

# Upper bounds and algorithms for parallel knock-out numbers

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## Abstract

We study parallel knock-out schemes for graphs. These schemes proceed in rounds in each of which each surviving vertex simultaneously eliminates one of its surviving neighbours; a graph is reducible if such a scheme can eliminate every vertex in the graph. We show that, for a reducible graph  $G$ , the minimum number of required rounds is  $O(\sqrt{\alpha})$ , where  $\alpha$  is the independence number of  $G$ . This upper bound is tight and the result implies the square-root conjecture which was first posed in MFCS 2004. We also show that for reducible  $K_{1,p}$ -free graphs at most  $p - 1$  rounds are required. It is already known that the problem of whether a given graph is reducible is NP-complete. For claw-free graphs, however, we show that this problem can be solved in polynomial time. We also pinpoint a relationship with (locally bijective) graph homomorphisms.

**Keywords:** parallel knock-out schemes, claw-free graphs, computational complexity

**AMS Subject Classifications:** 05C70, 05C85

## 1 Introduction

In this paper, we continue the study on *parallel knock-out schemes* for finite undirected simple graphs introduced in [9] and studied further in [3, 4, 5].

Such a scheme proceeds in rounds: in the first round each vertex in the graph selects exactly one of its neighbours, and then all the selected vertices are eliminated simultaneously. In subsequent rounds this procedure is repeated in the subgraph induced by those vertices not yet eliminated. The scheme continues until there are no vertices left, or until an isolated vertex is obtained (since an isolated vertex will never be eliminated).

A graph is *KO-reducible* if there exists a parallel knock-out scheme that eliminates the whole graph. The *parallel knock-out number* of a graph  $G$ , denoted by  $\text{pko}(G)$ , is the minimum number of rounds in a parallel knock-out scheme that eliminates every vertex of  $G$ . If  $G$  is not KO-reducible, then  $\text{pko}(G) = \infty$ .

Knock-out schemes have an obvious relationship with games on graphs, a topic which has received considerable attention in the last decades ([7]). But unlike many games on graphs, knock-out schemes can be motivated by practical settings, e.g., in which objects exchange entities that inactivate the receiving objects, like viruses that paralyse or block computers, or computational tasks that disable processors or sensors from other tasks. Especially in the relatively new area of sensor networks, knock-out schemes for the underlying graph structures can model practical situations in which sensors exchange data with neighbouring sensors that temporarily disables the receiving sensors from their main monitoring tasks. This happens, e.g., in situations where sensors have a low battery and limited computational power. They share measured and processed data with other sensors in their close vicinity as well as with more powerful PCs, laptops or mainframes at larger distances. Consider a setting with a number of sensors that perform simple measurements, for instance on temperature, humidity, smoke levels, movements, or the like. Data sharing is important for two reasons: in order to rule out erroneous data (by comparisons with data gathered at a neighbouring sensor) and in order to preprocess the data before sending it to a more powerful computer. During the preprocessing stage in a sensor no new data can be collected by that sensor, so the chosen neighbouring sensors are out of order for the time being, while the other sensors continue collecting data, sharing it with other active neighbouring sensors, and so on, until all sensors are out of order or run out of available neighbouring sensors. Then a new round of data collection and sharing starts. In the ideal case all sensors have shared their data with at least one neighbouring sensor and have performed some preprocessing of their data. In order to keep the time intervals between successive rounds of data collection as short as possible, the number of stages within one round should be kept to a minimum. This problem setting can be modelled by parallel knock-out schemes

and the parallel knock-out number comes up naturally.

Our main motivation for studying knock-out schemes, though, is the intimate relationship between this concept and well-studied structural graph theoretical concepts like perfect matchings, hamiltonian cycles and 2-factors (they all yield knock-out schemes of one round). Apart from these structural aspects, we are interested in complexity aspects. Whereas the classical complexity problems related to matchings and hamiltonian cycles have been settled many years ago, the analogous problems related to knock-out schemes have been resolved recently, and only for general graphs and graphs of bounded tree-width. For many interesting classes, however, these problems on knock-out schemes are still open [4].

## 1.1 Our results

In [4], a number of results, conjectures and questions on upper bounds for knock-out numbers were presented. For trees, the problem was resolved by showing that the knock-out number of a tree on  $n$  vertices was  $O(\log n)$  and by exhibiting a family of trees that met this bound. Also presented was a family of bipartite graphs whose knock-out numbers grow proportionally to the square root of the number of vertices, and it was conjectured that for any KO-reducible graph on  $n$  vertices the knock-out number is at most  $2\sqrt{n}$ . In this paper, in Section 3, we prove this conjecture.

In [4], a polynomial algorithm was also given that would determine the parallel knock-out number of any tree. In [5] it was shown that the problem of finding parallel knock-out numbers is, for general graphs, NP-complete. In this paper, in Section 4, we present a polynomial algorithm that finds the knock-out number of claw-free graphs, that is, graphs that do not contain an induced  $K_{1,3}$ ; these form a well-studied class of graphs, see [6] for a survey. We also give a tight bound on the knock-out number of KO-reducible  $K_{1,p}$ -free graphs, generalizing a result of [4] on claw-free graphs.

In Section 5, we give an upper bound on the parallel knock-out number of one graph in terms of the parallel knock-out number of another graph: we show that if a graph  $G$  allows a so-called locally bijective homomorphism to a graph  $H$  then  $\text{pko}(G) \leq \text{pko}(H)$ . Locally bijective homomorphisms, also called graph coverings, are well-studied and have several applications [1, 8].

## 2 Preliminaries

Graphs in this paper are denoted by  $G = (V, E)$ . An edge joining vertices  $u$  and  $v$  is denoted by  $uv$ . If not stated otherwise a graph is assumed to be

undirected and simple. If a graph  $G$  is directed then an arc from a vertex  $u$  to a vertex  $v$  is denoted by  $(u, v)$ . In the *null graph*,  $V = E = \emptyset$ . For graph terminology not defined below, we refer to [2].

For a vertex  $u \in V$  we denote its *neighbourhood*, that is, the set of adjacent vertices, by  $N(u) = \{v \mid uv \in E\}$ . The *degree* of a vertex is the number of edges incident with it, or, equivalently, the cardinality of its neighbourhood. A subset  $U \subseteq V$  is called an *independent set* of  $G$  if no two vertices in  $U$  are adjacent to each other. The *independence number*  $\alpha$  of a graph  $G$  is the number of vertices in a maximum independent set of  $G$ .

A *complete bipartite* graph  $K_{|X|,|Y|}$  is a bipartite graph with the maximum number of edges between its bipartite classes  $X$  and  $Y$ . If  $|X| = 1$ , then it is a *star* and the vertex in  $X$  is the *centre vertex* and the vertices in  $Y$  are *leaves*. If  $|X| = 1$  and  $|Y| = 1$  we arbitrarily choose one of the star's two vertices to be the centre vertex. A graph  $G$  that does not contain a  $K_{1,p}$  as an induced subgraph for some  $p \geq 1$  is said to be  $K_{1,p}$ -free. A  $K_{1,3}$ -free graph is also called *claw-free*.

For a graph  $G$ , a *KO-selection* is a function  $f : V \rightarrow V$  with  $f(v) \in N(v)$  for all  $v \in V$ . If  $f(v) = u$ , we say that vertex  $v$  *fires at* vertex  $u$ , or that vertex  $u$  *is knocked out* by vertex  $v$ . We also say that  $u$  is a *victim* of  $v$ . For each  $u \in f(V)$ , we denote the set of vertices that fire at  $u$  by  $K(u)$ , i.e.,  $v \in K(u)$  if and only if  $f(v) = u$ . If  $K(u) = \{v\}$ , that is, vertex  $v$  is the only vertex that fires at  $u$ , then we call  $u$  the *unique victim* of  $v$ . For a subset  $U \subseteq f(V)$  we use the shorthand notation  $K(U) = \bigcup_{u \in U} K(u)$ , and we say that such a subset  $U$  is *knocked out* by a subset  $W \subseteq V$  if  $K(U) \subseteq W$ , that is, if every vertex in  $U$  is knocked out by a vertex in  $W$ .

For a KO-selection  $f$ , we define the corresponding *KO-successor* of  $G$  as the subgraph of  $G$  that is induced by the vertices in  $V \setminus f(V)$ ; if  $H$  is the KO-successor of  $G$  we write  $G \rightsquigarrow H$ . Note that every graph without isolated vertices has at least one KO-successor. A graph  $G$  is called *KO-reducible*, if there exists a finite sequence

$$G \rightsquigarrow G^1 \rightsquigarrow G^2 \rightsquigarrow \dots \rightsquigarrow G^r,$$

where  $G^r$  is the null graph. If no such sequence exists, then  $\text{pko}(G) = \infty$ . Otherwise, the parallel knock-out number of  $G$ ,  $\text{pko}(G)$ , is the smallest number  $r$  for which such a sequence exists. A sequence  $S$  of KO-selections that transform  $G$  into the null graph is called a *KO-reduction scheme*. A single step in this sequence is called a *round* of the KO-reduction scheme. We denote the number of rounds in  $S$  by  $r(S) = r$ .

For a KO-reduction scheme  $S$  we denote the set of vertices that are

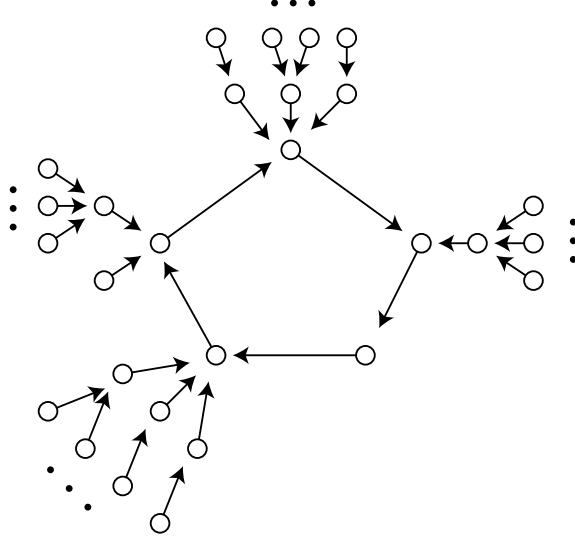


Figure 1: A component of a graph  $G_i$ .

victims of a vertex  $v$  by  $L(v)$ . For a subset  $W \subseteq V$ , we use the shorthand notation  $L(W) = \bigcup_{v \in W} L(v)$ .

An *in-tree* is a directed tree that contains a *root*  $u$  that can be reached from any other vertex by a directed path. Note that a graph containing only one vertex is an in-tree. For  $i = 1, \dots, r$ , we denote the subset of vertices knocked out in round  $i$  by  $R_i$ . Let  $G_i$  be the directed graph with vertex set  $R_i$  and an arc from a vertex  $u$  to a vertex  $v$  if and only if  $u$  fires at  $v$  in round  $i$ . We may also use  $G_i$  to denote the underlying undirected graph; it will always be clear which from the context). Also, observe that  $G_i$  and  $G^i$  denote two different graphs. As each vertex in a round has exactly one edge oriented away from it, we can make the following observation (which is illustrated in Fig. 1).

**Observation 1** *Let  $S$  be a KO-reduction scheme for a graph  $G$ . For  $i = 1, \dots, r$ , each component of  $G_i$  is formed by a directed cycle  $D$  on at least two vertices, such that each vertex on  $D$  is the root of some pendant in-tree.*

Another observation we will use is the following.

**Observation 2** *If a graph  $G$  contains two distinct vertices of degree 1 that share the same neighbour, then  $G$  is not KO-reducible.*

Note that when referring to, for example,  $G_i$ , it is implicit that we know with respect to which KO-reduction scheme this graph is defined (we wish

to avoid the cumbersome notation necessary to make it explicit). Sometimes we will be considering pairs of schemes and will write, for instance, that  $G_2$  has fewer vertices under  $S'$  than under  $S$ . The meaning of this should be clear.

### 3 Resolving the square-root conjecture

Let  $S$  be a KO-reduction scheme for a KO-reducible graph  $G$ . It turns out that the square-root conjecture can be solved by considering schemes that knock out vertices “as early as possible”. Hence, we define

$$w(S) = \sum_{i=1}^{r(S)} i|R_i|,$$

and we say that  $S$  is a *minimal* KO-reduction scheme for  $G$  if

$$w(S) = \min\{w(S) \mid S \text{ is a KO-reduction scheme for } G\}.$$

For a minimal KO-reduction scheme  $S$  of a graph  $G$ , we can make a number of further assumptions. We use the following terminology. If  $G_i$  has a component  $C$  that consists of two vertices  $u$  and  $v$  we call  $C$  a *two-component* of  $G_i$ . Note the existence of arcs  $(u, v)$  and  $(v, u)$  between the vertices  $u$  and  $v$  of a two-component  $C$ . If  $G_i$  has a component  $C$  that consists of vertices  $u, v_1, \dots, v_p$  for some  $p \geq 2$  with arcs  $(u, v_1), (v_1, u), (v_2, u), \dots, (v_p, u)$  then we call  $C$  a *star-component* of  $G_i$  with *centre vertex*  $u$ . The vertices  $v_1, \dots, v_p$  are called the *leaves* of  $C$ , and  $v_1$  is called the *centre-victim*, and the other leaves are called *centre-free*. Finally, if  $G_i$  has a component that is a directed cycle with an odd number of vertices then we call such a component an *odd cycle-component* of  $G_i$ .

**Lemma 3** *If  $G$  is KO-reducible, then  $G$  admits a minimal KO-reduction scheme  $S$  with the following properties:*

- (i) *Each component  $C$  of  $G_1$  is either a two-component, a star-component or an odd cycle-component.*
- (ii) *For  $2 \leq i \leq r - 1$ , every component of  $G_i$  is either a two-component or a star-component.*
- (iii) *Every component of  $G_r$  is a two-component.*
- (iv) *If  $C$  is an odd cycle-component (in  $G_1$ ) then no vertices of  $R_2, \dots, R_r$  fire at vertices of  $C$  in round 1.*

(v) For  $1 \leq i \leq r-1$ , there is no edge in  $G$  between any two leaves of the same star-component or of two different star-components in  $G_i$ .

**Proof:** Let  $G$  be a KO-reducible graph. Then  $G$  admits a KO-reduction scheme  $S$ . Let  $C$  be a component in  $G_i$  for some  $1 \leq i \leq r$ . We start the proof by showing that if  $S$  is minimal, then we can assume that  $C$  is either a two-component, a star-component or an odd cycle-component. By Observation 1,  $C$  is formed by a directed cycle  $D$  on vertices  $u_1, \dots, u_p$  for some  $p \geq 2$ , such that each  $u_i$  is the root of some pendant in-tree  $T_i$ .

Suppose  $p$  is even and  $p \geq 4$ . We adjust the firing by letting the vertices of  $V_D$  fire at each other according to a perfect matching of  $D$ . Hence, we may assume that this case does not occur.

Suppose  $p \geq 3$  is odd. If  $D$  contained a vertex that is knocked out by some vertex  $v$  in its corresponding pendant in-tree, then we can adjust the firing by letting the vertices of  $V_D \cup \{v\}$  fire at each other according to a perfect matching of this subgraph. Hence, we may assume that  $C = D$  is an odd cycle-component.

Suppose that  $p = 2$ . Then the underlying undirected graph of  $C$  is a tree, and it is obvious that it can be decomposed into two-components and star-components (and that we can let these components define the firing).

By Observation 2, we have that  $G_r$  cannot contain any star-components. To complete the proof of (i)–(iii), we must show that odd cycle-components only occur in  $G_1$ . To do this we shall first prove a claim which also immediately implies (iv): for any odd cycle-component  $D$  we may assume that  $K(D) = D$ ; that is, vertices in  $D$  are only knocked out by each other. Suppose  $D$  is an odd cycle-component on vertices  $u_1, \dots, u_p$  in some  $G_i$  for  $i \geq 1$ , such that there exists a vertex  $v \in K(D) \setminus D$  and  $v$  fires at  $u_1$ . We adjust the firing by replacing the arc  $(u_p, u_1)$  by  $(u_p, u_{p-1})$  and return to a previous case. Hence, we may assume that this case does not occur.

Now suppose that a graph  $G_i$ ,  $i \geq 2$ , contains an odd cycle-component  $D$ . First suppose that in round  $i-1$  all vertices in  $D$  fire at vertices in  $R_{i-1}$  that either are centre vertices of star-components, or else belong to two-components or odd cycle-components. Since we just saw that no vertices in  $R_{i+1} \cup \dots \cup R_r$  fire at  $D$ , we can move  $D$  to  $G_{i-1}$  (since all victims of  $D$  in  $R_{i-1}$  are not unique, it does not matter if the vertices of  $D$  fire at each other instead). This way we obtain a KO-reduction scheme  $S'$  with  $w(S') < w(S)$ . This contradicts the minimality of  $S$ . In the remaining case, there exists a vertex  $u$  in  $D$  that fires at a leaf  $w$  in a star-component in  $R_{i-1}$ . We let  $u$  and  $w$  fire at each other in round  $i-1$ , so we are able to move  $u$  to  $R_{i-1}$  as  $K(D) = D$ . We let the other vertices in  $D$  fire at each

other in round  $i$  according to a perfect matching of  $D - u$ . This way we again obtain a KO-reduction scheme  $S'$  with  $w(S') < w(S)$ , contradicting the minimality of  $S$ .

To finish the claim we prove (v). Suppose  $u$  and  $v$  are leaves in  $G_i$  for some  $1 \leq i \leq r - 1$ , such that  $u$  and  $v$  are adjacent in  $G$ . In case  $u$  and  $v$  are leaves of different star-components, we adjust the firing by letting  $u$  and  $v$  fire at each other, and, if necessary, changing the centre-victims to be vertices other than  $u$  and  $v$ . Suppose  $u$  and  $v$  are leaves of the same star-component  $C$ . Let  $z$  be the centre vertex of  $C$ . If  $C$  has a third leaf, then we again let  $u$  and  $v$  fire at each other and let another leaf be the centre-victim. Otherwise we can form an odd cycle-component and return to a previous case.  $\square$

We call a minimal KO-reduction scheme  $S$  of a graph  $G$  that satisfies the properties (i)-(v) of Lemma 3 a *simple* KO-reduction scheme of  $G$ . We will continue to find further properties of simple KO-reduction schemes.

**Observation 4** *Let  $S$  be a simple KO-reduction scheme for a graph  $G$ . Let  $u, v$  be, respectively, vertices of  $R_i$  and  $R_j$ ,  $i < j$ , such that  $u$  is the unique victim of  $v$ . Then  $u$  is a centre-free leaf of a star-component in  $G_i$ .*

**Proof:** By Lemma 3,  $u$  cannot be a vertex of an odd cycle-component. If  $u$  is in a two-component, or  $u$  is the centre vertex or centre-victim of a star-component, then there are at least two vertices firing at  $u$ . Hence  $u$  must be a centre-free leaf of a star-component.  $\square$

**Lemma 5** *Let  $S$  be a simple KO-reduction scheme for a graph  $G$  with  $r \geq 2$ . Let  $C$  be a two-component in  $G_r$ . Then in rounds  $1, \dots, r - 1$  all victims of one of the two vertices of  $G_r$  are not unique, and all victims of the other one are unique.*

**Proof:** For  $i = 1, \dots, r - 1$ , let  $x_i$  be the victim of  $u$  in round  $i$ , and let  $y_i$  be the victim of  $v$  in round  $i$ .

Suppose both  $x_{r-1}$  and  $y_{r-1}$  are not unique victims. We show that this means that it is possible to move  $u$  and  $v$  to  $R_{r-1}$ . If  $x_{r-1} \neq y_{r-1}$  or  $x_{r-1} = y_{r-1}$  is the victim of vertices other than  $u$  and  $v$ , then let  $u$  and  $v$  fire at each other in round  $r - 1$ . If  $x_{r-1} = y_{r-1}$  is fired at by only  $u$  and  $v$ , then it is a centre-free vertex of a star-component and we can adjust the firing to let  $u, v$  and  $x_{r-1}$  form an odd cycle-component in  $G_{i-1}$ . Either way we obtain a new KO-reduction scheme  $S'$  with  $w(S') < w(S)$ , contradicting the minimality of  $S$ . Hence we can assume that  $y_{r-1}$  is a unique victim.



We show that all victims of  $u$  are not unique by contradiction. Let  $h$  be the largest index such that  $x_h$  is unique. By Observation 4, vertices  $x_h$  and  $y_{r-1}$  are centre-free leaf vertices of star-components. Since centre vertices are not unique victims, we can let  $u$  and  $x_h$  fire at each other in round  $h$ , and we can let  $v$  and  $y_{r-1}$  fire at each other in round  $r-1$ . This way we obtain a new KO-reduction scheme  $S'$  with  $w(S') < w(S)$ . This contradicts the minimality of  $S$ .

Now we again find a contradiction to show that all victims of  $v$  are unique. Let  $h$  be the largest index such that  $y_h$  is not a unique victim. Then we let  $v$  fire at  $y_j$  in round  $j-1$  for  $j = h+1, \dots, r-1$  (so we move those vertices from  $R_j$  to  $R_{j-1}$ ), and  $v$  does not fire at  $y_h$  anymore. Since  $x_{r-1}$  is not a unique victim, we can then let  $u$  and  $v$  fire at each other in round  $r-1$ . This way we obtain a new KO-reduction scheme  $S'$  with  $w(S') < w(S)$ . This contradicts the minimality of  $S$  and completes the proof of the lemma.  $\square$

**Lemma 6** *Let  $S$  be a simple KO-reduction scheme for a graph  $G$  with  $r \geq 2$ . For each  $i \geq 2$ ,  $R_i$  contains a vertex  $v_i$  whose victims in round  $1, \dots, i-1$  are all unique. Let  $u_r$  be the (unique) neighbor of  $v_r$  in  $G_r$ . Then  $\bigcup_{i=2}^r L(v_i) \cup \{u_r\}$  is an independent set of cardinality  $\frac{r^2-r+2}{2}$  in  $G$ .*

**Proof:** Since  $R_r$  is non-empty, there exists a two-component  $C$  in  $G_r$ . Let  $u_r$  and  $v_r$  be the two vertices of  $C$ . By Lemma 5, we may assume that all victims of  $u_r$  in rounds  $i = 1, \dots, r-1$  are not unique, and all victims of  $v_r$  are unique. Denote the victims of  $v_r$  in rounds  $i = 1, \dots, r-1$  by  $y_1^r, \dots, y_{r-1}^r$ , respectively. By Observation 4, every  $y_i^r$  is a centre-free leaf vertex of a star-component  $C_i^r$ . For  $i = 2, \dots, r-1$ , let  $v_i$  be the centre vertex of  $C_i^r$  and for  $h = 1, \dots, i-1$ , let  $y_h^i$  be the victim of  $v_i$  in round  $h$ . We claim that these victims  $y_h^i$  are all unique. For  $i = r$ , this is already shown. We prove the rest of the statement by contradiction. Let  $2 \leq i \leq r-1$ . Let  $h$  be the largest index such that  $y_h^i$  is not a unique victim of  $v_i$ . We adjust the firing as follows. Since  $y_h^i$  is not a unique victim of  $v_i$ , we do not have to let  $v_i$  fire at it. Then we let  $v_i$  fire at  $y_j^i$  in round  $j-1$  for  $j = h+1, \dots, i-1$ , so we move  $y_j^i$  to  $R_{j-1}$  for  $j = h+1, \dots, i-1$ . In round  $i-1$  we let  $v_i$  fire at  $y_i^r$ , so we move  $y_i^r$  to  $R_{i-1}$ . Then we do not have to let  $v_r$  fire at  $y_i^r$ . Hence, we can let  $v_r$  fire at  $y_j^r$  in round  $j-1$  for  $j = i+1, \dots, r-1$ , so we move  $y_j^r$  to round  $j-1$  for  $j = i+1, \dots, r-1$ . Finally, we let  $u_r$  and  $v_r$  fire at each other in round  $r-1$ . This is possible, because the victim of  $u_r$  in round  $r-1$  is not unique, due to Lemma 5. This way we have obtained a new KO-reduction scheme  $S'$  with  $w(S') < w(S)$ , contradicting the minimality

of  $S$ .

We will now prove that

$$L = \bigcup_{i=2}^r L(v_i) = \bigcup_{i=2}^r \bigcup_{h=1}^{i-1} y_h^i$$

is an independent set. We first note that

$$|L| = \left| \bigcup_{i=2}^r \bigcup_{h=1}^{i-1} y_h^i \right| = \sum_{i=2}^r \sum_{h=1}^{i-1} 1 = \frac{r^2 - r}{2},$$

since all vertices in  $L$  are unique victims.

Because  $S$  is simple, by Lemma 3, there is no edge between any two vertices  $y_h^i$  and  $y_h^j$ . Suppose there were an edge  $y_h^i y_j^r$ , where  $h \neq j$ . If  $h < j$ , then we move  $y_j^r$  to  $R_h$ , each  $y_k^r$  for  $k = j+1, \dots, r-1$  to  $R_{k-1}$ , and finally  $u_r$  and  $v_r$  to  $R_{r-1}$ . We can adjust the firing and obtain a new KO-reduction scheme  $S'$  with  $w(S') < w(S)$ . This contradicts the minimality of  $S$ . If  $h > j$ , then we move  $y_h^i$  to  $R_j$ , each  $y_k^r$  for  $k = i, \dots, r-1$  to  $R_{k-1}$ , and finally  $u_r$  and  $v_r$  to  $R_{r-1}$ . We adjust the firing and obtain the same contradiction as before. Suppose there exists an edge between two vertices  $y_h^i$  and  $y_j^k$  with  $h < j$  and  $r \notin \{i, j\}$ . We move  $y_j^k$  to  $R_h$ , each  $y_\ell^r$  for  $\ell = j, \dots, r-1$  to  $R_{\ell-1}$ , and finally  $u_r$  and  $v_r$  to  $R_{r-1}$ . We adjust the firing and obtain the same contradiction as before.

Now suppose  $u_r$  is adjacent to a vertex  $y_h^i$  of  $L$ . By Lemma 5, all victims of  $u_r$  are not unique. Then we can let  $u_r$  fire at  $y_h^i$  in round  $i$ . Then  $y_h^i$  is no longer a unique victim and we find a KO-reduction scheme  $S'$  with  $w(S') < w(S)$  as before. This final contradiction completes the proof.  $\square$

We are now ready to state our main theorem, which proves (and strengthens) the square-root conjecture posed in [4].

**Theorem 7** *Let  $G$  be a KO-reducible graph. Then*

$$pko(G) \leq \min \left\{ -\frac{1}{2} + \sqrt{2n - \frac{7}{4}}, \frac{1}{2} + \sqrt{2\alpha - \frac{7}{4}} \right\}.$$

**Proof:** It is straightforward to check that the statement holds for a graph  $G$  with  $pko(G) = 1$ . Let  $S$  be a simple KO-reduction scheme for a graph  $G$  with  $r \geq pko(G) \geq 2$ . By Lemma 6, we find an independent set  $L'$  of  $G$  that has cardinality  $|L'| = \frac{1}{2}(r^2 - r + 2) \leq \alpha$ . Note that  $R_1$  contains a centre vertex of a star-component. This, together with Lemmas 5 and 6, implies

that  $n \geq |L'| + r - 1 + 1 = \frac{1}{2}(r^2 - r + 2) + r$ . Solving both inequalities gives us the required upper bound.  $\square$

We note that the bound mentioned in Theorem 7 is asymptotically tight. In [4], it has been proven that for all  $p \geq 1$ ,  $\text{pko}(K_{p,q}) = p = \Theta(\sqrt{n}) = \Theta(\sqrt{\alpha})$  for all complete bipartite graphs on  $n = p + q$  vertices with  $q = \frac{1}{2}p(p + 1)$ .

## 4 Claw-free graphs

It is known that claw-free graphs can be knocked out in at most two rounds [4] if they are KO-reducible (not all claw-free graphs are, take for example an isolated vertex or a path on three vertices). We generalize this result for  $K_{1,p}$ -free graphs for any  $p \geq 2$ . This solves a question in [4].

**Theorem 8** *Let  $p \geq 1$ . If a  $K_{1,p}$ -free graph  $G$  is KO-reducible then  $\text{pko}(G) \leq p - 1$ .*

**Proof:** The case  $p = 1$  is trivial. For  $p \geq 2$ , the statement follows directly from Lemma 6.  $\square$

This result is the best possible. In [4, Section 4], a tree  $Y_\ell$  is defined for each integer  $\ell \geq 1$ , and it is shown that  $\text{pko}(Y_\ell) = \ell$ . It is also easy to check that  $Y_\ell$  is  $K_{1,\ell+1}$ -free. We omitted the details.

In the rest of this section, we suppose that  $G = (V, E)$  is a claw-free graph and show that  $\text{pko}(G)$  can be determined in polynomial time. We need the following lemma.

**Lemma 9** *Let  $G$  be a connected claw-free graph with  $\text{pko}(G) = 2$ . Then there is a simple KO-reduction scheme in which only two vertices  $u$  and  $v$  survive to the second round.*

**Proof:** By Lemma 3 and claw-freeness, we know there is a simple two-round KO-reduction scheme  $S$  for  $G$  such that

- (i) each component of  $G_1$  is a two-component, star-component or odd cycle,
- (ii) each component of  $G_2$  is a two-component,
- (iii) in the first round the vertices of  $G_2$  do not fire at vertices that belong to odd cycles in  $G_1$ , and

(iv) the leaves of the star-components in  $G_1$  are not adjacent.

As the leaves of the star-components are not adjacent, we can, by claw-freeness and Lemma 3, further suppose that each star-component is a path on three vertices which we shall call a *three-component*.

Note that among all schemes that satisfy these properties,  $S$  is the one with the fewest number of components in  $G_2$  (as it is minimal). To prove the lemma, we show that if, for  $S$ ,  $G_2$  contains more than one component, then we can find a scheme  $S'$  that admits fewer components to  $G_2$ .

For  $S$ , let the vertex sets of the two-components of  $G_2$  be  $\{\{u_i, v_i\} \mid i = 1, \dots, q\}$ . By Lemma 5, we can assume that the victim of  $u_i$  in  $G_1$  is not unique, but that of  $v_i$  is unique. By Observation 4,  $v_i$  fires at the centre-free leaf of a three-component, say  $y_i$ . Let  $x_i$  be the victim of  $u_i$ . Suppose that  $x_i$  is the centre vertex of a three-component. Then there is also an edge from  $u_i$  to one of the leaves, say  $w$ , of the three-component (else, by (iv),  $x_i, u_i$  and the leaves of the three-component induce a claw). Let  $z$  be the other leaf of the three-component.

Suppose that  $y_i = w$ . Then let  $S'$  be a scheme identical to  $S$  except that in the first round

- $v_i$  fires at  $y_i$ ,
- $y_i$  fires at  $u_i$ ,
- $u_i$  fires at  $v_i$ ,
- $x_i$  and  $z$  fire at each other.

Thus  $S'$  has one fewer two-component in  $G_2$  than  $S$ .

Suppose that  $y_i = z$ . Then let  $S'$  be a scheme identical to  $S$  except that in the first round

- $v_i$  and  $y_i$  fire at each other,
- $u_i$  fires at  $x_i$ ,
- $x_i$  fires at  $w$ ,
- $w$  fires at  $u_i$ .

Thus  $S'$  has one fewer two-component in  $G_2$  than  $S$ .

Suppose  $y_i \notin \{w, z\}$ . Then let  $S'$  be a scheme identical to  $S$  except that in the first round

- $v_i$  and  $y_i$  fire at each other,
- $u_i$  and  $w$  fire at each other, and
- $x_i$  and  $z$  fire at each other.

Thus  $S'$  has one fewer two-component in  $G_2$  than  $S$ . Hence, we have proven that  $x_i$  is not the centre-vertex of a three-component.

Suppose that  $x_i$  is the leaf of a three-component. If  $y_i$  also belongs to this three-component, then, since  $x_i \neq y_i$ , we have that  $u_i, v_i$  and the three-component of their victims lie on a 5-cycle in  $G$ . Then let  $S'$  be a scheme identical to  $S$  except that in the first round these five vertices fire according to an orientation of this 5-cycle. Thus  $S'$  has one fewer two-component in  $G_2$  than  $S$ .

If  $x_i$  is the leaf of a three-component that does not contain  $y_i$ , then  $u_i, v_i$  and the components containing their first round victims lie on a path of length 8 in  $G$  so can be matched. So let  $S'$  be a scheme identical to  $S$  except that in the first round these eight vertices fire according to this matching. Thus  $S'$  has one fewer two-component in  $G_2$  than  $S$ .

Thus  $x_i$  is not the leaf of a three-component, and, by (iii),  $x_i$  belongs to a two-component.

Thus  $u_i$  and  $v_i$  combined with the components of  $G_1$  containing their victims lie on a path of length 7 in  $G$ . We call such a path a *seven-component*. Let us motivate this choice of name by showing that the seven-components are vertex-disjoint.

The vertices  $v_i$ ,  $1 \leq i \leq r$ , fire at distinct three-components in the first round (as their victims are unique and one of the leaves of each three-component is the centre-victim). We must also show that the victims  $x_i$  of the vertices  $u_i$ ,  $1 \leq i \leq r$ , belong to distinct two-components. Suppose that  $x_i$  and  $x_j$ ,  $i \neq j$ , are distinct but belong to the same two-component in  $G_1$ . Then let  $S'$  be a scheme identical to  $S$  except that in the first round

- $v_i$  and  $y_i$  fire at each other,
- $v_j$  and  $y_j$  fire at each other,
- $u_i$  and  $x_i$  fire at each other, and
- $u_j$  and  $x_j$  fire at each other.

Again  $S'$  has fewer two-components in  $G_2$  than  $S$ . Now suppose that  $x_i = x_j$ . If either  $u_i$  or  $u_j$  is adjacent to the other vertex in  $x_i$ 's two-component, then we have the previous case. Otherwise, there is an edge  $u_i u_j$  (else there is a claw). So let  $S'$  be a scheme identical to  $S$  except that in the first round

- $v_i$  and  $y_i$  fire at each other,
- $v_j$  and  $y_j$  fire at each other, and
- $u_i$  and  $u_j$  fire at each other.

Again  $S'$  has fewer two-components in  $G_2$  than  $S$ .

We have shown that the seven-components are vertex-disjoint. Note that all the three-components in  $G_1$  contain a victim of a vertex in  $G_2$  and so must be a subgraph of a seven-component. Thus we can represent  $S$  as a collection of vertex-disjoint seven-components, two-components and odd cycles that span  $G$ . We denote such a representation  $G^*$ . Note that the number of two-components in  $G_2$  is equal to the number of seven-components in  $G^*$ . Thus to prove the lemma we show that if for  $S$  there is more than one seven-component in  $G^*$ , then we can find another scheme with fewer seven-components.

Let  $A = a_1 \cdots a_7$  and  $B = b_1 \cdots b_7$  be a pair of seven-components in  $G^*$ . First we consider the case where, in  $G$ ,  $A$  and  $B$  are joined by an edge  $a_i b_j$  for some  $i, j$ . We shall show that this implies that the vertices of  $A$  and  $B$  admit a perfect matching; thus we can replace two seven-components by seven two-components.

If  $i$  and  $j$  are both odd, then we match  $a_i$  with  $b_j$  and the remaining vertices and edges of  $A$  and  $B$  form paths of even length, so can clearly be matched. If  $i$  is even and  $j$  is odd, then, if either  $a_{i-1}$  or  $a_{i+1}$  is adjacent to  $b_j$ , we have the previous case. Otherwise, by claw-freeness, there is an edge  $a_{i-1} a_{i+1}$  and we include both this and  $a_i b_j$  in the matching, and, again, what remains of  $A$  and  $B$  are paths of even length. Finally suppose that  $i$  and  $j$  are both even. If there are any other edges from a vertex in  $\{a_{i-1}, a_i, a_{i+1}\}$  to a vertex in  $\{b_{j-1}, b_j, b_{j+1}\}$ , then we have an earlier case. Otherwise, claw-freeness implies edges  $a_{i-1} a_{i+1}$  and  $b_{j-1} b_{j+1}$ , and we include these and  $a_i b_j$  in the matching to again leave only even length paths.

So we can assume that no pair of seven-components in  $S$  are joined by an edge in  $G$ . Now let us assume that  $S$  is such that we can find seven-components  $A$  and  $B$  such that the length of the shortest path in  $G$  between them is minimum (that is, there is no pair of seven-components in any other simple scheme separated by a shorter path).

Suppose a shortest path from  $A$  to  $B$  meets  $A$  at  $a_i$  and the next vertex along is  $w$ . In  $G^*$ ,  $w$  must belong to either a two-component or an odd cycle.

First suppose  $w$  is in a two-component  $C$  whose other vertex is  $z$ . We describe how to use the vertices of  $A$  and  $C$  to find a seven-component  $A'$  and two-component  $C'$  such that  $w$  is in  $A'$ ; thus  $A'$  is closer to  $B$  than  $A$

contradicting our choice of  $A$  and  $B$ . By symmetry, there are four cases according to which vertex of  $A$  neighbours  $w$ . Suppose  $a_1$  is adjacent to  $w$ . Then replace  $A$  and  $C$  with  $A' = zwa_1 \cdots a_5$  and  $C' = a_6a_7$ . If  $a_2$  is adjacent to  $w$ , then claw-freeness implies one of the edges  $a_1a_3$ ,  $a_1w$  or  $a_3w$  is present. Let  $C'$  be, respectively,  $a_6a_7$ ,  $a_6a_7$  or  $a_1a_2$ , and in each case we find a path of length 7 on the remaining vertices to be  $A'$ . If  $a_3$  is adjacent to  $w$ , then let  $A' = zwa_3 \cdots a_7$  and  $C' = a_1a_2$ . If  $a_4$  is adjacent to  $w$ , then one of  $a_3a_5$ ,  $a_3w$  or  $a_5w$  is present. Let  $C'$  be, respectively,  $a_1a_2$ ,  $a_1a_2$  or  $a_6a_7$ , and in each case we find a path of length 7 on the remaining vertices to be  $A'$ .

Finally suppose that  $w$  belongs to an odd cycle. If  $a_i$ ,  $i$  odd, is joined to  $w$ , then there is a perfect matching on the vertices of  $A$  and the cycle and we have a scheme with fewer seven-components. Suppose  $a_i$ ,  $i$  even, is adjacent to  $w$ . If either  $a_{i-1}$  or  $a_{i+1}$  is joined to  $w$ , then we have the previous case. Otherwise, there must be an edge  $a_{i-1}a_{i+1}$ , and if we match both this pair of vertices and  $a_i$  and  $w$ , then the remaining vertices of  $A$  and the cycle induce even-length paths and a perfect matching can again be found.  $\square$

**Theorem 10** *Computing the parallel knock-out number of a claw-free graph can be done in polynomial time.*

**Proof:** By Theorem 8, it is sufficient to present methods for checking whether or not  $\text{pko}(G)$  is equal to 1 or 2, since if it is neither it must be  $\infty$ . Deciding whether a graph can be knocked-out in a single round can be solved in polynomial time ([4]). So we need only show how to check whether  $G$  can be knocked out in two rounds.

Suppose that  $\text{pko}(G) = 2$ . By Lemma 9, we can assume that there is a two-round simple KO-reduction scheme for  $G$  in which only two vertices, say  $u$  and  $v$ , survive to the second round, and, by the proof of the lemma, there is exactly one three-component in  $G_1$ .

Let  $w$  be the first round victim of  $v$ . Then  $G - \{u, v, w\}$  has a spanning subgraph comprising two-components and odd cycles (that is,  $G_1 - w$ ) and can thus be knocked out in one round. Therefore the following is a necessary condition for  $\text{pko}(G) = 2$ : there are three vertices  $u$ ,  $v$  and  $w$  in  $V$  such that

- there are edges  $uv$  and  $vw$ ,
- $u$  and  $w$  have neighbours other than  $v$  and each other, and
- $\text{pko}(G - \{u, v, w\}) = 1$

It is easy to see that this condition is also sufficient. Therefore to decide whether or not  $\text{pko}(G) = 2$ , we look for a set of three vertices that satisfies this condition. This can be done in polynomial time.  $\square$

As noted before any graph with  $\text{pko}(G) = 1$  has a spanning subgraph consisting of a number of mutually disjoint matchings edges and disjoint cycles. For *claw-free* graphs we have found the following characterization, which directly follows from the proof of Lemma 9.

**Corollary 11** *Let  $G$  be a connected claw-free graph with  $\text{pko}(G) = 2$ . Then  $G$  has a spanning subgraph consisting of a number of vertex-disjoint matching edges, odd cycles and one path on seven vertices.*

## 5 Locally bijective homomorphisms

A *graph homomorphism* from  $G = (V_G, E_G)$  to  $H = (V_H, E_H)$  is a vertex mapping  $f : V_G \rightarrow V_H$  satisfying the property that for any edge  $(u, v)$  in  $E_G$ , we have  $(f(u), f(v))$  in  $E_H$  as well, i.e.,  $f(N_G(u)) \subseteq N_H(f(u))$  for all  $u \in V_G$ . For two graphs  $G$  and  $H$  we write  $G \xrightarrow{B} H$  if there exists a so-called *locally bijective* homomorphism  $f : V_G \rightarrow V_H$  satisfying:

$$\text{for all } u \in V_G : f(N_G(u)) = N_H(f(u)) \text{ and } |f(N_G(u))| = |N_G(u)|.$$

We compare the parallel knock-out numbers of two graphs  $G$  and  $H$  with  $G \xrightarrow{B} H$ . Then we find that  $\text{pko}(H)$  is an upper bound for  $\text{pko}(G)$ .

**Proposition 12** *If  $G \xrightarrow{B} H$  then  $\text{pko}(G) \leq \text{pko}(H)$ .*

**Proof:** If  $\text{pko}(H) = \infty$  the statement holds. Suppose  $\text{pko}(H) = k$  for some integer  $k$  and consider a parallel knock-out scheme that eliminates  $H$  in exactly  $k$  rounds. Let  $f : V_G \rightarrow V_H$  be a locally bijective homomorphism. For any pair  $x, y \in V_H$  with  $x$  firing at  $y$  in the first round we do as follows. In  $G$  we let each vertex  $u$  with  $f(u) = x$  fire at its (only) neighbor  $v$  with  $f(v) = y$ . Clearly there is a locally bijective homomorphism from the KO-successor of  $G$  to the KO-successor of  $H$  (the restriction of  $f$  to the remaining vertices is one). Thus we can, in the same way, decide how the vertices of  $G$  should fire in the second and subsequent rounds, and so a reduction scheme for  $G$  that also has  $k$  rounds is obtained.  $\square$

We note that the reverse implication is not true. Let  $P_n$  denote the path on  $n$  vertices. Then we can take  $G = P_2$  and  $H = P_3$ . Clearly, there



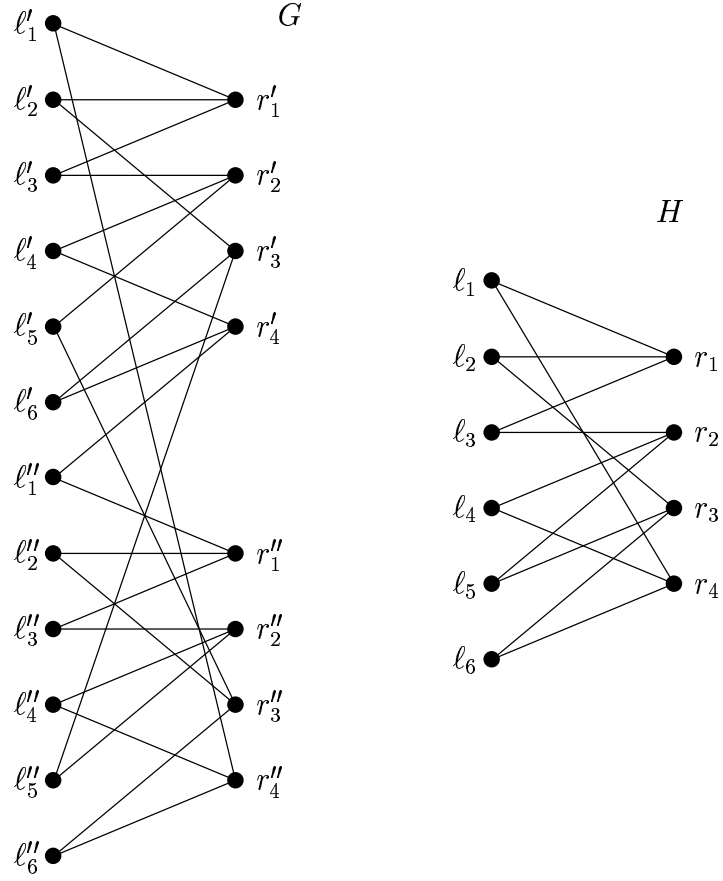


Figure 2: Two graphs  $G, H$  with  $G \xrightarrow{B} H$  and  $\text{pko}(G) < \text{pko}(H)$ .

does not exist a locally bijective homomorphism from  $G$  to  $H$ . However,  $\text{pko}(G) = 1 < \text{pko}(H) = \infty$ .

In Figure 2, we illustrate an example that shows that strict inequality may hold in the statement of Proposition 12: it displays two graphs  $G$  and  $H$  with  $G \xrightarrow{B} H$  and  $\text{pko}(G) < \text{pko}(H)$ . This can be seen as follows. The mapping  $f : V_G \rightarrow V_H$  defined by  $f(\ell'_i) = f(\ell''_i) = \ell_i$ , for  $1 \leq i \leq 6$ , and  $f(r'_j) = f(r''_j) = r_j$ , for  $1 \leq j \leq 4$ , is a locally bijective homomorphism from  $G$  to  $H$ . Below we show that  $\text{pko}(G) = 2 < \infty = \text{pko}(H)$ .

We first need some terminology. A bipartite graph  $G$  is called  $(r, s)$ -regular if all vertices in one class of the bipartition have degree  $r$  and all other vertices have degree  $s$ . Let  $F = (V, E)$  be a  $(2, 3)$ -regular bipartite graph. Let  $L$  denote the vertices with degree 2 (*left* vertices) and  $R$  the vertices with degree 3 (*right* vertices). Then  $|E| = 2|L| = 3|R|$ , so  $|R| = 2k$  and  $|L| = 3k$  for some positive integer  $k$ . We call a subset  $A$  of  $R$  with  $k$  vertices that has the whole set  $L$  as its neighbourhood a *star cover* of  $F$ . Note that both  $G$  and  $H$  are  $(2, 3)$ -regular bipartite graphs. Furthermore,  $G$  has a star cover  $\{r'_1, r'_4, r''_3, r''_2\}$  while  $H$  does not have a star cover. Then  $\text{pko}(G) = 2$  and  $\text{pko}(H) = \infty$  follow immediately from a result from [5] on  $(2, 3)$ -regular bipartite graphs that states that a  $(2, 3)$ -regular bipartite graph  $G$  is KO-reducible if and only if  $G$  has a star cover and in this case  $\text{pko}(G) = 2$ .

## 6 Conclusions

We solved the square-root conjecture of [4] by giving a tight upper bound on the parallel knock-out number of a KO-reducible graph  $G$ . We also showed that the parallel knock-out number of a KO-reducible  $K_{1,p}$ -free graph is at most  $p - 1$ , and that this bound is tight. For claw-free graphs we showed that their parallel knock-out number can be computed in polynomial time. The question of whether the parallel knock-out number for  $K_{1,p}$ -free graphs with  $p \geq 4$  can also be computed in polynomial time remains open.

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