STAT 433: HOMEWORK 2

DANIEL PFEFFER

Problem 1. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, and is 0.6 if at least one of the last two days was rainy. Let $\Omega = \{(S, S), (S, R), (R, S), (R, R)\}$, where S denotes sunny and R denotes rainy. Let $X_n = (W_n, W_{n-1})$, where W_n denotes the weather on the nth day.

(i) Prove that X_n is a Markov chain on Ω, and find its transition matrix P.

Solution. That X_n is a Markov chain on Ω follows from the fact that is satisfies the Markov property:

$$P(X_n = x_n \mid X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0)$$

$$= \begin{cases} 0.3 & \text{if } x_n = (R, S), x_{n-1} = (S, S) \\ 0.7 & \text{if } x_n = (S, S), x_{n-1} = (S, S) \\ 0.6 & \text{if } x_n = (R, S), x_{n-1} = (S, R) \\ 0.6 & \text{if } x_n = (R, R), x_{n-1} = (R, S) \text{ or } (R, R) \\ 0.4 & \text{if } x_n = (S, S), x_{n-1} = (S, R) \\ 0.4 & \text{if } x_n = (S, R), x_{n-1} = (R, S) \text{ or } (R, R). \end{cases}$$

These probabilities depend only on x_n and x_{n-1}. So we may write

$$P(X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = P(X_n = x_n \mid X_{n-1} = x_{n-1}),$$

and the Markov property is satisfied. It follows from this results that the transition matrix is

$$P = \begin{array}{c} \\ (S,S) \\ (S,R) \\ (R,S) \\ (R,R) \end{array} \overset{\begin{array}{cccc} (S,S) & (S,R) & (R,S) & (R,R) \end{array}}{\left(\begin{array}{cccc} 0.7 & 0 & 0.3 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.4 & 0 & 0.6 \end{array} \right)}$$

$\square$

(ii) Calculate the 2-step transition matrix P^2. Use it to find what is the probability that it will rain on Wednesday if it didn't rain on Sunday and Monday.

Solution. The 2-step transition matrix is

$$P^2 = \begin{pmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.28 & 0.24 & 0.12 & 0.36 \\ 0.16 & 0.24 & 0.24 & 0.36 \\ 0.16 & 0.24 & 0.24 & 0.36 \end{pmatrix}.$$

Let $X_0 = (W_{\text{Mon}}, W_{\text{Sun}})$, $X_1 = (W_{\text{Tue}}, W_{\text{Mon}})$, and $X_2 = (W_{\text{Wed}}, W_{\text{Tue}})$ and note that the initial distribution of X_0 is $\mu = (1, 0, 0, 0)$. So

$$\mu P^2 = (0.49, 0.12, 0.21, 0.18),$$

where each element corresponds to the probability (S, S), (S, R), (R, S), and (R, R), respectively. Therefore

$$P(W_{\text{Wed}} = R) = 0.21 + 0.18 = 0.39.$$

$\square$

Problem 2. Four white balls and four black balls are distributed in two urns in such a way that each urn contains four balls. At each step, we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.

Solution. Note first that the Markov property is satisfied since X_n depends only on the time n draw and the amount of white balls in remaining in the left urn at time $n - 1$. If there are i white balls in the left urn, there are $(4 - i)$ black balls in that urn. Moreover, there are $(4 - i)$ white balls and i black balls in the right urn. Then, for $1 \leq i, j \leq 4$, we have

$$P(X_n = j \mid X_{n-1} = i) = \begin{cases} i(4-i)/8 & \text{if } j = i \\ (4-i)^2/16 & \text{if } j = i + 1 \\ i^2/16 & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases}.$$

$\square$

Problem 3. Find the stationary distributions for the Markov chains with the following transition matrices.

(i)
$$\begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

Solution. Since each column sums to one, the matrix is doubly stochastic, and therefore has the uniform stationary distribution $\pi = (1/3, 1/3, 1/3)$. $\square$

(ii)
$$\begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.2 & 0.4 & 0.4 \\ 0 & 0.2 & 0.8 \end{pmatrix}.$$

Solution. Let $\pi = (x, y, z)$ and solve the linear system $\pi P = \pi$ to obtain $y = 2x$ and $z = 4x$. Then, since we require that $x + y + z = 1$, the stationary distribution is $\pi = (1/7, 2/7, 4/7)$. $\square$

Problem 4. Let X_n be a Markov chain with state space $E = \{r, w, b, g\}$ and transition matrix

$$P = \begin{matrix} r \\ w \\ b \\ y \end{matrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0.4 & 0.6 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \end{pmatrix}.$$

(i) Compute $P(X_5 = b, X_6 = r, X_7 = b, X_8 = b \mid X_4 = w)$.

Solution. We have
$$\begin{aligned} P(X_5 = b, X_6 = r, X_7 = b, X_8 = b \mid X_4 = w) &= p(w, b)p(b, r)p(r, b)p(b, b) \\ &= 0.6 \cdot 0.8 \cdot 1 \cdot 0.2 \\ &= 0.096. \end{aligned}$$

□

(ii) Compute $E(f(X_5), f(X_6) \mid X_4 = y)$ for the function with
$$f(r) = 2, \quad f(w) = 4, \quad f(b) = 7, \quad f(y) = 3.$$

Solution. First note that
$$P(X_5 = r, X_6 = b \mid X_4 = y) = 0.2 \cdot 1.0 = 0.20$$
$$P(X_5 = w, X_6 = w \mid X_4 = y) = 0.3 \cdot 0.4 = 0.12$$
$$P(X_5 = w, X_6 = b \mid X_4 = y) = 0.3 \cdot 0.6 = 0.18$$
$$P(X_5 = y, X_6 = r \mid X_4 = y) = 0.5 \cdot 0.2 = 0.10$$
$$P(X_5 = y, X_6 = w \mid X_4 = y) = 0.5 \cdot 0.3 = 0.15$$
$$P(X_5 = y, X_6 = y \mid X_4 = y) = 0.5 \cdot 0.5 = 0.25.$$

Then
$$E(f(X_5), f(X_6) \mid X_4 = y) = 0.2 \cdot 14 + 0.12 \cdot 16 + 0.18 \cdot 28 + 0.1 \cdot 6 + 0.15 \cdot 12 + 0.25 \cdot 9$$
$$= 14.41.$$

□

(iii) Let $T_{(w,b)}$ be the first time n that $X_n = b$ given $X_0 = w$. Let $T_{(b,b)}$ be the first time n that $X_n = b$ given $X_0 = b$. Find the distribution of $T_{(w,b)}$ and $T_{(b,b)}$, respectively. (You can find them by inspection without doing much calculation.)

Solution. For the chain started in w, it either stays at w with probability 0.4 or jump to b with probability 0.6, implying that $T_{(w,b)} \sim \text{Geometric}(0.6)$ and has distribution
$$P(T_{(w,b)} = n) = 0.6 \cdot 0.4^{n-1},$$

for $n = 1, 2, \ldots$. For the chain started in b, it stays in b with probability 0.2 and jumps to r with probability 0.8 from which it returns to b with probability one. Hence
$$P(T_{(b,b)} = 1) = 0.2$$
$$P(T_{(b,b)} = 2) = 0.8.$$

□