STAT 433: HOMEWORK 3

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Problem 1. Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains.

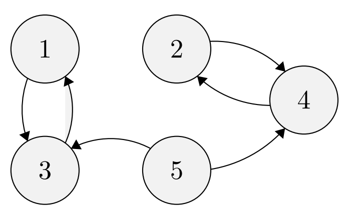
(a)	**1**	**2**	**3**	**4**	**5**
1	.4	.3	.3	0	0
2	0	.5	0	.5	0
3	.5	0	.5	0	0
4	0	.5	0	.5	0
5	0	.3	0	.3	.4

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	.1	0	0	.4	.5	0
2	.1	.2	.2	0	.5	0
3	0	.1	.3	0	0	.6
4	.1	0	0	.9	0	0
5	0	0	0	.4	0	.6
6	0	0	0	0	.5	.5

(c)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	.2	0	.8	0
3	.1	.2	.3	.4	0
4	0	.6	0	.4	0
5	.3	0	0	0	.7

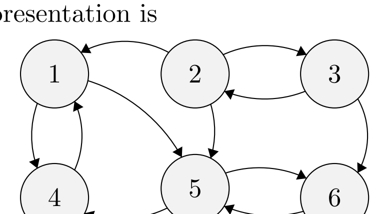
(d)	**1**	**2**	**3**	**4**	**5**	**6**
1	.8	0	0	.2	0	0
2	0	.5	0	0	.5	0
3	0	0	.3	.4	.3	0
4	.1	0	0	.9	0	0
5	0	.2	0	0	.8	0
6	.7	0	0	.3	0	0

(a) *Solution.* The directed graph representation is



Since $1 \to 2$ but $2 \not\to 1$, state 1 is transient. Since $5 \to 4$ but $4 \not\to 5$, state 5 is transient. Since $3 \to 2$ but $2 \not\to 3$, state 3 is transient. So, states 2 and 4 are recurrent. The closed irreducible set is $\{2, 4\}$. $\square$

(b) *Solution.* The directed graph representation is



Since $2 \to 1$ but $1 \not\to 2$, state 2 is transient. Since $3 \to 6$ but $6 \not\to 3$, state 3 is transient. So states 1, 4, 5, and 6 are recurrent. The closed irreducible set is $\{1, 4, 5, 6\}$. $\square$

(c) *Solution.* The directed graph representation is

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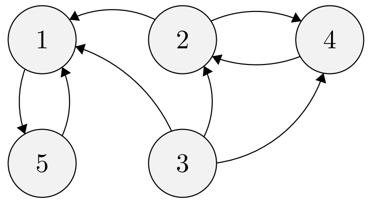
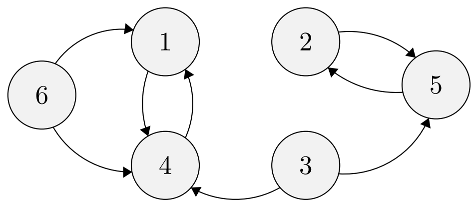


Since $3 \to 1$ but $1 \not\to 3$, state 3 is transient. So state 1, 2, 4, and 5 are recurrent. The closed irreducible sets are $\{1, 5\}$ and $\{2, 4\}$. □

(d) *Solution.* The directed graph representation is



Since $3 \to 4$ but $4 \not\to 3$, state 3 is transient, and since $6 \to 1$ but $1 \not\to 6$, state 6 is transient. So state 1, 2, 4, and 5 are recurrent. The closed irreducible sets are $\{1, 4\}$ and $\{2, 5\}$. □

Problem 2. Let G be a connected graph. Let X_n be a simple random walk on G. Show that the Markov chain $\{X_n\}$ is irreducible. (Hint: for any two vertices x and y in G, consider a path of consecutive nodes from x to y.)

Solution. Let x and y be any two vertices in G, let $T_y = \min\{n \geq 0 : X_n = y\}$ be the first hitting time of y, and let K be the smallest number of steps it takes to get from x to y. Since G is connected, x and y are connected by with a path of edges by definition, which implies that $p^K(x, y) > 0$. So, for the the chain started in x, there exists a sequence $\{y_1, y_2, \ldots, y_{K-1}\}$ such that

$$p(x, y_1) > 0, p(y_1, y_2) > 0, \ldots, p(y_{K-1}, y_K) > 0,$$

where $y_i \neq x$ for $i = 1, 2, \ldots, K - 1$. Then

$$P_x(T_y < \infty) = p(x, y_1)p(y_1, y_2) \cdots p(y_{K-1}, y_K)p(y_K, y) > 0.$$

By the same logic, for the chain started in y, we see that $\{X_n\}$ on G has exactly one communicating class, and hence is irreducible. □

Problem 3. Let G be a graph with two disjoint components G_1 and G_2. Let X_n be a simple random walk on G.

(i) Prove $\{X_n\}$ is not an irreducible Markov chain.

Solution. Since $G_1 \cap G_2 = \emptyset$, there is no edge between any node in G_1 and any node in G_2, and hence G_1 and G_2 are disconnected. Then $\{X_n\}$ on G is not irreducible.

If it were the case that G_1 and G_2 are connected, then there would exists a path between any two nodes $x_1, y_1 \in G_1$, i.e., $x_1 \leftrightarrow y_1$, and $x_2, y_2 \in G_2$, i.e., $x_2 \leftrightarrow y_2$. However, G_1 and G_2 are disconnected as previously noted, $x_1, y_1 \not\leftrightarrow x_2, y_2$. But $x_1 \leftrightarrow y_1$ and $x_2 \leftrightarrow y_2$. So, there exists two communicating classes, and hence $\{X_n\}$ on G does not satisfy the definition of an irreducible Markov chain. □

(ii) Let P be the transition matrix of X_n. Let P_1 and P_2 be the transition matrices for the SRW on G_1 and G_2, respectively. Let $V_1 = \{1, \dots, k\}$ and $V_2 = \{k+1, \dots, k+l\}$ be the set of vertices in G_1 and G_2. Show that P is of the following block diagonal form

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}. \tag{1}$$

Note a block diagonal matrix is a *reducible* matrix.

Solution. We have $P(i,j) = 0$ for all $[i] \neq [j]$ since $V_1 \cap V_2 = \emptyset$. If $[i] = [j]$ and $i \in V_1$, then X_n has transition matrix P_1, and if $i \in V_2$, X_n has transition matrix P_2. Hence, P is of the block diagonal form (1). □

(iii) Show the the SRW $\{X_n\}$ on G has infinitely many stationary distributions.

Solution. Since the disjoint graphs G_1 and G_2 are finite, P_1 has a stationary distribution, call it π_1, and P_2 has stationarity distribution, call it π_2. Note that π_1 has dimension $1 \times k$, π_2 has dimension $1 \times l$. Let

$$\pi = \begin{cases} (\pi_1, 0) & \text{if } X_0 \in G_1 \\ (0, \pi_2) & \text{if } X_0 \in G_2 \end{cases}$$

be the stationary distribution for the operators P on G, and note that π has dimension $1 \times k + l$. Note also that P has dimension $(k+l) \times (k+l)$. Now, for the chain started on any vertex in G_1, let $\pi = (\pi_1, 0)$ where 0 is a $1 \times l$ dimensional vector. Then

$$\begin{pmatrix} \pi_1 & 0 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} \pi_1 P_1 & 0 \end{pmatrix} = \begin{pmatrix} \pi_1 & 0 \end{pmatrix},$$

where the last equality follows since π_1 was defined to be the stationary distribution for P_1. For the chain started on any vertex in G_2, $\pi = (0, \pi_2)$ where here 0 is a $1 \times k$ vector. So

$$\begin{pmatrix} 0 & \pi_2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} 0 & \pi_2 P_2 \end{pmatrix} = \begin{pmatrix} 0 & \pi_2 \end{pmatrix}$$

For $0 \leq \lambda \leq 1$,

$$(\lambda \pi_1 + (1-\lambda)\pi_2) = (\lambda \pi_1 P + (1-\lambda)\pi_2 P)$$

is also stationary distribution for P since of the block diagonal form (1). Hence, there exists infinitely stationary for X_n on G. □

Problem 4. Consider a Markov chain with state space $S = \{1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix},$$

where $0 < a < 1$ and $0 < b < 1$.

(i) Find its stationary distribution π.

Solution. Let $\pi = (x, y)$. Then

$$\pi P = \pi \implies \begin{cases} (1-a)x + by = x \\ ax + (1-b)y = y \end{cases} \implies ax = by.$$

Recalling that $x + y = 1$ gives the unique stationary distribution

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b} \right).$$

□

(ii) Use the Markov property to show that

$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)\left[P(X_n=1)-\frac{b}{a+b}\right].$$

Solution. Start from the left hand side and condition on X_n to obtain

$$\begin{aligned}P(X_{n+1}=1)&=\sum_i P(X_{n+1}=1\mid X_n=i)P(X_n=i)\\&=(1-a)P(X_n=1)+b[1-P(X_n=1)]\\&=P(X_n=1)-aP(X_n=1)+b-bP(X_n=1)\\&=(1-a-b)P(X_n=1)+b\\&=(1-a-b)P(X_n=1)+\frac{b(a+b)}{a+b}\\&=(1-a-b)P(X_n=1)+\frac{ab}{a+b}+\frac{b^2}{a+b}.\end{aligned}$$

Next, subtract $b/(a+b)$ from both sides

$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)P(X_n=1)+\frac{ab}{a+b}+\frac{b^2}{a+b}-\frac{b}{a+b}.$$

Simplifying the above expression gives

$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)\left[P(X_n=1)-\frac{b}{a+b}\right].$$

□

(iii) Use mathematical induction to show that

$$P(X_n=1)=\frac{b}{a+b}+(1-a-b)^n\left[P(X_0=1)-\frac{b}{a+b}\right]$$

Proof (by induction on n). The base case $n=0$ holds since

$$\begin{aligned}P(X_0=1)&=\frac{b}{a+b}+(1-a-b)^0\left[P(X_0=1)-\frac{b}{a+b}\right]\\&=\frac{b}{a+b}+P(X_0=1)-\frac{b}{a+b}=P(X_0=1),\end{aligned}$$

and the base case $n=1$ holds since

$$\begin{aligned}P(X_1=1)&=\sum_i P(X_1=1\mid X_0=i)P(X_0=i)\\&=P(X_1=1\mid X_0=1)P(X_0=1)+P(X_1=1\mid X_0=2)P(X_0=2)\\&=(1-a)P(X_0=1)+b(1-P(X_0=1))\\&=\frac{b}{a+b}+(1-a-b)\left[P(X_0=1)-\frac{b}{a+b}\right]\end{aligned}$$

where the last equality uses the same logic as in (ii).

Assume the claim is true for any n. We want to show that the result is also true for $n+1$. From (i), we know that

$$P(X_{n+1}=1)=\frac{b}{a+b}+(1-a-b)\left[P(X_n=1)-\frac{b}{a+b}\right].$$

Substituting in for $P(X_n = 1)$ gives

$$\begin{aligned} P(X_{n+1} = 1) &= \frac{b}{a+b} + (1-a-b)\left[\left[(1-a-b)^n\left[P(X_0 = 1) - \frac{b}{a+b}\right]\right] - \frac{b}{a+b}\right] \\ &= \frac{b}{a+b} + (1-a-b)^{n+1}\left[P(X_0 = 1) - \frac{b}{a+b}\right]. \end{aligned}$$

Shifting the indices back one period gives

$$P(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n\left[P(X_0 = 1) - \frac{b}{a+b}\right].$$

□

(iv) Show that $P(X_n = 1)$ converges exponentially fast to $\pi(1)$ for the π you found in (i).

Solution. Since $0 < a < 1$ and $0 < b < 1$, the term $|1 - a - b| < 1$. Then, noting that $(1 - a - b)^n$ is the only term depending on n, we have

$$\begin{aligned} \lim_{n\to\infty} P(X_n = 1) &= \lim_{n\to\infty}\left[\frac{b}{a+b} + (1-a-b)^n\left[P(X_0 = 1) - \frac{b}{a+b}\right]\right] \\ &= \frac{b}{a+b} + \lim_{n\to\infty}(1-a-b)^n\left[P(X_0 = 1) - \frac{b}{a+b}\right] \\ &= \frac{b}{a+b} + O((1-a-b)^n). \end{aligned}$$

That is, convergence to the limiting distribution, $P(X_n = 1) \to \pi(1)$ as $n \to \infty$, is exponentially fast. □