On disconnected cuts and separators ⋆

Takehiro Ito[1], Marcin Kamiński[2], Daniël Paulusma[3]⋆⋆
and Dimitrios M. Thilikos[4]⋆⋆⋆

[1] Graduate School of Information Sciences, Tohoku University,
Aoba-yama 6-6-05, Sendai, 980-8579, Japan
takehiro@ecei.tohoku.ac.jp

[2] Computer Science Department, Université Libre de Bruxelles,
Boulevard du Triomphe CP212, B-1050 Brussels, Belgium
marcin.kaminski@ulb.ac.be

[3] Department of Computer Science, University of Durham,
Science Laboratories, South Road, Durham DH1 3LE, England
daniel.paulusma@durham.ac.uk

[4] Department of Mathematics, National and Kapodistrian University of Athens,
Panepistimioupolis, GR15784 Athens, Greece
sedthilk@math.uoa.gr

Abstract. For a connected graph $G = (V, E)$, a subset $U \subseteq V$ is called a disconnected cut if U disconnects the graph and the subgraph induced by U is disconnected as well. A natural condition is to impose that for any $u \in U$ the subgraph induced by $(V \setminus U) \cup \{u\}$ is connected. In that case U is called a minimal disconnected cut. We show that the problem of testing whether a graph has a minimal disconnected cut is NP-complete. We also show that the problem of testing whether a graph has a disconnected cut separating two specified vertices s and t is NP-complete.

Keywords. cut set; $2K_2$-partition; retraction; compaction.

1 Introduction

Graph connectivity is a fundamental graph-theoretic property that is well-studied in the context of network robustness. In the literature several measures for graph connectivity are known, such as requiring hamiltonicity, edge-disjoint spanning trees, or edge- or vertex-cuts of sufficiently large size.

Let $G = (V, E)$ be a connected simple graph. For a subset $U \subseteq V$, we denote by $G[U]$ the subgraph of G induced by U. We say that U is a *cut* of G if U disconnects G, that is, $G[V \setminus U]$ contains at least two (connected) components. A cut U is *connected* if $G[U]$ contains exactly one component, and *disconnected* if $G[U]$ contains at least two components. We observe that $G[U]$ is a disconnected cut if and only if $G[V \setminus U]$ is a disconnected cut.

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In our paper [6] we studied the following three problems. The Disconnected Cut problem is to test whether a connected graph has a disconnected cut. For a fixed integer k, the k-Cut problem is to test whether a connected graph $G = (V, E)$ has a cut U such that $G[U]$ contains exactly k components. For fixed integers k, ℓ, the (k, ℓ)-Cut problem is to test whether a connected graph $G = (V, E)$ has a cut U such that $G[U]$ and $G[V \backslash U]$ contain exactly k and ℓ components, respectively. We showed that the k-Cut problem is polynomial-time solvable if $k = 1$, and NP-complete if $k \geq 2$. We also showed that the (k, ℓ)-Cut problem is polynomial-time solvable if $k = 1$ or $\ell = 1$, and NP-complete otherwise.

The complexity of the Disconnected Cut problem is still open for general graphs, but we showed that the problem can be solved in polynomial time for planar graphs, claw-free graphs and chordal graphs [6]. In addition, Fleischner et al. [5] showed that Disconnected Cut is polynomial-time solvable for triangle-free graphs, graphs with bounded maximum degree, graphs with a dominating edge (including co-graphs) and graphs that are not locally connected. In particular, they show that every graph of diameter at least three has a disconnected cut.

The Disconnected Cut problem is equivalent to several other problems posed in the literature. A graph G has a disconnected cut if and only if G allows a vertex-surjective homomorphism to the reflexive 4-vertex cycle. Furthermore, if G has diameter two, then G has a disconnected cut if and only if G allows a compaction to the reflexive 4-vertex cycle if and only if G can be contracted to some biclique. We refer to our paper [6] for more details. Here, we also mention that a graph $G = (V, E)$ has a disconnected cut if and only if its complement $\overline{G} = (V, \{uv \mid uv \notin E\})$ has a spanning subgraph that consists of two bicliques [5].

The Disconnected Cut problem is also studied in the context of H-partitions as introduced by Dantas et al. [1]. A *model graph* H with $V_H = \{h_1, \ldots, h_k\}$ has two types of edges: solid and dotted edges, and an *H-partition* of a graph G is a partition of V_G into k (nonempty) sets $V_1, \ldots, V_k$ such that for all vertices $u \in V_i$, $v \in V_j$ and for all $1 \leq i < j \leq k$ the following two conditions hold. Firstly, if h_ih_j is a solid edge of H, then $uv \in E_G$. Secondly, if h_ih_j is a dotted edge of H, then $uv \notin E_G$. There are no such restrictions when h_i and h_j are not adjacent. Let $2K_2$ be the model graph with vertices $h_1, \ldots, h_4$ and two solid edges h_1h_3, h_2h_4, and $2S_2$ be the model graph with vertices $h_1, \ldots, h_4$ and two dotted edges h_1h_3, h_2h_4. Then a graph G has a disconnected cut if and only if G has a $2S_2$-partition if and only if its complement $\overline{G}$ has a $2K_2$-partition. The (equivalent) cases $H = 2K_2$ and $H = 2S_2$ are the only two cases of model graphs on at most four vertices whose computational complexity is still open. Especially, $2K_2$-partitions have been well studied, see e.g. two very recent papers of Dantas, Maffray and Silva [2] and Teixeira, Dantas and de Figueiredo [7]. The first paper [2] studies the $2K_2$-Partition problem for several graph classes and the second paper [7] defines a new class of problems called $2K_2$-hard.

In this manuscript, we study three natural variants of the Disconnected Cut problem in order to increase our understanding of this problem. Our study is also motivated by the following example. Let P_n denote the path on n vertices. We observe that $P_4 = p_1p_2p_3p_4$ has a disconnected cut $\{p_1, p_3\}$ and a disconnected cut $\{p_2, p_4\}$. We observe that both these cuts contain a vertex, namely p_1 and p_4, respectively, such that moving this vertex from the cut back into the graph keeps the graph disconnected. As such, the property of the cut being disconnected can be viewed to be somewhat artificial in this case. Therefore, we can define the following problem, where we call a disconnected cut U of a connected graph $G = (V, E)$ *minimal* if $G[(V \backslash U) \cup \{u\}]$ is connected for every $u \in U$.

Minimal Disconnected Cut
Instance: a connected graph G
Question: does G have a minimal disconnected cut U?

We can relax the minimality by defining a disconnected cut U of a connected graph $G = (V, E)$ to be *semi-minimal* if $G[(V \backslash U) \cup \{u\}]$ contains fewer components than $G[V \backslash U]$ for every $u \in U$. This leads to the problem:

Semi-Minimal Disconnected Cut
Instance: a connected graph G
Question: does G have a semi-minimal disconnected cut U?

We note that any minimal disconnected cut is semi-minimal. However, the reverse is not true; to illustrate the differences between these two problems and the Disconnected Cut problem we observe the following:

(i) The path P_k has a disconnected cut if and only if $k \geq 4$.
(ii) The path P_k has a semi-minimal disconnected cut if and only if $k \geq 5$.
(iii) The path P_k does not have a minimal disconnected cut for any $k \geq 1$.

Because a minimal disconnected cut of a graph G does not contain a cut vertex of G, we can generalize (iii) to the following statement: every connected graph that contains a cut-vertex in all its cuts has no minimal disconnected cut. We will show that the Minimal Cut and Semi-Minimal Cut problem are NP-complete.

An *s-t separator* of a connected graph G with two specified vertices s and t is a cut U such that s and t belong to two different components of $G[V \backslash U]$. We say that an s-t separator U is *disconnected* if U is a disconnected cut.

Disconnected Separator
Instance: a graph $G = (V, E)$ and two vertices $s, t \in V$
Question: does G have a disconnected s-t separator U?

We will prove that the Disconnected Separator problem is NP-complete.

2 Preliminaries

The graphs that we consider are undirected and without multiple edges. We assume that they may contain self-loops. For undefined (standard) graph terminology we refer to [3].

Let $G=(V,E)$ be a graph. Each maximal connected subgraph of G is called a *component* of G. For a vertex $u \in V$, we denote its neighborhood, *i.e.*, the set of its adjacent vertices, by $N(u) = \{v \mid uv \in E\}$. Two disjoint nonempty subsets $U, U' \subset V$ are *adjacent* if there exist vertices $u \in U$ and $u' \in U'$ with $uu' \in E$. The *distance* $d_G(u,v)$ between two vertices u and v in a graph G is the number of edges in a shortest path between them. The *diameter* $\mathrm{diam}(G)$ is defined as $\max\{d_G(u,v) \mid u,v \in V\}$. We say that $S \subset V$ is *separated* from $T \subset V$ by $W \subset V \setminus (S \cup T)$ if every path that starts in a vertex of S and that ends in a vertex of T uses at least one vertex from W.

Let U be a cut of a graph G. If $G[(V\backslash U) \cup \{u\}]$ is connected we say that u is a *minimal* vertex of U. If $G[(V\backslash U) \cup \{u\}]$ contains fewer components than $G[V\backslash U]$ we say that u is a *semi-minimal* vertex of U.

A graph is *reflexive* if it has a self-loop in every vertex. We denote the reflexive n-vertex cycle by $\mathcal{C}_n$. A graph with no self-loops is called *irreflexive*.

Let $f : V_G \to V_H$ be a *(graph) homomorphism* from a graph G to a graph H, i.e., $f(u)f(v) \in E_H$ whenever $uv \in E_G$. We say that f is *vertex-surjective* if $f(V_G) = V_H$. Here we used the shorthand notation $f(S) = \{f(u) \mid u \in S\}$ for a subset $S \subseteq V$. We say that f is a *compaction* if f is *edge-surjective*, *i.e.*, for every edge $xy \in E_H$ with $x \neq y$ there exist two adjacent vertices u, v with $f(u) = x$ and $f(v) = y$. We stress that the surjectivity condition only holds for edges $xy \in E_H$; there is no such condition on the self-loops $xx \in E_H$. If f is a compaction from G to H, we also say that G *compacts* to H.

Let H be an induced subgraph of a graph G. A homomorphism f from a graph G to H is a *retraction* from G to H if $f(h) = h$ for all $h \in V_H$. In that case we say that G *retracts to* H.

The H-Compaction problem asks if a graph G compacts to a fixed graph H, i.e., H is not part of the input. The H-Retraction problems asks if a graph G retracts to a fixed graph H. The following two results proven by Feder and Hell [4] and Vikas [8], respectively, are of importance to us.

Theorem 1 ([4]). *The $\mathcal{C}_4$-Retraction problem is NP-complete.*

Theorem 2 ([8]). *The $\mathcal{C}_4$-Compaction problem is NP-complete.*

3 Gadgets

In the remainder of this paper, the graph H denotes the reflexive 4-vertex cycle $h_0h_1h_2h_3h_0$ with self-loops h_ih_i for $i = 1, \ldots, 4$, and the graph $G = (V,E)$ denotes a graph that contains H as an induced subgraph.

For each vertex $v \in V_G \backslash V_H$ we add three new vertices u_v, w_v, y_v with edges $h_0u_v, h_0y_v, h_1u_v,\ h_2w_v, h_2y_v, h_3w_v, u_vv, u_vw_v, u_vy_v,\ vw_v, w_vy_v$. We also add all edges between any two vertices $u_v, u_{v'}$ and between any two vertices $w_v, w_{v'}$ with $v \neq v'$. For each edge vv' in $E_G \backslash E_H$ we choose one arbitrary direction, say from v to v', and then add a new vertex $x_{vv'}$ with edges $vx_{vv'}, v'x_{vv'}, u_vx_{vv'}, w_{v'}x_{vv'}$. We call the new graph G' obtained from G an *H-compactor* of G. See Figure 1 for an example. This figure does not depict any self-loops, although formally G

must have at least four self-loops, because G contains H as an induced subgraph. However, this is irrelevant for our problems, and we may just as well assume that G is irreflexive.

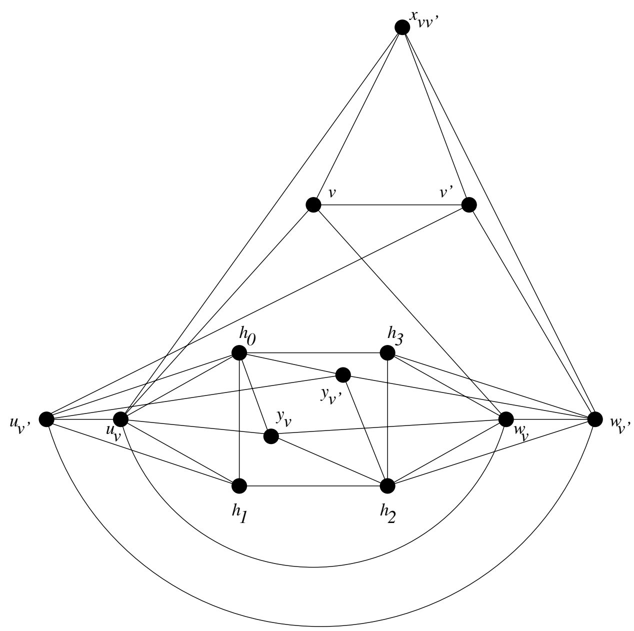

Fig. 1. The part of G' that corresponds to edge $vv' \in E_G \setminus E_H$ as displayed in [8].

Vikas [8] proves Theorem 2 by a reduction from H-Retraction, which is NP-complete by Theorem 1. In his proof he shows the following result, which we will use as well.

Lemma 1 ([8]). *Let G' be an H-compactor of a graph G that has H as an induced subgraph. Then the following statements are equivalent:*

(i) G retracts to H;
(ii) G' retracts to H;
(iii) G' compacts to H.

Below we explore the properties of a retraction f from an H-compactor G' to H. We call a subgraph of G', every vertex of which is mapped to the same vertex h_i by f *monochromatic*.

Lemma 2. *Let G' be an H-compactor of a graph G that has H as an induced subgraph. Any retraction f from G' to H satisfies:*

(i) for $i = 0, \ldots, 3$, the subgraph G'_i induced by $\{u \in V_{G'} \mid f(u) = h_i\}$ is connected;

(ii) for $i = 0, \ldots, 3$, each vertex u with $f(u) = h_i$ has a neighbor v with $f(v) = h_j$ for some $j \neq i$.

Proof. Let G' be an H-compactor of a graph G with H as an induced subgraph. Let f be a retraction from G' to H. We prove that (i) and (ii) hold.

Proof of (i). By definition, $f(h_i) = h_i$ for $i = 0, \ldots, 3$. This means that f maps u_v-vertices to h_0 and h_1, and w_v-vertices to h_2 and h_3. It also means that f maps y_v-vertices to h_1 or h_3.

We first prove the following claim.

Claim 1. For every $v \in V_G \setminus V_H$, if $f(v) \in \{h_0, h_1\}$ then $f(u_v) = f(v)$, and if $f(v) \in \{h_2, h_3\}$ then $f(w_v) = f(v)$.

We prove Claim 1 as follows. Suppose $f(v) \in \{h_0, h_1\}$ and $f(u_v) \neq f(v)$. Recall that $f(u_v) \in \{h_0, h_1\}$ and $f(w_v) \in \{h_2, h_3\}$. Then f maps u_v, v, w_v to three different vertices of H. This is not possible, because u_v, v, w_v form a triangle in G'. By the same argument we can show that $f(w_v) = f(v)$ if $f(v) \in \{h_2, h_3\}$. This proves Claim 1.

We now show that G'_0 is connected. Let V_0 denote the vertex set of G'_0. Let $z \neq h_0$ be a vertex in V_0, so $f(z) = h_0$. We show that z is in the same component of G'_0 as h_0. This means that G'_0 is connected as desired.

Suppose z is a u_v-vertex. Then z is adjacent to h_0. Note that z is neither a w_v-vertex nor a y_v-vertex, because such a vertex is mapped to a vertex in $\{h_2, h_3\}$ or $\{h_1, h_3\}$, respectively. Suppose $z = v$ for some $v \in V_G \backslash V_H$. By Claim 1, we find that $f(u_v) = f(v) = h_0$. Then v is in the same component of $G[V_0]$ as h_0 due to the monochromatic path vu_vh_0.

Finally suppose $z = x_{vv'}$ for two adjacent vertices $v, v' \in V_G \backslash V_H$. If $f(u_v) = h_0$, then $x_{vv'}$ is connected to h_0 in $G[V_0]$ due to the path $x_{vv'}u_vh_0$. If $f(u_v) \neq h_0$ then $f(u_v) = h_1$. Because v is adjacent to $x_{vv'}$ with $f(x_{vv'}) = h_0$ and to u_v with $f(u_v) = h_1$, we obtain $f(v) \in \{h_0, h_1\}$. Then by Claim 1, $f(v) = f(u_v) = h_1$. Because $f(x_{vv'}) = h_0$ and $f(w_{v'}) \in \{h_2, h_3\}$, we find that $f(w_{v'}) = h_3$. Then v' is adjacent to three vertices, namely $x_{vv'}, v, w_{v'}$, that are mapped to h_0, h_1, h_3, respectively. This means that $f(v') = h_0$. Consequently, $f(u_{v'}) = f(v') = h_0$ by Claim 1. Hence, $x_{vv'}$ is in the same component of $G[V_0]$ as h_0 due to the monochromatic path $x_{vv'}v'u_{v'}h_0$.

From the above we conclude that G'_0 is connected. By symmetry, we find that G'_2 is connected as well. We now show that G'_1 is connected.

Let $z \neq h_0$ be a vertex in V_1, so $f(z) = h_1$. We show that z is in the same component of G'_1 as h_1 by the same arguments as we used for $i = 0$; the only difference is the argument for the case in which z is a y_v-vertex. In that case z is connected to h_1 by the edge h_1z. Hence, we conclude that G'_1 is connected. By symmetry, we find that G'_3 is connected as well. Consequently, we have shown (i).

Proof of (ii). Let z be a vertex in G'. Suppose $f(z) = h_0$. Then z is neither a w_v-vertex nor a y_v-vertex, and z is not in $\{h_1, h_2, h_3\}$ either, because f does not map such vertices to h_0. If z is h_0 or a u_v-vertex, then z is adjacent to h_1 with $f(z) = h_1$. Otherwise, $z \in V_G \backslash V_H$ or $z = x_{vv'}$ for some $vv' \in E_G \backslash E_H$. In both cases, z is adjacent to a w_v-vertex, which f maps to h_2 or h_3. The case $f(z) = h_2$ follows by symmetry.

Suppose $f(z) = h_1$. We can use the same arguments as in the previous case; the only difference is when z is a y_v-vertex. In that case z is adjacent to h_0 with $f(h_0) = h_0$. The case $f(z) = h_3$ follows by symmetry. Consequently, we have shown (ii). This completes the proof of Lemma 2. □

The following lemma will be used later on as well, in order to strengthen our NP-hardness results. We note that it also strengthens Theorem 2, i.e., the H-Compaction problem is NP-complete, even for graphs of diameter 3.

Lemma 3. *Let G be a graph that has H as induced subgraph. The H-compactor of G has diameter three.*

Proof. Let G' be the H-compactor of G that has H as an induced subgraph. We choose that G' has diameter 3 by a straightforward case analysis.

Consider a vertex $h_i \in V_H$. By symmetry, we may assume $i \in \{0, 1\}$. As H is isomorphic to C_4, we have $d(h_i, h_j) \leq 2$ for all $h_j \in H \backslash \{h_i\}$. Suppose $v \in V_G \backslash V_H$. Then $d(h_i, v) \leq 2$ and $d(h_i, u_v) = 1$ due to the path $h_i u_v v$. We also deduce $d(h_i, w_v) = 2$ due to the path $h_i u_v w_v$, and $d(h_i, y_v) \leq 2$ due to the path $h_i y_v$ if $i = 0$ or $h_i h_{i-1} y_v$ if $i = 1$. Furthermore, $d(h_i, x_{v'v''}) = 2$ holds for any $v'v'' \in E_G \backslash E_H$ due to the path $h_i u_{v'} x_{v'v''}$.

Consider a vertex $v \in V_G \backslash V_H$. By construction, $d(v, u_v) = d(v, u_w) = 1$. We deduce $d(v, y_v) = 2$ due to the path $v u_v y_v$, and for all $vv' \in E_G \backslash E_H$ we have $d(v, x_{vv'}) = 1$ due to the edge $v x_{vv'}$. Suppose $v' \in V_G \backslash (V_H \cup \{v\})$. Then $d(v, v') \leq 3$ and $d(v, u_{v'}) = 2$ due to the path $v u_v u_{v'} v'$. Also, $d(v, w_{v'}) = 2$ due to the path $v w_v w_{v'}$, and $d(v, y_{v'}) \leq 3$ due to the path $v u_v u_{v'} y_{v'}$. Furthermore, $d(v, x_{v'v''}) \leq 3$ for all $v'v'' \in E_G \backslash E_H$ due to the path $v u_v u_{v'} x_{v'v''}$.

Consider a vertex u_v for some $v \in V_G \backslash V_H$. By construction, $d(u_v, w_v) = d(u_v, y_v) = 1$ and also $d(u_v, x_{vv'}) = 1$ for all $vv' \in E_G \backslash E_H$. Suppose $v' \in V_G \backslash (V_H \cup \{v\}$. Then $d(u_v u'_v) = 1$ by the edge $u_v u_{v'}$, and $d(u_v, w_{v'}) = 2$ due to the path $u_v u_{v'} w_{v'}$, and $d(u_v, y_{v'}) = 2$ due to the path $u_v u_{v'} y_{v'}$. Furthermore, $d(u_v, x_{v'v''}) = 2$ for all $v'v'' \in E_G \backslash E_H$ with $v' \neq v$ due to the path $u_v u'_v x_{v'v''}$.

Consider a vertex w_v for some $v \in V_G \backslash V_H$. By symmetry, we return to the previous case.

Consider a vertex y_v for some $v \in V_G \backslash V_H$. Then $d(y_v, x_{vv'}) \leq 2$ for all $vv' \in E_G \backslash E_H$ due to the path $y_v u_v x_{vv'}$. Suppose $v' \in V_G \backslash (V_H \cup \{v\})$. Then $d(y_v, y_{v'}) = 2$ due to the path $y_v h_0 y_{v'}$. Furthermore, $d(y_v, x_{v'v''}) \leq 3$ for all $v'v'' \in E_G \backslash E_H$ with $v' \neq v$ due to the path $y_v u_v u_{v'} x_{v'v''}$.

Consider a vertex $x_{vv'}$ for some $vv' \in E_G \backslash E_H$. Suppose $v''v^* \in E_G \backslash (E_H \cup \{vv'\})$. Then $d(x_{vv'}, x_{v''v^*}) \leq 3$ due to the path $x_{vv'} u_v u_{v''} x_{v''v^*}$ if $v \neq v''$; otherwise we can take the path $x_{vv'} u_v x_{v''v^*}$. This completes our case analysis, and we have proven Lemma 3. □

4 NP-completeness proofs

We first prove the following result on H-compactors.

Lemma 4. *Let G' be the H-compactor of a graph G that has H as an induced subgraph. Then the following three statements are equivalent.*

(i) G' compacts to H.
(ii) G' has a minimal disconnected cut.
(iii) G' has a semi-minimal disconnected cut.

Proof. Let G' be the H-compactor of a graph G that has H as an induced subgraph.

"$(i) \Rightarrow (ii)$" Suppose G' compacts to H. Then by Lemma 1 there exists a retraction f from G' to H. Then f partitions $V_{G'}$ into four classes $V_i = \{u \in V \mid f(u) = h_i\}$ for $i = 0, \ldots, 3$. By Lemma 2 (i), each V_i induces a connected subgraph of G'.

Consider V_0. We repeatedly perform the following operation as long as possible. Let $v \in V_0$. By Lemma 2 (ii), v has at least one neighbor in $V_1 \cup V_3$. If v is adjacent to a vertex in V_1 but not adjacent to any vertex in V_3, then put v in V_1. Similarly, if v is adjacent to a vertex in V_3 but not adjacent to any vertex in V_1, put v in V_3. Afterwards we end up with a subset $V_0' \subseteq V_0$ that only contains vertices that have a neighbor in both V_1 and V_3. We note that $h_0 \in V_0'$, because h_0 is in V_0, and h_0 is adjacent to $h_1 \in V_1$ and $h_3 \in V_3$. Hence, $V_0' \neq \emptyset$.

By the same arguments we modify V_2 into a nonempty set V_2' in which all vertices have a neighbor in V_1 and a neighbor in V_3. Note that the above operations do not introduce an edge between V_1 and V_3. They do not introduce an edge between V_0' and V_2' either. Furthermore, V_1 and V_3 still induce connected subgraphs of G'. Because every vertex in $V_0' \cup V_2'$ is adjacent to a vertex in V_1 and to a vertex in V_3, this means that $V_0' \cup V_2'$ is a minimal disconnected cut of G'.

"$(ii) \Rightarrow (iii)$" This follows directly from the two definitions.

"$(iii) \Rightarrow (i)$" Suppose G' has a semi-minimal disconnected cut U. Let the components of $G'[U]$ be $A_1, \ldots, A_k$ for some $k \geq 2$. Let the components of $G'[V \backslash U]$ be $B_1, \ldots, B_\ell$ for some $\ell \geq 2$. Because U is semi-minimal, every vertex $u \in A_1$ has a neighbor in at least two components B_i and B_j for some $1 \leq i < j \leq \ell$. By the same reasoning, every vertex $v \in A_2$ has a neighbor in at least two components $B_{i'}$ and $B_{j'}$ for some $1 \leq i' < j' \leq \ell$. Because $i \neq j$ and $i' \neq j'$, we may assume without loss of generality that $i \neq j'$ and $i' \neq j$; otherwise we swap two indices.

We define the function f that maps each vertex in A_1 to h_0, each vertex in $A_2 \cup \ldots \cup A_k$ to h_2, each vertex in $B_i \cup B_{i'}$ to h_1, and each vertex in $B_j \cup B_{j'}$ to h_3. We let f map all remaining vertices of $V_{G'} \backslash U$ to h_3 as well. By our choice of indices i, i', j, j', we find that f is a compaction from G' to H. This finishes the proof of Lemma 4. $\square$

We are now able to show the first main result of this section.

Theorem 3. *The* Minimal Disconnected Cut *and the* Semi-Minimal Disconnected Cut *problem are* NP*-complete, even for the class of graphs of diameter three.*

Proof. Note that both problems are in NP. To prove NP-completeness, we use a reduction from the $\mathcal{C}_4$-Retraction problem, which is NP-complete by Theorem 1. Let G be a graph that has H as an induced subgraph. Let G' be an H-compactor of G. By Lemma 3, G' has diameter three. By Lemma 1 and Lemma 4 we find that G retracts to H if and only if G' compacts to H if and only if G' has a minimal disconnected cut if and only if G' has a semi-minimal disconnected cut. This proves Theorem 3. □

Here is our second main result.

Theorem 4. *The* Disconnected Separator *problem is* NP*-complete even for the class of graphs of diameter 3.*

Proof. Note that this problem is in NP. To prove NP-completeness, we use a reduction from the $\mathcal{C}_4$-Retraction problem, which is NP-complete by Theorem 1. Let G be a graph that has H as an induced subgraph. Let G' be an H-compactor of G. By Lemma 3, G' has diameter three. We claim that G retracts to H if and only if G' has a disconnected h_0-h_2 separator.

Suppose G retracts to H. By Lemma 1, there exists a retraction f from G' to H. Let $V_i = \{x \in V_{G'} \mid f(x) = h_i \text{ for } i = 0, \ldots, 3\}$. By definition, $h_0 \in V_0$ and $h_2 \in V_2$, and there are no edges between V_0 and V_2, and no edges between V_1 and V_3. Because $h_1 \in V_1$ and $h_3 \in V_3$ by definition, V_1 is nonempty and V_3 is nonempty. Hence $V_1 \cup V_3$ is a disconnected h_0-h_2 separator of G'.

In order to prove the reverse implication, suppose G' has a disconnected h_0-h_2 separator U. Let $A_1, \ldots, A_k$ be the vertex sets of the components of $G[U]$ and let $B_1, \ldots, B_\ell$ be the vertex sets of the components of $G[V \backslash U]$. As U is an h_0-h_2 separator, we may without loss of generality assume that $h_0 \in B_1$ and $h_2 \in B_2$. Because h_1 and h_3 are each adjacent to both h_0 and h_1, we find that h_1 and h_3 are in $V \setminus U$, say $h_1 \in A_1$ and $h_3 \in A_i$ for some $i \geq 1$; note that we must consider the case $h_3 \in A_1$ as a possibility.

Define $f : V_G \to V_H$ as follows. Let f map each vertex of B_1 to h_0, each vertex of $B_2 \cup \cdots \cup B_\ell$ to h_2, each vertex of A_1 to h_1 and each vertex of $A_2 \cup \cdots \cup A_k$ to h_3. We observe that f is a homomorphism to H with $f(h_i) = h_i$ for $0 \leq i \leq 2$. Because $A_2 \cup \cdots \cup A_k$ is nonempty, it contains a vertex z, which is mapped to h_3. If we can show that z is adjacent to a vertex mapped to h_0 and to a vertex mapped to h_2, then we find that f is a compaction from G' to H. Then, by Lemma 1, G retracts to H, and we are done. Below we consider each possibility.

We first note that z cannot be a u_v-vertex. The reason is that a u_v-vertex is mapped to a vertex in $\{h_0, h_1\}$, because it is adjacent to h_0 with $f(h_0) = h_0$ and to h_1 with $f(h_1) = h_1$.

Suppose $z = h_3$. Then z is adjacent to h_0, which is mapped to h_0, and to h_2, which is mapped to h_2, as desired. Suppose we cannot choose z to be h_3. Then $h_3 \in A_1$, and consequently, $f(h_3) = h_1$.

Because $f(h_3) = h_1$, we find that z cannot be a w_v-vertex. The reason is that a w_v-vertex is also adjacent to h_2 with $f(h_2) = h_2$. Hence, it must be mapped to a vertex in $\{h_1, h_2\}$.

Suppose z is a y_v-vertex. Then z is adjacent to both h_0 with $f(h_0) = h_0$ and h_2 with $f(h_2) = h_2$, as desired. Suppose this is not the case.

Suppose $z = v$ for some $v \in V_G \backslash V_H$. Recall that u_v is adjacent to h_0, h_1, v. Because $f(h_0) = h_0$ and $f(h_1) = 1$, we then find that $f(u_v) = h_0$. Recall that w_v is adjacent to h_2, h_3, u_v, v, which are mapped to h_2, h_1, h_0, h_3, respectively. This is not possible. Hence, z cannot be in $V_G \backslash V_H$.

Suppose $z = x_{vv'}$ for some $vv' \in E_G \setminus E_H$. Recall that $x_{vv'}$ is adjacent to u_v and $w_{v'}$. Because u_v is also adjacent to h_0 with $f(h_0) = h_0$ and to h_1 with $f(h_1) = 1$, we find that $f(u_v) = h_0$. Because $w_{v'}$ is also adjacent to h_2 with $f(h_2) = h_2$ and to h_3 with $f(h_3) = h_1$, we find that $f(w_{v'}) = h_2$. Hence, z is adjacent to a vertex that is mapped to h_0, namely u_v, and to a vertex that is mapped to h_2, namely $w_{v'}$, as desired. This completes our case analysis. Hence, we have proven Theorem 4. □

5 Further Work

The main open problem is to determine the computational complexity of the Disconnected Cut problem. Graphs with diameter at least three have a disconnected cut [5]. Graphs with diameter one are complete graphs and do not have a disconnected cut. Hence, we may restrict ourselves to graphs of diameter two. For this reason the following result are of interest. It shows that the four problems Disconnected Cut, Minimal Disconnected Cut, Semi-Minimal Disconnected Cut and $\mathcal{C}_4$-Compaction are polynomially equivalent to each other for graphs of diameter two.

Proposition 1. *Let G be a graph of diameter two. Then the following statements are equivalent.*

(i) G has a disconnected cut;
(ii) G has a minimal disconnected cut;
(iii) G has a semi-minimal disconnected cut.
(iv) G compacts to $\mathcal{C}_4$.

Proof. By definition, any minimal disconnected cut is a semi-minimal disconnected cut, and any semi-minimal disconnected cut is a disconnected cut. The equivalence "$(i) \Leftrightarrow (iv)$" is straightforward and has been shown in [5]. Hence, we only need to prove "$(i) \Rightarrow (ii)$".

Suppose $G = (V, E)$ has a disconnected cut U. As long as U stays a disconnected cut we move vertices from U to $V \backslash U$. Denote the resulting disconnected cut by U'. We claim that U' is minimal. Suppose not. Then U' contains a vertex u that is not minimal. Then $G[U']$ consists of a component A and a component $\{u\}$; if not we would have added u to $V \backslash U'$. As u is not minimal, there exists a component B of $G[V \backslash U']$ such that u is not adjacent to V_B. Let v be a vertex in B. Then a shortest path from v to u must use at least one vertex from A and some other component $B' \neq B$ of $G[V \setminus U']$. Hence $d_G(u,v) \geq 3$. This is not possible as $\mathrm{diam}(G) = 2$. $\square$

The following two questions are of interest as well.

1. What is the computational complexity of the Disconnected Separator problem for graphs of diameter two?
2. What is the computational complexity of the C_4-Retraction problem for graphs of diameter two?

Regarding question 2, recall that the C_4-Retraction problem is NP-complete by Theorem 1. Below we show that C_4-Retraction problem is NP-complete even for graphs of diameter three.

Proposition 2. *The C_4-Retraction problem is NP-complete even for graphs of diameter three.*

Proof. We reduce from $\mathcal{C}_4$-Retraction for general graphs. Let $G = (V, E)$ be a graph that has H as an induced subgraph. Let $V = \{v_1, \ldots, v_n\}$. For each pair of different vertices v_i, v_j we add a new vertex a_{ij} only adjacent to v_i and v_j. We add a vertex b and edges $a_{ij}b$ for all $1 \leq i < j \leq n$. We denote the resulting graph by $G^* = (V^*, E^*)$.

We show that G^* has diameter 3. Consider a vertex $v_i \in V$. Then v_i is of distance at most two from each vertex $v_j \in V$ due to the path $v_i a_{ij} v_j$. Furthermore, v_i is of distance at most three from each vertex a_{jk} due to the path $v_i a_{ij} b a_{jk}$. As b is on this path, v_i has distance two to b. A vertex a_{ij} is of distance one from b due to the edge $a_{ij}b$ and of distance two from a vertex $a_{k\ell}$ due to the path $a_{ij} b a_{k\ell}$. Hence, G^* has diameter 3 indeed. Below we prove that G retracts to H if and only if G^* retracts to H.

Suppose G retracts to H via f. Consider a vertex a_{ij}. Suppose $h_0 \in f(\{v_i, v_j\})$. If $f(\{v_i, v_j\})$ does not contain h_2, then we map a_{ij} to h_0. Otherwise we map a_{ij} to h_1. Suppose $h_0 \notin f(\{v_i, v_j\})$ and $h_1 \in f(\{v_i, v_j\})$. If $f(\{v_i, v_j\})$ does not contain h_3, then we map a_{ij} to h_1. Otherwise we map a_{ij} to h_2. Suppose $\{h_0, h_1\} \cap f(\{v_i, v_j\}) = \emptyset$. Then we map a_{ij} to h_2. Finally, we map b to h_1. This way we have extended f to a homomorphism f^* from G^* to H with $f^*(h_i) = f(h_i) = h_i$ for $i = 0, \ldots, 3$. Hence G^* retracts to H.

Suppose G^* retracts to H. Because G is a subgraph of G^* and H is a subgraph of G, we find that G retracts to H. This completes the proof of Proposition 2. $\square$

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