

A complete complexity classification of the role assignment problem

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Abstract

In social network theory a simple graph G is called R -role assignable, if there is a locally surjective homomorphism from G to a fixed graph R , i.e., a vertex mapping $r : V_G \rightarrow V_R$, such that the neighborhood relation is preserved: $r(N_G(u)) = N_R(r(u))$. The decision problem whether such a mapping exists is called the R -role assignment problem. Kristiansen and Telle conjectured that the R -role assignment problem is an NP-complete problem for any simple connected graph R on at least three vertices. In this paper we prove the conjecture. Since in social network theory also nonsimple and disconnected role graphs are studied, we determine the computational complexity of the role assignment problem for these role graphs as well. We show further corollaries for related problems.

Keywords: computational complexity, graph homomorphism, role assignment

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1 Introduction

Given two graphs, say G and R , an R -role assignment for G is a vertex mapping $r : V_G \rightarrow V_R$, such that the neighborhood relation is maintained, i.e., all roles

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of the image of a vertex appear on the vertex's neighborhood. Such a condition can be formally expressed as

$$\text{for all } u \in V_G : r(N_G(u)) = N_R(r(u)),$$

where $N(u)$ denotes the set of neighbors of u in the corresponding graph.

Such assignments have been introduced by Everett and Borgatti [6], who called them role colorings. They originated in the theory of social behavior. The graph R , i.e., the *role graph*, models roles and their relationships, and for a given society we can ask whether its individuals can be assigned roles such that the relationships are preserved: Each person playing a particular role has among its neighbors exactly all necessary roles as they are prescribed by the model.

This way one can investigate whether large networks of individuals can be compressed into smaller ones that still give some description of the large network. Because persons of the same social role may be related to each other, the smaller network can contain loops. In other words: Given a *simple* instance graph G of n vertices does there exist a *possibly nonsimple* role graph R of $k < n$ vertices in such a way that G is R -role assignable?

From the computational complexity point of view it is interesting to know whether it is possible to decide quickly (i.e., in polynomial time) whether such an assignment exists. This problem was considered by Roberts and Sheng [15], who show that it is already NP-complete for $k = 2$.

In order to make a more precise study we consider a class of R -role assignment problems, $RA(R)$, parameterized by the role graph R . Here the instance is formed only by the graph G , and we ask whether an R -role assignment of G exists. (If both graphs G and R are part of the input, the problem is NP-complete already for $R = K_3$ [12].)

The complexity study of this class of problems is closely related to a similar approach for locally constrained graph homomorphism problems [9]. A graph homomorphism from G to H is a vertex mapping $f : V_G \rightarrow V_H$ satisfying the property that whenever an edge (u, v) appears in E_G , then $(f(u), f(v))$ belongs to E_H as well.

The adjective “locally constrained” expresses the condition that the mapping f restricted to the neighborhood of any vertex u must satisfy further properties. (See [14, 7] for a general model of such conditions.)

It may be required to be locally

- *bijective*, then the mapping is called a *full cover* of H , and the corresponding decision problem is called H -COVER [1, 13],
- *injective*, then it is called a *partial cover* of H , and the problem H -PCOVER [8, 9],
- *surjective*, then we get a *locally surjective cover* of H , and decision problem H -COLORDOMINATION [14].

All these problems are parameterized by a fixed graph H , and the instance is formed only by a graph G . The question is whether an appropriate graph homomorphism from G to H exists. Observe that the definition of a locally surjective cover is equivalent with the definition of an R -role assignment for $R = H$.

Full covers have important applications, for example in distributed computing [5], in recognizing graphs by networks of processors [2, 3], or in constructing highly transitive regular graphs [4]. Similarly, partial covers are used in distance constrained labelings of graphs [10].

Even if the first attempt to get some results on the computational complexity for the class of H -COVER problems was made a decade ago in [1], it is not fully classified yet neither for H -PCOVER nor for H -COLORDOMINATION ($RA(H)$) problems. However, several partial results are known. For example, if the H -COVER problem is NP-complete, then the corresponding H -PCOVER [9] and H -COLORDOMINATION problems [14] are NP-complete as well. Moreover, the H -COVER problem is known to be NP-complete for all k -regular graphs H of valency $k \geq 3$ [9], and the NP-hardness hence propagates for partial and locally surjective covers of such graphs as well.

The H -COLORDOMINATION problem was proven to be NP-complete for paths on more than two vertices, cycles and stars in [14]. It was conjectured there that for simple connected graphs the H -COLORDOMINATION problem is NP-complete if and only if H has at least three vertices. Our main theorem proves this conjecture.

Theorem 1 *For a role graph R , the $RA(R)$ problem is solvable in polynomial time if and only if*

- *either R has no edge,*
- *or one of its components consists of a single vertex incident with a loop,*
- *or R is simple and bipartite and has at least one component isomorphic to K_2 .*

In all other cases the $RA(R)$ problem is NP-complete.

The paper is organized as follows. The next section provides necessary definitions and basic observations. The third section provides technical lemmas used in the proof of the main theorem. The proof is split into two parts. The fourth section shows the construction for connected role graphs. The fifth section describes the complexity of the role assignment problem for disconnected role graphs. We finally apply the main theorem to prove NP-completeness for a generalized k -role assignment problem [15] in the sixth section.

2 Preliminaries

Through the paper we use terminology stemming from the role assignment problems.

We consider graphs, denoted by $G = (V_G, E_G)$, where V_G is a finite vertex set of vertices and E_G is a set of unordered pairs of vertices, called edges. It is possible that some edges connect a vertex to itself. Such edges are called *loops*. A loopless graph is called to be *simple*. Through the paper no multiple edges nor multiple loops appear in the considered graphs.

For a vertex $u \in V_G$ we denote its *neighborhood*, i.e., the set of adjacent vertices, by $N(u) = \{v \mid (u, v) \in E_G\}$. The *degree* $\deg(u)$ of a vertex u is the number of edges incident with it, or equivalently the size of its neighborhood. In particular, $u \in N(u)$ if and only if u is incident with a loop, and each loop increases the degree of the associated vertex by exactly one.

A graph is called *bipartite* if it is simple and its vertices can be partitioned into two sets A and B such that each edge has one of its endpoints incident with the set A and the other with B .

A *complete* graph is a graph with an edge between every pair of vertices. The complete graph on n nodes is denoted by K_n .

A graph G is called *connected* if for every pair of distinct vertices u and v , there exists a *path* connecting u and v , i.e., a sequence of distinct vertices starting by u and ending by v where each pair of consecutive vertices forms an edge of G . The *length* of the path is the number of its edges. A graph that is a path on n vertices is denoted by P_n .

The *distance* $\text{dist}(u, v)$ between two vertices u and v is the length of a shortest path between them. The maximum distance in a graph G is called the *diameter* diam_G of G , i.e., $\text{diam}_G = \max\{\text{dist}(u, v) \mid u, v \in V_G\}$. A vertex $u \in V_G$ is called a *maximum distance vertex* if there exists a vertex $v \in V_G$ with $\text{dist}(u, v) = \text{diam}_G$. We denote by D_G the set of all maximum distance vertices in G .

A graph that is not connected is called *disconnected*. Each maximal connected subgraph of a graph is called a *component*. A vertex whose removal causes a component of a graph to become disconnected is called a *cutvertex*.

Two graphs G and \tilde{G} are called *isomorphic*, denoted by $G \simeq \tilde{G}$, if there exists a one-to-one mapping f of vertices of G onto vertices of \tilde{G} such that $(u, v) \in E_G$ if and only if $(f(u), f(v)) \in E_{\tilde{G}}$. The mapping f is called an *isomorphism* between G and \tilde{G} .

In the sequel the symbol G denotes the instance graph and R the so-called *role graph*. If necessary, the above defined notions of neighborhood, etc., are distinguished by subscripts G or R to indicate whether they are related to the graph G or to R .

Definition We say that G is R -role assignable if a mapping $r : V_G \rightarrow V_R$ exists satisfying:

$$\text{for all } u \in V_G : r(N_G(u)) = N_R(r(u)),$$

where we use the notation $r(S) = \bigcup_{u \in S} r(u)$ for a set of vertices $S \subseteq V_G$. The function r is called an R -role assignment of G .

The goal of this paper is a full characterization of the computational complexity for the following class of problems:

R -ROLE ASSIGNMENT ($\text{RA}(R)$)

Instance: A graph G .

Question: Does the graph G allow an R -role assignment?

We continue with some observations that we use later in the paper. We first note that role assignments are closed under composition:

Observation 2.1 *If G is S -role assignable and S is R -role assignable, then G is R -role assignable.*

Proof: Let $s : V_G \rightarrow V_S$ be an S -role assignment for G and $r : V_S \rightarrow V_R$ be an R -role assignment for S . Then $t : V_G \rightarrow V_R$ defined by $t(u) = r(s(u))$ for all $u \in V_G$ is an R -role assignment for G . \square

Observation 2.2 *If G is R -role assignable with role assignment r , then $\deg_G(u) \geq \deg_R(r(u))$ for all vertices $u \in V_G$.*

Proof: $\deg_G(u) = |N_G(u)| \geq |r(N_G(u))| = |N_R(r(u))| = \deg_R(r(u))$. \square

From this we derive:

Lemma 2.3 *Let G be R -role assignable with role assignment r , and x, y be vertices of R connected by a path P_R . Then for each u with $r(u) = x$ a vertex $v \in V_G$ and a path P_G connecting u and v exist, such that r restricted to P_G is an isomorphism between P_G and P_R .*

Proof: We prove the statement by induction on the length of the path P_R . If x and y are adjacent, then the vertex u has a neighbor v mapping onto y , by the definition of the R -role assignment r .

Now assume that the path P_R is of length $k \geq 2$, and that the hypothesis is valid for all paths of length at most $k - 1$. Denote by y' the predecessor of y in P_R and by P'_R the truncation of P_R by the last edge, i.e., the path of length $k - 1$ connecting x and y' . By the induction hypothesis G contains a vertex v' and a path P'_G such that $P'_G \simeq P'_R$ under r . Then it is easy to find a neighbor v of v' satisfying $r(v) = y$ and tack it to P'_G to get the desired path P_G . \square

We get immediately the following claims:

Observation 2.4 *If G is R -role assignable and R is connected, then each vertex $v \in V_R$ appears as a role for some vertex $u \in V_G$.*

Observation 2.5 *Let u be a vertex in a graph G that is R -role assignable with role assignment r . If R is connected, then $r(\{v \mid \text{dist}(u, v) \leq \text{diam}_R\}) = V_R$.*

Proof: Let $r(u) = x$. Suppose y is an arbitrary role in R . Let P_R be a shortest path in R connecting x and y . By Lemma 2.3 a vertex $v \in V_G$ exists with $r(v) = y$ and a path P_G of length $\text{dist}_R(x, y) \leq \text{diam}_R$ connecting u and v . Hence $y \in r(\{v \mid \text{dist}(u, v) \leq \text{diam}_R\})$. \square

If $G \simeq R$ then G is R -role assignable, because every isomorphism satisfies the condition of the role assignment. Due to Observation 2.4 we have:

Observation 2.6 *Let R be a connected role graph. If G is R -role assignable and $|V_G| = |V_R|$, then $G \simeq R$.*

Lemma 2.7 *Let G be R -role assignable, $u \in V_G$ be a vertex of role x , and $z, y \in V_R$ be some other roles. If in G each path connecting u to a vertex of role y contains a vertex of role z , then the vertex z is a cutvertex in R .*

Proof: Since vertices of roles x and y are connected by a path in G , there exists a path in R connecting x to y . Moreover if z were not a cutvertex, then we can find such a path avoiding the role z . But then by Lemma 2.3 we can find a path in G from u to some vertex of role y avoiding any vertex of role z . \square

3 Gadgets

3.1 Product graphs

For the garbage collection in our NP-completeness proof for simple role graphs we want to construct a simple connected graph that allows two different role assignments. For our NP-completeness proof for nonsimple role graphs we want to make a reduction from an R -role assignment problem, in which R is simple. For these purposes we adapt a method of graph product from [9].

Let G and H be two graphs. The *product graph* $G \times H$ is the graph with vertex set $V_{G \times H} = V_G \times V_H$, and edges $((u, w), (x, y)) \in E_{G \times H}$ if and only if $(u, x) \in E_G$ and $(w, y) \in E_H$.

Lemma 3.1 *Let R be a role graph, and let H be a graph without isolated vertices. If a graph G is R -role assignable with role assignment r , then the mapping $s : V_{G \times H} \rightarrow V_R$ given by $s((u, w)) = r(u)$ is an R -role assignment of $G \times H$.*

Proof: Let (u, w) be a vertex of $F = G \times H$ with $s((u, w)) = r(u) = a$.

Suppose b is a role in $s(N_F(u, w))$. Then there exists a neighbor (x, y) of (u, w) that has role b . By definition of s we have $b = s((x, y)) = r(x)$, and (u, x) is an edge in G by definition of F . Because r is an R -role assignment of G , the role b must be a neighbor of $a = r(u) = s((u, w))$.

Now suppose b is a role in $N_R(s((u, w))) = N_R(r(u)) = N_R(a)$. Since r is an R -role assignment of G , vertex u must have a neighbor x with role b . Because H has no isolated vertices, vertex w in H has a neighbor y . Then (u, w) and (x, y) are adjacent vertices in F . Hence b is a role in $s(N_F(u, w))$. \square

For the garbage collection in our NP-completeness proof for simple role graphs we need the following lemma.

Lemma 3.2 *Let R be a simple role graph without isolated vertices. Then for any two roles x and y a simple connected graph H exists that has two R -role assignments r_1 and r_2 , such that a vertex u exists in H with $r_1(u) = x$, and $r_2(u) = y$. Moreover, H can be constructed in time being polynomial with respect to the size of R .*

Proof: Start with the product graph $R \times R$. By Lemma 3.1 it is clear that the projections $r_1 : (a, b) \rightarrow a$ and $r_2 : (a, b) \rightarrow b$ are R -role assignments. Hence vertex $u = (x, y)$ satisfies the statement of the lemma. Since R is simple, $R \times R$ does not contain loops as well. Then we can take H as the component of $R \times R$ containing the vertex $u = (x, y)$. \square

In general, a graph G will not be $R \times S$ -role assignable if G is both R -role assignable and S -role assignable. However, if $S = K_2$ this does hold. We will use this result in our NP-completeness proof for nonsimple role graphs.

Lemma 3.3 *Let R be a role graph. A graph G is $R \times K_2$ -role assignable if and only if G is bipartite and R -role assignable.*

Proof: Let $T = R \times K_2$.

Suppose G is T -role assignable. Since nonbipartite graphs cannot homomorphically map to any bipartite graph like T , the graph G must be bipartite. By Lemma 3.1 we deduce that T is R -role assignable. Hence the forward implication follows by Observation 2.1.

Now suppose G is bipartite and R -role assignable with role assignment r . Let $s : V_G \rightarrow V_{K_2}$ be the homomorphism derived from the bipartition of G . Then $t : V_G \rightarrow V_T$ given by $t(u) = (r(u), s(u))$ is the desired T -role assignment of G . This can be seen as follows. Let u be a vertex in G with $t(u) = (r(u), s(u)) = (a, i)$.

Suppose (b, j) is a role in $t(N_G(u))$. Then u has a neighbor v with $t(v) = (r(v), s(v)) = (b, j)$. Since u and v are neighbors, $r(v) = b$ is a neighbor of a in R and i is not equal to j . Hence (a, i) and (b, j) are adjacent vertices in T .

In the opposite direction suppose (b, j) is a role in $N_T(t(u))$. Then there is an edge between a and b in R , and i is not equal to j . Because b is a neighbor of a , the vertex u must have a neighbor v in G with $r(v) = b$. Since u and v are adjacent, s does not map v to i . Hence $t(v) = (r(v), s(v)) = (b, j)$. \square

3.2 Glued subgraphs

Definition *We say that a graph \tilde{R} is glued in a graph G by a vertex \tilde{x} , if G can be obtained from \tilde{R} and some other graph G' by identifying a vertex $v \in V_{G'}$ with the vertex \tilde{x} .*

See Figure 1 for a more intuitive picture of such a glued graph.

For our NP-completeness proof we would like to construct instance graphs that contain vertices, for which we know on which kind of roles they are mapped. For this purpose we utilize the maximum distance vertices. (Recall the notion of D_R from Section 2.)

Lemma 3.4 *Let R be a simple connected role graph and let x be a maximum distance vertex in the role graph R . Let further G be an R -role assignable graph with role assignment r , where $\tilde{R} \simeq R$ is glued in G by the vertex \tilde{x} , the isomorphic copy of x in \tilde{R} . Then r restricted to $V_{\tilde{R}}$ is an isomorphism between*

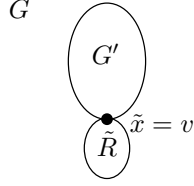


Figure 1: A role assignment of a glued subgraph.

\tilde{R} and R . In particular, r can be composed with a permutation π of V_R to an R -role assignment $s : V_G \rightarrow V_R$ such that $s(\tilde{x}) = x$.

Proof: Choose a role $y \in V_R$ such that $\text{dist}(x, y) = \text{diam}_R$. Then by Observation 2.5 all roles must appear on vertices at distance at most diam_R from \tilde{y} . Since there are exactly $|V_R|$ many such vertices, namely only the vertices in $V_{\tilde{R}}$, the mapping r is a one-to-one mapping when restricted to $V_{\tilde{R}}$.

Every edge-preserving bijective mapping between two graphs with the same number of edges must be an isomorphism. Now let the permutation $\pi : V_R \rightarrow V_R$ be defined by

$$\pi(y) = z \quad \Leftrightarrow \quad r(\tilde{z}) = y,$$

where \tilde{z} is the isomorphic copy of z in \tilde{R} . Then π is an R -role assignment of R . We have already observed in Observation 2.1 that the composition of two role assignments is a role assignment. Then $s = \pi \circ r$ is an R -role assignment of G satisfying $s(\tilde{x}) = \pi(r(\tilde{x})) = x$. \square

4 Connected role graphs

We assume that the instance graph G is simple, while role graphs R may contain loops. In this section we consider the case where R is simple as well. Below we prove the conjecture of Kristiansen and Telle [14].

Proposition 2 *Let R be a simple connected role graph. Then the R -role assignment problem is polynomially solvable if $|V_R| \leq 2$ and it is NP-complete if $|V_R| \geq 3$.*

Proof: First we show that $\text{RA}(R)$ is polynomially solvable for $|V_R| \leq 2$.

- $|V_R| = 1$. Clearly, a graph G is R -role assignable if and only if G contains only isolated vertices.
- $|V_R| = 2$, i.e., $R \simeq K_2$. Then a graph G is R -role assignable if and only if G is a bipartite graph that does not contain any isolated vertices.

So let $|V_R| \geq 3$. Since we can guess a mapping $r : V_G \rightarrow V_R$ and check in polynomial time if r is an R -role assignment, the problem $\text{RA}(R)$ is a

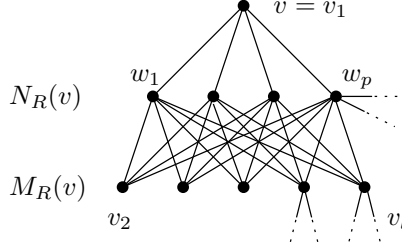


Figure 2: Neighborhood of a vertex v in R .

member of NP. We prove NP-completeness by reduction from HYPERGRAPH 2-COLORABILITY. This is a well-known NP-complete problem (cf. [11]).

HYPERGRAPH 2-COLORABILITY (H2C)

Instance: A set $Q = \{q_1, \dots, q_m\}$ and a set $\mathcal{S} = \{S_1, \dots, S_n\}$ with $S_j \subseteq Q$ for $1 \leq j \leq n$.

Question: Is there a 2-coloring of (Q, \mathcal{S}) , i.e., a partition of Q into $Q_1 \cup Q_2$ such that $Q_1 \cap S_j \neq \emptyset$ and $Q_2 \cap S_j \neq \emptyset$ for $1 \leq j \leq n$?

With such a hypergraph we associate its incidence graph I , which is a bipartite graph on $Q \cup \mathcal{S}$, where (q, S) forms an edge if and only if $q \in S$.

Let $p = \min\{\deg_R(u) \mid u \in D_R\}$ and let v be a maximum distance vertex with $\deg_R(v) = p$. Denote the neighbors of v by $N_R(v) = \{w_1, \dots, w_p\}$. Denote the second common neighborhood as $M_R(v) = \bigcap_{u \in N_R(v)} N_R(u) = \{v, v_2, \dots, v_l\}$. Choose v such that l is minimal, i.e., there does not exist a role v' in D_R with $|N_R(v')| = p$ and $|M_R(v')| < l$. See Figure 2 for a drawing of a possible situation. We distinguish four cases according to possible values of p and l .

Case 1: $p = 1, l = 1$. Then $R = K_2$, and we have already discussed this case above.

Case 2: $p = 1, l \geq 3$. We extend the incidence graph I as follows: According to Lemma 3.2 we construct a connected graph H for which two role assignments exist mapping a particular vertex u to v_2 and v_3 . We form an instance G as the union of the graph I and m disjoint copies of the graph H , where the vertex u of the i -th copy is identified with the vertex q_i of I . Finally we insert into G two extra copies \tilde{R}, R' of the role graph R , where \tilde{v} is the isomorphic copy of v in \tilde{R} and v'_k is the isomorphic copy of v_k in R' for $1 \leq k \leq l$. We add the following edges (cf. Figure 3):

- (\tilde{v}, S_j) for all $S_j \in \mathcal{S}$,
- (v'_k, S_j) for all $S_j \in \mathcal{S}$ and all $4 \leq k \leq l$ (this set may be empty).

We show that the graph G formed in this way allows an R -role assignment if and only if (Q, \mathcal{S}) is 2-colorable.

Assume first that G is R -role assignable. Then according to Lemma 3.4 we assume that the vertex \tilde{v} is assigned role v and all vertices S_j are mapped to role

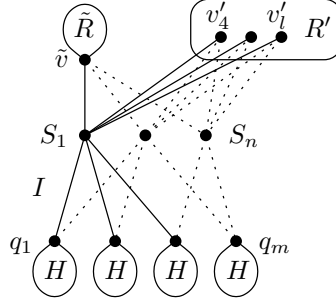


Figure 3: Construction of the graph G in Case 2.

w_1 . Since their neighborhoods are saturated by common $l-3$ roles on v'_4, \dots, v'_l , at least two distinct roles $v_a, v_b \in M_R(v) \setminus r(\{v'_4, \dots, v'_l\})$ exist that are used on some neighbors of each S_j in the set \mathcal{S} .

The partition $Q_1 = \{q_i \mid r(q_i) = v_a\}$ and $Q_2 = Q \setminus Q_1 \supseteq \{q_i \mid r(q_i) = v_b\}$ is the desired 2-coloring of (Q, \mathcal{S}) .

In the opposite direction, any 2-coloring $Q_1 \cup Q_2$ can be transformed into an R -role assignment r of G by letting $r(q_i) = v_a$ if $q_i \in Q_a$ for $a = 1, 2$ and by further extension according to the two projections of the graph H and graph isomorphisms $\tilde{R} \rightarrow R, R' \rightarrow R$.

Case 3: $p = 1, l = 2$.

First assume that R is not isomorphic to a path P_n on $n \geq 3$ vertices. Let y be the first vertex on the unique path $P_R(v)$ from v in R that has degree $\deg_R(y) \geq 3$. Now we can use the same construction as in Case 2 after a couple of modifications: We replace each edge (\tilde{v}, S_j) by a path from \tilde{v} to S_j of the same length as $P_R(v)$. Furthermore, we make sure to add the right edges from each S_j to R' (instead of edges (v'_k, S_j)) and to choose the right vertex $u \in H$.

If R is isomorphic to P_n for some $n \geq 3$, then we act as follows. (See also [14]. Since we want to use certain properties of the instance graphs later on, in Section 6, we have included a proof for these kind of role graphs in our paper as well.)

First assume that $n \neq 4$. We construct a graph G from I . First we insert n new vertices S'_1, \dots, S'_n and a copy \tilde{P}_n of the role graph P_n . We identify each q_i with the vertex u of an extra copy of the graph H as in the previous case, but here H is constructed such that u can be assigned v or v_2 .

These parts are linked as follows (cf. Figure 4):

- $(\tilde{v}, S'_j) \in E_G$ for all $j \in \{1, \dots, n\}$,
- $(q_i, S'_j) \in E_G$ if and only if $(q_i, S_j) \in E_I$.

Now the proof is similar to the proof of Case 2. If G is R -role assignable, then without loss of generality we may assume that \tilde{v} has role v . Then all S'_j have role w_1 since w_1 is the only neighbor of v . The roles of all q_i hence belong

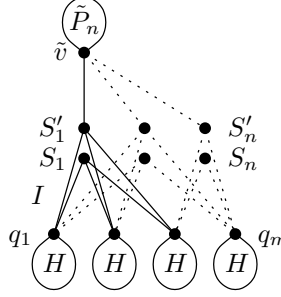


Figure 4: Construction of the graph G if $R = P_n$.

to $N_R(w_1) = \{v, v_2\}$. Each S'_j requires the role v_2 to be present among its neighbors in Q . Moreover, if all neighbors of some S'_j in Q are assigned the role v_2 , we get that S_j must be mapped to a neighbor of v_2 that is a leaf, which is only possible if $R = P_4$. We conclude that each S_j is mapped to w_1 . Hence both roles v, v_2 appear on its neighborhood and the partition $Q_1 = \{q_i \mid r(q_i) = v\}$ and $Q_2 = \{q_i \mid r(q_i) = v_2\}$ is a 2-coloring of (Q, \mathcal{S}) .

In the opposite direction, an R -role assignment of G can be constructed from a 2-coloring of (Q, \mathcal{S}) in a straightforward way as in the previous case.

If $n = 4$ we replace the edges between \tilde{v} and each S_j by paths of length 2, and we identify each q_i with the vertex $u \in V_H$ that can be assigned to both neighbors of v_2 . After these modifications the proof is similar to the case $n \neq 4$.

Case 4: $p \geq 2$. As above we first build the graph H , which allows two R -role assignments mapping a vertex u either to w_1 or to w_2 .

The graph G consists of the graph I , where each q_i is unified with the vertex u of an extra copy of H . We further include two copies of R denoted by \tilde{R} and R' . Finally we extend the set of edges by (cf. Figure 5):

- (\tilde{v}, q_i) for all $q_i \in Q$,
- (\tilde{v}, w'_k) for all $1 \leq k \leq p$,
- (S_j, w'_k) for all $3 \leq k \leq p$ (this set may be empty).

If an R -role assignment exists, then we assume that $r(\tilde{v}) = v$. For each S_j we have $N_G(S_j) \subseteq N_G(\tilde{v})$. So we know that S_j is assigned some role v_i for which $N_R(v_i) = N_R(v)$. Since v is a maximum distance vertex in R , S_j is mapped on a role in D_R as well. Because p is the smallest number of neighbors these roles can have, $r(S_j)$ has degree at least p .

However, only $p - 2$ roles appear on vertices w'_3, \dots, w'_p . So two distinct roles w_a and w_b are used on none of w'_3, \dots, w'_p . Then we define a 2-coloring of (Q, \mathcal{S}) by selecting $Q_1 = \{q_i \mid r(q_i) = w_a\}$ and $Q_2 = Q \setminus Q_1 \supseteq \{q_i \mid r(q_i) = w_b\}$.

Suppose a 2-coloring of (Q, \mathcal{S}) exist. Then an R -role assignment can be derived from this 2-coloring as in the previous cases. \square

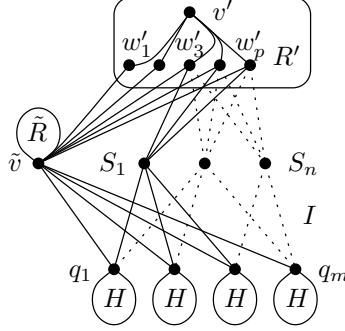


Figure 5: Construction of the graph G in Case 4.

Observe that all graphs G involved in our constructions had an isomorphic copy of the role graph glued in, and were connected, even if the incidence graph I was not connected.

We continue with the case of nonsimple connected role graphs. For this purpose we make use of the product graphs from Section 3.1.

Theorem 3 *Let R be a connected role graph. Then the R -role assignment problem is polynomially solvable if $|E_R| \leq 1$, and it is NP-complete if $|E_R| \geq 2$.*

Proof: The polynomially solvable cases follow from Proposition 2. Without loss of generality we may assume that R contains a loop, Let $S = R \times K_2$. Then S is connected, because R has a loop, and S is simple, because K_2 has a loop.

Since nonbipartite graphs cannot homomorphically map to any bipartite graph like S , we may assume that an instance G of $\text{RA}(S)$ is bipartite. Then NP-completeness of $\text{RA}(R)$ immediately follows from Lemma 3.3 and Proposition 2. \square

5 Disconnected role graphs

Up to now we have only considered role graphs that were connected. Due to this property we could easily derive that all roles appear as the image of the vertex in the instance graph (cf. Observation 2.4). We now focus our attention to the case of disconnected role graphs. Suppose R is a role graph with set of components $\mathcal{C} = \{R_1, \dots, R_m\}$. We order the components such that the latter have a higher number of vertices or a higher number of edges in the case of ties. (Formally, for all $i \leq j : |V_{R_i}| \leq |V_{R_j}|$.)

Note that the identity mapping $\pi : V_{R_1} \rightarrow V_R$ preserves the local constraint for role assignment, but Observation 2.4 is no longer valid here (take $G \simeq R_1$). Our argument guarantees that a locally surjective cover is globally surjective only for connected role graphs. Within some social network models it is natural to demand that all roles appear on the vertices of the instance graph. We show

below that the computational complexity of the role assignment problem for disconnected role graphs depends on whether such a property $r(V_G) = V_R$ is required or not.

We call an R -role assignment $r : V_G \rightarrow V_R$ a *globally R -role assignment* for G if r is an R -role assignment and $r(V_G) = V_R$ holds. Our generalized role assignment problem can now be formulated as

GLOBAL R -ROLE ASSIGNMENT (GRA(R))

Instance: A graph G .

Question: Is G globally R -role assignable?

With respect to the computational complexity we obtain the following result.

Theorem 4 *Let R be a disconnected role graph. Then the GRA(R) problem is polynomially solvable if all components have at most two vertices and it is NP-complete otherwise.*

Proof: Clearly the GRA(R) problem belongs to NP. For connected role graphs the statement immediately follows from Theorem 1.

Suppose R has $m \geq 2$ components ordered as shown above. If all components consist of only one vertex, then a graph G is R -role assignable if and only if G is a collection of at least m isolated vertices. Suppose R consists of k isolated vertices and $m-k$ isolated edges. Then a graph G is R -role assignable if and only if G contains at least k isolated vertices and at least $m-k$ bipartite components, each on at least two vertices.

Now suppose $|V_{R_m}| \geq 3$. We prove NP-completeness by reduction from RA(R_m). Without loss of generality we assume that the instance graph G for the RA(R_m) problem is connected. Let G' be the graph with components $G, \tilde{R}_1, \dots, \tilde{R}_{m-1}$, where \tilde{R}_i is isomorphic to R_i for $1 \leq i \leq m-1$. It is straightforward to see that G' is R -role assignable if G is R_m -role assignable.

Now assume that G' is R -role assignable. Observe that both G' and R have the same number of components, so each component of R provides roles for exactly one component of G' . It is impossible to make a role assignment from \tilde{R}_i to R_j when $|V_{\tilde{R}_i}| < |V_{R_j}|$. Hence the component G can only be assigned roles of one of the components of maximum size.

If the roles of the component G belong to R_m , then we are finished. Suppose the roles of the component G are in R_i with $i \neq m$. Then \tilde{R}_i maps to some other component R_j such that $|V_{\tilde{R}_i}| = |V_{R_j}|$ and by Observation 2.6 we get $\tilde{R}_i \simeq R_j$. Then G allows also an R_j -role assignment. If $j \neq m$ we repeat the argument, and after at most m iterations we find a desired R_m -role assignment of G . \square

Now we show that without the condition of global surjectivity “ $r(V_G) = V_R$ ”, some polynomially solvable RA(R) problems exist for role graphs R with large components.

Take any role graph R with bipartite components (of arbitrary size) but assure that at least one of these components is isomorphic to K_2 . For simplicity, assume that R has no isolated vertices. Clearly, any graph G is R -role assignable

if and only if G is bipartite without isolated vertices. This is because a nonbipartite graph G allow no homomorphism to any of the bipartite components of R , while a possible bipartition of G provides a natural K_2 -role assignment.

This observation leads us to the following proposition for simple role graphs:

Proposition 5 *For a simple role graph R , the $\text{RA}(R)$ problem is solvable in polynomial time if and only if either all components of R have at most two vertices, or R is bipartite and at least one component is isomorphic to K_2 . In all other cases the $\text{RA}(R)$ problem is NP-complete.*

Proof: We have already shown the proof for the polynomial part. So suppose R is a role graph with $m \geq 2$ components $\{R_1, \dots, R_m\}$ ordered in a non-decreasing sequence. Either R is nonbipartite or R is bipartite, contains at least one component that is not an isolated vertex, and does not have a component isomorphic to K_2 . (If R is nonbipartite, then R may have a component isomorphic to K_2 . That is the reason why we have to distinguish these two cases.)

First assume that R contains at least one nonbipartite component. Again we prove NP-completeness by reduction from H2C. Given an instance (Q, \mathcal{S}) we act as follows.

Choose the role graph $L = R_j$ to be the first nonbipartite component in the order, i.e., all components R_i with $i < j$ are bipartite. With respect to L we extend the incidence graph I corresponding to an instance (Q, \mathcal{S}) to an appropriate graph G as in the proof of Proposition 2. Note that G contains an isomorphic copy \tilde{L} glued in G by a vertex \tilde{v} that is the isomorphic copy of a vertex $v \in D_R$.

Starting with graph G we construct a new graph G^* . Let u, v be two vertices in L with distance $\text{dist}_L(u, v) = \text{diam}_L$.

Construct a graph F_0 that consists of two isomorphic copies L_0^1 and L_0^2 of $L = R_j$ glued together by the vertex $u_0^{1,2} = u_0^1 = u_0^2$, the isomorphic copy of u in *both* copies. (See Figure 6 for a picture of such a graph.) Glue F_0 to G by \tilde{v} in such a way that \tilde{v} is identified with the vertex $v_0^2 \in V_{F_0}$.

For each nonbipartite component R_k with $\text{diam}_{R_k} > \text{diam}_L = \text{diam}_{R_j}$ we construct an appropriate subgraph F_k as follows. Let a_k be the smallest even integer such that $\text{diam}_{R_k} < a_k \cdot \text{diam}_L$. The graph F_k contains a_k isomorphic copies of $L = R_j$ glued in a “chain”: Each odd numbered copy L_k^i is linked with the successive copy L_k^{i+1} by the common vertex $u_k^{i,i+1} = u_k^i = u_k^{i+1}$, while each even numbered copy shares with its successor the vertex $v_k^{i,i+1}$.

We finalize F_k as follows. Let $d_k = \text{diam}_L$. Since we have added enough copies, the set $D_k = \{\hat{t} \mid \text{dist}_{F_k}(\hat{t}, v_k^1) = d_k\}$ is non-empty. Let $\hat{t} \in D_k$ be the isomorphic copy of vertex t in $L = R_j$. We glue an isomorphic copy $L^{\hat{t}}$ of L to F_k in such a way that also in the new copy, \hat{t} is identified with vertex t . We do this for all \hat{t} in D_k . See Figure 6 for a picture of a graph F_k .

Finally we make the graph F_k connected to G by identifying vertices $\tilde{v} = v_k^{a_k}$. By repeating the above process for all components R_k with $\text{diam}_{R_k} > \text{diam}_L$ we have obtained graph G^* . See Figure 7 for a picture of this graph.

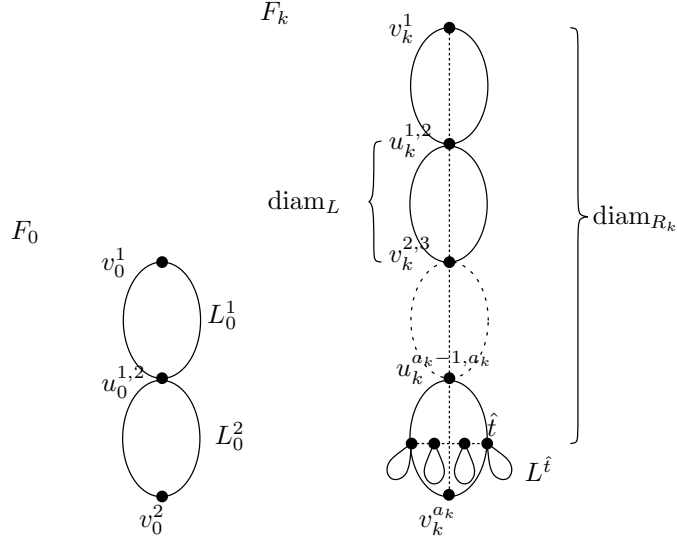


Figure 6: The graphs F_0 and F_k .

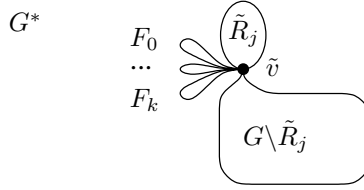


Figure 7: The graph G^* .

Claim: The graph G^* is R -role assignable if and only if it is L -role assignable.

We show this as follows. By definition G^* is R -role assignable, if G^* is L -role assignable. We prove the reverse statement by contradiction. Suppose G^* is R -role assignable but not L -role assignable. Then G^* must be R_k -role assignable for a certain component R_k in R with $k \neq j$. Because G contains a nonbipartite subgraph $\tilde{R}_j \simeq R_j$, G is nonbipartite and R_k cannot be bipartite too. Hence $|V_{R_k}| \geq |V_{R_j}| = |V_L|$.

First suppose R_k has diameter $\text{diam}_{R_k} \leq \text{diam}_L$. Let Z_0 be the set containing all vertices $z \in V_{G^*}$ with distance $\text{dist}_{G^*}(v_0^1, z) \leq \text{diam}_{R_k}$. By Observation 2.5 we deduce that $r(Z_0) = V_{R_k}$.

By construction of G^* , the distance $\text{dist}_{G^*}(v_0^1, z)$ between v_0^1 and any vertex z not in L_0^1 is greater than $\text{diam}_L \geq \text{diam}_{R_k}$. Then Z_0 is a subset of $V_{L_0^1}$. Together with $r(Z_0) = V_{R_k}$ this implies that all roles of R_k appear as an image of a vertex in $V_{L_0^1}$. This is only possible if R_k has no more vertices than L . Hence $|V_{R_k}| = |V_L|$ (and $\text{diam}_{R_k} = \text{diam}_L = \text{diam}_{R_j}$).

If R_k and L have the same number of vertices, each role of R_k appears exactly once as the image of a vertex in L_0^1 . If $r(u_0^{1,2})$ has a neighbor with role y only appearing as an image of a neighbor of $u_0^{1,2}$ outside L_0^1 , then $r(u_0^{1,2})$ must appear at least twice in L_0^1 . Hence L_0^1 is R_k -role assignable. By Observation 2.6 we deduce that $R_k \simeq L_0^1 \simeq L \simeq R_j$. This implies that G^* would be L -role assignable as well, a contradiction.

So we know that $\text{diam}_{R_k} > \text{diam}_L$ must hold. In that case G^* has a corresponding subgraph F_k . Let Z_k be the set containing all vertices $z \in V_{G^*}$ with distance $\text{dist}_{G^*}(v_k^1, z) \leq \text{diam}_{R_k}$. Again we use Observation 2.5 to deduce that $r(Z_k) = V_{R_k}$. Then, by construction of G^* , a vertex $\hat{t} \in D_k$ exists that is mapped on a maximum distance vertex $x \in D_{R_k}$.

Note that \hat{t} is a cutvertex in G^* . Because R_k has a strictly greater diameter than L , not all roles appear as the image of a vertex in $L^{\hat{t}} \simeq L$. Then applying Lemma 2.7 yields that $r(\hat{t}) = x$ is a cutvertex of R_k , contradicting the fact that maximum distance vertices in a graph cannot be cutvertices.

Hence G^* is R -role assignable if and only if G^* is L -role assignable. To obtain G^* we glued a number of graphs to G by \hat{v} that clearly are L -role assignable. Analogously to the proof of Theorem 2 we can show that G^* is L -role assignable if and only if (Q, \mathcal{S}) is 2-colorable. Hence, G^* is R -role assignable if and only if (Q, \mathcal{S}) is 2-colorable.

If R only contains bipartite components, then we choose for L the smallest component that is not an isolated vertex. (Recall that in this case R does not contain any K_2 nor any isolated vertex incident with a loop.) Our construction is exactly the same, only the reasoning differs at one point: Instead of showing that G^* cannot map onto a nonbipartite component R_k (see the first paragraph after the claim) we exclude – due to trivial reasons – the case when G is R_k -role assignable for R_k being an isolated vertex without a loop.

Hence we conclude that also in this case the R -role assignment problem is NP-complete. \square

Again we can use the notion of product graphs to prove the following.

Proposition 6 *Let R be a role graph with $|E_R| \geq 2$ that does not contain any component isomorphic to a K_2 or to a single loop-incident vertex. Then the $\text{RA}(R)$ problem is NP-complete, even if instances are restricted to the class of bipartite graphs.*

Proof: Consider the product graph $S = R \times K_2$, which is bipartite. Then the corresponding S -role assignment problem is NP-complete, because S contains no K_2 or loop-incident vertices as components. We get the NP-hardness of the R -role assignment problem exactly by the same argument as in Theorem 3. \square

What happens if S contains an isolated edge? Then the corresponding S -role assignment problem is polynomially solvable by Proposition 5. Hence we cannot make a reduction from $\text{RA}(S)$, and a little more work on the construction is required in order to complete the proof of Theorem 1.

Theorem 1 *For a role graph R , the $\text{RA}(R)$ problem is solvable in polynomial time if and only if*

- *either R has no edge,*
- *or one of its components consists of a single vertex incident with a loop,*
- *or R is simple and bipartite and has at least one component isomorphic to K_2 .*

In all other cases the $\text{RA}(R)$ problem is NP-complete.

Proof: The polynomially cases are easy to see, and we have discussed them before. So let R be a graph that is neither an edgeless graph, nor a graph with a component consisting of a single loop-incident vertex, nor bipartite simple with at least one component isomorphic to K_2 .

If R is simple, then the R -role assignment problem is NP-complete due to Proposition 5. Moreover if R is connected (yet not necessarily simple) we get the same from Theorem 3.

It remains to show the case when R is not simple and disconnected. Consider the product graph $S = R \times K_2$. If S does not contain a K_2 as a component, then we are done by Proposition 6.

If S has isolated edges, then the original graph R has isolated edges as well. Hence R must contain nonbipartite components. Let the graph L be isomorphic to the first nonbipartite component of R in the order defined at the beginning of this section. Let further R' be the subgraph of R consisting of all non-bipartite components of R . Denote $S' = R' \times K_2$. Since S' is bipartite and simple with no isolated edges, the S' -role assignment problem is NP-complete by Proposition 5. The smallest component in S' is $T = L \times K_2$.

Review the hardness reduction in Proposition 5, which for a given instance (Q, \mathcal{S}) of H2C constructs a connected and bipartite graph G^* such that G^* is S' -role assignable if and only if (Q, \mathcal{S}) is 2-colorable. In G^* a copy $\tilde{T} \simeq T$ is glued by a vertex \tilde{v} that is the isomorphic copy of a vertex $v \in D_T$.

We extend the graph G^* into a graph G^{**} such that G^* is S' -role assignable if and only if G^{**} is R -role assignable.

For this purpose we first extend the product graph $L \times K_2$ in such a way that it will contain the loopless part of L on both coordinates with respect to K_2 . Formally, the graph $L \bowtie K_2$ is defined as

$$\begin{aligned} V_{L \bowtie K_2} &= V_L \times \{1, 2\} \\ E_{L \bowtie K_2} &= \{(x_1, y_2) \mid (x, y) \in E_L\} \\ &\cup \{(x_i, y_i) \mid (x, y) \in E_R, x \neq y, i = 1, 2\}. \end{aligned}$$

See Figure 8 for an example of a product $L \bowtie K_2$ for a nonsimple role graph L .

Clearly, $L \bowtie K_2$ is simple and nonbipartite, and the projection to the first coordinate is an L -role assignment. We glue it into the graph G^* by identifying vertices $\tilde{v} \in V_{G^*}$ and $v_1 \in D_{L \bowtie K_2}$. This way we have obtained the graph G^{**} .

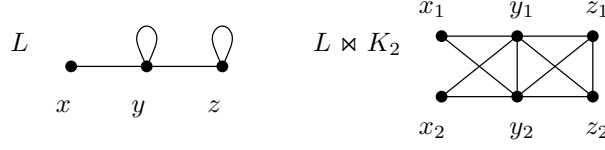


Figure 8: A nonsimple role graph L and its product $L \bowtie K_2$.

Suppose G^* is S' -role assignable. Then the construction of G^* implies that G^* must be T -role assignable with \tilde{v} mapped on v_1 (cf. the proof of Proposition 5). Then an R -role assignment of G^{**} can be composed from the projection to the first coordinates of an S' -role assignment of G^* and an L -role assignment of $L \bowtie K_2$.

Now suppose G^{**} is R -role assignable. Since G^{**} is nonbipartite, any R -role assignment r must choose roles from a nonbipartite component R_j of R . Let r^* be the restriction of r to the bipartite subgraph G^* . Now r^* behaves like an R_j -role assignment of G^* on $V_{G^*} \setminus \{\tilde{v}\}$.

Since G^* is bipartite, a homomorphism $b : G^* \rightarrow K_2$ exists. We define a mapping $s^* : V_{G^*} \rightarrow V_{S'}$ that is composed coordinate-wise of r^* together with b . This mapping behaves like an S' -role assignment of G^* on $V_{G^*} \setminus \{\tilde{v}\}$.

With the same arguments as in the proof of Proposition 5 we can show that R_j must be equal to T , and that s^* restricted to \tilde{T} is an isomorphism between \tilde{T} and T . Hence s^* is a T -role assignment of G^* . This implies that G^* is S' -role assignable. \square

6 k -Role Assignability

In this section we study a more general version of the role assignment problem. We call a graph G *k -role assignable* if there exists a role graph R on k vertices, such that G is globally R -role assignable.

k -ROLE ASSIGNMENT (k -RA)

Instance: A graph G .

Question: Is G k -role assignable?

This problem was studied by [15], and is of interest in social network theory where networks are modeled, in which individuals of the same social role relate to other individuals in the same way. The networks of individuals are represented by simple graphs. As above, in this new model two individuals that are related to each other may have the same role, hence role graphs that contain loops are allowed.

Again our aim is to fully characterize the computational complexity of the k -RA problem. Clearly the 1-RA problem is solvable in linear time, since it is sufficient to check whether G has no edges ($R = K_1$) or whether all vertices in G have degree at least one (R consists of one vertex with a loop). The 2-

RA problem is proven to be NP-complete in [15]. We generalize this result as follows:

Corollary 7 *The k -RA problem is polynomially solvable for $k = 1$ and it is NP-complete for all $k \geq 2$.*

Proof: We show that k -RA is NP-complete for $k \geq 3$. We prove NP-completeness by reduction from $\text{RA}(P_k)$, where P_k is a path on k vertices.

Let G be an instance of $\text{RA}(P_k)$ constructed in the proof of Proposition 2. Without loss of generality we may assume that G is connected and that a graph $\tilde{P}_k \simeq P_k$ is glued in G by vertex \tilde{v} , the isomorphic copy of one of the two leaves of P_k . Let G' be the graph obtained after linking a path P' on $2k - 2$ vertices to G via an edge from \tilde{v} to one of the leaves of P' . Our claim is that G is P_k -role assignable if and only if G' is k -role assignable.

Clearly, if G is P_k -role assignable, then G' is k -role assignable.

In the opposite direction, consider any k -role assignment of G with a connected role graph R on k vertices. Denote the set of vertices of \tilde{P}_k by $\{\tilde{v} = u_1, u_2, \dots, u_k\}$. Since u_k is a leaf, it must be mapped to a leaf, and then by downward induction each $u_i : 2 \leq i < k$ has neighbors of two distinct roles. Otherwise R cannot be connected and hence cannot be used for a global R -role assignment of G .

From the above we conclude that R must be isomorphic to P_k , or otherwise to a path on k vertices with a loop in one of its end points. However, the latter case leads to a contradiction, if we try to assign roles to P' . Hence G' can only be P_k -role assignable, if it is k -role assignable. Clearly, this implies that G is P_k -role assignable as well. □

7 Conclusion

We have fully characterized the computational complexity for the R -role assignment problem for all role graphs R (without multiple edges), proving the conjecture of Kristiansen and Telle from [14]. In particular, we have discussed two different approaches to disconnected role graphs as well.

Our characterization in Theorem 1 gives us only three component-wise minimal polynomial instances of the R -role assignment problem: a loopless isolated vertex, an isolated vertex incident with a loop, and an isolated edge. This is in contrary to the related H -COVER and H -PCOVER problem for locally bijective and locally injective homomorphisms. For these problems many nontrivial graphs H are known, for which the associated problems are polynomially solvable, and where up today no computational complexity classification is known (or even conjectured).

We hope that our study brought an insight into the relationship between full, partial and injective covers. Not surprisingly, the local surjectivity can be more variable than the other two local constraints. Hence the associated problems

are more likely to be NP-complete than the problems, in which the mappings are supposed to be locally injective or bijective.

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