On-line coloring of H-free bipartite graphs

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Abstract. We present a new on-line algorithm for coloring bipartite graphs. This yields a new upper bound on the on-line chromatic number of bipartite graphs, improving a bound due to Lovász, Saks and Trotter. The algorithm is on-line competitive on various classes of H-free bipartite graphs, in particular P_6-free bipartite graphs and P_7-free bipartite graphs, i.e., that do not contain an induced path on six, respectively seven vertices. We show that the number of colors the on-line algorithm uses in these particular cases is bounded by roughly twice, respectively roughly eight times the on-line chromatic number. In contrast, it is known that there exists no competitive on-line algorithm to color P_6-free (or P_7-free) bipartite graphs, i.e., for which the number of colors is bounded by any function only depending on the chromatic number.

1 Introduction

In static optimization problems one is often faced with the challenge of determining efficient algorithms that solve a particular problem (nearly) optimally for any given instance of the problem. This task is usually facilitated if the structure of the instances is pretty straightforward. As an example, it is a trivial exercise to determine an algorithm for finding a 2-coloring of a given bipartite graph.

In the area of dynamic optimization the situation gets more complicated. There, one often lacks the knowledge of the complete instances of the problems. As an illustration, compare the previous problem with the slightly changed situation in which the bipartite graph comes in on-line, i.e., vertex by vertex and the algorithm has to assign a color to a vertex as it comes in, i.e., only based on the knowledge of the subgraph that has been revealed so far. This slight change of the problem formulation makes it a lot more difficult: Whereas the static problem was trivial, no algorithm for the dynamic problem can guarantee an optimal solution for every instance. In [9] it has been shown that the worst-case performance ratio between on-line and off-line coloring of a known input graph on n vertices is at least $\Omega(n/log_2 n)$. It is even questionable whether one can expect to determine an on-line algorithm that does reasonably well, in the sense that the number of colors used is bounded in some other reasonable way. In this paper we will focus on particular questions of this type related to coloring bipartite graphs. This type of questions in a more general setting is at the heart of the areas of on-line algorithms and of approximation algorithms.

We first give a short historical excursion starting with a benchmark paper due to Gyárfás and Lehel [6]. They introduced the concept of on-line coloring as a general approach. This was motivated by their translation of a rectangle packing problem related to dynamical storage allocation appearing in [2] into an on-line coloring problem. The latter problem was to decide whether the on-line coloring algorithm known as *First-Fit (FF)* has a constant worst-case performance ratio on the family of interval graphs. We note that since [6] many papers on on-line (coloring) problems have appeared. We refer to [11] for a survey.

In order to have some measure of the performance of on-line algorithms, the notion of competitive algorithms has been introduced in [6]. Intuitively, an on-line coloring algorithm is said to be competitive for a family of graphs $\mathcal{G}$, if for any graph $G \in \mathcal{G}$, the number of colors used by the algorithm on G is bounded from above by a function only depending on the chromatic number of G. In [10] it is shown that *FF* is competitive for interval graphs, with a bounding function that is linear in the chromatic number, and in [3] competitiveness of *FF* for geometric intersection graphs has been proven. It is well-known that *FF* is not competitive for P_6-free bipartite graphs, i.e., bipartite graphs that do not contain an induced path on six vertices: If the vertices of a complete bipartite graph $K_{m,m}$ minus a perfect matching $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_m, v_m\}$ are presented in the ordering $u_1, v_1, u_2, v_2, \ldots, u_m, v_m$, then *FF* uses m colors. In fact, there are many families of graphs for which no competitive algorithms exist: Two examples given in [6] are the family of trees and the family of P_6-free bipartite graphs. These negative results have led to the definition of a weaker form of competitiveness in [4], although results of this type have been obtained before the term was formally introduced. An on-line coloring algorithm is said to be on-line competitive if the number of colors is bounded from above by a function only depending on the on-line chromatic number of G. It is shown in [7] that *FF* is on-line competitive for trees; it is even optimal for trees, in the sense that if *FF* uses k colors, then the on-line chromatic number of the tree is also k. In [4] it is shown that *FF* is on-line competitive with an exponential bounding function for graphs with girth at least five. There are very few existing results on on-line competitive coloring algorithms.

In the context of algorithmic graph theory it is rather natural to consider forbidden subgraph conditions, as many NP-hard problems turn out to be solvable in polynomial time when restricted to H-free graphs for particular choices of H. Therefore, these graph classes are well-studied throughout a range of NP-hard problems. In the context of coloring, e.g., 3-colorability is polynomially solvable for P_6-free graphs, while 4-colorability remains NP-hard for P_{12}-free graphs, and 5-colorability remains NP-hard for P_8-free graphs. We refer the reader to the survey paper [16] for more details. Note that also well-studied graph classes like chordal graphs can be characterized by forbidden subgraph conditions.

2 Results of this paper

One of the main open problems concerning on-line competitive coloring algorithms [4] is to decide whether for every k there exists an on-line competitive coloring algorithm for the family of graphs with on-line chromatic number k. Perhaps surprisingly, this is even open for bipartite graphs for $k = 4$, whereas it has been solved for general graphs for $k \leq 3$. (In both [5] and [14] it is proven that for the family of graphs with on-line chromatic number 3 at most 4 colors are needed.) The open problem on bipartite graphs seems to be very hard and emphasizes how much on-line coloring differs from off-line coloring. We are not aware of any recent developments towards settling this problem. Our results are motivated by a number of open problems, but most strongly by the above open problem for bipartite graphs. We solve the problem for several subclasses of bipartite graphs which are defined by forbidding a certain fixed graph H as an induced subgraph. For a relatively small graph H this is an easy exercise, but for larger graphs this gets difficult, in correspondence with the fact that the class of H-free graphs contains the class of H'-free graphs if H' is a subgraph of H. By combining known results and dealing with a few cases ourselves, we show that for every graph H with at most 5 vertices there exists an on-line competitive coloring algorithm for the class of H-free bipartite graphs. Since for P_4-free and P_5-free graphs there even exists a competitive algorithm [6, 8], and since P_6-free bipartite graphs do not admit a competitive algorithm [6], our natural starting point from there is the latter class. The main contribution of this paper is the proof that the on-line coloring algorithm we present for bipartite graphs is on-line competitive for P_6-free bipartite graphs; its bounding function is *linear* in the on-line chromatic number, namely roughly twice the on-line chromatic number. In fact, this gives a 2-approximation algorithm for on-line coloring P_6-free bipartite graphs. We can prove a similar result for the larger class of P_7-free bipartite graphs with a bounding function that is roughly eight times the on-line chromatic number. Due to page limitations we leave its proof for the full paper. Note that the on-line chromatic number for both these graph classes can be arbitrarily high, so these classes are definitely no subclasses of the class of bipartite graphs with on-line chromatic number 4. In this sense, our results have a broader appeal than just solving the aforementioned problem with $k = 4$ for the restricted classes of P_6-free and P_7-free bipartite graphs. It might be possible that our algorithm or variations on it can be used to prove similar results for larger subclasses of bipartite graphs, although we have not been able to do so yet. We will see that our algorithm is competitive for the class of P_5-free bipartite graphs.

The rest of the paper is organized as follows. Section 3 contains the basic notation and definitions. In Section 4 we start our exposition by proving the result on H-free bipartite graphs with $|V(H)| \leq 5$. Next we present the key algorithm of the paper called *BicolorMax*. We prove that it is on-line competitive for P_6-free bipartite graphs, and that the number of colors used by *BicolorMax* on any bipartite graph is bounded from above by the number of mutually remote subgraphs isomorphic to P_5. As a consequence we improve the best known upper bound for the on-line chromatic number of bipartite graphs given in [15] and [11].

3 Preliminaries

Throughout the paper we consider simple graphs, denoted by $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_{|V(G)|}\}$ is a set of vertices and $E(G)$ is a set of unordered pairs of vertices, called edges. For graph terminology not defined below we refer to [1]. If $S \subseteq V(G)$, then $G[S]$ denotes the subgraph of G with vertex set S and edge set $\{\{x, y\} \mid x \in S, y \in S\}$. A graph H is an *induced subgraph* of G if H is isomorphic to $G[S]$ for some nonempty $S \subseteq V(G)$. A graph G is *H-free* if it does not contain the graph H as an induced subgraph. We call two vertex-disjoint graphs *remote* if there are no edges joining them. A maximal connected subgraph of a graph G is called a *component* of G. For any two vertices x, y of a connected graph G we denote by P_{xy} a shortest path between x and y in G, and we define the *distance* $d(x, y, G)$ between x and y in G as $|E(P_{xy})|$. A *coloring* of a graph G is a function $c : V(G) \to \{1, 2, \ldots\}$ such that $c(v) \neq c(w)$ whenever $\{v, w\} \in E(G)$. We denote the set of all colorings of G by $\mathcal{C}(G)$. The smallest number of colors in a coloring of G is the *chromatic number* of G and denoted by $\chi(G)$.

We assume that the reader is familiar with the basic concept of an on-line coloring algorithm. For details we refer to [11]. Intuitively, an on-line coloring algorithm properly colors the vertices of a graph one by one, consistently using a fixed strategy, depending only on the subgraph induced by the revealed vertices and their colors, according to an externally determined ordering of the presented vertices.

We denote the (finite) set of all on-line coloring algorithms for a graph G by $AOL(G)$. Let $\Pi(G)$ denote the set of all permutations of the vertices of G. If $A \in AOL(G)$ and $\pi \in \Pi(G)$, we denote by $\chi_A(G, \pi)$ the number of colors used by A when the vertices of G are presented according to π. The largest number of colors used by the on-line algorithm A for G is called the *A-chromatic number* of G and denoted by $\chi_A(G)$. Hence $\chi_A(G) = \max_{\pi \in \Pi(G)} \chi_A(G, \pi)$. The smallest number of colors used by an on-line algorithm for G is the *on-line chromatic number* of G, and denoted by $\chi_{OL}(G)$ [6]. Hence $\chi_{OL}(G) = \min_{A \in AOL(G)} \chi_A(G)$. Let $\mathcal{G}$ denote a (possibly infinite) family of graphs. If $A \in AOL(G)$ for every $G \in \mathcal{G}$, we say that A is an on-line coloring algorithm for $\mathcal{G}$ and write $A \in AOL(\mathcal{G})$. An algorithm $A \in AOL(\mathcal{G})$ is said to be *competitive* for $\mathcal{G}$ if there exists a function f such that $\chi_A(G) \leq f(\chi(G))$ for every $G \in \mathcal{G}$; it is *on-line competitive* if $\chi_A(G) \leq f(\chi_{OL}(G))$ for every $G \in \mathcal{G}$.

4 On-line competitive coloring algorithms

As stated before, there does not exist a competitive on-line coloring algorithm for P_6-free bipartite graphs, but there exists a competitive on-line coloring algorithm for P_5-free bipartite graphs. In fact, combining results from [4, 8, 12, 13], and analyzing a few cases ourselves, we can show there exists an on-line coloring algorithm that is on-line competitive for the class of H-free bipartite graphs for any fixed graph H on at most five vertices.

Proposition 1. *Let H be a (bipartite) graph on at most five vertices. Then there exists an on-line coloring algorithm that is on-line competitive for the class of H-free bipartite graphs.*

Proof. The statement is trivial when H is not bipartite. We may further restrict ourselves to bipartite graphs on exactly five vertices, noting that an F-free bipartite graph with F bipartite on at most four vertices is also H-free for some bipartite graph H on five vertices. We use $H + H'$ to denote the disjoint union of two graphs H and H', and pH to denote the disjoint union of $p \geq 2$ copies of H. Before we make a case distinction we first make the following easy observation:

(1) Let F be a graph and A an on-line coloring algorithm that is on-line competitive for the class of F-free bipartite graphs. Then there exists an on-line coloring algorithm A' that is on-line competitive for the class of $F + K_1$-free bipartite graphs.

This claim can be seen as follows. Initially we use algorithm A to color the vertices of an $F + K_1$-free graph G. If G contains an induced F, then as soon as all vertices of F have been colored all vertices presented afterwards have a neighbor in F. Since G is bipartite, this means that the coloring of G can be finished using only two new colors at most. We now distinguish a number of cases depending on the value of $|E(G)| = m$.

Case I: $m = 0$. Then $H = 5K_1$ and clearly $\chi_{FF} \leq 5$, since FF only uses color 6 on a vertex that has already neighbors with colors 1 to 5. In a bipartite graph these neighbors form an independent set. On-line competitiveness also follows from applying (1) five times.

Case II: $m = 1$. Then $H = K_2 + 3K_1$. It is trivial to see that FF is on-line competitive for the class of K_2-free graphs. After applying (1) three times we get the desired result.

Case III: $m = 2$. Then $H = P_3 + 2K_1$ or $2K_2 + K_1$. For the first subcase we can proceed similarly as in Case II. For the second subcase we use the following result from [8]:

(2) If G is a P_5-free graph without triangles, then $\chi_{FF}(G) \leq 3$.

Noting that $2K_2$-free bipartite graphs are both P_5-free and triangle-free, and combining (1) and (2), yields the result.

Case IV: $m = 3$. Then $H = P_4 + K_1$, $K_{1,3} + K_1$, or $P_3 + K_2$. Noting that P_4-free bipartite graphs are both P_5-free and triangle-free, and combining (1) and (2), yields the desired result for the first subcase. For the second subcase we first observe that $\chi_{FF}(G) \leq 3$ for any $K_{1,3}$-free bipartite graph G (cf. Case I), and then we apply (1) to get the result. Since a $P_3 + K_2$-free bipartite graph is a

P_6-free bipartite graph, we can of course immediately apply Theorem 1 (which will be presented later) for the third subcase. It is also not difficult to give a direct proof that our algorithm *BicolorMax* is on-line competitive for this class of graphs.

Case V: $m = 4$. Then $H = K_{1,4}$, $C_4 + K_1$, P_5, or the unique connected graph with degree sequence 3,2,1,1,1 which we denote by $K_{1,3}^+$. For the first subcase we easily get that $\chi_{FF}(G) \leq 4$ in a similar way as in Case I. The *girth* of a graph G is the number of edges of a smallest cycle in G. For the second subcase we combine (1) with the following result from [4]:

(3) If G has girth at least five, then $\chi_{FF}(G) \leq \binom{2^{\chi_{OL}(G)}}{2}$.

For the third subcase we use (2). The *radius* of a graph G is defined as the minimum of $\max_v d(u, v, G)$ over all vertices u in G. For the fourth subcase we use the following result from [13]:

(4) For every tree T with radius 2, there is an on-line coloring algorithm A that is on-line competitive for the class of T-free graphs.

Case VI: $m = 5$. Then $H = K_{2,3} - e$ for an edge e of $K_{2,3}$. We need a separate proof for this case. We first prove the following claim:

Claim: Let G be bipartite and H-free and let C be a component of G such that C_4 is an induced subgraph of C. Then $C = K_{s,t}$ for some integers $s, t \geq 2$.

We prove this claim as follows. If $C = C_4 = K_{2,2}$, then the claim trivially holds. If not, let $C_4 = uvwxu$, and let $N(p)$ denote the neighbors of vertex p in C. If $N(u) \not\subseteq N(w)$, then G contains H as an induced subgraph. So, by symmetry, $N(u) = N(w)$, and similarly $N(v) = N(x)$. Let $y \in N(u) \cap N(w)$. Then $uvwyu$ is an induced C_4, so as before $N(y) = N(v) = N(x)$. Hence all neighbors of u and w are adjacent to all neighbors of v and x, and vice versa. By repeating the arguments for all induced C_4s, we obtain that $C = K_{s,t}$ for some $s, t \geq 2$.

Since $\chi_{FF}(K_{s,t}) = 2$, the above claim together with (3) implies that $\chi_{FF}(G) \leq \max\{\binom{2^{\chi_{OL}(G)}}{2}, 2\}$.

Case VII: $m = 6$. Then $H = K_{2,3}$. Kierstead and Penrice [12] showed that *FF* is on-line competitive for the class of H-free graphs. □

We conclude that the first open question with respect to the (non)existence of on-line competitive coloring algorithms for H-free bipartite graphs concerns bipartite graphs H on 6 vertices, in particular $H = P_6$. In 4.1 we present a new on-line algorithm for coloring general bipartite graphs. We analyze the behavior of this algorithm in 4.3 and 4.4. In 4.3 we present our main results: the algorithm is a linear on-line competitive algorithm for P_6-free bipartite graphs and for P_7-free bipartite graphs. For our proof of the P_6-free case we need a suitable new class of P_6-free bipartite graphs that will be introduced in 4.2. We will not prove the P_7-free case here due to the page limits. In 4.4 we give a new upper bound for the on-line chromatic number of bipartite graphs.

4.1 The algorithm BicolorMax

Let G be a bipartite graph on n vertices denoted by $1, 2, \ldots, n$. Let $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_p\}$ be two disjoint ordered sets of colors. For a fixed positive integer $i \leq p$, let $A(i) = \{a_1, a_2, \ldots, a_i\}$ and $B(i) = \{b_1, b_2, \ldots, b_i\}$.

We first give an informal description of our on-line algorithm. Suppose that G is presented to the algorithm. At some stage a new uncolored vertex v of G is revealed, together with its adjacencies to the set S of already colored vertices of G. If v is not adjacent to any previously revealed vertex of G, then v receives color a_1. Otherwise, the choice of the color for v is based on the present colors in the bipartition classes of the subgraph $G(k)$ of G induced by v and the vertices of S with colors in $A(k) \cup B(k)$ for some suitable $k \geq 1$. We first determine the largest k (if any) such that the color a_k appears in both bipartition classes of the component $C(k)$ of $G(k)$ containing v. If there is no such k, then we assign color a_1 or b_1 to v, depending on whether a_1 or b_1 appears in the same bipartition class as v in $C(1)$. If there is such a (largest) k, then we assign color a_{k+1} or b_{k+1} to v, depending on whether a_{k+1} or b_{k+1} appears in the same bipartition class as v in $C(k+1)$.

We need some definitions in order to give a more formal description. If $F \subseteq V(G)$, then the *hue* of F, denoted by $H(F)$, is the set of all colors used on vertices in F. Let $\pi(G)$ be a permutation of $V(G)$. Given a component $C(k)$ of G as above, we say that color a_k is *mixed* on $C(k)$ if in the bipartition of $V(C(k))$ into (I_1, I_2), there exist at least two vertices $v \in I_1$ and $w \in I_2$ that have been colored with a_k. We then call (v, w) a *k-mixed pair.*

The algorithm *BicolorMax* is defined inductively. The vertex $\pi(1)$ is colored with a_1. Suppose that vertices $\pi(1), \ldots, \pi(j-1)$ have already been colored and let $v = \pi(j)$ be the next vertex presented to the algorithm. If k is a positive integer, then $G_j(k, v)$ denotes the subgraph of $G_j = G[[v]]$ induced by v and all the vertices in $V(G_{j-1})$ that have been assigned colors from $A(k) \cup B(k)$. We denote by $C_j(k, v)$ the component of $G_j(k, v)$ containing v, and we use $C_j(k, v) := (I_1, I_2)$ to indicate the bipartition of its vertex set. Note that (I_1, I_2) is the unique bipartition of $C_j(k, v)$ (up to a reordering), because $C_j(k, v)$ is connected.

BicolorMax(G_{j-1}, v)

$m := \max(\{0\} \cup \{k : a_k \text{ is mixed on } C_j(k, v)\})$.
if $a_{m+1} \notin H(V(C_j(m+1, v)))$
    $C_j(m+1, v) := (I_1, I_2)$ such that $v \in I_1$
else
    $C_j(m+1, v) := (I_1, I_2)$ such that $a_{m+1} \in H(I_1)$.
if $v \in I_1$
    assign color a_{m+1} to v
else
    assign color b_{m+1} to v.

It is easy to check that *BicolorMax* is a polynomial time on-line coloring algorithm for bipartite graphs. We leave the details to the reader, but we illustrate the algorithm with the following example.

Example 1. Let G be a $K_{4,4}$ without a perfect matching, i.e., with $V(G) = \{1,2,3,4,5,6,7,8\}$, bipartition in $\{1,3,5,7\}$ and $\{2,4,6,8\}$, and only edges $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, and $\{7,8\}$ omitted. If the vertices are revealed in the order of increasing numbers, the algorithm assigns colors $a_1, a_1, b_1, b_1, a_2, b_2, a_2, b_2$, respectively. The last color is assigned since a_1 is mixed in the subgraph of G induced by $\{1,2,3,4,8\}$, while a_2 is assigned to a vertex in the other bipartition class of $C_8(2,8) = G$ than the vertex 6. Suppose that G is extended and a new vertex 9 is revealed. Then 9 is respectively assigned color a_1 if 9 is only adjacent to 7, color b_1 if 9 is adjacent to 1 and 7, color b_2 if 9 is adjacent to 1, 3 and 7, and color a_2 if 9 is adjacent to 2, 4 and 6. For a $K_{n,n}$ without a perfect matching with $n \geq 5$ the algorithm will continue assigning a_2 and b_2 if the vertices are presented in an order alternating between the two classes of the bipartition, as in the above example for $n = 4$. In contrast, recall that *FF* uses n colors in this case.

4.2 A class of P_6-free bipartite graphs

The objective is to show that *BicolorMax* is an on-line competitive algorithm for P_6-free bipartite graphs. As a first step, we inductively define a class of P_6-free bipartite graphs (see Figure 1). The members of this class will have the following useful property: The larger members contain pairwise remote copies of the smaller members with complementary adjacencies with respect to the bipartition. The latter property enables us to define a permutation which forces a large number of colors on any on-line coloring algorithm for the large members of this class. It will turn out that a member H_k from this class has on-line chromatic number at least k, and that if *BicolorMax* uses color a_k on a P_6-free bipartite graph G, then H_{k+1} is an induced subgraph of G.

Each graph H_i of the class has a root vertex $r(H_i)$, and:

- H_1 is a graph consisting of a single root vertex.
- H_2 is a graph consisting of an edge, one of whose end vertices is the root vertex.
- H_3 is a path on four vertices, one of whose internal vertices is the root vertex.
- H_k, $k \geq 4$ consists of a root vertex v and two disjoint copies H^1_{k-1} and H^2_{k-1} of H_{k-1} and edges joining v to all non-neighbors of $r(H^1_{k-1})$ (including $r(H^1_{k-1})$) in H^1_{k-1} and all neighbors of $r(H^2_{k-1})$ in H^2_{k-1}.

It is easy to check that for all $k \geq 1$ the graph H_k is bipartite and P_6-free. We note that the above defined class is different from the class of P_6-free bipartite graphs defined in [6]. The graphs H_k have the following useful properties.

Lemma 1. *The two remote copies H^1_{k-1} and H^2_{k-1} of H_{k-1} in H_k ($k \geq 4$) each contain:*

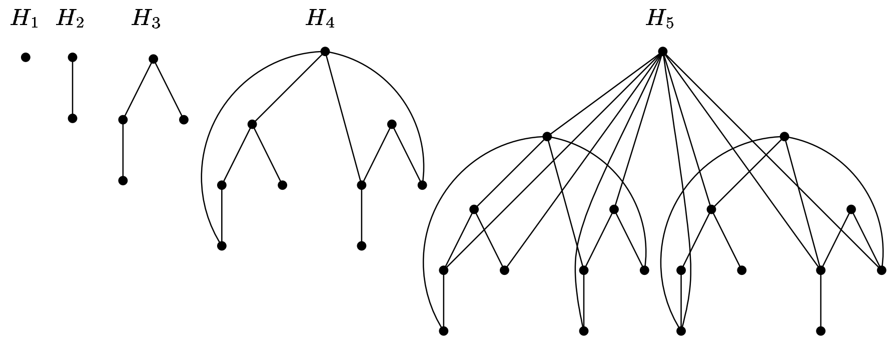

Fig. 1. The graphs H_1, H_2, H_3, H_4, H_5.

(i) a set of pairwise remote subgraphs isomorphic to $H_1, \ldots, H_{k-2}$ with all the vertices in the bipartition class containing their root vertex adjacent to $r(H_k)$;
(ii) a set of pairwise remote subgraphs isomorphic to $H_1, \ldots, H_{k-2}$ with all the vertices in the bipartition class not containing their root vertex adjacent to $r(H_k)$.

Proof. By induction on k. This can easily be checked. Note that a subgraph in (i) can use some vertices of H_k that a graph in (ii) also uses. □

The structural properties of H_k imply that its on-line chromatic number is at least k.

Proposition 2. *For any $k \geq 1$, $\chi_{OL}(H_k) \geq k$.*

Proof. By induction on k. It is routine to check this for $k = 1, 2, 3$. Suppose that $k \geq 4$ and that the result holds for H_k with $4 \leq k \leq t$. Consider H_{t+1} and an on-line algorithm A for coloring H_{t+1}. The first time the i^{th} color is used by A we identify it as color i. We choose an ordering on $V(H_{t+1})$ such that the vertices of pairwise remote copies of $H_1, \ldots, H_t$ are presented until color i is used on H_i ($i = 1, \ldots, t$); then, if $i \leq t - 1$, we immediately start presenting the vertices of H_{i+1}. By the adjacency relations from the definition of H_{t+1} and the properties of Lemma 1, the ordering of the vertices of $H_1, \ldots, H_t$ can be chosen in such a way that $r(H_{t+1})$ is adjacent to the (not necessarily root) vertices that received colors $1, \ldots, t$. Hence a new color $t + 1$ is forced upon A. □

4.3 BicolorMax is on-line competitive

Before we present our main result on the on-line competitiveness of *BicolorMax*, we make a number of useful observations in the following three lemmas.

Lemma 2. *Let G be a bipartite graph. Let BicolorMax color vertex $v = \pi(j)$ with a_m or b_m, $m \geq 2$. If (x, y) is a k-mixed pair in $C_j(k, v)$ with $k \leq m - 1$, then any path between x and y in $C_j(k, v)$ must pass through v.*

Proof. Suppose there exists a path in $C_j(k,v)$ between x and y not passing through v. Let $x=\pi(r)$ and let $y=\pi(s)$. We assume without loss of generality that y has been presented to *BicolorMax* after x, i.e., $s>r$. Suppose x belongs to $C_s(k,y)$, implying that $a_k \in H(V(C_s(k,y)))$. Since y is colored with a_k, color a_{k-1} is mixed on $C_s(k-1,y)$. Then *BicolorMax* would have colored y with color b_k. Hence x does not belong to the component $C_s(k,y)$. Suppose there exists an index i with $s<i<j$ such that x and y belong to the component $C_i(k,\pi(i))$. This means that a_k is mixed on $C_i(k,\pi(i))$. Then *BicolorMax* would never use a color a_h with $h \leq k$ to color $\pi(i)$. This implies that such an index i does not exist. We conclude that every path between x and y in $C_j(k,v)$ must pass through v. □

Lemma 3. *Let G be a P_6-free bipartite graph. Let BicolorMax color vertex $v=\pi(j)$ with a_m, $m \geq 2$. Let z be a vertex in $C_j(m-1,v)$ assigned color a_{m-1}. If z has odd distance from v in $C_j(m-1,v)$, then $d(v,z,C_j(m-1,v))=1$. Otherwise $d(v,z,C_j(m-1,v))=2$.*

Proof. Since *BicolorMax* uses a_m for v, color a_{m-1} is mixed on $C_j(m-1,v)$. This means that there exists a vertex z^* with color a_{m-1}, such that z and z^* are in different classes of the bipartition of $C_j(m-1,v)$. By Lemma 2, a shortest path P_{zz^*} must be formed by joining shortest paths P_{zv} and P_{vz^*}. Suppose $d(v,z,C_j(m-1,v))$ is odd. Then z^* has even distance from v in $C_j(m-1,v)$ implying that $d(v,z,C_j(m-1,v)) \geq 2$. If $d(v,z,C_j(m-1,v)) \geq 3$, then P_{zz^*} contains an induced P_6. Hence $d(v,z,C_j(m-1,v))=1$. Suppose $d(v,z,C_j(m,v))$ is even. If $d(v,z,C_j(m-1,v)) \geq 4$, then P_{zz^*} contains an induced P_6. Hence $d(v,z,C_j(m-1,v))=2$. □

Lemma 4. *Let G be a P_6-free bipartite graph. If BicolorMax uses color a_k on vertex $v=\pi(j)$, $k \geq 2$, then $C_j(k-1,v)$ contains H_{k+1} as an induced subgraph with $v=r(H_{k+1})$.*

Proof. By induction on k. The case $k=2$ is trivial. Let $k \geq 3$. Since *BicolorMax* uses color a_k on vertex v, there exists a $(k-1)$-mixed pair (x,y) in $C_j(k-1,v)$. Assume $x=\pi(r)$ and $y=\pi(s)$. By Lemma 2 the components $C_r(k-2,x)$ and $C_s(k-2,y)$ are remote. By the inductive hypothesis x is the root of an induced copy H_k^1 of H_k in $C_r(k-2,x)$ and y is the root of an induced copy H_k^2 of H_k in $C_s(k-2,y)$. Lemma 3 implies that we may without loss of generality assume that distance $d(x,v,C_j(k-1,v))=2$ and distance $d(y,v,C_j(k-1,v))=1$. We claim that v is adjacent to all neighbors of x in H_k^1 and to all non-neighbors of y in H_k^2. Suppose x has a neighbor x' in H_k^1 not adjacent to v. Let y' be a neighbor of y in H_k^2, and let z be a common neighbor of x and v in $C_j(k-1,v)$. Then the path $x'xzvyy'$ is an induced P_6 in G, which is a contradiction. By using similar arguments we can prove that v is adjacent to all non-neighbors of y in H_k^2. Hence, we obtain an induced H_{k+1} in $C_j(k-1,v)$ with $v=r(H_{k+1})$. □

We now present our main theorem showing that *BicolorMax* is a linear on-line competitive algorithm for the class of P_6-free bipartite graphs. Denote by $\chi_{Bm}(G)$ the maximum number of colors used by *BicolorMax* for coloring G.

Theorem 1. *If G is a P_6-free bipartite graph, then $\chi_{Bm}(G) \leq 2\chi_{OL}(G) - 1$.*

Proof. Let k be the highest index such that *BicolorMax* uses color a_k on a vertex in the P_6-free bipartite graph G. Since *BicolorMax* only uses b_i with $i \leq k$ if a_i has been used before, $\chi_{Bm}(G) \leq 2k$. For $k = 1$ the statement of the theorem obviously holds. Suppose $k \geq 2$. Due to Lemma 4 the graph G contains a copy of H_{k+1} as an induced subgraph. Proposition 2 implies that $\chi_{OL}(G) \geq \chi_{OL}(H_{k+1}) \geq k + 1$. □

Using a similar but more involved analysis, we were able to prove the following result, showing that *BicolorMax* is also on-line competitive for the class of P_7-free bipartite graphs. We will postpone the proof to the full paper.

Theorem 2. *If G is a P_7-free bipartite graph, then $\chi_{Bm}(G) \leq 8\chi_{OL}(G) + 8$.*

4.4 A new upper bound on χ_{OL} for bipartite graphs

In [15], Lovász, Saks and Trotter define an on-line coloring algorithm A for general graphs that has $\chi_A(G) \leq 2\log_2(n)$ when applied to any bipartite graph G on n vertices (See also [11]). Below we give a tighter upper bound for the on-line chromatic number of a bipartite graph in terms of subgraphs isomorphic to P_5. We note that it is not possible to prove an upper bound in terms of induced subgraphs isomorphic to P_6, since it follows from Proposition 2 and also from a result in [6] that no competitive algorithm exists for the class of P_6-free bipartite graphs.

Theorem 3. *Let G be a bipartite graph in which each component has at most s pairwise remote induced subgraphs isomorphic to P_5. If $s = 0$, then $\chi_{Bm}(G) \leq 4$. If $s > 0$, then $\chi_{Bm}(G) \leq 2\log_2(s) + 6$.*

Proof. We prove the theorem by showing that a component C of G contains at least 2^{k-3} pairwise remote induced subgraphs isomorphic to P_5, if *BicolorMax* uses color a_k on C with $k \geq 3$. We use induction on k. It is easy to check that a component C contains an induced P_5, if *BicolorMax* uses color a_3 on a vertex of C. Let $k \geq 4$. Suppose $v = \pi(j)$ is colored by a_k. Then there exists a $(k-1)$-mixed pair (x, y) in $C_j(k-1, v)$. By Lemma 2, x and y belong to two different components in $G_j(k-1, v) - v$ both containing 2^{k-4} pairwise remote induced subgraphs isomorphic to P_5. □

The above proof shows that if *BicolorMax* uses color a_3 on a bipartite graph G, then G contains an induced P_5. This implies that *BicolorMax* is competitive for the class of P_5-free bipartite graphs.

5 Conclusions and future work

We have introduced the new on-line coloring algorithm *BicolorMax* for bipartite graphs. We have shown that the number of colors used by this algorithm on a bipartite graph G is bounded from above by the number of remote induced subgraphs of G isomorphic to P_5. As a consequence we improved the best known upper bound for the on-line chromatic number of bipartite graphs given in [15]. For any P_6-free (respectively, P_7-free) bipartite graph G, *BicolorMax* has been shown to use at most twice (respectively, eight times) as many colors as any optimal on-line coloring algorithm for G. In a future continuation of this work, we would like to face the problem of deciding whether for any $n \geq 8$, a linear on-line competitive algorithm can be defined for the class of P_n-free bipartite graphs. We also consider analyzing *BicolorMax* and related algorithms for other classes of H-free bipartite graphs, in particular for graphs H with 6 vertices. A seemingly difficult and interesting open case is the (non)existence of an on-line competitive algorithm for the class of C_6-free bipartite graphs.

References

1. J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications,* Macmillan, London and Elsevier, New York (1976).
2. M. Chrobak and M. Ślusarek, Problem 84-23. *Journal of Algorithms 5* (1984), 588.
3. T. Erlebach and J. Fiala, On-line coloring of geometric intersection graphs. *Computational Geometry: Theory and Applications 23* (2002), 243–255.
4. A. Gyárfás, Z. Király, and J. Lehel, On-line competitive coloring algorithms. *Technical report TR-9703-1* (1997), available at http://www.cs.elte.hu/tr97/.
5. A. Gyárfás, Z. Király, and J. Lehel, On-line 3-chromatic graphs. II. Critical graphs. *Discrete Mathematics 177* (1997), 99–122.
6. A. Gyárfás and J. Lehel, On-line and first-fit colorings of graphs. *Journal of Graph Theory 12* (1988), 217–227.
7. A. Gyárfás and J. Lehel, First fit and on-line chromatic number of families of graphs. *Ars Combinatorica 29C* (1990), 168-176.
8. A. Gyárfás and J. Lehel, Effective on-line coloring of P_5-free graphs. *Combinatorica 11* (1991), 181–184.
9. M.M. Halldórsson, Online coloring known graphs. *Electronic Journal of Combinatorics 7* (2000), Research Paper 7, 9pp.
10. H.A. Kierstead, The linearity of first-fit coloring of interval graphs. *SIAM Journal on Discrete Mathematics 1* (1988), 526–530.
11. H.A. Kierstead, Coloring graphs on-line. In: Fiat and Woeginger, eds.: *Online algorithms: the state of the art* (1998), no. 1442 in Lecture Notes in Computer Science, Springer Verlag, 281–305.
12. H. Kierstead and S.G. Penrice Radius two trees specify χ-bounded classes. *Journal of Graph Theory 18* 1994, 119–129.
13. H. Kierstead, S.G. Penrice, and W. Trotter, On-line graph coloring and recursive graph theory. *SIAM Journal on Discrete Mathematics 7* 1994, 72–89.
14. K. Kolossa, On the on-line chromatic number of the family of on-line 3-chromatic graphs. *Discrete Mathematics 150* (1996), 205–230.
15. L. Lovász, M. Saks, and W.T. Trotter, An on-line graph coloring algorithm with sublinear performance ratio. *Discrete Mathematics 75* (1989), 319–325.
16. B. Randerath and I. Schiermeyer, Vertex colouring and forbidden subgraphs – a survey. *Graphs and Combinatorics 20* (2004), 1–40.

Appendix

Notes to the reader:
We would like to note here that the extended abstract is self-contained (except for the proof of the P_7-free case) and that this appendix is included for two reasons only:
- to convince the reviewer that we have a proof for the P_7-free case;
- to show that it is too long and too difficult to sketch to include in the extended abstract.

By a series of lemmas and propositions we prove that $\chi_{Bm}(G) \leq 8\chi_{OL}(G) + 8$ for any P_7-free bipartite graph G. If H is a *subgraph* of G, i.e., $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, we write $H \subseteq G$.

First note that Lemma 2 is valid for any bipartite graph. For the P_7-free case we need the statement below. The proof is analogous to Lemma 2.

Lemma 5. *Let G be a bipartite graph. Let BicolorMax color vertex $v = \pi(j)$ with a_m or b_m, $m \geq 2$. For some $k \leq m-1$ let x and y be two vertices in $C_j(k,v)$ colored with a_k and b_k respectively. If $d(x,y,C_j(k,v))$ is even, then any path between x and y in $C_j(k,v)$ must pass through v.*

Proof. Let G be a bipartite graph. Let v be a vertex in G that is colored with a_m or b_m, $m \geq 2$ by *BicolorMax*. For some $k \leq m-1$ let x and y be two vertices in $C_j(k,v)$ colored with a_k and b_k respectively. Suppose $d(x,y,C_j(k,v))$ is even. We will show that any path between x and y in $C_j(k,v)$ passes through v. In order to obtain a contradiction suppose there exists a path in $C_j(k,v)$ between x and y not passing through v. Let $x = \pi(r)$ and let $y = \pi(s)$.

Suppose x belongs to component $C_s(k,y)$ implying that $a_k \in H(V(C_s(k,y)))$. Since y is colored with b_k, color a_{k-1} is mixed on $C_s(k-1,y)$. Because the distance between x and y in $C_s(k,y)$ is even, *BicolorMax* would have colored y with color a_k. Hence x does not belong to the component $C_s(k,y)$.

Suppose y belongs to component $C_r(k,x)$. Since y is colored with b_k, color b_k is in $H(V(C_r(k,x)))$. Then, by definition of *BicolorMax*, there exists a vertex z in $C_r(k,x)$ with color a_k and with odd distance from y. Since x is colored with a_k, color a_{k-1} is mixed on $C_r(k-1,x)$. Because the distance between x and y in $C_r(k,x)$ is even, the distance between x and z in $C_r(k,x)$ is odd. Then *BicolorMax* would have colored x with color b_k. Hence, x does not belong to the component $C_s(k,y)$.

In the remaining case there exists an index i with $s < i < j$ such that x and y belong to the component $C_i(k,\pi(i))$. Since y is colored with b_k, by definition of *BicolorMax*, the component $C_s(k,y)$, which is a subgraph of $C_i(k,\pi(i))$, must contain a vertex w with color a_k and with odd distance from y. Since the distance between x and y is even, the distance between w and x is odd, and we find that a_k is mixed on $C_i(k,\pi(i))$. Then *BicolorMax* would never use a color a_h with $h \leq k$ to color $\pi(i)$. This implies that an index i with $s < i < j$, such that x and y belong to the component $C_i(k,\pi(i))$, does not exist. We conclude that every path between x and y in $C_j(k,v)$ must pass through v. □

Lemma 6. *Let G be a P_7-free bipartite graph. Let BicolorMax color vertex $v = \pi(j)$ with a_2 or b_2. Then there exists a 1-mixed pair (z^*, z) in $C_j(1, v)$ with $d(v, z^*, C_j(1, v)) = 1$ and $d(v, z, C_j(1, v)) = 2$.*

Proof. Let v be a vertex in a P_7-free bipartite graph G that has received color a_2 from *BicolorMax*. Then there exists a 1-mixed pair (u, w) in $C_j(1, v)$. By Lemma 2, any path P_{uw} from u to w in $C_j(1, v)$ goes through v. Besides v component $C_j(1, v)$ contains only vertices colored with a_1 or b_1. Hence, P_{uw} contains a path z^*vyz on four vertices, in which both z and z^* have color a_1. Note that the pair (z^*, z) is a 1-mixed pair in $C_j(1, v)$. □

Lemma 7. *Let G be a P_7-free bipartite graph. Let BicolorMax color vertex $v = \pi(j)$ with a_m or b_m, $m \geq 3$. Any vertex z in $C_j(m-1, v)$ with color a_{m-1} that is at even distance from v has $d(v, z, C_j(m-1, v)) = 2$. Any vertex z' in $C_j(m-1, v)$ with color a_{m-1} that is at odd distance from v has $d(v, z', C_j(m-1, v)) \leq 3$. Furthermore, there exists an $(m-1)$-mixed pair $(z^*, \hat{z})$ in $C_j(m-1, v)$ with $d(v, z^*, C_j(m-1, v)) = 1$ and $d(v, \hat{z}, C_j(m-1, v)) = 2$.*

Proof. Let G be a P_7-free bipartite graph. Let *BicolorMax* color vertex $v = \pi(j)$ with a_m or b_m, $m \geq 3$. Let z be a vertex with color a_{m-1} that is at even distance from v. Let z' be a vertex with color a_{m-1} that is of odd distance from v. Note that (z, z') is an $(m-1)$-mixed pair in $C_j(m-1, v)$. (By definition of *BicolorMax*, color a_{m-1} is mixed on $C_j(m-1, v)$. So $C_j(m-1, v)$ contains at least one $(m-1)$-mixed pair.) By Lemma 2, a shortest path $P_{zz'}$ from z to z' in $C_j(m-1, v)$ must be formed by joining shortest paths P_{zv} from z to v and $P_{vz'}$ from v to z'. First we show that $d(v, z, C_j(m-1, v)) = 2$.

Suppose $d(v, z, C_j(m-1, v)) \geq 4$, Then $d(v, z', C_j(m-1, v) = 1$, i.e., z' and v are adjacent. Otherwise $P_{zz'}$ contains an induced P_7. Let $z' = \pi(s)$ for some $s < j$. Since *BicolorMax* has used color $a_{m-1} \neq a_1$ (due to our assumption that $m \geq 3$) on vertex z', the component $C(m-2, z')$ contains an $(m-2)$-mixed pair. This means that z' has a neighbor $w \neq v$ in $C_s(m-2, z') \subset C_j(m-1, v)$. By Lemma 2, w is not adjacent to any vertex in P_{vz}. This implies that G contains an induced P_7, which is a contradiction. Hence $d(v, z, C_j(m-1, v)) = 2$. We now show that $d(v, z', C_j(m-1, v)) \leq 3$.

Suppose $d(v, z', C_j(m-1, v)) \geq 5$. Then $P_{zz'}$ would contain an induced P_7. Hence $d(v, z', C_j(m-1, v)) = 1$ or $d(v, z', C_j(m-1, v)) = 3$. Suppose $d(v, z', C_j(m-1, v)) = 3$. We will show that there exists a vertex on $P_{z'v}$ with color a_{m-1} that is adjacent to v.

Let $P_{z'v} = z'yz^*v$. Note that the distance between z^* and z' is even. Then, due to Lemma 5, vertex z^* has not been colored with b_{m-1}. We will show that z^* has not been colored with any color from $A(m-2) \cup B(m-2)$ either. First note that any neighbor of z' in $C_j(m-1, v)$ is adjacent to z^*. Otherwise, we could extend the path $P_{zz'}$ on six vertices with one extra vertex, and $C_j(m-1, v)$ would contain an induced P_7.

Suppose $z^* = \pi(r)$ for some $r < j$. Recall that $z' = \pi(s)$. We first consider the case $s > r$, i.e., vertex z' has appeared after z^*. Since z^* is adjacent to every neighbor of z' in $C_s(m-2, z') \subset C_j(m-1, v)$ and a_{m-2} is mixed on $C_s(m-2, z')$, Lemma 2 prevents that z^* is in $C_s(m-2, z')$. Otherwise, $C_s(m-2, z')$ would contain a path (using z^*) between two vertices u_1 and u_2 of an $(m-2)$-mixed pair (u_1, u_2) in $C_s(m-2, z')$ not going through z'. We already noted that z^* has not received color b_{m-1}. Then, since z^* is in $C_j(m-1, v)$, vertex z^* must have been colored with a_{m-1}.

Now assume $s < r$, i.e., vertex z^* has appeared after z'. Every neighbor of z' in $C_r(m-2, z^*) \subset C_j(m-1, v)$ is adjacent to z^*. Hence a_{m-2} is not only mixed on $C_s(m-2, z')$ but also on $C_r(m-2, z^*)$. Since b_{m-1} was not allowed while z^* is in $C_j(m-1, v)$, *BicolorMax* must have colored z^* with a_{m-1}. □

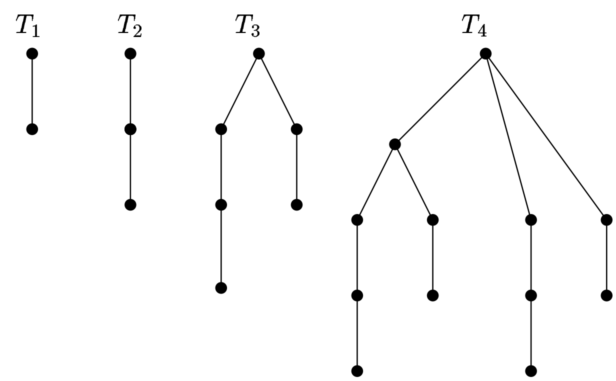

Fig. 2. The trees T_1, T_2, T_3, T_4.

We inductively define a class of trees (see Figure 2 and 3). Each tree T_k of the class has a root vertex $r(T_k)$, and:

- T_1 is a tree consisting of an edge, one of whose end vertices is the root vertex $r(T_1)$.
- T_2 is a path on three vertices, one of whose end vertices is the root vertex $r(T_2)$.
- T_k, $k \geq 3$ consists of a root vertex $r(T_k)$ that is adjacent to the root vertices of mutually disjoint copies of $T_1, T_2, \ldots T_{k-1}$ (one copy of each of these trees). These copies are then called the *child trees* of T_k.

Below we denote a copy of a tree T_k with root vertex v by $T_k(v)$. The child trees of $T_k(v)$ are denoted by $T_1^v, T_2^v, \ldots, T_{k-1}^v$.

Lemma 8. *Let G be a P_7-free bipartite graph. If BicolorMax uses color a_k or b_k on vertex $v = \pi(j)$ with $k \geq 2$, then $C_j(k-1, v)$ contains the tree $T_{k-1}(v)$ as a (not necessarily induced) subgraph in such a way that:*

(i) If there exists an edge in G between any two vertices x, y in $T_{k-1}(v)$ with $d(v, x, T_{k-1}(v)) \leq d(v, y, T_{k-1}(v))$, then x lies on the path from y to v in $T_{k-1}(v)$.

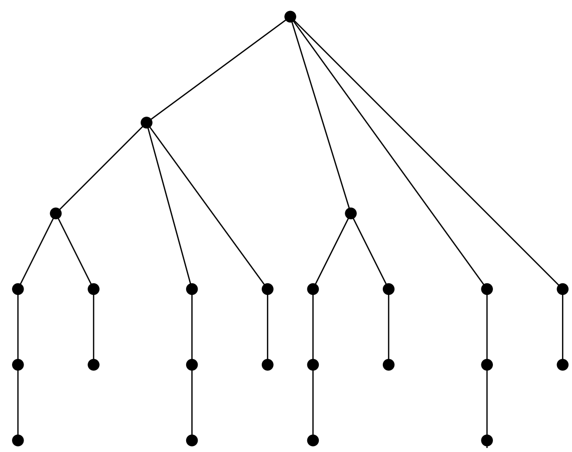

Fig. 3. The tree T_5.

(ii) The root of child tree T_i^v is colored with a_{i+1} or b_{i+1} for all $1 \le i \le k-2$.

Proof. By induction on k. Let $k = 2$, i.e., *BicolorMax* uses color a_2 or b_2 on vertex v. Then $C_j(1, v)$ contains a 1-mixed pair. This implies that v has a neighbor in $C_j(1, v)$, and the conditions of the lemma are trivially satisfied.

Let $k = 3$, i.e., *BicolorMax* uses color a_3 or b_3 on vertex v. By Lemma 7, vertex v has a neighbor z^* in $C_j(2, v)$ with color a_2. Let $z^* = \pi(q)$ for some $q < j$. Then $C_q(1, z^*)$ contains a 1-mixed pair. This implies that z^* has a neighbor not equal to v in $C_j(2, v)$. We conclude that the conditions of the lemma are satisfied.

Let $k \ge 4$. Since *BicolorMax* uses color a_k on vertex v, there exists a $(k-1)$-mixed pair (x, y) in $C_j(k-1, v)$. By Lemma 7 we may without loss of generality assume that $d(v, x, C_j(k-1, v)) = 2$ and $d(v, y, C_j(k-1, v)) = 1$. Assume $x = \pi(h)$ for some $h < j$ and $y = \pi(i)$ for some $i < j$. By the induction hypothesis, $C_h(k-2, x)$ contains the tree $T_{k-2}(x)$, and $C_i(k-2, y)$ contains the tree $T_{k-2}(y)$. Since (x, y) is a $(k-1)$-mixed pair in $C_j(k-1, v)$, every path from x to y in $C_j(k-1, v)$ must go through v due to Lemma 2. This implies that every path in $C_j(k-1, v)$ from a vertex in $C_h(k-2, x) \subset C_j(k-1, v)$ to a vertex in $C_i(k-2, y) \subset C_j(k-1, v)$ must go through v. Then we have also found that every path in $C_j(k-1, v)$ from a vertex in $T_{k-2}(x) \subset C_h(k-2, x)$ to a vertex in $T_{k-2}(y) \subset C_i(k-2, y)$ must go through v. We distinguish two cases: Either $C_h(k-2, x)$ contains a common neighbor of v and x, or $C_h(k-2, x)$ does not contain any common neighbors of v and x.

Case 1. Component $C_h(k-2, x)$ contains a common neighbor w of x and v. Again we need to distinguish two cases: Either w is in $T_{k-2}(x)$, or w is not in $T_{k-2}(x)$.

Case 1a. Vertex w is in $T_{k-2}(x)$. Then w is in a child tree T_p^x of $T_{k-2}(x)$. If $k = 4$, then $T_{k-2}(x) = T_2(x)$, and w must be the root of T_1^x. By the induction hypothesis w is colored with a_2 or b_2. Recall that every path in $C_j(3, v)$ from a vertex in $T_2(x)$ to a vertex in $T_2(y)$ goes through v, and that y is colored with $a_{k-1} = a_3$. This implies that v is the root of a copy of T_3 satisfying (i) and (ii).

Suppose $k \geq 5$. Then $T_{k-2}(x)$ has a child tree T_q^x with root u for some $q \neq p$. Note that for all $k \geq 2$ any child tree of a tree T_k consists of at least two vertices. Let u' be a neighbor of u in T_q^x. Let y' is a neighbor of y in $T_{k-2}(y)$. Note that u', u, x, w, v, y, y' are seven different vertices of G. This implies that $P_{u'y'} = u'uxwvyy'$ is a path on seven vertices in $C_j(k-1, v)$. Recall that every path from a vertex of $T_{k-2}(x)$ to a vertex of $T_{k-2}(y)$ goes through v. Then there are no edges between $\{u', u, x, w\}$ and $\{y, y'\}$. By the induction hypothesis T_p^x and T_q^x are remote. This implies that w is neither adjacent to u nor to u'. Then there must be an edge between u and v, otherwise $P_{u'y'}$ is an induced P_7 in G. Hence, we have found that v is adjacent to the root of all child trees of $T_{k-2}(x)$ that are not equal to T_p^x. However, also the root of T_p^x must be joined to v by an edge. This can be shown by using exactly the same arguments (in which vertex u takes over the role of vertex w).

From the above we conclude that v is adjacent to the roots of all child trees of $T_{k-2}(x)$. These trees together with tree $T_{k-2}(y)$ form the child trees of $T_{k-1}(v)$. Due to the fact that any path from a vertex in $T_{k-2}(x)$ to a vertex in $T_{k-2}(y)$ goes through v and our induction hypothesis, the child trees of $T_{k-1}(v)$ satisfy (i). Recall that the root vertex y of $T_{k-2}^v = T_{k-2}(y)$ is colored with a_{k-1}. The root vertices of the other child trees of $T_{k-1}(v)$ are colored with the desired colors due to the induction hypothesis. Hence, also condition (ii) of the lemma is satisfied.

Case 1b. Vertex w is not in $T_{k-2}(x)$. Since, $k \geq 4$, there exists a vertex s with color a_{k-2} in $C_h(k-2, x)$ with $d(x, s, C_h(k-2, x)) = 2$ due to Lemma 2. Since $d(x, w, C_h(k-2, x)) = 1$, vertex s is not equal to vertex w. Again, let y' be a neighbor of y in $T_{k-2}(y)$. We first show that we may without loss of generality assume that w is adjacent to s.

Suppose w is not adjacent to s. Since $d(x, s, C_h(k-2, x)) = 2$, vertex s and x have a common neighbor t in $C_h(k-2, x)$. Note that s, t, x, w, v, y, y' are seven different vertices of G. This implies that $P_{sy'} = stxwvyy'$ is a path on seven vertices in $C_j(k-1, v)$. Recall that every path from a vertex in $C_h(k-2, x)$ to a vertex in $T_{k-2}(y)$ goes through v. Then there are no edges between $\{s, t, x, w\}$ and $\{y, y'\}$. This together with $\{w, s\} \notin E(G)$ implies that v must be adjacent to t, otherwise $P_{sy'}$ is an induced P_7 in G. Then, we can pick vertex t instead of vertex w. So from now on we assume that w is a common neighbor of s and x in $C_h(k-2, x)$.

Let r be the root of child tree T_{k-3}^x of $T_{k-2}(x)$. Let r' be a neighbor of r in T_{k-3}^x. Note that r', r, x, v, w, y, y' are seven different vertices of G. This implies that $P_{r'y'} = r'rxwvyy'$ is a path on seven vertices in $C_j(k-1, v)$. By the induction hypothesis r is colored with a_{k-2} or b_{k-2}.

Suppose r is colored with a_{k-2}. Since $d(r, s, C_h(k-2, x))$ is odd, (r, s) is a $(k-2)$-mixed pair in $C_h(k-2, x)$. Lemma 7 implies that every path from s in $C_h(k-2, x)$, and hence every path from w in $C_h(k-2, x)$, to a vertex in $T_{k-3}^x \subset C_h(k-2, x)$ goes through x. Then there are no edges between $\{r, r'\}$

and w. Furthermore, recall that every path in $C_j(k-1,v)$ from a vertex in $C_h(k-2,x)$ to a vertex in $T_{k-2}(y) \subset C_i(k-2,y)$ goes through v. Then there are no edges between $\{r', r, x, w\}$ and $\{y, y'\}$ either. Since $P_{r'y'}$ may not be an induced P_7, these observations imply that there must be an edge between v and r. Hence, using vertex r instead of w brings us back to Case 1a.

Suppose r is colored with b_{k-2}. By Lemma 7, component $C_h(k-2,x)$ contains a vertex $\hat{s}$ that has received color a_{k-2} and that is adjacent to x. Let $\hat{s} = \pi(\ell)$ for some $\ell < j$. By our induction hypothesis, vertex $\hat{s}$ is the root of a tree $T_{k-3}(\hat{s})$ in $C_\ell(k-3,\hat{s})$.

Suppose every path in $C_h(k-2,x)$ from $\hat{s}$ to a vertex in any child tree T^x_p for $1 \leq p \leq k-4$ goes through x. Then in $T_{k-2}(x)$ we can replace the child tree $T_{k-3}(r)$ by the child tree $T_{k-3}(\hat{s})$. Then we can repeat the argument above in order to find that v is adjacent to $\hat{s}$, and we return to Case 1a.

Suppose $C_h(k-2,x)$ contains a path from z to child tree T^x_m for some $1 \leq m \leq k-4$ that does not go through x. Let z be the root of T^x_m and let z' be a neighbor of z in T^x_m. Then by the same argument we used for the case in which r was assigned color a_{k-2} we find that v is adjacent to z, and we return to Case 1a.

Case 2 Component $C_h(k-2,x)$ does not contain a common neighbor of x and v. Since x has distance two from v in $C_j(k-1,v)$, there exists a common neighbor v' of v and x in $C_j(k-1,v)$. We first prove that v' has received color b_{k-1}.

Since $k \geq 4$, Lemma 6 and Lemma 7 imply that component $C_h(k-2,x)$ contains a vertex s with color a_{k-2} at distance $d(x,s,C_h(k-2,x)) = 2$ from x, and $C_h(k-2,x)$ contains a vertex t with color a_{k-2} that is a neighbor of x. Since $d(x,s,C_h(k-2,x)) = 2$, vertex s and x have a common neighbor s' in $C_h(k-2,x)$. Since $k \geq 4$, vertex t has a neighbor t' in $C_h(k-2,x)$ that is not equal to x. Let y' be a neighbor of y in $T_{k-2}(y)$. Note that $s, s', t, t', x, v', v, y, y'$ are nine different vertices in G. This implies that both $P_{sy'} = ss'xv'vyy'$ and $P_{ty'} = t'txv'vyy'$ are paths on seven vertices in $C_j(k-1,v)$. Any path in $C_j(k-1,v)$ from a vertex in $T_{k-2}(y)$ to a vertex in $C_h(k-2,x)$ or to v' goes through v. Otherwise there exists a path in $C_j(k-1,v)$ from y to x that does not use v, which is not possible due to Lemma 2. Hence, there are no edges between $\{s, s', x, v'\}$ and $\{y, y'\}$, and there are no edges between $\{t, t', x, v'\}$ and $\{y, y'\}$ either. Since we assumed that $C_h(k-2,x)$ does not contain any common neighbor of v and x, there are no edges between v and $\{s', t\}$. Then v' must be adjacent to both s and t', otherwise G contains an induced P_7. Since v' is in $C_j(k-1,v)$ and adjacent to vertex x with color a_{k-1}, the color of v' is in $A(k-2) \cup B(k-1)$. If v' has not received color b_{k-1} but some color from $A(k-2) \cup B(k-2)$, then v' must have appeared after x, due to our assumption that v' is not in $C_h(k-2,x)$. However, in that case, v' has also appeared after s and t. Then (s,t) is a $(k-2)$-mixed pair of $C_{\pi^{-1}(v')}(k-2,v')$ implying that *BicolorMax* would never color v' with a color from $A(k-2) \cup B(k-2)$. Hence, v' has received color b_{k-1}.

Since $y = \pi(i)$ is assigned color a_{k-1}, by Lemma 7, component $C_i(k-2,y)$ contains a $(k-2)$-mixed pair (z^*, z) such that $d(y, z^*, C_i(k-2,y)) = 1$ and

$d(y, z, C_i(k-2, y)) = 2$. We first show that v is adjacent to z and every neighbor of z^* in $C_i(k-2, y)$.

Since $d(y, z, C_i(k-2, y)) = 2$, component $C_i(k-2, y)$ contains a common neighbor z' of y and z. Let x' be a neighbor of x in $C_h(k-2, x)$. Since v' with color b_{k-1} and neighbor v with color a_{k-1} are neither in $C_h(k-2, x)$ nor in $C_i(k-2, y)$, we find that $z, z'y, v, v', x, x'$ are seven different vertices. This implies that $P_{sy'} = zz'yvv'xx'$ is a path on seven vertices in $C_j(k-1, v)$. By Lemma 2, any path from y to x in $C_j(k-1, v)$ must go through v. This implies that there are no edges between vertices from $\{z, z', y\}$ and $\{v', x, x'\}$. Since we assume that v and x do not have a common neighbor in $C_h(k-2, x)$, vertices x' and v are not adjacent. Then v must be adjacent to z. By the same arguments we find that v is adjacent to every neighbor of z^* in $C_i(k-2, y)$.

Let $z^* = \pi(\ell^*)$ for some $\ell^* < j$. By our induction hypothesis, $C_{\ell^*}(k-3, z^*)$ contains the tree $T_{k-3}(z^*)$ such that conditions (i) and (ii) of the lemma are satisfied. Since v is adjacent to every neighbor of z^* in component $C_i(k-2, y) \supset C_{\ell^*}(k-3, z^*)$, vertex v is adjacent to the roots of all child trees $T_{k-4}^{z^*}, T_{k-3}^{z^*}, \ldots, T_1^{z^*}$. We also use our induction hypothesis with respect to vertex z, which has been assigned color a_{k-2}. Let $z = \pi(\ell)$ for some $\ell < j$. Then $C_\ell(k-3, z)$ contains the tree $T_{k-3}(z)$ such that conditions (i) and (ii) of the lemma are satisfied. Recall that z is adjacent to v. Finally, we consider vertex v' with color b_{k-1}. Let $v' = \pi(\ell')$ for some $\ell' < j$. Again by our induction hypothesis, $C_{\ell'}(k-2, v')$ contains the tree $T_{k-2}(v')$ such that conditions (i) and (ii) of the lemma are satisfied. Recall that v is adjacent to v'. Then we conclude that $C_j(k-1, v)$ contains the tree $T_{k-1}(v)$ as a subgraph such that condition (ii) is satisfied and that condition (i) will be satisfied as well if we can show the following statement: There are no edges between $T_{k-2}^v = T_{k-2}(v')$ and a child tree T_i^v for all $1 \le i \le k-3$, and there are no edges between $T_{k-3}^v = T_{k-3}(z^*)$ and a child tree T_i^v for all $1 \le i \le k-4$. This claim can be seen as follows. If there is an edge between T_{k-2}^v and a child tree T_i^v for $1 \le i \le k-3$, then $C_j(k-1, v)$ contains a path from x to y that does not go through v. Since (x, y) is a $(k-1)$-mixed pair in $C_j(k-1, v)$, this is not possible due to Lemma 2. If there is an edge between T_{k-3}^v and a child tree T_i^v for $1 \le i \le k-4$, then $C_i(k-2, y)$ contains a path from z to z^* that does not go through y. Since (z^*, z) is a $(k-2)$-mixed pair in $C_i(k-2, y)$, this is not possible, again due to Lemma 2. □

We also inductively define a class of P_6-free bipartite graphs F_i (see Figure 4 and 5). (Later on its purpose and its relation to the class of trees T_i will be made clear.) Each graph F_i of the class has a root vertex $r(F_i)$, and:

- F_1 is a graph consisting of a single root vertex.
- F_2 is a graph consisting of an edge, one of whose end vertices is the root vertex.
- F_k, $k \ge 3$ consists of a root vertex $r(F_k)$ that is adjacent to the root vertices of disjoint copies of $F_1, F_2, \ldots F_{k-1}$ (one copy of each of these trees). These copies are then called the *child graphs* of F_k. For all $1 \le j \le k-1$, we join vertex $r(F_k)$ also to every vertex in F_j that has distance 2 to $r(F_j)$. This implies that every vertex of F_k is at distance at most 2 from $r(F_k)$. Hence,

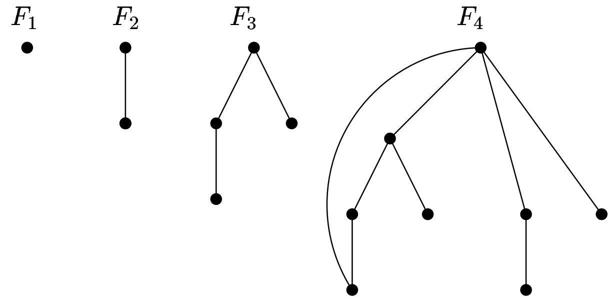

Fig. 4. The graphs F_1, F_2, F_3, F_4.

the maximum distance between two vertices in F_k is at most four and F_k is P_6-free.

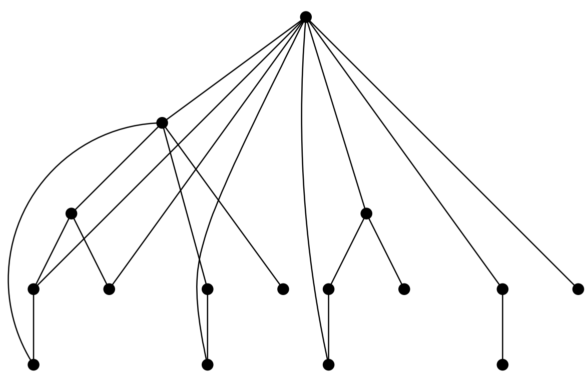

Fig. 5. The graph F_5.

A graph F_k has the following useful properties (see also Figure 6).

Lemma 9. *Graph F_k with $k \geq 4$ contains copies F_t^1 and F_t^2 of F_t for $2 \leq t \leq k-2$ such that*

(i) For all $1 \leq, i, j \leq 2$ and all $2 \leq s < t \leq k-2$, the graphs F_s^i and F_t^j are remote.
(ii) For all $2 \leq t \leq k-2$, any edge between F_t^1 and F_t^2 has $r(F_t^1)$ as one of its end vertices.
(iii) For all $2 \leq t \leq k-2$, the vertices of the graph F_t^1 in the bipartite class containing the root vertex of F_t^1 are adjacent to $r(F_k)$.
(iv) For all $2 \leq t \leq k-2$, the vertices of the graph F_t^2 in the bipartite class not containing the root vertex of F_t^2 are adjacent to $r(F_k)$.

Proof. One easily checks that child graph F_t of graph F_k contains the desired copies F^1_{t-1} and F^2_{t-1} for $3 \le t \le k-1$. □

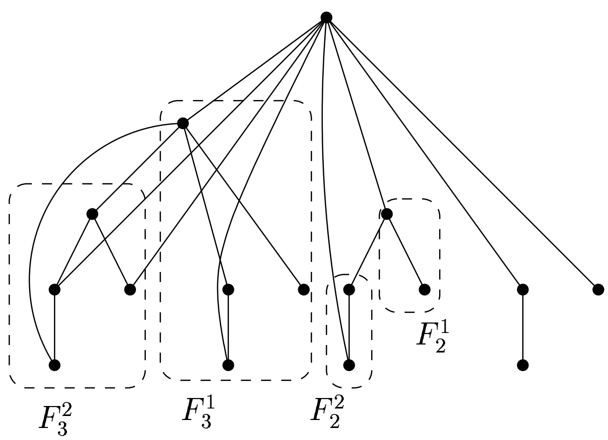

Fig. 6. The graph F_5 containing copies $F^1_3, F^2_3, F^1_2, F^1_3$.

Proposition 3. *For any $k \ge 1$, $\chi_{OL}(F_{2k}) \ge k$.*

Proof. By induction on k. The case $k = 1$ is trivial. Let $k \ge 2$. Consider F_{2k} and an on-line algorithm A for coloring F_{2k}. The first time the i^{th} color is used by A we identify it as color i. We choose an ordering on $V(F_{2k})$ such that the vertices of remote copies of $F_2, F_4 \ldots, F_{2k-2}$ are presented until color i is used on F_{2i} $(i = 1, \ldots, k-1)$, i.e., as soon as color 1 is used on F_2 we start presenting vertices of F_4, as soon as color 2 is used on F_4 we start presenting vertices of F_6 and so on, until color $k-1$ is used on F_{2k-2}. By the adjacency relations from the definition of F_{2k} and the properties of Lemma 9, the ordering of the presented vertices of $F_2, F_4, \ldots, F_{2k-2}$ can be chosen in such a way that $r(F_{2k})$ is adjacent to the (not necessarily root) vertices that received colors $1, \ldots, k-1$. Hence a new color k is forced upon A. □

Below we denote a copy of a graph F_k with root vertex v by $F_k(v)$. The child graphs of $F_k(v)$ are denoted by $F^v_1, F^v_2, \ldots, F^v_{k-1}$.

Lemma 10. *Let G be a P_7-free bipartite graph. If BicolorMax uses color a_k or b_k with $k \ge 3$ on vertex $v = \pi(j)$, then $C_j(k-1, v)$ contains the graph $F_{\lfloor \frac{k-1}{2} \rfloor}(v)$ as an induced subgraph.*

Proof. Due to Lemma 8 the component $C_j(k-1, v)$ contains the tree $T_{k-1}(v)$ as a subgraph in such a way that:

(i) If there exists an edge in G between any two vertices x, y in $T_{k-1}(v)$ with $d(v, x, T_{k-1}(v)) \le d(v, y, T_{k-1}(v))$, then x lies on the path from y to v in $T_{k-1}(v)$.

(ii) The root of child tree T_i^v is colored with a_{i+1} or b_{i+1} for all $1 \leq i \leq k-2$.

By induction on k we will show that the subgraph of G induced by $V(T_{k-1}(v))$ contains the graph $F_{\lfloor \frac{k-1}{2} \rfloor}(v)$ as an induced subgraph. The case $k = 3$ is trivial. Let $k \geq 4$.

Claim. There exists at most one child tree T_h^v of $T_{k-1}(v)$ for some $1 \leq h \leq k-2$ that contains a vertex x with $d(x, r(T_h^v), T_h^v) = 2$ and with $\{x, v\} \notin E(G)$. So, for all $1 \leq i \neq h \leq k-2$, vertex v is adjacent to every vertex in T_i^v that is at distance 2 from $r(T_i^v)$ in T_i^v.

In order to obtain a contradiction suppose there exists another child tree with the properties as described above: Let T_i^v be a child tree of $T_{k-1}(v)$ for some $1 \leq i \neq h \leq k-2$ that contains a vertex y with $d(y, r(T_i^v), T_i^v) = 2$ and with $\{y, v\} \notin E(G)$. Let y' be the common neighbor of y and $r(T_i^v)$ in T_i^v. Let x' be the common neighbor of x and $r(T_h^v)$ in T_h^v. Since $T_{k-1}(v)$ satisfies (i) and (ii), graph G contains an induced $P_7 = xx'r(T_h^v)vr(T_i^v)y'y$, which is a contradiction. Hence the above claim is proved.

By the induction hypothesis and condition (ii), for $2 \leq i \leq k-2$, the subgraph induced by $V(T_i^v)$ contains the graph $F_{\lfloor \frac{i}{2} \rfloor}(r(T_i^v))$ as an induced subgraph. This together with the fact that $T_{k-1}(v)$ satisfies condition (i) as well and the above claim immediately implies that $C_j(k-1, v)$ contains $F_{\lfloor \frac{k-1}{2} \rfloor}(v)$ as an induced subgraph. □

Theorem 4. *If G is a P_7-free bipartite graph, then $\chi_{Bm}(G) \leq 8\chi_{OL}(G) + 8$.*

Proof. Let k be the highest index such that *BicolorMax* uses color a_{4k+1} on a vertex in G. Note that it is possible that *BicolorMax* uses colors $a_{4k+2}, a_{4k+3}, a_{4k+4}$ or $b_{4k+2}, b_{4k+3}, b_{4k+4}$ to color G. Hence, every color used on a vertex of G is from $A(4k+4) \cup B(4k+4)$. Since *BicolorMax* only uses b_i if a_i has been used before, $\chi_{Bm}(G) \leq 2(4k+4) = 8k+8$. For $k = 0$ the statement obviously holds. Suppose $k \geq 1$. Due to Lemma 10, graph G contains a copy of F_{2k} as an induced subgraph. Proposition 3 implies that $\chi_{OL}(G) \geq \chi_{OL}(F_{2k}) \geq k$. □