STAT 433: HOMEWORK 1

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Problem 1. A poker hand consists of five cards drawn from a standard 52-card deck. Find the expected number of aces in a poker hand given that the first card drawn is an ace.

Solution. Let X be the number of aces in a poker hand, let A_i, for $i = 1, 2, \ldots, 5$, be the event that the ith card drawn is an ace, and let 1_{A_i} be an indicator function that equals one if the A_i occurs and zero otherwise. Then we may write

$$X = 1_{A_1} + 1_{A_2} + \cdots + 1_{A_5}.$$

Since we are conditioning on the event that the first card drawn is an ace, $1_{A_1} = 1$. It follows that the probability that one of the next four cars is an ace is $3/51 = 1/17$. By linearity of expectations, we obtain

$$\mathrm{E}\, X = \mathrm{E}\, 1_{A_1} + \frac{1}{17}(\mathrm{E}\, 1_{A_1} + \mathrm{E}\, 1_{A_2} + \cdots + \mathrm{E}\, 1_{A_5}) = 1 + \frac{4}{17} = \frac{21}{17}.$$

□

Problem 2. An urn has n balls. Balls are drawn one at a time and then put back in the urn. Let X denote the number of drawings required until some ball is drawn more than once. Find the probability distribution of X.

Solution. Since it is impossible for a ball to be drawn twice if only one ball has been drawn, $P(X = 1) = 0$. Note also that $P(X = 2) = 1/n$ because there is only one ball (out of n) that can be drawn again. To deduce the probability that $X = 3$, notice that the same ball is not drawn on the first trial with probability $1 - P(X = 1) = 1$ and on the second draw, the same ball is not drawn with probability $1 - P(X = 2) = 1 - 1/n$. Then, for the event $\{X = 3\}$ to occur, the repeat draw must occur on the third trial, which happens with probability $2/n$ since there are 2 balls in the urn containing n balls which can be drawn again:

$$P(X = 3) = \left(1 - \frac{1}{n}\right)\left(\frac{2}{n}\right).$$

Continuing with this reasoning, we obtain a probability distribution, for $1 \leq x \leq n + 1$, given by

$$P(X = x) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{x-2}{n}\right)\left(\frac{x-1}{n}\right)$$

□

Problem 3. A man with n keys wants to open his door. He tries the keys in a random manner. Let X be the number of trials required to open the door.

(i) Find $\mathrm{E}(X)$ and $\mathrm{var}(X)$ if unsuccessful keys are eliminated from further selection.

Solution. Note that, since the number of keys n decreases by 1 each trial, we have

$$P(X = 1) = \frac{1}{n}$$
$$P(X = 2) = \left(1 - \frac{1}{n}\right) \frac{1}{n-1}$$
$$\vdots$$
$$P(X = k) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n-1}\right) \cdots \left(1 - \frac{1}{n-(k-2)}\right) \frac{1}{n-(k-1)},$$

Then, the expectation of X is

$$\mathrm{E}(X) = \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

To find the variance of X, compute the second moment:

$$\mathrm{E}(X^2) = \frac{1}{n} + \frac{4}{n} + \cdots + \frac{n^2}{n} = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}.$$

Using the fact that $\mathrm{var}(X) = \mathrm{E}(X^2) - (\mathrm{E}\,X)^2$ gives

$$\mathrm{var}(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}.$$

□

(ii) Find $\mathrm{E}(X)$ and $\mathrm{var}(X)$ if unsuccessful keys are not eliminated from further selection.

Solution. If unsuccessful keys not are eliminated from further selection, then $X \sim \text{Geometric}(1/n)$, and hence

$$\mathrm{E}(X) = \frac{1}{1/n} = n$$
$$\mathrm{var}(X) = \frac{1 - 1/n}{1/n^2} = n(n-1).$$

□

Problem 4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $X_i \sim \mathcal{U}([0,t])$, and let $Y_n = \min\{X_1, \ldots, X_n\}$.

(i) Find $P(Y_n > x)$ for $x \in [0,t]$.

Solution. Since Y_n is greater than the minimum of $X_1, X_2, \ldots, X_n$, we know Y_n must also be greater than every $X_1, X_2, \ldots, X_n$. So,

$$\begin{aligned} P(Y_n > x) &= P(X_1 > x, X_2 > x, \ldots, X_n > x) \\ &= \prod_{i=1}^{n} \left(1 - \frac{x}{t}\right) \\ &= \left(1 - \frac{x}{t}\right)^n \end{aligned}$$

□

(ii) Let t be a function of n such that $\lim_{n\to\infty} n/t = \lambda$. Show that

$$\lim_{n\to\infty} P(Y_n > x) = e^{-\lambda x}.$$

Solution. Evaluating the limit gives

$$\begin{aligned}
\lim_{n\to\infty} P(Y_n > x) &= \lim_{n\to\infty} e^{n\log(1-x/t)^n} \\
&= \lim_{n\to\infty} e^{n(-x/t+o(1/t))} \\
&= \lim_{n\to\infty} e^{n(-x\lambda/n+o(1/n))} \\
&= e^{-\lambda x},
\end{aligned}$$

since by definition t satisfies $\lim_{n\to\infty} n/t = \lambda$. □