Overview of DCC-GARCH Methodology and Code Explanation

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1 Introduction

This document provides a brief overview of how the DCC-GARCH (Dynamic Conditional Correlation-Generalized Autoregressive Conditional Heteroskedasticity) methodology is employed here. DCC-GARCH improves on other multivariate GARCH models by not making the assumption that correlation between assets is constant over time. This allows the model to take into account the varying relationships between assets over time and in different market conditions.

2 Data Fetching and Preprocessing

- **Market Data Retrieval:** The `yfinance` library is used to pull asset price data for specified tickers over a given period (here, 5 years).
- **Log-Returns:** After retrieving daily closing prices P_t, log-returns are calculated.

3 Univariate GARCH(1,1)

Intuitively, GARCH allows us to model volatility as a combination of historic volatility, and historic shocks (i.e unexpected volatility). A GARCH(1,1) is fitted model for each asset to capture individual time-varying volatility. For each asset i, the model is as follows:

$$r_{t,i} = \mu_i + \epsilon_{t,i}, \quad \epsilon_{t,i} \sim \mathcal{N}(0, \sigma_{t,i}^2),$$

where $\sigma_{t,i}^2$ is represented by

$$\sigma_{t,i}^2 = \omega_i + \alpha_i \, \epsilon_{t-1,i}^2 + \beta_i \, \sigma_{t-1,i}^2.$$

ω_i, α_i, and β_i are estimated using maximum likelihood. This leaves us with the conditional volatility $\sigma_{t,i}$ for each asset at each time t.

3.1 Standardized Residuals

After fitting GARCH(1,1) for each asset i, residuals from the last model are standardised:

$$z_{t,i} = \frac{\epsilon_{t,i}}{\sigma_{t,i}}.$$

These standardized series $\{z_{t,i}\}$ are inputs to the DCC correlation model.

4 Dynamic Conditional Correlation (DCC)

The **DCC** component models the time-varying correlation of the standardized residuals. Let $\mathbf{z}_t$ be the vector of standardized residuals across all assets at time t.

4.1 Unconditional Covariance

First, we compute the unconditional (time invariant) covariance matrix of our standardised residuals $\mathbf{z}_t$:

$$\overline{\mathbf{Q}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{z}_t \mathbf{z}_t^\top.$$

4.2 Recursion for $\mathbf{Q}_t$

The DCC model is generated recursively, each Qt a quasi-correlation matrix for each time period, defined as:

$$\mathbf{Q}_t = \overline{\mathbf{Q}} + \alpha\Big(\mathbf{z}_{t-1}\mathbf{z}_{t-1}^\top - \overline{\mathbf{Q}}\Big) + \beta\Big(\mathbf{Q}_{t-1} - \overline{\mathbf{Q}}\Big),$$

where α and β are DCC parameters to be estimated. To ensure stationarity, we required $\alpha + \beta < 1$.

4.3 Correlation Matrices $\mathbf{R}_t$

$\mathbf{Q}_t$ is then converted into a correlation matrix $\mathbf{R}_t$ by normalizing with its diagonal:

$$\mathbf{D}_t = \sqrt{\mathrm{diag}(\mathbf{Q}_t)}, \quad \mathbf{R}_t = \mathbf{D}_t^{-1}\,\mathbf{Q}_t\,\mathbf{D}_t^{-1}.$$

Hence, $\mathbf{R}_t$ is guaranteed to be a correlation matrix with ones on the diagonal. Note here, $\mathbf{R}_t$ isn't the correlation of returns, but of standardized shocks, intuitively volatility.

4.4 Log-Likelihood and Estimation

Then, α and β are estimated by maximizing the DCC log-likelihood function:

$$\mathcal{L} = \sum_{t=1}^{T} \left[-\frac{1}{2} \Big(\ln|\mathbf{R}_t| + \mathbf{z}_t^\top \mathbf{R}_t^{-1} \mathbf{z}_t \Big) \right].$$

I use `scipy.optimize.minimize` with the constraints $0 \le \alpha, \beta \le 1$ and $\alpha + \beta < 1$.

5 Time-Varying Covariance $\mathbf{H}_t$

Once $\mathbf{R}_t$ is obtained, the final covariance matrix for the original returns at time t is

$$\mathbf{H}_t = \mathbf{D}_t^{(\mathrm{GARCH})}\,\mathbf{R}_t\,\mathbf{D}_t^{(\mathrm{GARCH})},$$

where $\mathbf{D}_t^{(\mathrm{GARCH})}$ is the diagonal matrix of univariate GARCH volatilities $\sigma_{t,i}$. Equivalently,

$$\mathbf{D}_t^{(\mathrm{GARCH})} = \mathrm{diag}(\sigma_{t,1}, \sigma_{t,2}, \dots).$$

6 Backtesting Methodology

Once $\mathbf{H}_t$ is estimated for all t, $\mathbf{H}_t$s are backtested by comparing their VaR (Value-at-Risk) forecasts to realized portfolio returns as follows:

- **Portfolio Returns:** Given an n-dimensional returns vector $\mathbf{r}_t$ at time t and portfolio weights $\mathbf{w}$, the portfolio return is:
$$r_{p,t} \;=\; \mathbf{w}^\top \mathbf{r}_t.$$

- **Forecasted VaR:** Under the assumption of normality, the daily VaR at level α is approximated by
$$\mathrm{VaR}_{\alpha,t} \;=\; z_\alpha \sqrt{\mathbf{w}^\top \mathbf{H}_t \, \mathbf{w}},$$
where z_α is the critical value from the standard normal distribution (for instance, $z_{0.95} \approx 1.645$).

- **Violation Check:** For each time t, a check is done to see if realized portfolio return $r_{p,t}$ is above or below the negative of the VaR threshold:
$$r_{p,t} \;<\; -\,\mathrm{VaR}_{\alpha,t}.$$
A violation (or "breach") occurs if $r_{p,t} < -\mathrm{VaR}_{\alpha,t}$.

- **Violation Rate:** Our model found violation rates of around 5% as expected, giving us good initial validation of our model.

7 High-Median-Low Volatility Periods

A rolling 7-day mean of the portfolio volatility series is taken, identifying the time periods with the **highest**, **lowest**, and **median** volatility levels. The average correlation, covariance and VaR of each period is then found.

8 Portfolio Volatility and Risk Metrics

- **Portfolio Weights:** $n \times 1$ vectors of asset weights within a portfolio are represented by $\mathbf{w}$.

- **Instantaneous Variance:** At time t, the portfolio variance is
$$\mathrm{Var}(r_{p,t}) = \mathbf{w}^\top \mathbf{H}_t \, \mathbf{w},$$
with volatility $\sqrt{\mathbf{w}^\top \mathbf{H}_t \, \mathbf{w}}$.

- **Value-at-Risk (VaR)**: VaR is computed using monte carlo simulations. 1000 random simulations of returns based on the calculated portfolio volatility are conducted and then sampled to find the 5% VaR. CVaR is found by taking the mean of the values in that bottom 5% represented the expected shortfall, or how much will be lost on average if in the losing domain.
$$\mathrm{VaR}_\alpha = z_\alpha \cdot \text{Volatility} \quad (\text{e.g., } z_{0.95} \approx 1.645).$$

- **Expected Shortfall (ES)/CVaR:** Computed as the average loss beyond the VaR threshold in a simulated normal or empirical distribution of returns.

9 Variance Minimizing Weights

Portfolio weights $\mathbf{w}$ that minimize portfolio variance at each time t are also found. Given the last periods covariance matrix $\mathbf{H}_t$ the following is solved to find the current, variance minimising weights:

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^\top \mathbf{H}_t \, \mathbf{w} \\ \text{subject to} \quad & \sum_{i=1}^{n} w_i \; = \; 1, \quad w_i \; \geq \; 0 \quad \forall i. \end{aligned}$$

Again, `scipy.optimize.minimize` is used giving weights $\mathbf{w}^*$.