Detecting induced star-like minors in polynomial time*

Jiří Fiala[1], Marcin Kamiński[2], and Daniël Paulusma[3]

[1] Charles University, Faculty of Mathematics and Physics,
DIMATIA and Institute for Theoretical Computer Science (ITI)
Malostranské nám. 2/25, 118 00, Prague, Czech Republic.
fiala@kam.mff.cuni.cz†

[2] Département d'Informatique, Université Libre de Bruxelles, Belgium
marcin.kaminski@ulb.ac.be

[3] School of Engineering and Computing Sciences, Durham University,
Science Laboratories, South Road, Durham DH1 3LE, United Kingdom
daniel.paulusma@durham.ac.uk‡

Abstract. The INDUCED MINOR problem is to test whether a graph G contains a graph H as an induced minor, i.e., if G can be modified into H by a sequence of vertex deletions and edge contractions. When H is fixed, i.e., not part of the input, this problem is denoted H-INDUCED MINOR. We provide polynomial-time algorithms for this problem in the case that the fixed target graph has a star-like structure. In particular, we show polynomial-time solvability for all forests H on at most seven vertices except for one such case.

1 Introduction

Whether or not a graph G *contains* a graph H depends on the notion of containment we use; in the literature several natural definitions have been studied such as containing H as a contraction, dissolution, immersion, (induced) minor, (induced) topological minor, (induced) subgraph, or (induced) spanning subgraph (cf. [13]). In this paper, we focus on the containment relation "induced minor". Before we give a survey of existing work and present our own results, we first state some basic terminology.

We consider undirected graphs with no loops and no multiple edges. We denote the vertex set and edge set of a graph G by V_G and E_G, respectively. If no confusion is possible, we may omit subscripts. We refer the reader to Diestel [5] for any undefined graph terminology.

Let $e = uv$ be an edge in a graph G. The *edge contraction* of e removes u and v from G, and replaces them by a new vertex adjacent to precisely those vertices to which u or v were adjacent. Let G and H be two graphs. Then G contains H as a *contraction*, *induced minor* or *minor* if G can be modified into H by a sequence of edge contractions, edge contractions and vertex deletions, or edge contractions, edge deletions and vertex deletions, respectively. The corresponding decision problems are called CONTRACTIBILITY, INDUCED MINOR and MINOR, respectively. All three problems are

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NP-complete even for pairs (G, H) where G and H are trees of bounded diameter, or trees, the vertices of which have degree at most 3 except for at most one vertex, as shown by Matoušek and Thomas [13]. It is therefore natural to fix the graph H (the *target graph*) in an ordered input pair (G, H) and consider only the graph G (the *host graph*) to be part of the input. We indicate this by adding "H-" to the names of the decision problems.

Known results. A celebrated result by Robertson and Seymour [14] states that the problem H-MINOR can be solved in cubic time for every fixed graph H. The computational complexity classifications of H-INDUCED MINOR and H-CONTRACTIBILITY are still open. Many partial results are known, in particular for special graph classes. Below we briefly survey these.

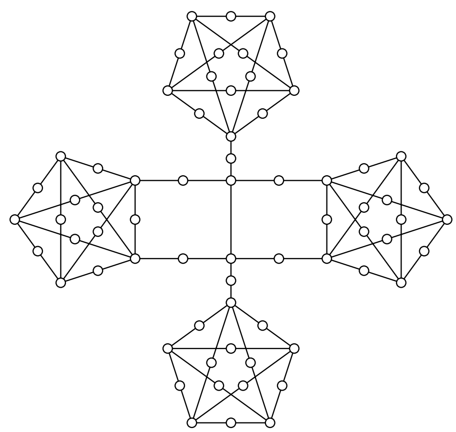

Fig. 1. The smallest graph H for which H-INDUCED MINOR is NP-complete [6].

Fellows et al. [6] showed that the H-INDUCED MINOR problem is NP-complete for a specific graph H on 68 vertices displayed in Figure 1. This is still the smallest known NP-complete case for H-INDUCED MINOR. They also showed that for every fixed graph H, the H-INDUCED MINOR problem can be solved in polynomial time on planar graphs. Later this result was extended by van 't Hof et al. [9] who showed that for every fixed planar graph H, the H-INDUCED MINOR problem is polynomial-time solvable on any minor-closed graph class not containing all graphs. Belmonte et al. [1] showed that for every fixed graph H, the H-INDUCED MINOR problem is polynomial-time solvable for chordal graphs, whereas for claw-free graphs partial results that only include polynomial-time solvable cases are known [7].

Brouwer and Veldman [4] gave polynomial-time solvable and NP-complete cases for the H-CONTRACTIBILITY problem. One of their results is that this problem is already NP-complete for a graph H on 4 vertices, namely when H is fixed to be the 4-vertex path or the 4-vertex cycle. This research was later extended by Levin, Paulusma and Woeginger [11,12] and van 't Hof et al. [9]. Kamiński, Paulusma and Thilikos [10] showed that for every fixed H, the H-CONTRACTIBILITY problem can be solved in polynomial time on planar graphs. By extending previous results [2,8], Belmonte et al. [1] showed that for every fixed graph H, the H-CONTRACTIBILITY problem is polynomial-time solvable for chordal graphs.

Our focus. We consider the H-INDUCED MINOR problem when H is a fixed forest. Our research is motivated by the following problem that was first posed at the AMS-IMS-SIAM Joint Summer Research Conference on Graph Minors in 1991.

Can H-INDUCED MINOR *be solved in polynomial time for any fixed tree* H*?*

In contrast to the H-CONTRACTIBILITY problem, which is already NP-compete when H is the 4-vertex path [4], the H-INDUCED MINOR problem is polynomial-time solvable when H is a path of arbitrary length. This is because in that case the problem is equivalent to checking if H appears as an induced subgraph in the host graph G. However, for other trees, the situation is considerably less clear, and the problem posed above is still open.

Our results. In Section 3 we show that H-INDUCED MINOR is polynomial-time solvable when H is any fixed star that may be subdivided or any fixed double star, one side of which contains exactly 2 leaves. See Figure 2 for an illustration of these star-like trees. In addition, we show a number of further consequences, which enable us to settle the complexity of H-INDUCED MINOR for any forest H on at most 7 vertices except when H is the 7-vertex tree H^* obtained by subdividing the centre edge in a double star, both sides of which contain exactly two leaves (also see Figure 2). In Section 4 we discuss a number of open problems.

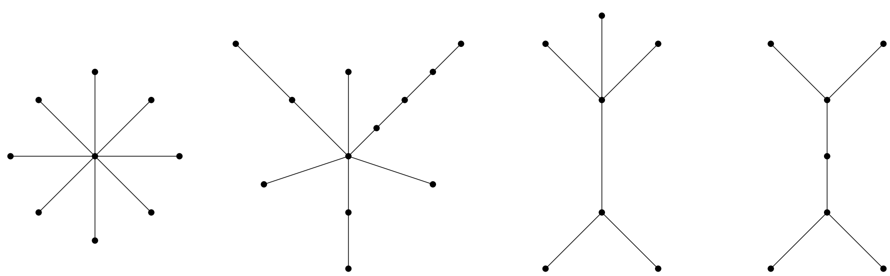

Fig. 2. From left to right: an example of a star, a subdivided star, a double star with 3 vertices on one side and 2 vertices on the other side, and the graph H^*.

2 Preliminaries

Let $G = (V, E)$ be a graph. We write $G[U]$ to denote the subgraph of G *induced* by $U \subseteq V$, i.e., the graph on vertex set U and an edge between any two vertices if and only if there is an edge between them in G. For a vertex u, the graph $G - u$ denotes the graph obtained from G after removing u. We say that U is an *independent set* if there is no edge in G between any two vertices of U. Two sets $U, U' \subseteq V$ are called *adjacent* if there exist vertices $u \in U$ and $u' \in U'$ such that $uu' \in E$. A vertex v is a *neighbor* of u if $uv \in E$. We let $N(u)$ denote the set of neighbors. The *degree* of a vertex u is its number of neighbors. We let C_n, K_n, and P_n denote the cycle, complete graph, and path on n vertices, respectively.

A graph $G = (V, E)$ is called *k-connected* if $G[V \backslash U]$ is connected for every set $U \subseteq V$ of at most $k-1$ vertices. A graph G that is not connected is called *disconnected*. A *k-vertex cut* is a subset $S \subseteq V$ of size k such that $G[V \backslash S]$ is disconnected. The vertex in a 1-vertex cut of a graph G is called a *cut vertex*. Each maximal 2-connected subgraph of a graph G is called a *block* of G. Note that by their maximality any two blocks of G have at most one vertex in common, and such a common vertex is a cut vertex of G. A *block* that contains at most one cut vertex is called a *leaf block*. We call a vertex of G that is not a cut vertex an *internal* vertex. Observe that every leaf block of G contains at least one internal vertex.

A *star* is a graph formed by joining each vertex of an independent set to an extra vertex called the *centre vertex*. A *double star* is formed by joining each vertex of an independent set to one of the two end-vertices of an extra edge called the *centre edge*.

Let G and H be two graphs. An *H-witness structure* $\mathcal{W}$ is a vertex partition of G into $|V_H|$ (nonempty) sets $W(x)$ called *H-witness bags*, such that

(i) each $W(x)$ induces a connected subgraph of G;
(ii) for all $x, y \in V_H$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in G if and only if x and y are adjacent in H;

By contracting all bags to single vertices we find that H is a contraction of G if and only if G has an H-witness structure. We note that G may have more than one H-witness structure. We call a bag that corresponds to a vertex of degree one in H a *leaf bag*.

The algorithm in the following lemma is not only useful for contractions but also for induced minors. The lemma is stated as Corollary 5 in the paper by Levin et al. [11], the proof of which explains that it follows from applying Robertson and Seymour's cubic-time algorithm [14] for finding a fixed graph minor at most $O(|V|^{k^2})$ times.

Lemma 1 ([14]). *Let $G = (V, E)$ be a graph and let $Z_1, \ldots, Z_p \subseteq V_G$ be p specified pairwise disjoint sets such that $\sum_{i=1}^{p} |Z_i| \leq k$ for some fixed integer $k \geq p$. The problem of deciding whether G contains K_p as a contraction with K_p-witness bags $W_1, \ldots W_p$ such that $Z_i \subseteq W_i$ for $i = 1, \ldots, p$ can be solved in $O(|V|^{k^2+3})$ time.*

We observe that a graph G contains a graph H as an induced minor if and only if G has an induced subgraph G' that contains H as a contraction. In that case we say that an H-witness structure of G' is an *H-semi-witness structure* of G and call the H-witness bags of G' *H-semi-witness bags* of G, or just *bags* if no confusion is possible. Just as for contractions, a bag that corresponds to a vertex of degree one in H is called a *leaf bag*.

3 Induced minors

In order to prove the results in this section we need the following lemma. Let G be a graph that contains H as an induced minor. Then we say that an H-semi-witness structure of G is *minimum* if the union of its bags has minimum size over all H-semi-witness structures of G.

Lemma 2. *If a graph G has a graph H as an induced minor, then every leaf bag in every minimum H-semi-witness structure $\mathcal{W}$ of G contains exactly one vertex.*

Proof. In order to obtain a contradiction, suppose that $\mathcal{W}$ is a minimum H-semi-witness structure of G that has a leaf bag $W(x)$ on more than one vertex. Then we can remove all vertices from $W(x)$ except a vertex adjacent to a vertex in the neighbor bag of $W(x)$. This is not possible. $\square$

We also need the next lemma which shows that every graph that contains $K_{1,3}$ as an induced minor has a $K_{1,3}$-semi-witness structure of bounded size, where bounded size means that its bags contain in total at most 6 vertices.

Lemma 3. *If G contains $K_{1,3}$ as an induced minor, then G has a $K_{1,3}$-semi-witness structure whose bags contain in total at most 6 vertices.*

Proof. Denote the centre vertex of $K_{1,3}$ by b and its leaves by a_1, a_2, a_3. Let G be a graph that contains $K_{1,3}$ as an induced minor. Let $\mathcal{W}$ be a minimum $K_{1,3}$-semi-witness structure for G. By Lemma 2, we may assume that each leaf bag $W(a_i)$ consists of exactly one vertex. Denote these vertices by u_1, u_2, u_3, respectively.

Consider a shortest path P from u_1 to u_2 in the subgraph of G induced by $W(a_1) \cup W(b) \cup W(a_2)$. Let Q be a shortest path from u_3 to a vertex $z \in V_P$ in the subgraph of G induced by $W(b) \cup W(a_3)$. Note that $z \notin \{u_1, u_2, u_3\}$, because by definition $z \in V_P \cap V_Q \subseteq W(b)$. We also observe that P and Q are induced paths in G. Moreover, the minimality of $\mathcal{W}$ combined with the observation that $G[V_P \cup V_Q]$ is connected implies that $(V_P \cup V_Q) \setminus \{u_1, u_2, u_3\} = W(b)$.

Case 1. Q only consists of u_3 and one other vertex.
Let ℓ be the neighbor of u_3 on P that is as close to u_1 as possible. Let r be the neighbor of u_3 on P that is as close to u_2 as possible. Note that $\ell = r$ is possible. By the minimality of $\mathcal{W}$, we find that u_1 is the left neighbor of ℓ on P and that u_2 is the right neighbor of r on P. If $\ell = r$, or ℓ is adjacent to r, then $W(b)$ contains no other vertex except ℓ and r. Hence $|W(b)| \leq 2$, and consequently, $\mathcal{W}$ is a desired $K_{1,3}$-semi-witness structure for G.

Now suppose that $\ell \neq r$ and that ℓ is not adjacent to r. Let $P' = \ell t_1 \cdots t_q r$ be the subpath of P from ℓ to r; note that $q \geq 1$. See Figure 3 for an illustration. If $q \geq 2$, then we find a $K_{1,3}$-semi-witness structure $\mathcal{W}'$ for G given by $W'(a_1) = \{u_1\}$, $W'(a_2) = \{t_1\}$, $W'(a_3) = \{r\}$, and $W'(b) = \{\ell, u_3\}$. This is a contradiction to the minimality of $\mathcal{W}$. Hence, $q = 1$. Then $W(b) = \{\ell, r, t_1\}$. We conclude that $\mathcal{W}$ is a desired $K_{1,3}$-semi-witness structure for G.

Case 2. Q consists of u_3 and at least two other vertices.
We denote the subpath of Q from u_3 to the vertex of Q that is adjacent to a vertex of P

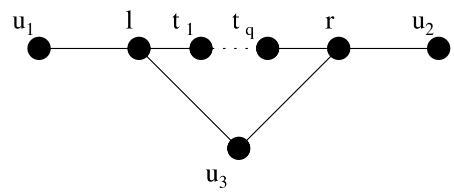

Fig. 3. An illustration of Case 1 of the proof of Lemma 3. Note that $q = 0$ is possible, and that if $q \geq 1$, then u_3 can be shown to be adjacent to every t_i. However, this is not relevant for our proof, and we did not draw such edges.

as $u_3 s_1 \cdots s_q$ for some $q \geq 1$. Let ℓ and r be the neighbors of s_q on P that are closest to u_1 and u_2, respectively; we note that $\ell = r$ is possible. See Figure 4 for an illustration.

First suppose that $\ell = u_1$. Consider the subpath P' of P that goes from the neighbor of u_1 to the left neighbor of r (i.e., which does not pass through r but just stops before). If P' is nonempty, then we can remove all vertices of P' in order to obtain a new $K_{1,3}$-semi-witness structure for G. This is a contradiction to the minimality of $\mathcal{W}$. Hence, P' is empty. This means that r is the neighbor of u_1 on P. The minimality of $\mathcal{W}$ also implies that r is the neighbor of u_2 on P; note that $r = u_2$ is not possible, because r is adjacent to u_1. Suppose that $q \geq 3$. If some s_i is adjacent to u_2, then we can remove r from $W(b)$ and obtain a new $K_{1,3}$-semi-witness structure for G. This is a contradiction to the minimality of $\mathcal{W}$. Hence no s_i is adjacent to u_2. This enables us to use the following argument. If some s_i is not adjacent to u_1, then we can remove u_3 from $W(a_3)$, the vertices $s_1, \ldots, s_{i-1}$ (if they exist) from $W(b)$ and move s_i from $W(b)$ to $W(u_3)$. This leads to a new $K_{1,3}$-semi-witness structure for G, which is a contradiction to the minimality of $\mathcal{W}$. Hence, all s_i are adjacent to u_1. However, recall that we assume that $q \geq 3$. Then we obtain a new $K_{1,3}$-semi-witness structure $\mathcal{W}'$ for G that is defined by $W'(a_1) = \{r\}$, $W'(a_2) = \{s_2\}$, $W'(a_3) = \{u_3\}$ and $W'(b) = \{s_1, s_q, u_1\}$. This is a contradiction to the minimality of $\mathcal{W}$. Hence, $q \leq 2$. Recall that $q \geq 1$. Then $W(b) = \{r, s_1\}$ if $q = 1$ and $W(b) = \{r, s_1, s_2\}$ if $q = 2$. We conclude that $\mathcal{W}$ is a desired $K_{1,3}$-semi-witness structure for G. Now suppose that $r = u_2$. Then we follow the same reasoning. Hence, from now on we may assume that $\ell \neq u_1$ and $r \neq u_2$.

The minimality of $\mathcal{W}$ implies that u_1 is the left neighbor of ℓ on P and that u_2 is the right neighbor of r on P. If $\ell = r$, then ℓ, u_1, u_2, s_q form an induced claw with centre ℓ, and as such a $K_{1,3}$-semi-witness structure for G. This is a contradiction to the minimality of $\mathcal{W}$.

Suppose that $\ell \neq r$. If ℓ is not adjacent to r, then let t be the neighbor of ℓ on P that is not equal to u_1. We define a new $K_{1,3}$-semi-witness structure $\mathcal{W}'$ for G by $W'(a_1) = \{u_1\}$, $W'(a_2) = \{u_2\}$, $W'(a_3) = \{t\}$ and $W'(b) = \{\ell, r, s_q\}$. This is a contradiction to the minimality of $\mathcal{W}$.

Finally, suppose that ℓ and r are two distinct vertices that are adjacent. First suppose that $q \geq 2$. Then we consider s_{q-1}. By the definition of Q, we know that s_{q-1} is not adjacent to any vertex of P except perhaps u_1 or u_2. If s_{q-1} is adjacent to both u_1 and u_2, then we may remove ℓ, r, s_q from $W(b)$ in order to obtain a new $K_{1,3}$-semi-

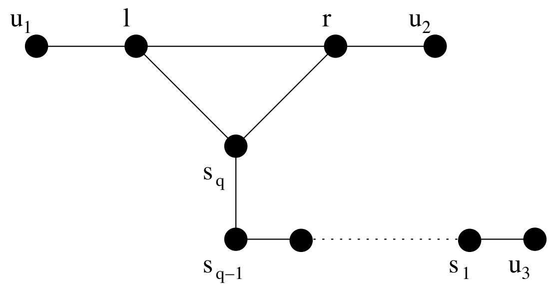

Fig. 4. An illustration of Case 2 of the proof of Lemma 3. Note that $q \geq 1$. It is possible that either $u_1 = \ell$ or $u_2 = r$. In the proof we show that either $\ell = r$, or ℓ and r are adjacent. Also, there may exist edges between some u_i with $1 \leq i \leq 2$ and some s_j with $1 \leq j \leq q$ but we did not draw them. However, by definition, there is no edge between any s_i with $1 \leq i \leq q-1$ and a vertex from $V_P \setminus \{u_1, u_2\} = \{\ell, r\}$.

witness structure for G. This is a contradiction to the minimality of $\mathcal{W}$. If s_{q-1} is neither adjacent to u_1 nor to u_2, then we may remove u_3 from $W(u_3)$, the vertices $s_1, \ldots, s_{q-2}$ (if they exist) from $W(b)$ and move s_{q-1} from $W(b)$ to $W(u_3)$ in order to obtain a new $K_{1,3}$-semi-witness structure for G. This is again a contradiction to the minimality of $\mathcal{W}$. Hence, we find that s_{q-1} is either adjacent to u_1 or to u_2, say s_{q-1} is adjacent to u_1 and thus non-adjacent to u_2. Then we define the $K_{1,3}$-semi-witness structure $\mathcal{W}'$ by $W'(a_1) = \{\ell\}$, $W'(a_2) = \{u_2\}$, $W'(a_3) = \{s_{q-1}\}$ and $W'(b) = \{r, s_q\}$. This is a contradiction to the minimality of $\mathcal{W}$. Hence $q \leq 1$. Recall that $q \geq 1$. We conclude that $q = 1$. This means that $W(b) = \{\ell, r, s_1\}$, and in that case, $\mathcal{W}$ is a desired $K_{1,3}$-semi-witness structure for G. This completes the proof of Lemma 3. □

We note that Lemma 3 only holds for stars with four vertices. A counterexample for the case $K_{1,4}$ is as follows: construct a graph G by taking an arbitrary long path $u_1u_2 \cdots u_{p-1}u_p$ and adding two new vertices v, w and edges u_2v and $u_{p-1}w$; see Figure 5. Then the only $K_{1,4}$-semi-witness structure of G uses all vertices of G. We also note that the bound of 6 on the total number of vertices in a minimum $K_{1,3}$-semi-witness structure in Lemma 3 is best possible. In order to see this we consider the graph G^* obtained from a path on five vertices $u_1v_1u_2v_2u_3$ after adding a new vertex u_2' that we make adjacent (only) to u_2, v_1, v_2; also see Figure 5. The graph G^* contains $K_{1,3}$ as an induced minor, but has only two $K_{1,3}$-semi-witness structures $\mathcal{W}_1$ and $\mathcal{W}_2$, where $\mathcal{W}_1$ is given by leaf bags $\{u_i\}$ for $i = 1, 2, 3$ and centre bag $\{u_2, v_1, v_2\}$, and $\mathcal{W}_2$ is obtained from $\mathcal{W}_1$ by swapping u_2 and u_2'. Both $\mathcal{W}_1$ and $\mathcal{W}_2$ use all vertices of G^*, and hence contain six vertices in total.

We use Lemma 3 to prove Proposition 1. The graph $G + H = (V_G \cup V_H, E_G \cup E_H)$ is the *disjoint union* of two vertex-disjoint graphs G and H.

Proposition 1. *Let H be a graph and F be the disjoint union of claws and paths. If H-*INDUCED MINOR *is polynomial-time solvable, then so is $(H + F)$-*INDUCED MINOR.

Proof. The result follows from Lemma 3 and the observation that a graph G contains a path as an induced minor if and only if it contains this path as an induced subgraph.

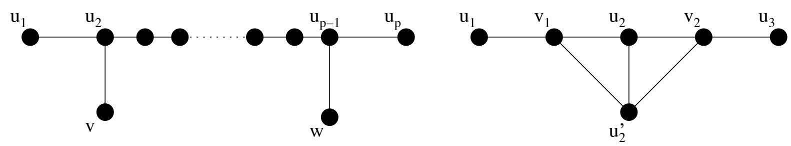

Fig. 5. The two counterexamples consisting of a graph G (left side) and a graph G^* (right side).

Consequently, we can guess the bags of an F-semi-witness structure in G, remove all vertices that are adjacent to at least one vertex of this copy from G and check if the remaining graph has H as an induced minor. Because the number of guesses is $O(|V_G|^{\frac{3|V_F|}{2}})$ and F is fixed (so $|V_F|$ is a constant) the result follows. $\square$

We note that Proposition 1 shows that F-INDUCED MINOR is polynomial-time solvable when F is the disjoint union of claws and paths; take as H the empty graph.

The *subdivision* of an edge uv in a graph replaces uv by two new edges uw and wv for some new vertex w. A *subdivided star* is a graph obtained from a star after performing a sequence of zero or more edge subdivisions.

Proposition 2. *The H-*INDUCED MINOR *problem is solvable in polynomial time for every fixed subdivided star H.*

Proof. First assume that H is a star. If H has p leaves $a_1, \ldots, a_p$, then we try all at most n^p choices for H-semi-witness bags $W(a_1), \ldots, W(a_p)$, where each $W(a_i)$ consist of only one vertex u_i. For each choice we check whether $\{u_1, \ldots, u_p\}$ forms an independent set, and whether the subgraph of G induced by $V_G \setminus \{u_1, \ldots, u_p\}$ contains a connected component that is adjacent to each vertex of $\{u_1, \ldots, u_p\}$. If one of these tries succeeds, we choose such a component as the H-semi-witness bag for the centre vertex of the star and find that H is an induced minor of G. Because the connected components can be found in $O(m)$ time, and also all adjacencies can be tested in the same time, where m is the number of edges of G, the total time complexity of this algorithm is $O(n^p m)$. This is polynomial because H is fixed, and consequently, p is a constant.

If H has one or more subdivided edges, we can use similar arguments after observing that every witness bag in an H-semi-witness structure except the centre bag may be assumed to have size one. $\square$

Let H and G be graphs such that G contains H as an induced minor. Let $\mathcal{W}$ be an H-semi-witness structure of G. We call the subset of vertices in a semi-witness bag $W(x_i)$ that are adjacent to vertices in some other semi-witness bag $W(x_j)$ an *interface*, denoted $I_\mathcal{W}(x_i, x_j)$. Observe that $I_\mathcal{W}(x_i, x_j) \cap I_\mathcal{W}(x_j, x_i) = \emptyset$ for $i \neq j$, because $I_\mathcal{W}(x_i, x_j) \subseteq W(x_i)$ and $I_\mathcal{W}(x_j, x_i) \subseteq W(x_j)$, and $W(x_i) \cap W(x_j) = \emptyset$ for $i \neq j$. We use this notion to simplify the semi-witness structures of graphs with an induced minor.

From now on, we denote the vertices in a double star as follows: the centre edge is bc where b is adjacent to a set of degree-one vertices $A = \{a_1, \ldots, a_p\}$ for some $p \geq 1$ and c is adjacent to a set of degree-one vertices $B = \{d_1, \ldots, d_q\}$ for some $q \geq 1$. If

H is a double star with $p = 1$ or $q = 1$, then H is a subdivided star, and we can apply Proposition 2. Hence, we assume that $p \geq 2$ and $q \geq 2$. We prove the following result.

Theorem 1. *For any fixed double star H with $p \geq 2$ and $q = 2$, the H-*INDUCED MINOR *problem can be solved in polynomial time.*

Proof. Let G be a graph and H be a double star with $p \geq 2$ and $q = 2$. We apply the following algorithm called DOUBLE STAR, the correctness of which we prove afterwards.

We choose $p + 2$ different vertices $u_1, \ldots, u_p, u'_1, u'_2$ that form an independent set of G. We remove any vertex that is adjacent to both some u-vertex and some u'-vertex. Afterwards, we contract any edge that has both its end-vertices in the neighborhood of some u-vertex, or both its end-vertices in the neighborhood of some u'-vertex. We do this repeatedly until this is no longer possible. We then check if H is an induced subgraph of the resulting graph G''. If so, then we return `yes`. Suppose not. We choose sets $S_1, \ldots, S_p$ of at most $4p + 1$ vertices each and sets T_1, T_2 of at most $p + 7$ vertices each; these sets must consist of neighbors of $u_1, \ldots, u_p$, u'_1, u'_2, respectively. Then we remove $u_1, \ldots, u_p$, u'_1, u'_2 together with all their other neighbors not in any S- or T-set. We check if $S_1, \ldots, S_p$, T_1, T_2 are all in the same connected component L of the remaining graph. If so, then we apply the algorithm of Lemma 1 on L with $Z_1 = S_1 \cup \cdots \cup S_q$ and $Z_2 = T_1 \cup T_2$, and if we find an H-witness structure, then we return `yes`. Otherwise, we adjust our choice of S-sets and T-sets, and if necessary also our choice of u-vertices and u'-vertices, unless we already considered all possible choices; in that case we return `no`. Our algorithm terminates because the number of different choices it makes during its execution is finite. For clarity, we give its pseudocode below.

DOUBLE STAR
Input: A graph G.
Output: `yes` or `no`.

While there are $p + 2$ distinct vertices $u_1, \ldots, u_p, u'_1, u'_2$ that form an independent set **do**
    Remove any vertex that is adjacent to a u-vertex and u'-vertex.
    Contract all edges that have both end-vertices in the neighborhood of a u- or u'-vertex.
If H is an induced subgraph of the resulting graph, **then return** `yes`.
For all sets $S_1 \subseteq N(u_1), \ldots, S_p \subseteq N(u_p)$ of at most $4p + 1$ vertices each and
sets $T_1 \subseteq N(u'_1), T_2 \subseteq N(u'_2)$ of at most $p + 7$ vertices each **do**
    Remove $u_1, \ldots, u_p, u'_1, u'_2$ together with all their other neighbors not in any S- or T-set.
    If $S_1, \ldots, S_p, T_1, T_2$ are in the same connected component L, **then**
        Apply the algorithm of Lemma 1 on L with $Z_1 = S_1 \cup \ldots \cup S_p$ and $Z_2 = T_1 \cup T_2$.
        If the algorithm finds an H-witness structure, **then return** `yes`.
Return `no`.

We now prove that our algorithm is correct, i.e., that it returns `yes` if and only if G contains H as an induced minor. First suppose that the algorithm returns `yes`. This will only happen when it finds that G'' contains H as an induced subgraph, or when it applies Lemma 1 on some sets $S_1, \ldots, S_p, T_1, T_2$ resulting from some choice of vertices $u_1, \ldots, u_p$, u'_1, u'_2. For the first case, we use the property that the induced minor relation is transitive. We first deduce that G contains G'' as an induced minor, because we only performed edge contractions and vertex deletions to obtain G'' from G. We then observe that G'' contains H as an induced subgraph, and consequently, G'' contains H as an induced minor. Hence, G contains H as an induced minor. In the second case, a K_2-witness structure of L has been found. Let W_B and W_C denote the two bags of this structure. Because the u-vertices together with the u'-vertices form an independent set, we can then define an H-semi-witness structure of G by setting $W(a_i) = \{u_i\}$ for $i = 1, \ldots, p$, $W(b) = W_B$, $W(c) = W_C$ and $W(d_i) = \{u'_i\}$ for $i = 1, 2$. Hence, G also contains H as an induced minor in this case.

Now suppose that G contains H as an induced minor. Then we can consider a minimum H-semi-witness structure $\mathcal{W}$ of G. By Lemma 2, we find that there exist $p + 2$ vertices $u_1, \ldots, u_p$, u'_1, u'_2 in G such that $W(a_1) = \{u_i\}$ for $i = 1, \ldots, p$ and $W(d_j) = \{u'_j\}$ for $j = 1, 2$. Because our algorithm considers all possibilities, it will consider these choices of u-vertices and u'-vertices at some moment (unless it has already outputted `yes` before). Hence, from now on, we may assume that our algorithm is processing this particular choice of u-vertices and u'-vertices.

Any vertex v that is adjacent to both some u_i and some u'_j is neither in $W(b)$ nor in $W(c)$; otherwise, in the first case, $W(b)$ would be adjacent to $W(u'_j)$, which is not possible, and in the second case, $W(c)$ would be adjacent to $W(u_i)$, which is not possible either. Hence, our algorithm may without loss of generality remove v from G. Let G' denote the resulting graph obtained after removing all such vertices. From the above, we find that $\mathcal{W}$ is an H-semi-witness structure of G' as well. The graph G' will be processed further, and we prove the following claim.

Claim 1. We may without loss of generality contract all edges vw whenever v, w are neighbors of the same u-vertex or neighbors of the same u'-vertex; this results in a graph G'' that has an H-semi-witness structure with the same leaf bags as $\mathcal{W}$.

We prove Claim 1 as follows. Let v and w be two adjacent neighbors of some u_i; the proof when v and w are neighbors of some u'_j goes the same. If v or w both belong to $W(b)$, then contracting vw results in a graph that still contains H as an induced minor. Suppose that v and w do not belong to $W(b)$. Then, v and w both do not belong to any bag of $\mathcal{W}$, as otherwise $W(u_i)$ is adjacent to some bag not equal to $W(b)$, which is not possible. Hence, also in this case, contracting vw results in a graph that still contains H as an induced minor. Finally, suppose that one of v, w, say v, belongs to $W(b)$, whereas w does not belong to $W(b)$. Then, w does not belong to any bag of $\mathcal{W}$, as otherwise $W(u_i)$ is adjacent to some bag not equal to $W(b)$, which is not possible. We also observe that w is neither adjacent to u'_1 nor to u'_2, because the algorithm already removed all vertices adjacent to both an u-vertex and an u'-vertex in the previous step. Furthermore, w is adjacent to $W(b)$ due to the edge vw. Hence, the collection $\mathcal{W}'$ obtained from $\mathcal{W}$ by adding w to $W(b)$ is an H-semi-witness structure of G'. We conclude that our algorithm may without loss of generality contract vw. By the same arguments it may continue contracting any other edges whose end-vertices are neighbors of the same u-vertex or the same u'-vertex. The resulting graph G'' has an H-semi-witness structure with the same leaf bags as $\mathcal{W}$. This proves Claim 1.

Note that the neighborhood of every u-vertex and every u'-vertex in G'' is an independent set by construction of G''. If G'' contains H as an induced subgraph, then our algorithm will detect this before it continues to the next step. In that case, it will return `yes`, as desired. From now on, assume that G'' does not contain H as an induced subgraph. By Claim 1, we find that G'' has an H-semi-witness structure $\mathcal{W}''$ such that $W''(a_i) = \{u_i\}$ for $i = 1, \ldots, p$ and $W''(d_j) = \{u'_j\}$ for $j = 1, 2$. We say that $\mathcal{W}''$ or any other H-semi-witness structure of G'' that has leaf bags $\{u_1\}, \ldots, \{u_p\}$, $\{u'_1\}, \{u'_2\}$ corresponding to $a_1, \ldots, a_p, d_1, d_2$, respectively, is *leaf-suitable*.

Let $\mathcal{W}^*$ be a leaf-suitable H-semi-witness structure such that $|W^*(b) \cup W^*(c)|$ is minimum over all leaf-suitable H-witness structures of G''; note that $\mathcal{W}^* = \mathcal{W}''$ is possible. Recall that we call a vertex of a graph that is not a cut vertex an internal vertex. We write $B = G''[W^*(b)]$ and prove the following claim.

Claim 2. The number of internal vertices of B is at most $p + 1$.

We prove Claim 2 as follows. Every internal vertex v of B has a private neighbor bag $W^*_v \in \{W^*(a_1), \ldots, W^*(a_p), W^*(c)\}$, i.e., a bag adjacent to v but not to any other vertex of B; otherwise we could remove v from $W^*(b)$ and obtain a new leaf-suitable witness structure with fewer vertices in the union of the two centre bags, contradicting the minimality of $|W^*(b) \cup W^*(c)|$. Because $|\{W^*(a_1), \ldots, W^*(a_p), W^*(c)\}| = p+1$, this means that B contains at most $p + 1$ internal vertices. This proves Claim 2.

We use Claim 2 to prove Claim 3, which is crucial for our algorithm.

Claim 3. Every u-vertex has at most $4p + 1$ neighbors in B.

We prove Claim 3 as follows. Suppose that some u_i has at least $4p + 2$ neighbors in B. Let $\{r_1, \ldots, r_m\}$ be the set of cut vertices in B that are adjacent to u. Then Claim 2 tells us that $m \geq 3p + 1$. Consider r_1. Because r_1 is a cut vertex of B, we find that r_1 has two neighbors s and t in B that are not adjacent to each other.

We claim that both s and t have at most p neighbors in $\{r_2, \ldots, r_m\}$. In order to see this, suppose that one of s, t, say s, is adjacent to $q \geq p+1$ vertices in $\{r_2, \ldots, r_m\}$. We may assume without loss of generality that s is adjacent to all vertices of $\{r_2, \ldots, r_{q+1}\}$. Consider a vertex r_j for some $1 \leq j \leq q + 1$. Because s is adjacent to all vertices in $\{r_1, \ldots, r_{q+1}\}$, every r_i with $1 \leq i \leq q+1$ and $i \neq j$ is in the same connected component C_j of $B - r_j$. Because r_j is a cut vertex of B, we find that $B - r_j$ has a connected component $C'_j \neq C_j$. This means that B contains a leaf block, all of its vertices belong to C'_j. Moreover, for any two distinct vertices r_i and r_j in $\{r_1, \ldots, r_{q+1}\}$, we find that C'_i is a subgraph of C_j; see Figure 6 for an example. As C_j and C'_j are vertex-disjoint, this means that C'_i and C'_j are vertex-disjoint. Hence the leaf blocks contained in the graphs $C'_1, \ldots, C'_{q+1}$ are mutually vertex-disjoint. This implies that B contains at least $q + 1 \geq p + 2$ distinct leaf blocks. Recall that each leaf block contains at least one internal vertex of B. Hence, B contains at least $p + 2$ internal vertices. However, this is not possible due to Claim 2. We conclude that both s and t have at most p neighbors in $\{r_2, \ldots, r_m\}$.

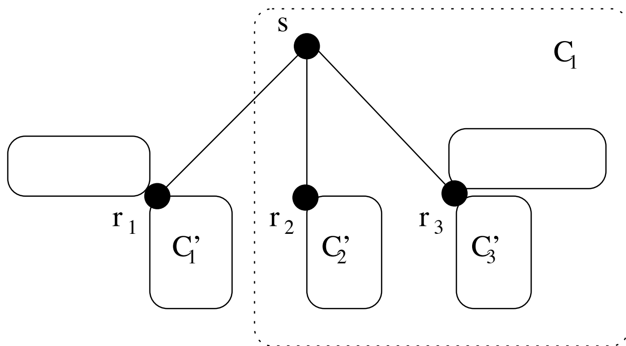

Fig. 6. An example of a graph B. Although $m \geq 3p+1$, for clarity we assume that $m = 3$ here. Also note that in this example $B - r_1$ and $B - r_3$ each consist of 3 components, whereas $B - r_2$ consists of two components. Hence, for $B - r_1$ and $B - r_3$ we have two choices for the components C'_1 and C'_3, respectively. We only indicated the components C_1, C'_1, C'_2, C'_3. Note that C_1 contains C'_2 and C'_3 as subgraphs.

Because $m \geq 3p+1$, the above implies that there exist $3p+1-(2p+1) = p$ vertices in $\{r_2, \dots, r_m\}$ that are neither adjacent to s nor to t. Denote this set of vertices by R', so $|R'| = p$. Recall that all neighbors of u_i form an independent set. This means that we can derive the following. First, u_i is neither adjacent to s nor to t, because s and t are adjacent to a neighbor, namely r_1, of u_i. Second, $R' \cup \{r_1\}$ is an independent set. Third, using that s and t are not adjacent to any vertex of R', we find that $R' \cup \{s, t\}$ is an independent set. However, then $R' \cup \{r_1, s, t, u_i\}$ induce a subgraph of G'' that is isomorphic to H, where the vertices in R' correspond to the p a-vertices, u_i to the b-vertex, r_1 to the c-vertex, and s, t to the two d-vertices. This is not possible, because we assume that in this stage of the algorithm, G'' does not contain H as an induced subgraph. Hence, we have proven Claim 3.

In the same way as for the u-vertices, we can show a bound on the number of neighbors that a u'-vertex has in $W^*(c)$; note that we assumed that $W^*(b) \cup W^*(c)$ had minimum size for exactly this reason. Because we only have two sets $W^*(d_1)$ and $W^*(d_2)$, we copy the proof of Claim 2 to find that the number of internal vertices of $W^*(c)$ is at most $2+1=3$. Then, analogously to Claim 3, we find that every u'-vertex has at most $p+7$ neighbors in $W^*(c)$.

For $i = 1, \dots, p$, let S_i denote the set of neighbors of u_i in $B = G''[W^*(b)]$. For $j = 1, 2$, let T_j denote the set of neighbors of u'_j in $G''[W^*(c)]$. Because each S_i has size at most $4p+1$ and each T_j has size at most $p+7$, the algorithm will consider these sets as a possible choice at some moment (unless it already has returned `yes` before). Hence, from now on, we may assume that our algorithm is processing this particular choice of S-sets and T-sets. Then, in the next step, the algorithm removes all u-vertices together with all their other neighbors, and both u'-vertices together with all their other neighbors from G''. We call the resulting graph $\tilde{G}$.

By definition, $W^*(b)$ contains all vertices of every S_i, and $W^*(c)$ contains all vertices of every T_j. Moreover, $W^*(b)$ and $W^*(c)$ are adjacent. Hence, the vertices of

$S_1 \cup \cdots \cup S_p \cup T_1 \cup T_2$ all belong to the same connected component L of $\tilde{G}$. This will be detected by our algorithm, when it will check this. Consequently, the next step of our algorithm will be to apply Lemma 1 on L with $Z_1 = S_1 \cup \cdots \cup S_p$ and $Z_2 = T_1 \cup T_2$. Because $\{W^*(b), W^*(c)\}$ is a K_2-witness structure of L with $Z_1 \subseteq W^*(b)$ and $Z_2 \subseteq W^*(c)$, the call on Lemma 1 will be positive, and our algorithm will return `yes`, as desired.

What remains for us to do is to analyze the running time of our algorithm. Let n be the number of vertices of G. Then the total number of choices of combinations of sets of u-vertices, sets of u'-vertices, collections of S-sets, and collections of T-sets is bounded by $n^p \cdot n^2 \cdot n^{p(4p+1)} \cdot n^{2(p+7)}$. This is a polynomial number, because we assume that p is fixed. For each choice, all operations of the algorithm take polynomial time; in particular every call on Lemma 1 takes polynomial time as Z_1 and Z_2 have fixed size, namely at most $p(4p+1)$ or at most $2(p+7)$, respectively. Hence, the total running time is polynomial. This completes the proof of Theorem 1. $\square$

We define a k*-subdivided double star* as the graph that is obtained from a double star after performing a subdivision of the edge $a_1b, \ldots, a_kb$, where $1 \leq k \leq p$. Then we can show the following result; note that the case $p \geq 1$ and $q = 1$ follows from Proposition 2.

Theorem 2. *For any fixed k-subdivided double star with $p \geq \max\{k, 2\}$ and $q = 2$, the H-*Induced Minor *problem can be solved in polynomial time.*

Proof. Let G be a graph and H be a k-subdivided double star with $p \geq \max\{k, 2\}$ and $q = 2$. We use an algorithm called k-Subdivided Double Star with $q = 2$ that is very similar to the algorithm in the proof of Theorem 1. In order to do so, we need the following extra terminology. Let F be a graph that is isomorphic to $kP_2 + (p-k+2)P_1$. In F, we specify one vertex of each connected component isomorphic to P_2 and call this vertex the *marked* vertex. Moreover, we partition the $(p-k+2)$ isolated vertices of F into one set of $(p-k)$ vertices called *left-unique* and one set of 2 vertices called *right-unique*. We call F as *semi-leaf* graph, that has become *ordered* after we made our choices of marked vertices, left-unique and right-unique vertices.

We are now ready to describe our algorithm. First, we check if G contains a *semi-leaf* graph F. If not, then we return `no`. Otherwise, we choose a semi-leaf graph F and order it. In F, let $u_1, \ldots, u_k$ be its marked vertices, $v_1, \ldots, v_k$ be the respective neighbors of $u_1, \ldots, u_k$, whereas $u_{k+1}, \ldots, u_p$ are its left-unique vertices and u'_1, u'_2 its right-unique vertices. Note that $V_F = \{u_1, \ldots, u_p, u'_1, u'_2, v_1, \ldots, v_k\}$. We remove all neighbors of each v_i not equal to u_i from G should there be any. We remove any vertex that is adjacent to both some u-vertex and some u'-vertex. Afterwards, we contract any edge that has both its end-vertices in the neighborhood of some u-vertex, or both its end-vertices in the neighborhood of some u'-vertex. We do this repeatedly until this is no longer possible. We then check if H is an induced subgraph of the resulting graph G''. If so, then we return `yes`. Suppose not. We remove $v_1, \ldots, v_k$. We then choose sets $S_1, \ldots, S_p$ of at most $p(k+2) + 2p + 1$ vertices each and sets T_1, T_2 of at most $2(k+2) + p + 3$ vertices each; these sets must consist of neighbors of $u_1, \ldots, u_p$, u'_1, u'_2, respectively. Then we remove $u_1, \ldots, u_p$, u'_1, u'_2 together with all their other neighbors. We check if $S_1, \ldots, S_p$, T_1, T_2 are all in the same connected component

L of the remaining graph. If so, then we apply the algorithm of Lemma 1 on L with $Z_1 = S_1 \cup \cdots \cup S_p$ and $Z_2 = T_1 \cup T_2$. If we find an H-witness structure, then we return `yes`. Otherwise, we adjust our choice of S-sets and T-sets, and if necessary our choice of ordering of F or even our choice of F, unless we already have considered all possible choices; in that case we return `no`. For clarity, we give the pseudo-code of this algorithm below.

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k-SUBDIVIDED DOUBLE STAR WITH $q = 2$
Input: A graph G.
Output: `yes` or `no`.

While there is an ordered semi-leaf graph F with marked vertices $u_1, \ldots, u_k$, their respective neighbors $v_1, \ldots, v_k$, left-unique vertices $u_{k+1}, \ldots, u_p$, and right-unique vertices u'_1, u'_2 **do**
    Remove any other vertex that is adjacent to both a u-vertex and a u'-vertex.
    Contract all edges that have both end-vertices in the neighborhood of a u- or u'-vertex.
If H is an induced subgraph of the resulting graph, **then return** YES.
Remove $v_1, \ldots, v_k$.
For all sets $S_1 \subseteq N(u_1), \ldots, S_p \subseteq N(u_p)$ of at most $p(k+2) + 2p + 1$ vertices each and sets $T_1 \subseteq N(u'_1), T_2 \subseteq N(u'_2)$ of at most $2(k+2) + p + 3$ vertices each **do**
  Remove $u_1, \ldots, u_p, u'_1, u'_2$ together with all their other neighbors not in any S- or T- set.
  If $S_1, \ldots, S_p, T_1, T_2$ are in the same connected component L, **then**
    Apply the algorithm of Lemma 1 on L with $Z_1 = S_1 \cup \ldots \cup S_p$ and $Z_2 = T_1 \cup T_2$.
    If the algorithm finds an H-witness structure, **then return** YES.
Return NO.

The correctness proof and running time analysis of this algorithm uses the same arguments as the proof of Theorem 1. The only difference lies in the proof of Claim 3, which changes into:

Claim 3'. Every u-vertex has at most $p(k+2) + 2p + 1$ neighbors in B.

This claim can be proven as follows. Suppose that some u_i has at least $p(k+2)+2p+2$ neighbors in B. Let $R_0 = \{r_1, \ldots, r_{|R_0|}\}$ be the set of cut vertices in B that are adjacent to u_i. Claim 2 tells us that B has at most $p + 1$ internal vertices. This means that $|R_0| \geq p(k + 2) + p + 1$. We assume that the vertices in R_0 are ordered in such a way that for $h = 2, \ldots, |R_0|$, vertices $r_1, \ldots, r_{h-1}$ are in the same connected component of $B - r_h$. Note that such an ordering of R_0 can be obtained as follows. Let $B_1, \ldots, B_s$ be the blocks of B, and let $z_1, \ldots, z_t$ be the cut vertices of B. Then we can define the *block tree* $\mathcal{T}$ of B as the tree that has vertices $B_1, \ldots, B_s, z_1, \ldots, z_t$ and edges $B_i z_j$ if and only if block B_i contains cut vertex z_j in the graph B. We choose z_1 to be the root of $\mathcal{T}$ and order $z_1, \ldots, z_t$ according to a breadth-first search performed on $\mathcal{T}$ that starts in z_j. This yields an ordering $z_{i_1}, \ldots, z_{i_t}$ with $z_{i_1} = z_1$. By definition of a breadth-first search, at the moment the breadth-first search algorithm visits a vertex z_j for some $1 \leq j \leq t$, it has not yet visited any children of z_j. Hence, for $h = 2, \ldots, t$, vertices

$z_{i_1}, \ldots, z_{i_{h-1}}$ are in the same connected component of $B - z_{i_h}$. The restriction of the ordering $z_{i_1}, \ldots, z_{i_t}$ to the vertices of R_0 gives us the desired ordering of $r_1, \ldots, r_{|R_0|}$.

For a vertex $s \in B \setminus R_0$ we define

$$\mathcal{I}(s) = \{i \mid s \text{ is adjacent to at least one vertex of } \{r_i\} \cup N_B(r_i)\}.$$

We claim that $|\mathcal{I}(s)| \leq p + 1$ for all $s \in B \setminus R_0$. This can be seen as follows. Suppose that $|\mathcal{I}(s)| = q \geq p + 2$ for some $s \in B \setminus R_0$. We may assume without loss of generality that $\mathcal{I}(s) = \{1, \ldots, q\}$. Consider a vertex r_j for some $1 \leq j \leq q$. By the definition of the set $\mathcal{I}(s)$, we find that s is adjacent to r_i or a neighbor r_i' of r_i in B for all $1 \leq i \leq q$. In the latter case, i.e., if s is adjacent to a neighbor r_i' of r_i, then $r_i' \notin \{r_1, \ldots, r_q\}$, because the vertices $r_1, \ldots, r_q$ form an independent set. Hence, for every r_j with $1 \leq j \leq q$, all vertices of $\{r_1, \ldots, r_q\} \setminus \{r_j\}$ are in the same connected component C_j of $B - r_j$. Because r_j is a cut vertex of B, we find that $B - r_j$ has a connected component $C_j' \neq C_j$. This means that B contains a leaf block, all of its vertices belong to C_j'. Moreover, for any two distinct vertices r_i and r_j in $\{r_1, \ldots, r_q\}$, we find that C_i' is a subgraph of C_j. As C_j and C_j' are vertex-disjoint, this means that C_i' and C_j' are vertex-disjoint. Hence the leaf blocks contained in the graphs $C_1', \ldots, C_q'$ are mutually vertex-disjoint. This implies that B contains at least $q \geq p+2$ distinct leaf blocks. Recall that each leaf block contains at least one internal vertex of B. Hence, B contains at least $p + 2$ internal vertices. However, this is not possible due to Claim 2. We conclude that $|\mathcal{I}(s)| \leq p + 1$ for all $s \in B \setminus R_0$.

We proceed as follows. We choose r_{i_1} to be the vertex in R_0 that has the lowest index over all vertices in R_0; note that $r_{i_1} = r_1$. Because r_{i_1} is a cut vertex of B, we find that r_{i_1} has two neighbors s_1 and s_1' in B that are not adjacent to each other. Moreover, because R_0 is an independent set, and s_1, s_1' are neighbors of r_{i_1}, we find that s_1 and s_1' are in $B \setminus R_0$. Hence, the sets $\mathcal{I}(s_1)$ and $\mathcal{I}(s_1')$ are defined. Because $\mathcal{I}(s_1) \leq p + 1$, $\mathcal{I}(s_1') \leq p+1$, and $|\mathcal{I}(s_1) \cap \mathcal{I}(s_1')| \geq 1$, there exists a set $R_1 \subseteq R_0 \setminus \{r_{i_1}\}$ of cardinality

$$|R_1| \geq |R_0| - (2p + 1) \geq p(k + 2) + p + 1 - 2p - 1 = pk + p,$$

such that neither s_1 nor s_1' is adjacent to any vertex of $\{r_j\} \cup N_B(r_j)$ for all $r_j \in R_1$.

We choose r_{i_2} to be the vertex in R_1 that has the lowest index over all vertices in R_1. We let s_2 be a neighbor of r_{i_2} that is in a connected component of $B - r_{i_2}$ that does not contain the vertices s_1 and s_1'. Such a choice is possible because of the following two reasons. First, $B - r_{i_2}$ has at least two connected components, because r_{i_2} is a cut vertex of B. Second, s_1 and s_1' belong to the same connected component of $B - r_{i_2}$, because s_1 and s_1' are both adjacent to r_{i_1}. Because R_0 is independent and s_2 is adjacent to r_{i_2}, we find that $s_2 \in B \setminus R_0$. Hence, the set $\mathcal{I}(s_2)$ is defined. Because $\mathcal{I}(s_2) \leq p+1$, there exists a set $R_2 \subseteq R_1 \setminus \{r_2\}$ of cardinality $|R_2| \geq |R_1| - (p+1) \geq pk + p - p - 1 = p(k - 1) + p - 1$, such that s_2 is not adjacent to any vertex of $\{r_j\} \cup N_B(r_j)$ for all $r_j \in R_2$.

We proceed in an inductive way. Suppose that for some $h \leq k$, we have defined sets $R_h \subseteq R_{h-1} \setminus \{r_{i_h}\} \subseteq \cdots \subseteq R_1 \setminus \{r_{i_2}\} \subseteq R_0 \setminus \{r_{i_1}\}$ with respect to $2h + 1$ distinct vertices $r_{i_1}, \ldots, r_{i_h}, s_1, s_1', s_2, \ldots, s_h$ in B that have the following two properties. First, s_g is adjacent to r_{i_g} for $g = 1, \ldots, h$, whereas s_1' is adjacent to r_{i_1}. Second, for $g = 2, \ldots, h$, the vertices $s_1', s_1, \ldots, s_{g-1}$ are in the same connected component of

$B - r_{i_g}$, whereas s_{i_g} does not belong to this connected component but to some other connected component of $B - r_{i_g}$. Moreover, $|R_g| \geq p(k - g + 1) + p - g + 1$ for $g = 1, \ldots, h$.

We now choose $r_{i_{h+1}}$ to be the vertex in R_h that has the lowest index over all vertices in R_h. We let s_{h+1} be a neighbor of $r_{i_{h+1}}$ that is in a connected component of $B - r_{i_{h+1}}$ that does not contain $s_1, s'_1, s_2, \ldots, s_h$. Such a choice is possible, because of the following arguments. Recall that we choose r_{i_g} to be the vertex with the smallest index in R_{g-1} for $g = 1, \ldots, h + 1$. Then, because $R_h \subset \cdots \subset R_0$, we obtain $i_1 < \cdots < i_{h+1}$. Hence, due to the way we ordered the vertices in R_0, we find that $r_{i_1}, \ldots, r_{i_h}$, and consequently, their neighbors $s'_1, s_1, \ldots, s_h$ are in the same connected component of $B - r_{i_{h+1}}$. Recall that $B - r_{i_{h+1}}$ has at least two connected components, because $r_{i_{h+1}}$ is a cut vertex of B. We conclude that we can make the choice of s_{h+1} as described above.

Because R_0 is independent and s_{h+1} is adjacent to $r_{i_{h+1}}$, we find that $s_{h+1} \in B \setminus R_0$. Because $\mathcal{I}(s_{h+1}) \leq p+1$, there exists a set $R_{h+1} \subseteq R_h \setminus \{r_{i_{h+1}}\}$ of cardinality $|R_{h+1}| \geq p(k - h) + p - h$, such that s_{h+1} is not adjacent to any vertex of $\{r_j\} \cup N_B(r_j)$ for all $r_j \in R_{h+1}$. Hence, after $k + 1$ steps, we have found sets $R_{k+1} \subseteq R_k \setminus \{r_{i_{k+1}}\} \subseteq \cdots \subseteq R_1 \setminus \{r_{i_2}\} \subseteq R_0 \setminus \{r_{i_1}\}$ with respect to $2k + 3$ distinct vertices $r_{i_1}, \ldots, r_{i_{k+1}}, s_1, s'_1, s_2, \ldots, s_{k+1}$ in B that have the following two properties. First, s_g is adjacent to r_{i_g} for $g = 1, \ldots, k + 1$, whereas s'_1 is adjacent to r_{i_1}. Second, for $g = 2, \ldots, k + 1$, vertices $s'_1, s_1, \ldots, s_{g-1}$ are in the same connected component of $B - r_{i_g}$, whereas s_{i_g} does not belong to this connected component but to some other connected component of $B - r_{i_g}$. The latter property, together with the property that s_1 and s'_1 are not adjacent, implies that $s'_1, s_1, \ldots, s_{k+1}$ form an independent set. By induction, we also have found that $|R_g| \geq p(k-g+1)+p-g+1$ for $g = 1, \ldots, k+1$.

Because $|R_{k+1}| \geq p - k$, there exist vertices $r_{i_{k+2}}, \ldots, r_{i_{p+1}}$ in R_{k+1}. Let G^* denote the subgraph of G induced by the vertices $u_1, r_{i_1}, \ldots, r_{i_{p+1}}, s_1, s'_1, s_2, \ldots, s_{k+1}$. We now show that G^* is isomorphic to H. We let $s_2, \ldots, s_{k+1}, r_{i_{k+2}}, \ldots, r_{i_{p+1}}$ correspond to the p a-vertices of H, $r_{i_2}, \ldots, r_{i_{k+1}}$ to the vertices of H obtained by subdividing the edges $a_i b$ for $i = 1, \ldots, k$, u_1 to the b-vertex of H, r_{i_1} to the c-vertex of H, and s_1, s'_1 to the two d-vertices of H. The edges $r_{i_1} s'_1$ and $r_{i_h} s_h$ for $h = 1, \ldots, k + 1$, together with the edges $r_{i_h} u_1$ for $h = 1, \ldots, p + 1$ ensure that G^* contains a spanning subgraph isomorphic to H. Because $\{r_{i_1}, \ldots, r_{i_{p+1}}\}$ and $\{s'_1, s_1, \ldots, s_{k+1}\}$ are independent sets, and s'_1 is not adjacent to any r_{i_h} with $2 \leq h \leq k+1$, and no s_g is adjacent to any r_{i_h} with $h \neq g$, this spanning subgraph of G^* is induced. Hence, G^* is isomorphic to H. However, in this stage of the algorithm we assume that G does not contain H as an induced subgraph. Hence, by this contradiction, we have proven Claim 3'.

Adapting the proof of Claim 3' with respect to the vertices u'_1 and u'_2, we find that u'_1 and u'_2 each have at most $2(k + 2) + p + 3$ neighbors in B. We also note that the number of different ordered semi-leaf graphs of G is bounded by n^{k+p+2}, which is a polynomial number, because $k \leq p$, and p is assumed to be fixed. As all other arguments are the same as in the proof of Theorem 1, the theorem follows. $\square$

Recall that H^* denote the graph obtained by subdividing the centre edge in a double star with $p = q = 2$ (see Figure 2). Our last result is a consequence of Propositions 1 and 2 and Theorems 1 and 2.

Corollary 1. *For any fixed forest $H \neq H^*$ on at most 7 vertices, H-*INDUCED MINOR *can be solved in polynomial time.*

Proof. Let H be a forest on at most 7 vertices that is not isomorphic to H^*. First suppose that H is a tree. Note that in our definition a path is a subdivided star. If H is a subdivided star, then we apply Proposition 2. Now suppose that H is not a subdivided star. Then H contain at least two vertices b and c of degree at least 3. Because $H \neq H^*$, this means that H is a double star with $2 \leq p \leq 3$ and $q = 2$, or H is a 1-subdivided double star with $p = q = 2$. In the first case we can apply Theorem 1. In the second case we can apply Theorem 2. Now suppose that H has at least two connected components. Then all but at most one of its connected components are paths. Hence we may apply Proposition 1. This completes our proof of Corollary 1. □

4 Future work

The following problem is open.

1. What is the computational complexity of H-INDUCED MINOR, when H is a double star with $p = 3$ and $q = 3$?

With respect to Problem 1, we note that the proof of Theorem 1 does not generalize in the sense that an induced copy of a double star with $p = 3$ and $q = 3$ seems hard to force in order to bound the number of vertices in the interfaces.

By Corollary 1 we have a polynomial-time algorithm for H-INDUCED MINOR if H is a forest on at most 7 vertices except when H is the graph H^*, which is the graph obtained by subdividing the centre edge in a double star with $p = q = 2$.

2. What is the computational complexity of H^*-INDUCED MINOR?

Also with respect to Problem 2, we note that our current techniques (bounding the size of some semi-witness bags or interfaces, or excluding the target as an induced subgraph) are not sufficient. The reason is that these techniques in combination with some brute force guessing of bags or interfaces do not forbid any non-adjacencies between bags, and an "induced" version of Lemma 1 does not exist due to NP-completeness of the corresponding decision problem [3]. In the case of H^* the two bags that correspond to the end-vertices of the centre edge that has been subdivided may no longer be adjacent.

We observe that Proposition 1 does not easily translate to cycles F. Because a graph contains the k-vertex cycle denoted C_k as an induced minor if and and only if it contains an induced cycle on at least k vertices, the C_k-INDUCED MINOR problem is polynomial-time solvable for any fixed $k \geq 3$. However, the following case is a notoriously open case, which also shows that a similar result as Corollary 1 for general target graphs H on at most 6 vertices is still far away. Let $2C_3$ denote the disjoint union of two 3-vertex cycles.

3. What is the computational complexity of $2C_3$-INDUCED MINOR?

We observe that $2C_3$-CONTRACTIBILITY is polynomial-time solvable. This can be seen as follows. A graph G contains $2C_3$ as a contraction if and only G consists of two connected components, each of which contains C_3 as a contraction. The latter can be tested in polynomial time by verifying if the two connected components are not trees. So far, there are no cases known for which H-CONTRACTIBILITY is polynomial-time solvable but H-INDUCED MINOR is NP-complete. On the other hand there are many cases for which H-INDUCED MINOR is polynomial-time solvable and H-CONTRACTIBILITY is NP-complete. Recall for instance that P_k-INDUCED MINOR is polynomial-time solvable for any fixed $k \geq 1$, whereas Brouwer and Veldman [4] showed that P_k-CONTRACTIBILITY is NP-complete for any fixed $k \geq 4$. The case $H = 2C_3$ illustrates that when the target graph becomes disconnected there might exist cases for which H-INDUCED MINOR is computationally harder than H-CONTRACTIBILITY. This brings us to the last open problem.

4. Does there exist a graph H for which H-CONTRACTIBILITY is polynomial-time solvable and H-INDUCED MINOR is NP-complete?

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