STAT 433: HOMEWORK 7

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Problem 1. The planets of the Galactic Empire are distributed in space according to a spatial Poisson process at an approximate density of one planet per cubic parsec. From the Death Star, let X be the distance to the nearest planet.

(a) Find the probability density function of X.

Solution. The event $\{X > r\}$ occurs iff there are no objects in a ball B_r with radius r around the Death State. Note that B_r has Lebesgue measure (volume) $|B_r| = -4\pi r^3/3$ and

$$P(X > r) = P(N_{B_r} = 0) = P(\text{Poisson}(4\pi r^3/3) = 0) = e^{-4\pi r^3/3}$$

Then notice that the cdf is given by

$$P(X < r) = 1 - e^{-4\pi r^3/3}.$$

Differentiating the cdf, we obtain the pdf

$$f(r) = 4\pi r^2 e^{-4\pi r^3/3},$$

for all $r \geq 0$. □

(b) Find the mean distance from the Death Star to the nearest planet. You can calculate the integral numerically.

Solution. The integral of X is

$$\mathrm{E}\, X = \int_0^\infty 4\pi r^3 e^{-4\pi r^3/3} dr = \frac{1}{36\sqrt[3]{6\pi}} \Gamma(1/3) \approx 0.55.$$

□

Problem 2. Customers arrive at a bank according to a Poisson process with rate 10 per hour. Given that 5 customers arrived in the first 30 minutes, answer the following questions.

(i) What is the probability that at least 3 arrived in the first 10 minutes?

Solution. Let $N_{1/2} = 5$ denote the number of arrivals after 30 minutes, let T_i denote the ith arrival time, note that

$$(T_1, T_2, T_3, T_4, T_5) \mid N_{1/2} = 5 \sim \mathcal{U}([0, 1/2]).$$

Then the number of arrivals in the first 10 minutes is Binomial$(5, 1/3)$, and so the event that the at least 3 customers arrived in the first 10 minutes is

$$\binom{5}{3} \frac{1}{3^3} \cdot \frac{2^2}{3^2} + \binom{5}{4} \frac{1}{3^4} \cdot \frac{2}{3} + \frac{1}{3^5}$$

□

(ii) What is the probability that 2 arrived in the first 10 minutes and 1 arrived in the next 5 minutes?

Solution. For a random variable $U_i \sim \mathcal{U}([0,1/2])$, the probability that it lies in the interval $[0,1/6]$ is $1/3$, the probability that it lies in the interval $(1/6,1/4]$ is $1/6$, and the probability that it lies in the interval $(1/4,1/2]$ is $1/2$. Then for the 5 i.i.d. random variables $U_1,\dots,U_5$, the probability that two of them lie in $[0,1/6]$, 1 lies in $(1/6,1/4]$, and 2 lie in $(1/4,1/2]$ is

$$\frac{5!}{2!1!2!}\cdot\frac{1}{3^2}\cdot\frac{1}{6}\cdot\frac{1}{2^2}=\frac{5}{36},$$

which follows directly from the multinomial distribution. □

(iii) What is the mean of the arriving time for the first customer?

Solution. Let T_1 be the arriving time for the first customer. Then,

$$P(T_1 \geq t) = P\left(\min_{1\leq i\leq 5} U_i \geq t\right) = (1-2t)^5.$$

for $0 \leq t \leq 1/2$, and the integral of T_1 is

$$\mathrm{E}\,T_1 = \int_0^{1/2} P(T_1\geq t)dt = \int_0^{1/2}(1-2t)^2 dt = \frac{1}{2}\int_0^1 x^5 dx = \frac{1}{12}\text{ hours.}$$

So the mean of the arriving time of the first customer is 5 minutes. □

Problem 3. Recall the long run car costs problem. Suppose that the lifetime of a car is a random variable with density function $f(t)$. Our methodical Mr. Brown buys a new car as soon as the old one breaks down or reaches T years. Suppose that a new car costs A dollars and that an additional cost of B dollars to repair the vehicle is incurred if it breaks down before time T. If $f(t)=\lambda e^{-\lambda t}$, show that for any A and B the optimal time is $T=\infty$. Can you give a simple explanation in words.

Solution. The cost of the ith cycle is

$$\mathrm{E}\,r_i = A + B\int_0^T \lambda e^{-\lambda t}dt = A + B(1-e^{-\lambda T}).$$

For the the duration of the ith cycle, we have

$$\mathrm{E}\,\tau_i = \int_0^T t\lambda e^{-\lambda t}dt + T\int_T^\infty \lambda e^{-\lambda t}dt = -te^{-\lambda t}\Big|_0^T + \int_0^T e^{-\lambda t}dt + Te^{-\lambda T} = \frac{1}{\lambda}(1-e^{-\lambda T}).$$

By the elementary renewal theorem tells, the long run reward per unit time is

$$\frac{\mathrm{E}\,r_i}{\mathrm{E}\,\tau_i} = \frac{A\lambda}{1-e^{-\lambda T}} + B\lambda.$$

Since the function is strictly decreasing in T, the optimal policy is to set $T=\infty$. This result can be understood intuitively through the memoryless property of exponential random variables: if Mr. Brown used the car for t years, then the probability that it can be used for another s years is the same as the probability for a new car to work for s years. □

Problem 4. A young doctor is working at night in an emergency room. Emergencies come in at times of a Poisson process with rate $\lambda = 0.5$ per hour. The doctor can only get to sleep when it has been $c = 36$ minutes (0.6 hours) since the last emergency. For example, if there is an emergency at 1:00 and a second one at 1:17 then she will not be able to get to sleep until at least 1:53, and it will be even later if there is another emergency before that time. We want to compute the long-run fraction of time the doctor spends sleeping with the following strategy.

(a) If $T \sim$ Exponential(λ), find $\mathrm{E}(T \mid T < c)$.

Solution. The desired conditional expectation is

$$\begin{aligned}
\mathrm{E}(T \mid T < c) &= \frac{1}{P(T<c)} \int_0^c t\lambda e^{-\lambda t} dt \\
&= \frac{1}{1-e^{-\lambda t}} \left(\frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda c} - ce^{-\lambda c} \right) \\
&= \frac{1}{\lambda} - \frac{ce^{-\lambda c}}{1-e^{-\lambda c}}.
\end{aligned}$$

□

(b) Let $J_n = \min\{j : \tau_j > c\}$, where $\tau_1, \tau_2, \ldots, \tau_n$ are the interarrival times of the Poisson process with rate λ. Use (a) to show that

$$\mathrm{E}(T_{J-1} + c) = \frac{e^{\lambda c} - 1}{\lambda}$$

Solution. Use the law of total expectation to write $\mathrm{E}\, T_{J-1} = \mathrm{E}\,\mathrm{E}(T_{J-1} \mid J)$. For $J = j$,

$$\begin{aligned}
\mathrm{E}(T_{J-1} \mid J = j) &= \mathrm{E}(T_{j-1} \mid J = j) \\
&= \mathrm{E}(\tau_1 + \cdots + \tau_{j-1} \mid \tau_1 < c, \cdots, \tau_{j-1} < c, \tau_j \geq c) \\
&= \sum_{k=1}^{j-1} \mathrm{E}(\tau_k \mid \tau_k < c) \\
&= (j-1) \left(\frac{1}{\lambda} - \frac{ce^{-\lambda c}}{1-e^{-\lambda c}} \right)
\end{aligned}$$

So

$$\mathrm{E}(T_{J-1} \mid J) = (J-1) \left(\frac{1}{\lambda} - \frac{ce^{-\lambda c}}{1-e^{-\lambda c}} \right)$$

Note that since $J \sim \mathrm{Geometric}(e^{-\lambda c})$, $\mathrm{E}\, J = 1/e^{-\lambda c}$. Then

$$\begin{aligned}
\mathrm{E}\, T_{J-1} &= \mathrm{E}(J-1) \left(\frac{1}{\lambda} - \frac{ce^{-\lambda c}}{1-e^{-\lambda c}} \right) \\
&= (e^{\lambda c} - 1) \left(\frac{1}{\lambda} - \frac{ce^{-\lambda c}}{1-e^{-\lambda c}} \right) \\
&= \frac{e^{\lambda c} - 1}{\lambda} - c.
\end{aligned}$$

Therefore,

$$\mathrm{E}(T_{J-1} + c) = \frac{e^{\lambda c} - 1}{\lambda}.$$

□

(c) The doctor alternates between sleeping for an amount of time s_i and being awake for an amount of time u_i. Use the result from (b) to compute $\mathrm{E}\, u_i$.

Solution. We have

$$\mathrm{E}\, u_i = \frac{e^{\lambda c} - 1}{\lambda} = \frac{e^{0.5 \cdot 0.6} - 1}{0.6} = 2(e^{0.3} - 1).$$

□

(d) Compute the long-run fraction of time the doctor spends sleeping.

Solution. Since $s_i \sim \text{Exponential}(\lambda)$, $\mathrm{E}\, s_i = 1/\lambda$ and the long-run fraction of time the doctor spends sleeping is

$$\frac{\mathrm{E}\, s_i}{\mathrm{E}\, s_i + \mathrm{E}\, u_i} = \frac{\lambda^{-1}}{\lambda^{-1} + \lambda^{-1}(e^{\lambda c} - 1)} = e^{-\lambda c} = e^{-0.3}.$$

□

(e) Model the process using a counter model, and compute (d) in another way using the formula on class.

Solution. This process can be modeled using a Type II counter model with sleeping corresponding to the alive period and being awake corresponding to the locked period. Then, using the result that the long-run fraction of alive time is

$$\lim_{t\to\infty} p_a(t) = e^{-\lambda Y_i},$$

and setting $Y_i = c$ for our problem, gives

$$\lim_{t\to\infty} p_a(t) = e^{-\lambda c},$$

the desired result. □