Diestel's Graph Theory 4th Edition Solutions

Daniel Oliveira

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Frequently used relations and techniques

- Let X be a maximal path, cycle, clique, co-clique, subgraph with something, and show you can increase it to get a contradiction.
- Let X be a minimal with or without something and remove an element to get a graph with or without the something.
- Assume G is connected.
- Let $x_1, \dots, v_k$ be an ordering of $X \subseteq V$.
- $\sum d(v) = 2|E|$.
- Handshake lemma, if each A sees b of B and each B sees a of A, $|A|b = |B|a$ (We count the number of handshakes in two ways).
- G has a path of length $\delta(G)$, and a cycle of length $\delta(G) + 1$.
- Bipartite iff no odd cycle.
- Eulerian if even degree on all vertices. Eulerian G has $E(G) \subseteq \mathcal{C}$
- Set of disjoint matchings = Colouring of $|E|$
- 2-connected iff cycle + H-paths
- $\chi(G)\alpha(g) \geq |G|$
- Kempe switching.
- $\delta(G) \geq n/2$ then Hamiltonian.

Chapter 1 - The basics

1.23) Let $\mathcal{F}$ be a set of subtrees of a tree T, and $k \in \mathbb{N}$.

(i) Show that if the trees in $\mathcal{F}$ have pairwise non-empty intersection then their overall intersection $\bigcap \mathcal{F}$ is non-empty.

(ii) Show that either $\mathcal{T}$ contains k disjoint trees or there is a set of at most $k-1$ vertices of T meeting every tree in $\mathcal{T}$.

Solution (i)

Proof. We prove by induction on $|T|$. For $|T| = 1$, (i) is clearly true.

Now let T be a tree of order greater than 1, $\mathcal{F}$ be a collection of subtrees of T with pairwise non-empty intersection, and assume (i) is true for all smaller trees. If any trivial tree $T' = \{v\}$ is in $\mathcal{F}$, then clearly $v \in \bigcap \mathcal{F}$ and $\mathcal{F}$ is non-empty. So assume all subtrees in $\mathcal{F}$ have order at least 2, and consider a leaf vertex l of T. By the induction hypothesis, by removing l from T and each subtree of $\mathcal{F}$, we have that the overall intersection of the resulting collection $\mathcal{F}'$ is non-empty. As any pair of subtrees in $\mathcal{F}$ that intersect on l also intersect on the single vertex adjacent to l in T, say l', then such pair also intersect in $\mathcal{F}'$, therefore $\bigcap \mathcal{F}' \subseteq \bigcap \mathcal{F}$, and $\bigcap \mathcal{F}$ is non-empty. □

Solution (ii)

Proof. Consider the set F of all edges $e \in T$ such that both components of $T - e$ contains a tree in $\mathcal{T}$, we prove the following:

Lemma 1. *The set F forms a subtree of T.*

Proof. Suppose F forms a forest and let T_1 and T_2 be two of its components. As T is a tree, there is a unique path P from T_1 to T_2 over the edges of T. So let $v_1 \in V(T_1)$ and $v_2 \in V(T_2)$ be respectively the first and last vertex in P, e_1, e_2 be respectively the edges of T_1 and T_2 incident to v_1 and v_2, and f be an edge of P. Then we have components C_1, C_1' in $T - e_1$, C_2, C_2' in $T - e_2$, all containing some subtree in $\mathcal{T}$ by the definition of F, and components C_f, C_f' in $T - f$. But then, w.l.o.g., we have $C_f \supseteq C_1$ and $C_f' \supseteq C_2'$, thus both C_f and C_f' contain some subtree in $\mathcal{T}$, contradicting $f \notin F$. □

We now prove (ii) by induction on $|\mathcal{F}|$, the case $|\mathcal{F}| = 1$ being trivial. So let $\mathcal{F}$, with $|\mathcal{F}| > 1$, be a collection of subtrees of T, and let T' be the subtree of T formed by F as above.

So consider one of the leaves v of T', let $uv \in F$ be the edge of T' incident to v, and C_v be the component of $T - uv$ containing v. As F is maximal, all subtrees of $\mathcal{F}$ contained in C_v are rooted at v, so let $\mathcal{F}_v$ be the set of subtrees in C_v.

Now consider the collection $\mathcal{F}'$ of subtrees of T that we get by deleting from $\mathcal{F}$ all subtrees containing v. By the induction hypothesis, we either have $k-1$ disjoint trees in $\mathcal{F}'$ or at most $k-2$ vertices meeting all trees in $\mathcal{F}'$. If we have $k-1$ disjoint trees in $\mathcal{F}'$, we can take any subtree in $\mathcal{F}_v$ as the k-th disjoint subtree in $\mathcal{F}$, and if we have at most $k-2$ vertices meeting all trees in $\mathcal{F}'$, we can include v and have at most $k-1$ vertices covering all subtrees in $\mathcal{F}$. □

1.37) Let G be a connected graph, we define $\mathcal{F}_1$ as the minimal edge sets containing an edge from every spanning tree, and set $\mathcal{F}_2$ as the set of bonds of G.

Solution

Proof. Let F be a minimal set of edges containing an edge from every spanning tree of G. So F contains a cut, otherwise $G - F$ is connected and contains some spanning tree of G with edges disjoint from F. As any cut has edges from all spanning trees of G, F has exactly one cut, say $F = E(A, B)$. And if some side of the cut were disconnected, say $A' \subset A$, as G is connected, we would have $E(A', B) \neq \emptyset$, so $E(A \setminus A', B \cup A')$ would be strictly contained in F, therefore a cut with fewer edges, a contradiction. So both sides of F are connected, thus by exercise 31, F is a bond.

Now let F be a bond of G. Any spanning tree of G has at least one edge from any cut, otherwise it is disconnected, so F contains edges from all spanning trees of G. Then we need to show no edge can be removed from F. From exercise 31, we know that both sides of $F = E(V_1, V_2)$ are connected in G. So $G[V_1]$ has some spanning tree T_1 and $G[V_2]$ has some spanning tree T_2 and T_1 joined to T_2 using any of the edges in F form a distinct spanning tree of G. Therefore F is a minimal set of edges containing an edge from every spanning tree of G. □

1.38) Let F be a set of edges in a graph G.

(i) Show that F extends to an element of $\mathcal{C}^*(G)$ if and only if it contains no odd cycle.

(ii) Show that F extends to an element of $\mathcal{C}$ if and only if it contains no odd cut.

Solution (i)

Proof. Let F be a set of edges in G containing an odd cycle. As any set of edges in $\mathcal{C}^*$ forms a bipartite graph with the sides of the cuts as the partitions, and a graph is bipartite if and only if it contains no odd cycle, F cannot be extended to a set of edges of a bipartite subgraph and consequently cannot be extended to an element of $\mathcal{C}^*$.

Now let F be a set of edges with no odd cycle, so it forms a bipartite subgraph of G. So it is possible to partition V into sets A, B such that $F \subseteq E(A, B)$, therefore we can extend F to the cut $\delta(A) = \delta(B) = E(A, B)$. □

Solution (ii)

Proof. Let F be a set of edges in G containing an odd cut, say $X \in \mathcal{C}^*$. Then no matter how many edges we add to F, we will always have an odd intersection with X, therefore it cannot be extended to an element of $\mathcal{C}$, as $\mathcal{C}^\perp = \mathcal{C}^*$, that is, any element of $\mathcal{C}$ has an even intersection with any element of $\mathcal{C}^*$.

Now let F be a set of edges, we show that if $F \notin \mathcal{C}$ and contains no odd cut, we can include a new edge in F without increasing the number of odd degree vertices and without creating an odd cut. As we have a finite number of vertices and by including all edges of G in F we either have all vertices with even degree or some odd cut, we can extend F to a element of $\mathcal{C}$.

So let $F \notin \mathcal{C}$ be a set of edges of G containing no odd cut. As $F \notin \mathcal{C}$, some vertex v has $d_F(v)$ odd, if all edges incident to v are in F, then F contains the odd cut $\delta(v)$, so assume some edge incident to v can still be included in F. Let $N_{\bar{F}}(v)$ denote the set of neighbours u of v with $uv \notin F$.

If some u in $N_{\bar{F}}(v)$ has with $d_F(u)$ odd, we can include uv in F and both u and v will have even degree in $(V, F \cup uv)$, and if we have some odd cut X in $F \cup uv$, necessarly $uv \in X$, say $X = E[A, B]$ with $v \in A$ and $u \in B$. □

Chapter 2 - Matching, Covering and Packing

2.1) Let M be a matching in a bipartite graph G. Show that if M is suboptimal, then G contains an augmenting path with respect to M. Does this fact generalize to matching in non-bipartite graphs?

Proof. Yes, the fact apply to general graphs, and we present a general proof.

Let G be a graph, M, N be respectively a suboptimal and an optimal matching in G.

Now consider the graph $H = (V(G), M\delta N)$. As at most one edge in M and one in N are incident to any vertex of G, H consists of a series of paths and even cycles, with edges alternating between edges in M and N. As $|N| > |M|$ and even cycles all have the same number of edges in M and in N, there is a path P in H with one extra edge in N, so P is an alternating path starting and ending at a M-exposed vertex, thus P is an augmenting path. □

2.2) Describe an algorithm that finds, as efficiently as possible, a matching of maximum cardinality in any bipartite graph.

By question 1, we have an optimal matching if no augmenting path exists in G. As given a matching M and an augmenting path P, we have a larger matching $M' = M\delta E(P)$. So we can start with $M =$ and search for augmenting paths until they no longer exist, increasing the size of M with each augmenting path found. And by showing no more augmenting paths exist, we know M is maximum.

A simple algorithm would be to, using a BFS, enumerate all paths starting at an M-exposed vertex in A, in order to find a path also ending at an M-exposed vertex in A.

2.5) Derive the marriage theorem from Konig's theorem.

This was a theorem in past editions of the book, and can be found also in Schrijver Combinatorial Optimization book.

Proof. We prove the non-trivial implication.

Let G be a bipartite graph with partitions $\{A, B\}$ and no matching of A. We need to show some vertex set S with $|S| > |N(S)|$ exists.

Let M be a maximum matching G, let U be a minimum vertex cover of G, by Konig's, $|M| = |U|$, and let $A' = A \cap U$, $B' = B \cap U$. As U cover E, no edges exist between $A \setminus A'$ and $B \setminus B'$, so $|N(A \setminus A') \subseteq B'$. We then have,

$$\begin{aligned}
|A \setminus A'| &= |A| - |A'| \\
&= |A| - (|U| - |B'|) \\
&= |A| - |M| + |B'| \\
&\leq |A| - |M| + |N(A \setminus A')| \\
&< |N(A \setminus A')|
\end{aligned}$$

□

2.6) Let G and H be defined as for the third proof of Hall's theorem. Show that $D_H(b) \leq 1$ for every $b \in B$, and deduce the marriage theorem.

Proof. Let H be an edge-minimal subgraph of G that satisfies the marriage condition and contains A.

Suppose for a contradiction that some vertex $b \in B$ with $d(b) \geq 2$ exists. We find a vertex set violating the marriage condition in H.

Let $a_1, a_2 \in A$ be two vertices adjacent to b. By the minimality of H, $H - a_1b$ and $H - a_2b$ respectively have sets A_1 and A_2 violating the marriage condition. Note this implies $|A_1| = |N_H(A_1)|$, and $|A_2| = |N_H(A_2)|$.

As $|N_{H-a_1b}(A_1)| < |N_H(A_1)|$, a_1 is the only vertex in A_1 adjacent to b. Analogously, the same is true for a_2 in A_2. So $b \notin N_H(A_1 \cap A_2)$, also $b \in N_H(A_1) \cap N_H(A_2)$, therefore $|N_H(A_1 \cap A_2)| < |N_H(A_1) \cap N_H(A_2)|$.

We then have,

$$\begin{aligned}
|N_H(A_1 \cup A_2)| &= |N_H(A_1) \cup N_H(A_2)| \\
&= |N_H(A_1)| + |N_H(A_2)| - |N_H(A_1) \cap N_H(A_2)| \\
&= |A_1| + |A_2| - |N_H(A_1) \cap N_H(A_2)| \\
&= |A_1 \cup A_2| + |A_1 \cap A_2| - |N_H(A_1) \cap N_H(A_2)| \\
&\leq |A_1 \cup A_2| + |N_H(A_1 \cap A_2)| - |N_H(A_1) \cap N_H(A_2)| \\
&< |A_1 \cup A_2| + |N_H(A_1) \cap N_H(A_2)| - |N_H(A_1) \cap N_H(A_2)| \\
&= |A_1 \cup A_2|,
\end{aligned}$$

a contradiction.

So $d_H(b) \leq 1$, that is, no two vertices in A share a neighbor in B. So as H have the marriage condition satisfied for all $S \subseteq A$, , every vertex in A has a neighbour in B, so $E(H)$ is a matching of A. □

2.8) Let k be an integer. Show that any two partitions of a finite set into $k-sets$ admit a common choice of representatives.

Proof. Let S be a finite set, k be an integer, and $\mathcal{P}_1$, $\mathcal{P}_2$ be two partitions of S into k-sets. We create a bipartite graph with partitions $\{A, B\}$ as follows. For each element of $\mathcal{P}_1$, create a vertex in A, and for each element of $\mathcal{P}_2$, create a vertex in B. And we create A-B edges in G such that, for each pair of partitions $P_1 \in \mathcal{P}_1, P_2 \in \mathcal{P}_2$, there is an edge between its corresponding vertices in G iff P_1 and P_2 have an element in common.

As we partition S into k-sets, $|\mathcal{P}_1| = |\mathcal{P}_2$, and $|A| = |B|$. We then need to prove all subsets of A satisfy the marriage condition.

Let $S \subseteq A$. By the construction of G, the sets in $\mathcal{P}_1$ corresponding to the vertices in S have $k|S|$ vertices in total, and therefore, as $\mathcal{P}_2$ is also a partition into k-sets, should be linked to at least $|S|$ elements in B, so $|N(S)| \geq |S|$. □

2.9) Let A be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n \in \mathbb{N}$. Show that there are disjoint subsets $D_k \subseteq A_k$, with $|D_k| = d_k$, for all $k \leq n$ if and only if

$$\left| \bigcup_{i \in I} A_i \right| \geq \sum_{i \in I} d_i$$

for all $I \subseteq \{1, \ldots, n\}$.

Proof. We construct a bipartite graph G with partitions $\{B, C\}$ as follows. We make $C := A$,and for each $d_i \in \{d_1, \ldots, d_n\}$, we include a set $B_i\{b_{i1}, \ldots, b_{id_i}\}$ of d_i vertices in B, each connected to all vertices in C corresponding to a vertex of A_i.

So a matching of B corresponds to disjoint subsets $D_k \subseteq A_k$, with $|D_k| = d_k$, for all $k \leq n$. Just take D_k as the set of vertices in C connected to all b_{ik} in B.

Since $N(S) = N(C_i)$ for all $S \subseteq C_i$, having the marriage condition satisfied for C_i implies it is satisfied for all of its subsets. This way, the marriage condition in G is equivalent to

$$\left| \bigcup_{i \in I} A_i \right| \geq \sum_{i \in I} d_i$$

for all $I \subseteq \{1, \ldots, n\}$, and the result follows from Hall's Theorem. □

2.11$^+$) Let G be a bipartite graph with bipartition $\{A, B\}$. Assume that $\delta(G) \geq 1$, and that $d(a) \geq d(b)$ for every edge ab with $a \in A$. Show that G contains a matching of A.

Proof. Suppose there is no matching of A, let M be a maximum matching, and let $A' \subseteq A$, $B' \subseteq B$ be the sets of M-covered vertices. We show an augmenting path exists in G.

Let $u \in A \setminus A'$, as $E(A \setminus A', B \setminus B') = \emptyset$, otherwise M is not maximal, and $\delta(G) \geq 1$, u is connected to a vertex $v \in B'$. So consider all trails $\mathcal{W}$ starting with uv, let $V_{\mathcal{W}}$ be the set of vertices reached by any trail in $\mathcal{W}$, and let $A'' := A' \cap V_{\mathcal{W}}$, $B'' := B' \cap V_{\mathcal{W}}$. Note that any trail reaching a vertex $b \in B'$, can also reach a vertex $a \in A'$ along $ab \in M$, as all trails in $\mathcal{W}$ start with uv, $|A''| = |B''|$, by the same reason, and the property on the degrees of G, we have $d(A'') \geq d(B'')$.

If all trails in $\mathcal{W}$ end at a vertex $v \in A' \cup B'$ without reaching $B \setminus B'$, as at least one $A \setminus A' - B'$ edge exists (namely uv), we would have $d(A'') < d(B'')$, a contradiction. So there exists a trail in $\mathcal{W}$ ending at $B \setminus B'$. As any trail contains a path between its ends, we have an augmenting path. □

2.12$^-$) Find a bipartite graph with a set of preferences such that no matching of maximum size is stable and no stable matching has maximum size. Find a non-bipartite graph with a set of preferences that has no stable matching.

Proof. For the bipartite graph, let $G = \{V, E\}$, with $V = \{a_1, a_2, b_1, b_2\}$ and $E = \{a_1b_1, a_1b_2, a_2b_2\}$, and a set of preferences $a_1b_1 \leq_{a_1} a_1b_2$ and $a_2b_2 \leq_{b_2} a_1b_2$. So the maximum size matching $M = \{a_1b_1, a_2b_2\}$ is not stable, since a_1b_2 is preferred both by a_1 and b_2. And the stable matching $M = \{a_1b_2\}$ is not maximum.

Now for the non-bipartite graph, let $H = \{V, E\}$, with $V = \{a, b, c\}$ and $E = \{ab, ac, bc\}$, and a set of preferences $ac \leq_a ab$, $ab \leq_b bc$, and $ab \leq_c ac$. So H has no stable matching, since the 3 possible matchings $\{ab\}, \{bc\}$, and $\{ac\}$ respectively have b preferring bc, c preferring ac and a preferring ab. □

Chapter 3 - Connectivity

3.1)

Chapter 4 - Planarity

4.22) A graph is called outerplanar if it has a drawing in which every vertex lies on the boundary of the outer face. Show that a graph is outerplanar if and only if it contains neither K^4 nor $K_{2,3}$ as a minor.

Solution

Proof. Let G be an outerplanar graph, so we can add a vertex to its outer face and connect it to $V(G)$ without crossing edges, call the resulting graph G'. As G' is planar, by Kuratowski's theorem, it has neither K^5 nor $K_{3,3}$ as a minor. So G has neither K^4 nor $K_{2,3}$ as a minor, as these would yield a K^5 or $K_{3,3}$ as a minor with the extra vertex and edges of G'.

Now for the converse, let G be graph with no K_4 nor $K_{2,3}$ as a minor. By adding a single vertex and connecting it to all vertices of G, we have a graph G' which does not have K^5 nor $K_{3,3}$ as a minor, therefore is planar by Kuratowski's theorem. So all vertices of G lied on its outer face, otherwise by the Jordan Curve theorem, we would have some edge crossing. □

Chapter 5 - Colouring

5.1) Show that the four colour theorem does indeed solve the map colouring problema stated in the first sentence of the chapter. Conversely, does the 4-colourability of every map imply the four colour theorem?

Solution: If we place one vertex at the center of each country, we are able to draw arcs between the vertices of every pair of countries that have a border in common. So a colouring of the graph translates to a colouring of the map.

Yes, since every planar graph can be formed from a map.

5.2) Show that,for the map colouring above, it suffices to consider maps such that no point lies on the boundary of more than three countries. How does this affect the proof of the four colour theorem?

Solution: If we have a map with a point lying on the border of k countries, we have that region of the map represented as a C_k in the corresponding plane graph. It is not hard to see that the more edges we have in a graph, the more colours we might need, so that region of the map is not easier to colour than a triangulation of the k vertices, which will translate to a map with no point lying on the boundary of more than 3 vertices.

5.3) Try to turn the proof of the five colour theorem into one of the four colour theorem, as follows. Defining v and H as before, assume inductively that H has a 4-colouring; then proceed as before. Where does the proof fail?

Solution

5.4) Calculate the chromatic number of a graph in terms of the chromatic numbers of its blocks.

Solution

Proof. Let G be a graph, assume G is connected, as otherwise $\chi(G)$ is clearly the higher of the chromatic number among its components. Let $\mathcal{B}$ be the collection of blocs of G. Consider the block graph of G, it is a tree. We colour each block of G independently of each other. Starting from any block B, we fix its colouring in G and look to any adjacent block B', by the maximality of each block, B and B' are joined by a single edge, say uv with $u \in B$ and $v \in B'$, if the colour of u and v are the same, we switch the colours of B', and fix its colouring. This way, as no cycle exists in the block graph, we never have to match the colours of more than a single pair of vertices at a time. So we can match the colouring of each pair of blocks in this way until G is properly coloured. So we don't need to use any new colour to colour G and $\chi(G) = \max_B \in \mathcal{B}\{\chi(B)\}$. □

5.5) Show that every graph G has a vertex ordering for which the greedy algorithm uses only $\chi(G)$ colours.

Solution

Proof. Let $f : V \to \{1, \dots, k\}$ be an optimal colouring of G, thus $k = \chi(G)$, and order the colours such that by taking maximal sets of vertices $V_1, \dots, V_k$ for each colour, we have $|V_1| \geq |V_2| \geq \dots \geq |V_k|$. Colouring the vertices greedily following such ordering, clearly yields an optimal colouring. □

5.6) For every $n > 1$, find a bipartite graph on $2n$ vertices, ordered in such a way that the greedy algorithm uses n rather that 2 colours.

Solution

Proof. Let $G = (A \cup B, E)$ be a bipartite graph on $2n$ with partitions $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, and each vertex a_i is connected to all b_j with $j > i$, for $i = 1, \dots, n-1$.

This way, if we colour the vertices of G greedly following the sequence $a_1, b_1, a_2, b_2, \dots, a_n, b_n$. We clearly need 2 colours to colour $\{a_1, b_1, a_2, b_2\}$, and when colouring any a_i, $i > 2$, it will receive colour i, as it is connected to all b_j with $j < i$, and they were all coloured before with exactly $i - 1$ colours. The same argument goes for each b_i. So we need n colours to colour G, as a_n, b_n both receives colour n. □

5.7) Consider the following approach to vertex colouring. First, find a maximal independent set of vertices and colour these with colour 1; then find a maximal independent set of vertices in the remaining graph and colour those 2, and so on. Compare this algorithm with the greedy algorithm: which is better?

Solution The given algorithm is equivalent to the greed colouring if we ordered the vertices according to the same criteria. When colouring a vertex in V_i, it receives colour i as it is adjacent to at least one vertex in each of the preceding sets $V_1, \dots, V_{i-1}$, otherwise it could be included in some of them, a contradiction to their maximality.

If the given algorithm yields a non-optimal colouring, we saw in question 5 that there exists some ordering of the vertices such that the greed algorithm yields an optimal colouring.

And last, in question 6 we saw that such algorithm would give an optimal colouring, while some ordering of the vertices for the greed algorithm would yield a worse colouring.

So the greedy algorithm can either perform worse, the same, or better than the given algorithm, based on the chosen sequencing of the vertices.

5.8) Show that the bound of Proposition 5.2.2 is always at least as sharp as that of Proposition 5.2.1.

Solution

Proof. We want to show that for any graph G with m edges,

$$\max_{H \subseteq G}\{\delta(G)\} \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}} \tag{1}$$

First, notice that the expression

$$\frac{1}{2} + \sqrt{2m + \frac{1}{4}} \tag{2}$$

denotes the number of vertices on a clique with m edges.

We prove by contradiction, so suppose we have a graph G with

$$\max_{H\subseteq G}\{\delta(G)\} > \frac{1}{2} + \sqrt{2m + \frac{1}{4}}, \tag{3}$$

and let H be the maximizing subgraph. So H has more vertices than a clique with m edges, and each of its vertices have more neighbours than those of a clique with m edges, so H has more than m, a contradiction.

More formally, given the lower bound on $\delta(H)$, we can lower bound its the number of edges as follows,

$$\begin{aligned}
|E(H)| &\geq \frac{1}{2}\delta(H)|H| \\
&\geq \frac{1}{2}\delta(H)(\delta(H)+1) \\
&= \frac{1}{2}(\delta(H)^2 + \delta(H)) \\
&> \frac{1}{2}\left(\frac{1}{4} + \sqrt{2m+\frac{1}{4}} + 2m + \frac{1}{4} + \frac{1}{2} + \sqrt{2m+\frac{1}{4}}\right) \\
&= \frac{1}{2}\left(1 + 2m + 2\sqrt{2m+\frac{1}{4}}\right) \\
&= \frac{1}{2} + 2m + \sqrt{2m+\frac{1}{4}},
\end{aligned}$$

thus H has more edges than G, a contradiction. □

5.9) Find a lower bound for the colouring number in terms of average degree.

Solution

Proof. Let G be a graph and $v_1, \dots, v_n$ be the sequence of its vertices as suggested in the book, that is, each v_i has minimum degree in $G_i := G[v_1, \dots, v_n]$. Such sequence can be easily obtained by working backwards, choosing v_n of degree $\delta(G)$, and then v_{n-1} of degree $\delta(G - v_n)$ and so on. We then have

$$\sum_{i=1}^{n} d_{G_i}(v_i) = \sum_{i=1}^{n} \delta(G_i) \leq n(col(G) - 1), \tag{4}$$

as by the definition of $col(G)$, any subgraph $H \subseteq G$ has $col(G) \leq \delta(H) + 1$.

And by summing the degrees of $V(G)$ as above, for each vertex v_i we are only counting the edges $v_i v_j$ with $j < i$. So each edge in G gets counted once and we have,

$$\sum_{i=1}^{n} d_{G_i}(v_i) = |E|. \tag{5}$$

□

Putting (4) and (5) together we get,

$$n(col(G)-1) \geq |E| \rightarrow col(G) \geq \frac{|E|}{n} + 1 = \frac{d(g)}{2} + 1.$$

5.10) Find a function f such that every graph of arboricity at least $f(k)$ has colouring number at least k, and a function g such that every graph of colouring number at least $g(k)$ has arboricity at least k, for all $k \in N$.
Solution (i) $f(k) = k - 1$

Proof. Let G be a graph and $v_1, \ldots, v_n$ be some ordering of its vertices. Consider the following algorithm, we initialize a set of edges $F =$ and for each vertex v_i, $i = n, \ldots, 2$, we take some edge $v_i v_j$ with $j < i$, include it in F and delete it from G. We make $F = \emptyset$ again and repeat until no edges are left in G. Clearly F was a forest at the end of each iteration.

As $\mathrm{col}(G) = \max_{H \subseteq G}\{\delta(H)\} + 1$, there are at most $\mathrm{col}(G) - 1$ edges $v_i v_j$ with $j < i$ for any vertex v_i, thus $\mathrm{col}(G) - 1$ iterations of the algorithm suffices and we have at most $\mathrm{col}(G) - 1$ disjoint forests in G, therefore $\mathrm{arb}(G) \leq \mathrm{col}(G) - 1$. So $\mathrm{arb} \geq k - 1$ implies $\mathrm{col}(G) \geq k$. □

Solution (ii) $g(k) = 2k + 1$

Proof. Let $col(G) \geq g(k)$, then by the definition of colouring number, G has an induced subgraph H with $\delta(H) + 1 \geq g(k)$. Given the lower bound on the minimum degree, we can also lower bound the number of edges,

$$||H|| \geq \frac{\delta(H)|H|}{2} \geq \frac{(g(k)-1)|H|}{2}.$$

We then have

$$\frac{g(k)-1}{2} \leq \frac{||H||}{|H|} \leq \frac{||H|||}{|H|+1} \leq \mathrm{arb}(G),$$

the last inequality due to Nash-Willians theorem. So making $g(k) = 2k+1$ satisfies the requisite. □

5.11) A k-chromatic graph G is called critically k-chromatic, or just critical, if $\chi(G - v) < k$ for every vertex $v \in V$. Show that every k-chromatic graph has a critical k-chromatic induced subgraph, and that any such subgraph has minimum degree at least $k - 1$.

Solution

Lemma 2. *For any vertex of a k-chromatic graph G, we have either $\chi(G - v) = k$ or $\chi(G - v) = k - 1$.*

Proof. Clearly $\chi(G - v) \leq k$, and if $\chi(G - v) \leq k - 2$, we could colour v in G with the $k - 1$-th colour, contradiction. □

Lemma 3. *Any critical k-chromatic graph G has $\delta(g) = k - 1$.*

Proof. Suppose some vertex v has at most $k - 2$ neighbours. As $G - v$ has a $k - 1$ colouring, and v sees at most $k - 2$ colours, we can colour v using the $k - 1$-th colour, yielding a $k - 1$ colouring of G, contradiction. □

So let G be a k-chromatic graph, as it is finite, $\chi(empty) = 0$, and using lemma 2, there is a maximal set of vertices V' such that $\chi(G' := G - V') = k$. As V' is maximal by lemma 2, $\chi(G' - v) = k - 1$ for all $v \in V \setminus V'$, so G' is k-critical, and by lemma 3, has $\delta(G') = k - 1$, and is induced by definition.

5.12) Determine the critical 3-chromatic graphs.

Solution The odd cycles C, as $C - v$ has no odd cycle for any $v \in C$, any odd cycle require 3 colours, a graph being 2-chromatic is equivalent to it being bipartite, and a graph is bipartite if and only if it has no odd cycle.

5.13)

Solution

5.14)

Solution

5.15)

Solution

5.16)

Solution

5.17)

Solution

5.18)

Solution

5.19)

Solution

5.20)

Solution

Chapter 6 - Flows

6.1)

Solution

Chapter 7 - Extremal Graph Theory

7.1)

Solution

Chapter 9 - Ramsey Theory for Graphs

9.1)

Solution

Chapter 10 - Hamilton Cycles

10.1) An oriented complete graph is called a tournament. Show that every tournament contains a (directed) Hamiltonian path.

Solution

Proof. We prove by induction on $|G|$, for $|G| = 2$, the single oriented edge is a directed hamiltonian path by itself. Now let G be a tournament with $n > 2$ vertices and assume all tournaments on fewer vertices contains a directed hamiltonian path.

If some vertex v has all the arcs incident to it oriented towards v, we can take any directed hamiltonian path of $G - v$, which exists by the induction hypothesis, and extend it to v. So assume no such vertex exist, let v_1 be some vertex of G, and consider the directed hamiltonian path $v_2v_3 \ldots v_n$ of $G - v_1$. By our choice of v_1, some arc v_1v_i, $i \in \{2, \ldots, n\}$, oriented away from v_1 exists, so let k be the lowest value for i. If $k = 2$, we have the directed hamiltonian path $v_1v_2 \ldots v_n$ for G, if $k = n$, we have the directed hamiltonian path $v_2v_3 \ldots v_nv_1$ for G, otherwise, we have the directed hamiltonian path $v_2 \ldots v_{k-1}v_1v_k \ldots v_n$ for G. □

10.2) Show that every uniquely 3-edge-colourable cubic graph is hamiltonian. ('Unique' means that all 3-edge-colourings induce the same edge partition.)

Solution

Proof. Let $A \cup B$ be the union of the edges in two colours classes, as G is cubic, each vertex has all 3 colours incident to it, so $A \cup B$ is a set of cycles in G. If we have a single cycle, it is a hamiltonian cycle and G is hamiltonian. If not, we can flip the colours of one of the cycles, inducing a distinct edge partition, contradiction. □

10.4) Prove or disprove the following strengthening of Proposition 10.1.2: "Every k-connected graph G with $|G| \geq 3$ adn $\chi(G) \geq |G|/k$ has a hamiltonian cycle.

Solution

Proof. The proposition is false, consider the counterexample K_2 with 3 indepedent paths of size 2 between its two vertices. By inspection it has no hamiltonian cycle, and

$$\frac{|G|}{k} = \frac{5}{2} \leq 3 = \chi(G).$$

□