

New Bounds for the Snake-in-the-Box Problem

David Allison¹ and Daniël Paulusma¹

School of Engineering and Computing Sciences, Durham University,
Durham, United Kingdom

{david.allison2,daniel.paulusma}@durham.ac.uk

Abstract. The Snake-in-the-Box problem is that of finding a longest induced path in an n -dimensional hypercube. We prove new lower bounds for the values $n \in \{11, 12, 13\}$. The Coil-in-the-Box problem is that of finding a longest induced cycle in an n -dimensional hypercube. We prove new lower bounds for the values $n \in \{12, 13\}$.

1 Introduction

In 1958, Kautz [9] observed that the vertices of an induced path in a hypercube can be used as code words in Gray codes detecting single-bit errors. Applications of such codes date back to the 19th century, where they were used in telegraphy. They are nowadays used in many areas of mathematics, computer science and engineering, such as error correction in digital communication, disk sector encoding, clock domain crossing, computer network topologies and in the design of genetic algorithms, just to name a few; we refer to Drapela [4] for background information. The effectiveness of a Gray code depends on its length. Hence, the central research question is:

How can we find a longest induced path in a hypercube?

Kautz coined the name “snake-in-the-box” for this problem. The snake-in-the-box problem has been extensively studied in the literature, just as its variant “coil-in-the-box”, where the goal is to determine a longest induced cycle instead of a path. As we shall discuss, the lengths of a longest snake and a longest coil are not known for any hypercube of dimension larger than 8. The problem of finding a longest induced path is NP-hard even for bipartite graphs [6], but its complexity for hypercubes is still unknown. As exhaustive search is too time consuming, a variety of computational and mathematical techniques have been used to find lower bounds for these problems. See Drapela [4] for details on these techniques.

In this note the main aim is to show three new lower bounds for snake-in-the-box and two new lower bounds for coil-in-the-box. We do this in Section 3 after first giving the required terminology in Section 2. Examples meeting our lower bounds are presented in the Appendix. Our methods are based on the stochastic beam search method, which is a general heuristic search method that was recently used by Meyerson et al. [12, 13] to improve lower bounds for both snake-in-the-box and coil-in-the-box. We will explain our modifications to their algorithm in full detail in a future paper.

2 Terminology

A graph H is an *induced subgraph* of another graph G if H can be obtained from G by a sequence of vertex deletions. For $r \geq 1$, the *path* P_r on r vertices is the graph with vertices u_1, \dots, u_r and edges $u_1u_2, \dots, u_{r-1}u_r$. The *cycle* C_r is the graph obtained from P_r by adding the edge u_ru_1 . We say that P_r and C_r are of *length* $r - 1$ and r , respectively. For $n \geq 1$, the n -dimensional hypercube Q_n is the graph that contains 2^n vertices, each of which is represented by a binary vector of length n , such that two vertices are adjacent if and only if their binary vectors differ by exactly one bit. A *snake* is an induced subgraph of Q_n that is a path, and a *coil* is an induced subgraph of Q_n that is a cycle. The SNAKE-IN-THE-BOX problem is that of finding a longest snake in Q_n for a given integer n and the COIL-IN-THE-BOX problem is that of finding a longest coil in Q_n for a given integer n . Both problems have also been investigated for other graph classes (see, for example, [19]).

3 Known Results

Besides snakes and coils, a number of variants and generalizations, such as symmetric coils [2] (also known as doubled coils), k -snakes and k -coils [7, 17], and single-track circuit codes [8] have been studied. However, in this section we will only focus on the original notions of snakes and coils and we restrict our overview to exact or lower bounds for SNAKE-IN-THE-BOX and COIL-IN-THE-BOX for hypercubes of small dimensions, that is, we will consider n -dimensional hypercubes for $n \leq 20$. We refer to, for example, [5, 18, 21] for upper bounds for these two problems.

For $n \leq 8$, both SNAKE-IN-THE-BOX and COIL-IN-THE-BOX have been solved exactly:

- For $n \leq 5$, Kautz [9] determined a longest coil. Using exhaustive search, Davies [3] found a longest snake for $n \leq 6$ and a longest coil for $n = 6$.
- For $n = 7$, Potter, Robinson, Miller and Kochut [16] used a genetic algorithm to find a longest snake and Kochut [11] used exhaustive search to find a longest coil.
- For $n = 8$, Östergård and Pettersson used canonical augmentation to find a longest snake [14] and a longest coil [15].

For $n \geq 9$, only lower bounds for the length of a longest snake or coil are known. These lower bounds have been improved over the years and the state of the art results are as follows:

- For $n = 9$, Wynn [20] proved that the length of a longest snake and coil is at least 190 and 188, respectively.
- For $n = 10$, Kinny [10] showed that a longest snake has length at least 370 and Meyerson et al. [12] showed that a longest coil has length at least 362.

- For $11 \leq n \leq 12$, the results of Meyerson et al. [12] from 2015 are known to give the best lower bounds for the lengths of a longest snake and coil.
- For $n = 13$, Meyerson et al. [12] showed that a longest snake has length at least 2520 and Abbott and Katchalski [1] showed that a longest coil has length at least 2468.
- For $14 \leq n \leq 20$, the results of Abbott and Katchalski [1] from 1991 are still unbeaten. The same authors also showed a general lower bound by showing that for each $n \geq 21$, a longest coil has length at least $\frac{77}{256}2^n$.

We refer to Table 1 for an overview of the currently best lower bounds on the length of a longest snake or coil for every $n \leq 20$. We note that the Snake-in-the-Box Records page <http://ai1.ai.uga.edu/sib/sibwiki/> at the University of Georgia maintains a list of records for the Snake-in-the-Box and Coil-in-the-Box problems for every $n \leq 13$.

Dimension	Snake Length	Coil Length
1	1*	0*
2	2* [3]	4* [9]
3	4* [3]	6* [9]
4	7* [3]	8* [9]
5	13* [3]	14* [9]
6	26* [3]	26* [3]
7	50* [16]	48* [11]
8	98* [14]	96* [15]
9	190 [20]	188 [20]
10	370 [10]	362 [12]
11	708 (707 [12])	668 [12]
12	1357 (1302 [12])	1340 (1276 [12])
13	2687 (2520 [12])	2584 (2468 [1])
14	4932 [1]	4934 [1]
15	9866 [1]	9868 [1]
16	19738 [1]	19740 [1]
17	39478 [1]	39480 [1]
18	78958 [1]	78960 [1]
19	157898 [1]	157900 [1]
20	315798 [1]	315800 [1]

Table 1. The lower bounds on the maximum length of a snake or coil in a hypercube of dimension $n = 1, \dots, 20$. A * indicates that the bound is optimal. The unreferenced results in bold are the new bounds proven in this paper; the previous records are placed between parentheses after our lower bound values. Note that the lower bounds on the provided snakes for $n \geq 14$ are deduced from the lower bound on the length of a longest coil.

4 Our Results

We prove that the length of a longest snake is at least 708 for $n = 11$, at least 1357 for $n = 12$ and at least 2687 for $n = 13$ and that the length of a longest coil is at least 1340 for $n = 12$ and at least 2584 for $n = 13$. See also Table 1. We do this by giving examples of snakes and coils of these lengths in the Appendix. We verified the correctness of our solutions using the verifier provided at <http://ai1.ai.uga.edu/sib/sibwiki/doku.php/checker>.

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A The New Snakes and Coils

Recall that two vertices in a hypercube are adjacent if and only if they differ in exactly one bit. A path $u_1u_2\cdots u_n$ in a hypercube Q_n can be represented by a *transition sequence* i_1, \dots, i_{n-1} where for $j = 1, \dots, n-1$, i_j denotes the vector entry of the bit in which vertices u_{j-1} and u_j differ. Similarly, a cycle in a hypercube can be represented by a transition sequence as well.

A.1 A Snake of Length 708 for $n = 11$

0,1,2,3,0,1,4,0,3,5,4,0,1,4,5,2,3,5,4,1,0,4,6,3,5,0,3,4,1,3,2,1,0,3,5,0,1,4,5,0,3,5,4,2,
3,7,1,5,3,1,2,3,0,1,4,0,3,5,4,0,1,4,5,2,3,5,1,6,5,0,3,5,4,1,2,4,5,3,0,5,4,1,0,5,3,0,1,2,
3,0,8,2,0,1,4,0,3,5,4,0,1,4,5,2,3,5,4,1,0,4,6,5,0,4,1,3,2,1,0,3,5,0,1,4,5,0,3,5,4,2,1,7,
2,4,5,3,0,5,2,4,1,3,2,4,5,2,0,4,2,3,5,4,2,6,4,3,2,4,5,3,0,5,4,1,0,5,3,0,1,2,3,1,4,0,3,5,
0,8,9,0,5,3,0,4,1,3,2,1,0,3,5,0,1,4,5,0,3,5,4,2,3,4,6,2,4,5,3,2,4,0,2,5,4,2,3,1,4,2,5,0,
3,5,4,2,7,1,2,4,5,3,0,5,4,1,0,5,3,0,1,2,3,1,4,0,5,6,4,0,1,3,2,1,0,4,5,3,0,4,1,0,3,2,1,0,
4,1,3,0,8,2,0,5,4,1,0,5,3,0,1,2,3,1,4,5,6,1,5,3,2,5,4,1,0,4,5,3,0,4,1,0,3,2,1,3,5,1,7,3,
2,4,5,3,0,5,4,1,0,5,3,0,1,2,3,1,4,3,0,5,3,6,4,0,1,4,5,3,2,5,4,1,0,4,5,3,0,4,1,0,2,3,1,0,
10,8,0,1,2,3,0,1,4,0,3,5,4,0,1,4,5,2,3,5,4,1,0,4,6,3,5,0,3,4,1,3,2,1,0,3,5,0,1,4,5,0,3,5,
4,2,3,7,1,5,3,1,2,3,0,1,4,0,3,5,4,0,1,4,5,2,3,5,1,6,5,0,3,5,4,1,2,4,5,3,0,5,4,1,0,5,3,0,
1,2,3,0,8,2,0,1,4,0,3,5,4,0,1,4,5,2,3,5,4,1,0,4,6,5,0,4,1,3,2,1,0,3,5,0,1,4,5,0,3,5,4,2,
1,7,2,4,5,3,0,5,2,4,1,3,2,4,5,2,0,4,3,2,5,4,2,6,4,3,2,4,5,3,0,5,4,1,0,5,2,4,3,5,0,7,6,9,
3,1,5,3,2,5,4,1,0,4,5,3,0,4,1,0,2,5,7,3,2,4,5,3,0,5,4,1,0,5,3,0,1,2,3,1,4,3,0,5,3,6,4,0,
1,4,5,3,2,5,4,1,0,4,5,3,0,4,1,0,2,8,0,3,2,1,0,3,5,0,1,4,5,0,3,5,4,2,1,4,5,3,0,5,6,4,0,5,
3,1,2,3,0,1,4,3,7,4,1,2,3,1,4,0,3,5,4,1,2,4,5,3,0,5,4,1,0,4,7,3,4,0,1,4,2,5,3,0,5,7,6,0,
4,1,0,5,3,1,4,0,2,5,4,1,0,4,5,3,0,4,1,0,5,8,1,0,4,3,5,4,0,1,4,3,2,1,0,4,1,3,5,1,0,4,1,9,
10,4,1,0

A.2 A Snake of Length 1357 for $n = 12$

0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,6,1,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,
7,6,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,4,3,1,0,6,1,2,5,4,3,2,5,1,2,4,5,2,0,1,2,5,4,3,2,
5,8,0,4,3,2,5,4,3,1,2,6,0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,6,0,7,1,2,3,1,
0,2,6,0,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,4,3,1,0,6,1,4,0,2,3,4,5,2,1,5,4,9,5,6,2,3,0,
1,3,2,5,4,3,2,1,3,5,2,3,0,1,3,2,5,4,3,2,1,0,6,2,1,3,4,5,2,3,4,0,8,5,2,3,4,5,2,1,0,2,5,4,
2,1,5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,7,8,0,1,4,3,2,5,4,3,1,
0,3,4,5,3,1,4,3,2,5,4,3,1,0,6,1,2,5,4,3,2,5,1,2,4,5,2,0,1,2,5,4,3,2,1,8,2,3,4,2,1,3,2,5,

4,3,2,1,0,2,3,6,1,4,5,1,3,5,2,0,1,2,5,4,3,2,0,6,2,3,4,5,2,3,1,0,10,3,1,2,5,4,3,2,5,8,0,4
 ,3,2,5,4,3,1,2,6,0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,6,5,8,0,1,5,2,3,4,5,2,
 1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,6,7,1,4,3,2,5,4,3,1,0,2,6,1,2,3,4,5,2,3,
 1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,6,1,2,3,4,1,3,5,2,8,5,1,2,5,4,3,2,0,4,1,6,0,1,3,4,5,2,
 3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,6,2,0,1,3,2,1,5,6,9,3,0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,5,
 4,1,0,3,2,6,1,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,7,6,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,
 4,3,1,0,6,1,2,5,4,3,2,5,1,2,4,5,2,0,1,2,5,4,3,2,1,8,6,1,2,3,4,5,2,3,1,0,6,1,3,2,5,4,3,2,
 7,0,2,3,4,5,2,3,1,4,6,0,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,4,3,1,0,6,2,3,4,2,5,1,2,3,1,
 7,6,1,4,5,1,3,2,1,4,5,1,0,6,1,5,4,1,2,5,3,1,5,11,0,2,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3
 ,4,5,2,3,4,1,6,7,1,4,3,2,5,4,3,1,0,2,6,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,6,
 1,2,3,4,1,0,7,3,2,0,1,3,4,8,0,4,3,2,5,4,3,1,2,6,0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,
 3,1,0,3,2,6,0,7,1,2,3,1,0,2,6,0,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,6,1,9,5,6,2,3,0,1,3,
 2,5,4,3,2,1,3,5,2,3,0,1,3,2,5,4,3,2,1,0,6,2,1,3,4,5,2,3,4,0,8,5,2,3,4,5,2,1,0,2,5,4,2,1,
 5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,7,8,0,1,4,3,2,5,4,3,1,0,3,
 4,5,3,1,4,3,2,5,4,3,1,0,6,1,2,5,4,3,2,5,1,2,4,5,2,0,1,2,5,4,3,2,1,8,0,2,1,3,4,5,2,3,1,6,
 2,5,1,2,4,0,2,3,4,5,2,1,5,0,10,3,6,1,2,5,4,3,2,5,1,2,4,5,2,0,1,2,5,4,3,2,5,8,0,4,3,2,5,4
 ,3,1,2,6,0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,8,1,5,4,3,2,5,1,3,5,4,3,0,1,3,
 4,5,2,3,4,1,6,7,1,4,3,2,5,4,3,1,0,2,6,1,2,3,4,5,2,3,1,0,3,2,5,3,1,5,4,3,2,5,1,9,5,3,1,4,
 3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,4,3,1,6,5,1,3,2,5,4,3,2,0,1,7,2,1,3,4,5,2,3,1,6,3,0,1,2,
 5,4,3,2,5,1,2,0,1,3,2,5,4,3,2,1,3,4,8,1,4,5,2,3,4,1,6,7,1,4,3,2,5,4,3,1,0,2,6,1,2,3,4,5,
 2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,6,1,9,6,5,1,2,3,0,1,3,4,5,2,3,4,1,0,6,4,3,6,2,3,1,
 5,2,1,0,6,1,4,0,2,5,9,4,1,3,5,9,11,5,3,1,5,2,3,11,8,3,2,5,1,0,8,7,0,1,5,2,1,3,4,2,9,11,4
 ,5,3,1,5,2,3,4,11,3,0,4,2,1,0,10,3,2,6,10,2,5,3,1,4,3,6,8,3,4,1,3,5,2,4,5,0,2,8,4,2,5,4,
 7,9,5,2,0,8,9,3,0,1,4,5,1,3,2,10,8,6,3,0,1,6,2,9,1,11,7,8,11,5,4,1,3,2,1,0,6,1,2,3

A.3 A Snake of Length 2687 for $n = 13$

0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,6,1,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,
 7,6,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,4,3,1,0,6,1,2,5,4,3,2,5,1,2,4,5,2,0,1,2,5,4,3,2,
 5,8,0,4,3,2,5,4,3,1,2,6,0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,6,0,7,1,2,3,1,
 0,2,6,0,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,4,3,1,0,6,1,4,0,2,3,4,5,2,1,5,4,9,5,6,2,3,0,
 1,3,2,5,4,3,2,1,3,5,2,3,0,1,3,2,5,4,3,2,1,0,6,2,1,3,4,5,2,3,4,0,8,5,2,3,4,5,2,1,0,2,5,4,
 2,1,5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,7,1,5,2,3,4,5,2,1,0,2,
 5,3,1,2,3,4,5,2,3,1,6,3,2,0,1,5,4,3,2,5,1,2,0,1,3,2,5,4,3,2,1,0,8,1,2,3,4,5,2,1,0,2,5,4,
 2,1,5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,3,10,4,3,0,1,3,2,5,4,3,2
 ,1,0,6,2,1,3,4,5,2,3,4,0,8,5,2,3,4,5,2,1,0,2,5,4,2,1,5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,4,1,3,
 5,4,3,0,1,3,4,5,2,3,4,1,6,7,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,1,6,2,3,0,1,4,5,1,3,5,2,3,
 0,1,3,2,5,4,3,2,1,0,3,8,4,3,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,6,4,1,3,2,5,4,
 3,2,0,4,2,5,4,3,2,1,4,2,5,4,9,1,4,5,2,3,4,1,5,6,0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,
 3,1,0,3,2,6,1,5,4,3,2,5,1,0,8,1,4,3,2,5,4,3,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,
 4,1,0,6,1,5,2,3,4,5,2,1,5,4,7,1,5,2,3,4,5,2,1,5,0,1,3,2,5,4,3,2,1,6,0,1,2,3,4,5,2,1,4,2,
 3,0,1,3,2,5,4,3,2,1,6,8,1,2,3,4,1,6,4,3,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,6,
 5,1,2,3,4,5,2,1,5,9,1,3,2,1,6,0,1,5,4,1,0,6,10,11,6,0,1,3,2,5,4,3,2,1,3,5,2,3,0,1,3,2,5,
 4,3,2,1,0,6,2,1,3,4,5,2,3,4,0,8,5,2,3,4,5,2,1,0,2,5,4,2,1,5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,
 4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,6,7,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,1,6,2,3,0,1,4,5,1,3,
 5,2,3,0,1,3,2,5,4,3,2,1,0,3,8,4,3,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,6,4,1,3,

2,5,4,3,2,0,4,2,5,4,3,2,1,4,2,5,4,9,1,4,5,2,3,4,1,5,6,0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,
 4,5,2,3,1,0,3,2,6,1,2,3,4,1,0,8,1,4,3,2,5,4,3,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,
 3,4,1,0,6,1,5,2,3,4,5,2,1,5,4,7,1,5,2,3,4,5,2,1,5,0,1,3,2,5,4,3,2,1,6,0,1,2,3,4,5,2,1,4,
 2,3,0,1,3,2,5,4,3,2,1,6,8,3,5,2,0,1,2,5,4,3,2,1,5,6,0,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,
 5,4,3,1,0,3,4,6,1,5,2,3,1,10,6,1,3,2,5,4,3,2,1,3,5,2,3,0,1,3,2,5,4,3,2,1,0,6,1,5,4,3,2,5,
 ,4,1,8,2,1,5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,6,7,1,4,3,2,5,4,3,
 1,0,3,4,5,3,1,4,3,2,1,6,2,3,0,1,4,5,1,3,5,2,3,0,1,3,2,5,4,3,2,1,0,3,8,5,3,1,4,3,2,5,4,3,
 1,0,3,4,5,3,1,4,3,2,5,4,3,1,0,6,1,2,5,4,3,1,4,2,0,1,2,5,4,3,2,1,5,2,9,6,0,1,3,2,5,4,3,2,
 1,3,5,2,3,0,1,3,2,5,4,3,2,1,0,6,2,1,3,4,5,2,3,4,0,8,5,2,3,4,5,2,1,0,2,5,4,2,1,5,2,3,4,5,
 2,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,7,1,5,2,3,4,5,2,1,0,2,5,3,1,2,3,4,5,
 2,3,1,6,3,2,0,1,5,4,3,2,5,1,2,0,1,3,2,5,4,3,2,1,0,8,5,0,6,1,4,3,2,5,4,3,1,0,3,4,5,3,1,5,
 2,1,0,3,4,6,1,4,3,2,1,3,5,2,0,1,2,5,4,3,5,9,3,2,5,4,3,2,1,7,9,2,1,3,4,1,5,6,7,10,6,5,1,4,
 ,3,1,2,3,5,12,6,5,4,3,0,1,3,4,5,2,3,4,1,0,6,4,11,6,1,3,2,5,4,3,2,1,0,6,2,1,3,4,5,2,3,4,0,
 ,8,5,2,3,4,5,2,1,0,2,5,4,2,1,5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,
 6,7,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,1,6,2,3,0,1,4,5,1,3,5,2,3,0,1,3,2,5,4,3,2,1,0,3,8,
 4,3,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,6,5,1,2,3,4,5,2,1,0,2,5,4,9,1,4,5,2,3,
 4,1,5,6,0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,2,6,1,2,3,4,1,0,8,1,4,3,2,5,4,3,
 1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,6,1,5,2,3,4,5,2,1,5,4,7,1,5,2,3,4,5,2,
 1,5,0,1,3,2,5,4,3,2,1,6,0,1,2,3,4,5,2,1,4,2,3,0,1,3,2,5,4,3,2,1,6,8,3,5,2,0,1,2,5,4,3,2,
 1,5,6,0,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,4,3,1,0,3,4,6,1,5,2,3,1,10,6,1,3,2,5,4,3,2,1,
 ,3,5,2,3,0,1,3,2,5,4,3,2,1,0,6,1,5,4,3,2,5,4,1,8,2,1,5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,4,1,3,
 5,4,3,0,1,3,4,5,2,3,4,1,6,7,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,1,6,2,3,0,1,4,5,1,3,5,2,3,
 0,1,3,2,5,4,3,2,1,0,3,8,5,3,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,4,3,1,0,6,1,2,5,4,3,1,4,
 2,0,1,2,5,4,3,2,1,5,2,9,6,0,1,3,2,5,4,3,2,1,3,5,2,3,0,1,3,2,5,4,3,2,1,0,6,2,1,3,4,5,2,3,
 4,0,8,5,2,3,4,5,2,1,0,2,5,4,2,1,5,2,3,4,5,2,1,6,0,1,3,4,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,
 4,1,0,7,1,5,2,3,4,5,2,1,0,2,5,3,1,2,3,4,5,2,3,1,6,3,2,0,1,5,4,3,2,5,1,2,0,1,3,2,5,4,3,2,
 1,0,8,5,0,6,1,4,3,2,5,4,3,1,0,3,4,5,3,1,4,6,5,3,4,5,2,1,0,2,5,3,1,2,3,4,1,6,4,3,0,1,3,4,
 5,1,7,6,5,2,11,6,3,0,1,3,4,5,2,3,4,1,0,6,5,8,2,0,1,3,4,5,2,3,4,1,7,6,1,4,3,2,5,4,3,1,0,3,
 ,4,5,3,1,4,3,2,5,4,3,1,0,6,1,2,5,4,3,2,5,1,2,4,5,2,0,1,2,5,4,3,2,5,8,0,4,3,2,5,4,3,1,2,6,
 0,1,2,3,4,5,2,3,1,0,3,2,5,3,1,2,3,4,5,2,3,1,0,3,6,2,5,1,4,3,1,2,5,9,6,3,5,4,3,0,1,3,4,5,
 2,3,4,1,6,0,5,1,2,3,4,5,2,1,0,2,5,3,1,5,4,3,2,5,1,6,4,3,0,1,3,4,5,1,7,2,3,0,1,4,5,1,3,5,
 2,3,0,1,3,2,5,4,3,2,1,0,6,2,3,1,4,5,2,3,4,0,2,3,9,6,2,5,4,3,1,0,3,4,8,3,1,0,2,3,4,5,2,3,
 1,0,3,2,5,3,1,5,4,1,0,3,2,6,1,5,10,6,5,2,3,4,1,3,5,4,3,0,1,3,4,5,2,3,4,1,0,6,1,8,0,4,3,2,
 ,5,4,2,11,4,3,2,5,4,1,11,5,2,6,1,0,2,3,4,5,2,3,1,0,3,4,1,2,3,4,5,1,3,4,9,1,4,5,2,3,4,1,0,
 ,6,2,7,6,0,1,3,2,5,4,3,2,1,3,5,2,3,0,1,3,2,5,4,3,2,1,0,6,2,1,3,4,5,2,3,4,0,8,5,2,1,6,0,1,
 3,4,5,2,3,4,1,0,10,4,1,3,2,5,4,3,2,0,1,6,2,1,3,4,5,2,3,1,2,10,6,0,1,4,3,2,5,4,3,1,0,6,1,
 3,4,1,6,7,3,1,0,2,3,4,5,2,3,1,0,3,2,5,3,1,5,4,1,6,2,1,4,3,2,5,4,1,9,2,5,1,2,3,4,1,2,7,1,
 3,2,5,4,3,1,2,6,0,1,2,3,4,5,2,1,0,2,5,3,1,2,3,4,5,2,3,1,0,6,1,4,8,6,5,2,3,4,5,2,6,5,3,2,
 0,4,2,1,5,6,11,1,5,2,3,4,5,2,1,0,6,7,11,2,5,6,9,1,5,2,3,4,5,1,3,5,2,6,1,2,5,4,3,2,0,4,2,
 5,4,3,2,1,10,9,2,8,9,1,3,4,5,2,4,0,2,3,4,5,2,3,1,4,2,7,1,4,5,1,6,0,1,5,4,3,2,1,7,3,1,4,3,
 ,2,5,4,3,1,0,3,4,5,3,1,4,3,2,5,4,3,1,0,3,6,4,0,8,4,1,3,2,9,0,5,1,6,0,3,6,8,5,2,10,5,1,2,
 5,3,2,7,1,2,3,9,8,4,2,1,4,3,1,7,8,2,5,1,4,3,2,1,7,2,0,9,8,5,2,4,0,11,1,3,2,1,7,10,9,7,2,
 1,3,4,5,2,1,12,0,1,3

A.4 A Coil of Length 1340 for $n = 12$

0,1,2,3,1,4,2,5,6,4,5,1,4,2,5,6,3,2,6,4,2,1,4,5,2,0,1,4,2,1,3,4,5,6,3,5,1,3,2,5,6,4,2,6,
7,4,6,5,3,1,5,4,1,2,4,6,5,4,2,1,3,4,2,5,6,4,5,3,0,5,4,6,5,2,1,3,2,5,6,4,2,5,1,4,5,6,4,3,
8,6,1,2,4,6,2,3,6,5,2,4,1,5,4,6,5,2,4,1,3,2,1,0,3,5,2,4,3,5,1,3,2,5,6,4,2,5,1,4,2,3,5,7,
3,6,5,2,4,1,3,2,4,5,2,3,6,5,2,4,1,3,0,1,4,2,6,1,5,3,6,5,4,1,5,2,4,6,5,4,1,5,3,9,5,1,2,3,
1,4,2,5,6,4,5,1,4,2,5,6,3,2,6,4,2,1,4,5,2,0,1,4,2,1,3,4,5,6,3,5,1,3,2,5,6,4,2,6,7,4,6,5,
3,1,5,4,1,2,4,6,5,4,2,1,3,4,2,5,6,4,5,3,0,5,4,6,5,2,1,3,2,4,1,5,2,4,6,5,4,1,3,8,0,1,3,4,
2,3,0,2,5,4,3,2,4,6,5,2,4,1,5,4,3,2,5,4,1,0,3,5,2,4,3,5,1,3,2,5,6,4,5,7,4,2,5,6,3,2,5,4,
2,3,1,4,2,5,6,3,0,6,5,2,1,6,2,4,1,2,3,1,5,2,4,6,5,4,1,5,2,3,0,2,10,0,5,1,2,3,1,4,2,5,6,4,
5,1,4,2,5,6,3,2,6,4,2,1,4,5,2,0,1,4,2,1,3,4,5,6,3,5,1,3,2,5,6,4,2,6,7,4,6,5,3,1,5,4,1,2,
4,6,5,4,2,1,3,4,2,5,6,4,5,3,0,5,4,6,5,2,1,3,2,5,6,4,2,5,1,4,5,6,4,3,8,6,1,2,4,6,2,3,6,5,
2,4,1,5,4,6,5,2,4,1,3,2,1,0,3,5,2,4,3,5,1,3,2,5,6,4,2,5,1,4,2,3,5,7,3,6,5,2,4,1,3,2,4,5,
2,3,6,5,2,4,1,3,0,1,2,3,1,4,5,6,4,2,5,1,4,2,3,1,4,9,3,4,2,5,6,4,5,1,4,2,5,6,3,2,6,4,2,1,
4,5,2,0,1,4,2,1,3,4,5,6,3,5,1,3,2,5,6,4,2,6,7,4,6,5,3,1,5,4,1,2,4,6,5,4,2,1,3,4,2,5,6,4,
5,3,0,5,4,6,5,2,1,3,2,5,6,4,2,5,1,4,5,6,4,2,8,7,2,4,6,5,4,1,5,2,4,6,5,2,3,1,2,5,6,4,5,0,
3,5,4,6,5,2,4,3,5,6,4,5,1,3,5,6,3,2,5,4,3,5,7,4,5,2,3,1,5,3,6,5,4,3,1,2,4,1,5,4,6,5,0,1,
4,5,6,4,2,1,4,5,1,3,5,6,4,5,1,4,2,6,11,2,4,1,5,2,4,6,5,2,3,1,2,5,6,4,5,0,3,5,4,6,5,2,4,3,
1,2,4,5,6,4,2,1,4,5,1,3,5,6,4,7,6,2,4,6,5,2,3,1,5,3,6,5,4,3,1,2,4,1,0,2,5,4,1,2,4,6,2,3,
6,5,2,4,1,5,4,6,5,2,4,1,3,2,1,8,10,1,2,3,1,4,2,5,6,4,5,1,4,2,5,6,3,2,6,4,2,1,4,5,2,0,1,4,
2,1,3,4,5,6,3,5,1,3,2,5,6,4,2,6,7,4,6,5,3,1,5,4,1,2,4,6,5,4,2,1,3,4,2,5,6,4,5,3,0,5,4,6,
5,2,1,3,2,4,1,5,2,4,6,5,4,1,3,8,0,1,3,4,2,3,0,2,5,4,3,2,4,6,5,2,4,3,2,0,5,3,2,4,3,5,1,3,
2,5,6,4,5,0,4,2,5,6,4,7,6,2,4,6,5,2,3,6,2,4,1,2,3,4,5,6,3,4,2,9,5,1,2,3,1,4,2,5,6,4,5,1,
4,2,5,6,3,2,6,4,2,1,4,5,2,0,1,4,2,1,3,4,5,6,3,5,1,3,2,5,6,4,2,6,7,4,6,5,3,1,5,4,1,2,4,6,
5,4,2,1,3,4,2,5,6,4,5,3,0,5,4,6,5,2,1,3,2,5,6,4,2,5,1,4,5,6,4,3,8,6,1,2,4,6,2,3,6,5,2,4,
1,5,4,6,5,2,4,1,3,2,1,0,3,5,2,4,3,5,1,3,2,5,6,4,2,5,1,4,2,3,5,7,3,6,5,2,4,1,3,2,4,5,2,3,
6,5,2,4,1,3,0,2,3,1,5,2,4,6,5,4,1,5,2,4,3,5,1,6,2,10,9,4,2,1,3,4,5,6,3,5,1,3,2,5,6,4,2,6,
7,4,6,5,2,4,1,5,4,6,5,2,4,3,1,2,4,5,6,4,2,1,4,5,2,0,3,2,5,6,3,5,1,3,2,5,6,4,2,5,3,1,9,8,
1,3,5,2,4,6,5,2,3,1,5,3,6,5,2,3,5,7,3,6,5,2,4,1,3,2,4,5,2,3,6,5,2,4,1,3,0,2,1,3,5,2,4,6,
5,2,3,1,2,5,8,7,5,2,1,3,2,5,6,4,2,5,1,4,5,6,4,2,5,8,4,5,6,4,2,3,4,5,1,3,2,4,3,8,1,5,7,9,
4,0,1,6,2,3,6,5,2,4,6,3,2,1,3,4,2,3,9,4,5,10,2,6,1,2,10,8,2,1,6,0,3,4,6,5,2,8,6,2,4,1,3,
2,5,3,0,4,5,2,3,6,4,3,5,1,6,4,3,6,0,1,7,0,6,3,4,5,1,4,2,5,6,4,5,1,4,3,6,4,5,9,1,3,7,8,3,
1,10,5,4,1,5,6,3,4,2,3,1,4,3,5,4,6,1,2,11

A.5 A Coil of Length 2584 for $n = 13$

0,1,2,0,3,2,4,0,5,4,2,1,5,4,0,5,1,3,0,1,2,6,1,4,2,1,5,0,3,5,1,2,4,1,5,0,4,1,2,4,0,3,2,7,
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