

## STAT 433: HOMEWORK 1

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**Problem 1.** A poker hand consists of five cards drawn from a standard 52-card deck. Find the expected number of aces in a poker hand given that the first card drawn is an ace.

*Solution.* Let  $X$  be the number of aces in a poker hand, let  $A_i$ , for  $i = 1, 2, \dots, 5$ , be the event that the  $i$ th card drawn is an ace, and let  $1_{A_i}$  be an indicator function that equals one if the  $A_i$  occurs and zero otherwise. Then we may write

$$X = 1_{A_1} + 1_{A_2} + \dots + 1_{A_5}.$$

Since we are conditioning on the event that the first card drawn is an ace,  $1_{A_1} = 1$ . It follows that the probability that one of the next four cards is an ace is  $3/51 = 1/17$ . By linearity of expectations, we obtain

$$E X = E 1_{A_1} + \frac{1}{17}(E 1_{A_2} + \dots + E 1_{A_5}) = 1 + \frac{4}{17} = \frac{21}{17}.$$

□

**Problem 2.** An urn has  $n$  balls. Balls are drawn one at a time and then put back in the urn. Let  $X$  denote the number of drawings required until some ball is drawn more than once. Find the probability distribution of  $X$ .

*Solution.* Since it is impossible for a ball to be drawn twice if only one ball has been drawn,  $P(X = 1) = 0$ . Note also that  $P(X = 2) = 1/n$  because there is only one ball (out of  $n$ ) that can be drawn again. To deduce the probability that  $X = 3$ , notice that the same ball is not drawn on the first trial with probability  $1 - P(X = 1) = 1$  and on the second draw, the same ball is not drawn with probability  $1 - P(X = 2) = 1 - 1/n$ . Then, for the event  $\{X = 3\}$  to occur, the repeat draw must occur on the third trial, which happens with probability  $2/n$  since there are 2 balls in the urn containing  $n$  balls which can be drawn again:

$$P(X = 3) = \left(1 - \frac{1}{n}\right) \left(\frac{2}{n}\right).$$

Continuing with this reasoning, we obtain a probability distribution, for  $1 \leq x \leq n + 1$ , given by

$$P(X = x) = \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{x-2}{n}\right) \left(\frac{x-1}{n}\right)$$

□

**Problem 3.** A man with  $n$  keys wants to open his door. He tries the keys in a random manner. Let  $X$  be the number of trials required to open the door.

- (i) Find  $E(X)$  and  $\text{var}(X)$  if unsuccessful keys are eliminated from further selection.

*Solution.* Note that, since the number of keys  $n$  decreases by 1 each trial, we have

$$\begin{aligned} P(X = 1) &= \frac{1}{n} \\ P(X = 2) &= \left(1 - \frac{1}{n}\right) \frac{1}{n-1} \\ &\vdots \\ P(X = k) &= \left(1 - \frac{1}{n}\right) \left(1 - \frac{1}{n-1}\right) \cdots \left(1 - \frac{1}{n-(k-2)}\right) \frac{1}{n-(k-1)}, \end{aligned}$$

Then, the expectation of  $X$  is

$$E(X) = \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

To find the variance of  $X$ , compute the second moment:

$$E(X^2) = \frac{1}{n} + \frac{4}{n} + \cdots + \frac{n^2}{n} = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}.$$

Using the fact that  $\text{var}(X) = E(X^2) - (E X)^2$  gives

$$\text{var}(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}.$$

□

- (ii) Find  $E(X)$  and  $\text{var}(X)$  if unsuccessful keys are not eliminated from further selection.

*Solution.* If unsuccessful keys are not eliminated from further selection, then  $X \sim \text{Geometric}(1/n)$ , and hence

$$\begin{aligned} E(X) &= \frac{1}{1/n} = n \\ \text{var}(X) &= \frac{1 - 1/n}{1/n^2} = n(n-1). \end{aligned}$$

□

**Problem 4.** Let  $X_1, X_2, \dots, X_n$  be i.i.d. random variables with  $X_i \sim \mathcal{U}([0, t])$ , and let  $Y_n = \min\{X_1, \dots, X_n\}$ .

- (i) Find  $P(Y_n > x)$  for  $x \in [0, t]$ .

*Solution.* Since  $Y_n$  is greater than the minimum of  $X_1, X_2, \dots, X_n$ , we know  $Y_n$  must also be greater than every  $X_1, X_2, \dots, X_n$ . So,

$$\begin{aligned} P(Y_n > x) &= P(X_1 > x, X_2 > x, \dots, X_n > x) \\ &= \prod_{i=1}^n \left(1 - \frac{x}{t}\right) \\ &= \left(1 - \frac{x}{t}\right)^n \end{aligned}$$

□

- (ii) Let  $t$  be a function of  $n$  such that  $\lim_{n \rightarrow \infty} n/t = \lambda$ . Show that

$$\lim_{n \rightarrow \infty} P(Y_n > x) = e^{-\lambda x}.$$

*Solution.* Evaluating the limit gives

$$\begin{aligned}\lim_{n \rightarrow \infty} P(Y_n > x) &= \lim_{n \rightarrow \infty} e^{n \log(1-x/t)^n} \\ &= \lim_{n \rightarrow \infty} e^{n(-x/t + o(1/t))} \\ &= \lim_{n \rightarrow \infty} e^{n(-x\lambda/n + o(1/n))} \\ &= e^{-\lambda x},\end{aligned}$$

since by definition  $t$  satisfies  $\lim_{n \rightarrow \infty} n/t = \lambda$ .

□