Parameterizing cut sets in a graph by the number of their components *

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Abstract. For a connected graph $G = (V, E)$, a subset $U \subseteq V$ is called a k-cut if U disconnects G and the subgraph induced by U contains exactly k (≥ 1) components. More specifically, a k-cut U is called a (k, ℓ)-cut if $V \backslash U$ induces a subgraph with exactly ℓ (≥ 2) components. We study two decision problems, called k-Cut and (k, ℓ)-Cut, which determine whether a given graph G has a k-cut or (k, ℓ)-cut, respectively. By pinpointing a close relationship to graph contractibility problems we first show that (k, ℓ)-Cut is in P for $k = 1$ and any fixed constant $\ell \geq 2$, while the problem is NP-complete for any fixed pair $k, \ell \geq 2$. We then prove that k-Cut is in P for $k = 1$, and is NP-complete for any fixed $k \geq 2$. On the other hand, for every fixed integer $g \geq 0$, we present an FPT algorithm that solves (k, ℓ)-Cut on graphs of Euler genus at most g when parameterized by $k + \ell$. By modifying this algorithm we can also show that k-Cut is in FPT (with parameter k). We also show that Disconnected Cut is solvable in polynomial time for minor-closed classes of graphs excluding some apex graph.

1 Introduction

Graph connectivity is a fundamental graph-theoretic property that is well-studied in the context of network robustness. In the literature several measures for graph connectivity are known, such as requiring hamiltonicity, edge-disjoint spanning trees, or edge- or vertex-cuts of sufficiently large size. Here, we study the problem of finding a vertex-cut, called a "disconnected cut" of a graph, such that the cut itself is disconnected. As we shall see in Section 3, this problem is strongly related to several other graph problems such as biclique vertex-covers. We give all further motivation later and first state our problem setting.

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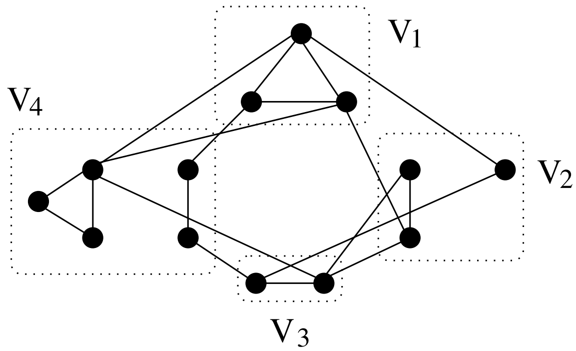

Fig. 1. A graph G with a disconnected cut $V_1 \cup V_3$ that is also a 2-cut and a $(2,4)$-cut and a disconnected cut $V_2 \cup V_4$ that is also a 4-cut and a $(4,2)$-cut.

Let $G = (V, E)$ be a connected simple graph. For a subset $U \subseteq V$, we denote by $G[U]$ the subgraph of G induced by U. We say that U is a *cut* of G if U disconnects G, that is, $G[V\backslash U]$ contains at least two components. A cut U is *connected* if $G[U]$ contains exactly one component, and *disconnected* if $G[U]$ contains at least two components. We observe that $G[U]$ is a disconnected cut if and only if $G[V\backslash U]$ is a disconnected cut. In Fig. 1, the subset $V_1 \cup V_3$ is a disconnected cut of G, and hence its complement $V_2 \cup V_4$ $\big(= V \setminus (V_1 \cup V_3)\big)$ is also a disconnected cut of G. This leads to the decision problem Disconnected Cut which asks if a given connected graph has a disconnected cut.

The complexity of Disconnected Cut is open. However, it is known that the problem can be solved in polynomial time for some restricted graph classes, as in the following theorem, which we will use in the proofs of some of our results. In particular, we mention that every graph of diameter at least three has a disconnected cut [12].

Theorem 1 ([12]). *The* Disconnected Cut *problem is solvable in polynomial time for the following classes of connected graphs*:

(i) *graphs of diameter not equal to two*;
(ii) *graphs with bounded maximum vertex degree*;
(iii) *graphs that are not locally connected*;
(iv) *triangle-free graphs*; *and*
(v) *graphs with a dominating edge* (*including cographs*).

Besides Disconnected Cut, we study two closely related problems in which we wish to find a cut having a prespecified number of components. For a fixed constant $k \geq 1$, a cut U of a connected graph G is called a *k-cut* of G if $G[U]$ contains exactly k components. Furthermore, for a pair (k, ℓ) of fixed constants $k \geq 1$ and $\ell \geq 2$, a k-cut U is called a *(k, ℓ)-cut* of G if $G[V\backslash U]$ consists of exactly ℓ components. Note that a k-cut and a (k, ℓ)-cut are connected cuts if $k = 1$; otherwise (when $k \geq 2$) they are disconnected cuts. It is obvious that, for a fixed pair $k, \ell \geq 2$, a (k, ℓ)-cut U of G corresponds to an (ℓ, k)-cut $V \setminus U$ of G. For example, the disconnected cut $V_1 \cup V_3$ in Fig. 1 is a 2-cut and a $(2, 4)$-cut, while its complement $V_2 \cup V_4$ is a 4-cut and a $(4, 2)$-cut. In this paper, we study the following two decision problems, where k and ℓ are fixed, *i.e.*, not part of the input. The k-Cut problem asks if a given connected graph has a k-cut. The (k,ℓ)-Cut problem asks if a given connected graph has a (k,ℓ)-cut.

Our results and the paper organization. Our three main results are as follows. First, we show that Disconnected Cut is strongly related to several other graph problems. In this way we determine the computational complexity of (k,ℓ)-Cut. Second, we determine the computational complexity of k-Cut. Third, for every fixed integer $g \geq 0$, we give an FPT algorithm that solves (k,ℓ)-cut for graphs of Euler genus at most g when parameterized by $k+\ell$. In the following, we explain our results in detail.

In Section 2 we define our terminology. Section 3 contains our first result. We state our motivation for studying these three types of cut problems. We then pinpoint relationships to other cut problems, and to graph homomorphism, biclique vertex-cover and vertex coloring problems. We show a strong connection to graph contractibility problems. In this way we prove that (k,ℓ)-Cut is solvable in polynomial time for $k=1, \ell \geq 2$, and is NP-complete otherwise.

Section 4 gives our second result: we classify the computational complexity of k-Cut. We show that k-Cut is solvable in polynomial time for $k=1$, while it becomes NP-complete for every fixed constant $k \geq 2$. Note that the NP-completeness of (k,ℓ)-Cut, shown in Section 3, does not imply this result, because ℓ is fixed and the subgraph obtained after removing a (k,ℓ)-cut must consist of *exactly* ℓ components.

In Section 5 we present our third result: an FPT algorithm that solves (k,ℓ)-Cut for graphs on surfaces when parameterized by $k+\ell$. We also show that k-Cut is FPT in k for graphs on surfaces and that Disconnected Cut is solvable in polynomial time for this class of graphs.

In Section 6 we state some further results and mention a number of open problems that are related to some other well-known graph classes, namely chordal, claw-free and line graphs.

2 Preliminaries

Without loss of generality, the graphs we consider are undirected and without multiple edges. Unless explicitly stated otherwise, they do not contain loops either. For undefined (standard) graph terminology we refer to Diestel [8].

Let $G=(V,E)$ be a graph. The vertex set V and the edge set E of G are often denoted by V_G and E_G, respectively. Each maximal connected subgraph of G is called a *component* of G. Let $N(u)$ denote the *neighborhood* of a vertex $u \in V$, that is, $N(u)=\{v \mid uv \in E\}$. Two disjoint nonempty subsets $U, U' \subset V$ are *adjacent* if there exist vertices $u \in U$ and $u' \in U'$ with $uu' \in E$. The *distance* $d_G(u,v)$ between two vertices u and v in G is the number of edges in a shortest path between them. The *diameter* $\text{diam}(G)$ is defined as $\max\{d_G(u,v) \mid u,v \in V\}$.

The cycle and path on n vertices are denoted by C_n and P_n, respectively. A graph $G=(V,E)$ is *complete p-partite* if V can be partitioned into p independent sets $V_1,\ldots,V_p$ such that $uv \in E$ if and only if $u \in V_i$ and $v \in V_j$ for some $1 \leq i < j \leq p$. For $p=2, |V_1|=k$, and $|V_2|=\ell$, we speak of a *biclique* $K_{k,\ell}$.

The *edge contraction* of an edge $e=uv$ in a graph G replaces the two end-vertices u and v with a new vertex adjacent to precisely those vertices to which u or v were adjacent. If a graph H can be obtained from G by a sequence of edge contractions, then G is said to be *contractible to* H, and G is called H*-contractible*. This is equivalent to saying that G has a so-called H*-witness structure* $\mathcal{W} = \{W(h_1), W(h_2), \ldots, W(h_{|V_H|})\}$, which is a partition of V_G into $|V_H|$ sets $W(h)$, called H*-witness sets*, such that each $W(h)$ induces a connected subgraph of G and for every two $h_i, h_j \in V_H$, witness sets $W(h_i)$ and $W(h_j)$ are adjacent in G if and only if h_i and h_j are adjacent in H. Clearly, by contracting the vertices in the witness sets $W(h)$ to a single vertex for every $h \in V_H$, we obtain the graph H. As an example, viewing each component of V_i, $1 \leq i \leq 4$, in the graph G in Fig. 1 as a witness set shows that G is $K_{2,4}$-contractible. In general, the witness sets $W(h)$ are not uniquely defined, since there may be different sequences of edge contractions that lead from G to H.

A *diagonal coloring* of G is a function $c : V \to \{1, 2, 3, 4\}$ such that all four *colors* $1, 2, 3, 4$ are used, and no edge has the colors $1, 3$ or $2, 4$ at its end-vertices. Note that a diagonal coloring does not have to be proper, because two adjacent vertices may receive the same color. Diagonal colorings are convenient for some of our proof techniques.

Below we give some background on parameterized complexity; for details we refer to Niedermeier [18]. In parameterized complexity theory, we consider the problem input as a pair (I, k), where I is the main part and k the parameter. A problem is *fixed parameter tractable* if an instance (I, k) can be solved in time $O(f(k)n^c)$, where f denotes a computable function and c a constant independent of k. The class FPT is the class of all fixed-parameter tractable decision problems.

3 Relationship to other problems

The Disconnected Cut problem can be formulated in several different ways as shown by Fleischner et al. [12]. We summarize these equivalent formulations and extend them in Proposition 1 after stating some additional terminology. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \{uv \notin E \mid u \neq v\})$.

A *model graph* is a simple graph with two types of edges: solid and dotted edges. Let H be a fixed model graph with vertex set $\{h_1, \ldots, h_k\}$. An H-partition of a graph G is a partition of V_G into k (nonempty) sets $V_1, \ldots, V_k$ such that for all vertices $u \in V_i$, $v \in V_j$ and for all $1 \leq i < j \leq k$ the following two conditions hold. Firstly, if h_ih_j is a solid edge of H, then $uv \in E_G$. Secondly, if h_ih_j is a dotted edge of H, then $uv \notin E_G$. Let $2K_2$ be the model graph with vertices $h_1, \ldots, h_4$ and solid edges h_1h_3, h_2h_4, and $2S_2$ be the model graph with vertices $h_1, \ldots, h_4$ and dotted edges h_1h_3, h_2, h_4.

A *homomorphism* from a graph G to a graph H is a vertex mapping $f : V_G \to V_H$ satisfying the property that $f(u)f(v) \in E_H$ whenever $uv \in E_G$. It is *vertex-surjective* if $f(V_G) = V_H$. Here we used the shorthand notation $f(S) = \{f(u) \mid u \in S\}$ for a subset $S \subseteq V$. A homomorphism f is called a *compaction* if f is *edge-surjective*, *i.e.*, for every edge $xy \in E_H$ with $x \neq y$ there exists an edge $uv \in E_G$ with $f(u) = x$ and $f(v) = y$. We then say that G *compacts to* H.

A graph G is called *reflexive* if every vertex i in G has a loop ii. We denote the reflexive cycle consisting of n vertices by $\mathcal{C}_n$.

Proposition 1. *Let G be a connected graph. Then, the following statements* (1)–(6) *are equivalent.*

(1) *G has a disconnected cut.*
(2) *G has a diagonal coloring.*

(3) *G has a $2S_2$-partition.*
(4) *G allows a vertex-surjective homomorphism to $\mathcal{C}_4$.*
(5) *$\overline{G}$ has a spanning subgraph that consists of two bicliques.*
(6) *$\overline{G}$ has a $2K_2$-partition.*

If $\text{diam}(G) = 2$*, then statements* (1)–(6) *above are also equivalent to the following statements* (7) *and* (8)*.*

(7) *G allows a compaction to $\mathcal{C}_4$.*
(8) *G is contractible to some biclique $K_{k,\ell}$ for some $k, \ell \geq 2$.*

Proof. In this paper, we only show that statements (1) and (8) are equivalent for a connected graph G with $\text{diam}(G) = 2$. The (straightforward) proofs of all other statements can be found in the paper by Fleischner et al. [12].

(1) $\Rightarrow$ (8): Let $G = (V, E)$ be a connected graph with $\text{diam}(G) = 2$. Suppose that G has a disconnected cut U. Let k be the number of components in $G[U]$, then $k \geq 2$. Let $X_1, X_2, \ldots, X_k$ be the vertex sets of the k components in $G[U]$. Then any two X_i and X_j are not adjacent. On the other hand, let ℓ be the number of components in $G[V \backslash U]$, and let $Y_1, Y_2, \ldots, Y_\ell$ be the vertex sets of the ℓ components in $G[V \backslash U]$. Then $\ell \geq 2$, and any two Y_i and Y_j are not adjacent.

We now show that the $k + \ell$ sets $X_1, X_2, \ldots, X_k, Y_1, Y_2, \ldots, Y_\ell$ form the $K_{k,\ell}$-witness sets for G. From the above it suffices to show that two sets X_i and Y_j are adjacent for every pair of indices i and j, $1 \leq i \leq k$, $1 \leq j \leq \ell$. Suppose for a contradiction that there exists a pair (X_i, Y_j) such that X_i and Y_j are not adjacent. Then, the distance from a vertex in X_i to a vertex in Y_i is at least three. This contradicts our assumption that $\text{diam}(G) = 2$.

(8) $\Rightarrow$ (1): Suppose that G is $K_{k,\ell}$-contractible for some $k, \ell \geq 2$. This means that G has a $K_{k,\ell}$-witness structure $\mathcal{W} = \{W(a_1), W(a_2), \ldots, W(a_k), W(b_1), W(b_2), \ldots, W(b_\ell)\}$, where $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_\ell\}$ are the two partite sets of $K_{k,\ell}$. Because both k and ℓ are at least 2, the set $W(a_1) \cup W(a_2) \cup \ldots \cup W(a_k)$ forms a disconnected cut of G. ☐

We now describe the different frameworks related to the equivalent statements in Proposition 1. As we shall see in all these problem settings the Disconnected Cut problem pops up as a missing case (often *the* missing case).

1. Cut sets. In the literature, various kinds of cut sets have been studied. For instance, a cut U of a graph $G = (V, E)$ is called a *k-clique cut* if $G[U]$ has a spanning subgraph consisting of k complete graphs [4, 21]; a *strict k-clique cut* if $G[U]$ consists of k components that are complete graphs [21]; a *stable cut* if U is an independent set [2, 16]; and a *matching cut* if $E_{G[U]}$ is a matching [5]. The problem that asks whether a graph has a k-clique cut is solvable in polynomial time for $k = 1$ [21] and $k = 2$ [4]. The latter paper also shows that deciding if a graph has a strict 2-clique cut can be solved in polynomial time. On the other hand, the problems that asks whether a graph has a stable cut [2] or a matching cut [5], respectively, are NP-complete. Recently, the problem that asks if a graph has a stable cut of size *at most* k has been shown to be in FPT [16].

2. H-partitions. Dantas et al. [7] proved that the problem of asking if a graph allows an H-partition can be solved in polynomial time for all fixed model graphs H with at most four vertices, except for $H = 2K_2$ or $H = 2S_2$. From statements (3) and (6) of Proposition 1, it is clear that these two cases correspond exactly to the Disconnected Cut problem. We note that the "list" version of the $2S_2$-Partition problem (and consequently the $2K_2$-Partition problem) is NP-complete. This follows directly from the result of Hell and Feder [10] who show that the problem of deciding whether a graph G retracts $\mathcal{C}_4$ is NP-complete. Here, a homomorphism f from G to an induced subgraph H of G is called a *retraction* if $f(h) = h$ for all $h \in V_H$, and in that case we say that G *retracts* to H. A variant on H-partitions that allows empty blocks V_i in an H-partition is studied by Feder et al. [11], and Cameron et al. [4] consider the list version of this variant.

3. Compactions. We note that any edge-surjective homomorphism from a graph G to a connected graph H is vertex-surjective (whereas the reverse is not necessarily true). Vikas [20] showed that the $\mathcal{C}_4$-Compaction problem, that asks if there exists a compaction from a graph G to $\mathcal{C}_4$, is NP-complete. By a modification of his proof, one can easily show that the $\mathcal{C}_4$-Compaction problem stays NP-complete for graphs of diameter three. As shown in statement (7) of Proposition 1, only for graphs of diameter two, the $\mathcal{C}_4$-Compaction problem is equivalent to the Disconnected Cut problem.

4. Contractibility. The H-Contractibility problem asks if a graph G is H-contractible. From the proof of Proposition 1, it follows that a graph with diameter two has a (k,ℓ)-cut if and only if it is $K_{k,\ell}$-contractible for two integers k and ℓ. Therefore, the (k,ℓ)-Cut problem is equivalent to the $K_{k,\ell}$-Contractibility problem for connected graphs. Brouwer and Veldman [3] show that $K_{k,\ell}$-Contractibility is solvable in polynomial time for $k = 1$, whereas it is NP-complete for each pair $k, \ell \geq 2$. The gadget in their construction has diameter two.[5] Hence, we obtain the following result.

Proposition 2. *The (k,ℓ)-Cut problem is in P for $k = 1$, and is NP-complete for each pair $k, \ell \geq 2$ even for graphs with diameter two.*

5. Vertex-covers. The problem of deciding if a graph has a spanning subgraph that consists of at most k mutually vertex-disjoint bicliques is called the k-Biclique Vertex-Cover problem. This problem is solvable in polynomial time if $k = 1$, and it is NP-complete if $k \geq 3$ [12]. The missing case is $k = 2$. Statement (5) of Proposition 1 shows that this case is equivalent to the Disconnected Cut problem. Due to Proposition 2 one can easily obtain the following.

Corollary 1. *The problem of deciding if a graph has a spanning subgraph consisting of two vertex-disjoint graphs, one of which is complete k-partite and the other one is complete ℓ-partite, is NP-complete for each pair $k, \ell \geq 2$.*

4 Cuts with a prespecified number of components

We determine the computational complexity of the k-Cut problem for any fixed $k \geq 1$.

Theorem 2. *The 1-Cut problem is solvable in polynomial time.*

Proof. Let $P_3 = p_1p_2p_3$ be the path on three vertices. We claim that a connected graph G has a connected cut (namely, a 1-cut) if and only if G is P_3-contractible. It is known that

[5] We show this in Appendix A.

a connected graph G is P_3-contractible if and only if G is neither a complete graph nor a cycle [3]. Therefore, the 1-Cut problem is solvable in polynomial time.

We now prove the claim. Suppose that G is P_3-contractible with a P_3-witness structure $\mathcal{W}$. Then, $U = W(p_2)$ is a connected cut of G. To prove the reverse implication, suppose that G has a connected cut U. Then, $G[V \backslash U]$ contains at least two components, and we arbitrarily choose two components $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ in $G[V \backslash U]$. We define three pairwise disjoint vertex-sets, as follows: $W(p_1) = V_1$, $W(p_2) = V \backslash (V_1 \cup V_2)$ and $W(p_3) = V_2$. Because $U \subseteq V \backslash (V_1 \cup V_2)$, we find that $G[V \backslash (V_1 \cup V_2)]$ is connected. Thus, $\mathcal{W} = \{W(p_1), W(p_2), W(p_3)\}$ forms a P_3-witness structure for G, and hence G is P_3-contractible. This completes the proof of Theorem 2. □

In contrast to the result in Theorem 2, the k-Cut problem becomes NP-complete for each $k \geq 2$, as we show in the following theorem. Note that Proposition 2 does *not* imply this theorem.

Theorem 3. *The k-Cut problem is NP-complete for each $k \geq 2$ even for graphs of diameter two.*

Proof. We first prove that the problem is NP-complete for $k = 2$ and then show that the proof for $k = 2$ can easily be modified to a proof for each $k \geq 3$. Clearly, this problem is in NP. Below we give a polynomial-time reduction from the problem Set Splitting.

Let $Q = \{q_1, q_2, \ldots, q_m\}$ be a set of m elements, and let $\mathcal{S}' = \{S_1, S_2, \ldots, S_{n'}\}$ be a collection of subsets $S_i \subseteq Q$. The Set Splitting problem has input $(Q, \mathcal{S}')$ and is to decide whether Q can be partitioned into two subsets Q_1 and Q_2 such that $Q_1 \cap S_i \neq \emptyset$ and $Q_2 \cap S_i \neq \emptyset$ for each i, $1 \leq i \leq n'$. This problem, also known as the Hypergraph 2-Colorability problem, is NP-complete (cf. [14]). We may assume without loss of generality that $S_i \neq \emptyset$ and $n' \geq 2$.

From a given instance $(Q, \mathcal{S}')$ of Set Splitting we construct an equivalent instance as follows. We do not modify Q but to $\mathcal{S}'$ we add a copy of each subset $S_i \in \mathcal{S}'$, and we also add the set $S_0 = Q$. This yields the *doubled* collection $\mathcal{S}$, which consists of $2n' + 1$ subsets. We call the instance $(Q, \mathcal{S})$ the *doubled* instance of $(Q, \mathcal{S})$. Clearly, solving the doubled instance is equivalent to solving the original instance of Set Splitting. Therefore, we consider only doubled instances in the following and simply write the doubled collection as $\mathcal{S} = \{S_0, S_1, \ldots, S_n\}$, where $S_0 = Q$ and $n = 2n' + 1 \geq 5$.

Reduction for $\mathbf{k = 2}$. We give a polynomial-time reduction from Set Splitting to 2-Cut. Let $(Q, \mathcal{S})$ be a doubled instance of Set Splitting with $Q = \{q_1, q_2, \ldots, q_m\}$ and $\mathcal{S} = \{S_0, S_1, \ldots, S_n\}$ for some $n \geq 5$. From this instance we construct a graph $G = (V, E)$, which will be our corresponding instance of 2-Cut, by performing the following steps (also see Fig. 2).

1. Regard each element $q_i \in Q$ and each subset $S_j \in \mathcal{S}$ as a vertex of G. Add edges to vertices in Q such that Q becomes a clique (whereas $\mathcal{S}$ stays an independent set).
2. For $i = 1, \ldots, m$ and $j = 1, \ldots, n$, add an edge between $q_i \in Q$ and $S_j \in \mathcal{S}$ if and only if $q_i \in S_j$.
3. Make a copy of the graph $G[Q \cup \mathcal{S}]$, where $Q' = \{q'_1, q'_2, \ldots, q'_m\}$ denotes the copy of Q such that q'_i corresponds to q_i for $i = 1, \ldots, m$, and $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$ denotes the copy of $\mathcal{S}$, such that T_j corresponds to S_j for $j = 1, \ldots, n$.

4. For $i = 1, \ldots, m$, add an edge between $q_i \in Q$ and its copy $q'_i \in Q'$.
5. Add 2 new independent vertices u_1 and u_2, and make each of them adjacent to each $S_j \in \mathcal{S}$ and to each $T_j \in \mathcal{T}$.
6. Add a new vertex v and make v adjacent to all vertices in G except u_1 and u_2.

We note that G has diameter two. This can be seen as follows. Firstly, v is adjacent to every vertex in $V \backslash \{u_1, u_2\}$, and v is of distance two from u_1 and u_2 via any vertex in $\mathcal{S} \cup \mathcal{T}$. Secondly, due to vertex v, all vertices in $V \backslash \{u_1, u_2, v\}$ are of distance 2 from each other. Thirdly, u_1 and u_2 are adjacent to every vertex in $\mathcal{S} \cup \mathcal{T}$, they are of distance two from each other and from every vertex in $Q \cup \{v\}$ via S_0, and they are of distance two from every vertex in Q via T_0.

We claim that Q has a desired partition (Q_1, Q_2) if and only if G has a 2-cut.

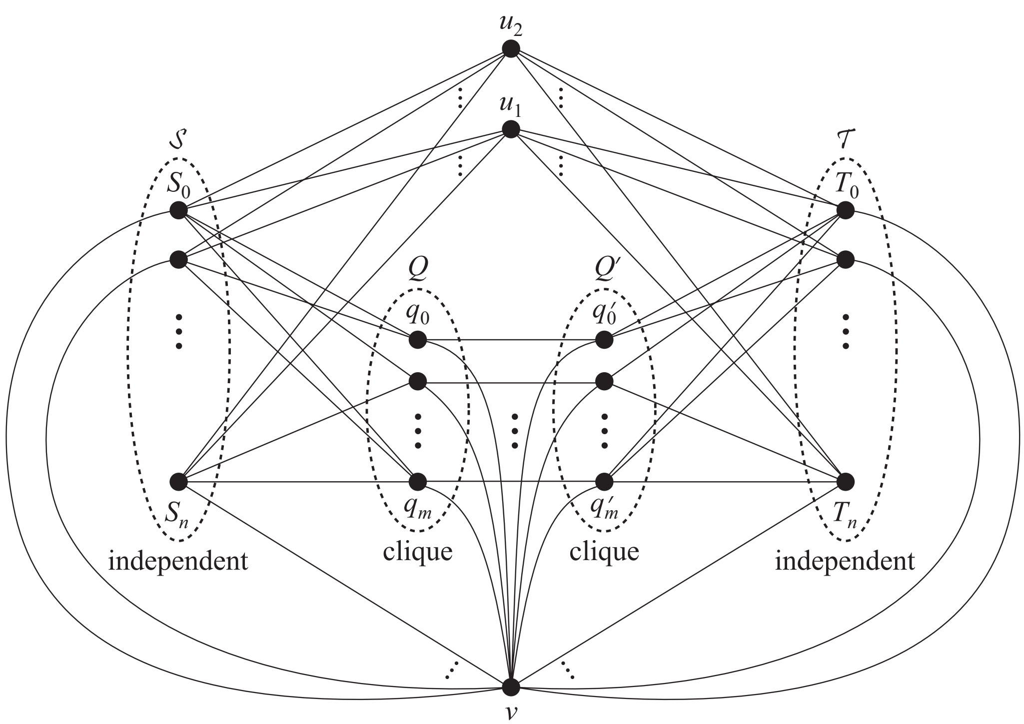

Fig. 2. The graph G that is obtained from an instance $(Q, \mathcal{S})$ of Set Splitting.

Necessity. Suppose that Q has a desired partition (Q_1, Q_2). Then, $Q_1 \cup Q'_2 \cup \mathcal{S} \cup \mathcal{T}$ forms a 2-cut of G, where Q'_2 is the set of copies of Q_2 in Q'.

Sufficiency. Suppose that G has a 2-cut X. Let X_1 and X_2 induce the two components of $G[X]$. Let $Y = V \setminus X$, and let $Y_1, Y_2, \ldots, Y_p$ induce the p (≥ 2) components of $G[Y]$. We will prove the following two claims (A) and (B):

Claim (A) $v \in Y_1$ and $u_1, u_2 \in Y \setminus Y_1$.

Claim (B) $\mathcal{S} \subset X_1$ and $\mathcal{T} \subset X_2$.

Before proving Claim (A) and (B), we first show that they imply the sufficiency. Let $Q_1 = Q \cap X_1$. By Claim (A) the three vertices v, u_1 and u_2 are not in X_1. Recall that $\mathcal{S}$ forms an independent set. Then, since $G[X_1]$ is connected and $\mathcal{S} \subset X_1$ by Claim (B), set X_1 must contain vertices in Q. Therefore, $Q_1 = Q \cap X_1 \neq \emptyset$ and every vertex in $\mathcal{S}$ is adjacent to at least one vertex in Q_1. This implies that each subset S_j in the given collection $\mathcal{S}$ satisfies $S_j \cap Q_1 \neq \emptyset$. Similarly, let $Q'_2 = Q' \cap X_2$. Then $Q'_2 \neq \emptyset$, and every vertex in $\mathcal{T}$ is adjacent to at least one vertex in Q'_2. Since $G[X_1]$ and $G[X_2]$ are different components in $G[X]$, there is no edge between Q_1 and Q'_2. Let $Q_2 = Q \setminus Q_1$. Then each vertex q'_i in Q'_2 has its corresponding vertex q_i in Q_2. Since $G[Q' \cup \mathcal{T}]$ is the copy of $G[Q \cup \mathcal{S}]$, every vertex in $\mathcal{S}$ is adjacent to at least one vertex in Q_2. This implies that each $S_j \in \mathcal{S}$ also satisfies $S_j \cap Q_2 \neq \emptyset$. Therefore, (Q_1, Q_2) forms a desired partition of Q.

Proof of Claim (A). We first show that $v \in Y_1$. Suppose for a contradiction that $v \in X$, say $v \in X_1$. Since v is adjacent to all vertices in G except u_1 and u_2, the set X_2 only contains vertices in $\{u_1, u_2\}$; otherwise $G[X]$ is connected. Then, there are the following three cases to consider:

(i) $X_2 = \{u_1, u_2\}$;
(ii) $X_2 = \{u_1\}$ and $u_2 \in X_1$;
(iii) $X_2 = \{u_1\}$ and $u_2 \in Y_j$ for some j, $1 \leq j \leq p$.

We first consider Case (i). Since u_1 and u_2 are not adjacent, $G[X_2]$ is not connected, a contradiction.

We then consider Case (ii). Since v and u_2 are not adjacent, X_1 must contain a neighbor of u_2. However, any such neighbor is adjacent to both v and u_1 as well. Then, the three vertices v, u_1 and u_2 are contained in the same component in $G[X]$, that is, $G[X_1 \cup X_2]$ is connected, a contradiction.

Finally, we consider Case (iii). Assume without loss of generality that $u_2 \in Y_2$. Because all vertices in $\mathcal{S} \cup \mathcal{T}$ are common neighbors of v and u_1, they do not belong to X; otherwise $G[X]$ would be connected. Because all vertices in $\mathcal{S} \cup \mathcal{T}$ are neighbors of u_2, they do not belong to $Y \setminus Y_2$ either. Hence, $\mathcal{S} \cup \mathcal{T}$ is a subset of Y_2. Because S_0 ($\in \mathcal{S} \subset Y_2$) is adjacent to all vertices in Q and T_0 ($\in \mathcal{T} \subset Y_2$) is adjacent to all vertices in Q', we obtain that $G[V \setminus X]$ ($= G[Y]$) is connected, a contradiction. Therefore, we have $v \notin X$, say $v \in Y_1$.

We now show that $u_1, u_2 \in Y \setminus Y_1$. Since $v \in Y_1$ and v is adjacent to all vertices in G except u_1 and u_2, we find that $Y \setminus Y_1$ must contain at least one of u_1 and u_2; otherwise $G[Y]$ would be connected. Assume without loss of generality that $u_1 \in Y \setminus Y_1$. Suppose for a contradiction that $u_2 \notin Y \setminus Y_1$. Then, there are the following two cases to consider:

(i) $u_2 \in Y_1$;
(ii) $u_2 \in X$.

We first consider Case (i). Since v and u_2 are not adjacent, Y_1 must contain a neighbor of u_2. However, any such neighbor is adjacent to both v and u_1 as well. Then, the three vertices v, u_1 and u_2 are contained in the same component in $G[Y]$. This contradicts that $u_1 \in Y \setminus Y_1$.

We now consider Case (ii). Because all vertices in $\mathcal{S} \cup \mathcal{T}$ are common neighbors of the three vertices v and u_1, they must all belong to X; otherwise v and u_1 are contained in the same component in $G[Y]$. Because S_0 ($\in \mathcal{S} \subset X$) is adjacent to all vertices in Q and T_0 ($\in \mathcal{T} \subset X$) is adjacent to all vertices in Q', we obtain that $G[X]$ is connected, a contradiction. This completes the proof of Claim (A).

Proof of Claim (B). We first show that either $\mathcal{S} \subset X_1$ or $\mathcal{S} \subset X_2$. Recall that all vertices in $\mathcal{S}$ are common neighbors of v and u_1. Because $v \in Y_1$ and $u_1 \in Y \backslash Y$ due to Claim (A), we then obtain that all vertices of $\mathcal{S}$ belong to $X(= X_1 \cup X_2)$. Suppose for a contradiction that $\mathcal{S} \cap X_1 \neq \emptyset$ and $\mathcal{S} \cap X_2 \neq \emptyset$. Since $|\mathcal{S}| = n \geq 5$, we may assume without loss of generality that $|\mathcal{S} \cap X_1| \geq 3$.

Recall that $G[\mathcal{S}]$ forms an independent set, and that, by Claim (A) $v, u_1, u_2 \in Y$. Then, X_1 must contain a vertex $q_i \in Q$; otherwise $G[X_1]$ would not be connected. This means that X_2 does not contain any vertex from Q, since such a vertex would be adjacent to q_i. Therefore, we have $X_2 = \{S_j\}$ for some $S_j \in \mathcal{S}$, and consequently, $\mathcal{S}\backslash\{S_j\} \subset X_1$. Because X_1 and X_2 are not adjacent, S_j is not adjacent to any vertex in $Q \cap X_1$. Because $q_i \in X_1$ and q_iS_0 is an edge, we then find that $S_j \neq S_0$. Because we are given a doubled instance of Set Splitting and $S_j \neq S_0$, we find that $\mathcal{S} \setminus \{S_j\}$ contains a vertex, namely S'_j, that is adjacent to exactly the same vertices in Q as S_j. Since $S'_j \in X_1$, $|\mathcal{S} \cap X_1| \geq 3$ and $G[X_1]$ is connected, we observe that X_1 must contain a vertex $q_h \in Q$ adjacent to S'_j. However, then q_h is adjacent to S_j as well. This means that $G[X_1]$ and $G[X_2]$ are not separate components in $G[X]$, a contradiction. Hence, we may assume without loss of generality that $\mathcal{S} \subset X_1$.

By similar arguments as above, we find that $\mathcal{T} \subset X_1$ or $\mathcal{T} \subset X_2$. We show that $\mathcal{T} \subset X_2$ as follows. Suppose for a contradiction that $\mathcal{T} \subset X_1$, and hence $\mathcal{S} \cup \mathcal{T} \subset X_1$. By Claim (A), we have $v, u_1, u_2 \in Y$. Since S_0 is adjacent to all vertices in Q and T_0 is adjacent to all vertices in Q', we then obtain that $G[X]$ is connected, a contradiction. This completes the proof of Claim (B).

Reduction for $\mathbf{k \geq 3}$. We slightly modify the construction for $k = 2$. Firstly, we construct the corresponding graph $G = (V, E)$ for the case $k = 2$. Secondly, we add $k - 2$ mutually nonadjacent vertices $w_1, \ldots, w_{k-2}$; each adjacent to each u_i and to v. Let G^+ be the resulting graph, then G^+ still has diameter two. It suffices to show that G^+ has a k-cut if and only if G has a 2-cut.

Necessity. Suppose that G^+ has a k-cut X, and let $Y = V(G^+)\backslash X$. By the same arguments as in the case $k = 2$, we find that $v, u_1, u_2 \in Y$. Then, since $w_1, w_2, \ldots, w_{k-2}$ are common neighbors of v, u_1 and u_2, we observe that each vertex w_i is in X. Furthermore, $\{w_i\}$ forms a component of $G^+[X]$. Hence, $X \setminus \{w_1, \ldots, w_{k-2}\}$ is a 2-cut of G.

Sufficiency. Suppose that G has a 2-cut X. By Claim (A), v, u_1, u_2 are not in X. Consequently, $X \cup \{w_1, \ldots, w_{k-2}\}$ is a k-cut of G^+. □

5 Graphs on surfaces

In this section, we focus on graphs embeddable on surfaces. For more information on graphs on surfaces (and for definitions not provided here), we refer the reader to Mohar and Thomassen [17].

We prove that for every fixed integer $g \geq 0$, in the class of graphs of Euler genus at most g, (k,ℓ)-Cut is fixed parameter tractable in $k+\ell$ and k-Cut is fixed parameter tractable in k. We also show that Disconnected Cut is solvable in polynomial time for minor-closed classes of graphs excluding some apex.

Our strategy is to apply a "win-win" approach. From Theorem 4 below, a graph embedded on some surface either has small treewidth or a large square grid as a surface minor. If the treewidth is small, we show that any of the three problems can be solved in polynomial time. If the graph has a large square grid, we show that it *always* has a (k,l)-cut (k-cut, or disconnected cut, respectively). We discuss each case separately, and refer to Courcelle [6] for the definitions of tree decomposition and treewidth, as we do not need them here. We do need the definition of a square grid and some related definitions, and we first state this terminology below.

An $r \times r$ square grid Γ is the graph with vertex set $V(G) = \{(i,j) \,|\, i,j = 0,\ldots,r-1\}$ and two vertices (i,j) and (i',j') of Γ are adjacent if and only if $i = i'$ and $|j-j'| = 1$, or else $|i-i'| = 1$ and $j = j'$.

Let G be a graph embedded on a surface. A *surface contraction* of an edge e of G is the operation of homeomorphically mapping the endpoints of e in G to a single vertex without any edge crossings and removing parallel edges and the loop. A *surface minor* of a graph G is a graph that can be obtained from G by a sequence of vertex and edge deletions and surface contractions.

Theorem 4 (Lemma 4 in [13]). *Let G be a graph embedded in a surface of Euler genus g. If $tw(G) \geq 12r(g+1)$, then G has the $r \times r$ square grid as a surface minor.*

A seminal result of Courcelle [6] is that in any class of graphs of bounded treewidth, every problem definable in monadic second-order logic can be solved in time *linear* in the number of vertices of the graph. We refer to Courcelle [6] for more details. For our purposes, we need the following proposition, the proof of which is straightforward.[6]

Proposition 3. *Let $k,\ell \geq 1$ be two fixed integers. Then the (k,ℓ)-Cut and the k-Cut problem can be defined in monadic second order logic.*

We also need the following two lemmas.

Lemma 1. *Let $r \geq 1$ be an integer. Every square grid Γ of size $3\cdot\lceil\sqrt{r}\rceil \times 3\cdot\lceil\sqrt{r}\rceil$ contains an independent set S of size at least r such that the graph $\Gamma[V(\Gamma)\setminus S]$ is connected.*

Proof. Let $S = \{(3i+1, 3j+1) \mid i,j = 0,\ldots,\lceil\sqrt{r}\rceil - 1\}$. It is easy to check that S is independent and that $\Gamma[V(\Gamma)\setminus S]$ is connected. □

Lemma 2. *Let $k,\ell \geq 1$ and $g \geq 0$ be three fixed integers. There exists a constant t such that each connected graph G of Euler genus at most g with $tw(G) \geq t$ has a (k,ℓ)-cut.*

Proof. Let $h = 3\cdot(\lceil\sqrt{\ell}\rceil + \lceil\sqrt{k}\rceil)$ and $t = 12h(g+1)$. By Theorem 4, if $\mathrm{tw}(G) \geq t$, then G contains an $h \times h$ square grid Γ as a surface minor. Let P be the unique facial cycle of Γ that is longer than 4 (or just the only cycle in case Γ is a 1×1 grid).

This means that there exists a function f that maps the vertices of a subgraph of G to the vertices of Γ in such a way that the preimage of every vertex of Γ is connected and

[6] See Appendix B for a proof.

when two vertices v, w of Γ are adjacent, there is a vertex in the preimage of v adjacent to a vertex in the preimage of w. Furthermore, the preimage in G of every connected subgraph of Γ is connected. In consequence, if there exists a (k,ℓ)-cut in Γ such that P belongs to one of the connected components of that cut, then the preimage of that (k,ℓ)-cut in Γ is a (k,ℓ)-cut in G. Hence, we are left to show that Γ has such an (k,ℓ)-cut.

There exists a cycle Q in Γ such that $\Gamma \backslash Q$ contains exactly two connected components: R containing the cycle P of Γ and T. Note that Q can be chosen in such a way that R contains a square grid of size at least $3 \cdot \lceil \sqrt{\ell-1} \rceil + 1$ and T contains a square grid of size at least $3 \cdot \lceil \sqrt{k-1} \rceil + 1$. From Lemma 1, R contains an independent set S_R of size $\ell - 1$ and T contains an independent set S_T of size $k-1$ such that $R \setminus S_R$ and $T \setminus S_T$ are connected. This gives a (k,ℓ)-cut in Γ with the components of $S_R \cup (T \setminus S_T) \cup Q$ on one side, and the components of $S_T \cup (R \setminus S_R)$ on the other side of the cut. This completes the proof of the lemma. □

We are now ready to prove the main result of this section.

Theorem 5. *Let $g \geq 0$ be an integer. For the class of connected graphs of genus at most g, the following statements hold*:

(i) *The (k,ℓ)-*Cut *problem is fixed parameter tractable in $k + \ell$;*

(ii) *The k-*Cut *problem is fixed parameter tractable in k.*

Proof. Let t be the constant from Lemma 2 which guarantees that a graph of Euler genus at most g with $\mathrm{tw}(G) \geq t$ is a Yes-instance of the (k,ℓ)-Cut problem. We first check if $\mathrm{tw}(G) < t$. We can do so as recognizing such graphs is fixed parameter tractable in t [1]. So, if $\mathrm{tw}(G) \geq t$ we are done. Suppose $\mathrm{tw}(G) < t$. By Proposition 3, the (k,ℓ)-Cut problem is expressible in monadic second order logic and therefore solvable on graphs of bounded treewidth [6].

For (ii) we take $\ell = 2$ and repeat all arguments. If $\mathrm{tw}(G) \geq t$ then G has a $(k,2)$-cut, and hence a k-cut. If $\mathrm{tw}(G) < t$ then we apply Proposition 3 on the k-Cut problem. □

Disconnected Cut **on apex-minor-free graphs**

A graph H is called an *apex* if it has a vertex v such that $G - v$ is planar. Eppstein [9] proved that if $\mathcal{F}$ is a family of graphs closed under taking minors and $\mathcal{F}$ does not contain all apex graphs, then there is a function f such that for every $G \in \mathcal{F}$, the treewidth of G is at most $f(\mathrm{diam}(G))$.

Let H be an apex, and let $\mathcal{F}$ be a minor-closed graph class that does not contain H. We will show how to solve Disconnected Cut on $\mathcal{F}$. By Theorem 1 (i) we can restrict ourselves to graphs from $\mathcal{F}$ that have diameter 2. By Eppstein's result the treewidth of these graphs is bounded by a constant. By Proposition 3, Disconnected Cut is expressible in monadic second order logic, and therefore solvable on graphs of bounded treewidth [6]. Hence, we obtained the following result.

Theorem 6. *The* Disconnected Cut *problem is solvable in polynomial time on any minor-closed class of graphs that does not contain all apex graphs.*

6 Further results and related open problems

The main open problem is to determine the computational complexity of the Disconnected Cut problem (on input graphs of diameter 2). In this section we consider this problem and its two variants for chordal graphs and claw-free graphs (in particular line graphs).

6.1. Chordal graphs. A *chordal* graph is a graph with no induced cycles of length larger than three. We can solve Disconnected Cut for the class of chordal graphs, as is shown in the following proposition.

Proposition 4. *The* Disconnected Cut *problem is solvable in polynomial time for chordal graphs.*

Proof. Suppose G is a chordal graph. If G has diameter not two we can efficiently decide if G has a disconnected cut due to Theorem 1. Suppose G has diameter two. Then G does not have a disconnected cut. In order to see this, suppose G does have a disconnected cut. By Proposition 1, G then has a diagonal coloring c with color classes V_i with $c(V_i) = i$ for $i = 1, \ldots, 4$. Let D_i be a component of $G[V_i]$ for $i = 1, \ldots, 4$. Since G has diameter two, each D_i contains vertices x_i and y_i with possibly $x_i = y_i$ such that $x_i y_{i+1} \in E$ for $i = 1, \ldots, 4$ (where y_5 must be interpreted as y_1). We now take a shortest path P_i from y_i to x_i for $i = 1, \ldots, 4$. If we find a vertex $u_i \neq x_i$ on some P_i with a neighbor v_{i+1} on P_{i+1}, then we replace x_i by u_i and y_{i+1} by v_{i+1}. Hence, we may assume that such a vertex does not exist. In this way we obtain an induced cycle $C = y_1 \overrightarrow{P_1} x_1 y_2 \overrightarrow{P_2} x_2 y_3 \overrightarrow{P_3} x_3 y_4 \overrightarrow{P_4} x_1 y_1$ on at least four vertices. This is not possible. $\square$

Is k-Cut or even (k, ℓ)-Cut problem polynomially solvable for chordal graphs?

6.2. Claw-free graphs and line graphs. A *claw-free* graph is a graph that does not contain $K_{1,3}$ as an induced subgraph. The *line graph* $L(G)$ of a graph G with edges $e_1, \ldots, e_p$ is the graph $L(G)$ with vertices $u_1, \ldots, u_p$ such that there is an edge between any two vertices u_i and u_j if and only if e_i and e_j share one end-vertex in G. Note that a line graph is claw-free.

An interesting open problem is whether Disconnected Cut is solvable in polynomial time for the class of claw-free graphs, or even for its subclass of line graphs. Recall that every graph of diameter at least three always has a disconnected cut [12]. Hence, we may assume our input graphs have diameter two, and then the following two connections become relevant.

Proposition 5. *For claw-free graphs of diameter two, the problems* Disconnected Cut*,* $(2,2)$-Cut *and* C_4-Contractibility *are equivalent.*

Proof. Let $C_4 = c_1c_2c_3c_4c_1$. Let $G = (V, E)$ be a claw-free graph of diameter two. We claim that G has a disconnected cut if and only if G is C_4-contractible. Observe that by Proposition 1 a graph with diameter two has a $(2,2)$-cut if and only if it is $K_{2,2}$-contractible. Hence, after proving the above claim we are done.

Suppose G has a disconnected cut. By Proposition 1, G has a diagonal coloring c with color classes V_i such that $c(V_i) = i$ for $i = 1, \ldots, 4$. First assume $G[V_1 \cup V_3]$ consists of components $D_1, \ldots, D_p$ for some $p \geq 3$.

We may without loss of generality assume that $G[V_2]$ consists of one component, as otherwise we recolor its other components by color 4. We perform the following procedure as long as $|V_2| \geq 2$ and $G[V_1 \cup V_3]$ consists of more than two components. If $|V_2| \geq 2$, then we can always find a vertex $u \in V_2$ that is not a cut-vertex of $G[V_2]$. As G has diameter two, u must have at least one neighbor v in $V_1 \cup V_3$ in order to be of distance two to each vertex in V_4. Say without loss of generality that v is in D_1. As G is claw-free and $p \geq 3$, u cannot be adjacent to every component of $G[V_1 \cup V_3]$. Say without loss of generality that u is not adjacent to D_p. We then (re)color $D_1 \cup \cdots \cup D_{p-1} \cup \{u\}$ by 1, and D_p by 3. As $|V_2| \geq 2$ we then obtain a new diagonal coloring of G. Suppose that at some moment $V_2 = \{a\}$ for some $a \in V$ while $p \geq 3$ still holds. As G is claw-free, we find that a is not adjacent to some component D_i of $G[V_1 \cup V_3]$. This means that the distance from a to any vertex in D_i is at least three. As G has diameter two, this is not possible. Hence, we may assume that in the end $G[V_1 \cup V_3]$ consists of two components.

Also, note that the recoloring we applied in the above procedure does not change the number of components in $G[V_2 \cup V_4]$. Hence if we apply the same procedure for $G[V_2 \cup V_4]$ instead of for $G[V_1 \cup V_3]$, the number of components of $G[V_1 \cup V_3]$ stays two, while the number of components of $G[V_2 \cup V_4]$ gets down to two as well. This means that in the end we have found a $\mathcal{C}_4$-witness structure of G with witness sets $W(c_i) = V_i$ for $i = 1, \ldots, 4$.

To prove the reverse implication, suppose G is C_4-contractible with C_4-witness structure $\mathcal{W}$. Then $W(c_1) \cup W(c_3)$ is a disconnected cut of G. □

Proposition 6. *Let $L(G)$ be the line graph of a graph G with diameter two. If the diameter of $L(G)$ is two, then the following holds: G has a disconnected cut if and only if $L(G)$ has a disconnected cut.*

Proof. Let G be a graph with diameter two whose line graph $L(G)$ has diameter two as well. Throughout the proof interpret integer 5 as 1. Suppose G has a disconnected cut. By Proposition 1, G has a diagonal coloring c with color classes V_i such that $c(V_i) = i$ for $i = 1, \ldots, 4$. We color the edges of G as follows. Let $e = uv$ be an edge. If $c(u) = c(v) = i$ for some $1 \leq i \leq 4$, then we assign color i to e. In the other case, $c(u) = i$ and $c(v) = i+1$ for some $1 \leq i \leq 4$. Then we assign color $i+1$ to e. We claim that in this way we obtain a diagonal coloring c' of $L(G)$ and consequently a disconnected cut by Proposition 1. In order to see this, suppose $L(G)$ contains two adjacent vertices e_1 and e_2 with forbidden colors: either 1,3 or 2,4. Suppose without loss of generality that $c'(e_1) = 1$ and $c'(e_2) = 3$. Let $e_1 = uv$ and $e_2 = vw$. As $c'(e_1) = 1$, at least one of the two vertices u, v must have received color 1. If $c(v) = 1$, then we would have chosen $c'(e_2) = 1$ or $c'(e_2) = 4$. So, $c(v) = 1$ is not possible. If $c(v) \neq 1$, then $c(u) = 1$. As $c(e_1) = 1$, this means that $c(v) = 4$. However, also in that case, we would not have colored e_2 with color 3. Hence, such a conflict does not occur. We are left to prove that c' is vertex-surjective, i.e., for $1 \leq i \leq 4$, there must be at least one vertex in $L(G)$ with color i. By Proposition 1, G allows a compaction to $\mathcal{C}_4$. In other words, c is edge-surjective. Hence G contains at least one edge whose end-vertices are colored with i and $i+1$ for each $i = 1, \ldots, 4$. By its definition, c' is then vertex-surjective.

Now suppose $L(G)$ has a disconnected cut. By Proposition 1, $L(G)$ has a diagonal coloring c' with color classes $E_i \subset V_{L(G)}$ such that $c'(E_i) = i$ for $i = 1, \ldots, 4$. We color the vertices of G as follows. If v is only incident with edges of color i for some $1 \leq i \leq 4$, then we give v color i. In the other case, v is incident with edges of color i and color $i+1$

for some $1 \leq i \leq 4$. In that case we give v color $i+1$. In this way we obtain a coloring c of V. Suppose G contains two adjacent vertices u, v with forbidden colors 1,3 or 2,4, say $c(u) = 1$ and $c(v) = 3$. If u is only incident with edges colored 1 then v has at least one edge colored 1, namely edge uv. Then we would have assigned $c(v) = 1$ or $c(v) = 4$. Hence u is incident with edges colored 1 and 4. In particular, $c(uv) = 4$. In that case, we would have colored v with color $c(v) = 1$ or $c(v) = 4$. So, such a conflict does not happen. By Proposition 1, c' is edge-surjective. This means that $L(G)$ contains an edge e_1e_2 with $c'(e_1) = i$ and $c'(e_2) = i+1$ for each $i = 1, \ldots, 4$. Let $e_1 = uv$ and $e_2 = vw$. Then $c(v) = i$. So c is vertex-surjective. This completes the proof of Proposition 6. □

Note that Proposition 6 does not hold if G has diameter at least three: take $G = P_4 = p_1p_2p_3p_4$, which has diameter three and disconnected cut $\{p_1, p_3\}$. However, $L(G) = P_3$ does not have a disconnected cut. The condition that $L(G)$ has diameter two is necessary as well. Take two triangles and merge them in one vertex in order to obtain the graph $(\{u, v, w, x, y\}, \{uv, vw, wu, ux, xy, yu\})$. This graph has diameter two but no disconnected cut. However, its line graph is a K_4 on vertices e_1, e_2, e_3, e_4 extended with two vertices d_1, d_2 and edges d_1e_1, d_1e_2 and d_2e_3, d_2e_4. This graph has diameter three and disconnected cut $\{d_1, e_3, e_4\}$.

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Here are the proofs that we left out. Unless a referee prefers otherwise we will not add them to the paper.

A Theorem of Brouwer and Veldman

For the sake of completeness, here is the construction of Brouwer and Veldman [3] which shows that the $K_{k,\ell}$-Contractibility problem is NP-complete for each pair $k, \ell \geq 2$, already for graphs of diameter two. The latter fact has not be noted in [3] but it will be clear from the construction.

Theorem 7. *The $K_{k,\ell}$-Contractibility problem is NP-complete for each pair $k, \ell \geq 2$, already for the class of graphs of diameter two.*

Proof. Brouwer and Veldman [3] use a reduction from the Hypergraph 2-Coloring problem. Let $(Q, \mathcal{S})$ be a hypergraph with $Q = \{q_1, \ldots, q_m\}$ for some $m \geq 1$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$ for some $n \geq 1$. We may without loss of generality assume that $\emptyset \notin \mathcal{S}$ and $S_n = Q$. From the incidence graph I of $(Q, \mathcal{S})$ we construct the following graph. Let $\mathcal{S}'$ consist of a copy S' of each $S \in \mathcal{S}$: add an edge qS' if and only if qS is an edge. Add all possible edges between $\mathcal{S}$ and $\mathcal{S}'$. Also add all possible edges between vertices in Q. Take a new biclique $K_{k-1,l-1}$ with partition classes $A = \{a_1, \ldots, a_{k-1}\}$ and $B = \{b_1, \ldots, b_{\ell-1}\}$. Finally add an edge between each a_i and each S_i, and an edge between each b_i and each S'_i. Then Brouwer and Veldman [3] show that G is $K_{k,\ell}$-contractible if and only if $(Q, \mathcal{S})$ has a 2-coloring.

Then G is of diameter two. This can be seen as follows. Every $a_i \in A$ is adjacent to all vertices in $B \cup \mathcal{S}$ and is connected to every vertex in $(A \backslash \{a_i\}) \cup Q \cup \mathcal{S}'$ via S_n. Every $S_j \in \mathcal{S}$ is adjacent to every vertex in $A \cup \mathcal{S}'$ and of distance at most two from any vertex in $B \cup Q \cup \mathcal{S} \backslash \{S_j\}$ via S'_n. Every $q_i \in Q$ is adjacent to every vertex in $Q \backslash \{q_i\} \cup \{S_n, S'_n\}$ and of distance at most two from any vertex in $A \cup B \cup (\mathcal{S} \backslash \{S_n\}) \cup (\mathcal{S}' \backslash \{S'_n\})$ via S_n or S'_n. The other two cases follow by symmetry. $\square$

B Proof of Proposition 3

Proposition 3. *Let k, ℓ be two fixed integers. Then the (k, ℓ)-*Cut*, the k-*Cut *and* Disconnected Cut *problem can be defined in monadic second order logic.*

Proof. Let G have a (k, ℓ)-cut U. We can express this property in monadic second order logic as follows. Let $\{U_1, \ldots, U_k\}$ be a partition of U that induces k components $G[U_i]$ of $G[U]$. Let $\{V_1, \ldots, V_\ell\}$ be a partition of $V \backslash U$ that induces ℓ components $G[V_i]$ of $G[V \backslash U]$. Then we can formulate the following set of conditions:

1. Every vertex of V must be in exactly one set from $\{U_1, \ldots, U_k\} \cup \{V_1, \ldots, V_\ell\}$.
2. Every U_i and every V_j must be nonempty.

3a. Every two sets U_i and U_j must be nonadjacent
3b. Every two sets V_i and V_j must be nonadjacent.
4. Every $G[U_i]$ and every $G[V_j]$ must be connected.

We first formulate the sentence "Every $x \in V$ must be in one of the sets $U_1, \ldots, U_k$, $V_1, \ldots, V_\ell$" as

$$\phi_0 = \forall x(U_1(x) \vee \ldots \vee U_k(x) \vee V_1(x) \vee \ldots \vee V_\ell(x)).$$

We then formulate the sentence "If $x \in V$ is in a set U_h it cannot be in a set U_i for $i \neq h$ or a set V_j" as $\phi^i_{1a}(x) =$

$$U_i(x) \rightarrow \neg(U_1(x) \vee \ldots \vee U_{i-1}(x) \vee U_{i+1}(x) \vee \ldots \vee U_k(x) \vee V_1(x) \vee \ldots \vee V_\ell(x)).$$

A similar sentence $\phi^j_{1b}(x)$ is formulated with respect to the sets V_j. Then Condition 1 can be formulated as

$$\phi_1 = \phi_0 \wedge \forall x(\phi^1_{1a}(x) \wedge \ldots \wedge \phi^k_{1a}(x) \wedge \phi^1_{1b}(x) \wedge \ldots \wedge \phi^\ell_{1b}(x)).$$

Condition 2 can be formulated as

$$\phi_2 = \exists x U_1(x) \wedge \ldots \wedge \exists x U_k(x) \wedge \exists x V_1(x) \wedge \ldots \wedge \exists x V_\ell(x).$$

We say that $E(x, y)$ holds if xy is an edge in G. Then Condition 3a for a single U_i can be formulated as

$$\phi^i_{3a} = \forall x \forall y((U_i(x) \wedge (U_1(y) \vee \ldots \vee U_{i-1}(y) \vee U_{i+1}(y) \vee \ldots \vee U_k(y)) \rightarrow \neg E(x, y)).$$

A similar formula ϕ^j_{3b} can be constructed with respect to Condition 3b. This leads to the formula

$$\phi_3 = \phi^1_{3a} \wedge \ldots \wedge \phi^k_{3a} \wedge \phi^1_{3b} \wedge \ldots \wedge \phi^k_{3b}.$$

Finally Condition 4 can be formulated as follows. We first consider a set U_i. We write $U^Z_i(x) = U_i(x) \wedge Z(x)$ and $U^{\neg Z}_i(x) = U_i(x) \wedge \neg Z(x)$. Then for U_i we take the negation of the formula $\phi^i_{4a} =$

$$\exists Z(\exists x U^Z_i(x) \wedge \exists x U^{\neg Z}_i(x) \wedge (\forall x \forall y((U^Z_1(x) \wedge U^{\neg Z}_1(y)) \rightarrow \neg E(x, y))).$$

as this formula is true if and only if $G[U_i]$ is disconnected. For each V_j we can construct a formula ϕ^j_{4b} analogously, and we take its negation. This leads to the following formula that expresses Condition 4:

$$\phi_4 = \neg\phi^1_{4a} \wedge \ldots \wedge \neg\phi^k_{4a} \wedge \neg\phi^1_{4b} \wedge \ldots \wedge \neg\phi^\ell_{4b}.$$

Then G has a (k, ℓ)-cut if and only if $\phi = \exists U_1 \ldots \exists U_k \exists V_1 \ldots \exists V_\ell(\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4)$ is true.

Expressing that G has a k-cut U comes down to encoding condition 1-4 with respect to U only. We can express the property that G has a disconnected cut in monadic second order logic as follows. By Proposition 1, G has a disconnected cut if and only if G has a diagonal coloring c with color classes V_i such that $c(V_i) = i$ for $i = 1, \ldots, 4$. By definition, we then obtain the following conditions.

1. Every vertex of V must be in exactly one set $\{V_1, \ldots, V_4\}$.
2. Every V_i must be nonempty.
3a. Sets V_1 and V_3 must be nonadjacent.
3b. Sets V_2 and V_4 must be nonadjacent.

As shown above, we can easily express such conditions in monadic second order logic. This finishes the proof of Proposition 3. □