

STAT 433: HOMEWORK 3

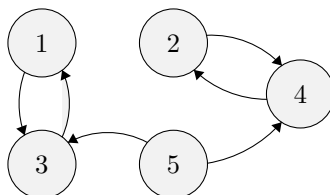
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Problem 1. Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains.

(a)	1	2	3	4	5	(b)	1	2	3	4	5	6
1	.4	.3	.3	0	0	1	.1	0	0	.4	.5	0
2	0	.5	0	.5	0	2	.1	.2	.2	0	.5	0
3	.5	0	.5	0	0	3	0	.1	.3	0	0	.6
4	0	.5	0	.5	0	4	.1	0	0	.9	0	0
5	0	.3	0	.3	.4	5	0	0	0	.4	0	.6
						6	0	0	0	0	.5	.5

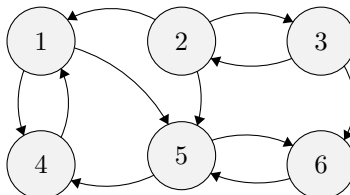
(c)	1	2	3	4	5	(d)	1	2	3	4	5	6
1	0	0	0	0	1	1	.8	0	0	.2	0	0
2	0	.2	0	.8	0	2	0	.5	0	0	.5	0
3	.1	.2	.3	.4	0	3	0	0	.3	.4	.3	0
4	0	.6	0	.4	0	4	.1	0	0	.9	0	0
5	.3	0	0	0	.7	5	0	.2	0	0	.8	0
						6	.7	0	0	.3	0	0

(a) *Solution.* The directed graph representation is



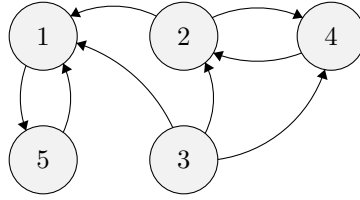
Since $1 \rightarrow 2$ but $2 \not\rightarrow 1$, state 1 is transient. Since $5 \rightarrow 4$ but $4 \not\rightarrow 5$, state 5 is transient. Since $3 \rightarrow 2$ but $2 \not\rightarrow 3$, state 3 is transient. So, states 2 and 4 are recurrent. The closed irreducible set is $\{2, 4\}$. \square

(b) *Solution.* The directed graph representation is



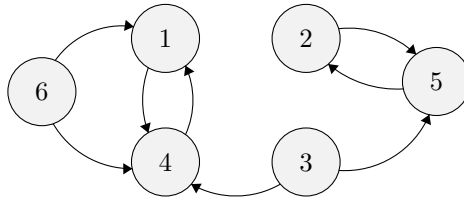
Since $2 \rightarrow 1$ but $1 \not\rightarrow 2$, state 2 is transient. Since $3 \rightarrow 6$ but $6 \not\rightarrow 3$, state 3 is transient. So states 1, 4, 5, and 6 are recurrent. The closed irreducible set is $\{1, 4, 5, 6\}$. \square

(c) *Solution.* The directed graph representation is



Since $3 \rightarrow 1$ but $1 \not\rightarrow 3$, state 3 is transient. So state 1, 2, 4, and 5 are recurrent. The closed irreducible sets are $\{1, 5\}$ and $\{2, 4\}$. \square

(d) *Solution.* The directed graph representation is



Since $3 \rightarrow 4$ but $4 \not\rightarrow 3$, state 3 is transient, and since $6 \rightarrow 1$ but $1 \not\rightarrow 6$, state 6 is transient. So state 1, 2, 4, and 5 are recurrent. The closed irreducible sets are $\{1, 4\}$ and $\{2, 5\}$. \square

Problem 2. Let G be a connected graph. Let X_n be a simple random walk on G . Show that the Markov chain $\{X_n\}$ is irreducible. (Hint: for any two vertices x and y in G , consider a path of consecutive nodes from x to y .)

Solution. Let x and y be any two vertices in G . Since G is connected, x and y are connected by with a path of edges formed by

$$x = x_0, x_1, x_2, \dots, x_n = y,$$

which implies that

$$\begin{aligned} \rho_{xy} &\geq \frac{1}{\deg(x)} \cdot \frac{1}{\deg(x_1)} \cdots \frac{1}{\deg(x_{n-1})} > 0 \\ \rho_{yx} &\geq \frac{1}{\deg(y)} \frac{1}{\deg(x_{n-1})} \cdots \frac{1}{\deg(x_1)} > 0. \end{aligned}$$

Therefore $x \leftrightarrow y$. \square

Problem 3. Let G be a graph with two disjoint components G_1 and G_2 . Let X_n be a simple random walk on G .

(i) Prove $\{X_n\}$ is not an irreducible Markov chain.

Solution. Since $G_1 \cap G_2 = \emptyset$, there does not exist an edge between some node $x \in G_1$ and some other node $y \in G_2$, which shows G_1 and G_2 are disconnected. So $x \not\leftrightarrow y$ and X_n on G is not irreducible. \square

(ii) Let P be the transition matrix of X_n . Let P_1 and P_2 be the transition matrices for the SRW on G_1 and G_2 , respectively. Let $V_1 = \{1, \dots, k\}$ and $V_2 = \{k+1, \dots, k+\ell\}$ be the set of vertices in G_1 and G_2 . Show that P is of the following block diagonal form

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}. \quad (1)$$

Note a block diagonal matrix is a *reducible* matrix.

Solution. We have $p(x, y) = 0$ if $1 \leq x \leq k$ and $k+1 \leq y \leq k+\ell$, or if $k+1 \leq x \leq k+\ell$ and $1 \leq y \leq k$, which shows that P is diagonal. We also have $p(x, y) = p_1(x, y)$ if $1 \leq x, y \leq k$ and $p(x, y) = p_2(x, y)$ if $k+1 \leq x, y \leq k+\ell$, which establishes the fact that $P = \text{diag}(P_1, P_2)$. \square

(iii) Show the the SRW $\{X_n\}$ on G has infinitely many stationary distributions.

Solution. Since the disjoint graphs G_1 and G_2 are finite, P_1 has a stationary distribution, call it π_1 , and P_2 has stationary distribution, call it π_2 . Note that π_1 has dimension $1 \times k$, π_2 has dimension $1 \times \ell$. Let

$$\pi = \begin{cases} (\pi_1, 0) & \text{if } X_0 \in G_1 \\ (0, \pi_2) & \text{if } X_0 \in G_2 \end{cases}$$

be the stationary distribution for the operator P on G , and note that π has dimension $1 \times k + \ell$. Note also that P has dimension $(k + \ell) \times (k + \ell)$. Now, for the chain started on any vertex in G_1 , let $\pi = (\pi_1, 0)$ where 0 is a $1 \times \ell$ dimensional vector. Then

$$(\pi_1, 0) \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = (\pi_1 P_1, 0) = (\pi_1, 0),$$

where the last equality follows since π_1 was defined to be the stationary distribution for P_1 . For the chain started on any vertex in G_2 , $\pi = (0, \pi_2)$ where here 0 is a $1 \times k$ vector. So

$$(0, \pi_2) \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = (0, \pi_2 P_2) = (0, \pi_2).$$

For $0 \leq \lambda \leq 1$, linear combinations of the form

$$(\lambda \pi_1 + (1 - \lambda) \pi_2) = (\lambda \pi_1 P + (1 - \lambda) \pi_2 P)$$

are also stationary distribution for P . Hence, there exists infinitely stationary for X_n on G . \square

Problem 4. Consider a Markov chain with state space $S = \{1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix},$$

where $0 < a < 1$ and $0 < b < 1$.

(i) Find its stationary distribution π .

Solution. Let $\pi = (x, y)$. Then

$$\pi P = \pi \implies \begin{cases} (1-a)x + by = x \\ ax + (1-b)y = y \end{cases} \implies \begin{cases} ax = by \\ x + y = 1 \end{cases} \implies \begin{cases} x = b/(a+b) \\ y = a/(a+b) \end{cases}.$$

\square

(ii) Use the Markov property to show that

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b) \left(P(X_n = 1) - \frac{b}{a+b} \right).$$

Solution. Start from the left hand side and condition on X_n to obtain

$$\begin{aligned}
 P(X_{n+1} = 1) &= \sum_i P(X_{n+1} = 1 \mid X_n = i)P(X_n = i) \\
 &= (1-a)P(X_n = 1) + b(1 - P(X_n = 1)) \\
 &= P(X_n = 1) - aP(X_n = 1) + b - bP(X_n = 1) \\
 &= (1-a-b)P(X_n = 1) + b \\
 &= (1-a-b)P(X_n = 1) + \frac{b(a+b)}{a+b} \\
 &= (1-a-b)P(X_n = 1) + \frac{ab}{a+b} + \frac{b^2}{a+b}.
 \end{aligned}$$

Next, subtract $b/(a+b)$ from both sides

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)P(X_n = 1) + \frac{ab}{a+b} + \frac{b^2}{a+b} - \frac{b}{a+b}.$$

Simplifying the above expression gives

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b) \left(P(X_n = 1) - \frac{b}{a+b} \right).$$

□

(iii) Use mathematical induction to show that

$$P(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n \left(P(X_0 = 1) - \frac{b}{a+b} \right)$$

Proof (by induction on n). The first base case $n = 0$ holds since

$$\begin{aligned}
 P(X_0 = 1) &= \frac{b}{a+b} + (1-a-b)^0 \left(P(X_0 = 1) - \frac{b}{a+b} \right) \\
 &= \frac{b}{a+b} + P(X_0 = 1) - \frac{b}{a+b} = P(X_0 = 1),
 \end{aligned}$$

and the second base case $n = 1$ holds since

$$\begin{aligned}
 P(X_1 = 1) &= \sum_i P(X_1 = 1 \mid X_0 = i)P(X_0 = i) \\
 &= P(X_1 = 1 \mid X_0 = 1)P(X_0 = 1) + P(X_1 = 1 \mid X_0 = 2)P(X_0 = 2) \\
 &= (1-a)P(X_0 = 1) + b(1 - P(X_0 = 1)) \\
 &= \frac{b}{a+b} + (1-a-b) \left(P(X_0 = 1) - \frac{b}{a+b} \right)
 \end{aligned}$$

where the last equality uses the same logic as in (ii).

Assume the claim is true for any n . We want to show that the result is also true for $n+1$. From part (i), we know that

$$P(X_{n+1} = 1) = \frac{b}{a+b} + (1-a-b) \left(P(X_n = 1) - \frac{b}{a+b} \right).$$

Substituting in for $P(X_n = 1)$ gives

$$\begin{aligned} P(X_{n+1} = 1) &= \frac{b}{a+b} + (1-a-b) \left(\left((1-a-b)^n \left(P(X_0 = 1) - \frac{b}{a+b} \right) \right) - \frac{b}{a+b} \right) \\ &= \frac{b}{a+b} + (1-a-b)^{n+1} \left(P(X_0 = 1) - \frac{b}{a+b} \right). \end{aligned}$$

Shifting the indices back one period gives

$$P(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n \left(P(X_0 = 1) - \frac{b}{a+b} \right).$$

□

- (iv) Show that $P(X_n = 1)$ converges exponentially fast to $\pi(1)$ for the π you found in (i).

Solution. Since $0 < a < 1$ and $0 < b < 1$, the term $|1-a-b| < 1$. Then, noting that $(1-a-b)^n$ is the only term depending on n , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} P(X_n = 1) &= \lim_{n \rightarrow \infty} \left(\frac{b}{a+b} + (1-a-b)^n \left(P(X_0 = 1) - \frac{b}{a+b} \right) \right) \\ &= \frac{b}{a+b} + \lim_{n \rightarrow \infty} (1-a-b)^n \left(P(X_0 = 1) - \frac{b}{a+b} \right), \end{aligned}$$

which shows that convergence to the limiting distribution, $P(X_n = 1) \rightarrow \pi(1)$ as $n \rightarrow \infty$, is exponential at rate $(1-a-b)^n$. □