Colouring of Graphs with Ramsey-Type Forbidden Subgraphs*

Konrad K. Dabrowski[1], Petr A. Golovach[2], and Daniel Paulusma[3]

[1] ESSEC Business School, Av. B. Hirsch, 95021 Cergy Pontoise, France
dabrowski@essec.edu
[2] Department of Informatics, Bergen University,
PB 7803, 5020 Bergen, Norway
petr.golovach@ii.uib.no
[3] School of Engineering and Computing Sciences, Durham University,
Science Laboratories, South Road, Durham DH1 3LE, United Kingdom
daniel.paulusma@durham.ac.uk

Abstract. A colouring of a graph $G = (V, E)$ is a mapping $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ if $uv \in E$; if $|c(V)| \leq k$ then c is a k-colouring. The COLOURING problem is that of testing whether a given graph has a k-colouring for some given integer k. If a graph contains no induced subgraph isomorphic to any graph in some family $\mathcal{H}$, then it is called $\mathcal{H}$-free. The complexity of COLOURING for $\mathcal{H}$-free graphs with $|\mathcal{H}| = 1$ has been completely classified. When $|\mathcal{H}| = 2$, the classification is still wide open, although many partial results are known. We continue this line of research and forbid induced subgraphs $\{H_1, H_2\}$, where we allow H_1 to have a single edge and H_2 to have a single non-edge. Instead of showing only polynomial-time solvability, we prove that COLOURING on such graphs is fixed-parameter tractable when parameterized by $|H_1| + |H_2|$. As a by-product, we obtain the same result both for the problem of determining a maximum independent set and for the problem of determining a maximum clique.

1 Introduction

Graph colouring involves the labelling of the vertices of some given graph by integers called colours such that no two adjacent vertices receive the same colour. The COLOURING problem is that of deciding whether or not a graph can be coloured with at most k colours for some given integer k. Because COLOURING is NP-complete for any fixed $k \geq 3$, its computational complexity has been widely studied for special graph classes, see for example the surveys of Randerath and Schiermeyer [33] and Tuza [36]. In this paper, we consider the COLOURING problem for graphs characterized by two forbidden induced subgraphs. Before we summarize related results and explain our new results, we first state the necessary terminology.

* The research in this paper was supported by EPSRC (EP/G043434/1), ERC (267959) and ANR (TODO ANR-09-EMER-010). An extended abstract of the paper will appear in the proceedings of WG 2013.

1.1 Basic Terminology

We only consider finite undirected graphs without loops or multiple edges. We refer to the textbook of Diestel [11] for any undefined graph terminology. Let $G = (V, E)$ be a graph. A *colouring* of G is a mapping $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. We call $c(u)$ the *colour* of u and $\{u \in V \mid c(u) = i\}$ for some $i \geq 1$ a *colour class* of c. A *k-colouring* of G is a colouring c of G with $1 \leq c(u) \leq k$ for all $u \in V$. The smallest integer k for which G has a k-colouring is called the *chromatic number* of G, denoted $\chi(G)$; a $\chi(G)$-colouring is said to be *optimal*. The k-Colouring problem is that of deciding whether a given graph admits a k-colouring. Here, k is *fixed*, that is, not part of the input. If k is part of the input, then we denote the problem as Colouring.

Let $G = (V, E)$ be a graph. A graph H is an *induced subgraph* of G if H can be obtained from G by deleting zero or more vertices. In this case we write $H \subseteq_i G$. For a set $S \subseteq V$, we let $G[S] = (S, \{uv \in E \mid u, v \in S\})$ denote the subgraph of G *induced by* S. Let $\{H_1, \ldots, H_p\}$ be a set of graphs. We say that G is *$(H_1, \ldots, H_p)$-free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write that G is H_1-free instead of (H_1)-free.

The *complement* of a graph $G = (V, E)$, denoted by $\overline{G}$, has vertex set V and an edge between two distinct vertices if and only if these vertices are not adjacent in G. The *disjoint union* of two graphs G and H with $V(G) \cap V(H) = \emptyset$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and we denote this by $G + H$. We denote the disjoint union of r copies of G by rG.

For $r \geq 1$, the graph P_r denotes the *path* on r vertices, that is, $V(P_r) = \{u_1, \ldots, u_r\}$ and $E(P_r) = \{u_i u_{i+1} \mid 1 \leq i \leq r - 1\}$. Adding the edge $u_1 u_r$ to this graph yields the *cycle* on r vertices, denoted C_r. The graph sP_1 denotes the *independent set* on s vertices.

The Independent Set problem is that of testing whether a given graph has an independent set of size at least k for some given integer k. The graph K_t denotes the *complete* graph on t vertices, that is, $V(K_t) = \{u_1, \ldots, u_t\}$ and $E(K_t) = \{u_i u_j \mid 1 \leq i < j \leq t\}$. The vertex set of a complete graph is called a *clique*. The Clique problem is that of testing whether a given graph has a clique of size at least k for some integer k. The graph $K_t - e$ denotes the graph obtained from K_t after removing exactly one edge.

The *clique-width* of a graph G is the minimum number of labels needed to construct G using the following four operations:

(i) Creating a new vertex v with label i (denoted by $i(v)$).
(ii) Taking the disjoint union of two labelled graphs G and H (denoted by $G \oplus H$).
(iii) Joining each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$).
(iv) Renaming label i to j (denoted by $\rho_{i \to j}$).

An algebraic term that represents such a construction of G and that uses k labels is called a *k-expression* of G (i.e. the clique-width of G is the minimum k for which G has a k-expression). For instance, an induced path on five consecutive vertices a, b, c, d, e has clique-width equal to 3, and a 3-expression can be defined as follows:

$$\eta_{3,2}(3(e) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))))))).$$

A class of graphs $\mathcal{G}$ has *bounded* clique-width if there is a constant c such that the clique-width of every graph in $\mathcal{G}$ is at most c; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

1.2 Our Results

We show fixed-parameter tractability results for three problems, namely for Colouring, Independent Set and Clique, restricted to $(sP_1 + P_2, K_t - e)$-free graphs. In parameterized complexity theory, the problem input consists of a pair (I, p), where I is the problem instance and p is the parameter. A problem is *fixed-parameter tractable* (fpt) with parameter p if it can be solved in time $f(p) \cdot |I|^{O(1)}$ for some function f that only depends on p. In our case, a natural parameter is $s + t$. In Section 2 we show that Colouring is fixed-parameter tractable with parameter $s + t$ when restricted to $(sP_1 + P_2, K_t - e)$-free graphs, that is, can be solved in time $f(s + t)(n + k)^{O(1)}$ for some function f that only depends on $s + t$. In the same section, we show that the Independent Set and Clique problems are also fixed-parameter tractable with parameter $s + t$ when restricted to $(sP_1 + P_2, K_t - e)$-free graphs. However, the main motivation for our research comes from the area of graph colouring, as we explain in Section 1.3. In Section 3 we give some directions for future research. There, we also show that Colouring is polynomial-time solvable for $(2P_2, K_t - e)$-free graphs.

Finally, it should be noted that many classes of (H_1, H_2)-free graphs are known to have bounded clique-width (see Section 1.3). It is well known that Colouring can be solved in polynomial time on any graph class of bounded clique-width by combining the following two results. First, for any constant k, the Colouring problem is polynomial-time solvable on any class of graphs that have clique-width at most k provided that a k-expression is given [22]. Second, a $(2^{3k+2} - 1)$-expression for any n-vertex graph with clique-width at most k can be found in $O(n^9 \log n)$ time [29]. However, the classes of $(sP_1 + P_2, K_t - e)$-free graphs only have bounded clique-width for small values of s and t, as we show in Section 4. Thus our results (which are also stronger than mere polynomial-time solvability) do not fall into this category.

1.3 Motivation and Related Work

The complexity of Maximum Independent Set restricted to H-free graphs has only been partially classified. For instance, the complexity status of this problem on P_5-free graphs is a notorious open case (see [34] for a subexponential algorithm). Because a graph has an independent set of size at least k if and only if its complement has a clique of size at least k, the complexity classification of the Clique problem on H-free graphs is also far from being settled. Our research on the Colouring problem, which is the main focus of our paper, is well embedded in the literature. As a starting point, Král' et al. [23] completely determined the computational complexity of Colouring for H-free graphs.

Theorem 1 ([23]). *Let H be a fixed graph. If H is a (not necessarily proper) induced subgraph of P_4 or of $P_1 + P_3$, then* Colouring *can be solved in polynomial time for H-free graphs; otherwise it is* NP*-complete for H-free graphs.*

Theorem 1 initiated a study of the computational complexity of the k-Colouring problem on H-free graphs. This classification is still open (see the paper of Golovach et al. [13] for a survey and the paper of Huang [20] for a number of additional results). Due to Theorem 1, the Colouring problem restricted to graph classes characterized by *two* forbidden induced subgraphs has received a significant amount of attention as well. We survey known results below.

We observe that Theorem 1 implies that Colouring is polynomial-time solvable for (H_1, H_2)-free graphs if one of H_1, H_2 is an induced subgraph of P_4 or of $P_1 + P_3$. Another straightforward case is as follows. For positive integers s and t, the *Ramsey number* $R(s, t)$ is the smallest number n such that all graphs on n vertices contain an independent set of size s or a clique of size t. Ramsey's Theorem [30] states that such a number exists for all positive integers s and t. As an immediate consequence, Colouring is polynomial-time solvable on (sP_1, K_t)-free graphs for all s and t. Hence, it is natural to consider graph classes that can be obtained by adding one edge to the first forbidden induced subgraph and removing one edge from the second. This leads to the class of $(sP_1 + P_2, K_t - e)$-free graphs; note that this class includes all $(sP_1, K_t - e)$-free graphs and all $(sP_1 + P_2, K_{t-1})$-free graphs. As explained in Section 1.2, we have an fpt algorithm with parameter $s + t$ for solving the Colouring problem on $(sP_1 + P_2, K_t - e)$-free graphs. This is also an fpt algorithm for $(sP_1, K_t - e)$-free graphs and $(sP_1 + P_2, K_t)$-free graphs, because any $(sP_1, K_t - e)$-free graph is $(sP_1 + P_2, K_t - e)$-free and any $(sP_1 + P_2, K_t)$-free graph is $(sP_1 + P_2, K_{t+1} - e)$-free. Our result adds to the body of existing work on Colouring for (H_1, H_2)-free graphs, which we further discuss below.

The following result, which we will use later on, is due to Gyárfás [16].

Theorem 2 ([16]). *Let $\ell, t \geq 1$ be two integers. Then every (P_ℓ, K_t)-free graph can be coloured with at most $(\ell - 1)^{t-2}$ colours.*

Theorem 2 was slightly improved by Gravier, Hoáng and Maffray [15], who showed that every (P_ℓ, K_t)-free graph that is not a complete graph can be coloured with at most $(\ell - 2)^{t-2}$ colours. Each of these two results implies that Colouring is polynomial-time solvable on (F, K_t)-free graphs, whenever F is the disjoint union of one or more paths such that k-Colouring is polynomial-time solvable on F-free graphs for all $k \geq 1$. Combining this observation with such existing results for k-Colouring [7,10,18] gives us a number of polynomial-time solvable cases [12].

Also, the fact that Colouring can be solved in polynomial time on graphs of bounded clique-width by combining the aforementioned results of Kobler and

Rotics [22] and Oum and Seymour [29] directly leads to polynomial-time results for Colouring restricted to $(K_{1,3}, C_3 + P_1)$-free graphs [1], $(P_1 + P_4, \overline{P_1 + P_4})$-free graphs [3], $(P_5, \overline{P_1 + P_4})$-free graphs [2] and $(P_1 + P_4, \overline{P_5})$-free graphs [2]. Here, the graph $K_{1,r}$ denotes the graph with vertices $u, v_1, \ldots, v_r$ and edges $uv_1, \ldots, uv_r$.

More results on Colouring for (H_1, H_2)-free graphs can be found in a number of other papers [4,5,10,19,23,26,31,32,35], all of which are summarized in Theorem 3 given below, together with the above results and a weaker formulation of our new result (Statement (ii)-8). In this theorem, the graph C_3^+ denotes the *paw*, which is the graph with vertices a, b, c, d and edges ab, ac, ad, bc; the graph C_3^{++} denotes the *bull*, which is the graph with vertices a, b, c, d, e and edges ab, ac, ad, bc, be and the graph C_3^* denotes the *hammer*, which is the graph with vertices a, b, c, d, e and edges ab, ac, ad, bc, de. For details we refer to Golovach and Paulusma [12], who formulate a similar theorem to Theorem 3, but without mentioning Statement (ii)-8 and four very recent results of Malyshev [28], namely that 3-Colouring is NP-complete for $(C_3^{++}, K_{1,4})$-free graphs, and that Colouring is polynomial-time solvable for (P_5, C_4)-free graphs, $(K_{1,3}, P_5)$-free graphs and $(K_{1,3}, C_3^*)$-free graphs.

Theorem 3. *Let H_1 and H_2 be two fixed graphs. Then the following holds:*

(i) Colouring *is NP-complete for (H_1, H_2)-free graphs if*

1. $H_1 \supseteq_i C_r$ *for some* $r \geq 3$ *and* $H_2 \supseteq_i C_s$ *for some* $s \geq 3$
2. $H_1 \supseteq_i K_{1,3}$ *and* $H_2 \supseteq_i K_{1,3}$
3. H_1 *and* H_2 *contain a spanning subgraph of* $2P_2$ *as an induced subgraph*
4. $H_1 \supseteq_i C_3^{++}$ *and* $H_2 \supseteq_i K_{1,4}$
5. $H_1 \supseteq_i C_3$ *and* $H_2 \supseteq_i K_{1,r}$ *for some* $r \geq 5$
6. $H_1 \supseteq_i C_r$ *for* $r \geq 4$ *and* $H_2 \supseteq_i K_{1,3}$
7. $H_1 \supseteq_i C_3$ *and* $H_2 \supseteq_i P_{164}$
8. $H_1 \supseteq_i C_r$ *for* $r \geq 5$ *and* H_2 *contains a spanning subgraph of* $2P_2$ *as an induced subgraph*
9. $H_1 \supseteq_i C_r + P_1$ *for* $3 \leq r \leq 4$ *or* $H_1 \supseteq_i \overline{C_r}$ *for* $r \geq 6$, *and* H_2 *contains a spanning subgraph of* $2P_2$ *as an induced subgraph*
10. $H_1 \supseteq_i K_4$ *or* $H_1 \supseteq_i K_4 - e$, *and* $H_2 \supseteq_i K_{1,3}$.

(ii) Colouring *is polynomial-time solvable for (H_1, H_2)-free graphs if*

1. H_1 *or* H_2 *is an induced subgraph of* $P_1 + P_3$ *or of* P_4
2. $H_1 \subseteq_i K_{1,3}$, *and* $H_2 \subseteq_i C_3^*$ *or* $H_2 \subseteq_i P_5$
3. $H_1 \neq K_{1,5}$ *is a forest on at most six vertices and* $H_2 \subseteq_i C_3^+$
4. $H_1 \subseteq_i sP_2$ *or* $H_1 \subseteq_i sP_1 + P_5$ *for* $s \geq 1$, *and* $H_2 \subseteq_i C_3^+$
5. $H_1 \subseteq_i sP_2$ *or* $H_1 \subseteq_i sP_1 + P_5$ *for* $s \geq 1$, *and* $H_2 = K_t$ *for* $t \geq 4$
6. $H_1 \subseteq_i P_1 + P_4$ *or* $H_1 \subseteq_i P_5$, *and* $H_2 \subseteq_i \overline{P_1 + P_4}$
7. $H_1 \subseteq_i P_1 + P_4$ *or* $H_1 \subseteq_i 2P_2$, *and* $H_2 \subseteq_i \overline{P_5}$
8. $H_1 \subseteq_i sP_1 + P_2$ *for* $s \geq 0$ *or* $H_1 = 2P_2$, *and* $H_2 \subseteq_i K_t - e$ *for* $t \geq 2$
9. $H_1 \subseteq_i P_5$ *and* $H_2 \subseteq_i C_4$.

We need some of the results listed in Theorem 3 for the proof of our result in Section 3. The following result, which we will also use, is the only parameterized result known for Colouring of H-free graphs. Recall that k is the number of colours permitted.

Theorem 4 ([8]). *The* Colouring *problem on* (sP_1+P_2)*-free graphs is fixed-parameter tractable with parameter* $k+s$*.*

Very few parameterized results on Independent Set are known [9]. In particular, we need the following one. Recall that k is the minimum number of independent vertices required.

Theorem 5 ([9]). *The* Independent Set *problem on* (K_t-e)*-free graphs is fixed-parameter tractable with parameter* $k+t$*.*

We remark that the running times of the algorithms of Theorems 4 and 5 are $f(k+s)n^{O(1)}$ and $g(k+t)n^{O(1)}$, respectively, with k in the exponent of both f and g, whereas in our setting k is part of the input.

2 The Proofs of Our Results

In this section, we show that the Colouring, Independent Set and Clique problems on (sP_1+P_2, K_t-e)-free graphs are fixed-parameter tractable with parameter $s+t$. We need the following additional terminology.

Let $G=(V,E)$ be a graph. Then $N(u)=\{v\in V \mid uv\in E\}$ is the *neighbourhood* of $u\in V$. For $S\subseteq V$, we write $N(S)=\{v\in V\setminus S \mid uv\in E \text{ for some } u\in S\}$. A subset $M\subseteq E$ is a *matching* if no two edges in M share an end-vertex. A matching M is A*-saturating* for some subset $A\subseteq V$ if every vertex of A is an end-vertex of some edge in M; if M is V-saturating, then M is a *perfect* matching. A graph is p*-partite* if its vertex set can be partitioned into at most p independent sets, which we call *partition classes*. If $p=2$, the graph is *bipartite*. The complement of a p-partite graph is called a *co-p-partite graph* (whose partition classes are cliques).

Before stating the proofs of our results, let us first give an outline. Because a graph is (sP_1+P_2, K_t-e)-free if and only if its complement is $((t-2)P_1+P_2, K_{s+2}-e)$-free, the results for the Clique problem follow immediately from those for the Independent Set problem. In Lemma 1 we show that every (sP_1+P_2, K_t-e)-free graph is $((s+1)P_1, K_t-e)$-free or $(sP_1+P_2, K_{s^2(t-3)+2})$-free. This enables us to do as follows. We first show our results for Colouring and Independent Set on (sP_1+P_2, K_t)-free graphs in Lemmas 2 and 3, respectively, and on (sP_1, K_t-e)-free graphs in Lemmas 8 and 4, respectively. To prove these lemmas, we will use Theorems 2, 4 and 5. We also use Lemma 2 to prove Lemma 3 and Lemma 4, along with some structural results (Lemmas 5–7) to prove Lemma 8. We then combine our intermediate steps to prove Theorem 6, in which we state our main results.

As noted, we start with Lemma 1.

Lemma 1. *Let G be a (sP_1+P_2, K_t-e)-free graph. Then G is $((s+1)P_1, K_t-e)$-free or $(sP_1+P_2, K_{s^2(t-3)+2})$-free.*

Proof. Let $G=(V,E)$ be a (sP_1+P_2, K_t-e)-free graph. Suppose that G is not $(s+1)P_1$-free. We will show that G must then be $K_{s^2(t-3)+2}$-free.

Because G is not $(s+1)P_1$-free, G contains an independent set S on at least $s+1$ vertices. We assume that S is maximal (with respect to set inclusion). Let $u_1,\ldots,u_p$ be the vertices of S for some $p \geq s+1$. Let $X_1 = N(u_1)$, and for $i=2,\ldots,p$, let X_i denote the set of vertices in $V \setminus S$ that are adjacent to u_i but not to any vertex in $\{u_1,\ldots,u_{i-1}\}$. By maximality of S, every vertex of G is either in S or in some X_i.

We claim that $X_i=\emptyset$ for $i \geq s+1$. Indeed, if $i \geq s+1$ and $x \in X_i$ then $G[\{u_1,\ldots,u_s,x,u_i\}]$ would be isomorphic to sP_1+P_2.

Now suppose that for some i, X_i contains a clique K on at least $s(t-3)+1$ vertices. If $t-2$ vertices of K were adjacent to u_j for some $j \neq i$, then these $t-2$ vertices, together with u_i and u_j, would induce a K_t-e in G. Therefore, for each $j \neq i$, at most $t-3$ vertices of K can be adjacent to u_j. Hence, there must be a vertex $x \in K$ that is not adjacent to any vertex in $\{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{s+1}\}$. However, then $G[\{u_1,\ldots,u_{s+1},x\}]$ is isomorphic to sP_1+P_2. This is a contradiction. Thus all sets $X_1,\ldots,X_s$ can only contain cliques of size at most $s(t-3)$. Because S is an independent set and $X_i=\emptyset$ for $i \geq s+1$, this means that the largest clique in G has size at most $s^2(t-3)+1$. We conclude that G is $K_{s^2(t-3)+2}$-free, as desired. □

We are now ready to prove Lemmas 2 and 3.

Lemma 2. *The* Colouring *problem on (sP_1+P_2, K_t)-free graphs is fixed-parameter tractable with parameter $s+t$.*

Proof. We observe that every (sP_1+P_2, K_t)-free graph is (P_{2s+2}, K_t)-free, and consequently can be coloured with at most $(2s+1)^{t-2}$ colours due to Theorem 2. Hence we can apply Theorem 4. In fact, Theorem 4 gives us an explicit optimal colouring, rather than just the chromatic number. Hence, even the problem of finding an optimal colouring in a (sP_1+P_2, K_t)-free graph is fixed-parameter tractable with parameter $s+t$. □

Lemma 3. *The* Independent Set *problem on (sP_1+P_2, K_t)-free graphs is fixed-parameter tractable with parameter $s+t$.*

Proof. Let G be a (sP_1+P_2, K_t)-free graph on n vertices. By (the proof of) Lemma 2, we can find an optimal colouring c of G in fpt time with parameter $s+t$. Because G is (P_{2s+2}, K_t)-free, c uses $k \leq (2s+1)^{t-2}$ colours due to Theorem 2. Let $C_1,\ldots,C_k$ be the colour classes of c. We may assume that C_1 is a maximal independent set (with respect to set inclusion) in G, and that for $i=2,\ldots,k$, the set C_i is a maximal independent set in $G \setminus (C_1 \cup \cdots \cup C_{i-1})$. Indeed, if some $x \in C_i$ has no neighbours in C_j for some $j<i$, we can move x to C_j in order to obtain another optimal colouring of G.

We branch by choosing an index b to be the largest index such that C_b contains a vertex of the maximum independent set I that we are searching for. There are at most $(2s+1)^{t-2}$ ways of doing this. We then branch further by choosing a vertex x in C_b that we assume will be in I. This leads to at most n branches altogether. By maximality, x has a neighbour in C_a for every $a < b$. Let J_a be the set of vertices in C_a that are not adjacent to x. Because G is $(sP_1 + P_2)$-free, J_a contains at most $s-1$ vertices. Let $H_x = J_1 \cup \cdots \cup J_{b-1}$. By the definition of the sets J_i and the choice of x, we find that $I \setminus H_x \subseteq C_b$. We branch further by choosing an independent set $I' \subseteq H_x$. Because H_x has size at most $\beta = (b-1)(s-1) \leq ((2s+1)^{t-2} - 1)(s-1)$, there are at most 2^β ways of doing this. We then extend I' by adding all vertices of C_b that do not have a neighbour in I'. The final independent set is a candidate for being a maximum independent set. After considering all independent sets obtained in this way, we choose one that has maximum size. Because we considered all possible ways of constructing a maximum independent set, the above algorithm is correct. Because our algorithm constructs at most $n2^\beta$ independent sets, it runs in fpt time when parameterized by $s+t$. □

Here is Lemma 4.

Lemma 4. *The* Independent Set *problem on $(sP_1, K_t{-}e)$-free graphs is fixed-parameter tractable with parameter $s+t$.*

Proof. Because any independent set in an sP_1-free graph has size at most $s-1$, the result follows from Theorem 5. In fact, Theorem 5 gives us an explicit independent set of maximum size, rather than just the size of such a set. Hence, even the problem of finding a maximum independent set in a $(sP_1, K_t - e)$-free graph is fixed-parameter tractable with parameter $s+t$. □

To prove Lemma 8 we need three structural lemmas, the first of which is well-known.

Lemma 5 (Hall's Marriage Theorem [17]). *A bipartite graph G with vertex partition $A \cup B$ has an A-saturating matching if and only if $|N(X)| \geq |X|$ for all $X \subseteq A$.*

We need Lemma 5 to prove Lemma 6. Note that in a bipartite graph with partition classes A and B an A-saturating matching is perfect if $|A| = |B|$.

Lemma 6. *Let G be a bipartite graph with partition classes A and B. Let p, q, n be integers such that $|A| = |B| = n \geq p+q$. If every vertex in A has degree at least $n-p$ and every vertex in B has degree at least $n-q$, then G contains a perfect matching.*

Proof. We use Lemma 5. Let $X \subseteq A$. If $|X| = 0$, then $|N(X)| \geq |X|$. Suppose $1 \leq |X| \leq n-p$. Let $x \in X$. Then $|N(X)| \geq |N(x)| \geq n-p \geq |X|$. Suppose $|X| \geq n-p+1$. Then $|X| \geq n-p+1 \geq q+1$. As every vertex in B has at most q non-neighbours in A, this means that $N(X) = B$. Hence, $|N(X)| = n \geq |X|$. □

We need Lemma 6 to prove Lemma 7. Lemma 7 is a key lemma. It gives us a sufficient condition on the number of edges that we may allow between mutually vertex-disjoint cliques without increasing the chromatic number of their union.

Lemma 7. *Let k, a, b be integers such that $k \geq 2a(b-1)$. Let G be a co-b-partite graph with partition classes $X_1, \ldots, X_b$, all of size at most k. If every vertex in X_i has at most a neighbours in X_j for all $1 \leq i, j \leq b$ when $i \neq j$, then G is k-colourable.*

Proof. Without loss of generality, we may assume that every clique X_i contains exactly k vertices; if a clique X_i has less than k vertices, we add $k - |X_i|$ new vertices to X_i, whose only neighbours are the vertices in X_i.

We use induction on b. The case $b = 1$ is trivial. Let $b \geq 2$. Let $G' = G[X_1 \cup \cdots \cup X_{b-1}]$. Because $k \geq 2a(b-1) \geq 2a(b-2)$, we can apply our induction hypothesis to find that G' is k-colourable. Let c be a k-colouring of G', and let $X_b = \{x_1, \ldots, x_k\}$. We construct an auxiliary bipartite graph F as follows. For each colour $1 \leq i \leq k$ we create a vertex u_i. This yields the set $U = \{u_1, \ldots, u_k\}$. For each vertex $x_j \in X_b$ we introduce a copy x'_j. This yields the set $X = \{x'_1, \ldots, x'_k\}$. The partition classes of F are U and X. We add an edge from u_i to x'_j in F if and only if c does not assign colour i to any neighbour of x_j. We observe that c can be extended to a k-colouring of G if and only if F has a perfect matching. Hence, it remains to show that this is indeed the case.

Because every X_i is a clique of size k, every colour of c occurs $b-1$ times. Recall that we assume that every vertex in X_i has at most a neighbours in X_j for all $1 \leq i, j \leq b$, where $i \neq j$. By combining these two facts we find that every u_i has degree at least $k - a(b-1)$ and that every x'_j has degree at least $k - a(b-1)$. As $k \geq 2a(b-1) = a(b-1) + a(b-1)$, we may apply Lemma 6 to find that F has a perfect matching. $\square$

We are now ready to prove Lemma 8.

Lemma 8. *The* Colouring *problem on $(sP_1, K_t - e)$-free graphs is fixed-parameter tractable with parameter $s + t$.*

Proof. Let G be an $(sP_1, K_t - e)$-free graph on n vertices. We may assume without loss of generality that $s \geq 2$ and $t \geq 3$, as the proof is straightforward for $s \leq 1$ or $t \leq 2$. We first find a maximum independent set S of G. According to (the proof of) Lemma 4, we can do this in fpt time with parameter $s + t$.

We may assume without loss of generality that S is of size $s - 1$; otherwise G is $((s-1)P_1, K_t - e)$-free. Let $u_1, \ldots, u_{s-1}$ be the vertices of S. For $1 \leq i < j \leq s - 1$, let $X_{i,j}$ be the set of vertices adjacent to both u_i and u_j. If some $X_{i,j}$ contains a clique on $t - 2$ vertices, then the vertices of this clique together with u_i, u_j form an induced $K_t - e$. Hence, every $X_{i,j}$ is (sP_1, K_{t-2})-free. Recall that $R(s,t)$ is the Ramsey number for integers s and t. Because $X_{i,j}$ is (sP_1, K_{t-2})-free, $X_{i,j}$ contains at most $R(s, t-2) - 1$ vertices. Let $D = \bigcup X_{i,j}$. Then $|D| \leq \binom{s-1}{2}(R(s,t-2) - 1)$. Hence, the size of D is bounded by a function of s and t.

Let X_i consist of u_i and those vertices that are adjacent to u_i but not to any other vertex in S. Each X_i must be a clique, since if $x, y \in X_i$ were non-adjacent, then $\{x, y\} \cup S \setminus \{u_i\}$ would be an independent set larger than S, contradicting the fact that S is a maximum independent set. Note that every vertex of G is either in D or in some X_i (see Figure 1). Hence, we find that the vertices of G can be partitioned into $s-1$ cliques $X_1, \ldots, X_{s-1}$ and a set D. However, from these s sets, we only know that D has bounded size (in terms of s and t).

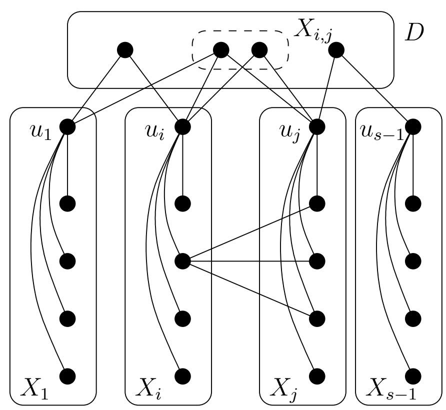

Fig. 1. Decomposition of the graph G into sets $X_1, \ldots, X_{s-1}, D$.

Now suppose that some vertex $x \in X_i$ had $t-2$ neighbours in X_j for some $j \neq i$. Then these $t-2$ neighbours, together with x and u_j, would induce a $K_t - e$ in G. Hence, every vertex in X_i has at most $t-3$ neighbours in every X_j with $j \neq i$. Consequently, each vertex in X_i has at most $(t-3)(s-1) + |D|$ neighbours outside of X_i.

We now start a branching procedure by first colouring the vertices of D in every possible way using colours from the set $C = \{1, \ldots, |D|\}$. After colouring D, for each $i = 1, \ldots, s-1$, we choose a subset $C_i \subseteq C$ of size $|C_i| \leq |X_i|$, which we let consist of exactly those colours from C that will occur on X_i. We branch over all possibilities for choosing such sets C_i. After choosing the sets C_i we branch further. For all cliques X_i of size at most $\gamma = (|D|+2)(s-2)(t-3) + (t-3)|D| + |D|$ (we explain this number later) we branch by trying every possible way of colouring a subset of X_i of size $|C_i|$ with the colours from C_i. This yields a partial colouring of G. The total number of these partial colourings is at most

$$|D|^{|D|} \prod_{i=1}^{s-1} 2^{|D|} \binom{\gamma}{|C_i|} |C_i|^{|C_i|},$$

which only depends on s and t, and which may be strictly less because whenever two adjacent vertices are assigned the same colour, we naturally cut the branch.

Now let c be a partial colouring that is obtained in this way. Assume that c uses k_c colours. We let $\texttt{total}(c)$ denote the smallest number of colours of a colouring c' of G subject to the following two conditions:

(i) c' extends c;
(ii) c' does not use any colours from $C \setminus C_i$ on vertices of X_i.

Let Z be the set of vertices that are not coloured by c. We may assume without loss of generality that $\beta = |X_1| - |C_1| = \max\{|X_i| - |C_i| \mid 1 \leq i \leq s-1\}$.

First suppose that X_1 has size at most γ, in which case Z has size at most $\sum_i |(X_i| - |C_i|) \leq (s-1)\gamma$. By definition of the sets C_i, we are required to use colours on Z that are not in C. In other words, we are to colour Z independently from the way we coloured the rest of G. Because in this case $|Z|$ only depends on s and t, we use brute force to compute the chromatic number $\chi_{G[Z]}$ of $G[Z]$. Hence we find that $\texttt{total}(c) = k_c + \chi_{G[Z]}$ in this case.

Now suppose that X_1 has size at least $\gamma + 1$. We observe the following. Let v be any vertex in D. If v is adjacent to all vertices of a clique X_i, then $c(v)$ does not appear in C_i, as otherwise we can cut the branch. If v is adjacent to a set X_i' of $t-2$ vertices of a clique X_i but not to some vertex $w \in X_i$, then $X_i' \cup \{v, w\}$ induce a $K_t - e$ in G, which is not possible. Hence, we may assume without of generality that every vertex in D is adjacent to at most $t-3$ vertices in every clique X_i. This means that the total number of vertices in X_i that have a neighbour in D is at most $(t-3)|D|$. Also recall that every vertex in X_i has at most $t-3$ neighbours in any X_j with $j \neq i$. We may assume without loss of generality that for some $h \leq s-1$, the cliques $X_1, \ldots, X_h$ are precisely those for which $|X_i| \geq \gamma + 1$. We apply the following greedy approach for assigning colours from C_i to every clique X_i with $1 \leq i \leq h$. We arbitrarily colour a set X_1^* of $|C_1|$ vertices of X_1 that are not adjacent to any vertices in D with colours from C_1. We then arbitrarily colour a set X_2^* of $|C_2|$ vertices of X_2 that are not adjacent to any vertices in $D \cup X_1^*$ with colours from C_2, and so on, until we have processed X_h. We can follow this greedy approach, because a clique X_i with $1 \leq i \leq h$ has a size that is sufficiently large, that is, at least $\gamma + 1 = (|D|+2)(s-2)(t-3) + (t-3)|D| + |D| + 1$. (Note that we could have chosen γ to be smaller here, but this value simplifies the arguments in the next paragraph.) Let c^* denote the resulting partial colouring. Note that c^* extends c.

Assume that c^* uses k_{c^*} colours. We claim that $\texttt{total}(c) = k_{c^*} + |X_1| - |C_1|$. Note that $\texttt{total}(c) < k_{c^*} + |X_1| - |C_1|$ is not possible, because of condition (ii) and the fact that X_1 is a clique. Hence, we are left to show that all uncoloured vertices of G can be coloured with at most $|X_1| - |C_1|$ colours. This follows immediately from Lemma 7 by taking $k = |X_1| - |C_1|$, $a = t-3$ and $b = s-1$. We may apply this lemma for the following two reasons. First, for $i = 1, \ldots, h$, we have $|X_i| - |C_i| \leq |X_1| - |C_1|$ by definition. Second, we have $|X_1| - |C_1| \geq \gamma + 1 - |C_1| \geq 2(s-2)(t-3)$.

As we branched in all possible ways, we find that the smallest $\texttt{total}(c)$ is the chromatic number of G. Note that our algorithm runs in fpt time with parameter $s + t$, as required, and that it also produces an optimal colouring of G. $\square$

We are now ready to state and prove our main theorem.

Theorem 6. *The* COLOURING*,* INDEPENDENT SET *and* CLIQUE *problems on* $(sP_1 + P_2, K_t - e)$*-free graphs are fixed-parameter tractable with parameter* $s + t$*.*

Proof. First recall the following. Because a graph is $(sP_1 + P_2, K_t - e)$-free if and only if its complement is $(P_2 + (t-2)P_1, K_{s+2} - e)$-free, we only have to consider the INDEPENDENT SET and COLOURING problems.

Let G be a $(sP_1 + P_2, K_t - e)$-free graph. We start by checking whether G contains an independent set on $s+1$ vertices. We can do this in fpt time with parameter $s+t$ by Theorem 5. If it does, then G is not $(s+1)P_1$-free. By Lemma 1, this means that G must be $(sP_1+P_2, K_{s^2(t-3)+2})$-free. In this case, we can solve COLOURING and INDEPENDENT SET by Lemmas 2 and 3, respectively. Otherwise, that is, if G contains no independent set on $s+1$ vertices, then G is $((s+1)P_1, K_t - e)$-free. In that case, we can solve INDEPENDENT SET and COLOURING by Lemmas 4 and 8, respectively. □

3 Final Remarks on Colouring for (H_1, H_2)-Free Graphs

The ultimate goal is to complete Theorem 3. To help with this, we are currently trying to characterize those classes of (H_1, H_2)-free graphs that have bounded clique-width. However, completing Theorem 3 will also require new proof techniques to deal with a number of non-trivial cases, such as when H_1 is the claw $K_{1,3}$ and H_2 is a long path. As regards our result it seems natural to settle, as a next step, the complexity status of COLOURING for $(sP_2, K_t - e)$-free graphs. Our next result shows that the case $s = 2$ is polynomial-time solvable.

Theorem 7. *The* COLOURING *problem on* $(2P_2, K_t-e)$*-free graphs can be solved in polynomial time for all* $t \geq 2$*.*

Proof. Let $G = (V, E)$ be a $(2P_2, K_t - e)$-free graph for some integer $t \geq 0$. We use induction on t. Due to Theorem 3 (ii)-6, we find that the statement of the theorem holds for $t \leq 4$.

Let $t \geq 5$. We first check if G is P_4-free; this can be done in $O(n^4)$ time by brute force. If so, then we apply Theorem 1. Otherwise, let $wxyz$ be an induced P_4 in G. We partition the vertices in $V \setminus \{x, y\}$ into four sets $W_\emptyset, W_x, W_y, W_{x,y}$ according to their neighbourhood in $\{x, y\}$.

We claim that $G[W_x]$ and $G[W_y]$ must be K_{t-1}-free. In order to obtain a contradiction, suppose that one of them, say $G[W_x]$, contains a K_{t-1}. Because $t \geq 5$, we can pick two distinct vertices a and b of this K_{t-1}. Then z must be adjacent to at least one of a or b, as otherwise $G[a, b, y, z]$ would be isomorphic to $2P_2$. By repeating this argument we find that in fact z must be adjacent to at least $t-2$ vertices of the K_{t-1}. However, these $t-2$ neighbours of z, together with z and x, induce a $K_t - e$ in G, which is a contradiction. Hence $G[W_x]$, and by symmetry, $G[W_y]$ are K_{t-1}-free.

If $a, b \in W_\emptyset$ were adjacent, then $G[a,b,x,y]$ would be isomorphic to $2P_2$, which is not possible. Hence, we find that $W_\emptyset$ is an independent set. We check whether or not $W_\emptyset \cup W_x \cup W_y$ is also an independent set.

First suppose that $W_\emptyset \cup W_x \cup W_y$ is not an independent set. We claim that $G[W_{x,y}]$ is K_{2t-4}-free. In order to obtain a contradiction, suppose that F is a subgraph of $G[W_{x,y}]$ isomorphic to K_{2t-4}. Because $W_\emptyset \cup W_x \cup W_y$ is not an independent set, $W_\emptyset \cup W_x \cup W_y$ contains an edge ab. Let c and d be two distinct vertices of F. Then at least one of a or b must be adjacent to c or d, as otherwise F, and thus G, contains an induced $2P_2$. By repeating this argument, we find that at least $2t-5$ vertices of F must be adjacent to at least one of a or b. Without loss of generality, we assume that a is adjacent to at least $t-2$ vertices of F. Note that a is non-adjacent to at least one of x or y. Without loss of generality, we assume that a is non-adjacent to x. Then $t-2$ neighbours of a in F, together with a and x induce a $K_t - e$ in G. This is a contradiction. Hence, indeed, $G[W_{x,y}]$ is K_{2t-4}-free. Thus the maximum clique size in G is at most $2t-5+t-2+t-2+2+1 = 4t-6$. Thus G is $(2P_2, K_{4t-5})$-free, and consequently also, (P_5, K_{4t-5})-free. Hence, we may apply Theorem 3 (ii)-5.

Now suppose that $W_\emptyset \cup W_x \cup W_y$ is an independent set. Consider any optimal colouring of G. In this colouring, x and y must have different colours. Let us call these colours 1 and 2, respectively. Then colours 1 and 2 cannot be used to colour any vertex in $W_{x,y}$. However, $W_\emptyset \cup W_y \cup \{x\}$ and $W_x \cup \{y\}$ are independent sets. Hence, if necessary, we can re-colour them with colours 1 and 2, respectively in order to obtain a new colouring that is still optimal.

Due to the above observation, we may colour $W_\emptyset \cup W_y \cup \{x\}$ and $W_x \cup \{y\}$ with colours 1 and 2, respectively, and moreover, we may remove all of these vertices from G. What remains is the graph $G[W_{x,y}]$. Because every vertex in $W_{x,y}$ is adjacent to both x and y, and G is $(2P_2, K_t - e)$-free, $G[W_{x,y}]$ is $(2P_2, K_{t-2} - e)$-free graph. By our induction hypothesis, we can solve Colouring in polynomial time on $G[W_{x,y}]$. Hence, we have proven Theorem 7. □

Another possible generalization of our result on Colouring is to consider the following variant of graph colouring. In Precolouring Extension we assume that a (possibly empty) subset $W \subseteq V$ of a graph $G = (V, E)$ is precoloured by a *precolouring* $c_W : W \to \{1, 2, \ldots, k\}$ for some given integer k, and the question is whether we can extend c_W to a k-colouring of G. The classification of Precolouring Extension on H-free graphs is known [14]. However, the classification of Precolouring Extension on (H_1, H_2)-free graphs is still open. In this respect, we note that our results cannot be generalized to another even more general variant of graph colouring called list colouring. A *list assignment* of a graph $G = (V, E)$ is a function $\mathcal{L}$ that assigns a list $L(u)$ of so-called *admissible* colours to each $u \in V$. We say that a colouring $c : V \to \{1, 2, \ldots\}$ *respects* $\mathcal{L}$ if $c(u) \in L(u)$ for all $u \in V$. The List Colouring problem is that of testing whether a given graph has a colouring that respects some given list assignment. Golovach and Paulusma [12] completely classified the complexity of the List Colouring problem for (H_1, H_2)-free graphs by showing that this problem is polynomial-time solvable for (H_1, H_2)-free graphs in the following three cases:

(i) $H_1 \subseteq_i P_3$ or $H_2 \subseteq_i P_3$, (ii) $H_1 \subseteq_i C_3$ and $H_2 \subseteq_i K_{1,3}$ and (iii) $H_1 = sP_1$ for $s \geq 3$ and $H_2 = K_t$ for $t \geq 3$, whereas it is NP-complete for all other pairs (H_1, H_2).

4 The Clique-Width of $(sP_1, K_t - e)$-free Graphs

In Section 1.2, we claimed that our results cannot be obtained by applying the results of Kobler and Rotics [22] and Oum and Seymour [29], which together imply that COLOURING is polynomial-time solvable on classes of graphs that have bounded clique-width. In this section, we will prove this claim by showing that the classes we consider have unbounded clique-width unless s or t is very small. We first state some useful facts for dealing with clique-width.

Fact 1: For a constant k and a class of graphs $\mathcal{G}$, let $\mathcal{G}_{[k]}$ denote the class of graphs obtained from $\mathcal{G}$ by deleting at most k vertices from each graph in $\mathcal{G}$. Then $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}_{[k]}$ has bounded clique-width [24].

Fact 2: For a graph G, the *subgraph complementation* is the operation that consists of replacing every edge of an induced subgraph of G by a non-edge, and vice versa. For a constant k and a class of graphs $\mathcal{G}$, let $\mathcal{G}^{(k)}$ be the class of graphs obtained from $\mathcal{G}$ by applying at most k subgraph complementations on each graph in $\mathcal{G}$. Then $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}^{(k)}$ has bounded clique-width [21].

Fact 3: The clique-width of every graph with maximum vertex degree at most 2 is at most 4 (see e.g. [6]).

Fact 4: The class of n by n grid graphs (see Figure 2 for an example) has unbounded clique-width (see e.g. [27]).

Fact 5: The class of walls (see Figure 3 for an example) has unbounded clique-width (see e.g. [21]).

We are now ready to prove that even for small fixed integers s and t, the class of $(sP_1 + P_2, K_t - e)$-free graphs has unbounded clique-width. In fact this statement even holds for the class of $(sP_1, K_t - e)$-free graphs, a proper subclass.

Theorem 8. *The class of $(sP_1, K_t - e)$-free graphs has bounded clique-width if and only if $s \leq 2$ or $t \leq 3$ or $s + t \leq 8$.*

Proof. We first prove the backward implication.

Suppose that $s \leq 2$. Because $2P_1$-free graphs are precisely those that consist of a single clique, they have clique-width at most 2.

Suppose that $t \leq 3$. The graph $K_3 - e$ is also known as P_3. Graphs which are P_3-free are precisely those that are the disjoint union of cliques and thus also have clique-width at most 2.

Suppose that $s + t \leq 8$ with $s \geq 3$ and $t \geq 4$. Consider an $(sP_1, K_t - e)$-free graph G. We may assume that the graph is sP_1-free, but contains an $(s-1)P_1$, say, on the vertices $x_1, \ldots, x_{s-1}$. Let $X = \{x_1, \ldots, x_{s-1}\}$. Let $W_{i,j}$ be the set of vertices of G adjacent to both x_i and x_j, and let $H = \cup_{i \neq j} W_{i,j}$. Since G

is $(K_t - e)$-free, every $W_{i,j}$ must be K_{t-2}-free. However, $W_{i,j}$ is also sP_1-free, so by Ramsey's Theorem, it must be bounded in size by a function of s and t. Therefore the set H is bounded in size, so by Fact 1, we may remove all the vertices from this set and the resulting $(sP_1, K_t - e)$-free graph G' will have clique-width bounded by a function of s and t if and only if G has.

Now let U_i be the set of vertices whose only neighbour in X is x_i. Since G' is sP_1-free, each U_i must be a clique (otherwise we could replace x_i in X by two non-adjacent vertices in U_i to form an sP_1). Now suppose that for some i, j with $i \neq j$, there is a vertex $y \in U_i$ with $t-2$ neighbours in U_j. Then these $t-2$ vertices, together with y and x_j, would induce a $K_t - e$ in G', which would be a contradiction. Therefore each vertex in U_i has at most $t-3$ neighbours in U_j for any i, j with $i \neq j$. We now apply the complementation operation with respect to the sets $U_i \cup \{x_i\}$ for each i, that is, we change these cliques into independent sets. By Fact 2, the resulting graph G'' will have clique-width bounded by a function of s and t if and only if G' has. In G'', every vertex will have degree at most $(s-2)(t-3)$. Since $s+t \leq 8, s \geq 3, t \geq 4$, we know that $(s-2)(t-3) \leq 2$. Hence the result follows from Fact 3.

We now prove the forward implication. In order to do this it suffices to present three graph classes of unbounded clique-width, namely the classes of $(3P_1, K_6 - e)$-free, $(4P_1, K_5 - e)$-free and $(5P_1, K_4 - e)$-free graphs, respectively.

First consider the class of $(3P_1, K_6 - e)$-free graphs. Lozin and Voltz [25] showed that the class of bipartite $(4P_1 + P_2)$-free graphs has unbounded clique-width. Consequently, the class of $(K_3, 4P_1 + P_2)$-free graphs has unbounded clique-width. By Fact 2, the class of $(3K_1, K_6 - e)$-free graphs has unbounded clique-width.

Now consider the class of $(4P_1, K_5 - e)$-free graphs. We construct a subclass of this class of graphs that has unbounded clique-width.

Let $\mathcal{G}$ be the class of n by n grid graphs. For every $G \in \mathcal{G}$ we do as follows. We colour each vertex of G with the sum of its x and y coordinates modulo 3; see Figure 2 for an example. This yields a 3-colouring of G, the colour classes of which correspond to three independent sets. We now replace each of these three independent sets by a clique. By Facts 2 and 4, the resulting graph class $\mathcal{G}'$ obtained in this way has unbounded clique-width.

We claim that every graph in $\mathcal{G}'$ is $(4P_1, K_5 - e)$-free. Let $G' \in \mathcal{G}'$. We first observe that G' is $4P_1$-free, because $V(G')$ can be partitioned into three cliques. Now suppose that G' contains an induced $K_5 - e$. By construction, G' contains no K_3 that consists of one vertex from each of the colour classes. Thus there must be two colour classes that contain all five of the vertices in the $K_5 - e$. However any vertex in any colour class can have at most two neighbours in any other colour class. This leads to a contradiction. Hence, we have found a subclass of the class of $(4P_1, K_5 - e)$-free graphs, namely the class $\mathcal{G}'$, that is of unbounded clique-width.

Finally, we consider the class of $(5P_1, K_4 - e)$-free graphs. We construct a subclass of this class of graphs that has unbounded clique-width.

$$
\begin{array}{ccccccccc}
1 & - & 2 & - & 0 & - & 1 & - & 2 \\
| & & | & & | & & | & & | \\
0 & - & 1 & - & 2 & - & 0 & - & 1 \\
| & & | & & | & & | & & | \\
2 & - & 0 & - & 1 & - & 2 & - & 0 \\
| & & | & & | & & | & & | \\
1 & - & 2 & - & 0 & - & 1 & - & 2 \\
| & & | & & | & & | & & | \\
0 & - & 1 & - & 2 & - & 0 & - & 1
\end{array}
$$

Fig. 2. A 5 by 5 grid graph coloured with three colours.

Let $\mathcal{G}$ be the class of walls. For each $G \in \mathcal{G}$ we do as follows. On the top "level" of G, we colour the vertices (in order) $1, 2, 3, 4, 1, 2, 3, 4$, and so on. On the second level, we colour them $3, 4, 1, 2, 3, 4, 1, 2$ and so on. On subsequent levels, we alternate between these two colourings; see Figure 3 for an example. Note that this is a 4-colouring of G, as no two vertices of the same colour are adjacent. Also, no vertex has two neighbours of the same colour. We replace each of the four independent sets that form the colour classes by a clique. By Facts 2 and 5, the resulting graph class $\mathcal{G}'$ has unbounded clique-width.

We claim that every graph in $\mathcal{G}'$ is $(5P_1, K_4 - e)$-free. Let $G' \in \mathcal{G}'$. Because $V(G')$ can be partitioned into four cliques, G' is $5P_1$-free. No vertex from one clique can have two neighbours in one of the other cliques. Therefore, if a $K_4 - e$ is present, it must consist of one vertex from each of the cliques. Consider a vertex x of degree 3 in the $K_4 - e$, and let i be the colour of this vertex. By construction, the vertices of colours present in the $K_4 - e$ that are not of colour i are uniquely determined by the choice of x, as they must be neighbours of x in G'. By construction, the neighbours of x in G' are pairwise non-adjacent. Hence, we obtain a contradiction. We have thus constructed a subclass of the class of $(5P_1, K_4 - e)$-free graphs, namely the class $\mathcal{G}'$, that is of unbounded clique-width. $\square$

References

1. A. Brandstädt, J. Engelfriet, H.-O. Le, and V.V. Lozin: Clique-Width for 4-Vertex Forbidden Subgraphs, Theory Comput. Syst. 39 (2006) 561–590.
2. A. Brandstädt and D. Kratsch, On the structure of (P_5, gem)-free graphs, Discrete Applied Mathematics 145 (2005) 155–166.
3. A. Brandstädt, H.-O. Le, R. Mosca, Gem- and co-gem-free graphs have bounded clique-width, Internat. J. Found. Comput Sci. 15 (2004), 163–185.
4. H.J. Broersma, P.A. Golovach, D. Paulusma and J. Song, Determining the chromatic number of triangle-free $2P_3$-free graphs in polynomial time, Theoretical Computer Science 423 (2012) 1–10.
5. H.J. Broersma, P.A. Golovach, D. Paulusma and J. Song, Updating the complexity status of coloring graphs without a fixed induced linear forest, Theoretical Computer Science 414 (2012) 9–19.

```
1 — 2 — 3 — 4 — 1 — 2 — 3
|       |       |       |
3 — 4 — 1 — 2 — 3 — 4 — 1 — 2
    |       |       |       |
1 — 2 — 3 — 4 — 1 — 2 — 3 — 4
|       |       |       |
3 — 4 — 1 — 2 — 3 — 4 — 1 — 2
    |       |       |       |
1 — 2 — 3 — 4 — 1 — 2 — 3 — 4
|       |       |       |
3 — 4 — 1 — 2 — 3 — 4 — 1
```

Fig. 3. A wall with a special 4-colouring.

6. B. Courcelle and S. Olariu, Upper bounds to the clique width of graphs, Discrete Applied Mathematics 101, (2000) 77–114.
7. J.F. Couturier, P.A. Golovach, D. Kratsch and D. Paulusma, List coloring in the absence of a linear forest, Proc. WG 2011, LNCS 6986 (2011) 119–130.
8. J.F. Couturier, P.A. Golovach, D. Kratsch and D. Paulusma, On the parameterized complexity of coloring graphs in the absence of linear forest, Journal of Discrete Algorithms 15 (2012) 56–62.
9. K. Dabrowski, V.V. Lozin, H. Müller and D. Rautenbach, Parameterized complexity of the weighted independent set problem beyond graphs of bounded clique number, Journal of Discrete Algorithms 14 (2012) 207–213.
10. K. Dabrowski, V.V. Lozin, R. Raman and B. Ries, Colouring vertices of triangle-free graphs without forests, Discrete Mathematics 312 (2012) 1372–1385.
11. R. Diestel, Graph Theory, Springer-Verlag, Electronic Edition, 2005.
12. P.A. Golovach and D. Paulusma, List coloring in the absence of two subgraphs, Proc. of CIAC 2013, LNCS, to appear.
13. P.A. Golovach, D. Paulusma and J. Song, 4-Coloring H-free graphs when H is small, Discrete Applied Mathematics 161 (2013) 140–150.
14. P.A. Golovach, D. Paulusma and J. Song, Closing complexity gaps for coloring problems on H-free graphs, Proc. ISAAC 2012, LNCS 7676 (2012)14–23.
15. S. Gravier, C.T. Hoáng and F. Maffray, Coloring the hypergraph of maximal cliques of a graph with no long path, Discrete Mathematics 272 (2003) 285–290.
16. A. Gyárfás, Problems from the world surrounding perfect graphs, Zastosowania Matematyki Applicationes Mathematicae XIX, 3–4 (1987) 413–441.
17. P. Hall, On Representatives of Subsets, J. London Math. Soc. s1–10 (1) (1935) 26–30.
18. C.T. Hoàng, M. Kamiński, V.V. Lozin, J. Sawada and X. Shu, Deciding k-colorability of P_5-free graphs in polynomial time, Algorithmica 57 (2010) 74–81.
19. C.T. Hoàng, F. Maffray and M. Mechebbek, A characterization of b-perfect graphs, Journal of Graph Theory 71 (2012) 95–122.
20. S. Huang, Improved complexity results on k-coloring P_t-free graphs, Proc. MFCS 2013, LNCS, to appear.
21. M. Kamiński, V.V. Lozin and M. Milanič, Recent developments on graphs of bounded clique-width, Discrete Applied Mathematics 157 (2009) 2747–2761.
22. D. Kobler and U. Rotics, Edge dominating set and colorings on graphs with fixed clique-width, Discrete Applied Mathematics 126 (2003) 197–221.

23. D. Král', J. Kratochvíl, Zs. Tuza, and G.J. Woeginger, Complexity of coloring graphs without forbidden induced subgraphs, Proc. WG 2001, LNCS 2204 (2001) 254–262.
24. V.V. Lozin and D. Rautenbach, On the band-, tree-, and clique-width of graphs with bounded vertex degree, SIAM Journal on Discrete Mathematics 18 (2004) 195–206.
25. V.V. Lozin and J. Voltz, The Clique-with of bipartite graphs in monogenic classes, Int. J. Found. Comput. Sci. 19 (2008) 477.
26. F. Maffray and M. Preissmann, On the NP-completeness of the k-colorability problem for triangle-free graphs, Discrete Mathematics 162 (1996) 313–317.
27. J.A. Makowsky and U. Rotics, On the clique-width of graphs with few P_4's, Int. J. Found. Comput. Sci. 10 (1999) 329–348.
28. D. Malyshev, The coloring problem for classes with two small obstructions, Manuscript, arXiv:1307.0278.
29. S.-I. Oum and P. Seymour, Approximating clique-width and branch-width, Journal of Combinatorial Theory, Series B 96 (2006) 514–528.
30. F.P. Ramsey, On a Problem of Formal Logic, Proc. London Math. Soc. s2–30 (1930) 264–286.
31. B. Randerath, 3-colorability and forbidden subgraphs. I., Characterizing pairs, Discrete Mathematics 276 (2004) 313–325.
32. B. Randerath and I. Schiermeyer, A note on Brooks' theorem for triangle-free graphs, Australas. J. Combin. 26 (2002) 3–9.
33. B. Randerath and I. Schiermeyer, Vertex colouring and forbidden subgraphs - a survey, Graphs and Combinatorics 20 (2004) 1–40.
34. B. Randerath, I. Schiermeyer, On maximum independent sets in P_5-free graphs, Discrete Applied Mathematics, 158(9) (2010) 1041–1044.
35. D. Schindl, Some new hereditary classes where graph coloring remains NP-hard, Discrete Mathematics 295 (2005)197–202.
36. Zs. Tuza, Graph colorings with local restrictions - a survey, Discuss. Math. Graph Theory 17 (1997) 161–228.