Characterizing Graphs of Small Carving-Width*

Rémy Belmonte[1,†], Pim van 't Hof[1,†], Marcin Kamiński[2], Daniël Paulusma[3,‡], and Dimitrios M. Thilikos[4,§]

[1] Department of Informatics, University of Bergen, Norway
{remy.belmonte,pim.vanthof}@ii.uib.no
[2] Département d'Informatique, Université Libre de Bruxelles, Belgium
marcin.kaminski@ulb.ac.be
[3] School of Engineering and Computing Sciences, Durham University, UK
daniel.paulusma@durham.ac.uk
[4] Department of Mathematics, National & Kapodistrian University of Athens, Panepistimioupolis, GR-15784, Athens, Greece.
sedthilk@math.uoa.gr

Abstract. We characterize all graphs that have carving-width at most k for $k = 1, 2, 3$. In particular, we show that a graph has carving-width at most 3 if and only if it has maximum degree at most 3 and treewidth at most 2. This enables us to identify the immersion obstruction set for graphs of carving-width at most 3.

1 Introduction

A *call routing tree* (or a *carving*) of a graph G is a tree T with internal vertices of degree 3 whose leaves correspond to the vertices of G. We say that the congestion of T is at most k if, for any edge e of T, the communication demands that need to be routed through e or, more explicitly, the number of edges of G that share endpoints corresponding to different connected components of $T \setminus e$, is bounded by k (we denote by $T \setminus e$ the graph obtained from T after the removal of e). The *carving-width* of a graph G is the minimum k for which there exists a call routing tree T whose congestion is bounded by k.

Carving-width was introduced by Seymour and Thomas [15] who proved that checking whether the carving-width of a graph is at most k is an NP-complete problem. In the same paper, they proved that there is a polynomial-time algorithm for computing the carving-width of planar graphs. Later, the problem of designing call routing trees of minimum congestion was studied by Khuller [10], who presented a polynomial-time algorithm for computing a call

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routing tree T whose congestion is within a $O(\log n)$ factor from the optimal. In [18] an algorithm was given that decides, in $f(k)\cdot n$ steps, whether an n-vertex graph has carving width at most k and, if so, also outputs a corresponding call routing tree. We stress that the values of $f(k)$ in the complexity of the algorithm in [18] are huge, which makes the algorithm highly impractical even for trivial values of k.

A graph G contains a graph H as an immersion if H can be obtained from some subgraph of G after lifting a number of edges (see Section 2 for the complete definition). Recently, the immersion relation attracted a lot of attention both from the combinatorial [1,5] and the algorithmic [8,9] point of view. It can easily be observed (cf. [18]) that carving-width is a parameter closed under taking immersions, i.e., the carving-width of a graph is not smaller than the carving-width of any of its immersions. Combining this fact with the seminal result of Robertson and Seymour in [13] stating that graphs are well-quasi-ordered with respect to the immersion relation, it follows that the set $\mathcal{G}_k$ of graphs with carving-width at most k can be completely characterized by forbidding a finite set of graphs as immersions. This set is called an *immersion obstruction set* for the class $\mathcal{G}_k$.

Identifying obstruction sets is a classic problem in structural graph theory, and its difficulty may vary, depending on the considered graph class. While obstructions have been extensively studied for parameters that are closed under minors (see [2,4,7,11,12,14,16,17] for a sample of such results), no obstruction characterization is known for any immersion-closed graph class. In this paper, we make a first step in this direction.

The outcome of our results is the identification of the immersion obstruction set for $\mathcal{G}_k$ when $k \leq 3$; the obstruction set for the non-trivial case $k = 3$ is depicted in Figure 3. Our proof for this case is based on a combinatorial result stating that $\mathcal{G}_3$ consists of exactly the graphs with maximum degree at most 3 and treewidth at most 2. A direct outcome of our results is a linear-time algorithm for the recognition of the class $\mathcal{G}_k$ when $k = 1, 2, 3$. This can be seen as a "tailor-made" alternative to the general algorithm of [18] for elementary values of k.

2 Preliminaries

We consider finite undirected graphs that have no self-loops but that may have multiple edges. For undefined graph terminology we refer to the text-book of Diestel [6].

Let $G = (V, E)$ be a graph. The set of *neighbors* of a vertex u is denoted by $N(u) = \{v \mid uv \in E\}$. We denote the number of edges incident with a vertex u by $\deg(u)$; note that $\deg(u)$ may be strictly greater than the number of neighbors of u because we allow G to have multiple edges. We let $\Delta(G) = \max\{\deg(u) \mid u \in V\}$. The n-vertex *path* is the graph with vertices $v_1, \ldots, v_n$ and edges v_iv_{i+1} for $i = 1, \ldots, n-1$. If v_nv_1 is also an edge, then we obtain the n-vertex *cycle*. The complete graph on k vertices is denoted by K_k.

Let $G = (V, E)$ be a graph. Then G is called *connected* if, for every pair of distinct vertices v and w, there exists a path connecting v and w. A maximal connected subgraph of G is called a *connected component* of G. A vertex u is called a *cut-vertex* of G if the graph obtained after removing u has more connected components than G. A connected graph is 2-*connected* if it does not contain a cut-vertex. A maximal 2-connected subgraph of G is called a *biconnected component* of G.

The *edge duplication* is the operation that takes two adjacent vertices u and v of a graph and adds a new edge between u and v. The *edge subdivision* is the operation that removes an edge uv of a graph and adds a new vertex w adjacent (only) to u and v. A *series-parallel graph* is a 2-connected graph that can be obtained from a graph consisting of two vertices with two edges between them by a sequence of edge duplications and edge subdivisions.

The *vertex dissolution* is the reverse operation of an edge subdivision; it removes a vertex u of degree 2 that has two distinct neighbors v and w, and adds an edge between v and w. A graph G contains a graph H as a *topological minor* if H can be obtained from G by a sequence of vertex deletions, edge deletions, and vertex dissolutions. Alternatively, G contains H as a topological minor if G contains a subgraph H' that is a *subdivision* of H, i.e., H' can be obtained from H by a sequence of edge subdivisions. We mention one more equivalent definition. The graph G has H as a topological minor if G contains a subset $S \subseteq V_G$ of size $|V_H|$ that has the following property: there exists a bijection f from V_H to S such that, for each edge $e \in E_H$, say with endpoints x and y, there exists a path P_e from $f(x)$ to $f(y)$, and such that for every two edges $e, e' \in E_H$, the paths P_e and $P_{e'}$ are internally vertex-disjoint.

Let u, v, w be three distinct vertices in a graph such that uv and vw are edges. The operation that removes the edges uv and vw, and adds the edge uw (even in the case u and w are already adjacent) is called a *lift*. A graph G contains a graph H as an *immersion* if H can be obtained from G by a sequence of vertex deletions, edge deletions, and lifts. Alternatively, G contains H as an immersion if G contains a subset $S \subseteq V_G$ of size $|V_H|$ that has the following property: there exists a bijection f from V_H to S such that, for each edge $e \in E_H$, say with endpoints x and y, there exists a path P_e from $f(x)$ to $f(y)$, and such that for every two edges $e, e' \in E_H$, the paths P_e and $P_{e'}$ are edge-disjoint. Note that since any two internally vertex-disjoint paths are edge-disjoint, G contains H as an immersion if G contains H as a topological minor.

The *edge contraction* is the operation that takes two adjacent vertices u and v and replaces them by a new vertex adjacent to exactly those vertices that are a neighbor of u or v. A graph G contains a graph H as a *minor* if H can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions.

A *tree* is a connected graph with no cycles and no multiple edges. A *leaf* of a tree is a vertex of degree 1. A vertex in a tree that is not a leaf is called an *internal* vertex. A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\mathcal{T}, \mathcal{X})$, where $\mathcal{X}$ is a collection of subsets of V, called *bags*, and $\mathcal{T}$ is a tree whose vertices, called *nodes*, are the sets of $\mathcal{X}$, such that the following three properties are satisfied:

(i) $\bigcup_{X \in \mathcal{X}} X = V$,
(ii) for each $uv \in E$, there is a bag $X \in \mathcal{X}$ with $u, v \in X$,
(iii) for each $u \in V$, the nodes containing u induce a connected subtree of $\mathcal{T}$.

The *width* of a tree decomposition $(\mathcal{T}, \mathcal{X})$ is the size of a largest bag in $\mathcal{X}$ minus 1. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width over all possible tree decompositions of G.

Let $G = (V, E)$ be a graph. Let $S \subset V$ be a subset of vertices of G. Then the set of edges between S and $V \setminus S$, denoted by $(S, V \setminus S)$, is called an *edge cut* of G. Let the vertices of G be in 1-to-1 correspondence to the leaves of a tree T whose internal vertices all have degree 3. The correspondence between the leaves of T and the vertices of G uniquely defines the following edge weighting w on the edges of T. Let $e \in E_T$. Let C_1 and C_2 be the two connected components of $T \setminus e$. Let S_i be the set of leaves of T that are in C_i for $i = 1, 2$; note that $S_2 = V \setminus S_1$. Then the weight $w(e)$ of the edge e in T is the number of edges in the edge cut (S_1, S_2) of G. The tree T is called a *carving* of G, and (T, w) is a *carving decomposition* of G. The *width* of a carving decomposition (T, w) is the maximum weight $w(e)$ over all $e \in E_T$. The *carving-width* of G, denoted by $\mathrm{cw}(G)$, is the minimum width over all carving decompositions of G. We define $\mathrm{cw}(G) = 0$ if $|V| = 1$. We refer to Figure 4 for an example of a graph and a carving decomposition.

3 The Main Result

The following observation is known and easy to verify by considering the number of edges in the edge cut $(\{u\}, V \setminus \{u\})$ of a graph $G = (V, E)$.

Observation 1 *Let G be a graph. Then* $\mathrm{cw}(G) \geq \Delta(G)$.

We also need the following two straightforward lemmas. The first lemma follows immediately from the observation that any subgraph of a graph is an immersion of that graph, combined with the observation that carving-width is a parameter that is closed under taking immersions (cf. [18]). We include the proof of the second lemma for completeness.

Lemma 1. *Let H be a subgraph of G. Then* $\mathrm{cw}(H) \leq \mathrm{cw}(G)$.

Lemma 2. *Let G be a graph with connected components $C_1, \ldots, C_p$ for some integer $p \geq 1$. Then* $\mathrm{cw}(G) = \max\{\mathrm{cw}(C_i) \mid 1 \leq i \leq p\}$.

Proof. Lemma 1 implies that $\max\{\mathrm{cw}(C_i) \mid 1 \leq i \leq p\} \leq \mathrm{cw}(G)$. Now let (T_i, w_i) be a carving decomposition of C_i of width $\mathrm{cw}(C_i)$ for $i = 1, \ldots, p$. We pick an arbitrary edge $e_i = x_i y_i$ in each T_i and subdivide it by replacing it with edges $x_i z_i$ and $z_i y_i$, where each z_i is a new vertex. We add edges $z_i z_{i+1}$ for $i = 1, \ldots, p-1$.

This results in a tree T. The corresponding carving decomposition (T, w) of G has width $\max\{\mathrm{cw}(C_i) \mid 1 \leq i \leq p\}$. Hence, $\mathrm{cw}(G) \leq \max\{\mathrm{cw}(C_i) \mid 1 \leq i \leq p\}$. We conclude that $\mathrm{cw}(G) = \max\{\mathrm{cw}(C_i) \mid 1 \leq i \leq p\}$. $\square$

The next lemma is the final lemma we need in order to prove our main result.

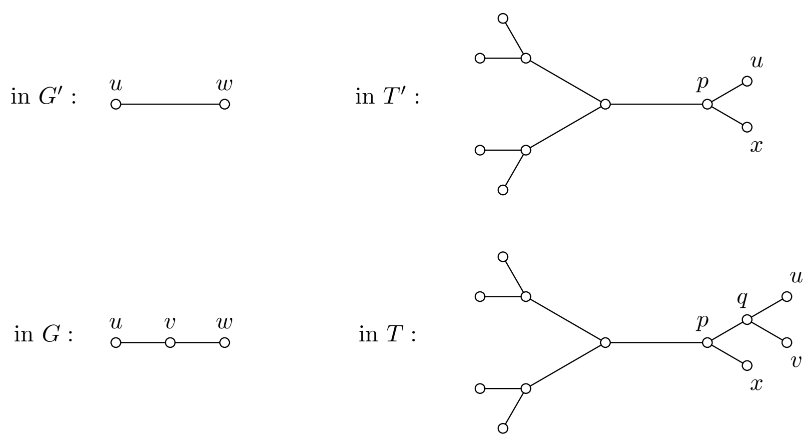

Fig. 1. A schematic illustration of how the tree T' in the carving decomposition of G' is transformed into a tree T in the proof of Lemma 3 when the edge uw in G' is subdivided. The vertex x is an arbitrary vertex of G', possibly w.

Lemma 3. *Let G' be a graph with carving-width at least 2, and let uw be an edge of G'. Let G be the graph obtained from G' by subdividing the edge uw. Then* $\mathrm{cw}(G) = \mathrm{cw}(G')$.

Proof. Let (T', w') be a carving decomposition of G' of width $\mathrm{cw}(G') \geq 2$, and let p be the unique neighbor of u in T'. Let v be the vertex that was used to subdivide the edge uw in G', i.e., the graph G was obtained from G' by replacing uw with edges uv and vw for some new vertex v. Let T be the tree obtained from T' by replacing the edge pu by edges pq, qu and qv for some new vertex q; see Figure 1 for an illustration. We first show that the resulting carving decomposition (T, w) of G has width at most $\mathrm{cw}(G')$, which implies that $\mathrm{cw}(G) \leq \mathrm{cw}(G')$.

Let e be an edge in T. Suppose that $e = pq$. By definition, $w(e)$ is the number of edges between $\{u, v\}$ and $V \setminus \{u, v\}$ in G, which is equal to the number of edges incident with u in G. The latter number is the weight of the edge up in T'. Hence, $w(e) \leq \mathrm{cw}(G')$. Suppose that $e = qu$. By definition, $w(e)$ is the number of edges incident with u in G, which is equal to the number of edges incident with u in G'. Hence $w(e) \leq \mathrm{cw}(G')$. Suppose that $e = qv$. By definition, $w(e)$ is the number of edges incident with v in G, which is 2. Hence $w(e) = 2 \leq \mathrm{cw}(G')$.

Finally, suppose that $e \notin \{pq, qu, qv\}$. Let C_1 and C_2 denote the subtrees of T obtained after removing e. Let S_i be the set of leaves of T in C_i for $i = 1, 2$. Then u and v either both belong to S_1 or both belong to S_2. Without loss of generality, assume that both u and v belong to S_1. By definition, $w(e)$ is the number of edges between S_1 and S_2 in G, which is equal to the number of edges between $S_1 \setminus \{v\}$ and S_2 in G'. The latter number is the weight of the edge e in T'. Hence, $w(e) \leq \text{cw}(G')$. We conclude that (T, w) has width at most $\text{cw}(G')$, and hence $\text{cw}(G) \leq \text{cw}(G')$.

It remains to show that $\text{cw}(G) \geq \text{cw}(G')$. Let (T^*, w^*) be a carving decomposition of G of width $\text{cw}(G)$. We remove the leaf corresponding to v from T^*. Afterwards, the neighbor of v in T^* has degree 2, and we dissolve this vertex. This results in a tree T''. It is easy to see that the corresponding carving decomposition (T'', w'') of G' has width at most $\text{cw}(G)$. Hence, $\text{cw}(G) \geq \text{cw}(G')$. This completes the proof of Lemma 3. □

We are now ready to show the main result of our paper.

Theorem 1. *Let G be a graph. Then the following three statements hold.*

(i) $\text{cw}(G) \leq 1$ *if and only if* $\Delta(G) \leq 1$.
(ii) $\text{cw}(G) \leq 2$ *if and only if* $\Delta(G) \leq 2$.
(iii) $\text{cw}(G) \leq 3$ *if and only if* $\Delta(G) \leq 3$ *and* $\text{tw}(G) \leq 2$.

Proof. Let $G = (V, E)$ be a graph. By Lemma 2 we may assume that G is connected. We prove the three statements separately.

(i) If $\text{cw}(G) \leq 1$, then $\Delta(G) \leq 1$ due to Observation 1. If $\Delta(G) \leq 1$, then G contains either one or two vertices. Clearly, $\text{cw}(G) \leq 1$ in both cases.

(ii) If $\text{cw}(G) \leq 2$, then $\Delta(G) \leq 2$ due to Observation 1. If $\Delta(G) = 1$, then the statement follows from (i). If $\Delta(G) = 2$, then G is either a graph consisting of two vertices with two edges between them, or a path, or a cycle. In all three cases, it is clear that $\text{cw}(G) \leq 2$.

(iii) First suppose that $\text{cw}(G) \leq 3$. Then $\Delta(G) \leq 3$ due to Observation 1. We need to show that $\text{tw}(G) \leq 2$. For contradiction, suppose that $\text{tw}(G) \geq 3$. It is well-known that any graph of treewidth at least 3 contains K_4 as a minor (see for example [6], p. 327). It is also well-known that every minor with maximum degree at most 3 of a graph is also a topological minor of that graph (see [6], p. 20). This means that G contains K_4 as a topological minor. Then, by definition, G contains a subgraph H such that H is a subdivision of K_4. Since $\text{cw}(K_4) = 4$, we have that $\text{cw}(H) = \text{cw}(K_4) = 4$ as a result of Lemma 3. Since H is a subgraph of G, Lemma 1 implies that $\text{cw}(G) \geq \text{cw}(H) = 4$, contradicting the assumption that $\text{cw}(G) \leq 3$.

For the reverse direction, suppose that $\Delta(G) \leq 3$ and $\text{tw}(G) \leq 2$. Bodlaender [3] showed that a graph has treewidth at most 2 if and only if all its biconnected components are series-parallel. Hence, we assume that $\Delta(G) \leq 3$ and that every biconnected component of G is series-parallel. We use induction on the number of vertices of G to prove that $\mathrm{cw}(G) \leq 3$. It is clear that this holds when $|V| \leq 2$, since we assumed $\Delta(G) \leq 3$.

Let $|V| \geq 3$. Suppose that G contains a vertex v of degree 2 that has two distinct neighbors u and w. Let $G' = (V', E')$ denote the (connected) graph obtained from G by dissolving v. Note that G' has maximum degree at most 3, and every biconnected component of G' is series-parallel. Hence, by the induction hypothesis, $\mathrm{cw}(G') \leq 3$. Because $|V| \geq 3$, we find that G' contains at least two vertices. If $\mathrm{cw}(G') = 1$, then $\Delta(G') = 1$ by Observation 1. This means that G' is a path on two vertices. Consequently, G is a path on three vertices, and hence $\mathrm{cw}(G) = 2 \leq 3$. If $2 \leq \mathrm{cw}(G') \leq 3$, then $\mathrm{cw}(G) = \mathrm{cw}(G') \leq 3$ as a result of Lemma 3.

From now on, we assume that G contains no vertex of degree 2 that has two distinct neighbors. Suppose that G contains two vertices u and v with at least two edges between them. First suppose that u and v are the only vertices of G. Then $\mathrm{cw}(G) \leq 3$, because the assumption $\Delta(G) \leq 3$ implies that u and v have at most three edges between them. Now suppose that at least one of u, v has at least one other neighbor outside $\{u, v\}$ in G, say v has a neighbor $t \neq u$. Then, because $\Delta(G) \leq 3$ and there exist at least two edges between u and v in G, we find that t and u are the only two neighbors of v in G and that the number of edges between u and v is exactly 2. Let G^* denote the graph obtained from G by deleting one edge between u and v. Let G' denote the graph obtained from G^* by dissolving v. Note that G' has maximum degree at most 3, and that every biconnected component of G' is series-parallel. Hence, by the induction hypothesis, $\mathrm{cw}(G') \leq 3$.

If $\mathrm{cw}(G') = 1$, then, for the same reasons as before, G' must a path on two vertices and G^* must be a path on three vertices, implying that $\mathrm{cw}(G^*) = 2$. Since G can be obtained from G^* by adding a single edge, $\mathrm{cw}(G) \leq 3$ in this case. Suppose $2 \leq \mathrm{cw}(G') \leq 3$. Then, by Lemma 3, $\mathrm{cw}(G^*) = \mathrm{cw}(G') \leq 3$. Moreover, from the proof of Lemma 3 it is clear that there exists a carving decomposition (T^*, w^*) of G^* of width $\mathrm{cw}(G^*)$ such that u and v have a common neighbor q in T^*. We consider the carving decomposition (T, w) of G with $T = T^*$. Let e be an edge in T. First suppose that $e = uq$ or $e = vq$. Then $w(e) \leq 3$, as both u and v have degree at most 3 in G. Now suppose that $e \notin \{uq, vq\}$. Then $w(e) = w^*(e) \leq \mathrm{cw}(G^*) \leq 3$. We conclude that the carving decomposition (T, w) of G has width at most 3, which implies that $\mathrm{cw}(G) \leq 3$.

From now on, we assume that G contains no multiple edges. Since we already assumed G not to contain any vertex of degree 2 that has two distinct neighbors, this implies that G contains no vertices of degree 2 at all. If G contains no cut-vertices, then G is 2-connected. Then G must be series-parallel, since we assumed that every biconnected component of G is series-parallel. Then, by definition, G contains either a vertex of degree 2 or two vertices with more than one edge between them. However, we assumed that this is not the case. We conclude that G contains at least one cut-vertex v.

Because v is a cut-vertex, it has degree at least 2. Since G contains no vertex of degree 2 and $\Delta(G) \leq 3$, we find that v has degree 3. Note that the graph

$G - v$ has either two or three connected components. Let D_1, D_2, D_3 denote the vertex sets of the connected components of $G - v$, where D_3 is possibly empty. Let G' be the subgraph of G induced by $D_1 \cup \{v\}$. Because v is a cut-vertex of G, the set of biconnected components of G' is a subset of the set of biconnected components of G. Hence, every biconnected component of G' is series-parallel. Moreover, since $\Delta(G) \leq 3$ and G' is a subgraph of G, we find that $\Delta(G') \leq 3$. Hence, by the induction hypothesis, G' has carving-width at most 3. Similarly, the subgraph G'' of G induced by $D_2 \cup D_3 \cup \{v\}$ has carving-width at most 3. Let (T', w') be a carving decomposition of G' of width $\mathrm{cw}(G') \leq 3$, and let (T'', w'') be a carving decomposition of G'' of width $\mathrm{cw}(G'') \leq 3$. From T' and T'', we construct a tree T as follows (see also Figure 2). We first identify the leaves of

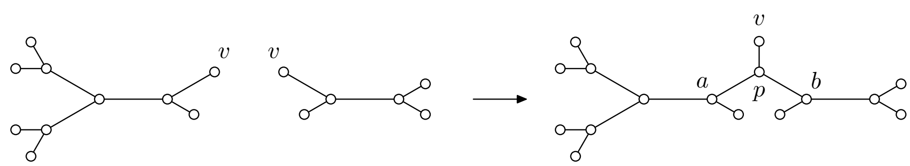

Fig. 2. A schematic illustration of how the tree T is constructed from the trees T' and T'' in the proof of Theorem 1.

T' and T'' that correspond to v. Let p denote the newly obtained vertex, and let a and b be the two neighbors of p, where a belongs to T' and b' belongs to T''. We then add a new leaf adjacent to the vertex p in T, and we let this leaf correspond to the vertex v of G. This completes the construction of T. Below we show that the corresponding carving decomposition (T, w) of G has width at most 3.

Let e be an edge of T. Let C_1 and C_2 be the two subtrees of the forest $T - e$. Let S_1 and S_2 be the set of leaves of T in C_1 and C_2, respectively. We will also use S_1 and S_2 to denote the vertices of G that correspond to the leaves in S_1 and S_2, respectively. Assume that $v \in S_1$. Suppose that $e = vp$. Then $w(e) = 3$, because there are three edges incident with v in G. Suppose that $e = ap$. Due to the fact that v is a cut-vertex of G, we find that v is the only vertex in S_1 that has at least one neighbor in S_2 in G. Since v has degree 3 and D_1 is not empty, v has at most two neighbors in S_2. Hence $w(e) \leq 2$. Suppose that $e = bp$. Then $w(e) \leq 2$ by a similar argument as in the previous case. Suppose that $e \in E_{T'} \setminus \{ap, bp, vp\}$. Then $w(e) = w'(e) \leq 3$, because $\mathrm{cw}(G') \leq 3$. Suppose that $e \in E_{T''} \setminus \{ap, bp, vp\}$. Then $w(e) = w''(e) \leq 3$, because $\mathrm{cw}(G'') \leq 3$. We conclude that $\mathrm{cw}(G) \leq 3$. This completes the proof of Theorem 1. $\square$

Since graphs of treewidth at most 2 can easily be recognized in linear time, Theorem 1 implies a linear-time recognition algorithm for graphs of carving-width at most 3.

Thilikos, Serna and Bodlaender [18] proved that for any k, there exists a linear-time algorithm for constructing the immersion obstruction set for graphs of carving-width at most k. For $k \in \{1, 2\}$, finding such a set is trivial. We now present an explicit description of the immersion obstruction set for graphs of carving-width at most 3.

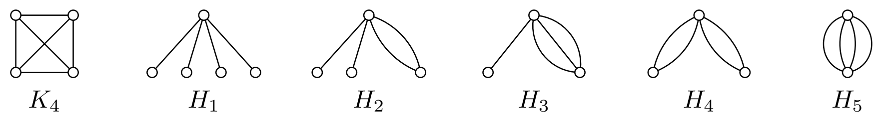

Fig. 3. The immersion obstruction set for graphs of carving-width at most 3.

Corollary 1. *A graph has carving-width at most* 3 *if and only if it does not contain any of the six graphs in Figure 3 as an immersion.*

Proof. Let G be a graph. We first show that if G contains one of the graphs in Figure 3 as an immersion, then G has carving-width at least 4. In order to see this, it suffices to observe that the graphs $K_4, H_1, \ldots, H_4$ all have carving-width 4. Hence, G has carving-width at least 4, because carving-width is a parameter that is closed under taking immersions (cf [18]).

Now suppose that G has carving-width at least 4. Then, due to Theorem 1, $\Delta(G) \geq 4$ or $\mathrm{tw}(G) \geq 3$. If $\Delta(G) \geq 4$, then G has a vertex v of degree at least 4. By considering v and four of its incident edges, it is clear that G contains one of the graphs $H_1, \ldots, H_5$ as a subgraph, and consequently as an immersion. Suppose that $\Delta(G) \leq 3$. Then $\mathrm{tw}(G) \geq 3$, which means that G contains K_4 as a minor [6]. Moreover, since K_4 has maximum degree 3, it is well-known that G also contains K_4 as a topological minor [6], and hence as an immersion. □

From the proof of Corollary 1, we can observe that an alternative version of Corollary 1 states that a graph has carving-width at most 3 if and only if it does not contain any of the six graphs in Figure 3 as a topological minor.

4 Conclusions

Extending Theorem 1 to higher values of carving-width remains an open problem, and finding the immersion obstruction set for graphs of carving-width at most 4 already seems to be a challenging task. We proved that for any graph G, $\mathrm{cw}(G) \leq 3$ if and only if $\Delta(G) \leq 3$ and $\mathrm{tw}(G) \leq 2$. We finish our paper by showing that the equivalence "$\mathrm{cw}(G) \leq 4$ if and only if $\Delta(G) \leq 4$ and $\mathrm{tw}(G) \leq 3$" does not hold in either direction.

To show that the forward implication is false, we consider the pentagonal prism F_5, which is displayed in Figure 4 together with a carving decomposition

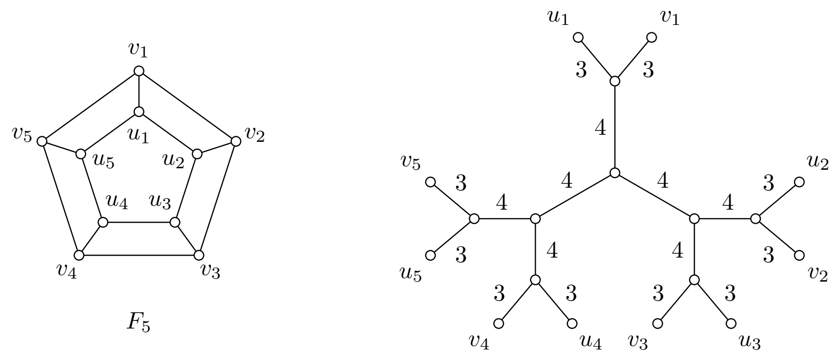

Fig. 4. The pentagonal prism F_5 and a carving decomposition (T, w) of F_5 that has width 4.

(T, w) of width 4. Hence, $\mathrm{cw}(F_5) \leq 4$. However, F_5 is a minimal obstruction for graphs of treewidth at most 3 [3]. Hence, $\mathrm{tw}(F_5) = 4$.

To show that the backward implication is false, we consider the graph K_5^-, which is the graph obtained from K_5 by removing an edge. Note that $\Delta(K_5^-) = 4$ and $\mathrm{tw}(K_5^-) = 3$. It is not hard to verify that $\mathrm{cw}(K_5) = 6$. Since removing an edge decreases the carving-width by at most 1, we conclude that $\mathrm{cw}(K_5^-) \geq 5$.

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