Temporal multidimensional item response theory

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March 23, 2013

a_{im}^t gives the mth ability for student i at timestep t. $x_{ij}^t \in \{0, 1\}$ is the correctness value of exercise j at timestep t for student i. The conditional probability of getting an exercise correct is

$$p\left(x_{ij}^t = 1 | \mathbf{a}_i^t\right) = \sigma\left(\mathbf{W}_j \mathbf{a}_i^t + b_j\right), \tag{1}$$

where W_{jm} is the coupling parameter between ability m and exercise j, and b_j is the bias associated with exercise j. r_i^t is the resource student i is exposed to at timestep t, where a resource might be an exercise, a video, or other content. An exercise answered correctly and the same exercise answered incorrectly are treated as two separate resources.

The prior distribution over the initial ability state is a unit norm Gaussian,

$$p\left(\mathbf{a}_i^1\right) = \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right). \tag{2}$$

The conditional distribution over the abilities at the next timestep given the current timestep is also given by a Gaussian,

$$p\left(\mathbf{a}_i^{t+1} | \mathbf{a}_i^t, r_i^t\right) = \mathcal{N}\left(\left(\mathbf{I} + \mathbf{\Phi}_{r_i^t}\right) \mathbf{a}_i^t + \theta_{r_i^t}, \mathbf{\Sigma}_{r_i^t}\right), \tag{3}$$

where the matrix $\mathbf{\Phi}_{r_i^t}$ and the bias $\theta_{r_i^t}$ are defined for each resource, and determine the mean prediction for the abilities vector at the next time step. The covariance matrix $\mathbf{\Sigma}_{r_i^t}$, also defined for each resource, determines the uncertainty in the abilities vector at the next time step.

The joint energy function is

$$\begin{aligned} E =& \frac{1}{2}\left(\mathbf{a}_i^1\right)^T \mathbf{a}_i^1 \\ &+ \frac{1}{2}\sum_{t=1}^{T-1}\left[\mathbf{a}_i^{t+1} - \left(\mathbf{I} + \mathbf{\Phi}_{r_i^t}\right)\mathbf{a}_i^t\right]^T \mathbf{\Sigma}_{r_i^t}^{-1}\left[\mathbf{a}_i^{t+1} - \left(\mathbf{I} + \mathbf{\Phi}_{r_i^t}\right)\mathbf{a}_i^t\right] \\ &+ \sum_{t=1}^{T-1} \log\det\left(\mathbf{\Sigma}_{r_i^t}\right) \\ &- \sum_{x_{ij}^t} \log\sigma\left(\mathbf{W}_j \mathbf{a}_i^t + b_j\right). \end{aligned} \tag{4}$$

It has gradients

$$\frac{\partial E}{\partial W_{km}} = -\sum_{x_{ij}^t} \left(1 - \sigma\left(\mathbf{W}_j \mathbf{a}_i^t + b_j\right)\right) a_{im}^t \delta_{kj}, \tag{5}$$

$$\frac{\partial E}{\partial \mathbf{\Phi}_{r_i^t}} = \sum_{t=1}^{T-1} \mathbf{\Sigma}_{r_i^t}^{-1} \left[\mathbf{a}_i^{t+1} - \left(\mathbf{I} + \mathbf{\Phi}_{r_i^t}\right) \mathbf{a}_i^t\right] \left(\mathbf{a}_i^t\right)^T, \tag{6}$$

$$\begin{aligned}\frac{\partial E}{\partial \mathbf{\Sigma}_{r_i^t}^{-1}} =& \frac{1}{2} \sum_{t=1}^{T-1} \left[\mathbf{a}_i^{t+1} - \left(\mathbf{I} + \mathbf{\Phi}_{r_i^t}\right) \mathbf{a}_i^t\right] \left[\mathbf{a}_i^{t+1} - \left(\mathbf{I} + \mathbf{\Phi}_{r_i^t}\right) \mathbf{a}_i^t\right]^T \\ &+ \sum_{t=1}^{T-1} \left(\mathbf{\Sigma}_{r_i^t}\right)^T \end{aligned} \tag{7}$$

The bias gradients can be computed by adding a “bias unit” to the abilities vector.