MAT-INF4170 Oblig 3

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Exercise 3.1

The augmented knot vector $\mathbf{t} = (0, 0, 0, 1, 3, 4, 5, 5)$ has 8 knots. Since the largest possible index i in $B_{i,p,\mathbf{t}}$ must then satisfy $i + p + 1 = 8$, we obtain $i = 4$ for $p = 3$. So we can only define 4 B-splines of degree 3:

$$\begin{aligned}
B_{1,3}(x) &= B[0, 0, 0, 1, 3](x), \\
B_{2,3}(x) &= B[0, 0, 1, 3, 4](x), \\
B_{3,3}(x) &= B[0, 1, 3, 4, 5](x), \\
B_{4,3}(x) &= B[1, 3, 4, 5, 5](x).
\end{aligned}$$

For $x \in [1, 3]$ that is $x \in [t_\mu, t_{\mu+1}]$, $\mu = 4$, these are all (the only) non-zero (active) B-splines. By lemma 3.7 they must then be linearly independent. We can show this easily using corollary 3.5.

From the corollary we know that there exists coefficients such that we can obtain the power basis $\{1, x, x^2, x^3\}$ for $x \in [t_{p+1}, t_{n+1}] = [t_4, t_5]$ by making linear combinations of the B-splines above. Moreover, it is well known that $\dim\{1, x, x^2, x^3\} = 4$ since this power basis is a linearly independent set of functions which can be used to define any real polynomial of degree 3 or less. It follows that $\dim\{B_{1,3}, ..., B_{4,3}\} = 4$ since they can represent the power basis, and therefore any real polynomial of degree 3 or less. Since there are only 4 of these B-splines, they must then be linearly independent.

Now we want to show $\hat{B}_{1,3}$ and $\hat{B}_{2,3}$ are linearly independent for the original knot vector $\hat{t} = (0, 0, 1, 3, 4, 5)$ for $x \in [1, 3]$. These are the only B-splines we can define of degree 3 for this knot vector, and they are both active in this domain. Since this knot vector is not a $p + 1$-extended knot vector we cannot use Theorem 3.9. However, for $x \in [1, 3]$ note that:

$$\begin{aligned}
\hat{B}_{1,3}(x) &= B[0, 0, 1, 3, 4](x) = B_{2,3}(x), \\
\hat{B}_{2,3}(x) &= B[0, 1, 3, 4, 5](x) = B_{3,3}(x),
\end{aligned}$$

which we just showed are linearly independent, and we are done.

Exercise 4.5 - Not assuming $p+1$ extended τ and t

We wish to prove lemma 4.2 for the more general case where τ and t are not $p+1$ regular knot vectors. In this case I have attempted to assume t and τ are not $p+1$ extended.

Suppose we have knot vectors $(\tau_j)_{j=1}^{n+p+1}$ and $(t_i)_{i=1}^{m+p+1}$ supporting at least one B-spline of degree p, so $m, n \geq 1$. Moreover, let $\tau \subseteq t$, in the sense of a subsequence, so $1 \leq n \leq m$. Other than the knot vectors being a sequence of non-decreasing real numbers we make no more assumptions on t and τ. Hence, at the ends, we can at most say $t_1 \leq \tau_1$ and $\tau_{n+p+1} \leq t_{m+p+1}$.

First, let us do away with some trivial cases. If $\tau_1 = \tau_{m+p+1}$, then any spline $f \in \mathbb{S}_{p,\tau}$ must be identically zero everywhere. Since all linear spline spaces have the zero element, we are done.

Consider now the case where we have at least one non-empty interval in τ. Suppose this interval occurs for some $\mu \in \mathbb{N}$ such that $\tau_\mu < \tau_{\mu+1}$. Then we add $p+1$ identical knots at each end of τ, but we choose them such that $a < t_1$ and $b > t_{n+p+1}$. We call this new knot vector τ'. We do the same for t and call it t'. Then for any $f \in \mathbb{S}_{p,\tau}$:

$$\begin{aligned} f &= \sum_{j=1}^{n+p+1} c_j B_{j,p,\tau} \\ f &= \sum_{j=1}^{p+1} 0 \cdot B_{j,p,\tau'} + \sum_{j=p+2}^{n+p+1} c_j B_{j,p,\tau'} + \sum_{j=n+p+2}^{n+2p+2} 0 \cdot B_{j,p,\tau'}, \end{aligned}$$

since $B_{j+p+1,p,\tau'} = B_{j,p,\tau}$ for $j = 1, ..n$, as these depend on exactly the same knots. Hence we have shown that by zeroing out the first $p+1$ and the last $p+1$ B-spline coefficients for a spline in $\mathbb{S}_{p,\tau'}$ we obtain a spline in $\mathbb{S}_{p,\tau}$ thus $\mathbb{S}_{p,\tau} \subseteq \mathbb{S}_{p,\tau'}$. The same argument applies to splines dependent on knot vectors t and t', so $\mathbb{S}_{p,t} \subseteq \mathbb{S}_{p,t'}$.

Now comes the part I struggled immensely with. Lemma 4.2 assumes that you have $p+1$ regular knot vectors. Having assumed at least one non-empty interval in τ, and thus also in t, the following is true for our modified $(t'_j)_{j=1}^{m'+p+1}$ and $(\tau')_{i=1}^{n'+p+1}$:

$$\begin{aligned} &n' = n + p + 1 \geq p + 2 > p + 1, \\ &m' = m + p + 1 \geq p + 2 > p + 1, \\ &t'_{p+1} = a < t_1, \\ &\tau'_{p+1} = a < t_1 \leq \tau_1, \\ &t'_1 = t'_{p+1} = a, \ t'_{m'+1} = t'_{m'+p+1} = b, \\ &\tau'_1 = \tau'_{p+1} = a, \ \tau_{n'+1} = \tau'_{n'+p+1} = b, \\ &t'_1 = a < t_1 = t'_{p+2}, \ t'_{m'} = t_{m+p+1} < b = t'_{m'+p+1}, \\ &\tau'_1 = a < t_1 \leq \tau_1 = \tau'_{p+2}, \ \tau'_{n'} = \tau_{n+p+1} \leq t_{m+p+1} < b = \tau'_{n'+p+1}, \end{aligned}$$

This takes care of almost everything we need to say that τ' and t' are $p+1$ regular. Specifically, we are lacking the inequalities for $j = 2, ..., n'-1$ and $i = 2, ..., n'-1$ saying that $B_{j,p,\tau'}$ and $B_{i,p,t'}$ are not identically zero. However, by assuming at least one non-empty interval in τ and therefore in t we know that there exists some $\mu, \nu \in \mathbb{N}$ such that $B_{\mu+p+1,p,\tau'}$ and $B_{\nu+p+1,p,t'}$ are not identically zero, i.e. equivalent to $B_{\mu,p,\tau}$ and $B_{\nu,p,t}$ not being identically zero. Then at least, we can speak of meaningful intervals:

$$\begin{aligned} a < \tau_1 \leq \tau_\mu < \tau_{\mu+1} \leq \tau_{n+p+1} < b, \\ a < t_1 \leq t_\nu < t_{\nu+1} \leq t_{m+p+1} < b. \end{aligned}$$

So no matter which index μ and ν are, we know that we have at least $p+1$ knots which are smaller than τ_μ and t_ν, and larger than $\tau_{\mu+1}$ and $t_{\nu+1}$ on the other side. Moreover, the number of active B-splines dependent on τ' and t' in the interval $x \in [\tau_\mu, \tau_{\mu+1})$ or $x \in [t_\nu, t_{\nu+1})$ will because of this be exactly $p+1$ since $\mu - p, ...\mu$ constitutes $p+1$ indices, and same for ν. So we have a full set of B-splines to play with! Zeroing out all other coefficients we obtain for two elements in each space:

$$f(x) = \sum_{j=\mu-p}^{\mu} c_j B_{j,p,\tau} = \sum_{j=\mu-p}^{\mu} c_{j+p+1} B_{j+p+1,p,\tau'},$$

$$g(x) = \sum_{j=\mu-p}^{\mu} c_j B_{j,p,\tau} = \sum_{j=\nu-p}^{\nu} b_{j+p+1} B_{j+p+1,p,t'}.$$

But then since $\tau_\mu > \tau'_{p+1}$ and $t_\nu > t'_{p+1}$ and also $\tau_{\mu+1} < \tau'_{n'+1}$ and $t'_{\nu+1} < t'_{m'+1}$ we can use Marsden's Identity, and Corollary 3.5 to construct the polynomial basis, and so these B-splines must be linearly independent, i.e. the result of Lemma 3.7. Then we can choose our $g(x)$ so that it is identically equal to $f(x)$ on this interval. However, since $\tau \subseteq t$, we may have more intervals to play with in t, and so can make functions which we cannot obtain using only the knots τ, and so $\mathbb{S}_{p,\tau'} \subseteq \mathbb{S}_{p,t'}$ and $\mathbb{S}_{p,\tau} \subseteq \mathbb{S}_{p,t}$ $\square$.

Exercise 4.5 - Assuming $p+1$ extended t and τ

We wish to prove Lemma 4.2 in the general case where τ and t are not required to be $p+1$ regular knot sequences. For clarity, we restate the definition. A knot sequence $(t_j)_{j=1}^{n+p+1}$ is said to be $p+1$ extended if:

$$
\begin{aligned}
& n \geq 1, \\
& t_{p+1} < t_{p+2}, \\
& t_n < t_{n+1}, \\
& t_j < t_{j+p+1}, j = 1, .., n
\end{aligned}
$$

and it is regular if it is both extended and

$$
\begin{aligned}
t_1 &= t_{p+1}, \\
t_{n+1} &= t_{n+p+1}.
\end{aligned}
$$

Suppose we are given some knot vector τ and its refinement t. That is, τ is a subsequence of t, and every real number in t occurs at least as many times in τ. Assume also that τ has at least $p+2$ knots ($n \geq 1$), where p is a positive integer. We now insert $p+1$ knots at either end of both t and τ, the numbers $a < t_1$ and $b > t_{m+p+1}$ ($m \geq 1$). Let us call these new knot sequences t' and τ'. If the original τ is a $p+1$ extended knot sequence then examining τ':

$$
\begin{aligned}
\tau' &= (a, ..., a, \underline{\tau_1, ..., \tau_{n+p+1}}, b, ..., b), \\
&= (\tau'_1, ..., \tau'_{p+1}, \underline{\tau'_{p+2}, ..., \tau'_{n+2p+2}}, \tau'_{n+2p+3}, ..., \tau'_{n+3p+3}),
\end{aligned}
$$

we see that:

$$
\begin{aligned}
& n' + p + 1 = n + 3p + 3 \Rightarrow n' = n + 2p + 2 > n \geq 1, \\
& \tau'_{p+1} = a < t_1 = \tau_1, \\
& \tau'_{n'} = \tau'_{n+2p+2} = \tau_{n+p+1} = t_{m+p+1} < b = \tau'_{n'+1}.
\end{aligned}
$$

Moreover for the first $p+1$ knots:

$$
\begin{aligned}
& \tau'_1 = a < \tau_1 = \tau'_{p+2} \\
& \tau'_2 = a < \tau_2 = \tau'_{p+3} \\
& ... \\
& \tau'_{p+1} = a < \tau_{p+1} = \tau'_{2p+2},
\end{aligned}
$$

and for the remaining knots (including the original knots are, which are $p+1$ extended):

$$
\begin{aligned}
& \tau'_{p+2} = \tau_1 < \tau_{p+2} = \tau'_{2p+3}, \\
& ... \\
& \tau'_{n+p+1} = \tau_n < \tau_{n+p+1} = \tau'_{n+2p+2}, \\
& \tau'_{n+p+2} = \tau_{n+1} < \tau'_{n+2p+3} = b \\
& ... \\
& \tau'_{n'} = \tau_{n+p+1} < \tau_{n'+p+1} = b
\end{aligned}
$$

hence by definition τ' is $p+1$ regular. The same argument applies to t' if t is a $p+1$ extended knot sequence.

The result of this is that Lemma 4.2 applies for t' and τ', so we have that:

$$\mathbb{S}_{p,\tau'} \subseteq \mathbb{S}_{p,t'}.$$

Let $f \in \mathbb{S}_{p,\tau}$, that is:

$$f = \sum_{i=1}^{n} c_i B_{i,p,\tau}.$$

Suppose also that we have some $g \in \mathbb{S}_{,p,\tau'}$. Then:

$$g = \sum_{i=1}^{n'} c_i' B_{i,p,\tau'}.$$

If we let the coefficients $c_i' = 0$ for $i = 1, ..., p+1$ and $i > n+p+1$ this g reduces to:

$$\begin{aligned} g &= \sum_{i=p+2}^{n+p+1} c_i' B_{i,p,\tau'}, \\ &= \sum_{i=p+2}^{n+p+1} c_i' B_{i-(p+1),p,\tau'}, \\ &= \sum_{j=1}^{n} c_{j+p+1}' B_{j,p,\tau}, \end{aligned}$$

where the last equality is a consequence of how we defined τ'. If we also let $(c_{p+2}', ..., c_{n+p+1}') = (c_1, ..., c_n)$ we obtain $g = f$. Hence $\mathbb{S}_{p,\tau} \subseteq \mathbb{S}_{p,\tau'}$. Similarly, given some $h \in \mathbb{S}_{p,t'}$

$$h = \sum_{i=1}^{m'} a_i' B_{i,p,t'},$$

then letting $a_i' = 0$ for $i = 1, ..p+1$ and $i > m+p+1$ we obtain:

$$h = \sum_{j=1}^{m} a_{j+p+1}' B_{j+p+1,p,t'},$$

which is an element of $\mathbb{S}_{p,t}$ with coefficients $(a_1, ..., a_m) = (a_{p+2}', ..., a_{m+p+1}')$ since $B_{j+p+1,p,t'} = B_{j,p,t}$ for these j. For the f in $\mathbb{S}_{p,\tau}$ we know by Lemma 4.2 and the previous result that we may write

$$f = \sum_{i=1}^{n} c_i B_{i,p,\tau} = \sum_{i=1}^{n} c_{i+p+1}' B_{i+p+1,p,\tau'} = \sum_{i=1}^{m'} d_i' B_{i,p,t'}.$$

If we evaluate f at any x where $a \leq x < \tau_1 = t_1$ or and $b \geq x > \tau_{n+p+1} = t_{m+p+1}$ the LHS must be zero. So restricting x to each interval, we get the following equalities for the B-splines dependent t' which are active here:

$$\begin{aligned} 0 &= \sum_{i=1}^{p+1} d_i' B_{i,p,t'}, \\ 0 &= \sum_{i=n+p+2}^{n'} d_i' B_{i,p,t'}. \end{aligned}$$

Now, if a set of functions are linearly independent, spanning some space, then any subset of that set must also be linearly independent, so the only solution is that $d_i' = 0$ for these i and so we must have that that $f \in \mathbb{S}_{p,t}$ as well, since we can choose our h to have coefficients to obtain f.

Exercise 4.6 - Old

We wish to to prove Theorem 4.6 in the general case, where τ and its refinement t are not $p+1$ regular knot vectors. Assume that they are at least $p+1$ extended. We augment τ and t by adding $p+1$ identical knots at both ends: the knots $(a_-, ..., a_-)$ and $(a_+, ..., a_+)$ where $a_- < t_1$ and $a_+ > t_{m+p+1}$. By the discussion in the previous exercise, we know that the augmented knot vectors, denoted t' and τ', are now $p+1$ regular knot vectors. Using this trick, we were able to conclude that $\mathbb{S}_{p,\tau} \subseteq \mathbb{S}_{p,t}$ for the original knot vectors. Specifically, we were able to show that the B-spline coefficients of a function $\mathbb{S}_{p,t}$ and in $\mathbb{S}_{p,\tau}$ had their first $p+1$ and last $p+1$ B-spline coefficients identically zero.

Now, consider Theorem 4.6 for the function $f \in \mathbb{S}_{p,\tau'}$ with B-spline coefficients $\mathbf{c}$ in this space, and B-spline coefficients $\mathbf{b}$ in $\mathbb{S}_{p,t'}$. Then:

$$\begin{aligned}
&\boldsymbol{\alpha}_p(i)^T = (\alpha_{\mu-p,p}(i), ..., \alpha_{\mu,p}(i)) = \begin{cases} 1 & p = 0 \\ \mathbf{R}_1(t'_{i+1}) \cdot ... \cdot \mathbf{R}_p(t'_{i+p}) & p > 0 \end{cases}, \\
&\boldsymbol{c}_p = (c_{\mu-p}, ..., c_\mu), \\
&b_i = \sum_{j=\mu-p}^{\mu} \alpha_{j,p}(i) c_j = \mathbf{R}_1(t'_{i+1}) \cdot ... \cdot \mathbf{R}_p(t'_{i+p}) \mathbf{c}_p,
\end{aligned}$$

for some $1 \leq i \leq m' = m+2p+2$ with μ such that $\tau'_\mu \leq t'_i < \tau'_{\mu+1}$. Suppose now that this $f \in \mathbb{S}_{p,\tau}$. Then we have already shown that the first $p+1$ and last $p+1$ coefficients are zero in both spaces. That is:

$$\begin{aligned}
c_j &= 0, \text{ for } j = 1, .., p+1, \text{ and } j > n+p+1, \\
b_i &= 0, \text{ for } i = 1, ..., p+1 \text{ and } i > m+p+1,
\end{aligned}$$

where m and n is the index of the last B-spline in $\mathbb{S}_{p,t}$ and $\mathbb{S}_{p,\tau}$ respectively.

First consider what values of μ that satisfy $\tau'_\mu < \tau'_{\mu+1}$, which the μ we need to find must satisfy. The first $p+1$ and last $p+1$ knots are equal, thus $\mu < p+1$ does not satisfy this inequality, nor does $\mu > n+2p+2$. Hence we are left with considering j for $\mu = p+1, ..., n+2p+2$, that is $j = 1, ..., n+2p+2$. For these μ the possible candidates of i must be in the set $\{1, ..., m+2p+2\}$. This leaves us with considering $\alpha_{j,p}(i)$ for $j = 1, ..., n+2p+2$, $i = 1, ..., m+2p+2$. If we can show that $\alpha_{j,p}(i) \equiv 0$ for all the i, j where either coefficient b_i, c_j is identically zero, the theorem reduces to case with the original knot vectors τ and t and their B-splines.

Note: Without showing this, we can actually see that for $i = 1, ..p+1$ and $i = m+2p+2$ the only μ that satisfies the inequalities are $\mu = p+1$ and $\mu = n+2p+2$, which in turn shows that the theorem gives correct b_i given that the c_j for $j = 1, ..., p+1$ and $j \geq n+p+2$ are zero. I have tried really hard to show that we always get $b_i = 0$ for remaining i but I haven't been able to prove that we obtain the same $\mu = n+2p+2$ for $m+2p+2 > i \geq m+p+2$, which is the only way to guarantee that no other c_j enters the equation for the remaining b_i, i.e. avoiding having to show that $\alpha_{j,p}(i) = 0$ for these i, j. It seems this would be the case if t has some specific properties regarding how many of each knot is added when refining from τ. Specifically, if $\tau'_{n+2p+2} \leq t'_{m+p+2} \leq t'_{m+p+3} \leq ... \leq t'_{m+2p+2} < a_+$ we obtain the correct results that $b_i = 0$ given the $c_j = 0$ for $j = p+2, ..., n+p+1$. However, this would imply that $t'_{m+p+2} = ...t'_{m+2p+2}$, so we have exactly $p+1$ of the last knot in τ in t, which as far as I can tell is not something that is true in general for a refinement of a $p+1$ extended knot vector τ. Indeed, consider the counter-example:

$$\tau = 1, 2, 3, 4, 5,$$

then clearly τ is $p+1$ extended, i.e. for instance with $n = 2, p = 2$. If we refine τ, but still require the resulting t to be $p+1$ extended we can use

$$t = 1, 1, 1, 2, 3, 3, 3, 4, 5, 5,$$

since again with $p = 2$, and now $m = 7$ we have $p+1$ extended knot vector: $m \leq 1$, $t_3 < t_4$, $t_7 < t_8$ and $t_j < t_{j+p+1}$ for $= 1, .., m$. But now adding say $p + 1$ identical knots to both ends:

$$\tau' = 0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6$$
$$t' = 0, 0, 0, 1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 6, 6, 6$$

we see that $t'_{m+p+2} = t'_{11} = 4$, and $\tau'_{n+2p+2} = \tau'_8 = 5$ so it doesn't hold. Because of this, we are forced to calculate the $\alpha_{j,p}(i)$.

The recurrence relation for the coefficients are:

$$\alpha_{j,p}(i) = \frac{t'_{i+p} - \tau'_j}{\tau'_{j+p} - \tau'_j}\alpha_{j,p-1}(i) + \frac{\tau'_{j+p+1} - t'_{i+p}}{\tau'_{j+1+p} - \tau'_{j+1}}\alpha_{j+1,p-1}(i),$$

and begins with $\alpha_{j,0}(i) = B_{j,0,\tau'}(t'_i)$. I have tried for quite some time, and have not been able to show that for $i = 1, .., p + 1$ and $i > m + p + 1$ and $j = 1, ..., p + 1$ and $j > n + p + 1$ these are all zero. If however that is the case, the theorem holds for t and τ.

I'll begin with $p = 1$, and try to use induction on p. Then we want the coefficients to be zero for $i = 1, 2$, $i > m + 2$, and $j = 1, 2$ $j > n + 2$. Let's check.

$$\alpha_{1,1}(1) = \frac{t'_2 - \tau'_1}{\tau'_2 - \tau'_1}\alpha_{1,0}(1) + \frac{\tau'_3 - t'_2}{\tau'_3 - \tau'_2}\alpha_{2,0}(1)$$

Now when $p = 1$ we have 2 identical knots a_- and 2 identical knots a_+ at each end of the knot vectors. That is $\tau'_1 = \tau'_2 = a_-$ and $t'_1 = t'_2 = a_-$ and at the other end $\tau'_{n+5} = \tau'_{n+6} = a_-$ and $t'_{m+5} = t'_{m+6} = a_+$. Our first coefficient has zero denominator in the first fraction, but its numerator is also 0. We use the zero rule for B-spline recurrences and set the fraction to 0. Indeed this holds since $\alpha_{1,0}(1) = B_{1,0}(a_-)$ and $B_{1,0}$ is a vanishing B-spline since its two knots are equal. We are left with

$$\alpha_{1,1}(1) = \frac{\tau'_3 - a_-}{\tau'_3 - a_-}\alpha_{2,0}(1) = 1 \cdot \alpha_{2,0}(1) = B_{2,0}(t'_1) = \chi_{[a_-,\tau'_3)}(a_-) = 1.$$

As we can see, this is not good, as it fails already at $p = 1$. After this (and many attempts which I haven't shown here), it seems like you cannot say that the $\alpha_{j,p}(i)$'s vanish as we want them to. Presumably you have to zero them all out, or ignore them entirely, so you only need to consider a sub matrix, to end up with something that never depends on the added knots and thus reduces to the original problem of τ and t, even though that seems to assume the conclusion. What I mean by this is that since, according to 4.1,

$$B_{j,p,\tau'}(x) = \sum_{i=1}^{m+2p+2} \alpha_{j,p}(i)B_{i,p,t'}(x),$$
$$\sum_{j=p+2}^{n+p+1} c_j B_j, p, \tau'(x) = \sum_{j=p+2}^{n+p+1} c_j \sum_{i=1}^{m+2p+2} \alpha_{j,p}(i)B_{i,p,t'}(x)$$
$$f(x) = \sum_{j=p+2}^{n+p+1} \sum_{i=1}^{m+2p+2} c_j\alpha_{j,p}(i)B_{j,p,t'}(x)$$

and I argued that if we evaluated f in $a_- \leq x \leq \tau_1$ or $a_+ \geq x > \tau_{n+p+1}$ then we get a linear combinations of linearly independent B-splines:

$$0 = \sum_{j=p+2}^{n+p+1} \sum_{i=1}^{p+1} c_j\alpha_{j,p}(i)B_{i,p,t'}(x) = \sum_{i=1}^{p+1}(\sum_{j=p+2}^{n+p+1} c_j\alpha_{j,p}(i) = d'_i)B_{i,p,t'}(x) \Rightarrow d'_i = 0 \text{ for } i = 1, .., p + 1$$
$$0 = \sum_{j=p+2}^{n+p+1} \sum_{i=m+p+2}^{m+2p+2} c_j\alpha_{j,p}(i)B_{i,p,t'}(x) = \sum_{i=m+p+2}^{m+2p+2} (\sum_{j=p+2}^{n+p+1} c_j\alpha_{j,p}(i) = d'_i)B_{i,p,t'}(x) \Rightarrow d'_i = 0 \text{ for } i \geq m + p + 2$$

But I don't know how to relate this to the Theorem 4.6, and am not really convinced this works.

Exercise 4.6 - New

We consider a sub-matrix A' of the basis transformation matrix A with elements $A_{ij} = \alpha_{j,p}(i)$ where the sub-matrix has i, j limited to $i = p+2, ...n+p+1$ and $j = p+2, ..., m+p+1$, Before we have argued that for the function $f \in \mathbb{S}_{p,\tau}$ and $g \in \mathbb{S}_{p,t}$, where t, τ and t', τ' are as before and $f = g$ it must be the case that

$$f = \sum_j c_j B_{j,p,\tau'} = \sum_i d_i B_{i,p,t'}$$

where $c_j = 0$ for $j < p+2$ and $j > n+p+1$ and $d_i = 0$ for $i < p+2$ and $i > m+p+1$. We know from equation 4.3 that:

$$\vec{d} = A\vec{c}$$

where $\vec{d} = (d_i)_{i=1}^{2m+2p+1}$ and $\vec{c} = (c_i)_{i=1}^{2n+2p+1}$. Now since for certain i, j these elements are zero in the corresponding vectors:

$$\begin{aligned} d_i &= \sum_j A_{ij} c_j = 0, \quad \text{for } j < p+1,\ j > n+p+1, \text{ and } i < p+1,\ i > m+p+1 \\ &= \sum_j A'_{ij} c_j, \end{aligned}$$

and we see that only considering the sub-matrix A' is equivalent due to the coefficients being zero. Since Theorem 4.6 must hold for the elements in the entire matirx, it must also hold for elements in the sub-matrix, with t' and τ'. Moreover,

$$f = \sum_j c'_j B_{j,p,\tau} = \sum_i d'_i B_{i,p,t}$$

for the original knot vectors, implying that

$$\begin{aligned} d'_i &= d_{i+p+1} \\ c'_j &= c_{j+p+1} \end{aligned}$$

and the theorem therefore holds (by shifting indices) for the orignal knot vectors as well, as long as we consider the sub-matrix.

Exercise 4.7

We want to show that if τ and t are $p+1$ regular knot vectors, and t is a refinement of τ whose knots agree at the ends, then $\sum_j \alpha_{j,p}(i) = 1$. Our hint is that $\sum_j B_{j,p} = 1$ and to look at the properties of discrete B-splines.

For the first proof, consider Lemma 4.5.1 and Theorem 4.6 (4.8). Given any $p, m \geq 1$, for each $i = 1, ..m$, for some μ such that $\tau_\mu \leq t_i < \tau_{\mu+1}$ where τ has $n+p+1$ elements and t has $m+p+1$ elements, we have from Lemma 4.5:

$$\alpha_{j,p}(i) = 0 \text{ for } j < \mu - p \text{ and } j > \mu.$$

Moreover, from Theorem 4.6, we have that:

$$[\alpha_{\mu-p,p}(i), ..., \alpha_{\mu,p}(i)] = \boldsymbol{\alpha}_p(i)^T = \begin{cases} 1 & p = 0 \\ \boldsymbol{R}_1(t_{i+1}) \cdot ... \cdot \boldsymbol{R}_p(t_{i+p}) & \text{otherwise} \end{cases}$$

In trying to prove $\sum_j \alpha_{j,p}(i) = 1$ we thus obtain that:

$$\sum_{j=1}^{n} \alpha_{j,p}(i) = \sum_{j=\mu-p}^{\mu} \alpha_{j,p}(i) = \sum_k (\boldsymbol{\alpha}_p(i))_k.$$

Consider the case $p = 1$:

$$\begin{aligned} \boldsymbol{\alpha}_1(i)^T &= \boldsymbol{R}_1(t_{i+1}), \\ &= \begin{bmatrix} \frac{\tau_{\mu+1} - t_{i+1}}{\tau_{\mu+1} - \tau_\mu} & \frac{t_{i+1} - \tau_\mu}{\tau_{\mu+1} - \tau_\mu} \end{bmatrix}, \\ &= \left[\alpha_{\mu-1,1}(i), \alpha_{\mu,1}(i).\right]. \end{aligned}$$

Hence:

$$\sum_{j=\mu-1}^{\mu} \alpha_{j,1}(i) = \frac{\tau_{\mu+1} - t_{i+1}}{\tau_{\mu+1} - \tau_\mu} + \frac{t_{i+1} - \tau_\mu}{\tau_{\mu+1} - \tau_\mu} = \frac{\tau_{\mu+1} - \tau_\mu}{\tau_{\mu+1} - \tau_\mu} = 1 \quad \forall i, \forall \mu.$$

Assume it now holds for $p-1$, $\forall i, \forall \mu, m \geq 1, n \geq 1$. Then:

$$\begin{aligned} \alpha_p(i)^T =& R_1(t_{i+1}) \cdot ... \cdot R_p(t_{i+p}) \\ &= \alpha_{p-1}(i)^T R_p(t_{i+p}) \\ &= \left[\sum_k (\alpha_{p-1}(i))_k \cdot (R_p(t_{i+p}))_{k,1} \quad \dots \quad \sum_k (\alpha_{p-1}(i))_k \cdot (R_p(t_{i+p}))_{k,p}\right], \end{aligned}$$

taking the sum and rearranging we obtain:

$$\sum_l (\alpha_p(i))_l = \sum_k \left((\alpha_{p-1})_k \left(\sum_{j=1}^{p} R_p(t_{i+p})_{k,j} \right) \right) = \sum_k (\alpha_{p-1})_k(1)) = 1,$$

where we have used that each row of the B-spline matrix sums to 1, which follows immediately from its definition. We have thus shown that it holds by induction on p.

For the second proof, let t and τ be $p+1$ regular knot vectors, where t is a refinement of τ such that τ is a subsequence of t. Uusing Corollary 3.5, we have that for $x \in [\tau_{p+1}, \tau_{n+1})$:

$$1 = \sum_{j=1}^{n} B_{j,p,\tau},$$

and simultaneously for $x \in [t_{p+1}, t_{m+1})$ we have that

$$1 = \sum_{i=1}^{m} B_{i,p,t}.$$

Let f be the function where $f(x) = 1$ for $x \in [\tau_{p+1}, \tau_{n+1})$. Clearly, it's coefficients in $\mathbb{S}_{p,\tau}$ must then be $c_j = 1$ for $j = 1, ..., n$. If also $f(x) = 1$ for $x \in [t_{p+1}, t_{m+1})$ then by Lemma 4.2 since $\mathbb{S}_{p,\tau} \subseteq \mathbb{S}_{p,t}$ its coefficients in $\mathbb{S}_{p,t}$ must be $b_i = 1, ..., m$. Then by Equation 4.3, we must have:

$$\begin{aligned} b_i &= \sum_{j=1}^{n} \alpha_{j,p}(i) c_j \\ 1 &= \sum_{j=1}^{n} \alpha_{j,p}(i) \text{ for } i = 1, ..., n, \end{aligned}$$

and we are done. I'm not sure whether or not I like first proof better, since there I don't have to make any assumptions on f to arrive at the same conclusion (given that the proof works of course).

Exercise 4.8

We wish to implement algorithm 4.10, and verify graphically that the control polygon converges to the spline as more and more knots are inserted. We are asked to do this without using the spline matrices explicitly. The calculations needed for $p > 0$ are:

$$b_i = \mathbf{R_1}(t_{i+1})...\mathbf{R_p}(t_{i+p})\mathbf{c_p} = \boldsymbol{\alpha_p}(i)^T\mathbf{c_p},$$

where $\mathbf{c_p} = (c_{\mu-p}, ..., c_\mu)$ and $\tau_\mu \leq t_i < \tau_{\mu+1}$. We can do this by calculating the $\boldsymbol{\alpha_p}(i)$ using recurrence relation 4.10. However, if we examine algorithm 2.20 which is:

$$\mathbf{c'_{p-k+1}} = \mathbf{R_k}(x)\mathbf{c'_{p-k}} \quad k = p, p-1, ..., 1$$

where $\mathbf{c'_0} = (c_{\mu-p}, ..., c_\mu)$, so it is quite similar. The starting vector here is the starting vector in algorithm 4.10. So if we allow $x_k = t_{i+k}$ then:

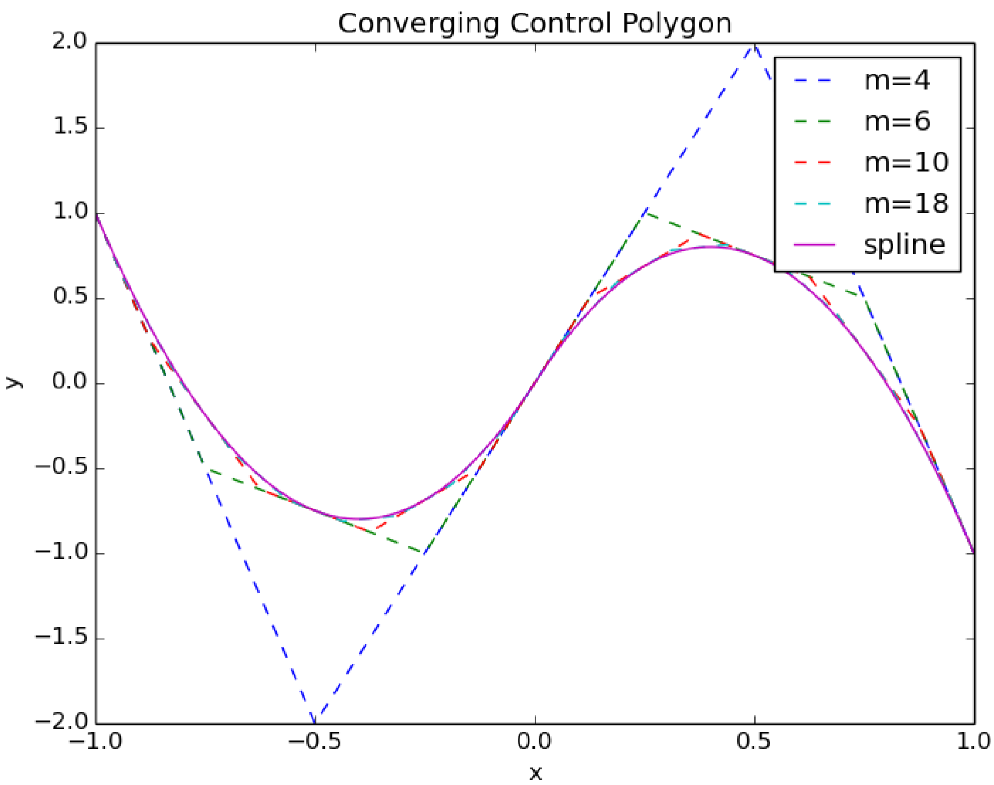
$$\begin{aligned}
&\mathbf{b_i} = \mathbf{c'_p} \\
&\mathbf{c'_{p-k+1}} = \mathbf{R_k}(x_k)\mathbf{c'_{p-k}} \quad k = p, p-1, ..., 1 \\
&\mathbf{c'_0} = \mathbf{c_p},
\end{aligned}$$

and we can obtain a recurrence relation for b_i by modifying 2.25:

$$(\mathbf{R_k}(x_k)\mathbf{c'_{p-k}})_j = \frac{\tau_{j+k} - x_k}{\tau_{j+k} - \tau_j}\mathbf{c'_{p-k}}_{j-1} + \frac{x_k - \tau_j}{\tau_{j+k} - \tau_j}\mathbf{c_{p-k}}_j, \quad j = \mu - k + 1, ..., \mu$$

Hence we run a modified algorithm 2.20 to obtain b_i for $i = 1, ..., m$. Once this is done, we can use the original algorithm 2.20 to evalute the spline in its new basis (or the old, they are the same), and calculate the knot averages to obtain the control polygon. The result is found in the figure below.

Figure 1: Example 4.7 for m coefficients, with uniform refinement on internal knots.



In addition I computed a numerical approximation to the Hausdorff distance between the 4 control polygons and the spline. The Hausdorff distance is given by:

$$d_H(X,Y) = \max(\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)),$$

where for this problem I chose d as the euclidean distance. In this finite case, the sup and inf become max and min, over all the possible pairwise distances of X, a partition from $\tau_1 = -1$ to $\tau_{n+p+1} = 1$ with 1000 points, to Y, evaluations of linear interpolations of the control polygons over X. The reason I did this is because it was suggested to me that the Hausdorff metric is the one that is proven convergence in, back in oblig 1. Letting Y_m denote the collection of points for the linear interpolant corresponding to m coefficients, then approximately:

$$\begin{aligned}
d_H(X, Y_4) &= 1.20,\\
d_H(X, Y_6) &= 0.233,\\
d_H(X, Y_{10}) &= 0.0764,\\
d_H(X, Y_{18}) &= 0.0184.
\end{aligned}$$

So both visually and w.r.t to the Hausdorff metric we appear to achieve numerical convergence.

4.10

We wish to find the cubic blossom $\mathcal{B}[g](x_1,x_2,x_3)$ of a function g.

Let $g(x) = x^3$.

$$\begin{aligned}
\mathcal{B}[x^3](x_1,x_2,x_3) &= s_3(x_1,x_2,x_3)\\
&= \left(\sum_{1\le i_1<i_2<i_3\le 3} x_{i_1}x_{i_2}x_{i_3}\right)\frac{3!(3-3)!}{3!}\\
&= x_1x_2x_3.
\end{aligned}$$

Let $g(x) = 1$. It is tempting to write $1 = x^0$ and use the elementary symmetric polynomials. However, if $x = 0$ then we have the indeterminate form 0^0, so we need to be careful and use definition 4.12. We make the anzats that the blossom $\mathcal{B}[1](x_1,x_2,x_3) = 1$. If this satisfies the properties of the definition, then since the blossom of a polynomial is unique, it must be the answer we are looking for. We immediately satisfy symmetry and diagonality. Moreover,

$$1(..., \alpha x + \beta y, ...) = 1 = (\alpha+\beta)1, \quad \text{for } \alpha+\beta = 1,$$

so it also satisfies affinity. Thus $\mathcal{B}[1](x_1,x_2,x_3) = 1$. We also see that this agrees with using $1 = x^0$ since the first (zero-order) elementary symmetric polynomial of any number of variables is always 1.

Let $g(x) = 2x + x^2 - 4x^3$. The blossom is linear in its polynomial argument so:

$$\begin{aligned}
\mathcal{B}[2x + x^2 - 4x^3] &= 2\mathcal{B}[x] + 1\mathcal{B}[x^2 - 4x^3],\\
&= 2\mathcal{B}[x] + \mathcal{B}[x^2] - 4\mathcal{B}[x^3].
\end{aligned}$$

We compute each of the terms in the sum using the elementary symmetric polynomials:

$$\begin{aligned}
&\mathcal{B}[x](x_1,x_2,x_3) = s_1(x_1,x_2,x_3) = (x_1+x_2+x_3)/3,\\
&\mathcal{B}[x^2](x_1,x_2,x_3) = s_2(x_1,x_2,x_3) = \left(\sum_{1\le i_1<i_2\le 3} x_{i_1}x_{i_2}\right)\frac{2!(3-2)!}{3!} = (x_1x_2 + x_1x_3 + x_2x_3)/3,\\
&B[x^3](x_1,x_2,x_3) = s_3(x_1,x_2,x_3) = x_1x_2x_3.
\end{aligned}$$

Thus we obtain:

$$\mathcal{B}[2x + x^2 - 4x^3] = \frac{2}{3}(x_1+x_2+x_3) + \frac{1}{3}(x_1x_2 + x_1x_3 + x_2x_3) - 4x_1x_2x_3$$

Let $g(x) = 0$. We once again make use of the linearity in the polynomial argument:

$$\mathcal{B}[0](x_1, x_2, x_3) = \mathcal{B}[1-1](x_1, x_2, x_3) = \mathcal{B}[1](x_1, x_2, x_3) - \mathcal{B}[1](x_1, x_2, x_3) = 0.$$

Let $g(x) = (x-a)^2$ where $a \in \mathbb{R}$.

$$\begin{aligned}
\mathcal{B}[(x-a)^2] &= \mathcal{B}[x^2 - 2ax + a^2] \\
&= \mathcal{B}[x^2] + \mathcal{B}[-2ax] + \mathcal{B}[a^2] \\
&= \mathcal{B}[x^2] - 2a\mathcal{B}[x] + a^2\mathcal{B}[1] \\
&= \frac{1}{3}(x_1x_2 + x_1x_3 + x_2x_3) - \frac{2a}{3}(x_1 + x_2 + x_3) + a^2
\end{aligned}$$