Graph labelings derived from models in distributed computing

Jérémie Chalopin[1] and Daniël Paulusma[2]

[1] LaBRI Université Bordeaux 1,
351 cours de la Libération, 33405 Talence, France
chalopin@labri.fr

[2] Department of Computer Science, Durham University,
Science Laboratories, South Road, Durham DH1 3LE, England.
daniel.paulusma@durham.ac.uk

Abstract. We discuss eleven well-known basic models of distributed computing: four message-passing models that differ by the (non-)existence of port-numbers and a hierarchy of seven local computations models. In each of these models, we study the computational complexity of the decision problem whether the leader election and/or naming problem can be solved on a given network. It is already known that this problem is solvable in polynomial time for two models and co-NP-complete for another one. Here, we settle the computational complexity for the remaining eight problems by showing co-NP-completeness. The results for six models and the already known co-NP-completeness result follow from a more general result on graph labelings.

1 Introduction

In distributed computing, one can find a wide variety of models of communication. These models reflect different system architectures, different levels of synchronization and different levels of abstraction. In this paper we consider eleven well-known basic models that satisfy the following two underlying assumptions. Firstly, a distributed system is represented by a simple (i.e., without loops or multiple edges), connected, undirected graph. Its vertices represent the processors, and its edges represent direct communication links. Secondly, the distributed systems we consider are anonymous, i.e., all the processors execute the same code to solve some problem and they do not have initial identifiers.

The eleven basic models can be divided into four *message-passing* models [6, 14, 16] and seven *local computations* models [1, 4, 5, 11, 12]. In a message-passing model, processors communicate by sending and receiving messages. In a local computations model, a computation step (encoded by a local relabeling rule) involves neighboring processors that synchronize, exchange information, and modify their states.

Understanding the computational power of various models, the role of structural network properties and the role of the initial knowledge enhances our understanding of distributed algorithms. For this purpose a number of standard problems in distributed computing are studied. The election problem is one of the paradigms of the theory of distributed computing. In our setting, a distributed algorithm solves the election problem if it always terminates and in the final configuration exactly one processor is marked as *elected* and all the other processors are *non-elected.* Elections constitute a building block of many other distributed algorithms, since the elected vertex can be subsequently used to make centralized decisions. A second important problem in distributed computing is the naming problem. Here, the aim is to arrive at a final configuration where all processors have been assigned unique identities. Again this is an essential prerequisite to many other distributed algorithms that only work correctly under the assumption that all processors can be unambiguously identified. For a reference book on distributed algorithms we refer to [13].

OUR RESULTS. Whether the naming or election problem can be solved on a given graph depends on the properties of the considered model. If it is possible to solve the election (naming) problem we call the graph a *solution graph* for the election (naming) problem. It is a natural question to ask how hard it is to check whether a given graph is a solution graph in a certain model. For two models this problem is known to be polynomially solvable [2] and for one model it is co-NP-complete [15]. What about the computational complexity of this problem for the other models? In this paper we solve this question by showing that this decision problem is co-NP-complete for all remaining models.

The paper is organized as follows. In Section 2 we define the necessary graph terminology. To obtain our results we translate known characterizations [1, 4–7, 11, 12, 14, 16] of solution graphs in terms of graph labelings. This is shown in Section 3 for the message-passing models and in Section 4 for the local computations models. In Section 5 we introduce a new kind of labeling that does not correspond to any model of distributed computing but that enables us to present a simpler co-NP-completeness proof for seven basic models including the already known model in [15]. In Section 6 we give the results for the remaining two models.

2 Preliminaries

For graph terminology not defined below we refer to [3]. A *labeling* of a graph $G = (V_G, E_G)$ is a mapping $\ell : V_G \to \{1, 2, 3, \ldots, \}$. For a set $S \subseteq V_G$ we use the shorthand notation $\ell(S)$ to denote the image set of S under ℓ, i.e., $\ell(S) = \{\ell(u) \mid u \in S\}$. A *labeling* ℓ of G is called *proper* if $|\ell(V_G)| < |V_G|$. For any *label* $i \geq 1$, the set $\ell^{-1}(i)$ is equal to $\{u \in V_G \mid \ell(u) = i\}$. The subgraph of G induced by a subset $S \subseteq V_G$ is denoted by $G[S]$. For a label $i \geq 1$ we write $G[i] = G[\ell^{-1}(i)]$. For two labels i, j, we let $G[i, j]$ be the bipartite graph obtained from $G[\ell^{-1}(i) \cup \ell^{-1}(j)]$ by deleting all edges $[u, v]$ with $\ell(u) = \ell(v) = i$ or with $\ell(u) = \ell(v) = j$.

For a vertex $u \in V_G$ in a graph $G = (V_G, E_G)$, we denote its *neighborhood* by $N_G(u) = \{v \mid [u, v] \in E_G\}$. A graph is *regular*, if all its vertices have the same number k of neighbors (i.e. are of *degree* $\deg_G(u) = k$), in that case we also say that the graph is k-regular. A graph is *regular bipartite* if it is regular and bipartite. A graph is *semi-regular bipartite* if it is bipartite and the vertices of one class of the bipartition are of degree k and all others are of degree l, in that case we also say that the graph is *(k, l)-regular bipartite.* In our context a *perfect matching* is a $(1, 1)$-regular bipartite graph.

3 Message-passing models

In [14–16], Yamashita and Kameda study four message-passing models. In the *port-to-port* model, each processor can send different messages to different neighbors (by having access to unique *port-numbers* that distinguish between neighbors), and each processor knows the neighbor each receiving message is coming from (again by using the port-numbers). In the *broadcast-to-mailbox* model, port-numbers do not exist. A processor can only send a message to all of its neighbors and all receiving messages arrive in a *mailbox*, so it never knows their senders. The two mixed models are called the *broadcast-to-port* model and the *port-to-mailbox* model. There exists an election (or naming) algorithm for a graph G if and only if the algorithm solves the problem on G whatever the port-numbers are.

In [16], Yamashita and Kameda characterize these four models: a graph G is a solution graph for the election and naming problem in the port-to-port model if and only if G does not have a proper *symmetric regular labeling*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular and contains a perfect matching if its vertices have odd degree, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is regular bipartite.

A graph G is a solution graph for the election and naming problem in the port-to-mailbox model if and only if G does not have a proper *regular labeling*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is regular bipartite.

A graph G is a solution graph for the naming problem in the broadcast-to-mailbox and the broadcast-to-port model if and only if G does not have a proper *semi-regular labeling*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is semi-regular bipartite.

In these two models, a graph G is a solution graph for the election problem if and only if there does not exist any semi-regular labeling ℓ of G such that for all $i \in \ell(V_G)$, $|\ell^{-1}(i)| > 1$.

In [1, 6], different characterizations for these models are obtained (based on fibrations and coverings of directed graphs). The problem of deciding whether a graph G is a solution graph for the election and naming problem in the port-to-port model is co-NP-complete [15]. On the other hand, in [2], it is shown that the problem of deciding whether a graph G is a solution problem for the election and naming problem is polynomially solvable in the broadcast-to-mailbox and the broadcast-to-port model (by computing the degree refinement of G).

4 Local computations models

In the local computations models, a computation step can be described by the application of some *local relabeling rule* that enables the modification of the states of the different vertices involved in the synchronization. Two local computation models are different in the types of relabeling rules that they allow, see Figure 1. In models $(5),(6)$ and (7) of Figure 1, a computation step occurs on a star, i.e., it involves some synchronization between one vertex and all its neighbors, whereas in models $(1),(2),(3)$ and (4), a computation step occurs on an edge, i.e., it involves some synchronization between two neighbors. All these models are asynchronous, in the sense that not all processors have to been involved in each computation step. In models $(5),(6)$ and (7) (resp. $(1),(2),(3)$ and (4)), two computations steps can occur concurrently if they occur on stars (resp. on edges) that do not share any vertex.

Mazurkiewicz [11] consider the model (7) of Figure 1, where in one computation step, a processor can modify its state and the states of its neighbors, according to the previous states of itself and its neighbors. In this model, a graph G is a solution graph for the election and naming problem if and only if G does not have a proper *perfect-regular coloring*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is edgeless, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i,j]$ is edgeless or else is a perfect matching.

Boldi et al. [1] consider the model (5) of Figure 1, where in one computation step one processor can modify its state, according to its previous state and to the states of its neighbors. In this model, a graph G is a solution graph for the naming problem if and only if G does not have a proper *semi-regular coloring*, i.e., a proper labeling ℓ of G such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is edgeless, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i,j]$ is semi-regular bipartite.

In this model, a graph G is a solution graph for the election problem if and only if there does not exist any semi-regular coloring ℓ of G such that for all $i \in \ell(V_G)$, $|\ell^{-1}(i)| > 1$.

In [5] the models (3), (4) and (6) of Figure 1 are considered; in these models, vertices and edges can be labelled. The model (6) differs from the model (5) by the fact that the processor that modifies its state can also modify the labels of the edges incident to it. In the models (3) and (4), a computation step occurs on an edge whose label can be modified. These two models differs by the fact that in model (3), only one endvertex of the edge can modify its state, whereas

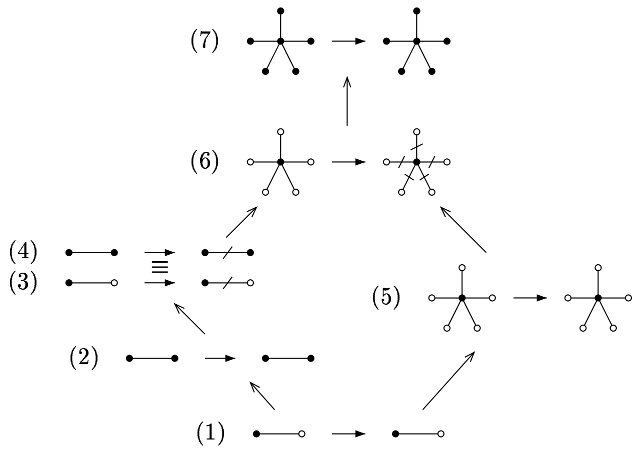

Fig. 1. A hierarchy of local computations models described by the different kinds of relabelling rules they use. Labels of black vertices can change when the local relabeling rule is applied. Labels of white vertices only enable to apply the rule but do not change. A rule can modify edge labels only in models (3), (4) and (6). If $r_i \to r_j$ for models r_i and r_j then r_j can simulate r_i but not vice versa, i.e., r_j has a greater computational power than r_i; this relation is transitive. If $r_i \equiv r_j$ then r_i and r_j have the same computational power. The computational power of the model (5) is incomparable with the power of the models (2), (3) and (4).

in model (4), the two endvertices of the edge can modify their states. In each of these models, a graph G is a solution graph for the election and the naming problem if and only if G does not have a proper *regular coloring*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is edgeless, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is regular bipartite.

We note that Mazurkiewicz [12] has given an equivalent characterization of model (4) in terms of equivalence relations over vertices and edges. The characterizations for model (6) can also be obtained from [1].

In [4], the model (2) of Figure 1 is considered: in one computation step, two neighbors modify simultaneously their states according only to their previous states (the edges are not labeled). In this model, a graph G is a solution graph for the election and naming problem if and only if G does not have a proper *pseudo-regular coloring*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is edgeless, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is edgeless or else contains a perfect matching.

In [7], the model (1) of Figure 1 is considered: in one computation step, one processor modify its state according to both its previous state and the state of one of its neighbor (the edges are not labeled). In this model, a graph G is a solution graph for the naming problem if and only if G does not admit any proper *connected coloring*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is edgeless, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is edgeless or else has minimum degree at least one.

We note that the hierarchy in Figure 1 is also reflected by the labelings, e.g., a perfect-regular coloring is also a regular coloring, and so on.

5 Pseudo-regular labelings

We call a labeling ℓ of a graph G a *pseudo-regular labeling* if

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is edgeless or else contains a perfect matching.

In this section we prove that the problem whether a given graph G has a proper pseudo-regular labeling is NP-complete. This gives us a number of co-NP-completeness results for various models of distributed computing discussed in the previous two sections. The following observation is useful.

Observation 1 *Let ℓ be a pseudo-regular labeling of a connected graph G. Then $|\ell^{-1}(i)| = \frac{|V_G|}{|\ell(V_G)|}$ for all $i \in \ell(V_G)$.*

Let $G = (V_G, E_G)$ and $H = (V_H, V_G)$ be two graphs. We write $V_H = \{1, 2, \ldots, |V_H|\}$. For a mapping $f : V_G \to V_H$ and a set $S \subseteq V_G$, we write $f(S) = \{f(u) \mid u \in S\}$. A *graph homomorphism* from G to H is a vertex mapping $f : V_G \to V_H$ satisfying the property that for any edge $[u, v]$ in E_G, we have $[f(u), f(v)]$ in E_H, in other words, $f(N_G(u)) \subseteq N_H(f(u))$ for all $u \in V_G$. A homomorphism f from G to H that induces a one-to-one mapping on the neighborhood of every vertex is called *locally bijective*, i.e., for all $u \in V_G$ it satisfies $f(N_G(u)) = N_H(f(u))$ and $|N_G(u)| = |N_H(f(u))|$. In that case we write $G \xrightarrow{B} H$, and call the vertices of H *colors* of G. Sometimes, we also say that the labels $\ell(i)$ of a labeling ℓ of G are *colors* of G.

The H-Cover problem asks whether there exists a locally bijective homomorphism from an instance graph G to a fixed graph H. In our NP-completeness proof we use reduction from the K-Cover problem, where K is the graph obtained after deleting an edge in the complete graph K_5 on five vertices. The K-Cover problem is NP-complete [10]. Note that the two non-adjacent vertices have degree three. The other three vertices are adjacent to two vertices of degree three and two vertices of degree four. Then the following observation immediately follows from the definition of a locally bijective homomorphism.

Observation 2 *Let G be a graph with $G \xrightarrow{B} K$. Then the following holds:*

(*) *$V_G = B_1 \cup B_2$ for two blocks B_1 and B_2 with $|B_1| = 2k$ and $|B_2| = 3k$ for some $k \geq 1$ such that*

- *for all $u \in B_1$, $|N_G(u) \cap B_1| = 0$ and $|N_G(u) \cap B_2| = 3$*
- *for all $u \in B_2$, $|N_G(u) \cap B_1| = 2$ and $|N_G(u) \cap B_2| = 2$.*

We call a graph satisfying the $(*)$ condition in Observation 2 a *K-candidate*. Since $(*)$ obviously can be checked in polynomial time, we may assume without loss of generality that any instance graph G of the K-Cover problem is a K-candidate.

For our NP-completeness structure we modify an instance graph G of the K-Cover as follows. Let u and v be vertices of G with $\deg_G(u) = 3$ and $\deg_G(v) = 4$. We replace the edge $[u, v]$ by a chain of $q \geq 1$ "diamonds" as described in Figure 2. We call the resulting graph G' a *diamond graph* of G *with respect to* the edge $[u, v]$. For $i = 1, \ldots, q$, the subgraph $D_i = G[\{a_i, b_i, c_i, d_i, e_i\}]$ is called a *diamond* of G'. The next lemma shows among others that a pseudo-regular

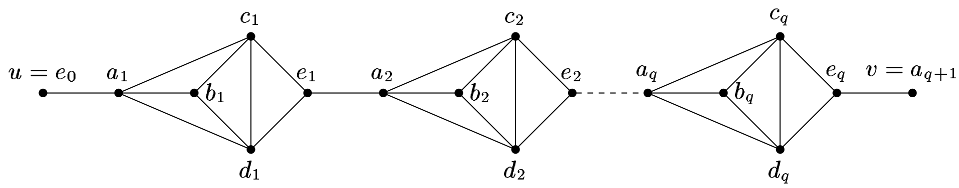

Fig. 2. The chain of q diamonds that replace the edge $[u, v]$.

labeling is injective on the neighborhood of any vertex in a diamond. Its proof involves a case analysis and will be presented in the journal version of our paper.

Lemma 3. *Let G be a K-candidate that contains adjacent vertices u, v with $\deg_G(u) = 3$ and $\deg_G(v) = 4$. Let G' be a diamond graph of G with respect to $[u, v]$ that has diamonds $D_1, \ldots D_q$, where $q > k + 2$ and $q + k$ is a prime number. If ℓ is a proper pseudo-regular labeling of G', then $|\ell(V_{D_i})| = 5$ and $\ell(e_{i-1}) \notin \ell(V_{D_i} \setminus \{e_i\})$ for all $1 \leq i \leq q$.*

The following lemma is a key result.

Lemma 4. *Let G be a K-candidate that contains adjacent vertices u, v with $\deg_G(u) = 3$ and $\deg_G(v) = 4$. Let G' be a diamond graph of G with respect to $[u, v]$ that has diamonds $D_1, \ldots D_q$, where $q > k+2$ and $q+k$ is a prime number. If ℓ is a proper pseudo-regular labeling of G' then $|\ell(V_{G'})| = 5$.*

Proof. We write $p = q + k$. Then $|V_{G'}| = 5p$ and p is a prime number. Hence we find that $|\ell(V_{G'})| = 5$ or $|\ell(V_{G'})| = p$, due to Observation 1.

Suppose $|\ell(V_{G'})| = p > 5$. By our choice of q, there exist a vertex u in a diamond D_i with the same color as a vertex v in a diamond D_j. By Lemma 3, we may assume that $i < j$. We choose u and v such that there do not exist two vertices in $G[D_i \cup \ldots \cup D_{j-1}]$ having the same color. By Lemma 3, we can write $\ell(a_i) = 1$, $\ell(b_i) = 2$, $\ell(c_i) = 3$, $\ell(d_i) = 4$ and $\ell(e_i) = 5$, and we find that $\ell(e_{i-1}) \notin \{1,2,3,4\}$. If $\ell(e_{i-1}) = 5$, then $\ell(a_{i+1}) = 1$ and consequently $|\ell(V_{G'})| = 5 < p$, so we write $\ell(e_{i-1}) = 6$.

By Observation 2 and the construction of G', every vertex of G has either degree 3 or 4. Note that, for each x in G' with $\ell(x) = 1$ (respectively $\ell(x) = 3$, $\ell(x) = 4$), we have that $\{2,3,4,6\} \subseteq \ell(N_{G'}(x))$ (respectively $\{1,2,4,5\} \subseteq \ell(N_{G'}(x))$, $\{1,2,3,5\} \subseteq \ell(N_{G'}(x))$). Consequently, each vertex x with $\ell(x) \in \{1,3,4\}$ has $\deg_{G'}(x) = 4$.

By our choice of D_i and D_j, vertex a_{i+1} belongs to some diamond. By Lemma 3, we know that $|\ell(N_{G'}(a_{i+1}))| = 4$. Then each vertex x with $\ell(x) = \ell(a_{i+1})$ has $\deg_{G'}(x) = 4$. Suppose now that there exists a vertex y such that $\deg_{G'}(y) = 4$ and $\ell(y) = 2$ (respectively $\ell(y) = 5$). Then $\ell(N_{G'}(y)) = \{1,3,4\}$ (respectively $\ell(N_{G'}(y)) = \{3,4,\ell(a_{i+1})\}$). Then y has three neighbors of degree four and this is not possible due to Observation 2. Consequently, each vertex y with $\ell(y) \in \{2,5\}$ has $\deg_{G'}(y) = 3$.

We show that $1 \notin \ell(D_j)$. Suppose $\ell(a_j) = 1$. From our choice of D_i and D_j, we know that $\ell(e_{j-1}) \notin \{2,3,4\}$. Then $\ell(\{b_j, c_j, d_j\}) = \{2,3,4\}$ and $\ell(e_{j-1}) = 6$. Then $\ell(V_G) = \ell(D_i \cup \ldots \cup D_{j-1})$ and since all colors are different on diamonds $D_i, D_{i+1}, \ldots, D_{j-1}$, we find that $p = |\ell(V_G)| = 5(j-i)$. Since p is a prime number not equal to 5, this is not possible. We already know that $1 \notin \ell(\{b_j, e_j\})$ since $\deg_{G'}(b_j) = \deg_{G'}(e_j) = 3$. Suppose $\ell(c_j) = 1$ (respectively $\ell(d_j) = 1$). Then $\ell(d_j) \in \{3,4\}$ (respectively $\ell(c_j) \in \{3,4\}$) and $\ell(\{b_j, e_j\}) = \{2,6\}$. Then a vertex with color in $\{3,4\}$ is adjacent to a vertex with color 6. This is not possible.

We show that $2 \notin \ell(D_j)$. We already know that the only vertices in D_j that can be mapped to 2 are b_j and e_j in D_j. If $\ell(b_j) = 2$, then $1 \in \ell(\{a_j, c_j, d_j\})$. If $\ell(e_j) = 2$, then either $1 \in \ell(\{c_j, d_j\})$ or $\ell(\{c_j, d_j\}) = \{3,4\}$ and in the second case $\ell(a_j) = 1$.

We show that $3 \notin \ell(D_j)$. We already know that only vertices a_j, c_j, d_j can be mapped to 3. If $\ell(a_j) = 3$ then 1, which does not occur on D_j, must be the color of $\ell(e_{j-1})$. This is not possible due to our choice of D_i and D_j. In the other two cases we find that $1 \in \ell(D_j)$. By symmetry, we deduce that $4 \notin \ell(D_j)$.

Finally, we show that $5 \notin \ell(D_j)$. We already know that only vertices b_j and e_j can be mapped to 5. In both cases, at least one of the colors $3,4$ is a color of a vertex in D_j. This finishes the proof of the lemma. □

Lemma 5. *Let G be a graph that contains adjacent vertices u, v with $\deg_G(u) = 3$ and $\deg_G(v) = 4$. Let G' be a diamond graph of G with respect to (u,v). Then $G \xrightarrow{B} K$ if and only if $G' \xrightarrow{B} K$.*

Proof. We denote the vertices of K by $1,2,3,4,5$ and its edges by $[1,2]$, $[1,3]$, $[1,4]$, $[1,5]$, $[2,3]$, $[2,4]$, $[3,4]$, $[3,5]$, $[4,5]$. Suppose $G \xrightarrow{B} K$. Without loss of generality we assume that u has color 5 and v has color 1. Then we assign color 1 to all a_i, color 2 all b_i, color 3 to all c_i, color 4 to all d_i and color 5 to all e_i.

Suppose $G' \xrightarrow{B} K$. The restriction of any locally bijective homomorphism $f' : V_{G'} \to V_K$ to V_G is a witness for $G \xrightarrow{B} K$. □

Theorem 1. *The problems that ask whether a given graph G allows a proper pseudo-regular coloring, a proper pseudo-regular labeling, a proper regular coloring, a proper regular labeling, a proper symmetric regular labeling, or a proper perfect-regular coloring, respectively, are* NP*-complete.*

Proof. Obviously, all problems are in NP. We use reduction from the NP-complete problem K-COVER [10]. Let G be an instance graph of this problem. Since we may assume that G is a K-candidate, graph G has $5k$ vertices for some $k \geq 1$ and contains adjacent vertices u of degree three and v of degree four. We construct the diamond graph G' with respect to $[u, v]$ that has q diamonds $D_1, \ldots, D_q$, where we chose q such that $q > k + 2$ and $p = q + k$ is a prime number. By Lemma 5 we can consider G' as our instance graph for the K-COVER problem.

Any locally bijective homomorphism is a proper perfect-regular coloring, which is a regular coloring, which is a symmetric regular labeling, which is a regular labeling, which is a pseudo-regular labeling, and any regular coloring is a pseudo-regular coloring, which is a pseudo-regular labeling.

So we are left to show that a proper pseudo-regular labeling of G' implies that $G' \xrightarrow{B} K$. Suppose G' allows a proper pseudo-regular labeling ℓ. By Lemma 3, $|\ell(D_1)| = 5$. Let $\ell(a_1) = 1$, $\ell(b_1) = 2$, $\ell(c_1) = 3$, $\ell(d_1) = 4$ and $\ell(e_1) = 5$. By Lemma 3, $\ell(e_0) \notin \{1, 2, 3, 4\}$. Since $|\ell(V_G)| = 5$ due to Lemma 4, we then find that $\ell(e_0) = 5$. This means that ℓ defines a locally bijective homomorphism from G to K. □

6 Connected colorings and semi-regular colorings

A *hypergraph* $(Q, \mathcal{S})$ is a set $Q = \{q_1, \ldots, q_m\}$ together with a set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of Q. A 2-*coloring* of a hypergraph $(Q, \mathcal{S})$ is a partition of Q into $Q_1 \cup Q_2$ such that $Q_1 \cap S_j \neq \emptyset$ and $Q_2 \cap S_j \neq \emptyset$ for $1 \leq j \leq n$. In our proofs we use reduction from the following, well-known NP-complete problem (cf. [9]).

HYPERGRAPH 2-COLORABILITY
Instance: A hypergraph $(Q, \mathcal{S})$.
Question: Does $(Q, \mathcal{S})$ have a 2-coloring?

With a hypergraph $(Q, \mathcal{S})$ we associate its *incidence graph* I, which is a bipartite graph on $Q \cup \mathcal{S}$, where $[q, S]$ forms an edge if and only if $q \in S$. From the incidence graph I we act as follows. Let C_k denote a cycle on k vertices. First we make a copy S' for each $S \in \mathcal{S}$. We add edges (S', q) if and only if $q \in S$. Let $\mathcal{S}' = \{S_1', \ldots, S_n'\}$. Then we glue a cycle C_{q_i} isomorphic to a C_{6i-3} in I by vertex q_i for $1 \leq i \leq m$. We add a new vertex v and edges from v to all vertices in $\mathcal{S}$. Finally we glue a cycle C_v isomorphic to C_{6m+3} in I by v. We call the resulting graph I^* the C_3-*minimizer* of $(Q, \mathcal{S})$. See Figure 3 for an example. The proof of the following lemma will be included in the journal version.

Lemma 6. *Let I^* be the C_3-minimizer of a hypergraph $(Q, \mathcal{S})$ with $S_j \neq S_k$ for all j, k. If ℓ is a proper connected coloring of I^* then $|\ell(V_{I^*})| = 3$.*

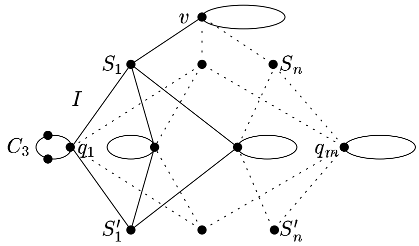

Fig. 3. Example of a C_3-minimizer I^* of a hypergraph $(Q, \mathcal{S})$.

Theorem 2. *The problem that asks whether a given graph G has a proper connected coloring is* NP*-complete.*

Proof. Obviously, this problem is in NP. We prove NP-completeness by reduction from the HYPERGRAPH 2-COLORABILITY problem. Let $(Q, \mathcal{S})$ be a hypergraph. We assume without loss of generality that $S_j \neq S_k$ for $j \neq k$. We claim that $(Q, \mathcal{S})$ has a 2-coloring if and only if its C_3-minimizer I^* admits a proper connected coloring.

Suppose $(Q, \mathcal{S})$ has a 2-coloring $Q_1 \cup Q_2$. Define $\ell(v) = 1$, $\ell(S) = 2$ for all $S \in \mathcal{S} \cup \mathcal{S}'$, $\ell(q) = 1$ for all $q \in Q_1$ and $\ell(q) = 3$ for all $q \in Q_2$. Finish the coloring in the obvious way.

Suppose I^* has a proper connected coloring ℓ. By Lemma 6 we find $|\ell(V_{I^*})| = 3$. Let $\ell(v) = 1$. Then $\ell(S_j) \in \{2, 3\}$ for all j. If $\ell(S'_j) = 1$ for some j, then S'_j needs a neighbor of color 2 and a neighbor of color 3, both are adjacent to S_j. Hence, $\ell(S'_j) \in \{2, 3\}$ for all j. We define $Q_1 = \{q \in Q \mid \ell(q) = 1\}$ and $Q_2 = Q \backslash Q_1$. Since each S'_j needs at least two neighbors with different colors and at least one neighbor with color 1, the partition $Q_1 \cup Q_2$ is a 2-coloring of $(Q, \mathcal{S})$. □

The proof of Theorem 3 uses arguments of the proofs of Theorem 1 and Theorem 2 but the NP-completeness construction is more involved. We postpone it to the journal version.

Theorem 3. *The problem that asks whether a given graph G has a proper semi-regular coloring is* NP*-complete.*

7 Conclusions

By Theorems 1, 2 and 3 we have determined the computational complexity of the question whether the election and/or naming problem can be solved on a given graph in eleven different models of distributed computing that all have been studied in the literature.

Corollary 1. *It is co-*NP*-complete to decide if on a given graph G we can solve*

(a) the election problem in the models described in Sections 3 and 4 except for the broadcast-to-port model, the broadcast-to-mailbox model and models (1), (5) of Figure 1;
(b) the naming problem in the models described in Sections 3 and 4 except for the broadcast-to-port and broadcast-to-mailbox model.

As a matter of fact the above decision problem is co-NP-complete for the election problem in models (1) and (5) of Figure 1 as well. In both cases, the NP-hardness of the decision problem can be derived from the proofs of corresponding theorems for the naming problem (i.e., Theorems 2 and 3). From the characterization of Boldi et al. [1] of solution graphs for the election problem in the model (5) of Figure 1, one can see that the corresponding decision problem is in co-NP. We do not know any characterization of graphs that admits an election algorithm in the model (1) that can be expressed in terms of graphs labelings, and then the proof that the corresponding decision problem is in co-NP is not so obvious. Both proofs are postponed to the journal version.

We note that the problem that asks whether a given connected graph G has a proper perfect-regular coloring is equivalent to the problem that asks whether $G \xrightarrow{B} H$ for some connected graph H with $|V_H| < |V_G|$. A graph homomorphism f from G to H satisfying $f(N_G(u)) = N_H(f(u))$ for all $u \in V_G$ is called *locally surjective*. If such a homomorphism exists, we write $G \xrightarrow{S} H$. The problem that asks whether a connected graph G has a proper connected coloring is equivalent to the problem that asks whether $G \xrightarrow{S} H$ for some connected graph H with $|V_H| < |V_G|$. Let $\mathcal{C}$ denote the set of connected graphs (up to isomorphism). In [8] it has been proven that $(\mathcal{C}, \xrightarrow{B})$ and $(\mathcal{C}, \xrightarrow{S})$ are partial orders. Theorem 1 and 2 imply that it is co-NP-complete to check whether a graph is minimal in $(\mathcal{C}, \xrightarrow{B})$ and $(\mathcal{C}, \xrightarrow{S})$, respectively. Also the other studied graph labeling problems can easily be formulated as problems that ask whether there exist a homomorphism f, that satisfies a few extra constraints, from a given graph G to a smaller graph H. In future research we will study the relations between these *constrained* homomorphisms more carefully.

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