Coloring graphs without short cycles and long induced paths *

Petr A. Golovach[1], Daniël Paulusma[2], and Jian Song[2]

[1] Department of Informatics, Bergen University,
PB 7803, 5020 Bergen, Norway
petr.golovach@ii.uib.no

[2] School of Engineering and Computing Sciences, Durham University,
Science Laboratories, South Road, Durham DH1 3LE, United Kingdom
{daniel.paulusma,jian.song}@durham.ac.uk

Abstract. For an integer $k \geq 1$, a graph G is k-colorable if there exists a mapping $c : V_G \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever u and v are two adjacent vertices. For a fixed integer $k \geq 1$, the k-COLORING problem is that of testing whether a given graph is k-colorable. The girth of a graph G is the length of a shortest cycle in G. For any fixed $g \geq 4$ we determine a lower bound $\ell(g)$, such that every graph with girth at least g and with no induced path on $\ell(g)$ vertices is 3-colorable. We also show that for all fixed integers $k, \ell \geq 1$, the k-COLORING problem can be solved in polynomial time for graphs with no induced cycle on four vertices and no induced path on ℓ vertices. As a consequence, for all fixed integers $k, \ell \geq 1$ and $g \geq 5$, the k-COLORING problem can be solved in polynomial time for graphs with girth at least g and with no induced path on ℓ vertices. This result is best possible, as we prove the existence of an integer ℓ^*, such that already 4-COLORING is NP-complete for graphs with girth 4 and with no induced path on ℓ^* vertices.

1 Introduction

Graph coloring involves the labeling of the vertices of some given graph by k integers called colors such that no two adjacent vertices receive the same color. Due to the fact that the corresponding decision problem k-COLORING is NP-complete for any fixed $k \geq 3$, there has been considerable interest in studying its complexity when restricted to certain graph classes, see e.g. the surveys of Randerath and Schiermeyer [32] and Tuza [36]. We focus on graph classes defined by forbidden induced subgraphs. Before we summarize the known results and explain our new results, we first state the necessary terminology and notations.

1.1 Terminology

We only consider finite undirected graphs with no loops and no multiple edges. We refer to the textbook by Bondy and Murty [3] for any undefined graph terminology. Let

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$G = (V, E)$ be a graph. We write $G[U]$ to denote the subgraph of G *induced by* the vertices in U, that is, the subgraph of G with vertex set U and an edge between two vertices $u, v \in U$ whenever $uv \in E$. The *length* of a path or cycle is the number of its edges. The graphs C_n and P_n denote the cycle and path on n vertices, respectively. The graph $K_{r,s}$ denotes the complete bipartite graph with partition classes of size r and s, respectively. The disjoint union of two graphs G and H is denoted $G + H$, and the disjoint union of r copies of G is denoted rG. A *linear forest* is the disjoint union of a collection of paths. Let G be a graph and $\{H_1, \dots, H_p\}$ be a set of graphs. We say that G is $(H_1, \dots, H_p)$*-free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \dots, H_p\}$; if $p = 1$, we sometimes write H_1-free instead of (H_1)-free. If G is C_3-free, then we also say that G is *triangle-free*. The *girth* $g(G)$ of G is the length of a shortest cycle in G. Note that G has girth at least p for some integer $p \geq 4$ if and only if G is $(C_3, \dots, C_{p-1})$-free.

A k*-coloring* of a graph $G = (V, E)$ is a mapping $\phi : V \to \{1, \dots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E$. We say that $\phi(u)$ is the *color* of u. If G has a k-coloring, then G is said to be k*-colorable*. The *chromatic number* $\chi(G)$ of G is the smallest k such that G is k-colorable. If $\chi(G) = k$, then we also say that G is k*-chromatic*. The COLORING problem is that of testing whether a given graph admits a k-coloring for some given integer k. If k is *fixed*, that is, not part of the input, then we denote this problem as k-COLORING. The problem k-PRECOLORING EXTENSION is that of deciding whether a given mapping $\phi_W : W \to \{1, \dots, k\}$ defined on a (possibly empty) subset $W \subseteq V$ of a graph $G = (V, E)$ can be extended to a k-coloring of G.

1.2 Related Work

Král' et al. [22] completely determined the computational complexity of COLORING for graph classes characterized by one forbidden induced subgraph H. They showed that COLORING can be solved in polynomial time for H-free graphs if H is an induced subgraph of P_4 or of $P_1 + P_3$, and that this problem is NP-complete if H is any other graph.

The computational complexity of the COLORING problem for (H_1, H_2)-free graphs where H_1 and H_2 are two distinct graphs is still open, although several partial results are known. In particular (C_3, H)-free graphs, or equivalently, H-free graphs with girth at least 4, are well studied. Král' et al. [22] showed that for any graph H that contains at least one cycle, 3-COLORING, and hence COLORING, is NP-complete for (C_3, H)-free graphs. Their work was extended by Schindl [34]. Maffray and Preissmann [28] showed that 3-COLORING and consequently, COLORING is NP-complete for $(C_3, K_{1,5})$-free graphs. Broersma et al. [8] showed that COLORING is polynomial-time solvable for $(C_3, 2P_3)$-free graphs, thereby completing a study of Dabrowski et al. [11], who considered the COLORING problem restricted to (C_3, H)-free graphs for graphs H with $|V_H| \leq 6$.

The computational complexity classification of k-COLORING for H-free graphs where k is a fixed integer and H is a fixed graph is also still open, but the following is known. Král' et al. [22] showed that 3-COLORING is NP-complete for graphs of girth at least g for any fixed $g \geq 3$. Kamiński and Lozin [21] used a similar reduction to show that k-COLORING is NP-complete for graphs of girth at least g for any fixed $k \geq 3$ and $g \geq 3$. Hence, for all $k \geq 3$, k-COLORING is NP-complete for H-free graphs if H contains a cycle. Holyer [18] showed that 3-COLORING is NP-complete for line graphs, whereas Leven and Galil [25] showed that k-COLORING is also NP-complete on line graphs for $k \geq 4$. Because every line graph is claw-free, that is, has no induced $K_{1,3}$, this means that for all $k \geq 3$, k-COLORING is NP-complete for H-free graphs if H is a forest that contains a vertex with degree at least 3. Hence, only the case in which H is a linear forest remains. Huang [19] proved that 4-COLORING is NP-complete for P_7-free graphs and that 5-COLORING is NP-complete for P_6-free graphs. In contrast to these hardness results, Couturier et al. [10] generalized a result for P_5-free graphs of Hoàng et al. [17] by proving that for any fixed integers k and r, the k-COLORING problem can be solved in polynomial time for $(P_5 + rP_1)$-free graphs, whereas Randerath and Schiermeyer [31] showed that 3-COLORING can be solved in polynomial time for P_6-free graphs. Broersma et al. [7] extended the latter result by showing that 3-COLORING is polynomial-time solvable for H-free graphs if H is a linear forest with $|V_H| \leq 6$ or $H = rP_3$ for any integer r. Moreover, it is known that 4-COLORING is polynomial-time solvable for $(P_2 + P_3)$-free graphs [14].

The k-COLORING problem has also been studied for (H_1, H_2)-free graphs where H_1 and H_2 are two distinct graphs. We refer to Randerath and Schiermeyer [32] for a detailed survey on so-called good Vizing-pairs (A, B) that satisfy the condition that every (A, B)-free graph is 3-colorable, in particular when $A = C_3$. Brandt [4] showed that every (C_3, sP_2)-free graph is $(2s - 2)$-colorable for any $s \geq 3$.

1.3 Our Results

We consider the relation between the girth of a graph and the length of a forbidden induced path for the k-COLORING problem. As a start, note that graphs with girth $g = \infty$ are forests, and consequently, these graphs are 2-colorable. What if g is finite? In Section 2 we determine, for any fixed girth $g \geq 4$, a lower bound $\ell(g)$ such that every $P_{\ell(g)}$-free graph with girth at least g is 3-colorable. This extends the result of Sumner [35] who showed that every P_5-free graph of girth at least 4 is 3-colorable complementing a result of Randerath and Schiermeyer [32], who showed that for all $\ell \geq 4$, every P_ℓ-free graph of girth at least 4 is $(\ell - 2)$-colorable. Our results lead to Table 1. Note that for the cases $g \in \{4, 5, 7\}$ the lower bounds are slightly worse than the lower bound for the general case $g \geq 8$; the difference between them is 1. The proofs of the results in Table 1 are constructive, that is, they yield polynomial-time algorithms for solving 3-COLORING on these graph classes.

In Section 3.1 we show that for all integers $k, \ell, r, s \geq 1$, the k-COLORING problem is polynomial-time solvable on $(K_{r,s}, P_\ell)$-free graphs by using a recent result of Atminas, Lozin and Razgon [1]. By taking $r = s = 2$, we obtain that for all integers $k, \ell \geq 1$, the k-COLORING problem is polynomial-time solvable on (C_4, P_ℓ)-free graphs. Consequently, for all integers $g \geq 5$ and $k, \ell \geq 1$, the k-COLORING problem is polynomial-time solvable on P_ℓ-free graphs of girth at least g. As every graph has girth at least 3, and 3-COLORING is NP-complete in general, the case $g = 4$ remains. We solve this case in Section 3.2 by showing that even 4-COLORING is NP-complete for (C_3, P_{164})-free graphs, that is, for P_{164}-free graphs of girth at least $g = 4$. This is a

g	forbidden induced path
4	P_5-free [35]
5	P_7-free
6	P_{10}-free
7	P_{12}-free
≥ 8	P_ℓ-free for $\ell = 2g + \lceil \frac{g-2}{4} \rceil - 3$

Table 1. 3-colorable P_ℓ-free graphs of given girth g.

new result as all the gadgets used in the proofs of the aforementioned NP-completeness results on k-COLORING for P_ℓ-free graphs are not triangle-free, that is, have girth equal to 3. We expect that $\ell = 164$ can be improved, but emphasize that our aim was to show the existence of such an integer ℓ rather than to minimize it. Moreover, this is also the first known NP-completeness result on COLORING for (C_3, P_ℓ)-free graphs. As such it also fits into the aforementioned complexity study of COLORING for (H_1, H_2)-free graphs initiated by Král' et al. [22] and Schindl [34]. Our hardness result complements the result of Kratochvíl [23] who showed that 5-PRECOLORING EXTENSION is NP-complete for P_{13}-free bipartite graphs.

In Section 3.2 we also show that for all $r \geq 5$, there exists a constant $\ell(r)$ such that 4-COLORING is NP-complete for $(C_5, \ldots, C_r, P_{\ell(r)})$-free graphs. In particular, we show that already 4-COLORING for (C_5, P_{23})-free graphs is NP-complete. Hence we have two complexity jumps when the length of the forbidden induced cycle increases, as can be seen in Table 2.

r	complexity
3	NP-complete for $k = 4$ and $\ell \geq 164$
4	polynomial-time solvable for $k \geq 1$ and $\ell \geq 1$
≥ 5	NP-complete for $k = 4$ and $\ell \geq \ell(r)$

Table 2. The computational complexity of k-COLORING for (C_r, P_ℓ)-free graphs. Here, $\ell(r)$ is a fixed constant that only depends on r.

Very recently, Hell and Huang [16] extended the results in Table 2 by showing that k-COLORING is NP-complete on (C_r, P_ℓ)-free graphs in the following cases:

(i) $k = 4$, $5 \leq r \leq 6$ and $\ell \geq 7$
(ii) $k = 4$, $r = 7$ and $\ell \geq 9$
(iii) $k = 4$, $r \geq 8$ and $\ell \geq 7$
(iv) $k \geq 5$, $r = 5$ and $\ell \geq 7$
(v) $k \geq 5$, $r \geq 6$ and $\ell \geq 6$.

1.4 Future Work

A classical result of Erdös [12] tells us that for every pair of integers k and g, there exists a k-chromatic graph G_k^g of girth g. This result immediately implies that there also exists a P_ℓ-free k-chromatic graph of girth g, as we can take $\ell = |V_{G_k^g}| + 1$. The proof of Erdös [12] is probabilistic, but we can obtain a P_ℓ-free k-chromatic graph of girth g by using the constructive proof of Lovász [26] or of Nešetřil and Rödl [29]. However, in general it is not trivial to construct, for given k and g, a k-chromatic P_ℓ-free graph of girth g for ℓ as small as possible, or, for given k and ℓ, a k-chromatic P_ℓ-free graph of girth g for g as large as possible. For example, the Grötzsch graph [15] is 4-chromatic, P_6-free and of girth 4. Hence, the bound of Sumner [35] is tight. Brinkmann and Meringer [6] constructed a 4-chromatic P_{10}-free graph with girth 5. Hence, the bound in Table 1 for P_{10}-free graphs is tight with respect to the girth. We are not aware of examples of 4-chromatic graphs of girth at least 6 without long induced paths and expect that some of our bounds in Table 1 can be improved.

The aforementioned results of Hell and Huang [16], which improve the results in Table 2, combined with our result for $k = 4$, $r = 3$ and $\ell = 164$ and the known polynomial-time results for k-COLORING on P_ℓ-free graphs (namely the cases $k \geq 1$, $\ell \leq 5$ [17] and $k = 3$, $\ell = 6$ [31]) leave a number of cases open in the computational complexity classification of k-COLORING for (C_r, P_ℓ)-free graphs. An intriguing question is to determine the computational complexity of 3-COLORING restricted to (C_3, P_ℓ)-free graphs for all integers ℓ. It is known that the class of (C_3, P_6)-free graphs have bounded clique-width [5]. Hence, even COLORING is polynomial-time solvable on (C_3, P_6)-free graphs (see e.g. [11]). The clique-width of the class of bipartite $2P_3$-free graphs [27], and hence of the class of (C_3, P_7)-free graphs, is unbounded. Very recently, Chudnovsky, Maceli and Zhong announced a polynomial-time algorithm for solving 3-COLORING on P_7-free graphs. Their result implies that (C_3, P_8)-free graphs form the first graph class to look at when trying to show polynomial-time solvability for (C_3, P_ℓ)-free graphs for some integer ℓ. On the other hand, there is no integer ℓ known for which 3-COLORING is NP-complete on (C_3, P_ℓ)-free graphs. In fact, an affirmative answer to this question would solve a well-known open problem, namely whether such an integer ℓ exists for 3-COLORING restricted to P_ℓ-free graphs [7, 17, 24, 31, 37].

2 The Lower Bounds for 3-Colorability

We start with some additional terminology. We say that a path between two vertices u and v in G is a (u, v)-*path*. The *distance* between u and v is the length of a shortest (u, v)-path in G and is denoted $\mathrm{dist}(u, v)$. For a vertex v and subset $U \subseteq V$ we define $\mathrm{dist}(v, U) = \min\{\mathrm{dist}(v, u) \mid u \in U\}$; note that $\mathrm{dist}(v, U) = 0$ if and only if $v \in U$. We denote the *neighborhood* of a vertex u by $N(u) = \{v \mid uv \in E\}$ and its *degree* by $\deg(u) = |N(u)|$. For a subset $U \subseteq V$ and an integer s, we define $N^s(U) = \{v \in V \mid \mathrm{dist}(v, U) = s\}$ and $N^s[U] = \{v \in V \mid \mathrm{dist}(v, U) \leq s\}$.

In order to prove the bounds in Table 1 we make two assumptions that are valid throughout this section. First of all, we may assume that the graphs we consider are connected. Second, we may assume that they have minimum degree at least 3 due to the following observation.

Observation. *Let G be a graph and u be a vertex of degree at most 2. Then G is 3-colorable if and only if $G - u$ is 3-colorable.*

Hence, we may remove vertices of degree at most 2 consecutively until the resulting graph has minimum degree at least 3. Note that removing a vertex of degree 2 from a graph may increase its girth. However, this is not a problem, because the bounds in Table 1 improve for increasing girth.

We show the four new bounds in Table 1 in Theorems 1–4, respectively.

Theorem 1. *Every P_7-free graph of girth at least* 5 *is* 3*-colorable.*

Proof. Let $G = (V, E)$ be a connected P_7-free graph with minimum degree at least 3 and girth at least 5. If $g(G) = 6$, $g(G) = 7$ or $g(G) \geq 8$, then we refer to Theorems 2, 3 or 4, respectively. Suppose that $g(G) = 5$. Consider $uv \in E$ and let $U = \{u, v\}$. We first prove four useful properties of the sets $N^s(U)$:

1. $N^1(U)$ is an independent set;
2. $N^2(U)$ induces a bipartite graph;
3. $N^3(U)$ is an independent set;
4. $N^s(U) = \emptyset$ for $s \geq 4$.

Property 1 immediately follows from the (C_3, C_4)-freeness of G. We prove property 2 as follows. In order to obtain a contradiction, suppose that $G[N^2(U)]$ contains an odd induced cycle $C_r = x_1x_2 \cdots x_rx_1$. Because $g(G) = 5$ and G is P_7-free, $r = 5$ or $r = 7$. Let w be a vertex in $N^1(U)$ adjacent to x_1 and assume without loss of generality that w is adjacent to u. Then $wv \notin E$. If $r = 5$, then w is not adjacent to $x_2, \ldots, x_5$, because $g(G) = 5$. Note that $x_1x_2x_3x_4$ is an induced path in G. Then $vuwx_1 \cdots x_4$ is an induced P_7. Hence $r = 7$. Because $g(G) = 5$, we find that w is adjacent to none of the vertices x_2, x_3, x_6, x_7. Note that $x_1x_2x_3x_4$ and $x_1x_7x_6x_5$ are induced paths in G. If w is not adjacent to x_4, then $vuwx_1 \cdots x_4$ is an induced P_7. Hence, w must be adjacent to x_4. If w is not adjacent to x_5, then $vuwx_1x_7x_6x_5$ is an induced P_7. Hence, w must be adjacent to x_5. However, now we have a triangle wx_4x_5w, which is not possible because $g(G) = 5$. We conclude that property 2 must hold.

We prove property 4 before property 3. Suppose that property 4 is not true. First suppose that $N^5(U) \neq \emptyset$. Then, for a vertex $x \in N^5(U)$, there is a path $w_1 \cdots w_4x$ where $w_i \in N^i(U)$. We assume without loss of generality that w_1 is adjacent to u. However, then $vuw_1 \cdots w_4x$ is an induced P_7. Hence $N^5(U) = \emptyset$, and consequently, $N^s(U) = \emptyset$ for $s \geq 5$. This means that for property 4 not to hold, we must have $N^4(U) \neq \emptyset$.

First suppose that $G[N^4(U)]$ contains an edge xy. There is a path $w_1w_2w_3$ such that $w_1 \in N^1(U)$, $w_2 \in N^2(U)$, $w_3 \in N^3(U)$ and $w_3x \in E$. Assuming that $uw_1 \in E$, we get the induced path $vuw_1w_2w_3xy$ isomorphic to P_7. Hence, such an edge xy does not exist, implying that $N^4(U)$ is an independent set. Suppose that a vertex $x \in N^4(U)$ is adjacent to at least two vertices z_1 and z_2 in $N^3(U)$ and let $uw_1w_2z_1x$ be a (u, x)-path in G. Because $g(G) = 5$, we find that $z_1z_2 \notin E$ and $w_2z_2 \notin E$. Then $vuw_1w_2z_1xz_2$ is an induced P_7. Hence, x is adjacent to exactly one vertex in $N^3(U)$. Recall that $N^5(U) = \emptyset$. Then we find that $d(x) = 1$. However, G has no such vertices, because G has minimum degree at least 3. We conclude that property 4 must hold.

For the proof of property 3, we first suppose that $x_1x_2x_3$ is an induced path in $G[N^3(U)]$. Then there are adjacent vertices $w_1 \in N^1(U)$ and $w_2 \in N^2(U)$ such that $w_2x_1 \in E$. Because w_1 is adjacent to a vertex in U, we assume without loss of generality that w_1 is adjacent to u and get the path $vuw_1w_2x_1x_2x_3$. Because $g(G) \geq 5$, this path is an induced P_7. Hence $G[N^3(U)]$ is P_3-free.

We now suppose that xy is an edge in $G[N^3(U)]$. Because G has minimum degree at least 3 and $G[N^3(U)]$ is P_3-free, y is adjacent to at least two vertices z_1 and z_2 in $N^2(U)$. By definition, there are adjacent vertices $w_1 \in N^1(U)$ and $w_2 \in N^2(U)$ such that $w_2x \in E$, and we can assume that $uw_1 \in E$. Because $g(G) = 5$, we find that z_1, z_2, w_2 are three different vertices that are pairwise non-adjacent, and that z_1w_1 and z_2w_1 cannot be both edges in G. We assume without loss of generality that $z_1w_1 \notin E$. Then $vuw_1w_2xyz_1$ is an induced P_7, which is not possible. We conclude that property 3 must hold, and we have proven all four properties.

Using these four properties we construct a 3-coloring of G as follows. We color the vertices u and v with colors 1 and 2 respectively, and all the vertices of the independent set $N^1(U)$ with color 3. The vertices of the bipartite graph $G[N^2(U)]$ are colored with colors 1 and 2. Finally, the vertices of the independent set $N^3(U)$ are colored with color 3. This completes the proof of Theorem 1. ☐

Theorem 2. *Every P_{10}-free graph of girth at least 6 is 3-colorable.*

Proof. Let $G = (V, E)$ be a connected P_{10}-free graph with minimum degree at least 3 and girth at least 6. If $g(G) = 7$ or $g(G) \geq 8$, then we refer to Theorems 3 or 4, respectively. Suppose that $g(G) = 6$. Let $U = \{x_1, \dots, x_6\}$ be the vertex set of a C_6 in G (vertices are enumerated in cyclic order). We observe that this 6-vertex cycle is induced, because $g(G) = 6$. Let X_i denote the set of vertices of $N^1(U)$ adjacent to x_i for $i = 1, \dots, 6$. Using the (C_3, C_4, C_5)-freeness of G, we observe the following:

1. X_i is independent for $1 \leq i \leq 6$;
2. $X_i \cap X_j = \emptyset$ for $1 \leq i < j \leq 6$;
3. if $y_iy_j \in E$ for $y_i \in X_i$, $y_j \in X_j$ and $1 \leq i < j \leq 6$, then $j - i = 3$.

Let $H_1, \dots, H_k$ be the connected components of $G[V \setminus N^1[U]]$. We need the following claim.

Claim 1. Each graph H_j is either an isolated vertex or a star $K_{1,r}$ for some $r \geq 1$.

We prove Claim 1 as follows. Consider a graph H_j for some $1 \leq j \leq k$. First we show that H_j is P_4-free. In order to obtain a contradiction, suppose that H_j contains an induced path $v_1v_2v_3v_4$. Let $z_1 \cdots z_s$ be a shortest path such that $z_1 \in U$ and $z_s \in \{v_1, \dots, v_4\}$. Without loss of generality we assume that $z_1 = x_1$, $z_2 \in X_1$ and either $z_s = v_1$ or $z_s = v_2$.

Because $g(G) = 6$, we find that z_{s-1} is adjacent to exactly one vertex of the path $v_1v_2v_3v_4$. If $z_s = v_1$ then $x_5x_4 \cdots x_1z_2 \cdots z_{s-1}v_2 \cdots v_4$ is an induced path with at least 10 vertices. Hence $z_s = v_2$, and moreover, any shortest path between U and $\{v_1, \dots, v_4\}$ contains neither v_1 nor v_4. If $s \geq 4$ then $x_5x_4 \cdots x_1z_2 \cdots z_{s-1}v_2v_3v_4$ is an induced path with at least 10 vertices. Hence $s = 3$, that is., v_2 is adjacent to a vertex of X_1. If v_4 is adjacent to a vertex y in some set X_j then x_jyv_4 is another shortest path between U and $\{v_1, \dots, v_4\}$. However, we already deduced that such paths do not contain v_4. Hence, $v_4 \notin N^2[U]$. The graph G has no vertices of degree one, and therefore v_4 is adjacent to a vertex $w \in V_{H_j}$. Because $g(G) = 6$, we find that w is not adjacent to v_1, v_2, v_3 or z_2. This means that $x_5x_4 \cdots x_1z_2v_2v_3v_4w$ is an induced P_{10}. This is not possible. Hence, H_j is P_4-free. Observe that every connected P_4-free graph without an induced C_3 or C_4 is either an isolated vertex or a star. This completes the proof of Claim 1.

We are now ready to construct a 3-coloring of G. Using properties 1–3, we color vertices x_1, x_3, x_5 and all vertices of X_2, X_4, X_6 with color 1, and x_2, x_4, x_6 and all vertices of X_1, X_3, X_5 with color 2. Now we color each H_j. If H_j consists of an isolated vertex, then we color this vertex with color 3. Suppose that H_j is a star $K_{1,r}$. Let w be its central vertex and $z_1, \ldots, z_r$ be its leaves.

If $w \notin N^2[U]$, then we color $z_1, \ldots, z_r$ with color 3 and w with color 1. Now let w be adjacent to a vertex of X_i for some $1 \leq i \leq 6$. In this case we color w with color 3. It remains to prove that each leaf z_s can be colored with color 1 or 2. Suppose that it is not so for some z_s. Then z_s is adjacent to two vertices in the sets $X_1, \ldots, X_6$ that have color 1 and 2, respectively. By symmetry, we assume that z_s is adjacent to $y_1 \in X_1$. Because $g(G) = 6$, we find that z_s is not adjacent to any vertices in X_2 or X_6, and therefore, z_s must be adjacent to some vertex $y_4 \in X_4$ in order to have a neighbor with color 1. Because $g(G) = 6$, we find that w is not adjacent to any vertices of $X_1 \cup X_4$. By symmetry, we can assume that $i = 2$, i.e., that w is adjacent to a vertex $y_2 \in X_2$. Because G has minimum degree at least 3 and $x_1x_2 \cdots x_6x_1$ is an induced cycle in G, we find that $X_3 \neq \emptyset$. Let $y_3 \in X_3$. However, then $y_2wz_sy_1x_1x_6 \cdots x_3y_3$ is an induced P_{10} due to $g(G) = 6$. This means that each z_s is adjacent either only to vertices colored 1 or only to vertices colored 2 in the sets $X_1, \ldots, X_6$. In the first case we can color z_s with color 2, and in the second case we can color z_s with color 1. This completes the proof of Theorem 2. □

Theorem 3. *Every P_{12}-free graph of girth at least 7 is 3-colorable.*

Proof. Let $G = (V, E)$ be a connected graph of girth at least 7. If $g(G) \geq 8$, then we refer to Theorem 4. Suppose that $g(G) = 7$. Let $U = \{x_1, \ldots, x_7\}$ be the vertex set of a C_7 in G (vertices are enumerated in cyclic order). We observe that this 7-vertex cycle is induced, because $g(G) = 7$. Let X_i denote the set of vertices of $N^1(U)$ adjacent to x_i for $i = 1, \ldots, 7$. Using the (C_3, C_4, C_5, C_6)-freeness of G, we observe the following:

1. $X_i \cap X_j = \emptyset$ for $1 \leq i < j \leq 7$;
2. $X_1 \cup \ldots \cup X_7$ is independent.

Let H be the subgraph of G induced by the set $V \setminus (\{x_1, \ldots, x_7\} \cup X_1 \cup \cdots \cup X_6)$; note that $X_7 \subseteq V_H$. We claim that H is bipartite. In order to obtain a contradiction, suppose that H contains an odd induced cycle $C_r = v_1v_2 \cdots v_rv_1$. Because $g(G) = 7$ and G is P_{12}-free, we deduce that $r \in \{7, 9, 11\}$. Let $y_1 \cdots y_s$ be a shortest path, such that $y_1 \in U$ and $y_s \in \{v_1, \ldots, v_r\}$. We may assume without loss of generality that $y_s = v_1$. By definition, $s \geq 2$.

Suppose that $r < 11$. Then y_{s-1} is not adjacent to any vertex of $\{v_2, \ldots, v_r\}$. If $y_1 = x_7$, then G has an induced path $x_2 \cdots x_6y_1 \cdots y_sv_2 \cdots v_{r-1}$ with at least 12 vertices, which is not possible. If $y_1 = x_1$, then we find that G has an induced path $x_6x_5 \cdots x_2y_1 \cdots y_sv_2 \cdots v_{r-1}$ with at least 13 vertices, which is not possible either. If $y_1 \in \{x_2, \ldots, x_6\}$, then we apply the same arguments as for the case $y_1 = x_1$.

Suppose that $r = 11$. If y_{s-1} is adjacent to exactly one vertex of $\{v_1, \dots, v_r\}$, then we use the same arguments as for the case $r < 11$. Let y_{s-1} be adjacent to at least two vertices of $\{v_1, \dots, v_r\}$. Because $g(G) = 7$, we deduce that y_{s-1} is adjacent to exactly two vertices, namely v_1, v_6 or v_1, v_7. By symmetry, we may assume that y_{s-1} is adjacent to v_1, v_7. If $y_1 = x_7$, then G has an induced path $x_2 \cdots x_6 y_1 \cdots y_s v_2 \cdots v_6$ on at least 12 vertices, which is not possible. If $y_1 = x_1$, then G has an induced path $x_6 x_5 \cdots x_2 y_1 \cdots y_s v_2 \cdots v_6$ on at least 13 vertices, which is not possible either. If $y_1 \in \{x_2, \dots, x_6\}$, then we apply the same arguments as for the case $y_1 = x_1$. We conclude that H is bipartite.

We now color G as follows. We color x_1, x_3, x_5 with color 1, x_2, x_4, x_6 with color 2, the vertices of $X_1 \cup \cdots \cup X_6$ and x_7 with color 3, and finally all the vertices of the (bipartite) graph H with colors 1 and 2. This completes the proof of Theorem 3. $\square$

Theorem 4. *Every P_k-free graph with $k = 2g + \lceil \frac{g-2}{4} \rceil - 3$ and girth $g \geq 8$ is 3-colorable.*

Proof. Let $G = (V, E)$ be a connected graph of girth $g \geq 8$. Let $U = \{x_1, \dots, x_g\}$ be the vertex set of a C_g in G (vertices are enumerated in cyclic order). We observe that this g-vertex cycle is induced. Let $s = \lceil \frac{g-2}{4} \rceil - 1$. We will prove the following two properties of the sets $N^t(U)$:

1. $N^t(U)$ is independent for $1 \leq t \leq s$;
2. each $x \in N^t(U)$ is adjacent to exactly one vertex in $N^{t-1}(U)$ for $1 \leq t \leq s$.

We first prove property 1. In order to obtain a contradiction, suppose that $N^t(U)$ contains two adjacent vertices y and z for some $1 \leq t \leq s$. By the definition of $N^t(U)$, we find that U contains two vertices x_i, x_j with $\mathrm{dist}(y, x_i) = t$ and $\mathrm{dist}(z, x_j) = t$. Because the distance between x_i and x_j in the cycle $G[U]$ is at most $\lfloor \frac{g}{2} \rfloor$, we find that $G[N^t[U]]$ contains a cycle of length at most $2t + 1 + \lfloor \frac{g}{2} \rfloor \leq 2\lceil \frac{g-2}{4} \rceil - 1 + \lfloor \frac{g}{2} \rfloor < g$. This is not possible. Hence, property 1 is valid.

The proof of property 2 is similar. Suppose that $x \in N^t(U)$ is adjacent to at least two different vertices in $N^{t-1}(U)$. This implies that there are two paths between x and U of length at most t. Therefore G has a cycle of length less than $g - 1$, which is not possible. Hence, property 2 is valid as well.

We now distinguish two cases; first we consider the case $g = 9$ and then the case $g \neq 9$.

Case 1. $g = 9$.
Then $k = 17$, so G is P_{17}-free, and $s = 1$. Let $X \subseteq N^1(U)$ be the set of vertices adjacent to x_9 and let $Y \subseteq N^2(U)$ be the set of vertices adjacent to the vertices of X. Because G has no cycles of length less than 9, we can deduce two extra properties in addition to properties 1 and 2:

3. Y is independent;
4. the vertices of Y are not adjacent to the vertices of $N^1(U) \setminus X$.

Let H be the subgraph of G induced by the set $V \setminus (N^1[U] \cup Y)$. We claim that H is bipartite. In order to obtain a contradiction, suppose that H contains an odd induced cycle $C_r = v_1v_2 \cdots v_rv_1$. Because $g = 9$ and G is P_{17}-free, we find that $9 \leq r \leq 17$. Let $y_1 \cdots y_t$ be a shortest path such that $y_1 \in U$ and $y_t \in \{v_1, \ldots, v_r\}$. Without loss of generality assume that $y_t = v_1$. We observe that $t \geq 3$. Because $g = 9$ and $r \leq 17$, we find that y_{t-1} is adjacent to at most one vertex of $\{v_2, \ldots, v_r\}$.

Suppose that y_{t-1} is adjacent to zero vertices of $\{v_2, \ldots, v_r\}$. If $y_1 = x_1$, then G has an induced path $x_8x_7 \cdots x_2y_1 \cdots y_tv_2 \cdots v_{r-1}$ on at least 17 vertices. This is not possible, because G is P_{17}-free. If $y_1 \in \{x_2, \ldots, x_9\}$, then we use the same arguments.

Suppose that y_{t-1} is adjacent to exactly one vertex of $\{v_2, \ldots, v_r\}$. Because $g = 9$ and r is odd, we find that $r = 15$ or $r = 17$. This implies that the 7-vertex sets $\{v_2, \ldots, v_8\}$ and $\{v_{r-6}, v_{r-5}, \ldots, v_r\}$ are disjoint. Because $g = 9$, we find that y_{t-1} is not adjacent to any vertex of $\{v_2, \ldots, v_7, v_{r-5}, v_{r-4}, \ldots, v_r\}$. Because y_{t-1} is adjacent to only one vertex of $\{v_2, \ldots, v_r\}$, we find in addition that y_{t-1} cannot be adjacent to both v_8 and v_{r-6}. By symmetry, we may assume that y_{t-1} is not adjacent to v_8. If $y_1 = x_1$, then $x_8x_7 \cdots x_2y_1 \cdots y_tv_2 \cdots v_8$ is an induced path in G with at least 17 vertices. This is not possible, because G is P_{17}-free. If $y_1 \in \{x_2, \ldots, x_9\}$, then we use the same arguments. We conclude that H is bipartite.

Using properties 1-4 and the fact that H is bipartite, we can color G as follows. We color vertices x_1, x_3, x_5, x_7 with color 1, vertices x_2, x_4, x_6, x_8 with color 2, vertex x_9 and the vertices of the (independent) set $N^1(U) \setminus X$ with color 3, all vertices in X and Y with color 1 and 3, respectively, and finally, all vertices of the (bipartite) graph H with colors 1 and 2.

Case 2. $g \neq 9$.
Let H be the subgraph of G induced by the set $V \setminus N^s[U]$. We claim that H is bipartite. In order to obtain a contradiction, suppose that H contains an odd induced cycle $C_r = v_1v_2 \cdots v_rv_1$. Because G is P_k-free for $k = 2g + \lceil \frac{g-2}{4} \rceil - 3$, we find that $g \leq r \leq 2g + \lceil \frac{g-2}{4} \rceil - 2$.

Let $y_1 \cdots y_t$ be a shortest path such that $y_1 \in U$ and $y_t \in \{v_1, \ldots, v_r\}$. We may assume without loss of generality that $y_1 = x_1$ and $y_t = v_1$. Recall that $s = \lceil \frac{g-2}{4} \rceil - 1$ and observe that $t \geq s + 2$. If y_{t-1} is adjacent to at least two vertices of $\{v_2, \ldots, v_r\}$, then $r \geq 3g - 6$. However, we also have $3g - 6 > 2g + \lceil \frac{g-2}{4} \rceil - 2 \geq r$ as $g \geq 8$. Hence, this is not possible, and consequently, y_{t-1} is adjacent to at most one vertex of $\{v_2, \ldots, v_r\}$.

First suppose that y_{t-1} is adjacent to zero vertices of $\{v_2, \ldots, v_r\}$. Then G has an induced path $x_{g-1}x_{g-2} \cdots x_2y_1 \cdots y_tv_2 \cdots v_{r-1}$ on $g + r + t - 4$ vertices. However, $g + r + t - 4 \geq 2g + s - 2 = 2g + \lceil \frac{g-2}{4} \rceil - 3 = k$, which is not possible as G is P_k-free.

Now suppose that y_{t-1} is adjacent to exactly one vertex of $\{v_2, \ldots, v_r\}$. By definition of g, y_{t-1} is not adjacent to any vertex of $\{v_2, \ldots, v_{g-2}\} \cup \{v_r, v_{r-1}, \ldots, v_{r-g+4}\}$. Note that $v_{g-1} \neq v_{r-g+3}$, as otherwise $r = 2g - 4$ would be even. Because y_{t-1} is adjacent to only one vertex of $\{v_2, \ldots, v_r\}$, we find that y_{t-1} cannot be adjacent to both v_{g-1} and v_{r-g+3}. We assume without loss of generality that y_{t-1} is not adjacent to v_{g-1}. Then we find that G has an induced path $x_{g-1}x_{g-2} \cdots x_2y_1 \cdots y_tv_2 \cdots v_{g-1}$

on $2g + t - 4$ vertices, which is not possible as $2g + t - 4 \geq 2g + s - 2 = k$ and G is P_k-free. We conclude that H is bipartite.

Using properties 1 and 2 and the fact that H is bipartite, we color G as follows. First suppose that g is even. For $1 \leq i \leq g/2$, we color x_{2i-1} with color 1 and x_{2i} with color 2. Then we color the vertices of the (independent) sets $N^1(U), \ldots, N^s(U)$ with colors 1 and 3, where we alternate the colors starting with color 3. Finally we color the vertices of the (bipartite) graph H with colors 1 and 2 if $N^s(U)$ was colored 3, and with colors 2 and 3 otherwise.

Now suppose that g is odd. Then $g \geq 11$. Let $X \subseteq N^1(U)$ be the set of vertices adjacent to x_g. By property 2, the vertices of X are not adjacent to any vertex of $\{x_1, \ldots, x_{g-1}\}$. Because $g \geq 11$, we have $s \geq 2$. For $1 \leq i \leq \lfloor g/2 \rfloor$, we color x_{2i-1} with color 1 and x_{2i} with color 2. Then we color x_g and the vertices of the (independent) set $N^1(U) \setminus X$ with color 3, and the vertices of X with color 1. Then we color the vertices of the (independent) sets $N^2(U), \ldots, N^s(U)$ with colors 2 and 3, where we alternate the colors starting with color 2. Finally, we color the vertices of the (bipartite) graph H with colors 1 and 3 if $N^s(U)$ was colored 2, and with colors 1 and 2 otherwise. This completes the proof of Theorem 4. □

3 The Computational Complexity Results

In this section we prove the results of Table 2. We show the polynomial-time results (row 2 of Table 2) in Section 3.1 and the NP-completeness results (rows 1 and 3 of Table 2) in Section 3.2.

3.1 The Polynomial-Time Results

We first prove the following theorem.

Theorem 5. *For all integers $k, \ell, r, s \geq 1$, the k-*COLORING *problem can be solved in linear time on $(K_{r,s}, P_\ell)$-free graphs.*

Proof. For positive integers p and q, the Ramsey number $a(p, q)$ is the smallest number of vertices n such that all graphs on n vertices contain an independent set of size p or a clique of size q. Ramsey's Theorem [30] states that such a number exists for all positive integers p and q. Atminas, Lozin and Razgon [1] showed that for any two integers ℓ and t, there exists an integer $b(\ell, t)$ such that any graph of treewidth at least $b(\ell, t)$ contains the path P_ℓ as an induced subgraph or the complete bipartite graph $K_{t,t}$ as a (not necessarily induced) subgraph. We will combine these two results in the following way. Let $k, \ell, r, s \geq 1$, and let G be a $(K_{r,s}, P_\ell)$-free graph. Let $a^* = a(\max\{k + 1, r, s\}, \max\{k + 1, r, s\})$. Using Bodlaender's algorithm [2] we can test in linear time whether the treewidth of G is at most $b(\ell, a^*) - 1$.

First suppose that the treewidth of G is at most $b(\ell, a^*) - 1$, which is a constant. In that case we can test in linear time whether G is k-colorable, because the k-COLORING problem can be expressed in monadic second-order logic, and hence the well-known result of Courcelle [9] for such problems may be applied.

Now suppose that the treewidth of G is at least $b(\ell, a^*)$. We claim that in this case G is not k-colorable. This can be seen as follows. Due to the aforementioned result of Atminas, Lozin and Razgon [1] and our assumption that G is P_ℓ-free, we find that G contains the complete bipartite graph K_{a^*,a^*} as a subgraph. Let S and T be the partition classes of this complete bipartite graph. By Ramsey's Theorem, we find that both S and T either contain an independent set of size $\max\{k+1, r, s\}$ or a clique of size $\max\{k+1, r, s\}$. If both of them contain an independent set of size $\max\{k+1, r, s\}$, then $G[S \cup T]$, and consequently, G contains an induced $K_{r,s}$, which is not possible. Hence, at least one of them, say S, contains a clique of size $\max\{k+1, r, s\}$. Then $G[S]$ is not k-colorable. Consequently, G is not k-colorable. □

By choosing $r = s = 2$, we find that for all $k \geq 1$, the k-COLORING problem can be solved in linear time on (C_4, P_ℓ)-free graphs due to Theorem 5. This proves the second row in Table 2.

Remark. We can extend Theorem 5 in the following way. A *list assignment* of a graph $G = (V, E)$ is a function $\mathcal{L}$ that assigns a list $L(u)$ of so-called admissible colors to each $u \in V$. If $L(u) \subseteq \{1, \ldots, k\}$ for each $u \in V$, then we also say that $\mathcal{L}$ a *k-list assignment*. We say that a coloring $c:\ V \to\ \{1, 2, \ldots\}$ *respects* $\mathcal{L}$ if $c(u) \in L(u)$ for all $u \in V$. For a fixed integer k, the LIST k-COLORING problem has as input a graph G with a k-list assignment $\mathcal{L}$ and asks whether G has a coloring that respects $\mathcal{L}$. Jansen and Scheffler [20] showed that LIST k-COLORING can be solved in time $O(nk^{t+1})$ on an n-vertex graph G with treewidth at most t that has a k-list assignment $\mathcal{L}$. In the proof of Theorem 5 we can replace the result of Courcelle [9] by this result in order to find that for all integers $k, \ell, r, s \geq 1$, the LIST k-COLORING problem can be solved in linear time on $(K_{r,s}, P_\ell)$-free graphs.

3.2 The NP-Completeness Results

We first show that 4-COLORING is NP-complete for (C_3, P_{164})-free graphs. As the problem is clearly in NP, we are left to prove NP-hardness. The gadgets used in the papers of Kamiński and Lozin [21] and Maffray and Preissmann [28] to prove NP-hardness of k-COLORING for graphs of girth g for any $g \geq 3$ may contain arbitrarily long induced paths. Similarly, the existing NP-hardness reductions for k-COLORING for P_ℓ-free graphs are based on the presence of triangles in the gadgets. Hence, the main task is to design a triangle-free gadget that can replace a number of edges of a graph G with no long induced paths in order to make G triangle-free while still bounding the maximum length of any induced path in G. We first present this gadget and its properties. We then show how to incorporate it in our final gadget that proves our NP-hardness reduction, which is from the NP-complete problem NOT-ALL-EQUAL-3-SATISFIABILITY (cf. [13]).

The edge-replacing gadget We define four independent sets A, B, C and D with $|A| = |B| = 10$ and $|C| = |D| = 5$. We add an edge between every vertex in A and every vertex in B. We also add an edge between every vertex in C and every vertex

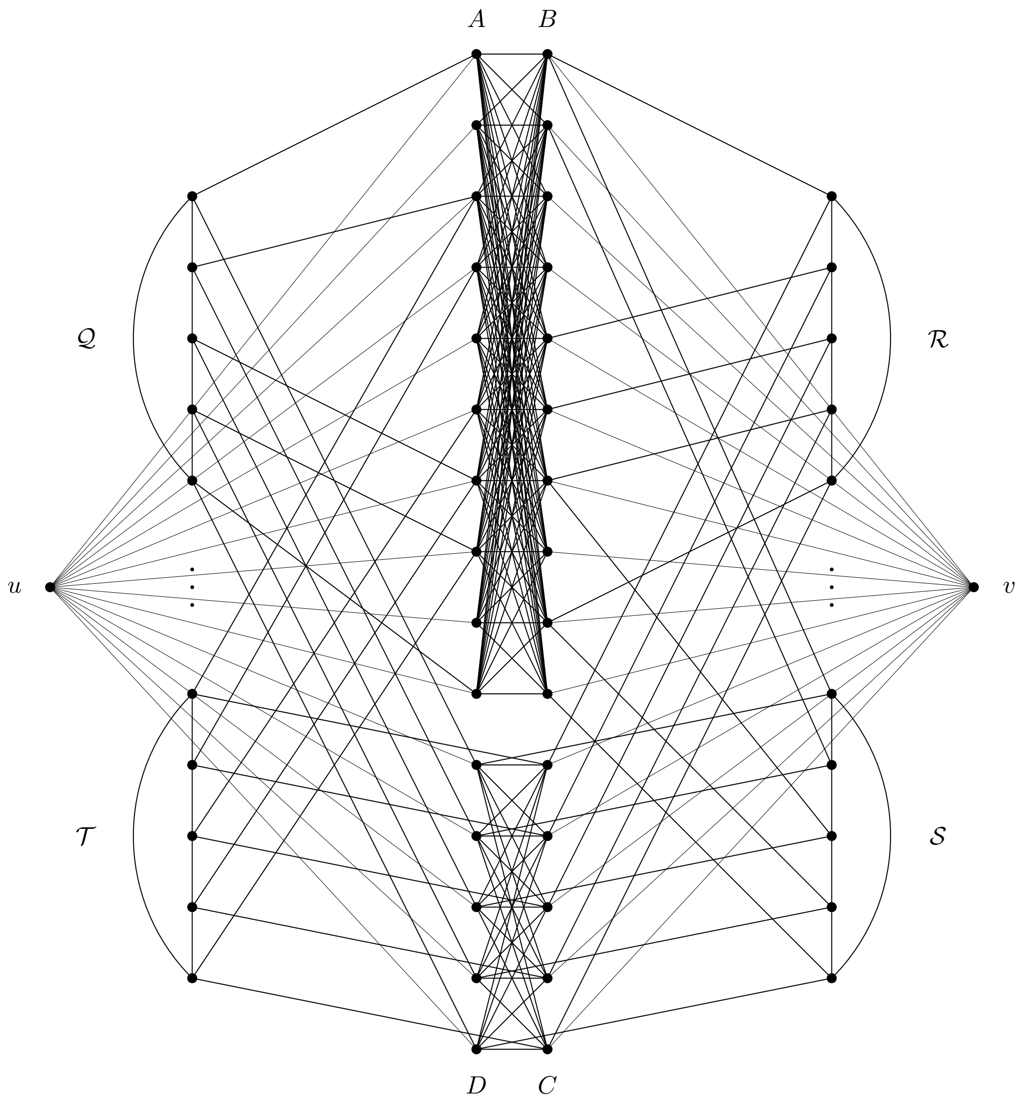

Fig. 1. The graph F; only one Q-cycle, R-cycle, S-cycle and one T-cycle are displayed.

in D. This leads to two vertex-disjoint complete bipartite graphs with partition classes A, B and C, D, respectively.

For every subset $A_i \subseteq A$ of five vertices, we create two cycles Q_i and T_i, each on five new vertices. We say that Q_i is a Q-cycle and that T_i is a T-cycle. We add five edges between the vertices of Q_i and A_i. We chose these edges arbitrarily subject to the condition that they form a matching. Similarly, we add five arbitrary matching edges between the vertices of T_i and A_i. We also add five arbitrary matching edges between the vertices of Q_i and D, and do the same for T_i and C. We let $\mathcal{Q}$ and $\mathcal{T}$ denote the set of all $\binom{10}{5}$ Q-cycles and $\binom{10}{5}$ T-cycles, respectively. Similarly, we define two sets $\mathcal{R}$ and $\mathcal{S}$ of $\binom{10}{5}$ R-cycles and $\binom{10}{5}$ S-cycles, respectively. Here, each R-cycle and each S-cycle correspond to exactly one subset $B_i \subseteq B$ of five vertices. For each such B_i we add five arbitrary matchings between its vertices and the vertices in its R-cycle and S-cycle, respectively. We also add five arbitrary matching edges between the vertices of each R-cycle and C, and between the vertices of each S-cycle and D. Finally, we add a new vertex u adjacent to every vertex of $A \cup D$, and a new vertex v adjacent to every vertex of $B \cup C$. The resulting graph, called F, is C_3-free; see Figure 1.

Lemma 1 states some useful properties of F that we will use later on.

Lemma 1. *The graph F is 4-colorable. Moreover, u and v are colored with different colors in every 4-coloring of F.*

Proof. We first show that F is 4-colorable. We choose a vertex $c \in C$ and a vertex $d \in D$, which we give colors 1 and 2, respectively. We give each vertex of $(A \cup D) \setminus \{d\}$ color 3 and each vertex of $(B \cup C) \setminus \{c\}$ color 4. We give u color 1 and v color 3. Consider a Q-cycle. We give its vertex adjacent to d color 1 and color its other four vertices with colors 2 and 4. Consider a T-cycle. We give its vertex adjacent to c color 4 and color its other four vertices with colors 1 and 2. By symmetry, we can also give the vertices of every R-cycle and S-cycle an appropriate color such that in the end we have obtained a 4-coloring of F.

We now prove that u and v are not colored alike in every 4-coloring of F. Let ϕ be a 4-coloring of F. First suppose that $|\phi(C)| \geq 2$ and $|\phi(D)| \geq 2$. Because C and D are partition classes of a complete bipartite graph, we then may without loss of generality assume that $\phi(C) = \{2, 3\}$ and $\phi(D) = \{1, 4\}$. This means that u can only get a color from $\{2, 3\}$ and v can only get a color from $\{1, 4\}$. Hence, u and v are not colored alike.

In the remaining case, we assume without loss of generality that $|\phi(D)| = 1$. If $|\phi(A)| = 4$, then we cannot color a vertex in B. Hence, $|\phi(A)| \leq 3$. If $|\phi(A)| = 3$, then every vertex of $B \cup \{u\}$ receives the same color. Because v is adjacent to the vertices of B, this means that v must receive a different color.

Suppose that $|\phi(A)| \leq 2$. Then A contains a subset A_i of five vertices that are colored alike, say with color 3. We observe that u does not get color 3. Consider the Q-cycle corresponding to A_i. Its five vertices can neither be colored with color 3 nor with the color in $\phi(D)$. Because this cycle needs at least three colors, this means that $\phi(D) = \phi(A_i) = \{3\}$. Because every vertex of A is adjacent to every vertex of B, color 3 is not used on B, so $|\phi(B)| \leq 3$.

First suppose that $|\phi(B)| \leq 2$. Then B contains a subset B_j of five vertices that are colored alike, say with color 4. We consider the S-cycle corresponding to B_j. Because every vertex of this cycle is not only adjacent to a vertex of B_j with color 4 but also adjacent to a vertex of D with color 3, we find that only colors 1 and 2 are available to color its five vertices. This is not possible. Hence, $|\phi(B)| = 3$, so $\phi(B) = \{1, 2, 4\}$. This means that v must receive color 3, whereas we already deduced that u does not get color 3. This completes the proof of Lemma 1. $\square$

Using the edge-replacing gadget We now present our reduction for showing that 4-COLORING is NP-complete for the class of (C_3, P_{164})-free graphs. This reduction is from the NOT-ALL-EQUAL 3-SATISFIABILITY problem with positive literals only. This problem is NP-complete [33] and is defined as follows. We are given a set $X = \{x_1, x_2, \ldots, x_n\}$ of logical variables, and a set $C = \{C_1, C_2, \ldots, C_m\}$ of three-literal clauses over X in which all literals are positive. The question is whether there exists a truth assignment for X such that each clause contains at least one true literal and at least one false literal.

We consider an arbitrary instance I of NOT-ALL-EQUAL 3-SATISFIABILITY with positive literals only that has variables $\{x_1, x_2, \ldots, x_n\}$ and clauses $\{C_1, C_2, \ldots, C_m\}$. We assume that no variable appears twice in a clause.[3] From I we first construct the P_7-free graph G from our previous paper [7]. We then explain how to incorporate our edge-replacing gadget F. This will yield a graph G'. In Lemma 2 we will show that G' is C_3-free, and in Lemma 3 it is stated that G' is P_{164}-free. In Lemma 4 we will show that G' is 4-colorable if and only if I has a satisfying truth assignment in which each clause contains at least one true literal and at least one false literal.

Here is the construction that defines the graph G.

- For each clause C_j we introduce a gadget with vertex set

$$\{a_{j,1}, a_{j,2}, a_{j,3}, b_{j,1}, b_{j,2}, c_{j,1}, c_{j,2}, c_{j,3}, d_{j,1}, d_{j,2}\}$$

  and edge set

$$\{a_{j,1}c_{j,1}, a_{j,2}c_{j,2}, a_{j,3}c_{j,3}, b_{j,1}c_{j,1}, c_{j,1}d_{j,1}, d_{j,1}c_{j,2}, c_{j,2}d_{j,2}, d_{j,2}c_{j,3}, c_{j,3}b_{j,2}, b_{j,2}b_{j,1}\},$$

  and a disjoint gadget called the *copy* that has vertex set

$$\{a'_{j,1}, a'_{j,2}, a'_{j,3}, b'_{j,1}, b'_{j,2}, c'_{j,1}, c'_{j,2}, c'_{j,3}, d'_{j,1}, d'_{j,2}\}$$

  and edge set

$$\{a'_{j,1}c'_{j,1}, a'_{j,2}c'_{j,2}, a'_{j,3}c'_{j,3}, b'_{j,1}c'_{j,1}, c'_{j,1}d'_{j,1}, d'_{j,1}c'_{j,2}, c'_{j,2}d'_{j,2}, d'_{j,2}c'_{j,3}, c'_{j,3}b'_{j,2}, b'_{j,2}b'_{j,1}\}.$$

  We say that all these vertices (so, including the vertices in the copy) are of a-type, b-type, c-type and d-type, respectively. We call the gadget and its copy *clause gadgets*.

[3] If there is a clause $C = \{x, x, x\}$, then $(X, \mathcal{C})$ is a no-instance. If there exists a clause $C = \{x, x, y\}$ with $x \neq y$, then we get an equivalent instance of size at most four times as large by replacing C by clauses $\{x, y, a\}$, $\{x, y, b\}$, $\{x, y, c\}$, $\{a, b, c\}$, where a, b, c are new variables.

- Every variable x_i is represented by a vertex in G, and we say that these vertices are of x-type.
- For every clause C_j we fix an arbitrary order of its variables $x_{i_1}, x_{i_2}, x_{i_3}$ and add edges $c_{j,h}x_{i_h}$ and $c'_{j,h}x_{i_h}$ for $h = 1, 2, 3$.
- We add an edge between every x-type vertex and every b-type vertex. We also add an edge between every x-type vertex and every d-type vertex.
- We add an edge between every a-type vertex and every b-type vertex. We also add an edge between every a-type vertex and every d-type vertex.

In Figure 2 we illustrate an example in which C_j is a clause with ordered variables $x_{i_1}, x_{i_2}, x_{i_3}$. The thick edges indicate the connection between the variables vertices and the c-type vertices of the two copies of the clause gadget. The dashed thick edges indicate the connections between the a-type and c-type vertices of the two copies of the clause gadget. We omitted the indices from the labels of the clause gadget vertices to increase the visibility.

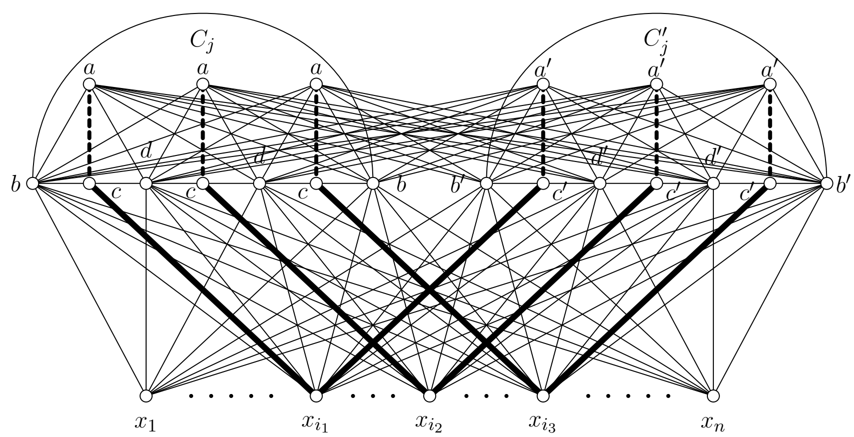

Fig. 2. The graph G for the clause $C_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$.

Before we show how to obtain the graph G', we introduce the following terminology. Let H be some graph. An *F-identification* of an edge $st \in E_H$ is the following operation. We remove the edge st from H but keep the vertices s and t. We take a copy of F and remove u and v from it. We then add an edge between s and $N_F(u)$ and an edge between t and $N_F(v)$. The resulting graph is a copy of F that is a subgraph of G'. The vertices in this copy not equal to s, t are called *inner* vertices of G'. Note that by symmetry we could reverse the role of u and v in this operation.

In order to obtain G' from G we first apply consecutive F-identifications on all edges between a-type and c-type vertices, on all edges between c-type and x-type vertices and on all edges between two b-type vertices. We take a complete graph on four new vertices $r_1, \ldots, r_4$ called r-type vertices, and apply consecutive F-identifications on each edge between them. This leads to a graph K. We connect K to the modified graph G by adding an edge between every $a_{i,j}$ and every vertex in $\{r_2, r_3, r_4\}$ and an edge between every $a'_{i,j}$ and every vertex in $\{r_1, r_3, r_4\}$. This completes the construction of G'.

We need the following lemma.

Lemma 2. *The graph G is C_3-free.*

Proof. In order to prove the lemma we must check if G' has an edge, the end vertices of which share a common neighbor. Recall that F is C_3-free and that u and v do not form an edge. Hence, we only have to consider edges in G' that are also in G. Such edges connect the following vertices: a-type with b-type, a-type with d-type, b-type with c-type, c-type with d-type, b-type with x-type, and d-type with x-type. By construction, the end vertices of all these edges have no common neighbor. We conclude that G' is C_3-free. □

We also need the following lemma.

Lemma 3. *The graph G' is P_{164}-free.*

The proof of Lemma 3 is straightforward to check but involves a lengthy case analysis, and therefore has been omitted from our paper. Moreover, it may be the case that the bound of 164 as stated in Lemma 3 can be slightly improved by a more careful analysis of the graph G'. However, we recall that our aim was to prove the existence of a constant ℓ, such that 4-COLORING is NP-complete for (C_3, P_ℓ)-free graphs rather than minimizing ℓ. It can be easily shown that G' is P_ℓ-free for *some* constant ℓ as we show below.

First we note that F is P_{186}-free, because the subgraph of F induced by $V_F \setminus (A \cup B \cup C \cup D)$ consists of connected components, each of which has at most five vertices, and consequently, every induced path in F contains (at most five) vertices of at most $|A| + |B| + |C| + |D| + 1 = 31$ such components besides at most 30 vertices of $A \cup B \cup C \cup D$. Because any induced path of G' that contains no vertices of G is contained in a copy of F, it has at most 185 vertices.

Let P be an arbitrary induced path of G' that contains at least one vertex of G. Let γ_a, γ_b, γ_c, γ_d, γ_r and γ_x denote the number of a-type, b-type, c-type, d-type, r-type and x-type vertices of P, respectively. Observe that $\gamma_r \leq 4$, because there are exactly four r-type vertices. Let $\mathcal{F}_P$ consist of those copies of F in G' that contain at least one vertex of P as an inner vertex. Every vertex of G' that is not of d-type belongs to at least one copy of F, whereas only vertices of c-type and x-type belong to more than one copy of F. Because P is a path and every copy of F contains exactly two vertices of G, no vertex of P can be in more than four different copies in $\mathcal{F}_P$; note that a vertex s is only contained in four copies if P starts at an inner vertex in a copy of F that contains s, passes through s via two other copies and ends in an inner vertex of yet another copy of F that contains s. Because F is P_{186}-free, we then find that

$$|V_P| \leq 4(\gamma_a + \gamma_b + \gamma_c + \gamma_x + 4) \cdot 185 + \gamma_d.$$

We make a case distinction based on the observation that the subgraph of G' induced by all a-type, b-type, d-type and x-type vertices has a spanning subgraph that is complete bipartite with partition classes Z_1 and Z_2, where Z_1 is formed by all a-type and x-type vertices, and Z_2 by all b-type and d-type vertices.

First suppose that P contains no vertex of Z_2. Let P' be a subpath of P that contains no r-type vertex. In order to connect any two vertices of Z_1 of P' via a subpath of P', a c-type vertex is needed. In G, a c-type vertex is adjacent to one x-type vertex and to one a-type vertex. Hence, by construction of G', we find that P' contains at most three vertices of Z_1 and at most two c-type vertices (if P' has exactly three vertices of Z_1, then two of them are of a-type, whereas the other one is of x-type and lies between the two a-type vertices on P', with two c-type vertices for the connectivity). In order to connect subpaths of P that do not contain r-type vertices, we must use r-type vertices. Because there are four r-type vertices, P contains at most $3 \cdot 5 = 15$ vertices of Z_1 and at most $2 \cdot 5 = 10$ vertices of c-type. Hence, $\gamma_a + \gamma_x \leq 15$, $\gamma_c \leq 10$. As $\gamma_b = \gamma_d = 0$, we find that P has at most $4 \cdot (15 + 10 + 4) \cdot 185 = 21460$ vertices.

Now suppose that P contains at least one vertex of Z_2 but no vertices of Z_1. Then P does not contain any vertices of K either. Consequently, by construction of G', P contains at most two b-type vertices, at most three c-type vertices and at most two d-type vertices (all of which belong to the same clause gadget in G). Hence $\gamma_a = \gamma_x = 0$, $\gamma_b \leq 2$, $\gamma_c \leq 3$ and $\gamma_d \leq 2$ implying that P has at most $4 \cdot (2+3+4) \cdot 185 + 2 = 6662$ vertices.

Finally suppose that P contains at least one vertex of Z_1 and at least one vertex of Z_2. Because Z_1 and Z_2 are the partition classes of a complete bipartite subgraph of G', we find that $|Z_1 \cup Z_2| \leq 3$, and moreover, that the vertices of $Z_1 \cup Z_2$ are consecutive in P. Any subpath of P containing no vertices of Z_2 has at most 21460 vertices. Any subpath of P containing no vertices of Z_1 has at most 6662 vertices. Hence, P has at most $21460 + 3 + 21460 = 42923$ vertices.

From the above we conclude that $\ell = 42923$ suffices (although this constant can be reduced considerably to the constant $\ell = 164$ from Lemma 3 by a more careful analysis that avoids all the double counting we have done here).

The following lemma is the last lemma we need in order to state our main result.

Lemma 4. *The graph G' is 4-colorable if and only if I has a satisfying truth assignment in which each clause contains at least one true literal and at least one false literal.*

Proof. We need the following claim from [7].

Claim 1. The graph G has a 4-coloring in which every $a_{j,h}$ has color 1 and every $a'_{j,h}$ has color 2 if and only if I has a truth assignment in which each clause contains at least one true and at least one false literal.

Suppose that G' is 4-colorable. By Lemma 1, the vertices of the edges on which we applied F-identifications do not have the same color. This means that G is 4-colorable.

It also means that the vertices $r_1, \ldots, r_4$ are not colored alike. We may assume without loss of generality that r_i gets color i for $i = 1, \ldots, 4$. Then, by construction, every $a_{j,h}$ has color 1 and every $a'_{j,h}$ has color 2 in G', and consequently in G. By Claim 1, I has a truth assignment in which each clause contains at least one true and at least one false literal.

Suppose that I has a truth assignment in which each clause contains at least one true and at least one false literal. By Claim 1, G has a 4-coloring in which every $a_{j,h}$ has color 1 and every $a'_{j,h}$ has color 2. By Lemma 1 we can extend the 4-coloring of G to a 4-coloring of G'. □

The main result of this section follows directly from Lemmas 2–4 after recalling that 4-COLORING is in NP and that NOT-ALL-EQUAL-3-SATISFIABILITY with positive literals only is NP-complete and observing that the construction of G' can be carried out in polynomial time.

Theorem 6. *The* 4-COLORING *problem is* NP*-complete even for* (C_3, P_{164})*-free graphs.*

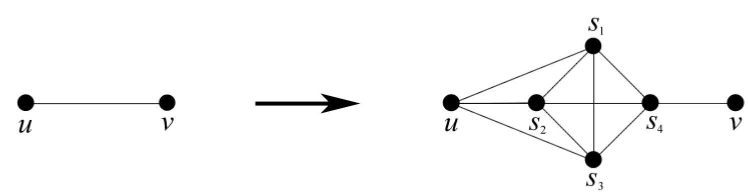

Fig. 3. Placing a generalized diamond on an edge uv of a graph.

Theorem 6 implies the first row of Table 2. We are left to prove the third row of this table. In order to do this we need the following terminology. We say that we place a *generalized diamond* on an edge uv of some graph if we remove uv and add four new mutually adjacent vertices $s_1, \ldots, s_4$ and edges us_i for $i = 1, 2, 3$ and the edge s_4v; see Figure 3. We say that the vertices $s_1, \ldots, s_4$ are *internal* vertices of the generalized diamond. Then the new graph is 4-colorable if and only if the original graph is 4-colorable. We are now ready to prove the following result which implies the third row of Table 2. In fact we show a slightly stronger result.

Theorem 7. *For all $r \geq 5$, there exists a constant $\ell(r)$ such that* 4-COLORING *is* NP*-complete for $(C_5, \ldots, C_r, P_{\ell(r)})$-free graphs.*

Proof. From an instance I of NOT-ALL-EQUAL 3-SATISFIABILITY with positive literals only we construct the P_7-free graph G as before (see also Figure 2). We introduce four new vertices $r_1, \ldots, r_4$ that are mutually adjacent, and we connect them to G by adding an edge between every $a_{h,i}$ and every vertex in $\{r_2, r_3, r_4\}$ and an edge between every $a'_{h,i}$ and every vertex in $\{r_1, r_3, r_4\}$. We denote the resulting graph by G^* and prove the following claim.

Claim 1. If C is an induced cycle in G^* with more than four vertices then

(i) $C = r_1r_2a_{h,i}sa'_{h',i'}r_1$ *for some* h, i, h', i'*, with* s *of* b*-type or* d*-type;*
(ii) $C = r_1r_2a_{h,i}c_{h,i}x_jc'_{h',i'}a'_{h',i'}r_1$ *for some* h, i, j, h', i'*;*
(iii) $C = r_istx_jt^*s^*r_i$ *for some* i, j*, with* s, s^* *of* a*-type and* t, t^* *of* c*-type;*
(iv) $C = b_{j,1}c_{j,1}d_{j,1}c_{j,2}d_{j,2}c_{j,3}b_{j,2}b_{j,1}$ *for some* j*;*
(v) $C = b'_{j,1}c'_{j,1}d'_{j,1}c'_{j,2}d'_{j,2}c'_{j,3}b'_{j,2}b'_{j,1}$ *for some* j*.*

We prove Claim 1 as follows. We say that an induced cycle of G^* is of *type* (i), $\ldots$, (v) if it is described in statements (i), $\ldots$, (v), respectively. Let C be an induced cycle of G^* that has more than four vertices. We observe that C can have at most two r-type vertices. We also recall that the subgraph of G, and consequently of G^*, induced by all a-type, b-type, d-type and x-type vertices has a spanning subgraph that is complete bipartite with partition classes Z_1 and Z_2, where Z_1 is formed by all a-type and x-type vertices, and Z_2 by all b-type and d-type vertices. We will use these two observations in our case analysis below.

Case 1. C contains no r-type vertices and no vertex of Z_1. Then C is a subgraph of a clause gadget. The only cycles in clause gadgets are of type (iv) and (v).

Case 2. C contains no r-type vertices and exactly one vertex $s \in Z_1$. Let t be one of the two neighbors of s on C. Let z be the other neighbor of t on C. If t is of c-type then z is of b-type or d-type. However, this is not possible as vertices of such types are adjacent to all vertices in Z_1, which would mean that C is the 3-vertex cycle $stzs$. Hence, t is of b-type or d-type. This means that z is of c-type or of b-type (note that the latter case can only occur if t is of b-type). However, because every vertex in Z_1 is adjacent to all vertices of b-type, z cannot be of b-type. Hence z is of c-type. This means that the other neighbor of z on C must be of b-type or d-type. Such a vertex is adjacent to s. Hence, C has only four vertices; a contradiction.

Case 3. C contains no r-type vertices and at least two vertices of Z_1. Let s, s^* be two vertices of Z_1 that are on C. Note that s, s^* are not adjacent. Because every vertex of Z_1 is adjacent to every vertex of Z_2, we find that C contains at most one vertex of Z_2. First suppose that C contains no vertex of Z_2. Then all vertices of C not in Z_1 are of c-type. We make the following three observations. First, Z_1 and the set of all c-type vertices are independent sets. Second, every a-type vertex has a unique c-type neighbor. Third, every c-type vertex has a unique a-type neighbor and a unique x-type neighbor. Consequently, C is not a cycle as C contains no subpath from s to s^*; a contradiction. Now suppose that C contains a vertex $t \in Z_2$. By the same arguments as before, we find that C contains only one subpath from s to s^* (namely the path sts^*). Hence C is not a cycle; a contradiction.

Case 4. C contains one r-type vertex. Let r_i be this r-type vertex. Let s, s^* be the two neighbors of r_i on C. Then s, s^* are of a-type. Let t, t^* be the other neighbors of s, s^* on C, respectively. Because C contains at least five vertices, t and t^* are two different vertices. Because vertices of b-type and d-type are common neighbors of s and s^*, this means that both t and t^* are neither of b-type nor of d-type. This means that both t and t^* are of c-type. Consequently, no vertex of C can be of b-type or d-type (as such a vertex would be adjacent to s and s', both of which cannot have more than two neighbors on C). Hence, because vertices of c-type have a unique neighbor of a-type, the other neighbor of t on C must be of x-type. Let x_j be this neighbor. The subpath P of C that starts at x_j, ends in t^* and that does not contain r_i can only contain vertices of a-type, x-type and c-type. Recall that every a-type vertex has a unique neighbor of c-type, and that every c-type vertex has a unique neighbor of a-type and a unique neighbor of x-type. Moreover, the union of vertices of a-type and x-type is an independent set. Hence, P must be the path $x_j t^*$. We conclude that C is of type (iii).

Case 5. C contains two r-type vertices. Then these vertices are neighbors on C. Because C contains at least five vertices, they both have their other neighbor on C of a-type. Hence, these r-type vertices must be r_1 and r_2, and their other neighbors on C are some $a'_{h',i'}$ and $a_{h,i}$, respectively. If $a_{h,i}$ and $a'_{h',i'}$ have a common neighbor on C, then this neighbor must be of b-type or d-type. In that case C is of type (i). Suppose that $a'_{h',i'}$ and $a_{h,i}$ have no common neighbor on C. Let s be the other neighbor of $a_{h,i}$ on C, and let s' be the other neighbor of $a'_{h',i'}$ on C. Then s and s' must be of C-type, and we use the same arguments as in Case 4, and find that C is of type (ii). This completes the proof of Claim 1.

We first consider the case $r = 5$. We place a generalized diamond on all edges $r_h a'_{i,j}$ and on all edges $r_h a_{i,j}$. We denote the resulting graph by G_5^*. We claim that G_5^* is C_5-free. This can be seen as follows. For contradiction, suppose that C is a cycle in G_5^* with exactly five vertices. If C contains no vertex of r-type, then C is a 5-vertex cycle in G'. However, all 5-vertex cycles in G' are of type (i) due to Claim 1. Hence, C must contain at least one vertex of r-type. If C contains no a-type vertex, then C only consists of vertices of r-type and internal vertices of generalized diamonds. However, then C is not a 5-vertex cycle. Hence, C must also contain an a-type vertex. However, every induced path between an a-type vertex and an r-type vertex has at least four vertices. Hence, this case is not possible either, and we conclude that G_5^* is C_5-free.

We find that G_5^* is 4-colorable if and only if G^* is 4-colorable if and only if G has a 4-coloring in which every $a_{j,h}$ has color 1 and every $a'_{j,h}$ has color 2 if and only if I has a truth assignment in which each clause contains at least one true and at least one false literal. The last two equivalences follow from the same arguments as used in Lemma 4.

Finally, we prove that G_5^* is P_{23}-free. Let P be an induced path of G_5^*. Then P contains at most two r-type vertices. Suppose P contains exactly two r-type vertices. Then these vertices are adjacent on P. We then find that P has maximum number of vertices if P can be decomposed into two parts in the following way. The first part of P starts with two inner vertices of a generalized diamond, followed by the vertices of an induced path in G^*, followed by two more inner vertices of a generalized diamond, followed by an r-type vertex. The second part of P contains the same types of vertices in reverse order. Hence, as G^* is P_7-free, we find that P has at most $2(2{+}6{+}2{+}1) = 22$ vertices. If P contains at most one r-type vertex, then P contains less than 22 vertices. We conclude that G_5^* is P_{23}-free. Hence, we define $\ell(5) = 23$ and have proven the case $r = 5$.

For $r \geq 6$ we place a sufficient number of generalized diamonds on all edges $r_1 a'_{ij}$ and on all edges $r_2 a_{ij}$. If $r = 6$, we have modified G^* into a (C_5, C_6)-free graph G_6 due to Claim 1. If $r \geq 7$, then we must get rid of all induced cycles on seven vertices as well. By Claim 1, these cycles are of type (ii), (iv) and (v). Hence, in order to modify G^* into a $(C_5, \ldots, C_r)$-free graph G_r^*, it suffices to do the following in addition to the generalized diamond placements that we placed already. We replace each edge $b_{j,1}b_{j,2}$ and each edge $b'_{j,1}b'_{j,2}$ by sufficiently long paths of odd length, whose inner vertices we connect to all a-type vertices and to all x-type vertices. These paths, which start at and end in a b-type vertex, are called *b-paths*.

We claim that G^*_r is $P_{\ell(r)}$-free for some constant $\ell(r)$. This can be seen as follows. Let P be an induced path in G^*_r. We show that the number of vertices of every type (a-type, b-type, c-type, d-type, r-type, x-type, inner vertex of generalized diamond, and inner vertex of b-path) present in P is bounded by a constant.

First of all, P contains at most two r-type vertices, and if P contains two r-type vertices then these vertices are adjacent. As before, this means that P contains inner vertices of at most four generalized diamonds.

We claim that P has at most four a-type vertices. In order to see this, suppose that P contains more than two a-type vertices. Then P contains no b-type vertices and no inner vertices of b-paths either. Recall that a-type vertices have a unique c-type neighbor, and that c-type vertices have a unique a-type neighbor and a unique x-type neighbor. Moreover, the union of the set of a-type vertices and the set of x-type vertices is an independent set. Hence, because P contains at most two r-type vertices, P has at most four a-type vertices (note that P can only have four a-type vertices if P starts at and ends in an a-type vertex and moreover contains exactly one r-type vertex).

By the same arguments as above, we find that P has at most two x-type vertices. Furthermore, P has at most $4 + 2 = 6$ c-type vertices, as the number of c-type vertices is not larger than the total number of a-type vertices and x-type vertices.

We claim that P has at most two b-type vertices and inner vertices of at most two b-paths. In order to see this, suppose that P has more than two b-type vertices or inner vertices of more than two b-paths. Then P contains no a-type vertices and no x-type vertices. This means that P only contains vertices from one clause gadget in G and one b-path; a contradiction. Hence, we have shown that the number of vertices of every type that is present in P is bounded by a constant. Consequently, the total number of vertices in P is bounded by a constant; note that this constant depends on the length of the b-paths, and thus on r.

Finally, we claim that G^*_r is 4-colorable if and only if G^* is 4-colorable. This can be seen as follows. We may assume without loss of generality that vertices of a-type are colored with colors 1 and 2 in any 4-coloring of G^*_r and in any 4-coloring of G^*. This means that all b-type vertices may be assumed to have color 3 or 4 in G^*. It also implies that we may assume that the colors of the vertices of every b-path in G^*_r alternate between colors 3 and 4. Because b-paths have odd length, adjacent b-type vertices in G^* have different colors in G^*_r. We conclude that 4-colorings of G^*_r and G^* correspond to each other. Hence, similar to the case $r = 5$, we have proven the case $r \geq 6$. $\square$

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