Detecting Induced Minors in AT-free Graphs *

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Abstract. The Induced Minor problem is that of testing whether a graph G can be modified into a graph H by a sequence of vertex deletions and edge contractions. If only edge contractions are permitted, we obtain the Contractibility problem. We prove that Induced Minor is polynomial-time solvable when G is AT-free and H is fixed, i.e., not part of the input. In addition, we show that Contractiblity is polynomial-time solvable when G is AT-free and H is a fixed triangle-free graph. We complement these two results by proving that both problems are W[1]-hard on AT-free graphs when parameterized by $|V_H|$.

1 Introduction

In this paper we study graph containment problems. Whether or not a graph contains some other graph depends on the notion of containment used. In the literature several natural definitions have been studied such as containing a graph as a contraction, dissolution, immersion, (induced) minor, (induced) topological minor, (induced) subgraph, or (induced) spanning subgraph. We focus on the containment relations "induced minor" and "contraction". A graph G contains a graph H as an *induced minor* if G can be modified into a graph H by a sequence of vertex deletions and edge contractions. Here, the operation *edge contraction* removes the end-vertices u and v of an edge from G and replaces them by a new vertex adjacent to precisely those vertices to which u or v were adjacent. A graph G contains H as a *contraction* if H can be obtained from G by edge contractions only. The decision problems that are to test whether a graph H is an induced minor or a contraction of a graph G are called Induced Minor and Contractibility, respectively. Both problems are known to be NP-complete even when G and H are trees of bounded diameter or trees, the vertices of which have degree at most 3 except for at most one vertex, as shown by Matoušek and Thomas [20]. It is therefore natural to fix the graph H and to consider only

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the graph G to be part of the input. We denote these variants as H-Induced Minor and H-Contractibility, respectively.

The computational complexity classifications of H-Induced Minor and H-Contractibility are far from being settled, although both polynomial-time and NP-complete cases are known. In contrast, the two related problems H-Minor and H-Topological Minor, which are to test whether a graph G contains a graph H as a minor or topological minor, respectively, can be solved in cubic time for any fixed graph H, as shown by Robertson and Seymour [22] and Grohe, Kawarabayashi, Marx, and Wollan [11], respectively. For H-Induced Minor, Fellows, Kratochvíl, Middendorf, and Pfeiffer [7] showed that there exists a graph H for which the problem is NP-complete. This specific graph H has 68 vertices and is yet the smallest H for which H-Induced Minor is known to be NP-complete. The question whether H-Induced Minor is polynomial-time solvable for any fixed tree H was posed as an open problem at the AMS-IMS-SIAM Joint Summer Research Conference on Graph Minors in 1991. So far this question could only be answered for trees on at most seven vertices except for one case [8]. Brouwer and Veldman [2] showed that H-Contractibility is NP-complete when H is a path or a cycle on four vertices. Other polynomial-time solvable and NP-complete cases, depending on H, can be found in [12, 18, 19].

Due to the notorious difficulty of solving H-Induced Minor and H-Contractibility for general graphs, the input has been restricted to special graph classes. Fellows, Kratochvíl, Middendorf, and Pfeiffer [7] showed that for every fixed graph H, the H-Induced Minor problem can be solved in linear time on planar graphs. Van 't Hof et al. [12] extended this result by proving that for every fixed planar graph H, the H-Induced Minor problem is linear-time solvable on any minor-closed graph class not containing all graphs. Kamiński and Thilikos [14] showed that H-Contractibility can be solved in cubic time for graphs of bounded genus. Belmonte et al. [1] showed that for every fixed graph H, the H-Induced Minor and H-Contractibility problems are polynomial-time solvable for chordal graphs. The H-Contractibility problem has also been studied for claw-free graphs, but only partial results are known for this graph class [9].

We consider H-Induced Minor and H-Contractibilty restricted to the class of *asteroidal triple-free* graphs, also known as *AT-free* graphs. An *asteroidal triple* is a set of three mutually non-adjacent vertices such that each two of them are joined by a path that avoids the neighborhood of the third, and AT-free graphs are exactly those graphs that contain no such triple. AT-free graphs, defined fifty years ago by Lekkerkerker and Boland [17], are well studied in the literature and contain many well-known classes, e.g., cobipartite graphs, cocomparability graphs, cographs, interval graphs, permutation graphs, and trapezoid graphs (cf. [3]). All these graph classes have geometric intersection models being extremely useful when designing polynomial-time algorithms for hard problems. No such model is available for AT-free graphs. Recently, Golovach, Paulusma and Van Leeuwen [10] showed that the H-Induced Topological Minor problem is polynomial-time solvable on AT-free graphs for every fixed H. This problem is to test if a graph G can be modified into a graph H by a sequence of vertex deletions and vertex dissolutions. The latter graph operation is the contraction of an edge incident to a degree-two vertex that is not in a triangle (cycle on three vertices). The same authors [10] also showed that this problem is W[1]-hard when parameterized by $|V_H|$.

Our Results. We show that H-Induced Minor can be solved in polynomial time on AT-free graphs for any fixed graph H. Consequently, on AT-free graphs, all four problems H-Minor, H-Induced Minor, H-Topological Minor and H-Induced Topological Minor are polynomial-time solvable for any fixed graph H. For H-Contractibility, we prove that the problem can be solved in polynomial time on AT-free graphs for any fixed graph H that is triangle-free. We complement these results by proving that that Induced Minor and Contractibilty is W[1]-hard when parameterized by $|V_H|$. This result indicates that we cannot expect to obtain FPT algorithms for these two problems, i.e., that run in time $f(|V(H)|)|V(G)|^{O(1)}$ unless FPT=W[1], which is considered to be unlikely [6]. Our W[1]-hardness proofs also show that Induced Minor and Contractibility are NP-complete for AT-free graphs; these results were not known before.

The celebrated result by Robertson and Seymour that H-Minor is FPT on general graphs [22] is closely connected to the fact that k-Disjoint Paths is FPT with parameter k. To solve H-Induced Topological Minor on AT-free graphs, Golovach et al. [10] considered the variant k-Induced Disjoint Paths, in which the paths must not only be vertex-disjoint but also mutually induced, i.e., edges between vertices of any two distinct paths are forbidden. Here we must consider another variant, which was introduced by Belmonte et al. [1]. A *terminal pair* in a graph $G = (V, E)$ is a specified pair of vertices s and t called *terminals*, and the *domain* of a terminal pair (s, t) is a specified subset $U \subseteq V$ containing both s and t. We say that two paths, each of which is between some terminal pair, are *vertex-disjoint* if they have no common vertices except possibly the vertices of the terminal pairs. This leads to the following decision problem, which is NP-complete on general graphs even when $k = 2$ [1].

Set-Restricted k-Disjoint Paths
Instance: a graph G, terminal pairs $(s_1, t_1), \dots, (s_k, t_k)$, and domains $U_1, \dots, U_k$.
Question: does G contain k mutually vertex-disjoint paths $P_1, \dots, P_k$ such that P_i is a path from s_i to t_i using only vertices from U_i for $i = 1, \dots, k$?

Note that the domains $U_1, \dots, U_k$ are not necessarily pairwise disjoint. If we let every domain contain all vertices of G, we obtain exactly the Disjoint Paths problem. We give an algorithm that solves Set-Restricted k-Disjoint Paths in polynomial time on AT-free graphs for any fixed integer k. We then use this algorithm as a subroutine in our polynomial-time algorithms for H-Induced Minor and H-Contractibility. We emphasize that we can not apply the algorithm for k-Induced Disjoint Paths on AT-free graphs [10] as a subroutine for solving H-Induced Minor on AT-free graphs. The techniques used in that algorithm are quite different from the techniques we use here to solve Set-

Restricted k-Disjoint Paths on AT-free graphs. Moreover, when k is in the input, k-Induced Disjoint Paths and Set-Restricted k-Disjoint Paths have a different complexity for AT-free graphs. Golovach et al. [10] proved that in that case k-Induced Disjoint Paths is polynomial-time solvable for AT-free graphs, whereas k-Disjoint Paths, and consequently Set-Restricted k-Disjoint Paths, are already NP-complete for interval graphs [21], which form a subclass of AT-free graphs.

We use our algorithm for solving Set-Restricted k-Disjoint Paths to obtain one additional results on AT-free graphs. We show that we can solve the problem Set-Restricted k-Disjoint Connected Subgraphs, also introduced by Belmonte et al. [1], in polynomial time on AT-free graphs for any fixed integer k. A *terminal set* in a graph $G = (V, E)$ is a specified subset $S_i \subseteq V$.

Set-Restricted k-Disjoint Connected Subgraphs
Instance: a graph G, terminal sets $S_1, \ldots, S_k$, and domains $U_1, \ldots, U_k$.
Question: does G have k pairwise vertex-disjoint connected subgraphs $G_1, \ldots, G_k$, such that $S_i \subseteq V_{G_i} \subseteq U_i$, for $1 \leq i \leq k$?

If $|S_i| = 2$ for all $1 \leq i \leq k$, then we obtain the Set-Restricted k-Disjoint Paths problem. If $U_i = V_G$ then we obtain the k-Disjoint Connected Subgraphs problem. The latter problem has been introduced by Robertson and Seymour [22] and is NP-complete on general graphs even when $k = 2$ and $\min\{|Z_1|, |Z_2|\} = 2$ [13].

2 Preliminaries

We only consider finite undirected graphs without loops and multiple edges. We refer to the textbook by Diestel [5] for any undefined graph terminology. Let G be a graph. We denote the vertex set of G by V_G and the edge set by E_G. The subgraph of G induced by a subset $U \subseteq V_G$ is denoted by $G[U]$. We say that $U \subseteq V_G$ is *connected* if $G[U]$ is a connected graph. The graph $G - U$ is the graph obtained from G by removing all vertices in U. If $U = \{u\}$, we also write $G - u$. The *open neighborhood* of a vertex $u \in V_G$ is defined as $N_G(u) = \{v \mid uv \in E_G\}$, and its *closed neighborhood* is defined as $N_G[u] = N_G(u) \cup \{u\}$. For $U \subseteq V_G$, $N_G[U] = \cup_{u \in U} N_G[u]$. The degree of a vertex $u \in V_G$ is denoted $d_G(u) = |N_G(u)|$. The *distance* $\text{dist}_G(u, v)$ between a pair of vertices u and v of G is the number of edges of a shortest path between them. Two sets $U, U' \subseteq V_G$ are called *adjacent* if there exist vertices $u \in U$ and $u' \in U'$ such that $uu' \in E_G$. A set $U \subseteq V_G$ *dominates* a vertex w if $w \in N_G[U]$, and U *dominates* a set $W \subseteq V_G$ if U dominates each vertex of W. In these two cases, we also say that $G[U]$ *dominates* w or W, respectively. A set $U \subseteq V_G$ is a *dominating set* of G if U dominates V_G.

The graph $P = u_1 \cdots u_k$ denotes the *path* with vertices $u_1, \ldots, u_k$ and edges $u_i u_{i+1}$ for $i = 1, \ldots, k - 1$. We also say that P is a (u_1, u_k)*-path*. For a path P with some specified end-vertex s, we write $x \prec_s y$ if $x \in V_P$ lies in P between s and $y \in V_P$; in this definition, we allow that $x = s$ or $x = y$. A pair of vertices $\{x, y\}$ is a *dominating pair* if the vertex set of every (x, y)-path is a dominating set of G. Corneil, Olariu and Stewart [3, 4] proved the following structural theorem.

Theorem 1 ([3, 4]). *Every connected AT-free graph has a dominating pair and such a pair can be found in linear time.*

Using these results, Kloks, Kratsch and Müller [15] gave the following tool for constructing dynamic programming algorithms on AT-free graphs. For a vertex u of a graph G, we call the sets $L_i(u) = \{v \in V_G \mid \text{dist}_G(u, v) = i\}$ $(i \geq 1)$ the *BFS-levels* of G. Note that the BFS-levels of a vertex can be determined in linear time by the Breadth-First Search algorithm (BFS).

Theorem 2 ([15]). *Every connected AT-free graph contains a dominating path $P = u_0 \cdots u_\ell$ that can be found in linear time such that*

(i) *ℓ is the number of BFS-levels of u_0;*
(ii) *$u_i \in L_i(u_0)$ for $i = 1, \ldots, \ell$;*
(iii) *each $z \in L_i(u_0)$ is adjacent to u_{i-1} or to u_i for all $1 \leq i \leq \ell$.*

3 Set-Restricted Disjoint Paths and Connected Subgraphs

In this section we show that Set-Restricted k-Disjoint Paths and its generalization Set-Restricted k-Disjoint Connected Subgraphs can be solved in polynomial time on AT-free graphs for any fixed integer k. For Set-Restricted k-Disjoint Paths, we first introduce some extra terminology and give a number of structural results. We then apply dynamic programming to solve this problem. Afterward, we solve Set-Restricted k-Disjoint Connected Subgraphs.

3.1 Structural Lemmas

Let G be a graph, and let $W \subseteq V_G$. Consider an induced path P in G. Then $V_P \cap W$ and $V_P \setminus W$ induce a collection of subpaths of P called W-*segments*, or *segments* if no confusion is possible. Segments induced by $V_P \cap W$ are said to lie *inside* W, whereas segments induced by $V_P \setminus W$ lie *outside* W. We need the following two lemmas.

Lemma 1. *Let P be an induced path in an AT-free graph G. Let $U \subseteq V_G$ be connected. Then P has at most three segments inside $N_G[U]$.*

Proof. To obtain a contradiction, assume that P has at least four segments inside $N_G[U]$. Then P has three segments P_1, P_2, P_3 outside $N_G[U]$ such that for each P_i, both end-vertices of P_i are adjacent to end-vertices of the segments inside $N_G[U]$. Let s be an end-vertex of P, and let x_i, y_i be the end-vertices of P_i for $i \in \{1, 2, 3\}$. Assume that P_1, P_2, P_3 and their end vertices are ordered in such a way that $x_1 \prec_s y_1 \prec_s x_2 \prec_s y_2 \prec_s x_3 \prec_s y_3$. Let $z_1, z_2 \in N_G[U]$, $z_1 \prec_s x_1, y_3 \prec_s z_2$

be the vertices adjacent to x_1 and y_3, respectively in P. Let $z_1', z_2' \in U$ be vertices adjacent to z_1, z_2, respectively. We claim that x_1, x_2, x_3 form an asteroidal triple. Since x_1, x_2, x_3 are vertices of different segments outside $N_G[U]$ and P is an induced path, x_1, x_2, x_3 are distinct and pairwise non-adjacent. Because P is an induced path, the (x_1, x_2) and (x_2, x_3)-subpaths of P avoid the neighborhoods of x_3 and x_1, respectively. Finally, the path obtained by the concatenation of the (x_1, z_1)-subpath of P, the path $z_1 z_1'$, a (z_1', z_2')-path in $G[U]$, $z_2' z_2$ and the (z_2, x_3)-subpath of P avoids the the neighborhood of x_2, as x_2 is not adjacent to z_1, z_2 and $x_2 \notin N_G[U]$. This gives us a contradiction. □

Lemma 2. *Let P be an induced path in an AT-free graph G. Let $U \subseteq V_G$ be connected. Then every segment of P outside $N_G[U]$ that contains no end-vertex of P has at most two vertices.*

Proof. To obtain a contradiction, assume that P has a segment P' with end-vertices x, y such that P' has at least three vertices and x, y have neighbors x_1, y_1 respectively in P, where $x_1, y_1 \notin V_{P'}$. Then $x_1, y_1 \in N_G[U]$. Let x_2, y_2 be neighbors of x_1, y_1 in U. We claim that x, y, x_2 is an asteroidal triple. Clearly, x, y, x_2 are distinct and pairwise non-adjacent. The (x, y)-path P' avoids $N_G[U]$ and, therefore, $N_G[x_2]$. Because P is an induced path, x_1 and y_1 are not adjacent, and they are not adjacent to y and x respectively. Hence $x x_1 x_2$ avoids $N_G[y]$. It remains to observe that the path obtained by the concatenation of a (x_2, y_2)-path in U and $y_2 y_1 u$ avoids $N_G[x]$. □

The next lemma follows directly from the property of being induced.

Lemma 3. *Let u be a vertex of an induced path P in a graph G. Then P has one segment inside $N_G[u]$ and this segment has at most three vertices.*

Let G be a graph with terminal pairs $(s_1, t_1), \dots (s_k, t_k)$ and corresponding domains $U_1, \dots, U_k$. Let $\{P_1, \dots, P_k\}$ be a set of mutually vertex-disjoint paths, such that P_i is a path from s_i to t_i using only vertices from U_i for $i = 1, \dots, k$. We say that $\{P_1, \dots, P_k\}$ is a *solution*. A solution $\{P_1, \dots, P_k\}$ is *minimal* if no P_i can be replaced by a shorter (s_i, t_i)-path P_i' that uses only vertices of U_i in such a way that $P_1, \dots, P_{i-1}, P_i', P_{i+1}, \dots, P_k$ are mutually vertex-disjoint. Clearly, every yes-instance of Set-Restricted k-Disjoint Paths has a minimal solution. We also observe that any path in a minimal solution is induced. We need Lemma 4, which gives some properties of minimal solutions.

Lemma 4. *Let G be a graph with terminal pairs $(s_1, t_1), \dots (s_k, t_k)$ and corresponding domains $U_1, \dots, U_k$. Let $u \in U_i$ for some $1 \le i \le k$, and let $\{P_1, \dots, P_k\}$ be a minimal solution with $u \notin \bigcup_{j=1}^{k} V_{P_j}$. Then P_i has at most two segments inside $N_G[u]$. Moreover, if P_i has one segment inside $N_G[u]$, then P_i has at most three vertices. If P_i has two segments Q_1 and Q_2 inside $N_G[u]$, then Q_1 and Q_2 each has precisely one vertex, and the segment Q' outside $N_G[u]$ that lies between Q_1 and Q_2 in P_i also has one vertex.*

Proof. Suppose that P_i has three segments Q_1, Q_2, Q_3 inside $N_G[u]$. Let x_j, y_j be the end vertices of Q_j for $j = 1, 2, 3$. Assume without loss of generality that $x_1 \prec_{s_i} y_1 \prec_{s_i} x_2 \prec_{s_i} y_2 \prec_{s_i} x_3 \prec_{s_i} y_3$. Clearly, we can replace the (x_1, y_3)-subpath of P_i of length at least four by $x_1 u y_3$ and obtain a shorter (s_i, t_i)-path. Hence, P_i has at most two segments inside $N_G[u]$. Now suppose that a (x, y)-path Q is a segment of P_i inside $N_G[u]$. If Q has at least four vertices, then we replace Q by xuy and obtain a shorter (s_i, t_i)-path. Finally, suppose that Q_1, Q_2 are segments of P_i inside $N_G[u]$. Let x_j, y_j be the end vertices of Q_j for $j = 1, 2$. Let also Q' be the segment outside $N_G[u]$ that lies between Q_1, Q_2 in P_i. Assume without loss of generality that $x_1 \prec_{s_i} y_1 \prec_{s_i} x_2 \prec_{s_i} y_2$. If one of the paths Q_1, Q_2, Q' has at least two vertices, then we can replace the (x_1, y_2)-subpath of P_i of length at least three by $x_1 u y_2$ and obtain a shorter (s_i, t_i)-path. $\square$

3.2 Dynamic programming for Set-Restricted *k*-Disjoint Paths

We apply dynamic programming to prove that Set-Restricted k-Disjoint Paths is polynomial-time solvable on AT-free graphs for every fixed integer k. Our algorithm solves the decision problem, but can easily be modified to produce the desired paths if they exist. It is based on the following idea. We find a shortest dominating path $u_0 \dots u_\ell$ in G as described in Theorem 2. For $0 \le i \le \ell$, we trace the segments of (s_j, t_j)-paths inside $N_G[\{u_0, \dots, u_i\}]$ by extending the segments inside $N_G[\{u_0, \dots, u_{i-1}\}]$ in $N_G[u_i] \setminus N_G[\{u_0, \dots, u_{i-1}\}]$. Note that if some path is traced from the middle, then we have to extend the corresponding segment in two directions, i.e., we have to trace two paths. The paths inside $N_G[u_i] \setminus N_G[\{u_0, \dots, u_{i-1}\}]$ are constructed recursively, as by Lemmas 3 and 4 we can reduce the number of domains by distinguishing whether u_i is used by one of the paths or not. Hence, it is convenient for us to generalize as follows:

Set-Restricted r-Group Disjoint Paths
Instance: A graph H, positive integers $p_1, \dots, p_r$, terminal pairs (s_i^j, t_i^j) for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, p_i\}$, and domains $U_1, \dots, U_r$.
Question: Does H contain mutually vertex-disjoint paths P_i^j, where $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, p_i\}$, such that P_i^j is a path from s_i^j to t_i^j using only vertices from U_i for $i = 1, \dots, r$?

Recall that the domains $U_1, \dots, U_k$ are not necessarily pairwise disjoint. Also note that for $p_1 = \ldots = p_r = 1$ we have the Set-Restricted r-Disjoint Paths problem. We say that for each $1 \le i \le r$, the pairs $(s_i^1, t_i^1), \dots, (s_i^{p_i}, t_i^{p_i})$ (or corresponding paths) form a *group*. We are going to solve Set-Restricted r-Group Disjoint Paths for induced subgraphs H of G and $r \le k$ recursively to obtain a solution that can be extended to a solution of Set-Restricted k-Disjoint Paths in such a way that $P_i^1, \dots, P_i^{p_i}$ are disjoint subpaths of the (s_i, t_i)-path P_i in the solution of Set-Restricted k-Disjoint Paths. Hence, we are interested only in some special solutions of Set-Restricted r-Group Disjoint Paths.

For $r = 1$, SET-RESTRICTED r-GROUP DISJOINT PATHS is the p_1-DISJOINT PATHS problem in $H[U_1]$. By the celebrated result of Robertson and Seymour [22], we immediately get the following lemma.

Lemma 5. *For $r = 1$ and any fixed positive integer p_1,* SET-RESTRICTED r-GROUP DISJOINT PATHS *can be solved in $O(n^3)$ time on n-vertex graphs.*

Now we are ready to describe our algorithm for SET-RESTRICTED r-GROUP DISJOINT PATHS. First, we recursively apply the following preprocessing rules.

Rule 1. If H has a vertex $u \notin \cup_{i=1}^{r} U_i$, then we delete it and solve the problem on $H - u$.

Rule 2. If there are $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, p_i\}$ such that s_i^j and t_i^j are in different components of $H[U_i]$, then stop and return `No`.

Rule 3. If H has components $H_1, \ldots, H_q$ and $q > 1$, then solve the problem for each component H_h for the pairs of terminals (s_i^j, t_i^j) such that $s_i^j, t_i^j \in V_{H_h}$ and the corresponding domains. We return `Yes` if we get a solution for each component H_h, and we return `No` otherwise.

Rule 4. If $r = 1$, then solve the problem by Lemma 5.

From now we assume that $r \geq 2$ and H is connected. Let $p = p_1 + \ldots + p_r$.

By Theorem 2, we can find a vertex $u_0 \in V_H$ and a dominating path $P = u_0 \ldots u_\ell$ in H with the property that for $i \in \{1, \ldots, \ell\}$, $u_i \in L_i$ and for any $z \in L_i$, z is adjacent to u_{i-1} or u_i, where $L_0, \ldots L_\ell$ are the BFS-levels of u_0. For $i \in \{0, \ldots, \ell\}$, let $W_i = N_H[\{u_0, \ldots, u_i\}]$, $W_{-1} = \emptyset$, and $S_i = N_G[u_i] \setminus W_{i-1}$. To simplify notations, we assume that for $i > \ell$, $S_i = \emptyset$, and $S_{-1} = \emptyset$. Note that by the choice of P, there are no edges $xy \in E_H$ with $x \in S_j$ and $y \in N_H[\{u_0, \ldots, u_i\}]$ if $j - i > 2$.

Our dynamic programming algorithm keeps a table for each $i \in \{0, \ldots, \ell\}$, $X_i \subseteq S_{i+1}$ and $Y_i \subseteq S_{i+2}$, where $|X_i| \leq 4p, |Y_i| \leq 4p$, and an integer $next_i \in \{0, \ldots, r\}$. The table stores information about segments of (s_j^h, t_j^h)-paths inside W_i. Recall that each path can have more than one segment inside W_i, but in this case by Lemma 1, there are at most three such segments, and by Lemma 2, the number of vertices of the segments outside W_i, that join the segments inside, is bounded. We keep information about these vertices in X_i, Y_i. If $next_i = 0$, then no path in the partial solution includes u_{i+1}, and if $next_i = j > 0$, then only $(s_j^h, \tilde{t}_j^h)$, $(\tilde{s}_j^h, t_j^h)$, $(\tilde{s}_j^h, \tilde{t}_j^h)$-paths can use u_{i+1} (if $i = \ell$, then we assume that $next_i = 0$). For each $i, X_i, Y_i, next_i$, the table stores a collection of records $\mathcal{R}(i, X_i, Y_i, next_i)$ with the elements

$$\{(State_j^h, R_j^h) | 1 \leq j \leq r, 1 \leq h \leq p_i\},$$

where R_j^h are ordered multisets of size at most two without common vertices except (possibly) terminals $s_1, \ldots, s_k, t_1, \ldots, t_k$ of the original instance of SET-RESTRICTED k-DISJOINT PATHS, $R_j^h \subseteq U_j$, and where each $State_j^h$ can have one of the following five values:

Not initialized, *Started from s*, *Started from t*, *Started from middle*, *Completed.*

These records correspond to a partial solution of Set-Restricted r-Group Disjoint Paths for $H_i = H[W_i \cup X_i \cup Y_i]$ with the following properties.

- If $State_j^h = \textit{Not initialized}$, then (s_j^h, t_j^h)-paths have no vertices in H_i in the partial solution and $R_j^h = \emptyset$.
- If $State_j^h = \textit{Started from } s$, then $s_j^h \in W_i$, $t_j^h \notin V_{H_i}$ and R_j^h contains one vertex. Let $R_j = (\tilde{t}_j^h)$. Then $\tilde{t}_j^h \in S_{i-1} \cup S_i$ and the partial solution contains an $(s_j^h, \tilde{t}_j^h)$-path.
- If $State_j^h = \textit{Started from } t$, then $s_j^h \notin V_{H_i}$, $t_j^h \in W_i$ and R_j^h contains one vertex. Let $R_j^h = (\tilde{s}_j^h)$. Then $\tilde{s}_j^h \in S_{i-1} \cup S_i$ and the partial solution contains an $(\tilde{s}_j^h, t_j^h)$-path.
- If $State_j^h = \textit{Started from middle}$, then $s_j^h, t_j^h \notin V_{H_i}$ and R_j^h contains two vertices. Let $R_j^h = (\tilde{s}_j^h, \tilde{t}_j^h)$ (it can happen that $\tilde{t}_j^h = \tilde{s}_j^h$). Then $\tilde{s}_j^h, \tilde{t}_j^h \in S_{i-1} \cup S_i$ and the partial solution contains an $(\tilde{s}_j^h, \tilde{t}_j^h)$-path.
- If $State_j^h = \textit{Completed}$, then $s_j^h, t_j^h \in W_i$, $R_j^h = \emptyset$, and it is assumed that the partial solution contains an (s_j^h, t_j^h)-path.

We consequently construct the tables for $i = 0, \ldots, \ell$. The algorithm returns `Yes` if $\mathcal{R}(\ell, X_\ell, Y_\ell, next_\ell)$ for $X_\ell = Y_\ell = \emptyset$ contains the record $\{(State_j^h, R_j^h) | 1 \leq j \leq r, 1 \leq h \leq p_i\}$, where each $State_j^h =$(Completed). The tables are constructed and updated as follows.

Constructing the table for $i = 0$. First, we guess the collection of terminal pairs (s_j^h, t_j^h), $j \in \{1, \ldots, r\}$, $h \in \{1, \ldots, p_j\}$, such that the (s_j^h, t_j^h)-paths have vertices in $W_0 = N_H[u_0]$. For simplicity, assume that it holds for the pairs (s_j^h, t_j^h), where $j \in \{1, \ldots, r'\}$ and $h \in \{1, \ldots, p'_j\}$, $r' \leq r$ and each $p'_j \leq p_j$. For each pair (s_j^h, t_j^h) of this type, we guess the first vertex $\tilde{s}_j^h$ and the last vertex $\tilde{t}_j^h$ of the (s_j^h, t_j^h)-path in W_0, $\hat{s}_j^h \prec_{s_j^h} \hat{t}_j^h$. Note that $\tilde{s}_j^h = s_j^h$ ($\tilde{t}_j^h = t_j^h$ respectively) if $s_j^h \in W_0$ ($t_j^h \in W_0$ respectively). For $j \in \{1, \ldots, r'\}$, $h \in \{1, \ldots, p'_j\}$, denote by P_j^h the corresponding $(\tilde{s}_j^h, \tilde{t}_j^h)$-subpath of the solution.

Clearly, we can assume that these paths are induced. By Lemma 1, P_j^h has at most two segments outside W_0 and by Lemma 2, each of these segments contains at most two vertices. Then the total number of the segments of paths P_j^h outside W_0 is at most $2p$. Because the vertices of these segments are adjacent to the vertices of W_0, they are in $S_1 \cup S_2$. We guess the set X'_0 of at most $4p$ vertices of them in S_1 and the set Y'_0 of at most $4p$ vertices of them in S_2.

Now we do guesses for u_0. We decide whether u_0 is included in some P_j^h or is not included in any of the paths. If u_0 is a vertex of P_j^h, then by Lemma 3, P_j^h has at most three vertices. Moreover, recall that we are interested in solutions of Set-Restricted r-Group Disjoint Paths that could be extended to solutions of Set-Restricted r-Disjoint Paths and conclude that by Lemma 3, $h = 1$ and $p'_j = 1$. Otherwise we discard the choice. Then we guess the set of vertices $Z = V_{P_j^h}$. Suppose now that u_0 is not included in any P_j^h. Assume that $u_0 \in U_j$. Since we can assume that the final solution of Set-Restricted r-Disjoint Paths is minimal, by Lemma 4 we can guess the paths (if any) P_j^h for $h \in \{1, \ldots, p'_i\}$

or discard the choice. Let now Z be the set of the vertices of all such paths (Z can be empty). Observe that by Lemma 4, $|Z| \leq 3$.

Now we are ready to construct $\mathcal{R}(0, X_0, Y_0, next_0)$ for each of these choices. We include a record $\{(State_j^h, R_j^h) | 1 \leq j \leq r, 1 \leq h \leq p_i\}$ in $\mathcal{R}(0, X_0, Y_0, next_0)$ if the following holds.

- $X_0' \subseteq X_0$ and $Y_0' \subseteq Y_0$.
- For $State_j^h$ and R_j^h,
  - if $s_j^h, t_j^h \in W_0$, then $State_j^h$ =*Completed*, $R_j^h = \emptyset$;
  - if $s_j^h \in W_0, t_j^h \notin W_0$, then $State_j^h$ =*Started from s*, $R_j^h = (\tilde{t}_j^h)$;
  - if $s_j^h \notin W_0, t_j^h \in W_0$, then $State_j^h$ =*Started from t*, $R_j^h = (\tilde{s}_j^h)$;
  - if $s_j^h \notin W_0, t_j^h \notin W_0$ and $j \leq r', h \leq p_i'$, then $State_j^h$ =*Started from middle*, $R_j^h = (\tilde{s}_j^h, \tilde{t}_j^h)$; and
  - $State_j^h$ =*Not initialized*, $R_j^h = \emptyset$ otherwise.
- If $next_0 = 0$, then we exclude u_1 from all domains U_j, else we exclude u_1 from domains U_j such that $next_0 \neq j$. Note that the value of $next_0$ should be consistent with our guesses, i.e., if u_1 is already chosen to be included in P_j^h, then $next_0 = j$. Then we solve Set-Restricted r'-Group Disjoint Paths for the pairs $(\tilde{s}_j^h, \tilde{t}_j^h)$, where $j \in \{1, \ldots, r'\}$ and $h \in \{1, \ldots, p_j'\}$ for $H[W_0 \cup X_0' \cup Y_0']$. Observe that by the choice of Z, we already guessed the paths from one group. Hence, it remains to find the paths P_j^h from at most $r-1$ groups, and we do it by calling our algorithm recursively for $H[W_0 \cup X_0' \cup y_0'] - Z$ and the corresponding terminal pairs. We include the record in $\mathcal{R}(0, X_0, Y_0, next_0)$ if we get a `Yes`-answer.

Constructing the table for $i > 0$**.** We assume that $\mathcal{R}(i-1, X_{i-1}, Y_{i-1}, next_{i-1})$ is already constructed. We consider each element $\{(\tilde{S}tate_j^h, \tilde{R}_j^h) | 1 \leq j \leq r, 1 \leq h \leq p_i\}$ of $\mathcal{R}(i-1, X_{i-1}, Y_{i-1}, next_{i-1})$ and we then construct their successors in $\mathcal{R}(i, X_i, Y_i, next_i)$.

First, we guess the collection of terminal pairs (s_j^h, t_j^h), $j \in \{1, \ldots, r\}$, $h \in \{1, \ldots, p_j\}$, such that the (s_j^h, t_j^h)-paths have vertices in S_i. For simplicity, assume that it holds for the pairs (s_j^h, t_j^h), where $j \in \{1, \ldots, r'\}$ and $h \in \{1, \ldots, p_j'\}$, $r' \leq r$ and each $p_j' \leq p_j$. Suppose also that for $j \in \{1, \ldots, r''\}$ and $h \in \{1, \ldots, p_j''\}$, where $r'' \leq r'$ and each $p_j'' \leq p_j'$, the (s_j^h, t_j^h)-paths have no vertices in W_{i-1}, i.e., $\tilde{S}tate_j^h$ = *Not initialized*.

For $j \in \{1, \ldots, r''\}$ and $h \in \{1, \ldots, p_j''\}$, we guess the first vertex $\hat{s}_j^h$ and the last vertex $\hat{t}_j^h$ of the (s_j^h, t_j^h)-path in $S_i \setminus (X_{i-1} \cup Y_{i-2})$, $\hat{s}_j^h \prec_{s_j^h} \hat{t}_j^h$, (it is assumed that $Y_{-1} = \emptyset$). Recall here that the vertices of $X_{i-1} \cup Y_{i-2}$ are already assumed to be included in paths that have vertices in W_{i-1}. Note that $\hat{s}_j^h = s_j^h$ ($\hat{t}_j^h = t_j^h$ respectively) if $s_j^h \in S_i$ ($t_j^h \in S_i$ respectively). For $j \in \{1, \ldots, r''\}$, $h \in \{1, \ldots, p_j''\}$, denote by P_j^h the corresponding $(\hat{s}_j^h, \hat{t}_j^h)$-subpath of the solution. Clearly, we can assume that these paths are induced. By Lemma 1, P_j^h has at most two segments outside S_i and by Lemma 2, each of these segments contains at most two vertices. Then by Lemma 1, the total number of the segments of paths P_j^h outside S_i is at most $2p''$, where $p''=p''_1+\ldots+p''_{r''}$. Because the vertices of these segments are adjacent to the vertices of S_i, they are in $S_{i+1}\cup S_{i+2}$. We guess the set X'_i with at most $4p''$ vertices of them in $S_{i+1}\setminus Y_{i-1}$ and the set Y'_i with at most $4p''$ vertices of them in S_{i+2}.

Consider now $j\in\{r''+1,\ldots,r'\}$ and $h\in\{p''_j+1,\ldots,p'_j\}$. We have the following cases.

Case 1. $State_j^h$ = Started from s. Then $s_j^h\in W_{i-1}$ and the partial solution in $\mathcal{R}(i-1,X_{i-1},Y_{i-1},next_{i-1})$ contains a $(s_j^h,\tilde{t}_j^h)$-path. Recall that $\tilde{R}_j^h=(\tilde{t}_j^h)$. We guess the first vertex $\hat{s}_j^h$ and the last vertex $\hat{t}_j^h$ of the (s_j^h,t_j^h)-path in $S_i\setminus(X_{i-1}\cup Y_{i-2})$, $\hat{s}_j^h\prec_{s_j^h}\hat{t}_j^h$, where $\hat{s}_j^h$ is adjacent to $\tilde{t}_j^h$. Note that $\hat{t}_j^h=t_j^h$ if $t_j^h\in S_i$. Denote by P_j^h the corresponding $(\hat{s}_j^h,\hat{t}_j^h)$-subpath of the solution. We say that P_j^h is obtained by the *extension from* $(\tilde{t}_j^h)$.

Case 2. $State_j^h$ = Started from t. Then $t_j^h\in W_{i-1}$ and the partial solution in $\mathcal{R}(i-1,X_{i-1},Y_{i-1},next_{i-1})$ contains a $(\tilde{s}_j^h,t_j^h)$-path. Recall that $\tilde{R}_j^h=(\tilde{s}_j^h)$. We guess the first vertex $\hat{s}_j^h$ and the last vertex $\hat{t}_j^h$ of the (s_j^h,t_j^h)-path in $S_i\setminus(X_{i-1}\cup Y_{i-2})$, $\hat{s}_j^h\prec_{s_j^h}\hat{t}_j^h$, where $\hat{t}_j^h$ is adjacent to $\tilde{s}_j^h$. Note that $\hat{s}_j^h=s_j^h$ if $s_j^h\in S_i$. Denote by P_j^h the corresponding $(\hat{s}_j^h,\hat{t}_j^h)$-subpath of the solution. We say that P_j^h is obtained by the *extension from* $(\tilde{s}_j^h)$.

Case 3. $State_j^h$ = Started from middle. Then $s_j^h,t_j^h\notin W_{i-1}$ and the partial solution in $\mathcal{R}(i-1,X_{i-1},Y_{i-1},next_{i-1})$ contains a $(\tilde{s}_j^h,\tilde{t}_j^h)$-path. Recall that $\tilde{R}_j^h=(\tilde{s}_j^h,\tilde{t}_j^h)$ in this case. We have three possibilities to extend this path.

Case 3.1. We guess the first vertex $\hat{s}_j^h$ and the last vertex $\hat{t}_j^h$ of the (s_j^h,t_j^h)-path in $S_i\setminus(X_{i-1}\cup Y_{i-2})$, $\hat{s}_j^h\prec_{s_j^h}\hat{t}_j^h$, where $\tilde{t}_j^h\prec_{s_j^h}\hat{s}_j^h$ and $\tilde{t}_j^h$ is adjacent to $\hat{s}_j^h$. Denote by P_j^h the corresponding $(\hat{s}_j^h,\hat{t}_j^h)$-subpath of the solution. We say that P_j^h is obtained by the *extension from* $(\tilde{t}_j^h)$. Note that $\hat{t}_j^h=t_j^h$ if $t_j^h\in S_i$.

Case 3.2. We guess the first vertex $\hat{s}_j^h$ and the last vertex $\hat{t}_j^h$ of the (s_j^h,t_j^h)-path in $S_i\setminus(X_{i-1}\cup Y_{i-2})$, $\hat{s}_j^h\prec_{s_j^h}\hat{t}_j^h$, where $\hat{t}_j^h\prec_{s_j^h}\tilde{s}_j^h$ and $\hat{t}_j^h$ is adjacent to $\tilde{s}_j^h$. Denote by P_j^h the corresponding $(\hat{s}_j^h,\hat{t}_j^h)$-subpath of the solution. We say that P_j^h is obtained by the *extension from* $(\tilde{s}_j^h)$. Note that $\hat{s}_j^h=s_j^h$ if $s_j^h\in S_i$.

Case 3.3. We guess the first vertex $\hat{s}_j^{h(1)}$ and the last vertex $\hat{t}_j^{h(1)}$ of the (s_j^h,t_j^h)-path in $S_i\setminus(X_{i-1}\cup Y_{i-2})$, $\hat{s}_j^{h(1)}\prec_{s_j^h}\hat{t}_j^{h(1)}$, where $\hat{t}_j^{h(1)}$ is adjacent to $\tilde{s}_j^h$. Then we guess the first vertex $\hat{s}_j^{h(2)}$ and the last vertex $\hat{t}_j^{h(2)}$ of the (s_j^h,t_j^h)-path in $S_i\setminus(X_{i-1}\cup Y_{i-2})$, $\hat{s}_j^{h(2)}\prec_{s_j^h}\hat{t}_j^{h(2)}$, where $\hat{s}_j^{h(2)}$ is adjacent to $\tilde{t}_j^h$. Note that $\hat{s}_j^{h(1)}=s_j^h$ if $s_j^h\in S_i$ and $\hat{t}_j^{h(2)}=t_j^h$ if $t_j^h\in S_i$. Denote by $P_j^{h(1)},P_j^{h(2)}$ the corresponding $(\hat{s}_j^{h(1)},\hat{t}_j^{h(1)})$ and $(\hat{s}_j^{h(2)},\hat{t}_j^{h(2)})$-subpath of the solution respectively. We say that $P_j^{h(1)},P_j^{h(2)}$ are obtained by the *extension from* $(\tilde{s}_j^h)$ and $(\tilde{t}_j^h)$ respectively.

Clearly, we can assume that all the paths $P_j^h, P_j^{h(1)}, P_j^{h(2)}$ are induced. By Lemma 1, each path has at most two segments outside S_i and by Lemma 2, each of these segments contains at most two vertices. Then the total number of segments of the paths $P_j^h, P_j^{h(1)}, P_j^{h(2)}$ outside S_i is at most $2(p' - p'')$, where $p' = P_1' + \ldots + p_{r'}'$. Because the vertices of these segments are adjacent to the vertices of S_i, they are in $S_{i+1} \cup S_{i+2}$. We guess the set X_i'' of at most $4(p' - p'')$ vertices of them in $S_{i+1} \setminus (Y_{i-1} \cup X_i')$ and the set Y_i'' of at most $2(p' - p'')$ their vertices in $S_{i+2} \setminus Y_i'$.

Observe additionally that again by Lemma 1, it should hold that $|Y_{i-1} \cup X_i' \cup X_i''| \leq 4p$; otherwise we discard our current choice.

Now we consider u_i. We have two cases.

Case 1. $next_{i-1} = 0$. Then u_i is not included in any path in the partial solution. Assume that $u_0 \in U_j$. Since we can assume that the final solution of Set-Restricted r-Disjoint Paths is minimal, by Lemma 4 we can guess the paths (if any) $P_j^h, P_j^{h(1)}, P_j^{h(2)}$ for $h \in \{1, \ldots, p_i'\}$ or discard the choice. Let now Z be the set of the vertices of all such paths (Z can be empty). Observe that by Lemma 4, $|Z| \leq 3$.

Case 2. $next_{i-1} = j > 0$. By Lemma 3, P_j^h (or $P_j^{h(1)}$ or $P_j^{h(2)}$) has at most three vertices. Moreover, we again recall that we are interested in solutions of Set-Restricted r-Group Disjoint Paths that could be extended to solutions of Set-Restricted r-Disjoint Paths and conclude by Lemma 3, that we should have at most one path for j; otherwise we discard the choice. Then we guess the set of vertices Z of this path.

Now we are ready to construct $\mathcal{R}(i, X_i, Y_i, next_i)$ for each of these choices. Recall that the pairs $(\tilde{State}_j^h, \tilde{R}_j^h)$ are elements of $\mathcal{R}(i-1, X_{i-1}, Y_{i-1}, next_{i-1})$. We include a record $\{(State_j^h, R_j^h) | 1 \leq j \leq r, 1 \leq h \leq p_i\}$ in $\mathcal{R}(i, X_i, Y_i, next_i)$ if the following holds.

- If $\tilde{State}_j^h =$*Completed*, then $State_j^h =$*Completed*, $R_j^h = \emptyset$.
- If $\tilde{State}_j^h =$*Not initialized* and $j > r'$, then $State_j^h =$*Not initialized*, $R_j^h = \emptyset$.
- If $\tilde{State}_j^h =$*Started from s* or *Started from t* or *Started from middle* and $z \in \tilde{R}_j^h \cap S_{i-2}$, then $r'' + 1 \leq j \leq r'$ and the path obtained by extension from z is added.
- If $\tilde{State}_j^h =$*Started from s* and $\tilde{R}_j^h = (\tilde{t}_j^h)$, then
  - if $j > r'$, then $State_j^h =$*Started from s*, $R_j^h = (\tilde{t}_j^h)$;
  - if $r'' + 1 \leq j \leq r'$, then if $t_j^h \in S_i$, then $State_j^h =$*Completed*, $R_j^h = \emptyset$; else if $t_j^h \notin S_i$, then $State_j^h =$*Started from s*, $R_j^h = (\hat{t}_j^h)$.
- If $\tilde{State}_j^h =$*Started from t* and $\tilde{R}_j^h = (\tilde{s}_j^h)$, then
  - if $j > r'$, then $State_j^h =$*Started from t*, $R_j^h = (\tilde{s}_j^h)$;
  - if $r'' + 1 \leq j \leq r'$, then if $s_j^h \in S_i$, then $State_j^h =$*Completed*, $R_j^h = \emptyset$; else if $s_j^h \notin S_i$, then $State_j^h =$*Started from s*, $R_j^h = (\hat{s}_j^h)$.
- If $\tilde{State}_j^h =$*Started from middle* and $\tilde{R}_j^h = (\tilde{s}_j^h, \tilde{t}_j^h)$, then

  - if $j > r'$, then $State_j^h$ =*Started from middle*, $R_j^h = (\tilde{s}_j^h, \tilde{t}_j^h)$;
  - if $r'' + 1 \leq j \leq r'$ and only the path obtained by the extension from $\tilde{t}_j^h$ is included in the partial solution by our guesses, then if $t_j^h \in S_i$, then $State_j^h$ =*Started from s*, $R_j^h = (\tilde{s}_j^h)$; else if $t_j^h \notin S_i$, then $State_j^h$ =*Started from middle*, $R_j^h = (\tilde{s}_j^h, \hat{t}_j^h)$;
  - if $r'' + 1 \leq j \leq r'$ and only the path obtained by the extension from $\tilde{s}_j^h$ is included in the partial solution by our guesses, then if $s_j^h \in S_i$, then $State_j^h$ =*Started from t*, $R_j^h = (\tilde{t}_j^h)$; else if $s_j^h \notin S_i$, then $State_j^h$ =*Started from middle*, $R_j^h = (\hat{s}_j^h, \tilde{t}_j^h)$;
  - if $r'' + 1 \leq j \leq r'$ and two path obtained by the extension from $\tilde{s}_j^h$ and $\tilde{t}_j^h$ are included in the partial solution by our guesses, then if $s_j^h, t_j^h \in S_i$, then $State_j^h$ =*Completed*, $R_j^h = \emptyset$; else if $s_j^h \in S_i, t_j^h \notin S_i$, then $State_j^h$ =*Started from s*, $R_j^h = (\hat{t}_j^{h(2)})$; else if $s_j^h \notin S_i, t_j^h \in S_i$, then $State_j^h$ =*Started from t*, $R_j^h = (\hat{s}_j^{h(1)})$; else if $s_j^h, t_j^h \notin S_i$, then $State_j^h$ = *Started from middle*, $R_j^h = (\hat{s}_j^{h(1)}, \hat{s}_j^{h(2)})$.
- For $j \leq r''$,
  - if $s_j^h, t_j^h \in S_i$, then $State_j^h$ =*Completed*, $R_j^h = \emptyset$;
  - if $s_j^h \in S_i, t_j^h \notin S_i$, then $State_j^h$ =*Started from s*, $R_j^h = (\hat{t}_j^h)$;
  - if $s_j^h \notin S_i, t_j^h \in S_i$, then $State_j^h$ =*Started from t*, $R_j^h = (\hat{s}_j^h)$;
  - if $s_j^h \notin S_i, t_j^h \notin S_i$ and $j \leq r', h \leq p_i'$, then $State_j^h$ =*Started from middle*, $R_j^h = (\hat{s}_j^h, \hat{t}_j^h)$; and
  - $State_j^h$ =*Not initialized*, $R_j^h = \emptyset$ otherwise.
- $X_i' \cup X_i'' \cup Y_i \subseteq X_i$ and $Y_i' \cup Y_i'' \subseteq Y_i$.
- If $next_i = 0$, then we exclude u_{i+1} from all domains U_j, else we exclude u_{i+1} from domains U_j such that $next_i \neq j$. Note that the value of $next_i$ should be consistent with our guesses, i.e., if u_{i+1} is already chosen to be included in P_j^h, then $next_0 = j$. Then we solve Set-Restricted r'-Group Disjoint Paths for the pairs $(\hat{s}_j^h, \hat{t}_j^h)$, $(\hat{s}_j^{h(1)}, \hat{t}_j^{h(1)})$ and $(\hat{s}_j^{h(2)}, \hat{t}_j^{h(2)})$ obtained by our guesses, where $j \in \{1, \ldots, r'\}$ and $h \in \{1, \ldots, p_j'\}$ for $H' = H[S_i \setminus (X_{i-1} \cup Y_{i-2}) \cup X_i' \cup X_i'' \cup Y_i' \cup Y_i'']$. Observe that by the choice of Z, we already guessed the paths from one group. Hence, it remains to find the paths from at most $r - 1$ groups, and we do it by calling our algorithm recursively for $H' - Z$ and the corresponding terminal pairs. We include the record in $\mathcal{R}(i, X_i, Y_i, next_i)$ if we get a `Yes`-answer.

Now we are ready to prove our main theorem.

Theorem 3. Set-Restricted k-Disjoint Paths *can be solved in* $O(n^{f(k)})$ *time for* n*-vertex AT-free graphs for some function* $f(k)$ *that only depends on* k*.*

Proof. We apply the algorithm for Set-Restricted k-Group Disjoint Paths described above for G, $p_1 = \ldots = p_k = 1$, terminal pairs $(s_j^1, t_j^1) = (s_j, t_j)$, and domains $U_1, \ldots, U_k$.

Correctness of the algorithm follows from its construction. It is sufficient to observe that if the `Yes`-answer was given by the algorithm then we have our disjoint paths. From another side, if a given instance of Set-Restricted k-Disjoint Paths has a solution, then we can assume that this solution is minimal, and in each step of the algorithm we can always make guesses that correspond to these paths. Clearly, these choices would lead us to a `Yes`-answer.

It remains to prove that the algorithm runs in polynomial time for any fixed k. To produce each $\mathcal{R}(i, X_i, Y_i, next_i)$, we guess some vertices. The total number of guesses is at most $O(p_1+\ldots+p_r)$. Note that initially $r = k$ and $p_1 = \ldots = p_k = 1$. We call our algorithm recursively and for each recursive call, we reduce the number of groups and we can multiply the number of pairs in a group by at most two. Hence, $p_1 + \ldots + p_r \leq k2^{k-1}$. If $r = 1$, then we solve the problem using Lemma 5 in time $g(k2^{k-1})n^3$ for some function $g(k)$ not depending on n. Then the running time of our algorithm is $O(n^{f(k)})$ for some function $f(k)$ not depending on n. □

3.3 Set-restricted k-Disjoint Connected Subgraphs

We show that our algorithm for Set-Restricted k-Disjoint Paths can be applied to solve the more general problem Set-restricted k-Disjoint Connected Subgraphs on AT-free graphs.

Theorem 4. Set-Restricted k-Disjoint Connected Subgraphs *can be solved in polynomial time on AT-free graphs for any fixed integer k.*

Proof. Clearly, we can assume that for $i \in \{1, \ldots, k\}$, $S_i \subseteq U_i$, and for any $j \in \{1, \ldots, k\}$, $j \neq i$, $S_i \cap U_j = \emptyset$.

Observe that if G contain k pairwise vertex-disjoint connected subgraphs $G_1, \ldots, G_k$ such that $S_i \subseteq V_{G_i} \subseteq U_i$, for $1 \leq i \leq k$, then each G_i is AT-free and, therefore, has a dominating pair (u_i, v_i) by Theorem 1. For each $i \in \{1, \ldots, k\}$, we guess this pair (u_i, v_i) (it can happen that $u_i = v_i$), and guess at most six vertices of a shortest (u_i, v_i)-path P_i in G_i as follows: if P_i has at most five vertices, then we guess all vertices of P_i, and if P_i has at least six vertices, then we guess the first three vertices $x_1^i, x_2^i, x_s^i \in U_i$ and the last three vertices $y_1^i, y_2^i, y_3^i \in U_i$, $u_i = x_1^i$, $v_i = y_3^i$ and $x_1^i \prec_{u_i} x_2^i \prec_{u_i} x_3^i \prec_{u_i} y_1^i \prec_{u_i} y_2^i \prec_{u_i} y_3^i$ in P_i. Observe that P_i is an induced path.

If we guess all vertices of P_i, then we check whether P_i dominates S_i, and if it is so, then we solve Set-Restricted Disjoint Connected Subgraphs for the graph $G - (V_{P_i} \cup S_i)$ and the sets S_j, $j \neq i$ with their domains. Otherwise we discard our choice.

Now we can assume that for each $i \in \{1, \ldots, k\}$, P_i has at least six vertices. We modify domains U_i and sets S_i: $U_i' = (U_i \setminus N_G[\{x_i^1, x_i^2, y_i^2, y_i^3\}]) \cup \{x_i^1, x_i^2, x_i^3, y_1^1, y_i^2, y_i^3\}$ and $S_i' = U_i \setminus N_G[\{x_i^1, x_i^2, y_i^2, y_i^3\}]$ for $i \in \{1, \ldots, k\}$. The vertices of S_i' should be in the same component of $G[U_i']$; otherwise we discard our guess, since P_i should dominate S_i. Denote by U_i'' the set of vertices of the component of $G[U_i']$ with $S_i \subseteq U_i''$.

We claim that (x_1^i, y_3^i) is a dominating pair in $G[U_i'']$ for $i \in \{1, \dots, k\}$. To show it, consider a dominating pair (x, y) in $G[U_i'']$. Any (x, y)-path P dominates x_1^i and y_3^i. It follows that one vertex of the pair is in $\{x_1^i, x_2^i\}$ and another is in $\{y_2^i, y_3^i\}$. It remains to observe that if x_2^i (y_2^i respectively) is in the pair, then it can be replaced by x_1^i (y_3^i respectively).

It follows that any (x_1^i, y_3^i)-path in dominates $G[U_i'']$. We can find disjoint (x_1^i, y_3^i)-paths (if exist), by solving Set-Restricted k-Disjoint Paths for the pairs of terminals (x_1^i, y_3^i) with domains U_i'' for $i \in \{1, \dots, k\}$. Since by Theorem 3, it can be done in polynomial time, and we guess at most $6k$ vertices, the claim of the theorem follows. □

4 Induced Minors

In this section we consider the H-Induced Minor problem. It is convenient for us to represent this problem in the following way. An *H-witness structure* of G is a collection of $|V_H|$ non-empty mutually disjoint sets $W(x) \subseteq V_G$, one set for each $x \in V_H$, called *H-witness sets*, such that

(i) each $W(x)$ is a connected set; and
(ii) for all $x, y \in V_H$ with $x \neq y$, sets $W(x)$ and $W(y)$ are adjacent in G if and only if x and y are adjacent in H.

Observe that H is an induced minor of G if and only if G has an H-witness structure.

Theorem 5. *H-Induced Minor can be solved in polynomial time on AT-free graphs for any fixed graph H.*

Proof. Suppose that H is an induced minor of G. Then G has an H-witness structure, i.e., sets $W(x) \subseteq V_G$ for $x \in V_H$. For each $x \in V_H$, $G[W(x)]$ is a connected AT-free graph. Hence, by Theorem 1, $G[W(x)]$ has a dominating pair (u_x, v_x).

For each $x \in V_H$, we guess the pair (u_x, v_x) (it can happen that $u_x = v_x$), and guess at most six vertices of a shortest (u_x, v_x)-path P_x in $G[W(x)]$ as follows: if P_x has at most five vertices, then we guess all vertices of P_x, and if P_x has at least six vertices, then we guess the first three vertices u_1^x, u_2^x, u_3^x and the last three vertices v_1^x, v_2^x, v_3^x such that $u_x = u_1^x$, $v_x = v_3^x$ and $u_1^x \prec_{u_x} u_2^x \prec_{u_x} u_3^x \prec_{u_x} v_1^x \prec_{u_x} v_2^x \prec_{u_x} v_3^x$ in P_x. Observe that P_x is an induced path. We denote by X_1, X_2 the partition of V_H (one of the sets can be empty), where for $x \in X_1$, all at most five vertices of P_x were chosen, and for $x \in X_2$, we have the vertices $u_1^x, u_2^x, u_3^x, v_1^x, v_2^x, v_3^x$. Further, for each edge $xy \in E_H$, we guess adjacent vertices $s_{xy}, s_{yx} \in V_G$, where $s_{xy} \in W(x)$ and $s_{yx} \in W(y)$. Note that the vertices s_{xy} are not necessarily distinct, and some of them can coincide with the vertices chosen to represent P_x. Let $S(x) = \{s_{xy} | xy \in E_H\}$. All the guesses should be consistent with the witness structure, i.e., vertices included in distinct $W(x)$ should be distinct, and if $xy \notin E_H$, then the vertices included in $W(x)$ and $W(y)$ should be non-adjacent in G.

For $x \in X_1$, we check whether the guessed path P_x dominates $S(x)$, and if it is so, then we let $W'(x) = V_{P_x} \cup S(x)$. Otherwise we discard our choice.

Recall that we already selected some vertices, and that we cannot use these vertices and also not their neighbors in case non-adjacencies in H forbid this. Hence, for each $x \in X_2$, we obtain the set

$$\begin{aligned}U_x = V_G \setminus \big(& (\cup_{y \in X_1, xy \in E_H} W'(y)) \cup (\cup_{y \in X_1, xy \notin E_H} N_G[W'(y)]) \cup \\ & \cup (\cup_{y \in X_2, xy \in E_H} (S(y) \cup \{u_1^y, u_2^y, u_3^y, v_1^y, v_2^y, v_3^y\}) \cup \\ & \cup (\cup_{y \in X_2 \setminus \{x\}, xy \notin E_H} N_G[S(y) \cup \{u_1^y, u_2^y, u_3^y, v_1^y, v_2^y, v_3^y\}] \cup \\ & \cup N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]\big) \cup \{u_1^x, u_2^x, u_3^x, v_1^x, v_2^x, v_3^x\}.\end{aligned}$$

Then for each $x \in X_2$, we check whether $S'(x) = S(x) \setminus N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$ is included in one component of $G[U_x]$. If it is not so, then we discard our choice, since we cannot have a path with the first vertices u_1^x, u_2^x, u_3^x and the last vertices v_1^x, v_2^x, v_3^x that dominates $S(x)$. Otherwise we denote by U'_x the set of vertices of the component of $G[U_x]$ that contains $S'(x)$. Note that (u_1^x, v_3^x) is a dominating pair in $G[U'_x]$ for $x \in X_2$. To show it, consider a dominating pair (u, v) in $G[U'_x]$. Any (u, v)-path P dominates u_1^x and v_3^x. It follows that one vertex of the pair is in $\{u_1^x, u_2^x\}$ and another is in $\{v_2^x, v_3^x\}$. It remains to observe that if u_2^x (v_2^x respectively) is in the pair, then it can be replaced by u_1^x (v_3^x respectively). We solve Set-Restricted $|X_2|$-Disjoint Paths for the pairs of terminals (u_1^x, v_3^x) with domains U'_x for $x \in X_2$. If we get a `No`-answer, then we discard our guess since there are no P_x that satisfy our choices. Otherwise, let P'_x be the (u_1^x, v_3^x)-path in the obtained solution for $x \in X_2$. We let $W'(x) = P'_x \cup S(x)$.

We claim that the sets $W'(x)$ compose an H-witness structure. To show it, observe first that by the construction of these sets, $W'(x)$ are disjoint. If $xy \in E_H$, then as $s_{xy} \in W'(x)$ and $s_{yx} \in W'(y)$, $W'(x)$ and $W'(y)$ are adjacent. It remains to prove that if $xy \notin E_G$, then $W'(x)$ and $W'(y)$ are not adjacent. To obtain a contradiction, assume that $W'(x)$ and $W'(y)$ are adjacent for some $x, y \in V_H$, i.e., there is $uv \in E_G$ with $u \in W'(x)$ and $v \in W'(y)$, where $xy \notin E_H$. By the construction of $W'(x), W'(y)$, $x, y \in X_2$. Moreover, $u \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$ or $v \notin N_G[\{u_1^y, u_2^y, v_2^y, v_3^y\}]$. If $u \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$, then we consider u_1^x, v_3^x, u_1^y and observe that these vertices compose an asteroidal triple. Clearly, the (u_1^x, v_3^x)-path P'_x avoids $N_G[u_1^y]$, because $N_G[u_1^y] \cap U_x = \emptyset$. Because $u \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$, u is either in P'_x or adjacent to a vertex in P'_x and v is either in P'_y or adjacent to a vertex in P'_y, $G[W'(x) \cup W'[y]] - N_G[u_1^x]$ and $G[W'(x) \cup W'[y]] - N_G[v_3^x]$ are connected. Hence, there are (u_1^x, u_1^y) and (v_3^x, u_1^y)-paths that avoid $N_G[v_3^x]$ and $N_G[u_1^x]$ respectively. By symmetry, we conclude that if $v \notin N_G[\{u_1^y, u_2^y, v_2^y, v_3^y\}]$, then u_1^y, v_3^y, u_1^x is an asteroidal triple. This contradiction proves our claim.

To complete the proof, note that we guess at most $6|V_H| + 2|E_H|$ vertices of G, and we can consider all possible choices in time $n^{O(|V_H|+|E_H|)}$, where $n = |V_G|$. If for one of the choices we get an H-witness structure, then H is an induced minor of G, otherwise we return `No`. As we can solve Set-Restricted $|X_2|$-Disjoint Paths in time $n^{f(|V_H|)}$ by Theorem 3, the claim follows. $\square$

We complement Theorem 5 as follows. A graph is *cobipartite* (and consequently AT-free) if its vertex set can be partitioned into two cliques. Clearly, any cobipartite graph is AT-free.

Theorem 6. *The H-Induced Minor problem is* NP*-complete for cobipartite graphs, and* $\mathsf{W}[1]$*-hard for cobipartite graphs when parameterized by* $|V_H|$.

Proof. We first give a $\mathsf{W}[1]$-hardness proof that also serves as an NP-hardness proof. We reduce from the well-known $\mathsf{W}[1]$-complete Clique problem [6]. For a graph G and a parameter k, this problem asks whether G has a clique of size k. Let (G, k) be an instance of Clique. Without loss of generality we assume that $k \geq 4$. We construct the cobipartite graph G' as follows:

- create a copy of V_G and construct a clique U on these vertices;
- for each edge $uv \in E_G$, create a vertex e_{uv} adjacent to all vertices of $U \setminus \{u, v\}$;
- construct the clique $W = \{e_{uv} \mid uv \in E_G\}$.

Now we construct the graph H:

- create a clique X of size k with vertices $x_1, \ldots, x_k$;
- create a clique Y with $\frac{1}{2}k(k-1)$ vertices y_{ij}, $1 \leq i < j \leq k$;
- for each pair i, j with $1 \leq i < j \leq k$, make x_h adjacent to y_{ij} for $h \in \{1, \ldots, k\}$, $h \neq i, j$.

We prove that G has a clique of size k if and only if H is an induced minor of G'. First suppose that G has a clique $\{u_1, \ldots, u_k\}$. Then the subgraph of G' induced by this clique and the set of vertices $\{e_{u_i u_j} \mid 1 \leq i < j \leq k\}$ is isomorphic to H, i.e, H is an induced subgraph of G', and therefore an induced minor of G'.

Now suppose that H is an induced minor of G'. Let $W(z) \subseteq V_{G'}$ for $z \in V_H$ be an H-witness structure. Note that because $k \geq 4$, each vertex $x_i \in U$ is not adjacent to at least three vertices of H. Since for each vertex $e_{uv} \in W$ $d_G(e_{uv}) = |V_G| - 3$, we find that $W \cap (\cup_{i=1}^{k} W(x_i)) = \emptyset$ and $\cup_{i=1}^{k} W(x_i) \subseteq U$. Suppose that $U \cap W(y_{ij}) \neq \emptyset$ for some $1 \leq i < j \leq k$. Then $W(y_{ij})$ is adjacent to each set $W(x_h)$ for $h \in \{1, \ldots, k\}$, but y_{ij} is not adjacent to two vertices of $\{x_1, \ldots, x_k\}$. This gives us a contradiction, and, therefore, $\cup_{1 \leq i < j \leq k} W(y_{ij}) \subseteq W$. Now we prove that $|W(z)| = 1$ for $z \in V_H$. To obtain a contradiction, assume that $|W(z)| \geq 2$ for some $z \in V_H$. We have two cases.

Case 1. $e_{u_1 u_2}, e_{v_1 v_2} \in W(y_{ij})$ for some $1 \leq i < j \leq k$. Observe that $d_G(e_{u_1 u_2}) = d_G(e_{v_1 v_2}) = |V_G| - 3$. Also for $u_1 u_2 \neq v_1 v_2$, $N_G[e_{u_1 u_2}] \neq N_G[e_{v_1 v_2}]$. Hence, $d_H(y_{ij}) \geq |V_H| - 2$, but H has no such vertices; a contradiction.

Case 2. $u, v \in W(x_i)$ for some $i \in \{1, \ldots, k\}$. Let $j \in \{1, \ldots, k\}$, $j \neq i$. Assume without loss of generality that $i < j$ (otherwise consider further the vertex y_{ji} instead of y_{ij}). The vertex y_{ij} is not adjacent to x_i, y_j. Hence, $W(y_{ij})$ is not adjacent to $W(x_i)$ and $W(x_j)$, i.e., the single vertex from W in $W(y_{ij})$ is not adjacent to at least three vertices of U. It remains to observe that any vertex in W is not adjacent to at most two vertices in U; a contradiction.

We conclude that H is an induced subgraph of G', and, moreover, the vertices of X are in U, and the vertices of Y are in W for the copy of H in G'. Hence, X is a clique in G. This completes the $\mathsf{W}[1]$-hardness proof.

To complete the proof of Theorem 6, it remains to observe that our $\mathsf{W}[1]$-hardness proof immediately implies NP-hardness, as Clique is NP-complete and our reduction is polynomial. $\square$

5 Contractibility

In this concluding section we give another applications the algorithm for Set-Restricted k-Disjoint Paths. We show that it can be used to obtain an algorithm for H-Contractibility for the case when H has no triangles.

As with Induced Minor, we represent H-Contractibility as a partition problem. An *H-contraction witness structure* of G is a partition of V_G into $|V_H|$ non-empty disjoint sets $W(x) \subseteq V_G$, one set for each $x \in V_H$, called *H-contraction witness sets*, such that

(i) each $W(x)$ is a connected set; and
(ii) for all $x, y \in V_H$ with $x \neq y$, sets $W(x)$ and $W(y)$ are adjacent in G if and only if x and y are adjacent in H.

Clearly, H is a contraction of G if and only if G has an H-contraction witness structure: H can be obtained from G by contracting the edges in each H-contraction witness set until a single vertex remains in each of them.

Let H be an induced minor of G with the witness structure $W(x)$ for $x \in V_H$ such that $\cup_{x \in V_H} W(x)$ is a dominating set of G. We say that the witness structure W *can be extended* to an H-contraction witness structure if there is an H-contraction witness structure $W'(x)$, where $W(x) \subseteq W'(x) \subseteq N_G[W(x)]$ for $x \in V_H$. We need the following lemma.

Lemma 6. *Let a triangle-free graph H be an induced minor of G with the corresponding witness structure $W(x)$ for $x \in V_H$ such that $\cup_{x \in V_H} W(x)$ is a dominating set of G. Then it can be decided in polynomial time whether W can be extended to an H-contraction witness structure.*

Proof. Let $U = V_G \setminus \cup_{x \in V_H} W(x)$. For a vertex $u \in U$, we say that we *assign* u to $W(x)$ if we put u in $W'(x)$. We assign vertices to the sets $W'(x)$ recursively using the following rules.

Rule 1. If for a non-assigned vertex $u \in U$ there is a unique $x \in V_H$ such that $u \in N_G[W(x)]$, then u is assigned to $W'(x)$.

Let $X(u) = \{x \in V_H | u \in N_G[W(x)]$ for $u \in U$. Now we can assume that $|X(u)| \geq 2$ for non-assigned vertices of U.

Rule 2. If for a non-assigned $u \in U$, $|X(u)| \geq 3$, then if $H[X(u)]$ is a star with the central vertex x, then we assign x to $W'(x)$. Otherwise, we stop and return a `No`-answer. To see the correctness of this rule, note that if $u \in W'(x)$, then x should be adjacent to all other vertices of $X(u)$. Then because H has no triangles, $H[X(u)]$ is a star with the central vertex x.

Now we can assume that $X(u) = \{x_1(u), x_2(u)\}$ for non-assigned vertices of U.

Rule 3. If for a non-assigned $u \in U$, $x_1(u), x_2(u)$ are not adjacent in H, then we stop and return a `No`-answer.

Rule 4. If for $u, v \in U$ such that $uv \in E_G$, u is not assigned and v is assigned to $W'(y)$, then if $y \notin N_H[x_1(u)]$ ($y \notin N_H[x_2(u)]$ respectively), then u is assigned to $W'(x_2(u))$ (to $W'(x_1(u))$ respectively).

After this rule, it is safe to use the next one.

Rule 5. If for a non-assigned $u \in U$, each $v \in U$ adjacent to u is assigned, then assign u arbitrary to either $W'(x_1(u))$ or $W'(x_2(u))$.

Rule 6. If for $u, v \in U$ such that $uv \in E_G$, u is assigned to $W'(x)$ and v is assigned to $W'(y)$, then if either $x \neq y$ or $xy \notin E_H$, then we stop and return a `No`-answer.

The remaining assignments are done as follows. Note that any non-assigned $u \in U$ is adjacent to non-assigned vertices of U, and for any assigned neighbor $v \in U$, any assignment of u is safe. For each non-assigned $u \in U$ with $X(u) = \{x_1(u), x_2(u)\}$, we introduce a Boolean variable z_u and assume that $z_u = \mathit{true}$ if u is assigned to $W'(x_1(u))$, and $z_u = \mathit{false}$ otherwise. For adjacent non-assigned $u, v \in U$, we define a Boolean function $f_{uv}(z_u, z_v)$ such that $f_{uv}(z_u, z_v) = \mathit{true}$ if and only if for the assignments of u, v to $W'(x_i(u)), W'(x_j(v))$ respectively that correspond to the values of z_u, z_v, either $x_i(u) = x_j(v)$ or $x_i(u)x_j(v) \in E_H$. Clearly, $f_{uv}(z_u, z_v)$ can be written in the conjunctive normal form where each clause contains at most two literals. Consider the formula $\phi = \wedge_{uv \in A} f_{uv}(z_u, z_v)$, where $A = \{uv \in E_G | u, v \in U \text{ and } u, v \text{ are non-assigned}\}$. Note that ϕ gives an instance of 2-Satisfiability that can be solved in linear time [16]. If we get a solution, then we assign u according to the value of z_u. Otherwise, we return a `No`-answer. □

Now we are ready to prove the main theorem of this section.

Theorem 7. *H-Contractibility can be solved in polynomial time on AT-free graphs for any fixed triangle-free graph H.*

Proof. We check whether H is a minor of G with an H-witness structure that can be extended to an H-contraction witness structure. To do it we use the same approach as in the proof of Theorem 5.

Suppose that H is a contraction of G. Then G has an H-contraction witness structure, i.e., a partition $W(x)$ for $x \in V_H$ of the set V_G. For each $x \in V_H$, $G[W(x)]$ is a connected AT-free graph. Hence, by Theorem 1, $G[W(x)]$ has a dominating pair (u_x, v_x).

For each $x \in V_H$, we guess the pair (u_x, v_x) (it can happen that $u_x = v_x$), and guess at most six vertices of a shortest (u_x, v_x)-path P_x in $G[W(x)]$ as follows:

if P_x has at most five vertices, then we guess all vertices of P_x, and if P_x has at least six vertices, then we guess the first three vertices u_1^x, u_2^x, u_3^x and the last three vertices v_1^x, v_2^x, v_3^x, $u_x = u_1^x$, $v_x = v_3^x$ such that $u_1^x \prec_{u_x} u_2^x \prec_{u_x} u_3^x \prec_{u_x} v_1^x \prec_{u_x} v_2^x \prec_{u_x} v_3^x$ in P_x. Observe that P_x is an induced path. We denote by X_1, X_2 the partition of V_H (one of the sets can be empty), where for $x \in X_1$, all at most five vertices of P_x were chosen, and for $x \in X_2$, we have the vertices $u_1^x, u_2^x, u_3^x, v_1^x, v_2^x, v_3^x$. Further, for each edge $xy \in E_H$, we guess adjacent vertices $s_{xy}, s_{yx} \in V_G$, where $s_{xy} \in W(x)$ and $s_{yx} \in W(y)$. Note that the vertices s_{xy} are not necessary distinct, and some of them can coincide with the vertices chosen to represent P_x. Let $S(x) = \{s_{xy} | xy \in E_H\}$. All the guesses should be consistent with the witness structure, i.e., vertices included in distinct $W(x)$ should be distinct, and if $xy \notin E_H$, then the vertices included in $W(x)$ and $W(y)$ should be nonadjacent in G.

For $x \in X_1$, we check whether the guessed path P_x dominates $S(x)$, and if it is so, then we let $W'(x) = V_{P_x} \cup S(x)$. Otherwise we discard our choice.

Recall that we already selected some vertices, and that we cannot use these vertices and also not their neighbors in case non-adjacencies in H forbid this. For each $x \in X_2$, we then define the set

$$\begin{aligned} U_x = V_G \setminus \big(& (\cup_{y \in X_1, xy \in E_H} W'(y)) \cup (\cup_{y \in X_1, xy \notin E_H} N_G[W'(y)]) \cup \\ & \cup (\cup_{y \in X_2, xy \in E_H} (S(y) \cup \{u_1^y, u_2^y, u_3^y, v_1^y, v_2^y, v_3^y\}) \cup \\ & \cup (\cup_{y \in X_2 \setminus \{x\}, xy \notin E_H} N_G[S(y) \cup \{u_1^y, u_2^y, u_3^y, v_1^y, v_2^y, v_3^y\}] \cup \\ & \cup N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]\big) \cup \{u_1^x, u_2^x, u_3^x, v_1^x, v_2^x, v_3^x\}. \end{aligned}$$

Then for each $x \in X_2$, we check whether $S'(x) = S(x) \setminus N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$ is included in one component of $G[U_x]$. If it is not so, then we discard our choice, since we cannot have a path with the first vertices u_1^x, u_2^x, u_3^x and the last vertices v_1^x, v_2^x, v_3^x that dominates $S(x)$. Otherwise we denote by U'_x the set of vertices of the component of $G[U_x]$ that contains $S'(x)$. Note that (u_1^x, v_3^x) is a dominating pair in $G[U'_x]$ for $x \in X_2$. To show it, consider a dominating pair (u, v) in $G[U'_x]$. Any (u, v)-path P dominates u_1^x and v_3^x. It follows that one vertex of the pair is in $\{u_1^x, u_2^x\}$ and another is in $\{v_2^x, v_3^x\}$. It remains to observe that if u_2^x (v_2^x respectively) is in the pair, then it can be replaced by u_1^x (v_3^x respectively). We solve Set-Restricted $|X_2|$-Disjoint Paths for the pairs of terminals (u_1^x, v_3^x) with domains U'_x for $x \in X_2$. If we get the `No`-answer, then we discard our guess since there are no P_x that satisfy our choices. Otherwise, let P'_x be the (u_1^x, v_3^x)-path in the obtained solution for $x \in X_2$. We let $W'(x) = P'_x \cup S(x)$.

We claim that the sets $W'(x)$ compose an H-witness structure. To show it we use exactly the same arguments as in the proof of Theorem 5. Observe first that by the construction of these sets, $W'(x)$ are disjoint. If $xy \in E_G$, then as $s_{xy} \in W'(x)$ and $s_{yx} \in W'(y)$, $W'(x)$ and $W'(y)$ are adjacent. It remains to prove that if $xy \notin E_H$, then $W'(x)$ and $W'(y)$ are not adjacent. To obtain a contradiction, assume that $W'(x)$ and $W'(y)$ are adjacent for some $x, y \in V_H$, i.e., there is $uv \in E_G$ with $u \in W'(x)$ and $v \in W'(y)$, where $xy \notin E_H$. By the construction of $W'(x), W'(y)$, $x, y \in X_2$. Moreover, $u \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$ or $v \notin N_G[\{u_1^y, u_2^y, v_2^y, v_3^y\}]$. If $u \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$, then we consider u_1^x, v_3^x, u_1^y and observe that these vertices compose an asteroidal triple. Clearly, the (u_1^x, v_3^x)-path P'_x avoids $N_G[u_1^y]$, because $N_G[u_1^y] \cap U_x = \emptyset$. Because $u \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$, u is either in P'_x or adjacent to a vertex in P'_x and v is either in P'_y or adjacent to a vertex in P'_y, $G[W'(x) \cup W'[y]] - N_G[u_1^x]$ and $G[W'(x) \cup W'[y]] - N_G[v_3^x]$ are connected. Hence, there are (u_1^x, u_1^y) and (v_3^x, u_1^y)-paths that avoid $N_G[v_3^x]$ and $N_G[u_1^x]$ respectively. By symmetry, we conclude that If $v \notin N_G[\{u_1^y, u_2^y, v_2^y, v_3^y\}]$, then u_1^y, v_3^y, u_1^x is an asteroidal triple. This contradiction proves our claim.

Suppose now that all our guesses were correct, i.e., the chosen vertices belong to the paths P_x and the sets $W(x)$ as it was described.

Now we claim that $\cup_{x \in V_H} W'(x)$ dominates G. Clearly, for $x \in X_1$, P_x dominates $W(x)$. For $x \in X_2$, by the construction of U'_x, we have that $W(x) \subseteq U'_x \cup N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$. Hence, $W'(x)$ dominates $W(x)$. Since the sets $W(x)$ form a partition of V_G, the claim follows. Hence, the next step of our algorithm is to check whether $\cup_{x \in V_H} W'(x)$ is a dominating set. If it is not so, then we discard our guesses.

Our next claim is that if the guesses were correct, then the H-witness structure $W'(x)$ can be extended to an H-contraction witness structure. Let $W''(x) = W'(x) \cup ((V_G \setminus \cup_{x \in V_H} W'(x)) \cap W(x))$. Since $W'(x)$ dominates $W(x)$, $W'(x) \subseteq W''(x) \subseteq N_G[W'(x)]$.

We show that $W''(x)$ is an H-contraction witness structure. Observe that the sets $W''(x)$ form a partition of V_G, and each $W''(x)$ is connected by the definition. To obtain a contradiction, assume that there is $uv \in E_G$ such that $u \in W''(x)$, $v \in W''(y)$, $x \neq y$ and $xy \notin E_H$. Note that $u \notin W(x)$ or $v \notin W(y)$. If $u \notin W(x)$ or $v \notin W(y)$, then $u \in W'(x)$ and $v \in W'(y)$, but this leads to a contradiction, since W' is an H-witness structure. Therefore, we can assume that $u \in W'(x) \setminus W(x)$ and $v \in W(y)$. Then $v \in W(y) \setminus W'(y)$. If $x \in X_1$, then $W'(x) \subseteq W(x)$ and $W'(x) \setminus W(x) = \emptyset$. Hence, $x \in X_2$. Since $\{u_1^x, u_2^x, v_2^x, v_3^x\} \subseteq W(x)$, $v \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\} \cup S(x)]$, and, moreover, $G[W(y)]$ contains a (u_1^y, v)-path P that avoids $N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\} \cup S(x)]$. Now we prove that u_1^x, v_3^x, u_1^y is am asteroidal triple. The (u_1^x, v_3^x)-path P_x avoids $N_G(u_1^y)$. If $u \in V_{P_x}$, then $u \neq u_1^x, u_2^x, v_2^x, v_3^x$ because v is adjacent to u. Then the concatenation of P, vu and (u, u_1^x)-subpath of P'_x ((u, v_3^x)-subpath of P'_x respectively) gives a (u_1^y, u_1^x)-path that avoids $N_G[v_3^x]$ ($N_G[v_3^x]$ respectively). If $u \notin V_{P_x}$, then $u \in S(x)$. But then $v \notin N_G[S(x)]$, a contradiction.

Using this claim, we check whether the H-witness structure $W'(x)$ can be extended to an H-contraction witness structure using Lemma 6. If we get an H-contraction witness structure, then we conclude that H is a contraction of G.

To complete the proof of the theorem, note that we guess at most $6|V_H| + 2|E_H|$ vertices of G, and we can consider all possible choices in time $n^{O(|V_H|+|E_H|)}$, where $n = |V_G|$. If for one of the choices we get an H-witness structure that can be extended to an H-contraction witness structure, then H is a contraction of G, otherwise we return `No`. Since, we can solve SET-RESTRICTED $|X_2|$-DISJOINT PATHS in time $n^{f(|V_H|)}$ by Theorem 3 and we can decide in polynomial time whether an H-witness structure can be extended to an H-contraction witness structure by Lemma 6, the claim follows. □

The *join* of two vertex-disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \bowtie G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv \mid u \in V_1, v \in V_2\})$. Van 't Hof et al. in [12] showed that a graph G contains a graph H as an induced minor if and only if $K_1 \bowtie G$ contains $K_1 \bowtie H$ as a contraction [12]. This fact together with Theorem 6 yields Corollary 1.

Corollary 1. *H-*Contractibility *is* NP*-complete for cobipartite graphs, and* W[1]*-hard for cobipartite graphs when parameterized by $|V_H|$.*

Determining the complexity classification of H-Contractibility on AT-free graphs when H is a fixed graph that is not triangle-free is an open problem.

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