STAT 433: HOMEWORK 3

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Problem 1. Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains.

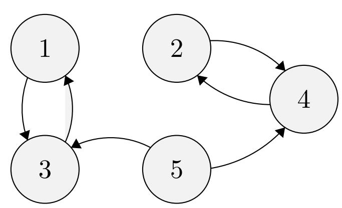
(a)	**1**	**2**	**3**	**4**	**5**
1	.4	.3	.3	0	0
2	0	.5	0	.5	0
3	.5	0	.5	0	0
4	0	.5	0	.5	0
5	0	.3	0	.3	.4

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	.1	0	0	.4	.5	0
2	.1	.2	.2	0	.5	0
3	0	.1	.3	0	0	.6
4	.1	0	0	.9	0	0
5	0	0	0	.4	0	.6
6	0	0	0	0	.5	.5

(c)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	.2	0	.8	0
3	.1	.2	.3	.4	0
4	0	.6	0	.4	0
5	.3	0	0	0	.7

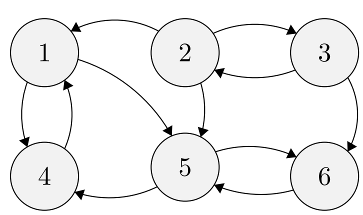
(d)	**1**	**2**	**3**	**4**	**5**	**6**
1	.8	0	0	.2	0	0
2	0	.5	0	0	.5	0
3	0	0	.3	.4	.3	0
4	.1	0	0	.9	0	0
5	0	.2	0	0	.8	0
6	.7	0	0	.3	0	0

(a) *Solution.* The directed graph representation is



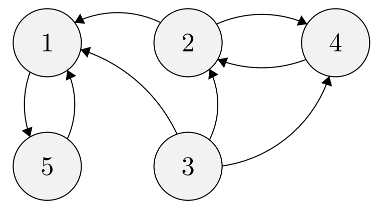
Since $1 \to 2$ but $2 \not\to 1$, state 1 is transient. Since $5 \to 4$ but $4 \not\to 5$, state 5 is transient. Since $3 \to 2$ but $2 \not\to 3$, state 3 is transient. So, states 2 and 4 are recurrent. The closed irreducible set is $\{2, 4\}$. □

(b) *Solution.* The directed graph representation is



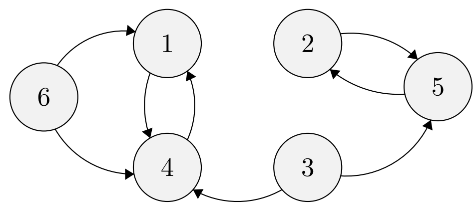
Since $2 \to 1$ but $1 \not\to 2$, state 2 is transient. Since $3 \to 6$ but $6 \not\to 3$, state 3 is transient. So states 1, 4, 5, and 6 are recurrent. The closed irreducible set is $\{1, 4, 5, 6\}$. □

(c) *Solution.* The directed graph representation is



Since $3 \to 1$ but $1 \not\to 3$, state 3 is transient. So state 1, 2, 4, and 5 are recurrent. The closed irreducible sets are $\{1, 5\}$ and $\{2, 4\}$. □

(d) *Solution.* The directed graph representation is



Since $3 \to 4$ but $4 \not\to 3$, state 3 is transient, and since $6 \to 1$ but $1 \not\to 6$, state 6 is transient. So state 1, 2, 4, and 5 are recurrent. The closed irreducible sets are $\{1, 4\}$ and $\{2, 5\}$. □

Problem 2. Let G be a connected graph. Let X_n be a simple random walk on G. Show that the Markov chain $\{X_n\}$ is irreducible. (Hint: for any two vertices x and y in G, consider a path of consecutive nodes from x to y.)

Solution. Let x and y be any two vertices in G. Since G is connected, x and y are connected by with a path of edges formed by

$$x = x_0, x_1, x_2, \dots, x_n = y,$$

which implies that

$$\rho_{xy} \geq \frac{1}{\deg(x)} \cdot \frac{1}{\deg(x_1)} \cdots \frac{1}{\deg(x_{n-1})} > 0$$
$$\rho_{yx} \geq \frac{1}{\deg(y)} \frac{1}{\deg(x_{n-1})} \cdots \frac{1}{\deg(x_1)} > 0.$$

Therefore $x \leftrightarrow y$. □

Problem 3. Let G be a graph with two disjoint components G_1 and G_2. Let X_n be a simple random walk on G.

(i) Prove $\{X_n\}$ is not an irreducible Markov chain.

Solution. Since $G_1 \cap G_2 = \emptyset$, there does not exist an edge between some node $x \in G_1$ and some other node $y \in G_2$, which shows G_1 and G_2 are disconnected. So $x \not\to y$ and X_n on G is not irreducible. □

(ii) Let P be the transition matrix of X_n. Let P_1 and P_2 be the transition matrices for the SRW on G_1 and G_2, respectively. Let $V_1 = \{1, \dots, k\}$ and $V_2 = \{k+1, \dots, k+\ell\}$ be the set of vertices in G_1 and G_2. Show that P is of the following block diagonal form

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}. \tag{1}$$

Note a block diagonal matrix is a *reducible* matrix.

Solution. We have $p(x,y) = 0$ if $1 \leq x \leq k$ and $k+1 \leq y \leq k+\ell$, or if $k+1 \leq x \leq k+\ell$ and $1 \leq y \leq k$, which shoes that P is diagonal. We also have $p(x,y) = p_1(x,y)$ if $1 \leq x, y, \leq k$ and $p(x,y) = p_2(x,y)$ if $k+1 \leq x, y, \leq k+\ell$, which establishes the fact that $P = \operatorname{diag}(P_1, P_2)$. □

(iii) Show the the SRW $\{X_n\}$ on G has infinitely many stationary distributions.

Solution. Since the disjoint graphs G_1 and G_2 are finite, P_1 has a stationary distribution, call it π_1, and P_2 has stationarity distribution, call it π_2. Note that π_1 has dimension $1 \times k$, π_2 has dimension $1 \times \ell$. Let

$$\pi = \begin{cases} (\pi_1, 0) & \text{if } X_0 \in G_1 \\ (0, \pi_2) & \text{if } X_0 \in G_2 \end{cases}$$

be the stationary distribution for the operator P on G, and note that π has dimension $1 \times k + \ell$. Note also that P has dimension $(k+\ell) \times (k+\ell)$. Now, for the chain started on any vertex in G_1, let $\pi = (\pi_1, 0)$ where 0 is a $1 \times \ell$ dimensional vector. Then

$$(\pi_1, 0) \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = (\pi_1 P_1, 0) = (\pi_1, 0),$$

where the last equality follows since π_1 was defined to be the stationary distribution for P_1. For the chain started on any vertex in G_2, $\pi = (0, \pi_2)$ where here 0 is a $1 \times k$ vector. So

$$(0, \pi_2) \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = (0, \pi_2 P_2) = (0, \pi_2).$$

For $0 \leq \lambda \leq 1$, linear combinations of the form

$$(\lambda \pi_1 + (1-\lambda)\pi_2) = (\lambda \pi_1 P + (1-\lambda)\pi_2 P)$$

are also stationary distribution for P. Hence, there exists infinitely stationary for X_n on G. □

Problem 4. Consider a Markov chain with state space $S = \{1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix},$$

where $0 < a < 1$ and $0 < b < 1$.

(i) Find its stationary distribution π.

Solution. Let $\pi = (x, y)$. Then

$$\pi P = \pi \implies \begin{cases} (1-a)x + by = x \\ ax + (1-b)y = y \end{cases} \implies \begin{cases} ax = by \\ x + y = 1 \end{cases} \implies \begin{cases} x = b/(a+b) \\ y = a/(a+b). \end{cases}$$

□

(ii) Use the Markov property to show that

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)\left(P(X_n = 1) - \frac{b}{a+b}\right).$$

Solution. Start from the left hand side and condition on X_n to obtain

$$\begin{aligned} P(X_{n+1}=1) &= \sum_i P(X_{n+1}=1 \mid X_n=i)P(X_n=i) \\ &= (1-a)P(X_n=1)+b(1-P(X_n=1)) \\ &= P(X_n=1)-aP(X_n=1)+b-bP(X_n=1) \\ &= (1-a-b)P(X_n=1)+b \\ &= (1-a-b)P(X_n=1)+\frac{b(a+b)}{a+b} \\ &= (1-a-b)P(X_n=1)+\frac{ab}{a+b}+\frac{b^2}{a+b}. \end{aligned}$$

Next, subtract $b/(a+b)$ from both sides

$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)P(X_n=1)+\frac{ab}{a+b}+\frac{b^2}{a+b}-\frac{b}{a+b}.$$

Simplifying the above expression gives

$$P(X_{n+1}=1)-\frac{b}{a+b}=(1-a-b)\left(P(X_n=1)-\frac{b}{a+b}\right).$$

$\square$

(iii) Use mathematical induction to show that

$$P(X_n=1)=\frac{b}{a+b}+(1-a-b)^n\left(P(X_0=1)-\frac{b}{a+b}\right)$$

Proof (by induction on n). The first base case $n=0$ holds since

$$\begin{aligned} P(X_0=1) &= \frac{b}{a+b}+(1-a-b)^0\left(P(X_0=1)-\frac{b}{a+b}\right) \\ &= \frac{b}{a+b}+P(X_0=1)-\frac{b}{a+b}=P(X_0=1), \end{aligned}$$

and the second base case $n=1$ holds since

$$\begin{aligned} P(X_1=1) &= \sum_i P(X_1=1 \mid X_0=i)P(X_0=i) \\ &= P(X_1=1 \mid X_0=1)P(X_0=1)+P(X_1=1 \mid X_0=2)P(X_0=2) \\ &= (1-a)P(X_0=1)+b(1-P(X_0=1)) \\ &= \frac{b}{a+b}+(1-a-b)\left(P(X_0=1)-\frac{b}{a+b}\right) \end{aligned}$$

where the last equality uses the same logic as in (ii).

Assume the claim is true for any n. We want to show that the result is also true for $n+1$. From part (i), we know that

$$P(X_{n+1}=1)=\frac{b}{a+b}+(1-a-b)\left(P(X_n=1)-\frac{b}{a+b}\right).$$

Substituting in for $P(X_n = 1)$ gives

$$\begin{aligned} P(X_{n+1} = 1) &= \frac{b}{a+b} + (1-a-b)\left\{\left((1-a-b)^n\left(P(X_0 = 1) - \frac{b}{a+b}\right)\right) - \frac{b}{a+b}\right\} \\ &= \frac{b}{a+b} + (1-a-b)^{n+1}\left(P(X_0 = 1) - \frac{b}{a+b}\right). \end{aligned}$$

Shifting the indices back one period gives

$$P(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n\left(P(X_0 = 1) - \frac{b}{a+b}\right).$$

□

(iv) Show that $P(X_n = 1)$ converges exponentially fast to $\pi(1)$ for the π you found in (i).

Solution. Since $0 < a < 1$ and $0 < b < 1$, the term $|1-a-b| < 1$. Then, noting that $(1-a-b)^n$ is the only term depending on n, we have

$$\begin{aligned} \lim_{n\to\infty} P(X_n = 1) &= \lim_{n\to\infty}\left\{\frac{b}{a+b} + (1-a-b)^n\left(P(X_0 = 1) - \frac{b}{a+b}\right)\right\} \\ &= \frac{b}{a+b} + \lim_{n\to\infty}(1-a-b)^n\left(P(X_0 = 1) - \frac{b}{a+b}\right), \end{aligned}$$

which shows that convergence to the limiting distribution, $P(X_n = 1) \to \pi(1)$ as $n \to \infty$, is exponential at rate $(1-a-b)^n$. □