Squares of Low Maximum Degree [*]

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Abstract. A graph H is a square root of a graph G if G can be obtained from H by adding an edge between any two vertices in H that are of distance 2. The SQUARE ROOT problem is that of deciding whether a given graph admits a square root. This problem is only known to be NP-complete for chordal graphs and polynomial-time solvable for non-trivial minor-closed graph classes and a very limited number of other graph classes. We prove that SQUARE ROOT is $O(n)$-time solvable for graphs of maximum degree 5 and $O(n^4)$-time solvable for graphs of maximum degree at most 6.

1 Introduction

The *square* H^2 of a graph $H = (V_H, E_H)$ is the graph with vertex set V_H, such that any two distinct vertices $u, v \in V_H$ are adjacent in H^2 if and only if u and v are of distance at most 2 in H. In this paper we study the reverse concept: a graph H is a *square root* of a graph G if $G = H^2$. There exist graphs with no square root (such as graphs with a cut-vertex), graphs with a unique square root (such as squares of cycles of length at least 7) as well as graphs with more than one square root (such as complete graphs).

In 1967 Mukhopadhyay [21] characterised the class of connected graphs with a square root. However, in 1994, Motwani and Sudan [20] showed that the decision problem SQUARE ROOT, which asks whether a given graph admits a square root, is NP-complete. As such, it is natural to restrict the input to special graph classes in order to obtain polynomial-time results. For several well-known graph classes the complexity of SQUARE ROOT is still unknown. For example, Milanic and Schaudt [19] posed the complexity of SQUARE ROOT restricted to split graphs and cographs as open problems. In Table 1 we survey the known results.

The last two rows in Table 1 correspond to the results in this paper. More specifically, we prove in Section 3 that SQUARE ROOT is linear-time solvable for graphs of maximum degree at most 5 via a reduction to graphs of bounded treewidth and in Section 4 that SQUARE ROOT is $O(n^4)$-time solvable for graphs of maximum degree at most 6 via a reduction to graphs of bounded size.

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graph class $\mathcal{G}$	complexity
planar graphs [17]	linear
non-trivial and minor-closed [22]	linear
K_4-free graphs [10]	linear
(K_r, P_t)-free graphs [10]	linear
3-degenerate graphs [10]	linear
graphs of maximum degree ≤ 5	linear
graphs of maximum degree ≤ 6	polynomial
graphs of maximum average degree $< \frac{46}{11}$ [9]	polynomial
line graphs [18]	polynomial
trivially perfect graphs [19]	polynomial
threshold graphs [19]	polynomial
chordal graphs [12]	NP-complete

Table 1. The known results for SQUARE ROOT restricted to some special graph class $\mathcal{G}$. Note that the row for planar graphs is absorbed by the row below. The two unreferenced results are the results of this paper.

Also the $\mathcal{H}$-SQUARE ROOT problem, which is that of testing whether a given graph has a square root that belongs to some specified graph class $\mathcal{H}$, has also been well studied. We refer to Table 2 for a survey of the known results on $\mathcal{H}$-SQUARE ROOT.

graph class $\mathcal{H}$	complexity
trees [17]	polynomial
proper interval graphs [12]	polynomial
bipartite graphs [11]	polynomial
block graphs [15]	polynomial
strongly chordal split graphs [16]	polynomial
ptolemaic graphs [13]	polynomial
3-sun-free split graphs [13]	polynomial
cactus graphs [8]	polynomial
graphs with girth at least g for any fixed $g \geq 6$ [7]	polynomial
graphs of girth at least 5 [6]	NP-complete
graphs of girth at least 4 [7]	NP-complete
split graphs [12]	NP-complete
chordal graphs [12]	NP-complete

Table 2. The known results for $\mathcal{H}$-SQUARE ROOT restricted to various graph classes $\mathcal{H}$. The result for 3-sun-free split graphs has been extended to a number of other subclasses of split graphs in [14].

Finally both SQUARE ROOT and $\mathcal{H}$-SQUARE ROOT have been studied under the framework of parameterized complexity. The generalization of SQUARE ROOT that takes as input a graph G with two subsets R and B of edges that need to be included or excluded, respectively, in any solution (square root)[5] has a kernel of size $O(k)$ for graphs that can be made planar after removing at most k vertices [9]. The problems of testing whether a connected n-vertex graph with m edges has a square root with at most $n - 1 + k$ edges and whether such a graph has a square root with at least $m - k$ edges are both fixed-parameter tractable when parameterized by k [3].

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[5] We give a formal definition of this generalization in Section 4, as we need it for proving that SQUARE ROOT is $O(n^4)$-time solvable for graphs of maximum degree at most 6.

2 Preliminaries

We only consider finite undirected graphs without loops or multiple edges. We refer to the textbook by Diestel [5] for any undefined graph terminology.

Let G be a graph. We denote the vertex set of G by V_G and the edge set by E_G. The *length* of a path or a cycle is the number of edges of the path or cycle, respectively. The *distance* $\text{dist}_G(u, v)$ between a pair of vertices u and v of G is the number of edges of a shortest path between them. The diameter $\text{diam}(G)$ of G is the maximum distance between two vertices of G. The *neighbourhood* of a vertex $u \in V_G$ is defined as $N_G(u) = \{v \mid uv \in E_G\}$. The *degree* of a vertex $u \in V_G$ is defined as $d_G(u) = |N_G(u)|$. The *maximum degree* of G is $\Delta(G) = \max\{d_G(v) \mid v \in V_G\}$. A vertex of degree 1 and the (unique) edge incident to it are said to be a *pendant* vertex and *pendant* edge of G respectively. A vertex subset of G that consists of mutually adjacent vertices is called a *clique*.

A *tree decomposition* of a graph G is a pair (T, X) where T is a tree and $X = \{X_i \mid i \in V_T\}$ is a collection of subsets (called *bags*) of V_G such that the following three conditions hold:

i) $\bigcup_{i \in V_T} X_i = V_G$,
ii) for each edge $xy \in E_G$, $x, y \in X_i$ for some $i \in V_T$, and
iii) for each $x \in V_G$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V_T\}, T)$ is $\max_{i \in V_T}\{|X_i| - 1\}$. The *treewidth* $\mathbf{tw}(G)$ of a graph G is the minimum width over all tree decompositions of G. If T restricted to be a path, then we say that (X, T) is a *path decomposition* of a graph G. The *pathwidth* $\mathbf{pw}(G)$ of G is the minimum width over all path decompositions of G. A class of graphs $\mathcal{G}$ has *bounded* treewidth (pathwidth) if there exists a constant p such that the treewidth (pathwidth) of every graph from $\mathcal{G}$ is at most p.

3 Graphs of Maximum Degree at Most 5

In this section we prove that SQUARE ROOT can be solved in linear time for graphs of maximum degree at most 5 by showing that squares of maximum degree at most 5 have bounded pathwidth. The latter enables us to use the following two lemmas. The first lemma can either be proven via a dynamic programming algorithm or by a non-constructive proof based on Courcelle's theorem [4] (see [9] for details). The second lemma is due to Bodlaender.

Lemma 1. *The* SQUARE ROOT *problem can be solved in time $O(f(k)n)$ for n-vertex graphs of treewidth (or pathwidth) at most k.*

Lemma 2 ([1]). *For any fixed constant k, it is possible to decide in linear time whether the treewidth (or pathwidth) of a graph is at most k.*

We now prove the key result of this section.

Lemma 3. *If G is a graph with $\Delta(G) \leq 5$ that has a square root, then $\mathbf{pw}(G) \leq 27$.*

Proof. Without loss of generality we assume that G is connected; otherwise, we can consider the components of G separately. Let H be a square root of G. Let $u \in V_G$. In H we apply a breadth-first search (BFS) starting at u. This yields the levels $L_0, \ldots, L_s$

for some $s \geq 0$, where $L_i(u) = \{v \in V_G \mid \operatorname{dist}_G(u, v) = i\}$ for $i = 0, \ldots, s$. Note that $L_0 = \{u\}$ and that $L_0(u) \cup \cdots \cup L_s$ is a partition of $V_H = V_G$. Let T be the corresponding BFS-tree of H rooted in u. Note that T also defines a parent-child relation on the vertices of G.

We prove the following two claims.

Claim A. *Let $i \geq 2$. Then $x \in L_i$ implies that*

i) *x has at most three children in T, and*
ii) *for any $j \in \{i+1, \ldots, s-1\}$, x has at most four descendants in $L_j \cup L_{j+1}$.*
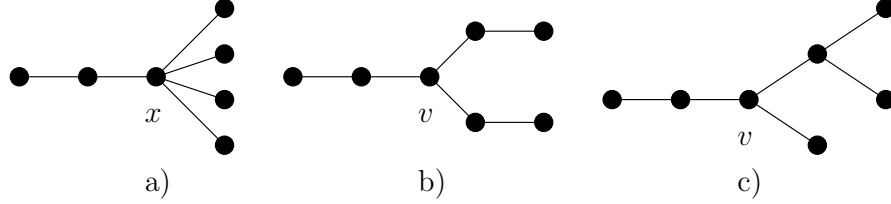
Fig. 1. Forbidden subgraphs for square roots of graphs of maximum degree at most 5.

We prove Claim A as follows. First we show i) by observing that if x had at least four children in T, then H contains the subgraph shown in Fig. 1 a) and, therefore, $d_G(x) \geq 6$, which is a contradiction.

We now prove ii). First assume that x has exactly one descendant $y \in L_j$. Then y has at most three children in L_{j+1} due to i), and hence the total number of descendants of x in $L_j \cup L_{j+1}$ is at most 4. Now assume that x has at least two descendants in L_j, say y_1 and y_2. Let v denote the lowest common ancestor of y_1, y_2 in T. Note that v is a descendant of x or $v = x$, so $v \in L_k$ for some $i \leq k \leq j - 1$. Suppose $k < j - 1$. Then H contains the subgraph shown in Fig. 1 b) and hence $d_G(v) \geq 6$, a contradiction. Hence, $v \in L_{j-1}$ and, moreover, we may assume without loss of generality that v is the parent in T of all the descendants of x in L_j (as otherwise there exists a vertex $v' \in L_{j-1}$ with a neighbour $y_3 \in L_j$, which means that the lowest common ancestor of v and v' has six neighbours in G).

By i), we find that v has at most three children. Hence, x has at most three descendants in L_j. To obtain a contradiction, assume that x has at least two descendants z_1, z_2 in L_{j+1}. If z_1, z_2 have distinct parents we again find that H contains the forbidden subgraph shown in Fig. 1 b). If z_1, z_2 have the same parent, H contains the subgraph shown in Fig. 1 c) and hence $d_G(v) \geq 6$, a contradiction. Hence, x has at most one descendant in L_{j+1}. We conclude that the total number of descendants of x in $L_j \cup L_{j+1}$ is at most 4. This completes the proof of ii). Consequently we have proven Claim A.

Claim B. $\mathbf{pw}(G) \leq \max\{|L_i \cup L_{i+1} \cup L_{i+2}| \mid 0 \leq i \leq s\} - 1$.

We prove Claim B as follows. Let P be the path on vertices $0, \ldots, s$ (in the order of the path). We set $X_i = L_i \cup L_{i+1} \cup L_{i+2}$ for all $i \in \{0, \ldots, s\}$ and define $X = \{X_1, \ldots, X_s\}$. We claim that (X, P) is a path decomposition of G. This can be seen as follows. Since the sets $L_0, \ldots, L_s$ form a partition of V_G, we find that $\bigcup_{i=0}^{s} X_i = V_G$. Moreover, for every edge $xy \in E_G$ with $x \in L_i$ and $y \in L_j$ we see that $|i - j| \leq 2$. Hence, for each

edge $xy \in E_G$, $x, y \in X_i$ for some $i \in \{0, \ldots, s\}$. Finally, if $x \in X_i \cap X_j$ such that $i + 1 < j$, then $i + 2 = j$ and $x \in L_{i+2} \subseteq X_{i+1}$. Therefore, the set $\{i \mid x \in X_i\}$ induces a subpath of P for each $x \in V_G$. It remains to observe that the width of (X, P) is $\max\{|L_i \cup L_{i+1} \cup L_{i+1}| \mid 0 \le i \le s\} - 1$. This completes the proof of Claim B.

Because $d_G(u) \le 5$, $|L_1 \cup L_2| \le 5$ and thus $|L_2| \le 4$ and $|L_0 \cup L_1 \cup L_2| \le 6$. By Claim A, each vertex of L_2 has at most three children in T and at most four descendants in $L_3 \cup L_4$. Hence, $|L_1 \cup L_2 \cup L_3| \le 17$ and $|L_2 \cup L_3 \cup L_4| \le 20$. For $j \in \{3, \ldots, s\}$, each vertex of L_2 has at most four descendants in $L_j \cup L_{j+1}$ and also at most four descendants in $L_{j+1} \cup L_{j+2}$ by Claim A ii). Therefore, each vertex of L_2 has at most seven descendants in $L_j \cup L_{j+1} \cup L_{j+2}$. As $|L_2| \le 4$, this means that $|L_j \cup L_{j+1} \cup L_{j+2}| \le 28$. We conclude that $|L_i \cup L_{i+1} \cup L_{i+2}| \le 28$ for all $i \in \{0, \ldots, s\}$. Consequently, Claim B implies that $\mathbf{pw}(G) \le 27$. $\qquad\square$

We are now ready to prove the main theorem of this section.

Theorem 1. SQUARE ROOT *can be solved in time* $O(n)$ *for n-vertex graphs of maximum degree at most* 5.

Proof. Let G be a graph with $\Delta(G) \le 5$. By Lemma 2 we can check in $O(n)$ time whether $\mathbf{pw}(G) \le 27$. If $\mathbf{pw}(G) > 27$, then G has no square root by Lemma 3. Otherwise, we solve SQUARE ROOT in $O(n)$ time by using Lemma 1. $\qquad\square$

Remark 1. The above approach cannot be extended to graphs of maximum degree at most 6. In order to see this, take a wall (see Figure 2) and subdivide each edge, that is, replace each edge uw by a path uvw where v is a new vertex. This gives us a graph H, such that H^2 has degree at most 6. A wall of height h has treewidth $\Omega(h)$ (see, for example, [5]). As subdividing an edge and adding edges does not decrease the treewidth of a graph, this means that the graph H^2 can have an arbitrarily large treewidth.
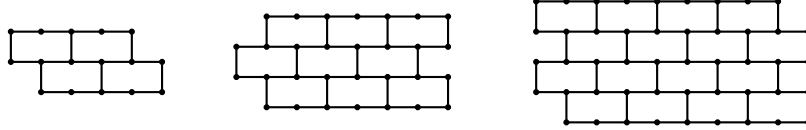
Fig. 2. Walls of height 2, 3, and 4, respectively.

4   Graphs of Maximum Degree at Most 6

In this section we show that the SQUARE ROOT problem can be solved in $O(n^4)$ time for n-vertex graphs of maximum degree at most 6. In order to do this we need to consider the aforementioned generalization of SQUARE ROOTS, which is defined as follows.

SQUARE ROOT WITH LABELS
Input: a graph G and two sets of edges $R, B \subseteq E_G$.
Question: is there a graph H with $H^2 = G$, $R \subseteq E_H$ and $B \cap E_H = \emptyset$?

Note that SQUARE ROOT is indeed a special case of SQUARE ROOT WITH LABELS: choose $R = B = \emptyset$.

The main idea behind our proof is to reduce to graphs with a bounded number of vertices by using the reduction rule that we recently introduced in [9] (the proof in [2] used a different and less general reduction rule, namely, the so-called path reduction rule, which only ensured boundedness of treewidth). In order to explain our new reduction we need the following definition. An edge uv of a graph G is said to be *recognizable* if $N_G(u) \cap N_G(v)$ has a partition (X, Y), where $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, y_q\}$, for some $p, q \geq 1$, such that the following conditions are satisfied:

a) X and Y are disjoint cliques in G;
b) $x_i y_j \notin E_G$ for $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$;
c) for any $w \in N_G(u) \setminus (X \cup Y \cup \{v\})$, $wy_j \notin E_G$ for $j \in \{1, \ldots, q\}$;
d) for any $w \in N_G(v) \setminus (X \cup Y \cup \{u\})$, $wx_i \notin E_G$ for $i \in \{1, \ldots, p\}$;
e) for any $w \in N_G(u) \setminus (X \cup Y \cup \{v\})$, there is an $i \in \{1, \ldots, p\}$ such that $wx_i \in E_G$;
f) for any $w \in N_G(v) \setminus (X \cup Y \cup \{u\})$, there is a $j \in \{1, \ldots, q\}$ such that $wy_j \in E_G$.

This leads to an algorithmic and a structural lemma, which we both need.

Lemma 4 ([9]). *For an instance (G, R, B) of* SQUARE ROOT WITH LABELS *where G has n vertices and m edges, it takes $O(n^2 m^2)$ time to either solve the problem or to obtain an instance (G', R', B') that has no recognizable edges and that is a yes-instance if and only if (G, R, B) is a yes-instance.*

Lemma 5 ([9]). *Let H be a square root of a graph with no recognizable edges. Then every non-pendant edge of H lies on a cycle of length at most 6.*

We are now ready to prove the aforementioned structural result. Its proof relies on Lemma 5.

Lemma 6. *Let G be a connected graph with $\Delta(G) \leq 6$ that has no recognizable edges. If G has a square root, then G has at most 103 vertices.*

Proof. Let G be a connected graph with $\Delta(G) \leq 6$ that has no recognizable edges. Assume that G has a square root H. We select a vertex u for which there exists a vertex v, such that $\text{dist}_H(u, v) = \text{diam}(H) = s$. We apply a breadth-first search on H starting at u to obtain levels $L_0, \ldots, L_s$, where $L_i = \{w \in V_G \mid \text{dist}_G(u, w) = i\}$ for $i = 0, \ldots, s$. Note that $L_0 = \{u\}$ and that $L_0 \cup \cdots \cup L_s$ is a partition of $V_H = V_G$. We say that a vertex y is a *child* of a vertex x, and that x is a *parent* of y if $xy \in E_G$, $x \in L_i$ and $y \in L_{i+1}$ for some $i \in \{0, \ldots, s-1\}$. It is worth mentioning that this parent-child relation differs from the relation defined by the corresponding BFS-tree. In particular, a vertex may have several parents. We also say that a vertex z is a *grandchild* of a vertex x and that x is a *grandparent* of z if there is a vertex y such that x is a parent of y and y is a parent of z. A vertex y is a *descendant* of a vertex x if $x \in L_i$, $y \in L_j$ for some i, j with $i < j$ and there is an (x, y)-path of length $|j - i|$.

We now prove a sequence of claims.

Claim A. *Let $i \geq 2$ and $x \in L_i$ such that x has at least two grandchildren. Then x has a child that is the parent of all grandchildren of x, while no other child of x is the parent of a grandchild of x.*

We prove Claim A as follows. For contradiction, assume there exists a vertex $x \in L_i$ for some $i \geq 2$ that has two distinct children y_1, y_2, such that y_1 and y_2 have distinct
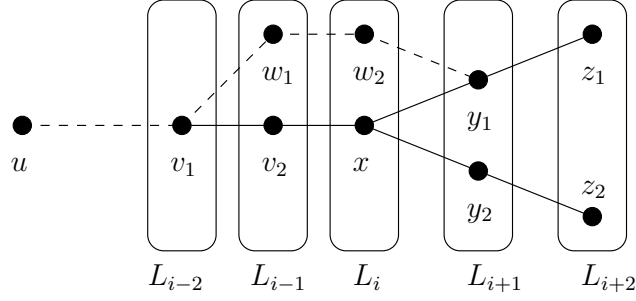
Fig. 3. The configuration for x and its relatives in the graph H of the proof of Claim A.

children z_1, z_2 respectively, as shown in Fig. 3. Notice that possibly z_1 (z_2 respectively) is a child of y_2 (y_1 respectively) as well. As $i \geq 2$, x has a parent v_2, and v_2 has a parent v_1. By Lemma 5, xv_2 is contained in a cycle C of H of length at most 6. If v_2 is adjacent to a vertex $z \notin \{v_1, x\}$ in H, then $z \in L_{i-2} \cup L_{i-1} \cup L_i$ and hence $z \notin \{y_1, y_2, z_1, z_2\}$. Consequently $d_G(x) \geq 7$; a contradiction. Therefore, $d_H(v_2) = 2$ and thus $v_1 v_2 \in E_C$. If x has a neighbour $r \notin \{v_2, y_1, y_2\}$ in H, then $d_G(x) \geq 7$, because $r \in L_{i-1} \cup L_i \cup L_{i+1}$ and hence r is distinct from v_1, z_1, z_2. This is again a contradiction. Therefore C contains one of xy_1, xy_2, say C contains xy_1. Hence $C = v_1 v_2 x y_1 w_2 w_1 v_1$ for some $w_1 \in L_{i-1}$ and $w_2 \in L_i$; see also Fig. 3. However, then x is adjacent to $v_1, v_2, w_2, y_1, y_2, z_1, z_2$ in G. Hence $d_G(x) \geq 7$, a contradiction. This completes the proof of Claim A.

Claim B. *Let $i \geq 2$ and $x \in L_i$. Then the number of descendants of x in L_j is at most four for every $j > i$.*

We prove Claim B as follows. Note that for all $i \geq 2$, every vertex $x \in L_i$ has at least two neighbours in G that belong to $L_{i-1} \cup L_{i-2}$. Hence the fact that $d_G(x) \leq 6$ implies that x has at most four neighbours in G belonging to $L_{i+1} \cup L_{i+2}$. In other words, the total number of children and grandchildren of x is at most four.

  We use induction on i. Let $i = s - 1$ or $i = s - 2$, As the total number of children and grandchildren of x is at most four, the claim holds. Let $i < s - 2$. Recall that x has at most four children. Hence the claim holds for $j = i + 1$. Let $j > i + 1$. If x has no grandchildren the claim holds. Now suppose that x has exactly one grandchild z. If $j = i + 2$, then the number of descendants of x in L_j is one. If $j > i + 2$, then by the induction hypothesis the number of descendants of x in L_j is at most four, since these vertices are descendants of z as well. Finally suppose that x has at least two grandchildren. Then, by Claim A, there is a unique child y of x that is the parent of all grandchildren of x, and no other child of x has children. By the induction hypothesis, the number of descendants of y in L_j is at most four. As all descendants of x in L_j are descendants of y, the claim holds. This completes the proof of Claim B.

Recall that v is a vertex that is of distance s of x.

Claim C. *Let $P = x_0 \cdots x_s$ with $x_0 = u$ and $x_s = v$ be a shortest (u, v)-path in H. Then for every $i \in \{3, \ldots, s - 4\}$, x_{i+1} is the unique child of x_i.*

We prove Claim C as follows. To obtain a contradiction, we assume that x_i has another child $y \neq x_{i+1}$ for some $i \in \{3 \ldots s - 3\}$, so $y \in L_{i+1}$. Since G has no recognizable

edge, Lemma 5 tells us that in H every non-pendant edge is contained in a cycle of length at most 6.

Let us first assume that there is no cycle of length at most 6 in H that contains both $x_{i-2}x_{i-1}$ and $x_{i-1}x_i$; this case will be studied later. Let C_1 and C_2 be two cycles in H of length at most 6 that contain $x_{i-2}x_{i-1}$ and $x_{i-1}x_i$, respectively. As $x_{i-1}x_i \notin E_{C_1}$ and $x_{i-2}x_{i-1} \notin E_{C_2}$, C_1 has an edge $x_{i-1}w$ and C_2 has an edge $x_{i-1}w'$ for some $w, w' \notin \{x_{i-2}, x_i\}$. Note that w and w' both belong to $L_{i-2} \cup L_{i-1} \cup L_i$. If $w \neq w'$, then in G we see that x_{i-1} is adjacent to x_{i-3}, x_{i-2}, x_i, x_{i+1}, w, w' and y. Hence $d_G(x_{i-1}) \geq 7$ contradicting $\Delta(G) \leq 6$. It follows that $w = w'$.

Let $z_1 w$ be an edge of C_1 and let $z_2 w$ be an edge of C_2, such that $x_{i-1} \notin \{z_1, z_2\}$. If z_1 or z_2 does not belong to $\{x_{i-3}, x_{i-2}, x_i, x_{i+1}, y\}$, then $d_G(x_{i-1}) \geq 7$, a contradiction. Hence, $\{z_1, z_2\} \subset \{x_{i-3}, x_{i-2}, x_i, x_{i+1}, y\}$ Recall that there is no cycle of length at most 6 that contains both $x_{i-2}x_{i-1}$ and $x_{i-1}x_i$. This implies that $z_1 \notin \{x_i, x_{i+1}, y\}$ (in C_1 remove w in the first case and replace w by x_i in the latter two cases) and $z_2 \notin \{x_{i-3}, x_{i-2}\}$ (in C_2 replace w by x_{i-2} in the first case and remove w in the second case). As both z_1 and z_2 are adjacent to v, this implies that $w \in L_{i-1}$, $z_1 = x_{i-2}$ and $z_2 = x_i$. However, this means that $x_{i-2}x_{i-1}x_iwx_{i-2}$ is a cycle of length 4 in H that contains both $x_{i-2}x_{i-1}$ and $x_{i-1}x_i$, a contradiction. Consequently, we may assume that there is a cycle C in H of length at most 6 that contains both $x_{i-2}x_{i-1}$ and $x_{i-1}x_i$. Since $x_{i-2} \in L_{i-2}$, we need to distinguish three cases.

Case 1. $x_i x_{i+1} \in E_C$.
Then C has an edge $x_{i+1}w$ for some $w \in L_i$ such that $w \neq x_i$ (see Fig. 4). Then, C has an edge wz for some $z \in L_{i-1}$. However, then we obtain $d_G(x_{i+1}) \geq 7$, a contradiction.
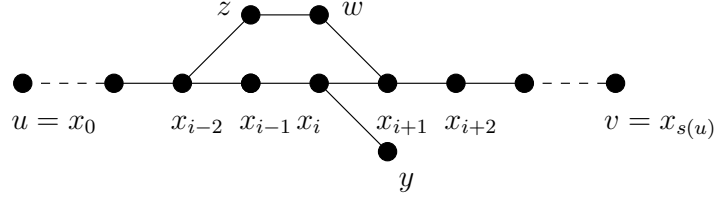

Fig. 4. Case 1.

Case 2. $x_i y \in E_C$.
Then C has an edge yw for some $w \in L_i$ such that $w \neq x_i$ (see Fig. 5). Similarly to Case 1, C has an edge wz for some $z \in L_{i-1}$. Note that $zx_{i-2} \in E_C$ as C has length at most 6. Hence, $C = x_{i-2}x_{i-1}x_iywzx_{i-2}$ (note that this is not in contradiction with Claim A as $i - 2 = 1$ may hold).

If $x_i w \in E_H$, then $d_G(x_{i-1}) \geq 7$, a contradiction. If $x_{i+1}w \in E_H$, then $d_G(x_{i+1}) \geq 7$. If $x_{i+2}y \in E_H$, then $d_G(y) \geq 7$. If $x_{i-1}z \in E_H$ or $x_i z \in E_H$, then $d_G(x_i) \geq 7$. Hence, $x_i w, x_{i+1}w, x_{i+2}y, x_{i-1}z, x_i z \notin E_H$.

The edge $x_{i+1}x_{i+2}$ is non-pendant and thus included in a cycle C' of length at most 6 by Lemma 5. Assume that C' is a shortest cycle of this type. Let $x_{i+1}h$ be the other edge in C' incident with x_{i+1}. We distinguish three subcases.

Case 2a. $h \neq x_i$.
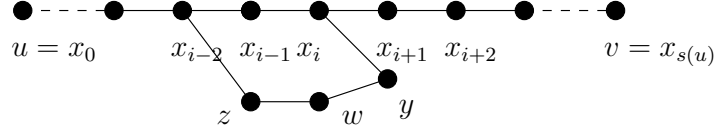If $h \neq y$, then $d_G(x_i) \geq 7$, a contradiction. Hence, $h = y$. Then, C' has an edge yg

Fig. 5. Case 2.

for some $g \neq x_{i+1}$. As C' is a shortest cycle with $x_{i+1}x_{i+2}$, we find that $g \neq x_i$ (as otherwise the cycle obtained from C' after removing y would be shorter) . Recall that $x_{i+2}y \notin E_H$. Hence $g \neq x_{i+2}$. If $g \neq w$, this means that $d_G(x_i) \geq 7$, a contradiction. Therefore, $g = w$. Let f be the next vertex of C', so $wf \in E_{C'}$. As C' has length at most 6, $f \notin L_{i-1}$, so $f \notin \{x_{i-1}, z\}$. Recall that x_iw and $x_{i+1}w$ are not in E_H, thus $f \notin \{x_i, x_{i+1}\}$ either. As $w \in L_i$, $z \neq x_{i+2}$. Hence, $d_G(y) \geq 7$, a contradiction.

Case 2b. $h = x_i$ and $x_ig \in E_{C'}$ for some $g \notin \{x_{i-1}, x_{i+1}\}$.
Recall that x_iw and x_iz are not in E_H. Hence $g \notin \{w, z\}$ either. If $g \neq y$, this means that $d_G(x_i) \geq 7$. Hence, $g = y$, that is, $x_iy \in E_{C'}$. Let f be the next vertex of C', so $yf \in E_{C'}$. Recall that $x_{i+2}y \notin E_H$. If $f \neq w$, this means that $d_G(x_i) \geq 7$. It follows that $f = w$, that is, $yw \in E_{C'}$. Let f' be the next vertex of C', so $wf' \in E_{C'}$. As C' has length at most 6, we find that $f' \in L_{i+1}$. As C' has length at most 6, we find that $f'x_{i+2} \in E_H$ and thus $C' = x_{i+2}x_{i+1}x_iywf'x_{i+2}$ (see Fig. 6).

If x_{i+2} has a neighbour in H distinct from x_{i+1}, x_{i+3}, f', then $d_G(x_{i+2}) \geq 7$, a contradiction. Hence, $N_H(x_{i+2}) = \{x_{i+1}, x_{i+3}, f'\}$. If x_{i+1} has a neighbour in H distinct from x_i and x_{i+2}, then that neighbour cannot be in $\{y, w, f'\}$, as C' is a shortest cycle containing $x_{i+1}x_{i+2}$. Consequently, we find that $d_G(x_{i+1}) \geq 7$, a contradiction. Hence, $N_H(x_{i+1}) = \{x_i, x_{i+2}\}$. Recall that $x_iz \notin E_H$. Moreover, as C' is a shortest cycle containing $x_{i+1}x_{i+2}$, we find that $x_iw, x_if' \notin E_H$. Consequently, if x_i has a neighbour in H distinct from x_{i-1}, x_{i+1}, y, then $d_G(y) \geq 7$, a contradiction. Hence, $N_H(x_i) = \{x_{i-1}, x_{i+1}, y\}$. As C' is a shortest cycle containing $x_{i+1}x_{i+2}$, we find that $yf' \notin E_H$. Consequently, if y has a neighbour in H distinct from x_i, w, then $d_G(y) \geq 7$, a contradiction. Hence, $N_H(y) = \{x_i, w\}$. If w has a neighbour distinct from y, z, f', then $d_G(y) \geq 7$, a contradiction. Hence, $N_H(w) = \{y, z, f'\}$. If f' has a neighbour distinct from w, x_{i+2}, then $d_G(w) \geq 7$, a contradiction. Hence, $N_H(f') = \{x_{i+2}, w\}$.

Now consider the (non-pendant) edge $x_{i+2}x_{i+3}$, which must be in a cycle C'' of length at most 6 due to Lemma 5. As $x_{i+3} \in L_{i+3}$ and $|E_{C''}| \leq 6$, we find that C'' contains no vertex of L_{i-1}. Now, by traversing C'' starting at x_{i+2} in opposite direction from x_{i+3}, we find that C'' contains the cycle $x_{i+2}x_{i+1}x_iywf'x_{i+2}$. Hence $d_{C''}(x_{i+2}) \geq 3$. which means that C'' is not a cycle, a contradiction.
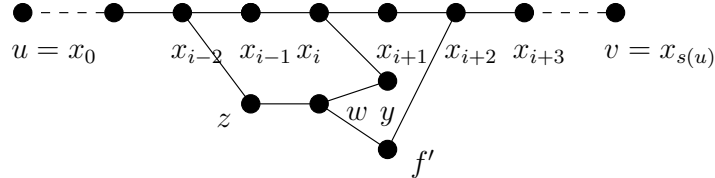

Fig. 6. Case 2b.

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Case 2c. $h = x_i$ and $x_i x_{i-1} \in E_{C'}$

Let g be the next vertex of C', so $x_{i-1}g \in E_{C'}$. As C' has length at most 6, we find that $g \in L_i$. Recall that $x_{i-1}w \notin E_H$. Then we find that $d_G(x_i) \geq 7$, a contradiction.

Case 3. $x_i x_{i+1}, x_i y \notin E_C$.

Then C has an edge $x_i w$ for some $w \notin \{x_{i-1}, x_{i+1}, y\}$ (see Fig. 7). Let z be the next vertex of C', so $wz \in E_C$. As C has length at most 6, we find that $z \notin L_{i+1} \cup L_{i+2}$. Hence, if $z \neq x_{i-2}$, then $d_G(x_i) \geq 7$, a contradiction. This means that $z = x_{i-2}$, and consequently $C = x_{i-2}x_{i-1}x_i w, x_{i-2}$ and $w \in L_{i-1}$.

Again let C' be a shortest cycle amongst all cycles that contains $x_{i+1}x_{i+2}$. Recall that the length of C' is at most 6 by Lemma 5. Let $x_{i+1}h$ be the other edge in C' incident with x_{i+1}. As in Case 2, we distinguish three subcases.
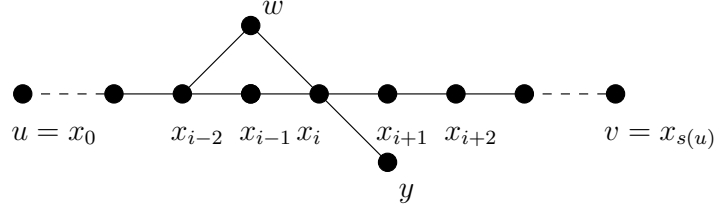

Fig. 7. Case 3.

Case 3a. $h \neq x_i$.

If $h \neq y$, then $d_G(x_i) \geq 7$. Hence, $h = y$, that is, $x_{i+1}y \in E_{C'}$. Let g be the next vertex of C', so $yg \in E_{C'}$. As C' is a shortest cycle containing $x_{i+1}x_{i+2}$, we find that $g \neq x_i$ (otherwise remove y from C' to obtain a shorter cycle). As $yx_{i+2} \notin E_H$ due to Claim A, we find that $g \neq x_{i+2}$. Since $y \in L_{i+1}$ and $w \in L_{i-1}$, we find that $g \neq w$ either. Then $d_G(x_i) \geq 7$ holds, a contradiction.

Case 3b. $h = x_i$ and $x_i g \in E_{C'}$ for some $g \notin \{x_{i-1}, x_{i+1}\}$.

If $g \notin \{w, y\}$, then $d_G(x_i) \geq 7$. First suppose $g = w$, so $x_i w \in E_{C'}$. Let f be the next vertex of C', so $wf \in E_{C'}$. As $w \in L_{i-1}$ and C' has length at most 6, we find that $f \in L_i$. However, then $d_G(x_i) \geq 7$, a contradiction. Now suppose, $g = y$, so $x_i y \in E_{C'}$. Let f' be the next vertex of C', so $yf' \in E_{C'}$. Recall that $yx_{i+2} \notin E_H$ due to Claim A. Then $d_G(x_i) \geq 7$ holds, a contradiction.

Case 3c. $h = x_i$ and $x_i x_{i-1} \in E_{C'}$

As C' has length at most 6, we find that $h \in L_i$. Then $d_G(x) \geq 7$, a contradiction.

We considered all possibilities, and in each case we obtained a contradiction. Hence we have proven Claim C.

Claim D. *Let $P = x_0 \cdots x_s$ with $x_0 = u$ and $x_s = v$ be a shortest (u,v)-path in H. Then for every $i \in \{4, \ldots, s-3\}$, x_{i-1} is the unique parent of x_i.*

To prove Claim D, it suffices to run a breadth-first search in H from v, to consider the resulting levels $L'_0, \ldots, L'_s$, where where $L'_i = \{w \in V_G \mid \text{dist}_G(v, w) = i\}$ for $i = 0, \ldots, s$, and to apply Claim C.

We are now ready to complete the proof. We do this by first showing that $s \leq 8$. To obtain a contradiction, assume that $s \geq 9$. By Lemma 5, H has a cycle C of length at

most 6 that contains $x_5 x_6$. As x_6 is the unique child of x_5 by Claim C and x_5 is the unique parent of x_6 by Claim D, we find that C has an edge yz, where $y \in L_5$, $z \in L_6$, such that C contains an (x_5, y) path Q of length at most 3 with $V_Q \subseteq L_4 \cup L_5$.

Suppose that $x_5 y \in E_Q$. Then y has a parent $h \in L_4$. By Claim C, $h \neq x_4$. Therefore, $d_G(x_5) \geq 7$, a contradiction. Hence $x_5 y \notin E_Q$. Suppose that $V_Q \subseteq L_5$. Then Q has edges $x_5 w, wh$ such that $w, h \in L_5$. Again, w has a parent $g \in L_4$ and $g \neq x_4$ due to Claim C. It follows that $d_G(x_5) \geq 7$; a contradiction. Hence, Q has a vertex of L_4.

As Q contains a vertex of L_4, Q has length at least 2. If Q has length 2, then $Q = x_5 x_4 y$. However, this is a contradiction, as x_5 is the unique child of x_4 by Claim C. Hence, Q has length 3, which implies that $Q = x_5 x_4 z y$ for some $z \in L_4$ or $Q = x_5 w w' y$ for some $w \in L_5$ and $w' \in L_4$.

First suppose that $Q = x_5 x_4 z y$ for some $h \in L_4$. Let $g \in L_3$ be a parent of h. As x_4 is the unique child of x_3 by Claim C, we find that $g \neq x_4$. Then $d_G(x_4) \geq 7$, a contradiction. Now suppose that $Q = x_5 w w' y$ for some $w \in L_5$ and $w' \in L_4$. Note that $w' \neq x_4$, as x_5 is the unique child of x_4 by Claim C. As C has length at most 6, we find that $C = x_6 x_5 w w' y z x_6$. This means that in G, x_5 is adjacent to $x_3, x_4, x_6, x_7, w, w', z$, so $d_G(x_5) \geq 7$, a contradiction. We conclude that $s \leq 8$.

As $L_0 = \{u\}$, we find that $|L_0| = 1$. Since $d_G(u) \leq 6$, $|L_1 \cup L_2| \leq 6$. As $|L_1| \geq 1$, this means that $|L_2| \leq 5$. If $|L_2| = 5$, then $L_1 = \{v\}$ for some $v \in V_H$ and each vertex of L_2 is a child of v. As $d_G(v) \leq 5$, this means that $V_G = V_H = L_0 \cup L_1 \cup L_2$, so $|V_H| = 1 + 1 + 5 = 7$. Suppose $|L_2| \leq 4$. By Claim B, $|L_i| \leq 4|L_2| \leq 16$ for $i \geq 3$. Because $s \leq 8$, $|V_G| = |V_H| = |L_0| + |L_1 \cup L_2| + |L_3| + \cdots + |L_8| \leq 1 + 6 + 6 \cdot 16 = 103$. □

We are now ready to prove our main result. Its proof uses Lemmas 4 and 6.

Theorem 2. SQUARE ROOT *can be solved in time* $O(n^4)$ *for n-vertex graphs of maximum degree at most* 6.

Proof. Let G be a graph of maximum degree at most 6. We construct an instance (G, R, B) of SQUARE ROOT WITH LABELS from G by setting $R = B = \emptyset$. Then we preprocess (G, R, B) using Lemma 4. In this way we either solve the problem (and answer **no**) or obtain an equivalent instance (G', R', B') of SQUARE ROOT WITH LABELS that has no recognizable edges. In the latter case we know from Lemma 6 that if G' has a square root, then each component of G' has at most 103 vertices. Hence, if G' has a component with at least 100 vertices, then we stop and return **no**. Otherwise, we solve the problem for each component of G' in constant time by brute force. Applying Lemma 4 takes time $O(n^2 m^2) = O(n^4)$, as $\Delta(G) \leq 6$. The remainder of our algorithm takes constant time. Hence, its total running time is $O(n^4)$. □

Remark 2. We cannot extend this approach for graphs of maximum degree at most 7. In particular, the following example shows that we cannot obtain an analog of Lemma 6 for graphs of maximum degree at most 7. Let H be the graph obtained from two paths $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ by adding the edge $u_i v_i$ for $i \in \{1, \ldots, n\}$. Notice that $G = H^2$ has no recognizable edges while $\Delta(G) = 7$. However, its size $|V_G| = |V_H| = 2n$ can be arbitrarily large.

5 Conclusion

We showed that SQUARE ROOT can be solved in polynomial time for graphs of maximum degree at most 6. We pose two open problems. First, can we solve SQUARE

ROOT in polynomial time on graphs of maximum degree at most 7; this will require new techniques. Second, does there exist an integer k such that SQUARE ROOT is NP-complete for graphs of maximum degree at most k?

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