ECON 706: PROBLEM SET 5

DANIEL PFEFFER

Problem 1. Vector Autoregressions

Consider the following VAR

$$y_t = \Phi_0 + \Phi_1 y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \text{iid } N(0, \Sigma),$$

where y_t is an $n \times 1$ vector.

(i) Show that the conditional maximum likelihood estimation of Φ is equivalent to equation-by-equation OLS estimation:

$$y_{it} = \Phi_{0i} + \Phi_{1i\cdot} y_{it-1} + \varepsilon_{it},$$

where $\Phi_{1i\cdot}$ is the ith row of the $n \times n$ matrix Φ_1.

Solution. Let $X_t = (1, y_{t-1})'$, $\Phi = (\Phi_0', \Phi_1')'$, and $\Phi^i = (\Phi_{0i}', \Phi_{1i}')'$. The least squares estimator for row (equation) i is given by

$$\hat{\Phi}^i = \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} \sum_{t=1}^{T} X_t y_{it}',$$

and maximum likelihood estimator of Φ is

$$\begin{aligned}
\hat{\Phi} &= \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} \sum_{t=1}^{T} X_t y_t' \\
&= \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} \sum_{t=1}^{T} X_t \begin{pmatrix} y_{1t} & y_{2t} & \cdots & y_{nt} \end{pmatrix} \\
&= \left(\sum_{t=1}^{T} X_t X_t' \right)^{-1} \begin{pmatrix} \sum_{t=1}^{T} X_t y_{1t} & \sum_{t=1}^{T} X_t y_{2t} & \cdots & \sum_{t=1}^{T} X_t y_{nt} \end{pmatrix} \\
&= \begin{pmatrix} (\sum_{t=1}^{T} X_t X_t')^{-1} \sum_{t=1}^{T} X_t y_{1t} \\ (\sum_{t=1}^{T} X_t X_t')^{-1} \sum_{t=1}^{T} X_t y_{2t} \\ \vdots \\ (\sum_{t=1}^{T} X_t X_t')^{-1} \sum_{t=1}^{T} X_t y_{nt} \end{pmatrix} \\
&= \begin{pmatrix} \hat{\Phi}^1 & \hat{\Phi}^2 & \cdots & \hat{\Phi}^n \end{pmatrix} \\
&= \begin{pmatrix} \hat{\Phi}_{01}' & \hat{\Phi}_{02}' & \cdots & \hat{\Phi}_{0n}' \\ \hat{\Phi}_{11}' & \hat{\Phi}_{12}' & \cdots & \hat{\Phi}_{1n}' \end{pmatrix}.
\end{aligned}$$

Hence the conditional maximum likelihood estimation of Φ_1 is equivalent to equation-by-equation OLS estimation: $\hat{\Phi}_1 = (\hat{\Phi}_{11}, \hat{\Phi}_{12}, \ldots, \hat{\Phi}_{1n})'$. □

(ii) Under the improper prior distribution

$$p(\Phi, \Sigma) \propto |\Sigma|^{-(n+1)/2},$$

can the posterior means of the VAR coefficients be computed equation-by-equation? Are coefficients of equation i *a posteriori* uncorrelated with coefficients from equation j?

Solution. Define the matrices $Y = (y_1', \dots, y_T')'$ and $X = (X_1', \dots, X_T')'$, and let y_0 be an $n \times 1$ vector with the initial observation. Then the likelihood function is

$$p(Y \mid y_0, \Phi, \Sigma) \propto |\Sigma|^{-T/2} \exp\left(-\frac{1}{2}\operatorname{tr}\left(\Sigma^{-1}(Y - X\Phi)'(Y - X\Phi)\right)\right).$$

Define

$$\hat{\Phi} = (X'X)^{-1}X'Y$$
$$S = (Y - \hat{\Phi})'(Y - \hat{\Phi}),$$

and write

$$p(Y \mid y_0, \Phi, \Sigma) \propto |\Sigma|^{-T/2} \exp\left(-\frac{1}{2}\operatorname{tr}\left(\Sigma^{-1}(S + (\Phi - \hat{\Phi})'X'X(\Phi - \hat{\Phi})\right)\right) \tag{1}$$

$$\propto |\Sigma|^{-T/2} \exp\left(-\frac{1}{2}\operatorname{tr}(S\Sigma^{-1}) - \frac{1}{2}\operatorname{tr}\left(\Sigma^{-1}(\Phi - \hat{\Phi})'X'X(\Phi - \hat{\Phi})\right)\right). \tag{2}$$

Let us also define $\beta = \operatorname{vec}(\Phi)$ and $\hat{\beta} = \operatorname{vec}(\hat{\Phi})$. Then

$$p(Y \mid y_0, \Phi, \Sigma) \propto |\Sigma|^{-T/2} \exp\left(-\frac{1}{2}\operatorname{tr}\left((\beta - \hat{\beta})'\Sigma^{-1} \otimes X'X(\beta - \hat{\beta})\right)\right),$$

which shows that the likelihood is characterized by a multivariate normal distribution with mean $\hat{\beta}$ and variance $\Sigma \otimes (X'X)^{-1}$.

For the joint posterior, we have

$$p(\Phi, \Sigma \mid Y, y_0) \propto |\Sigma|^{-(T+n+1)/2} \exp\left(-\frac{1}{2}\operatorname{tr}\left(\Sigma^{-1}(S + (\Phi - \hat{\Phi})'X'X(\Phi - \hat{\Phi})\right)\right),$$

Equation (2) leads to a factorization of the form

$$p(\Phi, \Sigma \mid Y) = p(\Phi \mid \Sigma, Y)p(\Sigma \mid Y),$$

and so the conditional posterior is

$$p(\Phi \mid \Sigma, Y) \propto |\Sigma|^{-T/2} \exp\left(-\frac{1}{2}\operatorname{tr}\left((\beta - \hat{\beta})'\Sigma^{-1} \otimes X'X(\beta - \hat{\beta})\right)\right),$$

which we identify with a Gaussian distribution of the (vectorzied) form

$$\operatorname{vec}(\Phi) \mid (\Sigma, Y) \sim N(\operatorname{vec}(\hat{\Phi}), \Sigma \otimes (X'X)^{-1}).$$

So under this prior, we see that the posterior mean is MLE estimator and from (i), which coincides with equation-by-equation least squares estimator. The equation-by-equation posterior means of the VAR coefficients are not uncorrelated since the covariance depends on Σ. □

(iii) Download observations on real GDP growth and inflation from FRED for the period 1985:I to 2019:IV. Estimate a VAR(1) with intercept by OLS. Conditional on the OLS estimate, compute the eigenvalues of Φ_1 and the unconditional mean and variance of y_t. Compare your estimate of the unconditional mean and variance obtained from the estimated VAR to sample means and variances of y_t. Would you expect them to be approximately the same? Are they approximately the same?

Solution. The following code estimates the VAR(1) and the results are presented in Table 1.

```python
import numpy as np
import pandas as pd
import statsmodels.api as sm
from statsmodels.tsa.api import VAR

gdp = pd.read_csv('GDPC1.csv', index_col='DATE')
infl = pd.read_csv('GDPDEF.csv', index_col='DATE')

ts = gdp.join(infl, how='outer')

# Convert to growth rates
ts = np.log(ts).diff(axis=0)
ts = ts.rename(columns={'GDPC1':'gdp_growth', 'GDPDEF': 'infl_growth'})
ts = ts.dropna()

mod = VAR(ts)
res = mod.fit(maxlags=1)
```

Table 1. VAR(1) results.

	GDP	Inflation
Constant	0.005***	0.004***
	(0.001)	(0.000)
Lag GDP	0.372***	0.079**
	(0.080)	(0.027)
Lag Inflation	-0.164	0.612***
	(0.192)	(0.065)

* $p < .1$, ** $p < .05$, *** $p < .01$

The eigenvalues of Φ_1 are 0.53 and 0.45, a strong indication of stability.
Table 2 shows the sample mean and mean implied by the model, which are nearly the same.

Table 2. Means.

	Sample	Model
GDP	0.006	0.006
Inflation	0.005	0.005

The sample covariance matrix is

$$\begin{array}{cc} & \begin{array}{cc} \text{GDP} & \text{Inflation} \end{array} \\ \begin{array}{c} \text{GDP} \\ \text{Inflation} \end{array} & \begin{pmatrix} 0.000032 & 0.000001 \\ 0.000001 & 0.000006 \end{pmatrix} \end{array}.$$

and the covariance matrix implied by the model is

$$\begin{array}{ll} & \begin{array}{cc}\text{GDP} & \text{Inflation}\end{array} \\ \begin{array}{l}\text{GDP} \\ \text{Inflation}\end{array} & \begin{pmatrix} 0.000032 & 0.000001 \\ 0.000001 & 0.000006 \end{pmatrix}. \end{array}$$

Since the model implied estimates are computed using coefficients, which from (i) we know are equivalent to equitation-by-equation least squares estimates, these similarities noted above are not surprising. □

(iv) Write a program that generates draws from the posterior $(\Phi, \Sigma) \mid Y$ using direct sampling from the MNIW distribution. For each coefficient compute the posterior mean and a 90% credible interval. Tabulate your posterior estimates along with the OLS estimates.

Solution. The following code conducts the desired posterior analysis with an improper prior.

```
from numpy.random import multivariate_normal

# Function to generate inverse wishart deviates
def inv_wishart(df, S):
    n = S.shape[0]
    Z = multivariate_normal(np.zeros(n), np.linalg.inv(S), df)
    ZTZ = Z.T @ Z
    return np.linalg.inv(ZTZ)

# Data
ts = sm.add_constant(ts)
X = ts.values
XTX_inv = np.linalg.inv(X.T @ X)

# Vectorize coefficient vector
Phi_00 = res.intercept[0]
Phi_01 = res.coefs.flatten()[0]
Phi_02 = res.coefs.flatten()[1]
Phi_10 = res.intercept[1]
Phi_11 = res.coefs.flatten()[2]
Phi_12 = res.coefs.flatten()[3]
vec_Phi = [Phi_00, Phi_01, Phi_02, Phi_10, Phi_11, Phi_12]

# Direct sampling from MNIW
N = 100
post = []
for i in range(N):
    S_iw = inv_wishart(df=137, S=S_hat)
    S_mn = np.kron(S_iw, XTX_inv)
    post.append(multivariate_normal(vec_Phi, S_mn).tolist())

# Save posterior sample
post_df = pd.DataFrame(np.array(post))
post_df = post_df.rename(columns={0:'Phi_00', 1:'Phi_01', 2:'Phi_02',
```

```
                                    3:'Phi_10', 4:'Phi_11', 5:'Phi_12'})

 # Generate posterior mean and 90\% interval
from scipy.stats import bayes_mvs

for i in post_df.columns:
    print(bayes_mvs(post_df[i])[0])
```

The results of the above computations are summarized in Table 3. The "closeness" of the estimated posterior means to the OLS estimates is a consequence fact that the true posterior mean is the OLS estimator.

TABLE 3. Posterior Results

	Mean	Lower	Upper	OLS
Φ_{00}	0.005	0.005	0.005	0.005
Φ_{01}	0.372	0.370	0.373	0.372
Φ_{02}	-0.165	-0.168	-0.162	-0.164
Φ_{10}	0.002	0.002	0.002	0.004
Φ_{11}	0.079	0.079	0.080	0.079
Φ_{12}	0.612	0.612	0.613	0.612

$\square$