Exercise 8.1: The AKLT model

In this exercise we will consider the construction of a MPS of a non-trivial quantum state. Namely, the ground state of the *Affleck-Kennedy-Lieb-Tasaki* model introduced in 1987 [1]. The **spin-1** Hamiltonian is given by

$$\hat{H} = \sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{3}\left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\right)^2. \tag{1}$$

This model has by construction a ground state in which all nearest neighboring spins share a valence bond, i.e. a spin-$\frac{1}{2}$ singlet

$$\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}. \tag{2}$$

This means that each spin-1 can be thought to be broken up in two spin-$\frac{1}{2}$, and each of the spin-$\frac{1}{2}$ forms a singlet with a spin-$\frac{1}{2}$ on the adjacent site (see also Fig. 1 (d) and (e)).

a) First construct a dimerized **spin-$\frac{1}{2}$** chain of singlets, i.e. we consider a **product of singlets** $\left(\frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle\right) \otimes \cdots \otimes \left(\frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle\right)$ on neighboring sites. Convince yourself that this state can be written in the MPS framework with 1×2 matrices on odd sites and 2×1 matrices on even sites given by

$$M^{[2n-1]\uparrow} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad M^{[2n-1]\downarrow} = \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}, \quad M^{[2n]\uparrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad M^{[2n]\downarrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

This is shown in Fig. 1 (c) in which the bold lines represent the singlet bonds with bond dimension 2. We will work with open boundary conditions, and we will put single spins-$\frac{1}{2}$ on the edges (to make the AKLT projection later on) but they do not matter. So you can just for instance choose them up.

b) Use the overlap function between two MPS that you wrote in the exercise last week to check the norm of this singlet MPS, and to compute the spin-correlation function $\langle \sigma_i^z \sigma_j^z \rangle$. You should observe that this gives always 0 for $|i-j| > 1$.

c) Construct the spin-1 projector and apply it to the singlet MPS as shown in Fig. 1 (d) and (e). This gives the MPS representation of the AKLT ground state.

d) Check the normalization of your AKLT MPS ground state (again using the overlap function), make sure it is normalized.

e) Calculate the correlation function $|\langle S_i^z S_j^z \rangle|$ (again using the overlap function), and plot it as a function of the distance $|i-j|$, what do you observe ?

[1] see Affleck, Kennedy, Lieb, Tasaki. "Rigorous results on valence-bond ground states in antiferromagnets". PRL 59 (7): 799–802, 1987.

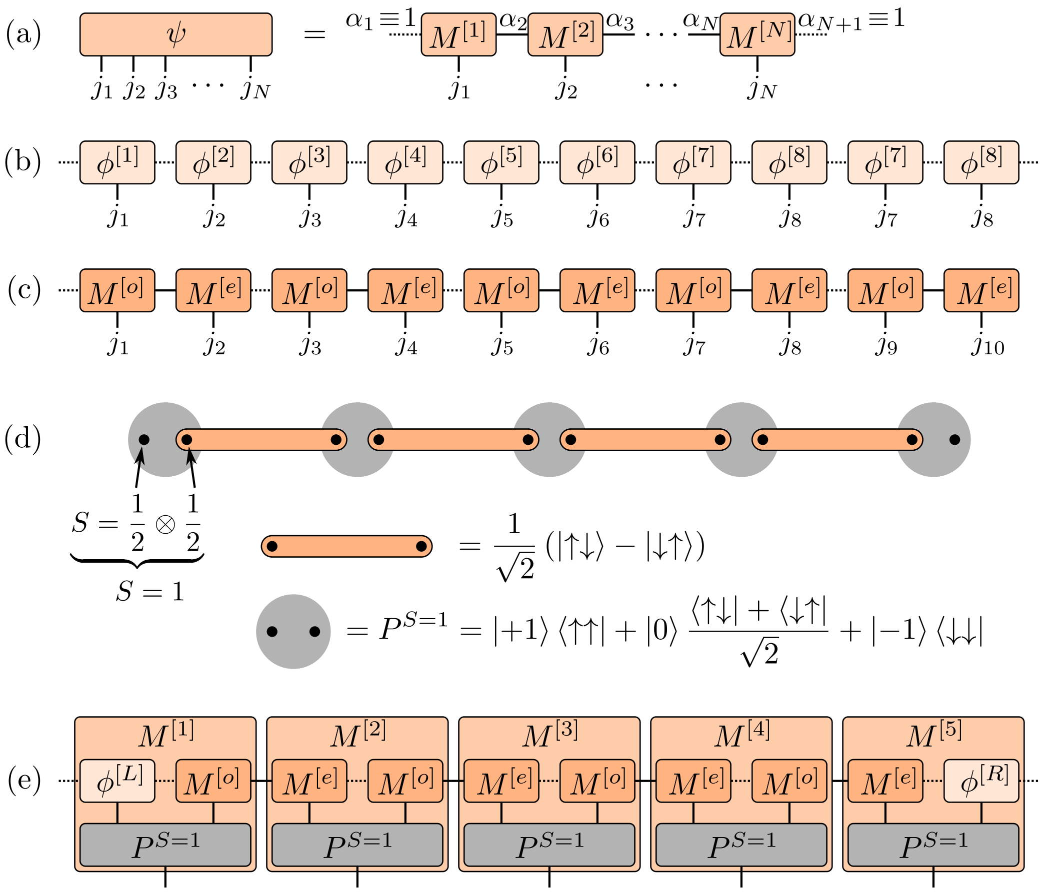

Figure 1: (a) In an MPS, the amplitude of the wave function is decomposed into a product of matrices $M^{[n]j_n}$. The indices α_1 and α_{N+1} are trivial, which we indicate by dashed lines. (b) A product state can be written as a trivial MPS with bond dimensions $\chi = 1$. (c) The MPS for a product of singlets on neighboring sites, with $M^{[1]}, M^{[2]}$ given in eq. (3). (d) Diagrammatic representation of the AKLT state. The $S = 1$ sites (grey circles) are decomposed into two $S = \frac{1}{2}$ that form singlets between neighboring sites. With open boundary conditions, the $S = \frac{1}{2}$ spins on the left and right are free edge modes leading to a four-fold degeneracy of the ground state. (e) The AKLT state can be represented by an MPS with bond dimension $\chi = 2$. - Figure from Johannes lecture notes, see `https://arxiv.org/abs/1805.00055`.

Optional 1: Calculate the ground state energy density by acting on it with the Hamiltonian.

Optional 2: Calculate the string correlation $\left\langle S_i^z \prod_{i<k<j} e^{i\pi S_k^z} S_j^z \right\rangle$. If you did this correctly you should observe that this value is finite $(-4/9)$ in the large L and large $|i-j|$ limit, unlike the spin-spin correlation. This implies the presence of hidden order, in the sense that it can only be detected by global operators.

Exercise 8.2: Q & A session

Go through the lecture notes and tutorials and see if you have any open questions. Prepare to ask them in the upcoming Q & A session during the next tutorial.