Lift-contractions*

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Abstract

We introduce and study a partial order on graphs – lift-contractions. A graph H is a lift-contraction of a graph G if H can be obtained from G by a sequence of edge lifts and edge contractions. We give sufficient conditions for a connected graph to contain every n-vertex graph as a lift-contraction and describe the structure of graphs with an excluded lift-contraction.

Keywords: edge contractions, edge lifts, immersions, treewidth.

1 Introduction

All graphs in this paper are undirected, loopless, and without multiple edges (unless mentioned otherwise). $V(G)$ and $E(G)$ denote the vertex and edge set of a graph G, respectively. The *degree* of a vertex $v \in V(G)$ is the number of edges incident with it. K_n is the complete graph on n vertices. Given an edge e of a graph G, the result of the *contraction* of e in G is the graph obtained by removing e from G and then identifying its endpoints to a single vertex v_e. For notions and notations not defined here, we refer the reader to the monograph [5].

Given two edges $e_1 = \{x, x_1\}$ and $e_2 = \{x, x_2\}$ of G, incident with the same vertex x, and such that $x_1 \neq x_2$, we define the *lift* of e_1 and e_2 in G as the graph obtained by removing e_1 and e_2 from G and then adding the edge $\{x_1, x_2\}$. If a contraction or lift creates multiple edges, we reduce their multiplicity to one and keep the graph simple.

Partial orders. The study of partial orders on graphs is one of the basic research avenues in graph theory. One of the most comprehensive studies of partial orders is

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the theory of Graph Minors by Robertson and Seymour [11] (see also the last chapter of [5]). A graph H is a *minor* of another graph G ($H \leq_{\mathrm{m}} G$) if H can be obtained from G by a sequence of vertex deletions, edge removals, and edge contractions. Some more restricted graph containment relations than graph minors, like contractions [3] or induced minors [9] have also been studied.

Graph immersions form another partial order that has been considered in the literature [4]. A graph H is an *immersion* of G if H can be obtained from G by a sequence of vertex deletions, edge removals, and lifts. The last operation was introduced by Lovász under the name of *splitting* as a reduction method to maintain edge connectivity [8].

In this paper, we introduce and study lift-contractions. We say that a graph H is a *lift-contraction* of a graph G if H can be obtained from G by a sequence of lifts and edge contractions. We also define lift-minors. We say that a graph H is a *lift-minor* of a graph G if H can be obtained from G by a sequence of vertex and edge deletions, lifts and contractions.

Being a lift-contraction (lift-minor) is a partial relation between graphs and we denote it by $H \leq_{\mathrm{lc}} G$ ($H \leq_{\mathrm{lm}} G$). If a graph H can be obtained from G by a sequence of contractions, we say that H is a contraction of G and we denote this by $H \leq_{\mathrm{c}} G$. Clearly, $H \leq_{\mathrm{c}} G \Rightarrow H \leq_{\mathrm{lc}} G \Rightarrow H \leq_{\mathrm{lm}} G$ and $H \leq_{\mathrm{m}} G \Rightarrow H \leq_{\mathrm{lm}} G$.

Forcing complete graphs. When studying a partial order $\leq$ on graphs, it is interesting to know under what conditions on G, for a fixed graph H, $H \leq G$. Kostochka [7] and Thomason [13] independently proved that if the average degree of G is at least $cn\sqrt{\log n}$, then G contains K_n as a minor (for some constant $c > 0$). Bollobás [2] showed that if the average degree of G is at least cn^2, then G contains K_n as a topological minor[1] (for some constant $c > 0$). Recently, DeVos et al. [4] proved that if the minimum degree of G is at least 200n, then G contains K_n as an immersion. For all these three partial orders, containing K_n implies containing any n-vertex graph.

In this paper, we identify three conditions on a connected graph G that force any n-vertex graph as a lift-contraction of G.

Theorem 1.1. *There exists a constant c such that every connected graph G of treewidth at least $c \cdot n^4$ contains every n-vertex graph as a lift-contraction.*

Theorem 1.2. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every 2-connected graph of pathwidth at least $f(n)$ contains every n-vertex graph as a lift-contraction.*

Theorem 1.3. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every connected graph with at least $f(n)$ vertices and minimum degree at least 3 contains every n-vertex graph of as a lift-contraction.*

We note that none of the three conditions above is alone enough to force all n-vertex graphs as a lift or as a contraction. In order to see this, consider a complete graph K with an arbitrarily large number of vertices. Because a lift does not change

[1] H is a *topological minor* of G, when some subdivision of H is a subgraph of G.

the number of vertices, we cannot obtain a graph with fewer vertices than K by taking lifts only. Because contracting an edge in K yields a new complete graph, we cannot obtain any non-complete graph by performing edge contractions only.

Structural theorem. Another point of focus, when studying partial orders on graphs, is to understand the structure of nontrivial ideals in this order. The best known example is the structural theorem on graphs with an excluded minor by Robertson and Seymour [11]. Recently, a structural description of graphs with an excluded topological minor was discovered by Grohe and Marx [6] and with an excluded immersion by Wollan [14].

Here we obtain, as a consequence of Theorem 1.3, a structural description of graphs with a forbidden lift-contraction. Informally, for a fixed graph H, any graph G that does not contain H as a lift-contraction contains a set of vertices R whose size depends only on the excluded graph H such that every connected component of $G[V \setminus R]$ is of treewidth at most 2 and has at most two neighbors in R. A simple corollary of our structural result is that graphs with an excluded lift-contraction are of bounded treewidth and thus of bounded chromatic number.

Paper structure. We start with some preliminary results in Section 2 which also includes the proofs of Theorems 1.1 and 1.2. The proof of Theorem 1.3 is presented in Section 3. In Section 4 we describe the structure of graphs with an excluded lift-minor.

2 Preliminary results

We prove below auxiliary results that will be useful later in the next sections, give some definitions and prove Theorems 1.1 and 1.2.

Lemma 2.1. *For every n-vertex graph H, $H \leq_{\mathrm{lc}} K_{2n}$.*

Proof. We prove that every n-vertex graph H is a lift-contraction of K_{2n}. Let $H^+ = K_2 \times H$. First we prove that H^+ is a lift of K_{2n}. Let $V(H) = \{v_1, \ldots, v_n\}$ and $V(H^+) = \{v'_1, \ldots, v'_n, v''_1, \ldots, v''_n\}$. Let us assume that $V(K_{2n}) = V(H^+)$ and observe that H^+ is a spanning subgraph of K_{2n}. Let R be the set of non-edges of H, i.e., $R = \{\{u, v\} \mid u, v \in V(H), u \neq v\} \setminus E(H)$. Notice that each $\{v_i, v_j\} \in R$ corresponds to the vertices $v'_i, v'_j, v''_i, v''_j \in V(H^+)$ such that the edges $\{v'_i, v''_j\}, \{v''_i, v'_j\}, \{v'_i, v'_j\}, \{v''_i, v''_j\}$ are present in K_{2n} but not in H^+. We use lifts to remove those edges. For every $\{v_i, v_j\} \in R$, we lift the pairs of edges $\{v'_i, v''_j\}, \{v''_j, v''_i\}$ and $\{v''_i, v'_j\}, \{v'_j, v'_i\}$. The result is H^+. Now we contract edges $\{v'_i, v''_i\}$ for all $i = 1, \ldots, n$ and obtain H as claimed. □

The following observation can be easily proved by induction on r.

Observation 2.2. *For every $r \geq 2$, the complete r-partite graph, where each of its parts has $r - 1$ vertices, has a perfect matching M such that for every two of its parts there is exactly one edge in M intersecting both of them.*

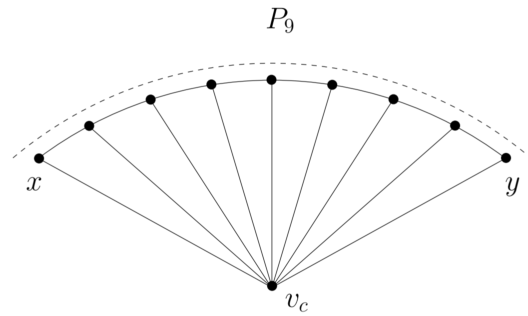

Figure 1: The graph F_9

For an integer $k > 1$, the *k-fan* is the graph obtained from the path P_k on k vertices by adding a dominating vertex v_c. We denote the k-fan by F_k and say that P_k is its *spine* and v_c is its *center* (see Figure 1). The *extreme vertices* of a k-fan are the endpoints of the path (i.e., the vertices x and y in Figure 1).

Lemma 2.3. *For any connected graph G and $n \geq 2$, if $F_{n(n-1)} \leq_{\mathrm{lm}} G$, then $K_n \leq_{\mathrm{lc}} G$.*

Proof. If $F_{n(n-1)} \leq_{\mathrm{lm}} G$, then it is possible to obtain $F_{n(n-1)}$ from G by a sequence of vertex deletions, edge removals, edge contractions and lifts. We modify this sequence as follows:

- a removal of an edge e such that e is a bridge in the already constructed graph is replaced by the contraction of e, all other edge removals are deleted from the sequence;
- a removal of a vertex v is replaced by the contraction of an edge incident with v;
- a lift operation for edges $\{u, v\}, \{v, w\}$ such that v has degree two in the already constructed graph is replaced by the contraction of $\{u, v\}$.

By the resulting sequence of contractions and lift operations, we obtain a graph $G' \leq_{\mathrm{lc}} G$ such that G' contains $F_{n(n-1)}$ as a spanning subgraph. Let the spine of this $n(n-1)$-fan in G' be a path P with $V_P = \{v_1^1, \dots, v_{n-1}^1, v_1^2, \dots, v_{n-1}^2, \dots, v_1^n, \dots, v_{n-1}^n\}$. Let J be an n-partite graph with V_P as its vertex set and let M be a perfect matching of J as in Observation 2.2. We choose an arbitrary edge $\{v_s^t, v_{s'}^{t'}\} \in M$. For each edge $\{v_i^j, v_{i'}^{j'}\} \in M$, where $\{v_s^t, v_{s'}^{t'}\} \neq \{v_i^j, v_{i'}^{j'}\}$, we lift the pair of edges $\{v_i^j, v_c\}$ and $\{v_{i'}^{j'}, v_c\}$ in G'. Then we contract $\{v_s^t, v_c\}$. In the resulting graph, we contract, for each $i \in \{1, \dots, n\}$, all the edges in $\{\{v_j^i, v_{j+1}^i\} \mid j \in \{1, \dots, n-2\}\}$ to a single vertex u^i. Observe that the resulting graph is a complete graph with the vertex set $\{u^1, \dots, u^n\}$. Hence, $K_n \leq_{\mathrm{lc}} G' \leq_{\mathrm{lc}} G$ as claimed. $\square$

A *tree decomposition* of a graph G is a pair $(\mathcal{X}, T)$ where T is a tree and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a collection of subsets of $V(G)$ (called bags) such that:

1. $\bigcup_{i \in V(T)} X_i = V(G)$;

2. for each edge $\{x, y\} \in E(G)$, $\{x, y\} \subseteq X_i$ for some $i \in V(T)$, and

3. for each $x \in V(G)$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *adhesion* if a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ is $\max\{|X_i \cap X_j| \mid i, j \in V(t), i \neq j\}$ and its width is $\max\{|X_i| - 1 \mid i \in V(T)\}$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G. A *path decomposition* of a tree decomposition where the tree T is a path. The *pathwidth* of a graph G is the minimum width of a path decomposition of it.

Proof of Theorem 1.1. From Lemmas 2.1 and 2.3, G does not contain F_{2n^2-2n} as a minor; otherwise we are done. A graph with no $K_2 \times C_k$ minor, where C_k is a cycle on k vertices, has treewidth at most $60k^2 - 120k + 63$ [1]. As F_k is a minor of $K_2 \times C_k$, the same bound holds for graphs with no F_k minor. The result follows by taking $k = 2n^2 - 2n$. □

Proof of Theorem 1.2. According to a result mentioned in [12], for any pair of graphs G and H such that G is an outerplanar graph and H has a vertex whose removal leaves a tree, there is a constant $c_{G,H}$ such that every 2-connected graph of pathwidth at least $c_{H,G}$ contains G or H as a minor. By taking both G and H to be a k-fan, we conclude that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every 2-connected graph of pathwidth at least $f(k)$ contains F_k as a lift-minor. Then Lemma 2.3 yields the result. □

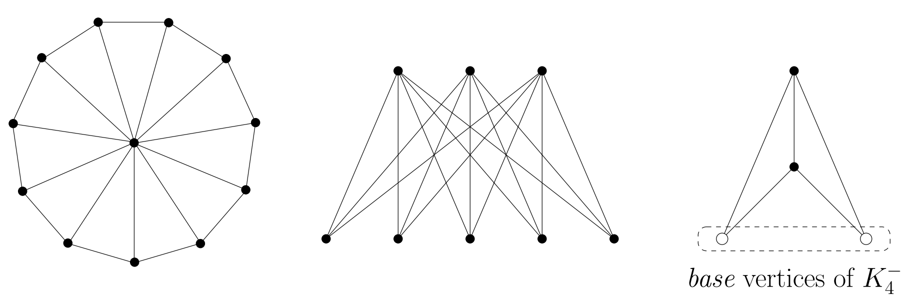

Figure 2: The graphs W_{11}, $K_{3,5}$, and K_4^-.

3 Proof of Theorem 1.3

Let W_k be the graph obtained from F_k by adding an edge between its extreme vertices (assuming that $k \geq 3$). Let $K_{3,k}$ be the complete bipartite graph whose parts have exactly 3 and k vertices. We denote by K_4^- the graph obtained from K_4 by removing an edge, and we call the vertices of degree 2 in it *base vertices*. Examples of these graphs are shown in Figure 2. We let $\Gamma_r = K_2 \times P_r$. We denote by M_r the graph obtained if we take r copies of K_4, pick a vertex in each of them, and then identify all chosen vertices to a single vertex. We denote by N_r the graph obtained as follows. We take r copies of K_4^-. In each copy we choose an arbitrary base vertex and call it a *left* base vertex, and say that another base vertex is *right*. Then we identify all left vertices and all right vertices. Finally, we denote by L_r the graph obtained if we take r copies of K_4^-, pick a left and right base vertices in each copy, and then identify the right base vertex of the $(i-1)$-th copy and the left base vertex of the i-th copy for $i \in \{2, \ldots, r\}$. See Figure 3 for examples.

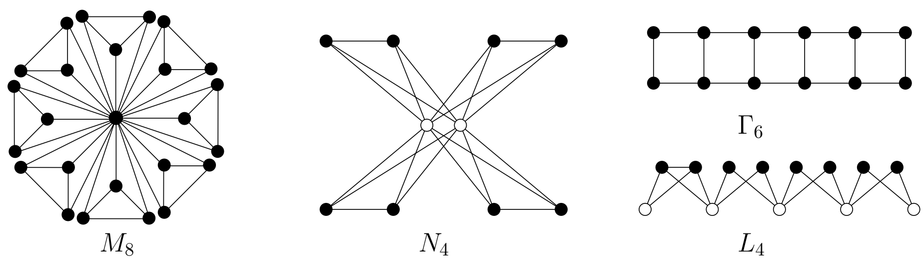

Figure 3: The graphs M_8, N_4, Γ_6, and L_4.

We need the following lemmas.

Lemma 3.1. *For each $k \geq 1$, it holds that $F_k \leq_{\mathrm{lm}} W_k$, $F_k \leq_{\mathrm{lm}} K_{3,k}$, $F_k \leq_{\mathrm{lm}} M_k$, $F_k \leq_{\mathrm{lm}} N_k$, $F_k \leq_{\mathrm{lm}} \Gamma_k$, and $F_k \leq_{\mathrm{lm}} L_k$.*

Proof. Clearly, $F_k \leq_{\mathrm{lm}} W_k$. Because Γ_k consists of two paths on k vertices joined by a matching, it is straightforward to see that we obtain F_k by contracting all the edges of one path.

For $K_{3,k}$, denote by u_1, u_2, u_3 and $v_1, \ldots, v_k$ the vertices of the respective partition sets. For $i \in \{1, \lfloor k/2 \rfloor\}$, we lift the edges $\{v_{2i-1}, u_1\}, \{u_1, v_{2i}\}$, and for $i \in \{1, \lceil k/2 \rceil - 1\}$, $\{v_{2i}, u_2\}, \{u_2, v_{2i+1}\}$ are lifted. Now F_k with the center u_3 is a subgraph of the obtained graph.

Recall that M_k is obtained from k copies of K_4 by identifying vertices chosen in each copy. Let x_i, y_i, z_i and v be the vertices of the i-th copy (v is a common vertex) for $i \in \{1, \ldots, k\}$. We obtain F_k as follows: for $i \in \{1, \ldots, k-1\}$, we lift $\{x_i, v\}, \{v, y_{i+1}\}$, and then x_i, y_i, z_i are contracted to a single vertex for all $i \in \{1, \ldots, k\}$.

Consider now N_k obtained from k copies of K_4^-. Let x_i, y_i, v_l, v_r be the vertices of the i-th copy where v_l, v_r are the common base vertices for $i \in \{1, \ldots, k\}$. For

$i \in \{1, \ldots, k-1\}$, we lift $\{x_i, v_l\}, \{v_l, y_{i+1}\}$ and observe that F_k is a subgraph of the obtained graph.

Finally, assume that L_k consists of k copies of K_4^- with the vertices x_i, y_i, u_i, v_i where u_i, v_i are base vertices and $v_i = u_{i+1}$ for $i \in \{1, \ldots, k-1\}$. We lift the edges $\{x_i, v_i\}, \{u_{i+1}, x_{i+1}\}$ and also the edges $\{y_i, v_i\}, \{u_{i+1}, y_{i+1}\}$ for $i \in \{1, \ldots, k-1\}$. It remains to observe that L_k is a subgraph of the obtained graph and $F_k \leq_{\mathrm{lm}} L_k$. □

Lemma 3.2. *Let G be a 3-connected graph with at least four vertices, $\{u, v\} \in E(G)$. Then G can be contracted to K_4 in such a way that $\{u, v\}$ is an edge of the obtained graph.*

Proof. The graph G has at least three internally vertex disjoint (u, v)-paths. Hence, there are at least two vertex disjoint (u, v)-paths P_1, P_2 that avoid the edge $\{u, v\}$. The set $\{u, v\}$ does not separate $V(P_1) \setminus \{u, v\}$ and $V(P_2) \setminus \{u, v\}$. Therefore, there is a path that joins these sets, and the claim follows. □

We also need the following proposition.

Proposition 3.3 ([10])**.** *There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that every graph excluding W_k and $K_{3,k}$ as a minor has a tree-decomposition of width at most $g(k)$ and adhesion at most two.*

Recall that for two vectors of integers $x = (x_w, \ldots, x_1)$ and $y = (y_w, \ldots, y_1)$, $x < y$ *lexicographically*, if there is $k \in \{1, \ldots, w-1\}$ such that $x_i = y_i$ for $i \in \{k+1, \ldots, w\}$ and $x_k < y_k$. For a tree decomposition $(\mathcal{X}, T)$ of width w, denote by b_i the number of bags of size i for $i \in \{1, \ldots, w+1\}$. We say that such a three decomposition $(\mathcal{X}, T)$ with adhesion at most two is *minimal*, if the vector $b = (b_w, \ldots, b_1)$ is lexicographically minimal, where the minimum is taken over all tree decompositions of width at most w and adhesion at most two. We need the following property of minimal tree decompositions.

Lemma 3.4. *Let $\mathcal{X} = \{X_i \mid i \in V(T)\}$ be a minimal tree decomposition of a connected graph G of minimum degree at least three. For a bag X_i denote by $\overline{G}[X_i]$ the graph obtained from $G[X_i]$ by the addition of (non-existing) edges $\{u, v\}$ for the pairs if vertices $u, v \in X_i$ such that there is another bag X_j with $X_i \cap X_j = \{u, v\}$. Then the following holds.*

a) No bag is a subset of another bag.

b) For each bag X_i, either i) $G[X_i]$ is a bridge in G and i is not a leaf of T, or ii) $\overline{G}[X_i]$ is a triangle and for each $u \in V(X_i)$, there is another bag X_j with $u \in X_j$, or iii) $\overline{G}[X_i]$ is a 3-connected graph with at least four vertices.

c) If X_i, X_j are distinct bags, $X_i \cap X_j = \{x, y\}$, then there is a (x, y)-path in G that avoids the vertices of $X_i \setminus \{x, y\}$.

Proof. The first statement follows directly from the minimality. Notice that it implies that there are no bags of size one, since G is a connected graph of minimum degree at least three.

Let $X_i = \{u, v\}$ be a bag of size two. We claim that v and u are adjacent. To see this, assume to the contrary that u and v are not adjacent. Let also e_v and e_u be the first and the last edge of a path in the connected graph G starting from v and finishing at u. Let also X_{i_u} (resp. X_{i_v}) be a bag where the edge e_u (resp. e_v) is contained. As G is connected, i cannot be in the path of T connecting i_v and i_u. Therefore, we may assume that either i_v is in the path of T connecting i and i_u or that i_u is in the path of T connecting i and i_v. In both cases, the third condition of the definition of a tree decomposition implies that either $u \in X_{i_v}$ or that $v \in X_{i_u}$, a contradiction to a).

Hence, u, v are adjacent, and because u, v are not included in another bag, $\{u, v\}$ is a bridge. Clearly, i cannot be a leaf of T, as G has no vertices of degree one. Let $X_i = \{u, v, w\}$ be a bag of size three. Since the minimum degree of G is at least three, each vertex of X_i is included to another bag. We have to prove that $\overline{G}[X_i]$ is a triangle. To obtain a contradiction, assume that u and v are not adjacent. Then there is no bag X_j, $j \neq i$, with $u, v \in X_j$. We modify the tree decomposition as follows. The node i is replaced by two adjacent nodes i', i''. Let $X_{i'} = \{u, w\}$ and $X_{i''} = \{v, w\}$. For each j such that $X_j \cap X_i \neq \emptyset$, we join j with i' by an edge if $X_i \cap X_j \subseteq \{u, w\}$, and we join j with i' if $X_i \cap X_j = \{v\}$ or $X_i \cap X_j = \{v, w\}$. We obtain a tree decomposition, where a bag of size tree is replaced by two bags of size two, but it contradicts the minimality of the original tree decomposition.

Suppose now that $X_i = \{u_1, \ldots, x_p\}$ is a bag of size $p \geq 4$. To obtain a contradiction, assume that $H = \overline{G}[X_i]$ is not 3-connected. Then it has a cut set S of size at most two. Let X be the set of vertices of a component of the graph obtained from H by the removal of S. Let $Y = X \cup S$ and $Z = V(H) \setminus X$. Notice that for any bag X_j, $X_i \cap X_j \subseteq Y$ or $X_i \cap X_j \subseteq Z$. We modify the tree decomposition as follows. The node i is replaced by two adjacent nodes i', i''. Let $X_{i'} = Y$ and $X_{i''} = Z$. For each j such that $X_j \cap X_i \neq \emptyset$, we join j with i' by an edge if $X_i \cap X_j \subseteq Y$ and $X_i \cap X_j \cap X \neq \emptyset$, and we join j with i' if $X_i \cap X_j \subseteq Z$. We obtain a tree decomposition, where a bag of size p is replaced by two bags of size at most $p - 1$, but it contradicts the minimality of the original tree decomposition.

Now we prove c). Suppose that X_i, X_j are distinct bags, $X_i \cap X_j = \{x, y\}$ and x, y are not adjacent. To obtain a contradiction, assume that there is no (x, y)-paths in G that avoids the vertices of $X_i \setminus \{x, y\}$. Let T be rooted in i. The root defines the parent-child relation on $V(T)$. Clearly, j is a child of i. Denote by p the last descendant of i with the same property as i, i.e., p has a child q, $X_p \cap X_q = \{u, v\}$ and there is no (u, v)-paths in G that avoids the vertices of $X_p \setminus \{u, v\}$, and no child of p satisfies this condition. Then X_q has at least three vertices, and the graph obtained from $\overline{G}[X_q]$ by the removal of the edge $\{u, v\}$ is disconnected, but it contradicts b). $\square$

Now we are in position to prove Theorem 1.3.

Proof of Theorem 1.3. We set $k = 2n(2n-1)$ and assume that G does not contain F_k as a lift-minor. Also, keep in mind that $k > 2$. From Lemmas 2.1 and 2.3 it is enough to prove that $|V(G)|$ cannot be bigger than $f(k)$ where f is a function that will be determined later in the proof.

Notice that by Lemma 3.1, W_k and $K_{3,k}$ both contain F_k as a lift-minor. Hence, by Proposition 3.3, G has a tree-decomposition of width at most $g(k)$ and adhesion at most two. We assume that $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a minimal and, subject to this condition, for a given $\mathcal{X}$, a tree T with the maximum number of leaves is chosen.

Let $\mathcal{L} \subseteq \mathcal{X}$ be the set of bags corresponding to the leaves of T. Our strategy is to observe that if the size of G is big enough, then either T has many leaves or there is a long path in T with all vertices of degree two in T. Then we construct a lift-minor F_k using either the leaf-bags or the path-bags. For this, we first bound the number of leaves in T. Then we take the size of G to be sufficiently big so that, given that both the treewidth of $\mathcal{X}$ and the number of leaves in T are bounded, we can force the existence of a path in T. We need the following claim.

Claim 1. There is a function f_1 such that if $F_k \not\leq_{\rm lm} G$, then $|\mathcal{L}| < f_1(k)$.

Proof of Claim 1. Let us assume that $|\mathcal{L}| \geq f_1(k)$ for some f_1 that will be determined in the end of the proof of this claim and consider the graphs $L_X = G[X]$, for each $X \in \mathcal{L}$. There are at most two vertices in each L_X that have neighbors outside X in G. Let S_X be the set of such vertices for each $X \in \mathcal{L}$. Denote by $\overline{L}_X$ the graph obtained from L_X by joining vertices of S_X by an edge (if $|S_X| = 1$, then $\overline{L}_X = L_X$). By Lemma 3.4, X has at least four vertices and $\overline{L}_X$ is 3-connected for each $X \in \mathcal{L}$. We set

$$\mathcal{S}_1 = \{S_X \mid X \in \mathcal{L} \text{ and } |S_X| = 1\} \text{ and } \mathcal{S}_2 = \{S_X \mid X \in \mathcal{L} \text{ and } |S_X| = 2\}.$$

If $S_X \in \mathcal{S}_1$, then the 3-connectivity of L_X implies that K_4 is a contraction of $\overline{L}_X = L_X$ by the Tutt's theorem (see c.f. [5]). In case $|S_X| = 2$, we define L_X^- as the graph taken from L_X by removing, if exists, the edge with endpoints in S_X. The 3-connectivity of $\overline{L}_X$ and Lemma 3.2 imply that L_X^- can be contracted to K_4^- in a way that the two base vertices are the vertices of S_X.

We now construct an auxiliary graph J by taking $G^- = G[\bigcup_{X \in \mathcal{X} \setminus \mathcal{L}} X]$ and then, for every $S \in \mathcal{S}_2$, adding an edge connecting the two vertices of S (if such an edge already exists, then do not add it). Let us call *essential* the edges connecting in J the two vertices of some $S \in \mathcal{S}_2$ (notice that an essential edge of J is not necessarily present in G).

We assign weights to the vertices and edges of J: each vertex $v \in V(J)$ receives weight $|\{L \in \mathcal{L} \mid \{v\} = S_L\}|$ and each edge $e \in E(J)$ receives weight $|\{L \in \mathcal{L} \mid e = S_L\}|$. Observe that the essential edges are *exactly* those with positive weights and recall that the sum of the weights of the edges and vertices of J is at least $f_1(k)$. We prove a series of subclaims.

Subclaim 1.1. The sum of the weights of the vertices in J is less than k.

Proof of Subclaim 1.1. Suppose that is not correct. Then contract in G all edges that do not belong to some of the graphs in $\{L_X \mid S_X \in \mathcal{S}_1\}$ and obtain a graph that, in turn, can be contracted to M_k. But then, from Lemma 3.1, G should contain F_k as a lift-minor, a contradiction.

Subclaim 1.2. There is a function f_2 such that J does not have more than $f_2(k)$ blocks with essential edges.

Proof of Subclaim 1.2. Notice that for each block B of J, there is a unique block B' of G such that $V(B) \subseteq V(B')$, and for different blocks B_1, B_2, $V(B_1)$ and $V(B_2)$ are included in distinct blocks of G. Observe also that if B is a block of J with at least one essential edge, then the corresponding block B' in G can be contracted to K_4 by Lemma 3.4. That way, we have that G can be contracted to a bridgeless graph W where each of its blocks is a K_4 and such that the number of blocks in W is equal to the number of blocks in J with essential edges. Notice that W cannot contain a cut vertex w with the property that $W - w$ has k or more connected components, otherwise W could be contracted to M_k and therefore $F_k \leq_{\text{lm}} M_k \leq_{\text{lm}} G$ by Lemma 3.1; a contradiction. Moreover, the diameter of W should be less than k, otherwise W contains L_k as a minor. Then $F_k \leq_{\text{lm}} L_k \leq_{\text{lm}} G$ by Lemma 3.1, a contradiction. It is now easy to verify that the number of blocks in W is bounded by some function f_2 of k. The subclaim follows.

From Subclaim 1.1, the sum of weights of the edges of J is more than $f_1(k) - k$. From Subclaim 1.2, one of the blocks of J denoted by B should have total-edge weight at least $\frac{f_1(k)-k}{f_2(k)}$.

We now construct the graph B^* from B by repetitively removing or contracting non-essential edges: for a non-essential edge $\{u, v\}$, if $\{u, v\}$ is a cut set in the already constructed graph, then we remove the edge, else we contract the edge. Notice that a non-essential edge can be identified with an essential one after one of these operations (in such a case, such a new edge is essential). Observe also that these operations maintain 2-connectivity. Hence, B^* is 2-connected. If during such a contraction two edges become one, the weight of the new edge is the sum of the weights of the two edges. Notice that the total edge-weight of B^* is the same as in B, that is at least $\frac{f_1(k)-k}{f_2(k)}$. Notice also that at most two edges of zero weight may survive in of B^* and this may happen only when B^* is a triangle where two or one of its edges have positive weights. Clearly, none of the edges in B^* may have weight at least k as, then, the same sequence of edge contractions and removals in G would create a graph that contains N_k as a minor. Then $F_k \leq_{\text{lm}} N_k \leq_{\text{lm}} G$ by Lemma 3.1; a contradiction. We obtain that the total weight of the edges in B^* is lower bounded by $\frac{f_1(k)-k}{k \cdot f_2(k)}$. In what follows, we will take f to be big enough so that this lower bound is greater than 2 and therefore, we may assume that all edges of B^* have positive weight. This implies that

$$|E(B^*)| \quad \geq \quad \frac{f_1(k) - k}{k \cdot f_2(k)} \tag{1}$$

Our next step is to observe that the maximum degree of B^* is less than k. Suppose towards a contradiction that some vertex y of B^* is incident with at least k edges. Recall that B^* is 2-connected and thus $B^* - y$ is connected. Therefore, if we contract in B^* all edges that are not incident to y, we create a single edge with total weight at least k. As before, this implies that $F_k \leq_{\text{lm}} N_k \leq_{\text{lm}} G$; a contradiction.

Our next observation is that every path in B^* has length at most $k-1$. Indeed, a path of length at least k in B^* would imply the existence in G of L_k as a minor, a contradiction, since by Lemma 3.1 $F_k \leq_{\text{lm}} L_k \leq_{\text{lm}} G$.

According to the two observations above, B^* has at most $f_3(k)$ edges for some function f_3. This, combined with (1), implies that $f_3(k) \geq \frac{f_1(k)-k}{k \cdot f_2(k)}$ for some specific choice of the functions f_2 and f_3. If we now take f_1 to be big enough so that this inequality is violated, we have a contradiction and the claim follows.

Notice that the fact that each bag of $\mathcal{X}$ has at most $g(n(n-1))$ vertices implies that $\mathcal{X}$ has at least $f(n)/g(k)$ bags. Therefore, the tree T has $\geq f(n)/g(k)$ vertices and from the above claim, less than $f_1(k)$ of them are leaves (recall that $k = n(n-1)$). But then we can choose the function f such that T contains a path P of $24(k+1)^3+3$ vertices such that all internal vertices of P have degree two in T. By the fact that the minimum degree of G is at least three, we obtain that at most the half of the graphs induced by the bags corresponding to the vertices of P are bridges. We call the bags corresponding to these bridges of G *bridge edges* of G and we may assume that this path P has at least $12(k+1)^3$ internal vertices that correspond to bags that are not inducing bridges in G. Let H be the graph obtained from $G[\bigcup_{i \in V(P)} X_i]$ by contracting all bridge edges. Our aim is to arrive to a contradiction by showing that H (and therefore G as well) contains either L_k, or F_k, or Γ_k as a minor. Notice that $\mathcal{X}$ gives rise to a path decomposition $\mathcal{X}' = \{X_0, \ldots, X_{r+1}\}$ of H containing at least $12(k+1)^3+2$ bags (we first crop from $\mathcal{X}$ the bags corresponding to P and then we suppress bridge bags). Recall that the number of leaves in T is maximum. Then each bag X_i of $\mathcal{X}'$ can be of one of the following types:

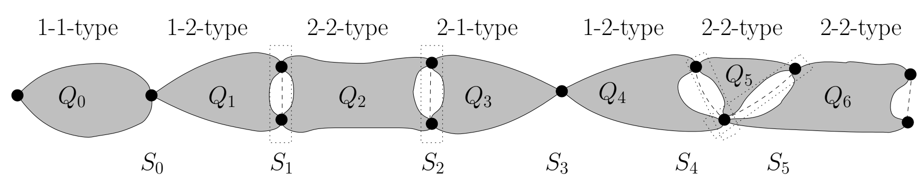

Figure 4: The types of bags in $\mathcal{X} = \{X_0, \ldots, X_{r+1}\}$.

- (1-1-type) $Q_i = H[X_i]$ is a 3-connected graph. Moreover, if $i \in \{1, \ldots, r\}$ then such a X_i contains two vertices x_l^i and x_r^i such that $\{x_l^i\} = X_i \cap X_{i-1}$ and $\{x_r^i\} = X_i \cap X_{i+1}$.
- (1-2-type) $Q_i = H[X_i]$ contains three vertices x_l^i, x_{ru}^i and x_{rd}^i such that the addition in $H[X_i]$ of the edge $\{x_{ru}^i, x_{rd}^i\}$ makes it 3-connected or a triangle (we

denote this enhanced graph by $\overline{Q}_i$). Moreover, if $i \in \{1, \dots, r\}$, then $\{x_l^i\} = X_i \cap X_{i-1}$, $\{x_{ru}^i, x_{rd}^i\} = X_i \cap X_{i+1}$ and $x_l^i \notin \{x_{ru}^i, x_{rd}^i\}$.

- (2-1-type) $Q_i = H[X_i]$ contains three vertices x_{lu}^i, x_{ld}^i and x_r^i such that the addition in $H[X_i]$ of the edge $\{x_{lu}^i, x_{ld}^i\}$ makes it 3-connected or a triangle (we denote this enhanced graph by $\overline{Q}_i$). Moreover, if $i \in \{1, \dots, r\}$, then $\{x_{lu}^i, x_{ld}^i\} = X_i \cap X_{i-1}$, $\{x_r^i\} = X_i \cap X_{i+1}$ and $x_r^i \notin \{x_{lu}^i, x_{ld}^i\}$.

- (2-2-type) $Q_i = H[X_i]$ contains the vertices x_{lu}^i, x_{ld}^i, x_{ru}^i and x_{rd}^i where $x_{lu}^i \neq x_{ld}^i$, $x_{ru}^i \neq x_{rd}^i$, $|\{x_{lu}^i, x_{ld}^i, x_{ru}^i, x_{rd}^i\}| \in \{3, 4\}$, and the addition in $H[X_i]$ of the edges $\{x_{ru}^i, x_{rd}^i\}$ and $\{x_{lu}^i, x_{ld}^i\}$ makes it 3-connected (we denote this enhanced graph by $\overline{Q}_i$). Moreover, if $i \in \{1, \dots, r\}$, then $\{x_{lu}^i, x_{ld}^i\} = X_i \cap X_{i-1}$ and $\{x_{ru}^i, x_{rd}^i\} = X_i \cap X_{i+1}$.

Notice that for each $i \in \{0, \dots, r\}$, if X_i is of x-y-type and X_{i+1} is of x'-y'-type, then $y = x'$ and that y is the cardinality of the set $S_i = X_i \cap X_{i+1}$. Observe also that for any $i \in \{1, \dots, r-1\}$, $S_{i-1} \setminus S_i \neq \emptyset$ and $S_i \setminus S_{i-1} \neq \emptyset$, since otherwise if $S_{i-1} \subseteq S_i$ or $S_i \subseteq S_{i-1}$, then the node i of T could be made a leaf, but this contradicts the choice of T. Notice that each $\overline{Q}_i$ is either a triangle or 3-connected by Lemma 3.4. For $1 \leq i \leq j \leq r$, we let $H_{ij} = H[\bigcup_{h \in \{i, \dots, j\}} X_h]$. We need some properties of H_{ij} given in the next four claims.

Claim 2. Suppose that $|S_{i-1}| = 1$, $|S_j| = 1$, and for $h \in \{i, \dots, j-1\}$, $|S_h| = 2$. Then the graph H_{ij} contains three paths P_1, P_2, P^*, where P_1, P_2 are internally vertex disjoint (x_l^i, x_r^j)-paths, P^* joins an internal vertex of P_1 with some internal vertex of P_2 and avoids x_l^i, x_r^j.

Proof of Claim 2. The graph H_{ij} is 2-connected. Hence, there are two internally vertex disjoint (x_l^i, x_r^j)-paths P_1 and P_2. Notice that for each $h \in \{i, \dots, j-1\}$, each of the paths P_1 and P_2 contains exactly one vertex of S_h.

If $j = i$, then H is 3-connected, and because the minimum degree of G is at least three, H_{ij} has at least four vertices. Moreover, as H is 3-connected we may assume that P_1, P_2 have internal vertices. Observe also that $\{x_l^i, x_r^j\}$ is not a cut set of H. Hence, there is a path P^* that joins an internal vertex of P_1 with some internal vertex of P_2.

Suppose that $j > i$. Notice that P_1, P_2 have internal vertices because $x_l^i \notin \{x_{ru}^i, x_{rd}^i\}$ and $x_r^j \notin \{x_{lu}^j, x_{ld}^j\}$. If for every $h \in \{i, \dots, j\}$, $|X_h| = 3$, then there is $h \in \{i, \dots, j-1\}$ such that x_{lu}^h, x_{ld}^h are adjacent in G, since G has no vertices of degree two. Then such an edge forms a path between an inner vertex of P_1 and an inner vertex of P_2. Assume that there is an $h \in \{i, \dots, j\}$ such that $|X_h| \geq 4$. By Lemma 3.4, $\overline{Q}_h$ is 3-connected. If $h = i$, then x_{ru}^h, x_{rd}^h are joined in $\overline{Q}_h$ by at least three internally vertex disjoint paths. At least one of these paths avoids x_r^h and the edge $\{x_{ru}^h, x_{rd}^h\}$ and we have P^*. If $h = j$, then we find P^* by the same arguments using the symmetry. Let $i < h < j$. Then x_{ru}^h, x_{rd}^h are joined in $\overline{Q}_h$ by at least three internally vertex disjoint paths, and at least one of these paths avoids the edges $\{x^h_{lu}, x^h_{ld}\}$ and $\{x^h_{ru}, x^h_{rd}\}$.

Claim 3. Suppose that $i < r$, $|S_{i-1}| = |S_i| = |S_{i+1}| = 2$ and $S_{h-1} \cap S_h \cap S_{h+1} = \{u\}$. Then the graph H_{ii+1} contains two paths P, P^*, where P joins the unique vertices $S_{i-1} \setminus \{u\}, S_{i+1} \setminus \{u\}$ and avoids u, P^* joins an internal vertex of P with u and avoids vertices of $S_{i-1} \cup S_{i+1} \setminus \{u\}$.

Proof of Claim 3. Assume without loss of generality that $u = x^i_{ld} = x^i_{rd} = x^{i+1}_{rd}$ and $S_{i-1} \setminus (S_i \cup S_{i+1}) = \{x^i_{lu}\}$, $S_{i+1} \setminus (S_{i-1} \cup S_i) = \{x^{i+1}_{ru}\}$.

The graph $\overline{H}_{ii+1}$ obtained from H_{ii+1} by the addition of edges $\{x^i_{lu}, u\}$ and $\{x^{i+1}_{ru}, u\}$ is 2-connected. Hence, there is a (x^i_{lu}, x^{i+1}_{ru})-path P in $\overline{H}_{ii+1}$ that avoids u. Clearly, P is a path in H_{ii+1}. This path contains at least one internal vertex x^i_{ru}. If $|X_i| = |X_{i+1}| = 3$, then x^i_{ru}, x^i_{rd} are adjacent in G, since x^i_{ru} has degree at least three. Then this edge forms a path between an inner vertex of P and u. Assume that $|X_i| \geq 4$. By Lemma 3.4, $\overline{Q}_i$ is 3-connected. Then x^i_{ru}, x^i_{rd} are joined in $\overline{Q}_i$ by at least three internally vertex disjoint paths, and at least one of these paths avoids the edges $\{x^i_{lu}, x^i_{ld}\}$ and $\{x^i_{ru}, x^i_{rd}\}$. We take this path as P^*. If $|X_{i+1}| \geq 4$, then we find P^* by symmetrically applying the same arguments.

Claim 4. Suppose that for $h \in \{i-1, \ldots, j\}$, $|S_h| = 2$, and $S_{i-1} \cap S_i = \emptyset$. Then the graph H_{ij} contains two disjoint paths P_1, P_2 joining the vertices in $\{x^i_{ld}, x^i_{lu}\}$ with the vertices in $\{x^j_{rd}, x^j_{ru}\}$.

Proof of Claim 4. The graph $\overline{H}_{ij}$ obtained from H_{ij} by the addition of edges $\{x^i_{ld}, x^i_{lu}\}$ and $\{x^j_{rd}, x^j_{ru}\}$ is 2-connected. If we subdivide the edges $\{x^i_{rd}, x^i_{ru}\}$, $\{x^j_{rd}, x^j_{ru}\}$, and denote the obtained vertices of degree two by u and v respectively, then the obtained graph contains two internally vertex disjoint (u, v)-paths. The claim follows immediately.

Claim 5. Suppose that for $t \in \{i+1, \ldots, j-1\}$, $S_i \cap S_t = \emptyset$, $S_t \cap S_j = \emptyset$ and for $h \in \{i-1, \ldots, j\}$, $|S_h| = 2$. Then the graph H_{ij} contains paths P_1, P_2, P^*, where P_1, P_2 are disjoint paths joining the vertices in $\{x^i_{ld}, x^i_{lu}\}$ with the vertices in $\{x^j_{rd}, x^j_{ru}\}$, and P^* joins a vertex of P_1 with some vertex of P_2.

Proof of Claim 5. The paths P_1 and P_2 exists by Claim 4. Without loss of generality we assume that P_1 is a (x^i_{ld}, x^j_{rd})-path and P_2 is a (x^i_{lu}, x^j_{ru})-path.

If for every $h \in \{i, \ldots, j\}$, $|X_h| = 3$, then there is $h \in \{i, \ldots, j\}$ such that x^h_{lu}, x^h_{ld} are adjacent in G, since otherwise the vertices of S_t would have degree two. Then such an edge forms a path between P_1 and P_2. Assume that there is an $h \in \{i, \ldots, j\}$ such that $|X_h| \geq 4$. By Lemma 3.4, $\overline{Q}_h$ is 3-connected. Then x^h_{ru}, x^h_{rd} are joined in $\overline{Q}_h$ by at least three internally vertex disjoint paths, and at least one of these paths avoids the edges $\{x^h_{lu}, x^h_{ld}\}$ and $\{x^h_{ru}, x^h_{rd}\}$. Clearly, this path joins P_1 and P_2 in H_{ij}.

Now we are ready to complete the proof of Theorem 1.3. Consider the sequence $S_1, \ldots, S_r$. Recall that $r \geq 12(k+1)^3$.

First, we show that the sequence $|S_1|, \dots, |S_r|$ contains at most k 1's. Suppose that $|S_{h_1}| = \ldots = |S_{h_{k+1}}|$ for some $1 \le h_1 < \ldots < h_{k+1} \le r$ and for any $h_1 \le h \le h_{k+1}$, $h \ne h_1 \ldots, h_{k+1}$, $|S_h| = 2$. Then we apply Claim 2 for $H_{h_1 h_2}, \dots, H_{h_k h_{k+1}}$ and conclude that H contains L_k as a minor which gives a contradiction because of Lemma 3.1. As a consequence of this, the sequence $|S_1|, \dots, |S_r|$ contains a subsequence $|S_i|, \dots, |S_j|$ formed by at least $12(k+1)^2$ consecutive 2's (in Figure 4, this holds for $i = 1$ and $j = 2$). This also means that for all $h \in \{i+1, \dots, j\}$, X_h is a 2-2-type bag.

Now we prove that the sequence $S_i, \dots, S_j$ does not contain any subsequence $S_{i'}, \dots, S_{j'}$ of more than $2k$ consecutive elements such that $\cap_{h=1}^{j'} S_h = \{u\}$. Otherwise, we apply Claim 3 for $H_{i'+1\,i'+2}, H_{i'+3\,i'+4} \dots, H_{i'+2k-1\,i'+2k}$, and it follows that F_k is a minor of H; a contradiction.

We have that the sequence $S_i, \dots, S_j$ contains a subsequence $S_{h_1}, \dots, S_{h_{3k}}$ of $3k$ pairwise disjoint (not necessarily consecutive) elements. We apply Claim 5 for $H_{h_1 h_3}, H_{h_4 h_6}, \dots, H_{h_{3k-2} h_{3k}}$ and Claim 4 for $H_{h_3 h_4}, H_{h_6 h_7}, \dots,\ H_{h_{3k-3} h_{3k-2}}$ and observe that Γ_k is a minor of H, a contradiction. □

4 On the structure of lift-contraction-free graphs

Given a graph G and a subset S of $V(G)$, we denote by $N_G(S)$ the set of vertices not in S that are neighbors of vertices in S. We also define $\overline{N}_G(S) = N_G(S) \cup S$. Theorem 1.3 implies the following structural theorem on the graphs excluding some graph H as a lift-contraction. We call a vertex set R of a graph G *2-central* if for every connected component C of $G \setminus R$, it holds that $G[\overline{N}_G(V(C))]$ has treewidth at most two and $|N_G(V(C))| \le 2$. We need the following observation.

Observation 4.1. *Let G be a graph with a 2-central set R and let G^+ be the graph obtained from G by the consecutive application of the following operations: i) edge subdivisions and ii) additions of a new vertex adjacent to either a single vertex or two adjacent vertices in the already constructed graph. Then R is a 2-central set in G^+ as well.*

Theorem 4.2. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every connected graph G that does not contain an h-vertex graph H as a lift-contraction contains a 2-central set R of at most $f(h)$ vertices.*

Proof. Let f be the function that exists by Theorem 1.3. Assume that there is a minimum size counterexample G and let $n = |V(G)|$. Clearly, $n > f(h)$ as any graph of at most $f(h)$ vertices satisfies trivially the property of the theorem. However, from Theorem 1.3, a graph with more than $f(h)$ vertices that does not contain H as a lift-contraction, should contain some vertex v of degree at most two. We contract an edge incident with v in the connected graph G, and denote by G' the obtained graph. Notice that the graph G' also excludes H as a lift-contraction and, as $|V(G')| < n$, G' contains a 2-central set R of at most $f(h)$ vertices. From Observation 4.1, R is also a 2-central set in G. □

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