STAT 433: HOMEWORK 4

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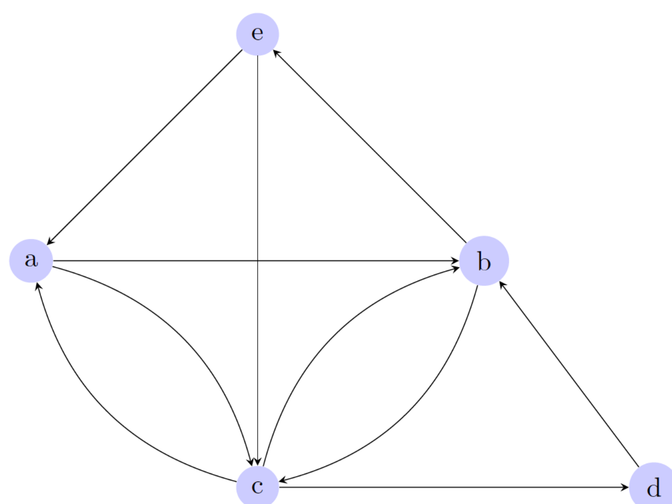
Problem 1. Consider the following directed webgraph, where each edge has a direction. The web surfer can only jump along the assigned direction of the edge – jumping in the reverse direction is prohibited. Let X_n be a simple random walk of a web surfer on this graph. More precisely, if the surfer is at site i, then it will randomly pick one of the outgoing edges and jump to its neighborhood along this edge. For example, if the surfer is at site c, then it will choose one of the three outgoing edges, and jump to site a, b or d with probability 1/3.



(i) Write down the transition matrix of this Markov chain.

Solution. Let $G = (V, E)$ be the webgraph where the vertex set $V = \{a, b, c, d, e\}$ and the edge set $E = \{(a, b), (a, c), (b, c), (b, e), (c, a), (c, b), (c, d), (d, b), (e, a), (e, c)\}$. Here the edge set contains ordered pairs where for any two vertices $i, j \in G$, (i, j) represents the edge from i to j. Then the transition matrix is

$$p(i,j) = P(X_{n+1} = i \mid X_n = j) = \begin{cases} 1/2 & \text{if } i = a, j = b, c \\ 1/2 & \text{if } i = b, j = c, e \\ 1/3 & \text{if } i = c, j = a, b, d \\ 1 & \text{if } i = d, j = b \\ 1/2 & \text{if } i = e, j = a, c. \end{cases}$$

Each entry is nonnegative and each row sums to one. Hence, P is a transition matrix. □

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(ii) Show that the Markov chain is irreducible.

Solution. To demonstrate that this chain is irreducible, we show by inspection that for any initial state i it is there is a nonzero probability of reaching any other state, i.e., the vertex set (state space) V is a single communicating class. For the chain started at a,

$$\begin{aligned} p(a,b) &= 1/2 > 0 \\ p(a,c) &= 1/2 > 0 \\ p(a,d) &= p(a,b)p(b,c)p(c,d) = 1/2 \cdot 1/2 \cdot 1/3 \cdot > 0 \\ p(a,e) &= p(a,b)p(b,e) = 1/2 \cdot 1/2 > 0. \end{aligned}$$

For the chain started at b,

$$\begin{aligned} p(b,a) &= p(b,c)p(c,a) = 1/2 \cdot 1/3 > 0 \\ p(b,c) &= 1/2 > 0 \\ p(b,d) &= p(b,c)p(c,d) = 1/2 \cdot 1/3 > 0 \\ p(b,e) &= 1/2 > 0. \end{aligned}$$

For the chain started in c,

$$\begin{aligned} p(c,a) &= 1/3 > 0 \\ p(c,b) &= p(c,a)p(a,b) = 1/2 \cdot 1/2 > 0 \\ p(c,d) &= 1/3 > 0 \\ p(c,e) &= p(a,c)p(a,b)p(b,e) = 1/3 \cdot 1/2 \cdot 1/2 > 0. \end{aligned}$$

For the chain started in d,

$$p(d,a) = p(d,b)p(b,e)p(e,a) = 1 \cdot 1/2 \cdot 1/2 > 0,$$

and as just shown, every state is accessible from b and hence every state is accessible from d. For the chain started in e,

$$p(e,a) = 1/2 > 0,$$

and again since every state is accessible from state a, every state is also accessible from d. Since every state leads to every other states, all states communicate with each other, and so the chain is irreducible. □

(iii) Find the stationary distribution π.

Solution. More explicitly, the transition matrix is

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \end{pmatrix}$$

It has stationary distribution $\pi = (6/35, 2/7, 3/10, 1/10, 1/7)$. □

(iv) Determine the PageRank for the sites and explain the reasons.

Solution. The PageRank is c, b, a, e, and then d, which corresponds to the frequencies from the limiting distribution in (iii). □