ECON 706: PROBLEM SET 1

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Problem 1. Prove that for a weakly stationary process, $|\gamma(\tau)| \leq \gamma(0)$ for all τ.

Solution. Let $\{y_t\}$ be a weakly stationary process, and without loss of generality assume the mean of y_t is zero for all t. Then, by the Cauchy-Schwartz inequality,

$$|\gamma(\tau)| = \mathrm{E}\, y_t y_{t-\tau} \leq \sqrt{\mathrm{E}\, y_t^2\, \mathrm{E}\, y_{t-\tau}^2} \leq \gamma(0).$$

□

Problem 2. Consider an AR(2) process

(i) Characterize the conditions for stationarity.

Solution. Write the AR(2) process as $(1 - \phi_1 L - \phi_2 L^2)y_t = \varepsilon_t$ where $\varepsilon_t \sim WN(0, \sigma^2)$, and consider the characteristic polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. Then the $\{y_t\}$ process is non-explosive if the characteristic equation

$$z^2 - \phi_1 z - \phi_2 = 0$$

has root pairs (z_1, z_2) outside the locus $\{z : |z| > 1\}$ called the unit circle:

$$\left| \frac{-\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} \right| > 1.$$

From this restriction on the roots of $\phi(z) = 0$ we can deduce restrictions on the autoregressive parameters. Substituting these roots into the characteristic polynomial for the coefficients gives the factorization

$$\phi(z) = (1 - z_1^{-1}z)(1 - z_2^{-1}z).$$

Here, it is identically the case that $\phi(z_1) = \phi(z_2) = 0$. Next rewrite the AR(2) process with the lag operator using this factorization: $(1 - z_1^{-1}L)(1 - z_2^{-1}L)y_t = \varepsilon_t$, which implies that $\phi_1 = z_1^{-1} + z_2^{-1}$ and $\phi_2 = -(z_1 z_2)^{-1}$. Then, together with the restriction that $|z_1| > 1$ and $|z_2| > 1$, we obtain the values for which $\{y_t\}$ is non-explosive:

$$\phi_1 + \phi_2 < 1, \qquad \phi_2 - \phi_1 < 1, \qquad |\phi_2| < 1,$$

a convex set in parameter space. □

(ii) Assuming stationarity, derive its Wold representation, i.e., the coefficients ψ_i for $i = 1, 2, \ldots$.

Solution. To obtain the Wold representation of an AR(2) process, invert then autoregressive lag polynomial using the partial fraction decomposition:

$$\frac{1}{(1 - \lambda_1 L)(1 - \lambda_2 L)} = \frac{c_1}{(1 - \lambda_1 L)} + \frac{c_2}{(1 - \lambda_2 L)} = \frac{c_1(1 - \lambda_2 L) + c_2(1 - \lambda_1 L)}{(1 - \lambda_1 L)(1 - \lambda_2 L)}$$

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where c_1 and c_2 are coefficients to be determined. We require that $c_1 + c_2 = 1$ and $\lambda_2 c_1 + \lambda_1 c_2 = 0$. Hence, the solution is

$$c_1 = \frac{\lambda_1}{\lambda_1 - \lambda_2}, \quad c_2 = \frac{-\lambda_2}{\lambda_1 - \lambda_2}.$$

Substituting this solution into the partial fraction decomposition gives

$$\frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} = \frac{\lambda_1}{(\lambda_1 - \lambda_2)(1-\lambda_1 L)} + \frac{\lambda_2}{(\lambda_2 - \lambda_1)(1-\lambda_2 L)}.$$

Now apply this expression for the inverted lag polynomial operator to our AR(2) process to obtain the Wold representation:

$$\begin{aligned}
y_t &= \frac{\lambda_1}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} \lambda_1^i \varepsilon_{t-i} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \sum_{i=0}^{\infty} \lambda_2^i \varepsilon_{t-i} \\
&= \sum_{i=0}^{\infty} \left(\frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_1^i + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_2^i \right) \varepsilon_{t-i} \\
&= \sum_{i=0}^{\infty} \left(c_1 \lambda_1^i + c_2 \lambda_2^i \right) \varepsilon_{t-i} = \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i},
\end{aligned}$$

where $\psi_i = c_1 \lambda_1^i + c_2 \lambda_2^i$, for $i = 0, 1, 2, \ldots$. □

(iii) Verify that $\psi_0 = 1$.

Solution. To verify that $\psi_0 = 1$, first note that $\psi_0 = c_1 \lambda_1^0 + c_2 + \lambda_2^0 = c_1 + c_2$, and then substitute in the expressions for c_1 and c_2 derived in (b) to obtain

$$\psi_0 = \frac{\lambda_1}{\lambda_1 - \lambda_2} + \frac{-\lambda_2}{\lambda_1 - \lambda_2} = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} = 1,$$

the desired result. □

(iv) Verify that the square summability conditional holds, i.e., $\sum_{i=0}^{\infty} \psi_i^2 < \infty$.

Solution. Write the infinite sum

$$\sum_{i=0}^{\infty} \psi_i^2 = \sum_{i=0}^{\infty} \left(c_1 \lambda_1^i + c_2 \lambda_2^i \right)^2 = c_1^2 \sum_{i=0}^{\infty} \lambda_1^{2i} + c_2^2 \sum_{i=0}^{\infty} \lambda_2^{2i} + c_1 c_2 \sum_{i=0}^{\infty} (\lambda_1 \lambda_2)^{2i}.$$

We know from the stationarity conditions derived in (i) that λ_1 and λ_2 have modulus less than unity, which implies that each term is a convergent geometric series. Hence $\sum_{i=0}^{\infty} \psi_i^2 < \infty$. □

Problem 3. Is the following process stationary?

$$y_t = 0.5 y_{t-1} + 0.9 y_{t-2} - 0.1 y_{t-3} + 0.3 y_{t-4} + 0.5 \varepsilon_{t-1} + \varepsilon_t \tag{1}$$

Solution. First rewrite the process in lag operator notation as

$$(1 - 0.5L - 0.9L^2 + 0.1L^3 - 0.3L^4) y_t = (1 + 0.5L) \varepsilon_t.$$

Notice that we can factor autoregressive operator and moving average operator as

$$\begin{aligned}
\phi(L) &= (1 - 0.5L - 0.9L^2 + 0.1L^3 - 0.3L^4) = -0.1(L+2)(3L^3 - 7L^2 + 5L - 5) \\
\theta(L) &= (1 + 0.5L) = 0.5(L+2).
\end{aligned}$$

which have a common factor $(L+2)$ and therefore a redundant parameter. So we can rewrite this process as an AR(3) model of the form $-0.1(3L^3 - 7L^2 + 5L - 5)y_t = 0.5\varepsilon_t$. This process is stationary if the roots of the characteristic polynomial

$$\phi(z) = -0.3z^3 + 0.7z^2 - 0.5z + 0.5.$$

are outside the unit circle. Factoring the characteristic equation as

$$-0.3(z - 1.91744)(z^2 - 0.415894z + 0.869215) = 0$$

immediately gives the real root $z_1 = 1.91744$, which is outside the unit circle. To obtain the other two roots, use the quadratic equation

$$z_2, z_3 = \frac{0.415894 \pm \sqrt{(-0.415894)^2 + 4(0.869215)}}{2}$$

which gives the complex roots (that occur in a complex conjugate pair) as $z_2 = 0.207947 + 0.908831i$ and $z_3 = 0.207947 - 0.908831i$. Their modulus is $0.932317 < 1$, and so the process is not stationary. □

Problem 4. Consider

$$\begin{aligned}
x_t &= \phi_1 x_{t-1} + \phi_2 x_{t-2} + \phi_3 x_{t-3} + \varepsilon_t \\
y_t &= \theta_1 \varepsilon_1 + \theta_2 \varepsilon_{t-2} + \varepsilon_t \\
z_t &= \phi_1 z_{t-1} + \phi_2 z_{t-2} + \phi_3 z_{t-3} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-2} + \varepsilon_t.
\end{aligned}$$

Take $(\phi_1, \phi_2, \phi_3, \theta_1, \theta_2) = (0.6, 0.2, 0.1, 0.5, -0.2)$ and assume $\varepsilon_t \sim \mathrm{N}(0, 1)$.

(i) Starting from arbitrary initial values, simulate a plot of a series of $T = 10000$ realizations for each process.

Solution. Below is the code used to simulate the processes and plot the resulting series. Figure 1 contains simulated series for the x_t, y_t and z_t processes, respectively.

```python
import numpy as np
import matplotlib.pyplot as plt
from statsmodels.graphics.tsaplots import plot_acf, plot_pacf

# Simulate the series
np.random.seed(42)
T = 10000
eps = np.random.normal(0,1,size=T)
x, y, z = np.empty_like(eps), np.empty_like(eps), np.empty_like(eps)
for t in range(T):
    x[t] = 0.6*x[t-1] + 0.2*x[t-2] + 0.1*x[t-3] + eps[t]
    y[t] = 0.5*eps[t-1] - 0.2*eps[t-2] + eps[t]
    z[t] = 0.6*z[t-1] + 0.2*z[t-2] + 0.1*z[t-3]\
           + 0.5*eps[t-1] - 0.2*eps[t-2] + eps[t]

# Plot results
plt.figure(figsize=(12,4))

plt.subplot(131)
plt.plot(x)
```

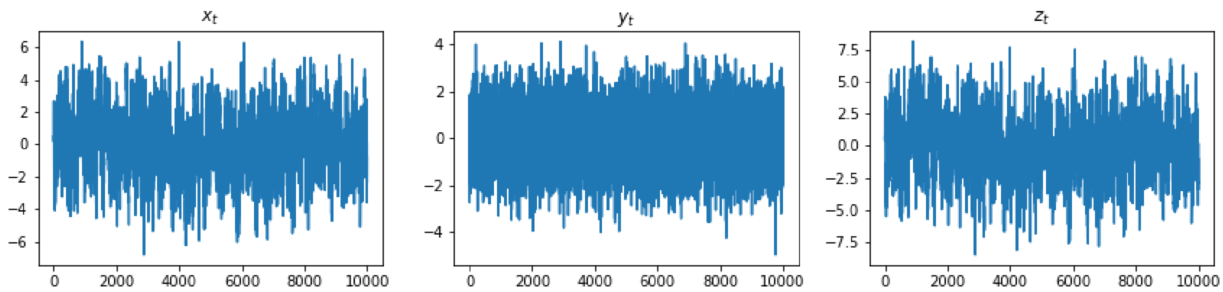

FIGURE 1

```
plt.ylabel('$x_t$')

plt.subplot(132)
plt.plot(y)
plt.ylabel('$y_t$')

plt.subplot(133)
plt.plot(z)
plt.ylabel('$z_t$')

plt.tight_layout()
plt.show()
```

□

(ii) Compute and plot the (empirical) ACF and PACF for each process. Discuss.

Solution. Below is the code used to plot the ACF and PACF for each process. Figure 2 contains the sample autocorrelation and partial autocorrelation plots for the x_t, y_t and z_t processes, respectively. The first row of Figure 2 (AR(3)) displays a geometrically decaying autocorrelation function and a partial autocorrelation function that becomes zero after 3 lags. The second row of Figure 2 (MA(2)) has an autocorrelation function that becomes zero after two lags and partial autocorrelation function that tends to zero only gradually. The third row of Figure 2 (ARMA(3,2)) has an autocorrelation function resembling an AR(3) process since it decays geometrically and a partial autocorrelation function that resembles an MA(2) process because it converges to zero gradually.

```
fig, ax = plt.subplots(3,2,figsize=(12,8))

plot_acf(x, ax=ax[0,0])
plot_pacf(x, ax=ax[0,1])

plot_acf(y, ax=ax[1,0], title='')
plot_pacf(y, ax=ax[1,1], title='')

plot_acf(z, ax=ax[2,0], title='')
plot_pacf(z, ax=ax[2,1], title='')
```

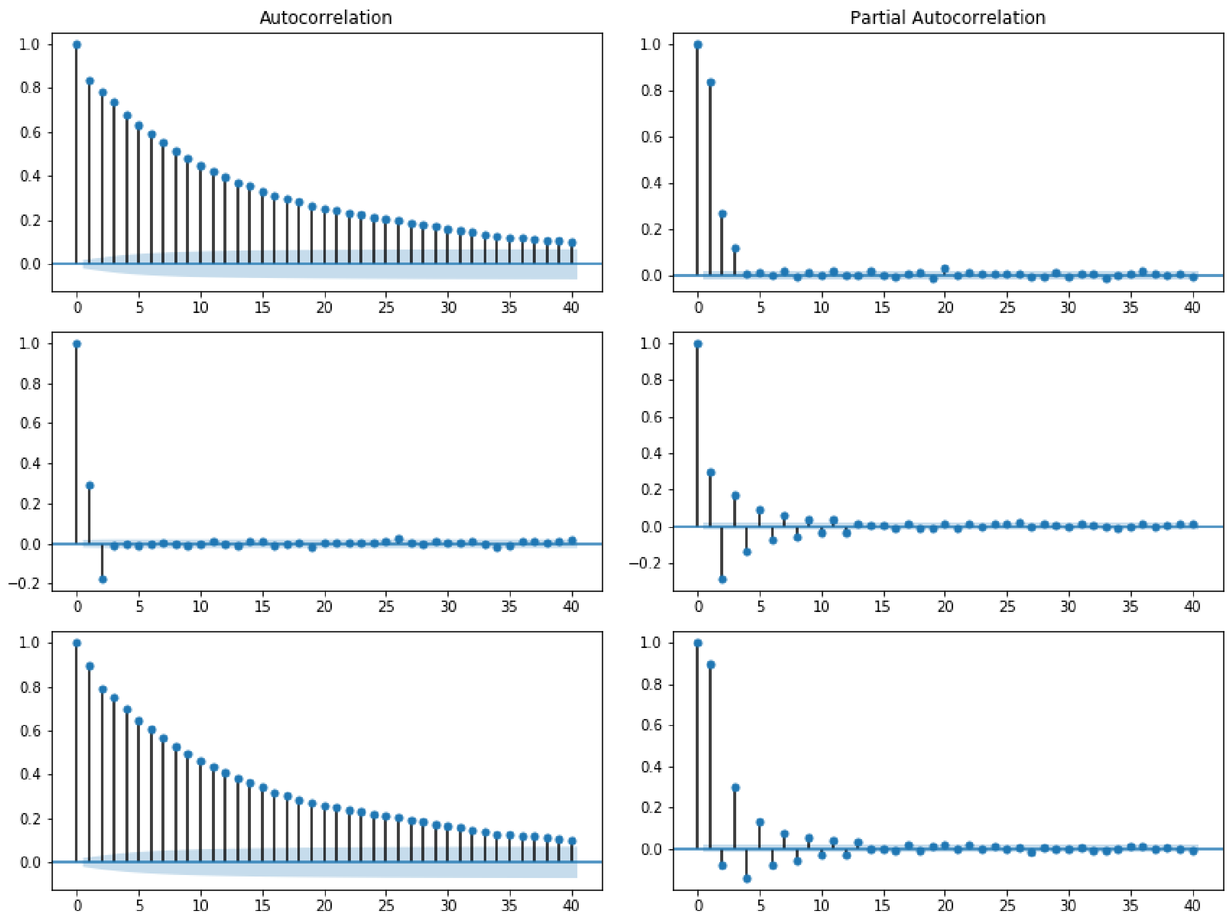

FIGURE 2

```
plt.tight_layout()
plt.show()
```

□