z$ .

*Solution.*

- (1) From Problem (1) we know that the PGF of  $N$  is

$$G_N(s) = e^{\lambda(s-1)},$$

and since  $X_i \sim \text{Bernoulli}(p)$  for all  $i$ , we also have that

$$G_{X_i}(s) = 1 - p + ps.$$

So

$$G_Z(s) = G_N \circ G_{X_i}(s) = e^{\lambda p(s-1)},$$

- (2) By the uniqueness property of PGFs,  $Z \sim \text{Poisson}(\lambda p)$ .

**Problem 3.** Consider a branching process with off-spring distribution

$$p = (t^2, 2t(1-t), (1-t)^2), \quad 0 < t < 1,$$

i.e., it is binomial with parameters 2 and  $1-t$ . Find the extinction probability.

*Solution.* Since the offspring distribution is Binomial(2,  $1 - t$ ), we know its PGF is given by

$$G(s) = (1 - (1 - t) + (1 - t)s)^2 = (t + s - ts)^2$$

The mean is

$$\mu = G'(1) = 2 - 2t.$$

When  $t \geq 1/2$ , the mean  $\mu \leq 1$ , which corresponds to the subcritical and critical cases, the population goes extinct with probability 1.

For  $t < 1/2$ , the mean is  $\mu > 1$  and the process is supercritical, in which case the extinction probability is determined by the equation

$$G(s) = s \iff (t + s - ts)^2 = s.$$

Solving for  $s$  gives the extinction probability in the subcritical case:

$$s = \frac{t^2}{(1 - t)^2}.$$

#### Problem 4.

- (1) A discrete random variable  $X$  taking values in  $\{0, 1, 2, \dots\}$  is said to be memoryless, if for any  $m, n \geq 0$

$$P(X > m + n \mid X > m) = P(X > n).$$

Prove that  $X$  is memoryless iff it is a geometric random variable.

- (2) A continuous random variable  $X$  taking values in  $[0, \infty)$  is said to be memoryless, if for any  $t, s \geq 0$

$$P(X > t + s \mid X > s) = P(X > t).$$

Prove that  $X$  is memoryless if it is an exponential random variable.

- (3) Suppose a continuous random variable  $X$  taking values in  $[0, \infty)$  is memoryless. Define the function  $f(t) = P(X > t)$ . Then  $f(0) = 1$ , and by the memoryless property we have

$$f(s + t) = f(s)f(t) \implies \frac{f(s + t) - f(t)}{s} = \frac{f(s) - f(0)}{s} f(t).$$

Show that by taking  $s \rightarrow 0$ , we obtain a differential equation

$$f'(t) = f'(0)f(t).$$

Solve the equation to show that form  $e^{-\lambda t}$ , which means that  $X$  is an exponential random variable.

*Proof.*

- (1) Let  $X$  be discrete nonnegative integer valued random variable satisfying the memoryless property. Then, by the definition of conditional probability,

$$P(X > m + n \mid X \geq m) = \frac{P(X > m + n)}{P(X \geq m)}$$

together with the memoryless property implies that we have the functional equation

$$P(X > m + n) = P(X \geq m)P(X > n)$$

holds for any nonnegative integers  $m$  and  $n$ . Now the functional equation remains to be solved. If we let  $m = 1$  and let  $1 - p = P(X \geq 1)$  and observe that

$$P(X = 1 + n \mid X > 1) = \frac{P(X = n + 1, X > 1)}{P(X > 1)} = \frac{P(X = n + 1)}{1 - p} = P(X = n).$$

we deduce the probabilities

$$\begin{aligned} P(X = 2) &= (1 - p)P(X = 1) \\ P(X = 3) &= (1 - p)P(X = 2) = (1 - p)^2 P(X = 1) \\ &\vdots \\ P(X = k) &= (1 - p)^k P(X = 1) \\ &\vdots \end{aligned}$$

for  $k = 0, 1, 2, \dots$ , which is the PMF of a geometric random variable, and hence  $X \sim \text{Geometric}(p)$ .

Now let  $X \sim \text{Geometric}(p)$  for  $0 \leq p \leq 1$ , which has tail probabilities given by

$$P(X > n) = \sum_{k=n}^{\infty} P(X = k) = \sum_{k=n}^{\infty} (1 - p)^{k-1} p.$$

Letting  $i = k - n$  gives

$$P(X > n) = \sum_{i=0}^{\infty} (1 - p)^{i+n-1} p = p(1 - p)^{n-1} \sum_{i=0}^{\infty} (1 - p)^i = (1 - p)^{n-1}.$$

Now

$$P(X > m + n \mid X > m) = \frac{P(X > m + n)}{P(X > m)} = \frac{(1 - p)^{m+n-1}}{(1 - p)^{m-1}} = (1 - p)^n$$

and hence  $X$  satisfies the memoryless property.

- (2) Let  $X \sim \text{Exponential}(\lambda)$ . Then, by the definition of conditional probabilities (as in part (1)) and the fact that  $P(X > t) = e^{-\lambda t}$ ,

$$P(X > t + s \mid X > s) = \frac{P(X > t + s)}{P(X > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}.$$

- (3) Let  $X$  be continuous nonnegative integer valued random variable satisfying the memoryless property, and let  $f(t) = P(X > t)$ . Let  $s \rightarrow 0$ :

$$\lim_{s \rightarrow 0} \frac{f(s) - f(0)}{s} f(t).$$

The fact that  $X$  is continuous on  $[0, \infty)$  implies that  $f$  is almost everywhere differentiable, and the above limit exists and equals

$$f'(t) = f'(0)f(t).$$

This ordinary differential equation, together with the condition that  $f(0) = 1$ , has solution

$$f(t) = e^{-\lambda t}.$$

□