STAT 433: HOMEWORK 1

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Problem 1. A poker hand consists of five cards drawn from a standard 52-card deck. Find the expected number of aces in a poker hand given that the first card drawn is an ace.

Solution. Let X be a random variable denoting the number of aces in a poker hand, and let 1_A be an indicator function that equals 1 if the first card drawn is an ace and 0 otherwise. Then,

$$\begin{aligned} \mathrm{E}(X \mid 1_A) &= \frac{\mathrm{E}(X 1_A)}{P(A)} \\ &= \frac{1}{P(A)} \sum_{x=1}^{4} x P(\{X = x\} \cap A) \\ &= \sum_{x=1}^{4} x P(X = x \mid A) \\ &= \frac{\binom{3}{x-1}\binom{48}{5-x}}{\binom{51}{4}} = \frac{21}{17}. \end{aligned}$$

□

Problem 2. An urn has n balls. Balls are drawn one at a time and then put back in the urn. Let X denote the number of drawings required until some ball is drawn more than once. Find the probability distribution of X.

Solution. On the first draw, a ball cannot be drawn more than once, so

$$P(X = 1) = 0.$$

On the second draw, the probability of drawing the same ball, i.e., the ball from the first trial, is

$$P(X = 2) = \frac{1}{n}.$$

To deduce the probability that $X = 3$, note that with probability 1 the same ball is not drawn on the first trial (i.e., $1 - P(X = 1)$), on the second draw, the same ball is not drawn with probability $1 - 1/n$ (i.e., $1 - P(X = 2)$), and on the third drawn, there are 2 balls in the urn containing n ball which can be drawn again. Hence

$$P(X = 3) = \left(1 - \frac{1}{n}\right)\frac{2}{n}.$$

Continuing in this fashion, we obtain a probability distribution given by

$$P(X = k) = \prod_{i=0}^{k} \left(1 - \frac{i}{n}\right)\frac{k-1}{n}$$

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□

Problem 3. A man with n keys wants to open his door. He tries the keys in a random manner. Let X be the number of trials required to open the door.

(i) Find $\mathrm{E}(X)$ and $\mathrm{var}(X)$ if unsuccessful keys are eliminated from further selection.

Solution. Note that, since the number of keys n decreases by 1 each trial, we have

$$\begin{aligned} P(X=1) &= \frac{1}{n} \\ P(X=2) &= \left(1-\frac{1}{n}\right)\frac{1}{n-1} \\ &\vdots \\ P(X=k) &= \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right)\cdots\left(1-\frac{1}{n-(k-2)}\right)\frac{1}{n-(k-1)}, \end{aligned}$$

which can be expressed more compactly as

$$P(X=k) = \prod_{i=0}^{k-1}\left(1-\frac{1}{n-i}\right).$$

Then, compute

$$\mathrm{E}(X) = \frac{1}{n}+\frac{2}{n}+\cdots+\frac{n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

and

$$\mathrm{E}(X^2) = \frac{1}{n}+\frac{4}{n}+\cdots+\frac{n^2}{n} = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}.$$

So, using the fact that $\mathrm{var}(X) = \mathrm{E}\,X^2 - (\mathrm{E}\,X)^2$, we get

$$\mathrm{var}(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}.$$

□

(ii) Find $\mathrm{E}(X)$ and $\mathrm{var}(X)$ if unsuccessful keys are not eliminated from further selection.

Solution. If unsuccessful keys not are eliminated from further selection, then $X \sim \mathrm{Geom}(1/n)$, and hence

$$\begin{aligned} \mathrm{E}(X) &= \frac{1}{1/n} = n \\ \mathrm{var}(X) &= \frac{1-1/n}{1/n^2} = n(n-1). \end{aligned}$$

□

Problem 4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \mathcal{U}([0,t])$, and let $Y_n = \min\{X_1,\ldots,X_n\}$.

(i) Find $P(Y_n > x)$ for $x \in [0,t]$.

Solution. Since Y_n is greater than the minimum of $X_1, X_2, \ldots, X_n$, we know Y_n must also be greater than every $X_1, X_2, \ldots, X_n$. So,

$$\begin{aligned} P(Y_n > x) &= P(X_1 > x, X_2 > x, \ldots, X_n > x) \\ &= \prod_{i=1}^{n} \left(1 - \frac{x}{t}\right) \\ &= \left(1 - \frac{x}{t}\right)^n \end{aligned}$$

□

(ii) Let t map $n \mapsto t(n)$ such that $\lim_{n\to\infty} n/t(n) = \lambda$. Show that

$$\lim_{n\to\infty} P(Y_n > x) = e^{-\lambda x}.$$

Solution. Evaluating the limit gives

$$\begin{aligned} \lim_{n\to\infty} P(Y_n > x) &= \lim_{n\to\infty} \left(1 - \frac{x}{t(n)}\right)^n \\ &= \lim_{n\to\infty} \left(1 - \frac{n}{n}\frac{x}{t(n)}\right)^n \\ &= \lim_{n\to\infty} \left(1 - \frac{(n/t(n))x}{n}\right)^n \\ &= \lim_{n\to\infty} \left(1 - \frac{\lambda x}{n}\right)^n \\ &= e^{-\lambda x}, \end{aligned}$$

since by definition t satisfies $\lim_{n\to\infty} n/t(n) = \lambda$. □