STAT 433: HOMEWORK 7

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Problem 1. The planets of the Galactic Empire are distributed in space according to a spatial Poisson process at an approximate density of one planet per cubic parsec. From the Death Star, let X be the distance to the nearest planet.

(a) Find the probability density function of X.

Solution. The event $\{X > r\}$ occurs iff there are no objects in a ball B_r with radius r around the Death State. Note that B_r has Lebesgue measure (volume) $|B_r| = -4\pi r^3/3$ and

$$P(X > r) = P(N_{B_r} = 0) = P(\text{Poisson}(4\pi r^3/3) = 0) = e^{-4\pi r^3/3}$$

Then notice that the cdf is given by

$$P(X < r) = 1 - e^{-4\pi r^3/3}.$$

Differentiating the cdf, we obtain the pdf

$$f(r) = 4\pi r^2 e^{-4\pi r^3/3},$$

for all $r \geq 0$. □

(b) Find the mean distance from the Death Star to the nearest planet. You can calculate the integral numerically.

Solution. The integral of X is

$$\mathrm{E}\, X = \int_0^\infty 4\pi r^3 e^{-4\pi r^3/3} dr = \frac{1}{36\sqrt[3]{6\pi}}\Gamma(1/3) \approx 0.55.$$

□

Problem 2. Customers arrive at a bank according to a Poisson process with rate 10 per hour. Given that 5 customers arrived in the first 30 minutes, answer the following questions.

(i) What is the probability that at least 3 arrived in the first 10 minutes?

Solution. Let $N_{1/2} = 5$ denote the number of arrivals after 30 minutes, let T_i denote the ith arrival time, note that

$$(T_1, T_2, T_3, T_4, T_5) \mid N_{1/2} = 5 \sim \mathcal{U}([0, 1/2]).$$

Then the number of arrivals in the first 10 minutes is Binomial$(5, 1/3)$, and so the event that the at least 3 customers arrived in the first 10 minutes is

$$\binom{5}{3}\frac{1}{3^3} \cdot \frac{2^2}{3^2} + \binom{5}{4}\frac{1}{3^4} \cdot \frac{2}{3} + \frac{1}{3^5}$$

□

(ii) What is the probability that 2 arrived in the first 10 minutes and 1 arrived in the next 5 minutes?

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Solution. For a random variable $U_i \sim \mathcal{U}([0, 1/2])$, the probability that it lies in the interval $[0, 1/6]$ is $1/3$, the probability that it lies in the interval $(1/6, 1/4]$ is $1/6$, and the probability that it lies in the interval $(1/4, 1/2]$ is $1/2$. Then for the 5 i.i.d. random variables $U_1, \dots, U_5$, the probability that two of them lie in $[0, 1/6]$, 1 lies in $(1/6, 1/4]$, and 2 lie in $(1/4, 1/2]$ is

$$\frac{5!}{2!1!2!} \cdot \frac{1}{3^2} \cdot \frac{1}{6} \cdot \frac{1}{2^2} = \frac{5}{36},$$

which follows directly from the multinomial distribution. □

(iii) What is the mean of the arriving time for the first customer?

Solution. Let T_1 be the arriving time for the first customer. Then,

$$P(T_1 \geq t) = P\left(\min_{1 \leq i \leq 5} U_i \geq t\right) = (1 - 2t)^5.$$

for $0 \leq t \leq 1/2$, and the integral of T_1 is

$$\mathrm{E}\, T_1 = \int_0^{1/2} P(T_1 \geq t) dt = \int_0^{1/2} (1 - 2t)^2 dt = \frac{1}{2} \int_0^1 x^5 dx = \frac{1}{12} \text{ hours.}$$

□

Problem 3. Recall the long run car costs problem. Suppose that the lifetime of a car is a random variable with density function $f(t)$. Our methodical Mr. Brown buys a new car as soon as the old one breaks down or reaches T years. Suppose that a new car costs A dollars and that an additional cost of B dollars to repair the vehicle is incurred if it breaks down before time T. If $f(t) = \lambda e^{-\lambda t}$, show that for any A and B the optimal time is $T = \infty$. Can you give a simple explanation in words.

Solution. The cost of the ith cycle is

$$\mathrm{E}\, r_i = A + B \int_0^T \lambda e^{-\lambda t} dt = A + B e^{-\lambda T}.$$

For the the duration of the ith cycle, we have

$$\begin{aligned} \mathrm{E}\, \tau_i &= \int_0^T t \lambda e^{-\lambda t} dt + T \int_T^\infty \lambda e^{-\lambda t} dt \\ &= \lambda \int_0^T t e^{-\lambda t} dt + T e^{-\lambda T}. \end{aligned}$$

Integrate by parts, $\int u dv = uv - \int v du$, with $u = t$ and $dv = e^{-\lambda t} dt$, where $du = dt$ and $v = e^{-\lambda t}/t$,

$$\begin{aligned} \int_0^T t e^{-\lambda t} dt &= -\frac{t e^{-\lambda t}}{\lambda}\bigg|_0^T + \frac{1}{\lambda} \int_0^T e^{-\lambda t} dt \\ &= -\frac{T e^{-\lambda T}}{\lambda} + \frac{1 - e^{-\lambda T}}{\lambda} \\ &= \frac{1 - \lambda e^{-\lambda T}(\lambda T + 1)}{\lambda}. \end{aligned}$$

Combining terms gives the duration of the ith cycle as

$$\mathrm{E}\, \tau_i = \frac{1 - \lambda e^{-\lambda T}(\lambda T + 1)}{\lambda} + T e^{-\lambda T} = \frac{e^{-\lambda T}(-\lambda + \lambda T + \lambda T - 1)}{\lambda}$$

The elementary renewal theorem tells us the long run reward per unit time is

$$\frac{\mathrm{E}\, r_i}{\mathrm{E}\, \tau_i} = \frac{A + Be^{-\lambda T}}{[e^{-\lambda T}(-\lambda + \lambda T + \lambda T - 1)]/\lambda} = -\frac{A + Be^{-\lambda T}}{\lambda^2 + \lambda - 2\lambda T}.$$

Differentiating with respect to T gives

$$\frac{\partial}{\partial T} \frac{A - Be^{-\lambda T}}{\lambda^2 + \lambda - 2\lambda T} = \frac{2\lambda(A - Be^{-\lambda T})}{(\lambda^2 + \lambda - 2\lambda T)^2} + \frac{B\lambda e^{-\lambda T}}{\lambda^2 + \lambda - 2\lambda T}$$

The optimal policy is to set $T = \infty$, since the exponential the numerator beats the quadratic in the denominator.

Since the lifetime of the car satisfies the memoryless property, given that a breakdown occurred, the probability that another one will occur is the same as if the original breakdown never happened. That is, after a breakdown, the car "forgets" that it broke is effectively becomes a new car. □

Problem 4. A young doctor is working at night in an emergency room. Emergencies come in at times of a Poisson process with rate $\lambda = 0.5$ per hour. The doctor can only get to sleep when it has been $c = 36$ minutes (0.6 hours) since the last emergency. For example, if there is an emergency at 1:00 and a second one at 1:17 then she will not be able to get to sleep until at least 1:53, and it will be even later if there is another emergency before that time. We want to compute the long-run fraction of time the doctor spends sleeping with the following strategy.

(a) If $T \sim \text{Exponential}(\lambda)$, find $\mathrm{E}(T \mid T < c)$.

Solution. Note first that by definition

$$\mathrm{E}\, T = P(T < c)\, \mathrm{E}(T \mid T < c) + P(T > c)\, \mathrm{E}(T \mid T > c).$$

Then

$$\mathrm{E}(T \mid T < c) = \frac{\mathrm{E}\, T - P(T > c)\, \mathrm{E}(T \mid T > c)}{P(T < c)} = \frac{2 + e^{-0.5 \cdot 36}}{1 - 0.5e^{-0.5 \cdot 36}}$$

□

(b) Let $J_n = \min\{j : \tau_j > c\}$, where $\tau_1, \tau_2, \dots, \tau_n$ are the interarrival times of the Poisson process with rate λ. Use (a) to show that

$$\mathrm{E}(T_{J-1} + c) = \frac{e^{\lambda c} - 1}{\lambda}$$

Solution.

$$\mathrm{E}(T_{J-1} + c) = \frac{ce^{-\lambda c} - ce^{-\lambda} + (1 - e^{-\lambda c})/\lambda}{P(\tau_1 > c)} = \frac{(1 - e^{-\lambda c})/\lambda}{e^{-\lambda c}} = \frac{e^{\lambda c} - 1}{\lambda}$$

□

(c) The doctor alternates between sleeping for an amount of time s_i and being awake for an amount of time u_i. Use the result from (b) to compute $\mathrm{E}\, u_i$.

Solution. By the memoryless property, $\mathrm{E}\, u_i = \mathrm{E}(T \mid T + c)$ □

(d) Compute the long-run fraction of time the doctor spends sleeping.

Solution. □

(e) Model the process using a counter model, and compute (d) in another way using the formula on class.

Solution. □