Critical Vertices and Edges in H-free Graphs*

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Abstract. A vertex or edge in a graph is critical if its deletion reduces the chromatic number of the graph by 1. We consider the problems of deciding whether a graph has a critical vertex or edge, respectively. We give a complexity dichotomy for both problems restricted to H-free graphs, that is, graphs with no induced subgraph isomorphic to H. Moreover, we show that an edge is critical if and only if its contraction reduces the chromatic number by 1. Hence, we also obtain a complexity dichotomy for the problem of deciding if a graph has an edge whose contraction reduces the chromatic number by 1.

Keywords. edge contraction, vertex deletion, chromatic number.

1 Introduction

For a positive integer k, a *k-colouring* of a graph $G = (V, E)$ is a mapping $c : V \to \{1, 2, \ldots, k\}$ such that no two end-vertices of an edge are coloured alike, that is, $c(u) \neq c(v)$ if $uv \in E$. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k for which G has a k-colouring. The well-known Colouring problem is to test if $\chi(G) \leq k$ for a given graph G and integer k. If k is not part of the input, then we call this problem k-Colouring instead. Lovász [15] proved that 3-Colouring is NP-complete.

Due to its computational hardness, the Colouring problem has been well studied for special graph classes. We refer to the survey [11] for an overview of the results on Colouring restricted to graph classes characterized by one or two forbidden induced subgraphs. In particular, Král', Kratochvíl, Tuza, and Woeginger [14] classified Colouring for *H-free graphs*, that is, graphs that do not contain a single graph H as an induced subgraph. To explain their result we need the following notation. For a graph F, we write $F \subseteq_i G$ to denote that F is an induced subgraph of a graph G. The *disjoint union* of two graphs G_1 and G_2 is the graph $G_1 + G_2$, which has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We write rG for the disjoint union of r copies of G by rG. The graphs P_r and C_r denote the induced path and cycle on r vertices, respectively. We can now state the theorem of Král et al.

Theorem 1 ([14]). *Let H be a graph. If $H \subseteq_i P_4$ or $H \subseteq_i P_1 + P_3$, then* Coloring *restricted to H-free graphs is polynomial-time solvable, otherwise it is* NP*-complete.*

For a vertex u or edge e in a graph G, we let $G - u$ and $G - e$ be the graph obtained from G by deleting u or e, respectively. Note that such an operation may

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reduce the chromatic number of the graph by at most 1. We say that u or e is *critical* if $\chi(G-u) = \chi(G)-1$ or $\chi(G-e) = \chi(G)-1$, respectively. A graph is vertex-critical if every vertex is critical and edge-critical if every edge is critical. To increase our understanding of the COLOURING problem and to obtain certifying algorithms that solve COLOURING for special graph classes, vertex-critical and edge-critical graphs have been studied intensively in the literature, see for instance [4–6, 8, 10, 12, 13, 16] for certifying algorithms for (subclasses of) H-free graphs and in particular P_r-free graphs.

In this paper we consider the problems CRITICAL VERTEX and CRITICAL EDGE, which are to test if a graph has a critical vertex or critical edge, respectively. In addition we also consider the edge contraction variant of these two problems. We let G/e denote the graph obtained from G after contracting $e = vw$, that is, after removing v and w and replacing them by a new vertex made adjacent to precisely those vertices adjacent to v or w in G (without creating multiple edges). Contracting an edge may reduce the chromatic number of the graph by at most 1. An edge e is *contraction-critical* if $\chi(G/e) = \chi(G)-1$. This leads to the CONTRACTION-CRITICAL EDGE problem, which is to test if a graph has a contraction-critical edge.

1.1 Our Results

We prove the following complexity dichotomies for CRITICAL VERTEX, CRITICAL EDGE and CONTRACTION-CRITICAL EDGE restricted to H-free graphs.

Theorem 2. *If a graph $H \subseteq_i P_4$ or of $H \subseteq_i P_1 + P_3$, then* CRITICAL VERTEX*,* CRITICAL EDGE *and* CONTRACTION-CRITICAL EDGE *restricted to H-free graphs are polynomial-time solvable, otherwise they are* NP*-hard or co-*NP*-hard.*

We note that the classification in Theorem 2 coincides with the one in Theorem 1. The polynomial-time cases for CRITICAL VERTEX and CONTRACTION-CRITICAL EDGE can be obtained from Theorem 1. The reason for this is that a class of H-free graphs is not only closed under vertex deletions, but also under edge contractions whenever H is a *linear forest*, that is, a disjoint union of a set of paths (see Section 5 for further details). However, no class of H-free graphs is closed under edge deletion. We get around this issue by proving, in Section 2, that an edge is critical if and only if it is contraction-critical. Hence, CRITICAL EDGE and CONTRACTION-CRITICAL EDGE are equivalent.

The NP-hardness constructions of Theorem 1 cannot be used for proving the hard cases for CRITICAL VERTEX, CRITICAL EDGE and CONTRACTION-CRITICAL EDGE. Instead we construct new hardness reductions in Sections 3 and 4. In Section 3 we prove that the three problems are NP-hard for H-free graphs if H contains a claw or a cycle on three or more vertices. In the remaining case H is a linear forest. In Section 4 we prove that the three problems are co-NP-hard even for $(C_5, 4P_1, 2P_1+P_2, 2P_2)$-free graphs. In Section 5 we combine the known cases with our new results from Sections 2–4 in order to prove Theorem 2.

1.2 Consequences

Our results have consequences for the computational complexity of two graph blocker problems. Let S be some fixed set of graph operations, and let π be some fixed graph parameter. Then, for a given graph G and integer $k \geq 0$, the S-BLOCKER(π) problem asks if G can be modified into a graph G' by using at most k operations from

S so that $\pi(G') \leq \pi(G) - d$ for some given *threshold* $d \geq 0$. Over the last few years, the S-Blocker(π) problem has been well studied, see for instance [1–3, 7, 9, 19–23]. If S consists of a single operation that is either a vertex deletion or edge contraction, then S-Blocker(π) is called Vertex Deletion Blocker(π) or Contraction Blocker(π), respectively. By taking $d = k = 1$ and $\pi = \chi$ we obtain the problems Critical Vertex and Contraction-Critical Edge, respectively. We showed in [20] how the results for Critical Vertex and Contraction-Critical Edge can be extended with other results to get complexity dichotomies for Vertex Deletion Blocker(χ) and Contraction Blocker(χ) for H-free graphs.

1.3 Future Work

A graph G is *$(H_1, \ldots, H_p)$-free* for some family of graphs $\{H_1, \ldots, H_p\}$ and integer $p \geq 2$ if G is H-free for every $H \in \{H_1, \ldots, H_p\}$. As a direction for future research we propose classifying the computational complexity of our three problems for $(H_1, \ldots, H_p)$-free graphs for any $p \geq 2$. We note that such a classification for Coloring is still wide open even for $p = 2$ (see [11]). Hence, research in this direction might lead to an increased understanding of the complexity of the Coloring problem.

2 Equivalence

We prove the following result, which implies that the problems Critical Edge and Contraction-Critical Edge are equivalent.

Proposition 1. *An edge is critical if and only if it is contraction-critical.*

Proof. Let $e = uv$ be an edge in a graph G. First suppose that e is critical, so $\chi(G - e) = \chi(G) - 1$. Then u and v are colored alike in any coloring of $G - e$ that uses $\chi(G - e)$ colors. Hence, the graph G/e obtained from contracting e in G can also be colored with $\chi(G - e)$ colors. Indeed, we simply copy a $(\chi(G - e))$-coloring of $G - e$ such that the new vertex in G/e is colored with the same color as u and v in $G - e$. Hence $\chi(G/e) = \chi(G - e) = \chi(G) - 1$, which means that e is contraction-critical.

Now suppose that e is contraction-critical, so $\chi(G/e) = \chi(G) - 1$. By copying a $\chi(G/e)$-coloring of G/e such that u and v are colored with the same color as the new vertex in G/e, we obtain a coloring of $G - e$. So we can color $G - e$ with $\chi(G/e)$ colors as well. Hence $\chi(G - e) = \chi(G/e) = \chi(G) - 1$, which means that e is critical. □

3 Forbidding Claws or Cycles

The *claw* is the 4-vertex star $K_{1,3}$ on vertices a, b, c, d and edges ab, ac and ad. In this section we prove that the problems Critical Vertex, Critical Edge and Contraction-Critical Edge are NP-hard for H-free graphs whenever the graph H contains a claw or a cycle on at least three vertices.

Let $\mathcal{G}$ be a graph class with the following property: if $G \in \mathcal{G}$, then so are $2G$ and $G + K_r$ for any $r \geq 1$. We call such a graph class *clique-proof*.

Theorem 3. *If* Coloring *is* NP*-complete for a clique-proof graph class $\mathcal{G}$, then both* Critical Vertex *and* Contraction-Critical Edge *are* NP*-hard for $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be a graph class that is clique-proof. From a given graph $G \in \mathcal{G}$ and integer $\ell \geq 1$ we construct the graph $G' = 2G + K_{\ell+1}$. Note that $G' \in \mathcal{G}$ by definition and that $\chi(G') = \max\{\chi(G), \ell + 1\}$. We first prove that $\chi(G) \leq \ell$ if and only if G' contains a contraction-critical edge.

Suppose that $\chi(G) \leq \ell$. Then $\chi(G') = \chi(K_{\ell+1}) = \ell+1$. In G' we contract an edge of the $K_{\ell+1}$. This yields the graph $G^* = 2G+K_\ell$, which has chromatic number $\chi(G^*) = \ell$, as $\chi(K_\ell) = \ell$ and $\chi(G) \leq \ell$. As $\chi(G') = \ell + 1$, this means that $\chi(G^*) = \chi(G') - 1$. Hence G' contains a contraction-critical edge.

Now suppose that G' contains a contraction-critical edge. Let G^* be the resulting graph after contracting this edge. Then $\chi(G^*) = \chi(G') - 1$. As contracting an edge in one of the two copies of G in G' does not lower the chromatic number of G', the contracted edge must be in the $K_{\ell+1}$, that is, $G^* = 2G + K_\ell$. As this did result in a lower chromatic number, we conclude that $\chi(G') = \chi(K_{\ell+1}) = \ell + 1$ and $\chi(G^*) = \chi(2G + K_\ell) = \max\{\chi(G), \ell\} = \ell$. The latter equality implies that $\chi(G) \leq \ell$.

From the above we conclude that Contraction-Critical Edge is NP-hard. We can prove that Critical Vertex is NP-hard by using the same arguments. □

We also need a result of Maffray and Preissmann as a lemma.

Lemma 1 ([17]). *The* 3-Coloring *problem is* NP*-complete for C_3-free graphs.*

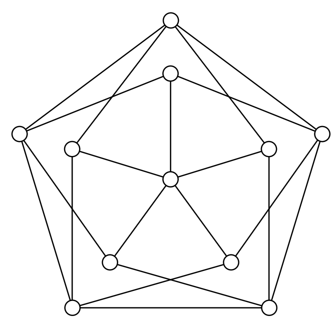

Fig. 1. The Grötzsch graph.

We are now ready to prove the main result of this section.

Theorem 4. *Let H be a graph such that $H \supseteq_i K_{1,3}$ or $H \supseteq_i C_r$ for some $r \geq 3$. Then the problems* Critical Vertex*,* Critical Edge *and* Contraction-Critical Edge *are* NP*-hard for H-free graphs.*

Proof. By Proposition 1 it suffices to consider Critical Vertex and Contraction-Critical Edge. If H is not a clique, then the class of H-free graphs is clique-proof. Hence, in this case, we can use Theorems 1 and 3 to obtain NP-hardness.

Suppose H is a clique. It suffices to show NP-completeness for $H = C_3$. We reduce from 3-Coloring restricted to C_3-free graphs. This problem is NP-complete by Lemma 1. Let G be a C_3-free graph that is an instance of 3-Coloring. We obtain an instance of Critical Vertex or Contraction-Critical Edge as follows. Take the disjoint union of two copies of G and the Grötzsch graph F (see Figure 1), which is known to be 4-colorable but not 3-colorable (see [24]). Call the resulting graph G', so $G' = 2G + F$. As G and F are C_3-free, G' is C_3-free. We claim that G is 3-colorable if and only if G' has a critical vertex if and only if G' has a contraction-critical edge. This can be proven via similar arguments as used in the proof of Theorem 3, with F playing the role of $K_{\ell+1}$. □

Note that in Theorem 4 we cannot prove membership in NP, as Coloring is NP-complete for the class of H-free graphs if $H \supseteq_i K_{1,3}$ or $H \supseteq_i C_r$ for some $r \geq 3$ due to Theorem 1. As such, it is not clear if there exists a certificate.

4 Forbidding Linear Forests

In this section we prove our second hardness result needed to show Theorem 2. We first introduce some additional terminology.

Let G be a graph. The graph $\overline{G}$ denotes the *complement* of G, that is, the graph with vertex set $V(G)$ and an edge between two vertices u and v if and only if u and v are not adjacent in G. A subset K of vertices in G is a *clique* if any two vertices in K are adjacent to each other. A *clique cover* of a graph G is a set $\mathcal{K}$ of cliques in G, such that each vertex of G belongs to exactly one clique of $\mathcal{K}$. The *clique covering number* $\sigma(G)$ is the size of a smallest clique cover of G. Note that $\chi(G) = \sigma(\overline{G})$. The size of a largest clique in a graph G is denoted by $\omega(G)$.

The hardness construction in the proof of our next result uses clique covers. Král et al. [14] proved that Coloring is NP-hard for $(C_5, 4P_1, P_1 + 2P_2, 2P_2)$-free graphs. This does not give us hardness for Critical Vertex or Critical Edge, but we can use some elements of their construction. For instance, we reduce from a similar NP-complete problem as they do, namely the NP-complete problem Monotone 1-in-3-SAT, which is defined as follows. Let Φ be a formula with clause set C of size m and variable set X of size n, so that each clause in C consists of three distinct positive literals, and each variable in X occurs in exactly three clauses. The question is whether Φ has a truth assignment, such that each clause is satisfied by exactly one variable. In that case we say that Φ is *1-satisfiable*. Note that $m = n$. Moore and Robson proved that this problem is NP-complete.

Lemma 2 ([18]). Monotone 1-in-3-SAT *is* NP*-complete.*

We are now ready to prove the main result of this section.

Theorem 5. *The problems* Critical Vertex*,* Critical Edge *and* Contraction-Critical Edge *are co-*NP*-hard for* $(C_5, 4P_1, 2P_1 + P_2, 2P_2)$*-free graphs.*

Proof. By Proposition 1 it suffices to consider Critical Vertex and Critical Edge. We will first consider Critical Vertex and show that the equivalent problem whether a graph has a vertex whose deletion reduces the clique covering number by 1 is co-NP-hard for $(C_4, C_5, K_4, \overline{2P_1 + P_2})$-free graphs. We call such a vertex *critical* as well. The complement of a $(C_4, C_5, K_4, \overline{2P_1 + P_2})$-free graphs is $(C_5, 4P_1, 2P_1 + P_2, 2P_2)$-free. Hence by proving this co-NP-hardness result we will have proven the theorem for Critical Vertex.

As mentioned, we reduce from Monotone 1-in-3-SAT, which is NP-complete due to Lemma 2. Given an instance Φ of Monotone 1-in-3-SAT with clause set C and variable set X, we construct a graph $G = (V, E)$ as follows. For every clause $c \in C$, the clause gadget $G_c = (V_c, E_c)$ is a cycle of length 7. For $c = (x, y, z)$, we let three pairwise non-adjacent vertices $c(x), c(y), c(z)$ of G_c correspond to the three variables x, y, z. We denote the other four vertices of G_c by $a_i^c, 1 \leq i \leq 4$, so that $G_c = c(x)a_1^c a_2^c c(y)a_3^c c(z)a_4^c c(x)$. For each variable $x \in X$ we let the variable gadget Q_x consist of the triangle $c(x)c'(x)c''(x)c(x)$, where c, c', c'' are the three clauses containing x. See Figure 2 for an illustration of the construction. We observe that $|V(G)| = 7n$ and that G is $(C_4, C_5, K_4, \overline{2P_1 + P_2})$-free with $\omega(G) = 3$.

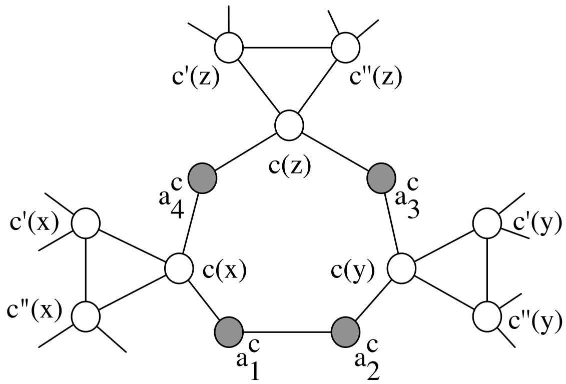

Fig. 2. The clause gadget G_c and three variable gadgets Q_x, Q_y and Q_z.

In order to prove co-NP-hardness we first need to deduce a number of properties of our gadget. We do this via a number of claims.

Claim 1. There exists a minimum clique cover of G, in which each a_i^c is covered by a clique of size 2, and moreover, every two vertices a_1^c and a_2^c belong to the same (2-vertex) clique.

We prove Claim 1 as follows. Let $\mathcal{K}$ be a minimum clique cover of G. Suppose two vertices a_1^c and a_2^c belong to two different cliques K, K' of $\mathcal{K}$. If one of K, K' has size 1, say $K = \{a_c^1\}$, then we can replace K and K' by $\{a_1^c, a_2^c\}$ and $K' \setminus \{a_2^c\}$. This yields a new minimum size clique cover of G, in which a_1^c and a_2^c belong to the same clique. Alternatively, if K and K' each have size 2, then $K = \{a_1^c, c(x)\}$ and $K' = \{a_2^c, c(y)\}$. Then, by construction, $\mathcal{K}$ contains a clique that either consists of a_3^c or of a_4^c , say a_4^c. We replace the cliques $\{a_4^c\}$ and K by $\{a_4^c, c(x)\}$ and $\{a_1^c\}$, respectively, and return to the previous situation. Hence we may assume without loss of generality that $\{a_1^c, a_2^c\}$ is a clique in K. This means that if a_3^c or a_4^c forms a 1-vertex clique in $\mathcal{K}$, then we can safely add $c(y)$ or $c(x)$, respectively, to it. This proves Claim 1.

Now let $\mathcal{K}$ be a minimum clique cover. By Claim 1, we may assume without loss of generality that each a_i^c is covered by a clique of size 2, and moreover, that every two vertices a_1^c and a_2^c belong to the same (2-vertex) clique. Since the clause gadgets G_c are pairwise non-intersecting and isomorphic to C_7, it takes at least four cliques to cover the vertices of every G_c. This means that exactly $3n$ cliques are needed to cover the $4n$ vertices a_i^c. By construction, we also find that $2n$ vertices $c(x)$ are covered by these cliques. Since $\omega(G) = 3$, at least $n/3$ other cliques are necessary to cover the n

remaining vertices $c(x)$. Hence, $\mathcal{K}$ has size at least $\frac{10}{3}n$, that is,

$$\sigma(G) \geq \frac{10}{3}n.$$

We now prove three more claims.

Claim 2. Φ is 1-satisfiable if and only if $\sigma(G) = \frac{10}{3}n$.

We prove Claim 2 as follows. First suppose Φ is 1-satisfiable. We construct a clique cover $\mathcal{K}$ in the following way. If x is true, then we let $\mathcal{K}$ contain the triangle Q_x. Since each clause c contains exactly one true variable for each G_c, exactly one vertex of G_c is covered by a variable gadget. Then $\mathcal{K}$ contains three cliques of size 2 covering the six other vertices of G_c. Hence $\mathcal{K}$ has size $\frac{10}{3}n$. As $\sigma(G) \geq \frac{10}{3}n$, this implies that $\sigma(G) = \frac{10}{3}n$.

Now suppose $\sigma(G) = \frac{10}{3}n$. Let $\mathcal{K}$ be a minimum clique cover of G. By Claim 1, we may assume without loss of generality that each a_i^c is covered by a clique of size 2, and moreover, that every two vertices a_1^c and a_2^c belong to the same (2-vertex) clique. Then at least $n/3$ other cliques are necessary to cover the vertices $c(x)$ that are not in a 2-vertex clique with a vertex a_i^c. Hence, as $\sigma(G) = \frac{10}{3}n$, these vertices are covered by exactly $n/3$ triangles, each one corresponding to one variable x (these are the only triangles in G). We assign the value true to a variable $x \in X$ if and only if its corresponding triangle Q_x is in the clique cover. Then, for each $c \in C$, exactly one variable is true, namely the one that corresponds to the unique vertex of G_c covered by a triangle. So Φ is 1-satisfiable. This completes the proof of Claim 2.

Claim 3. If G has a clique cover $\mathcal{K} = \{K_1, \dots, K_{\frac{10}{3}n}\}$, then each $K_i \in \mathcal{K}$ consists of either two or three vertices.

We prove Claim 3 as follows. As $\sigma(G) \geq \frac{10}{3}n$ and $|\mathcal{K}| = \frac{10}{3}n$, we find that $\mathcal{K}$ is a minimum clique cover. With each $v \in V$, we associate a *weight* $w_v \geq 0$ as follows. For $K_i \in \mathcal{K}$ and $v \in K_i$, we define $w_v = 1/|K_i|$. Since $\omega(G) = 3$ we have $w_v \in \{\frac{1}{3}, \frac{1}{2}, 1\}$. So we have

$$\sum_{G_c} \sum_{v \in V_c} w_v = \sum_{v \in V} w_v = \sum_{i=1}^{\frac{10}{3}n} \sum_{v \in K_i} w_v = \frac{10}{3}n,$$

where the first equality holds, because the clause gadgets G_c are vertex-disjoint. We show that for every c we have $\Sigma_{v \in V_c} w_v \geq \frac{10}{3}$. Since every a_i^c has exactly two neighbours and these neighbours are not adjacent, we have $w_{a_i^c} \in \{\frac{1}{2}, 1\}$. If there exists an index i such that $w_{a_i^c} = 1$, then

$$\sum_{v \in V_c} w_v \geq 1 + 3 \times \frac{1}{2} + 3 \times \frac{1}{3} = \frac{7}{2} > \frac{10}{3}.$$

Now if a_i^c has weight $w_{a_i^c} = \frac{1}{2}$ for each $1 \leq i \leq 4$, then a_i^c is covered by a clique of size 2 and the second vertex of this clique has weight $\frac{1}{2}$ as well by definition. Thus if $w_{a_i^c} = \frac{1}{2}$, exactly two among $c(x), c(y), c(z)$ have weight $\frac{1}{2}$. It follows that

$$\sum_{v \in V_c} w_v \geq 4 \times \frac{1}{2} + 2 \times \frac{1}{2} + \frac{1}{3} = \frac{10}{3}.$$

Hence $\sum_{v \in V_c} w_v = \frac{10}{3}$ if and only if each vertex of G_c is in a clique of size 2 or 3. Since $\sum_{G_c} \sum_{v \in V_c} w_v = \frac{10}{3}n$, we obtain $\sum_{v \in V_c} w_v = \frac{10}{3}$ for every $c \in C$. We conclude that each clique in $\mathcal{K}$ is of size 2 or 3. This completes the proof of Claim 3.

Claim 4. If $\sigma(G) > \frac{10}{3}n$, then G has a minimum clique cover $\mathcal{K}$ that contains a clique of size 1.

We prove Claim 4 as follows. Suppose $\sigma(G) > \frac{10}{3}n$. For contradiction, assume that every minimum clique cover of G has no clique of size 1. Let $\mathcal{K}$ be a minimum clique cover of G. By Claim 1, we may assume without loss of generality that each a_i^c is covered by a clique of size 2, and moreover, that every two vertices a_1^c and a_2^c belong to the same (2-vertex) clique. Hence the remaining vertex $c(x)$ is covered by some clique $K_i \in \mathcal{K}$, such that either $K_i = \{c(x), c'(x)\}$ or $K_i = \{c(x), c'(x), c''(x)\}$.

If $K_i = \{c(x), c'(x)\}$, then $c''(x)$ is covered by some clique $K_j = \{c''(x), a\}$. However, then we can take $K_i = \{c(x), c'(x), c''(x)\}$ and $K_j = \{a\}$ to obtain a minimum clique cover with $|K_j| = 1$, a contradiction. Hence $K_i = \{c(x), c'(x), c''(x)\}$. As this holds for every G_c we find that $\sigma(G) = \frac{10}{3}n$, a contradiction. This completes the proof of Claim 4.

We claim that Φ is a 1-satisfiable if and only if G has no critical vertex. First suppose that Φ is 1-satisfiable. By Claims 2 and 3 we find that $\sigma(G) = \frac{10}{3}n$ and every clique in any minimum clique cover of G has size greater than 1. Hence, there is no vertex u of G with $\sigma(G-u) \leq \sigma(G) - 1$, that is, G has no critical vertex.

Now suppose that Φ is not 1-satisfiable. By Claims 2 and 4 we find that $\sigma(G) > \frac{10}{3}n$ and that there exists a minimum clique cover that contains a clique $\{u\}$ of size 1. This means that $\sigma(G-u) = \sigma(G) - 1$. So u is a critical vertex.

We are left to consider the Critical Edge problem. We use the same construction as before except that the cycles G_c are isomorphic to C_{11}. To be more precise, we let $G_c = c(x)a_1^c a_2^c c(y) a_3^c a_4^c a_5^c c(z) a_6^c a_7^c a_8^c c(x)$. Again the resulting graph G is $(C_4, C_5, K_4, \overline{2P_1 + P_2})$-free. By using the same arguments as before we find that if Φ is 1-satisfiable, then every clique in any minimum clique cover of G has size greater than 1. Hence, as G is K_4-free, every clique in any minimum clique cover of G has size 2 or 3. Since G is $\overline{2P_1 + P_2}$-free, we cannot merge two cliques into one by adding a new edge. So $\overline{G}$ has no critical edge.

Now suppose that Φ is not 1-satisfiable. Then using the previous arguments we can prove that there exists a minimum clique cover $\mathcal{K}$ that contains a clique $\{u\}$ of size 1. By the adjusted construction of G_c we find that u is adjacent to exactly one vertex of a 2-vertex clique $\{v, w\}$ of $\mathcal{K}$, say u is adjacent to v but not to w. Then by adding the edge uw, which yields the graph $G + uw$, we merge two cliques into one, meaning that $\sigma(G + uw) = \sigma(G) - 1$. So uv is a critical edge of $\overline{G}$. This completes the proof of Theorem 5. □

5 The Proof of Theorem 2

We are now ready to prove Theorem 2, which we restate below.

Theorem 2. *If a graph $H \subseteq_i P_4$ or of $H \subseteq_i P_1 + P_3$, then* Critical Vertex, Critical Edge *and* Contraction-Critical Edge *restricted to H-free graphs are polynomial-time solvable, otherwise they are* NP*-hard or co-*NP*-hard.*

Proof. Let $H \subseteq_i P_1 + P_3$ or $H \subseteq_i P_4$. Let G be an H-free graph. By Theorem 1 we can compute $\chi(G)$ in polynomial time. We note that any vertex deletion results in a graph that is H-free as well. Hence in order to solve Critical Vertex we can compute the chromatic number of $G - v$ for each vertex v in polynomial time and compare it with

$\chi(G)$. As $(P_1 + P_3)$-free graphs and P_4-free graphs are closed under edge contraction as well, we can follow the same approach for solving Contraction-Critical Edge. By Proposition 1 we obtain the same result for Critical Edge.

Now suppose that neither $H \subseteq_i P_1+P_3$ nor $H \subseteq_i P_4$. If H has a cycle or an induced claw, then we use Theorem 4. Assume not. Then H is a disjoint union of r paths for some $r \geq 1$. If $r \geq 4$ we use Theorem 5. If $r = 3$ then either $H = 3P_1 \subseteq_i P_1 + P_3$, which is not possible, or $H \supseteq_i 2P_1 + P_2$ and we can apply Theorem 5 again. Suppose $r = 2$. If both paths contain an edge, then $2P_2 \subseteq_i H$. If at most one path has edges, then it must have at least four vertices, as otherwise $H \subseteq_i P_1 + P_3$. This means that $2P_1 + P_2 \subseteq_i H$. In both cases we apply Theorem 5. If $r = 1$, then H is a path on at least five vertices, which means $2P_2 \subseteq_i H$. We apply Theorem 5 again. ◻

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