STAT 433: HOMEWORK 5

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Problem 1. Let X and Y be two independent Poisson random variables with $X \sim \text{Poisson}(\lambda)$ and $X \sim \text{Poisson}(\mu)$. Use probability generating functions to find the distribution of $X + Y$.

Solution. The PGF of X is

$$G_X(s) = \mathrm{E}\, s^X = \sum_{k=0}^{\infty} s^k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda^k)}{k!} = e^{-\lambda} e^{s\lambda} = e^{\lambda(s-1)},$$

and by the same logic, the PGF of Y is $G_Y(s) = e^{\mu(s-1)}$. By independence of X and Y,

$$G_{X+Y}(s) = G_X(s) G_Y(s) = e^{\lambda(s-1)} e^{\mu(s-1)} = e^{(\lambda+\mu)(s-1)}.$$

which, by uniqueness of the PGFs, $X + Y \sim \text{Poisson}(\lambda + \mu)$. □

Problem 2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. Bernoulli random variables with parameter p. Let N be a Poisson random variables with $N \sim \text{Poisson}(\lambda)$, which is independent of the X_i's.

(i) Find the probability generating function of $Z = \sum_{i=1}^{N} X_i$.

Solution. By the law of total expectation, we have

$$G_Z(s) = \mathrm{E}\, s^Z = \mathrm{E}\,\mathrm{E}(s^Z \mid N)$$

For $N = n$,

$$\begin{aligned} \mathrm{E}(s^Z \mid N = n) &= \mathrm{E}(s^{\sum_{i=1}^{N} X_i} \mid N = n) = \mathrm{E}(s^{\sum_{i=1}^{n} X_i} \mid N = n) \\ &= \mathrm{E}\, s^{\sum_{i=1}^{n} X_i} = \prod_{i=1}^{n} \mathrm{E}\, s^{X_i} = \prod_{i=1}^{n} (ps + (1-p)s) = (1 + p(s-1))^n, \end{aligned}$$

and so $\mathrm{E}(s^Z \mid N) = (1 + p(s-1))^N$. Then, using the result from Problem 1,

$$G_Z(s) = \mathrm{E}(1 + p(s-1))^N = G_N((1 + p(s-1))^N) = e^{\lambda(1+p(s-1)-1)} = e^{\lambda p(s-1)}.$$

□

(ii) Use (1) to identify the probability distribution of Z.

Solution. By the uniqueness property of PGFs, $Z \sim \text{Poisson}(\lambda p)$. □

Problem 3. Consider a branching process with off-spring distribution

$$\mathbf{p} = \left(t^2, 2t(1-t), (1-t)^2\right), \qquad 0 < t < 1,$$

i.e., it is binomial with parameters 2 and $1 - t$. Find the extinction probability.

Solution. Since the offspring distribution is Binomial$(2, 1-t)$, we know its PGF is given by

$$G(s) = (1 - (1-t) + (1-t)s)^2 = (t + s - ts)^2$$

The mean is $\mu = G'(1) = 2 - 2t$. When $t \geq 1/2$, the mean $\mu \leq 1$, which corresponds to the subcritical and critical cases, the population goes extinct with probability 1.

For $t < 1/2$, the mean is $\mu > 1$ and the process is supercritical, in which case the extinction probability is determined by the smallest positive root of the fixed point equation

$$G(s) = s \iff (1-t)^2 s^2 + (2t - 2t^2 - 1)s + t^2 = 0,$$

which has solution

$$s = \frac{-(2t - 2t^2 - 1) \pm \sqrt{(2t - 2t^2 - 1)^2 - 4t^2(1-t)^2}}{2(1-t)^2} = \frac{t^2}{(1-t)^2}.$$

So, the extinction probability in the subcritical case is $t^2(1-t)^2$ for $t > 1/2$. □

Problem 4.

(i) A discrete random variable X taking values in $\{0, 1, 2, \dots\}$ is said to be memoryless, if for any $m, n \geq 0$

$$P(X > m + n \mid X > m) = P(X > n).$$

Prove that X is memoryless iff it is a geometric random variable.

Solution. Let X be discrete nonnegative integer valued random variable satisfying the memoryless property. Then, by the definition of conditional probability,

$$P(X > m + n \mid X \geq m) = \frac{P(X > m + n)}{P(X \geq m)}$$

together with the memoryless property implies that we have the functional equation

$$P(X > m + n) = P(X \geq m)P(X > n)$$

holds for any nonnegative integers m and n. Now the functional equation remains to be solved. If we let $m = 1$ and let $1 - p = P(X \geq 1)$ and observe that

$$P(X = 1 + n \mid X > 1) = \frac{P(X = n + 1, X > 1)}{P(X > 1)} = \frac{P(X = n + 1)}{1 - p} = P(X = n).$$

we deduce the probabilities

$$\begin{aligned}
P(X = 2) &= (1-p)P(X = 1) \\
P(X = 3) &= (1-p)P(X = 2) = (1-p)^2 P(X = 1) \\
&\vdots \\
P(X = k) &= (1-p)^k P(X = 1),
\end{aligned}$$

and so on, for $k = 0, 1, 2, \dots$, which is the PMF of a geometric random variable, and hence $X \sim$ Geometric(p).

Now let $X \sim$ Geometric(p) for $0 \leq p \leq 1$, which has tail probabilities given by

$$P(X > n) = \sum_{k=n}^{\infty} P(X = k) = \sum_{k=n}^{\infty} (1-p)^{k-1} p.$$

Letting $i = k - n$ gives

$$P(X > n) = \sum_{i=0}^{\infty} (1-p)^{i+n-1} p = p(1-p)^{n-1} \sum_{i=0}^{\infty} (1-p)^i = (1-p)^{n-1}.$$

Now

$$P(X > m+n \mid X > m) = \frac{P(X > m+n)}{P(X > m)} = \frac{(1-p)^{m+n-1}}{(1-p)^m} = (1-p)^{n-1}$$

and hence X satisfies the memoryless property. □

(ii) A continuous random variable X taking values in $[0, \infty)$ is said to be memoryless, if for any $t, s \geq 0$

$$P(X > t+s \mid X > s) = P(X > t).$$

Prove that X is memoryless if it is an exponential random variable.

Solution. Let $X \sim \text{Exponential}(\lambda)$. Then, by the definition of conditional probabilities (using the same logic as in part (1)) together with the fact that $P(X > t) = e^{-\lambda t}$, we have

$$P(X > t+s \mid X > s) = \frac{P(X > t+s)}{P(X > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}.$$

□

(iii) Suppose a continuous random variable X taking values in $[0, \infty)$ is memoryless. Define the function $f(t) = P(X > t)$. Then $f(0) = 1$, and by the memoryless property we have

$$f(s+t) = f(s)f(t) \implies \frac{f(s+t) - f(t)}{s} = \frac{f(s) - f(0)}{s} f(t).$$

Show that by taking $s \to 0$, we obtain a differential equation

$$f'(t) = f'(0)f(t).$$

Solve the equation to show that form $e^{-\lambda t}$, which means that X is an exponential random variable.

Solution. Let X be continuous nonnegative integer valued random variable satisfying the memoryless property. Letting $s \to 0$ gives

$$\lim_{s \to 0} \frac{f(s+t) - f(t)}{s} = f'(t),$$

and

$$\lim_{s \to 0} \frac{f(s) - f(0)}{s} = f'(0).$$

This leads to an ordinary differential equation

$$f'(t) = f'(0)f(t).$$

Since this ordinary differential equation is separable, we may write

$$\frac{f'(t)}{f(t)} = f'(0),$$

which implies

$$\frac{d}{dt} \log f(t) = f'(0)$$

or

$$\log f(t) = f'(0)t + c.$$

If we impose the initial condition that $f(0) = 1$, then $c = 0$ and

$$f(t) = e^{f'(0)t}.$$

Finally, setting $\lambda = -f'(0)$ gives

$$f(t) = e^{-\lambda t},$$

and therefore, X is an exponential random variable. $\square$