STAT 433: HOMEWORK 6

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Problem 1. Suppose N_t Poisson process with rate 3. Let T_n denote the time of the n-th arrival.

(i) Find $\mathrm{E}(T_{12})$,

Solution. The integral of T_{12} is

$$\mathrm{E}(T_{12}) = \int t f_{T_{12}}(t)dt = \int 3te^{-3t}\frac{(3t)^{11}}{11!}dt = 4.$$

□

(ii) Find $\mathrm{E}(T_{12} \mid N_2 = 5)$.

Solution. Since iterarrival times τ_i are i.i.d. Exponential(3) random variables,

$$\mathrm{E}(T_{12} \mid N_2 = 5) = \mathrm{E}(T_7) + 2 = 7\,\mathrm{E}\left(\sum_{i=1}^{12} \tau_i\right) + 2 = \frac{7}{3} + 2 = \frac{13}{3}.$$

□

(iii) Find $\mathrm{E}(N_5 \mid N_2 = 5)$.

Solution. First note that $N_5 = N_2 + (N_5 - N_2)$. Then

$$\mathrm{E}(N_2 \mid N_2 = 5) + \mathrm{E}(N_5 - N_2 \mid N_2 = 5) = 5 + \mathrm{E}(N_3) = 5 + 3 \cdot 3 = 14.$$

□

Problem 2. Starting at 9 a.m., patients arrive at a doctor's office according to a Poisson process. On average, three patients arrive every hour.

(i) Find the probability that at least two patients arrive by 9:30 a.m.

Solution. We have $N_t \sim \text{Poisson}(3t/2)$. Note first that in 30 minutes, $\lambda 30 = (3/60)30 = 3/2$, and so $N_{3/2} \sim \text{Poisson}(3/2)$. Since the probability that at least 2 patients arrive by 9:30 a.m. is equivalent to the complement of the event that no patients arrive by 9:30 a.m. or one patient arrives by 9:30 a.m., we have

$$\begin{aligned} P(N_{3/2} \geq 2) &= 1 - P(N_{3/2} = 0) - P(N_{3/2} = 1) = 1 - e^{-3/2} - \frac{3}{2}e^{-3/2} \\ &= 1 - \frac{5}{2}e^{-3/2} = 0.442. \end{aligned}$$

□

(ii) Find the probability that 10 patients arrive by noon and eight of them come to the office before 11 a.m.

Solution. We have, by stationary increments and independent increments,

$$\begin{aligned} P(N_2 = 8, N_3 = 10) &= P(N_2 = 8, N_3 - N_2 = 2) \\ &= P(N_2 = 8)P(N_1 = 2) \\ &= e^{-6}\frac{6^8}{8!} \cdot e^{-3}\frac{3^2}{2!} = 0.023. \end{aligned}$$

□

(iii) If six patients arrive by 10 a.m., find the probability that only one patient arrives by 9:15 a.m.

Solution. Again by stationary increments and independent increments, we have

$$\begin{aligned} P(N_{1/4} = 1 \mid N_1 = 6) &= \frac{P(N_{1/4} = 1, N_1 = 6)}{P(N_1 = 6)} \\ &= \frac{P(N_{1/4} = 1, N_1 - N_{1/4} = 5)}{P(N_1 = 6)} \\ &= \frac{P(N_{1/4} = 1)P(N_{3/4} = 5)}{P(N_1 = 6)} \\ &= \frac{(3/4)e^{-3/4}e^{-9/4}(9/4)^5/5!}{e^{-3}(3^6/6!)} = 0.356. \end{aligned}$$

□

Problem 3. Starting at 6 a.m., cars, buses, and motorcycles arrive at a highway toll booth according to independent Poisson processes. Cars arrive about once every 5 minutes. Buses arrive about once every 10 minutes. Motorcycles arrive about once every 30 minutes.

(i) Find the probability that in the first 20 minutes, exactly three vehicles – two cars and one motorcycle – arrive at the booth.

Solution. Let $(C_t)_{t\geq 0}$, $(M_t)_{t\geq 0}$, and $(B_t)_{t\geq 0}$ denote the three independent Poisson processes corresponding to cars, buses, and motorcycles, respectively. Note that $C_t \sim \text{Poisson}(t/5)$, $M_t \sim \text{Poisson}(t/10)$, and $B_t \sim \text{Poisson}(t/30)$. By independent increments, we have

$$\begin{aligned} P(C_{20} = 2, M_{20} = 1, B_{20} = 0) &= P(C_{20} = 2)P(M_{20} = 1)P(B_{20} = 0) \\ &= e^{-20/5}\frac{(20/5)^2}{2!}e^{-20/30}\frac{(20/30)^1}{1!}e^{-20/10}\frac{(20/10)^0}{0!} \\ &= 0.006787. \end{aligned}$$

□

(ii) At the toll booth, the chance that a driver has exact change is 1/4, independent of vehicle. Find the probability that no vehicle has exact change in the first 10 minutes.

Solution. The superposition of each process is $C_t + M_t + B_t = N_t \sim \text{Poisson}(1/5 + 1/10 + 1/30 = 1/3)$. This arrival process is then thinned according to the random variable indicating whether a car has exact change. The resulting Poisson process has parameter $1/4 \cdot 1/3 = 1/12$. Hence

$$P(\text{car has exact change}) = e^{-10(1/12)} = e^{-5/6}.$$

□

(iii) Find the probability that the 6th motorcycle arrives within 45 minutes of the third motorcycle.

Solution. Let $S(t)$ be the sum of motorcycles that arrive at the toll booth. Then

$$P(S(7) - S(3) < 45) = P(M_4 + M_5 + M_6 + M_7 < 45)$$
$$= \int_0^{45} \frac{1}{30} e^{-45/30} \frac{(45/30)^3}{3!} dt = 0.19,$$

which follows since the sum of exponential random variables follows a gamma distribution. □

(iv) Find the probability that at least one other vehicle arrives at the toll booth between the third and fourth car arrival.

Solution. Observe that $M_t + B_t \sim \text{Poisson}(1/10 + 1/30 = 2/15)$. Let τExponential$(1/5)$ be the interarrival time between the 3rd and 4th car arrival. Then

$$P(M_t + B_t > 0) = \int_0^{\infty} P(M_t + B_t > 0 \mid \tau = t) \frac{1}{5} e^{-t/5} dt$$
$$= \frac{1}{5} \int_0^{\infty} \left(1 - e^{-2t/15}\right) e^{-t/5} dt = \frac{2}{5}.$$

□

Problem 4. Let S_t be the price of a stock at time t and suppose that at times of a Poisson process with rate λ the price is multiplied by a random variable $X_i > 0$ with mean μ and variance σ^2. That is

$$S_t = \prod_{i=1}^{N_t} X_i,$$

where the product is 1 if $N_t = 0$. Find $\mathrm{E}\, S_t$ and $\operatorname{var} S_t$.

Solution. By the law of total expectation, we may write $\mathrm{E}\, S_t = \mathrm{E}\,\mathrm{E}(S_t \mid N_t)$ and $\mathrm{E}\, S_t^2 = \mathrm{E}\,\mathrm{E}(S_t^2 \mid N_t)$ Then condition on $N_t = n$ and use linearity of the expectation operator to obtain,

$$\mathrm{E}(S_t \mid N_t = n) = \mathrm{E}\left(\prod_{i=1}^{n} X_i \mid N_t = n\right) = \mathrm{E}\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} \mathrm{E}\, X_i = \mu^n,$$

and, for the conditional second moment,

$$\mathrm{E}(S_t^2 \mid N_t = n) = \mathrm{E}\left(\prod_{i=1}^{n} X_i^2 \mid N_t = n\right) = \mathrm{E}\left(\prod_{i=1}^{n} X_i^2\right) = \prod_{i=1}^{n} \mathrm{E}\, X_i^2 = (\mu^2 + \sigma^2)^n$$

We now have

$$\mathrm{E}\, S_t = \mathrm{E}\,\mathrm{E}(S_t \mid N_t) = \mu^{N_t},$$

and

$$\mathrm{E}\, S_t^2 = \mathrm{E}\,\mathrm{E}(S_t^2 \mid N_t) = (\mu^2 + \sigma^2)^{N_t},$$

The mean is

$$\mathrm{E}\, S_t = \mathrm{E}\, \mu^{N_t} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{-\lambda t} e^{-\mu \lambda t} = e^{(\mu - 1)\lambda t}.$$

For the second moment, we have

$$\mathrm{E}\, S^2 = \mathrm{E}(\mu^2 + \sigma^2)^{N_t} = e^{-\lambda t} e^{(\mu^2 + \sigma^2)\lambda t} = e^{(\mu^2 + \sigma^2 - 1)\lambda t},$$

and so the variance is

$$\operatorname{var} S_t = \mathrm{E}\, S_t^2 - (\mathrm{E}\, S_t)^2 = e^{(\mu^2 + \sigma^2 - 1)\lambda t} - e^{2(\mu - 1)\lambda t}.$$

□