Coloring Graphs Characterized by a Forbidden Subgraph *

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Abstract. The Coloring problem is to test whether a given graph can be colored with at most k colors for some given k, such that no two adjacent vertices receive the same color. The complexity of this problem on graphs that do not contain some graph H as an induced subgraph is known for each fixed graph H. A natural variant is to forbid a graph H only as a subgraph. We call such graphs strongly H-free and initiate a complexity classification of Coloring for strongly H-free graphs. We show that Coloring is NP-complete for strongly H-free graphs, even for $k = 3$, when H contains a cycle, has maximum degree at least five, or contains a connected component with two vertices of degree four. We also give three conditions on a forest H of maximum degree at most four and with at most one vertex of degree four in each of its connected components, such that Coloring is NP-complete for strongly H-free graphs even for $k = 3$. Finally, we classify the computational complexity of Coloring on strongly H-free graphs for all fixed graphs H up to seven vertices. In particular, we show that Coloring is polynomial-time solvable when H is a forest that has at most seven vertices and maximum degree at most four.

1 Introduction

Graph coloring involves the labeling of the vertices of some given graph by integers called colors such that no two adjacent vertices receive the same color. The corresponding Coloring problem is to decide whether a graph can be colored with at most k colors for some given integer k. Due to the fact that Coloring is NP-complete for any fixed $k \geq 3$, there has been considerable interest in studying its complexity when restricted to certain graph classes. One of the most well-known results in this respect is due to Grötschel, Lovász, and Schrijver [8] who show that Coloring is polynomial-time solvable on perfect graphs.

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A well-known structural result that is useful for the design of algorithms for special graph classes is Brooks' Theorem (Theorem 5.2.4 in [5]), which states that any connected graph G that is neither complete nor an odd cycle can be colored with at most $\Delta(G)$ colors where $\Delta(G)$ is the maximum degree of G. General motivation, background and related work on coloring problems restricted to special graph classes can be found in several surveys [13, 14].

We study the complexity of the COLORING problem restricted to graph classes defined by forbidding a graph H as a (not necessarily induced) subgraph. So far, COLORING has not been studied in the literature as regards to such graph classes. Before we summarize some related results and present our results, we first state the necessary terminology and notations.

1.1 Terminology

We consider finite undirected graphs without loops and multiple edges. We refer to the textbook of Diestel [5] for any undefined graph terminology. Let $G = (V, E)$ be a graph. The subgraph of G induced by a subset $U \subseteq V$ is denoted $G[U]$. The graph $G - u$ is obtained from G by removing vertex u. For a vertex u of G, its *open neighborhood* is $N(u) = \{v \mid uv \in E\}$, its *closed neighborhood* is $N[u] = N(u) \cup \{u\}$, and its degree is $d(u) = |N(u)|$. The maximum degree of G is denoted by $\Delta(G)$ and the minimum degree by $\delta(G)$. The *distance* $\mathrm{dist}(u, v)$ between two vertices u and v of G is the number of edges of a shortest path between them. The *girth* $g(G)$ is the length of a shortest cycle in G. We say that G is *(strongly) H-free* for some graph H if G has no subgraph isomorphic to H; note that this is more restrictive than forbidding H as an induced subgraph. A *subdivision* of an edge $uv \in E$ is the operation that removes uv and adds a new vertex adjacent to u and v. A graph H is a *subdivision* of G if H is obtained from G by a sequence of edge subdivisions. A *coloring* of G is a mapping $c : V \to \{1, 2, \ldots\}$, such that $c(u) \neq c(v)$ if $uv \in E$. We call $c(u)$ the *color* of u. A *k-coloring* of G is a coloring c of G with $1 \leq c(u) \leq k$ for all $u \in V$. If G has a k-coloring, then G is called *k-colorable*. The *chromatic number* $\chi(G)$ is the smallest integer k such that G is k-colorable. The k-COLORING problem is to test whether a graph admits a k-coloring for some fixed integer k. If k is in the input, then we call this problem COLORING. The graph P_n is the path on n vertices.

1.2 Related Work

Král', Kratochvíl, Tuza and Woeginger [11] completely determined the computational complexity of COLORING for graph classes characterized by a forbidden *induced* subgraph and achieved the following dichotomy. Here, $P_1 + P_3$ denotes the disjoint union of P_1 and P_3.

Theorem 1 ([11]). *If some fixed graph H is a (not necessarily proper) induced subgraph of P_4 or of $P_1 + P_3$, then* COLORING *is polynomial-time solvable on graphs with no induced subgraph isomorphic to H; otherwise it is* NP*-complete on this graph class.*

The complexity classification of the k-Coloring problem for graphs with no induced subgraphs isomorphic to some fixed graph H is still open. For $k = 3$, it has been classified for graphs H up to six vertices [3], and for $k = 4$ for graphs H up to five vertices [7]. We refer to the latter paper for a survey on the complexity status of k-Coloring for graph classes characterized by a forbidden induced subgraph.

1.3 Our Results

Recall that a strongly H-free graph denotes a graph with no subgraph isomorphic to some fixed graph H. Forbidding a graph H as an induced subgraph is equivalent to forbidding H as a subgraph if and only if H is a complete graph (a graph with an edge between any two distinct vertices). Hence, Theorem 1 tells us that Coloring is NP-complete for strongly H-free graphs if H is a complete graph. We extend this result by proving the following two theorems in Sections 2 and 3, respectively; note that the case when H is a complete graph is covered by condition (a) of Theorem 2. The trees $T_1, \ldots, T_6$ are displayed in Figure 1. For an integer $p \geq 0$, the graph T_2^p is the graph obtained from T_2 after subdividing the edge st p times; note that $T_2^0 = T_2$.

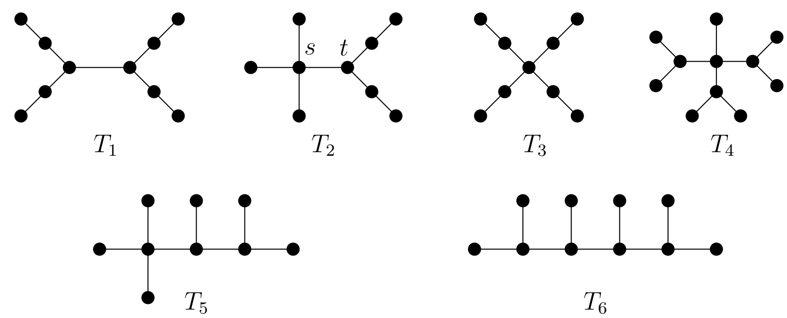

Fig. 1. The trees $T_1, \ldots, T_6$.

Theorem 2. 3-Coloring *(and hence* Coloring*) is* NP*-complete for strongly H-free graphs if*

(a) H contains a cycle, or
(b) $\Delta(H) \geq 5$, or
(c) H has a connected component with at least two vertices of degree four, or
(d) H contains a subdivision of the tree T_1 as a subgraph, or
(e) H contains the tree T_2^p as a subgraph for some $0 \leq p \leq 9$, or
(f) H contains one of the trees T_3, T_4, T_5, T_6 as a subgraph.

Theorem 3. Coloring *is polynomial-time solvable for strongly H-free graphs if*

(a) H is a forest with $\Delta(H) \leq 3$, such that each connected component has at most one vertex of degree 3, or
(b) H is a forest with $\Delta(H) \leq 4$ and $|V_H| \leq 7$.

Theorems 1–3 tell us that the COLORING problem behaves differently on graphs characterized by forbidding H as an induced subgraph or as a subgraph. As a consequence of Theorems 2 and 3(b) we can classify the COLORING problem on strongly H-free graphs for graphs H up to 7 vertices. The problem is NP-complete if H is not a forest or $\Delta(H) \geq 5$, and polynomial-time solvable otherwise.

1.4 Future Work

The aim is to complete the computational complexity classification of COLORING for strongly H-free graphs. Our current proof techniques are rather diverse, and a more unifying approach may be required.

2 The proof of Theorem 2

In the remainder of the paper we write H-free instead of strongly H-free as a shorthand notation. Here is the proof of Theorem 2.

(a) Maffray and Preissmann [12] showed that 3-COLORING is NP-complete for triangle-free graphs. This result has been extended by Kamiński and Lozin [10], who proved that k-COLORING is NP-complete for the class of graphs of girth at least p for any fixed $k \geq 3$ and $p \geq 3$. Suppose that H contains a cycle. Then $g(H)$ is finite. Let $p = g(H) + 1$. It remains to observe that any graph of girth at least p does not contain H as a subgraph, and (a) follows.

(b) It is well known that 3-COLORING is NP-complete for graphs of maximum degree at most four [6]. Then, because any graph G with $\Delta(G) \leq 4$ does not contain a graph H with $\Delta(H) \geq 5$ as a subgraph, (b) holds.

(c) As before, we reduce from 3-COLORING for graphs of maximum degree at most four. Let $G = (V, E)$ be a graph of maximum degree at most four. We define a useful graph operation. In order to do this, we need the graph displayed in Figure 2. It has vertex set $\{x, y, z, t\}$ and edge set $\{xz, xt, yz, yt, zt\}$ and is called a *diamond* with *poles* x, y. We observe that in any 3-coloring of a diamond with poles x, y, the vertices x and y are colored alike.

The graph operation that we use is displayed in Figure 2. For a vertex $u \in V$ with four neighbors $v_1, \ldots, v_4$, we do as follows. We delete the edges uv_i for $i = 1, \ldots, 4$. We then add 4 diamonds with poles x_i, y_i for $i = 1, \ldots, 4$ and identify u with each y_i. Finally, we add the edges $v_i x_i$ for $i = 1, \ldots, 4$. We call this operation the *vertex-diamond* operation. Note that this operation is only defined on vertices of degree four. Because any 3-coloring gives the poles of a diamond the same color, the resulting graph is 3-colorable if and only if G is 3-colorable. We also observe that this operation when applied on a vertex u

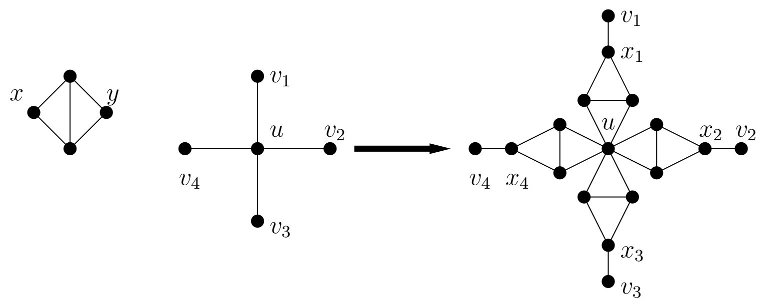

Fig. 2. A diamond with poles x, y and the vertex-diamond operation.

increases the distance between u and any other vertex of G by 2. Moreover, the new vertices added have degree three.

To complete the proof of (c), let H be a graph that has a connected component D with at least two vertices of degree four. Let α denote the maximum distance between two such vertices in D. Then we apply α vertex-diamond operations on each vertex of degree four in G. By our previous observations, the resulting graph G^* is D-free, and consequently, H-free, and in addition, G^* is 3-colorable if and only if G is 3-colorable. Hence (c) holds.

(d) As before, we reduce from 3-Coloring for graphs of maximum degree at most four. Let $G = (V, E)$ be a graph of maximum degree at most four. We define the following graph operation displayed in Figure 3. For an edge $x_0y_0 \in E$, we do as follows. We delete the edge x_0y_0 (but we keep the vertices x_0 and y_0) and add vertices $x_1, y_1, \ldots, x_\ell, y_\ell$. We then construct diamonds with poles x_{i-1}, x_i and y_{i-1}, y_i respectively, for $i = 1, \ldots, \ell$. Finally, we add the edge $x_\ell y_\ell$. We call this operation the *edge-diamond* operation *of type* ℓ. We let G_ℓ be the graph obtained from G after applying an edge-diamond operation of type ℓ on each of its edges. Because any 3-coloring gives the poles of a diamond the same color, G_ℓ is 3-colorable for any $\ell \geq 1$ if and only if G is 3-colorable.

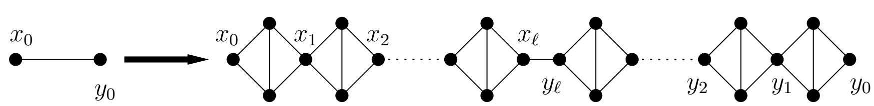

Fig. 3. The edge-diamond operation.

To complete the proof of (d), let H be a graph that contains a subdivision of T_1, which we will denote by T'. Let u, v be the vertices of degree three in T'. We choose $\ell = dist_{T'}(u, v)$. Then G_ℓ is H-free, and (d) holds.

subcases $\mathbf{p = 0}$ and $\mathbf{p = 1}$ of (e) and subcase $\mathbf{H = T_5}$ of (f). As before, we reduce from 3-Coloring for graphs of maximum degree at most four. Let

$G = (V, E)$ be a graph of maximum degree at most four. We construct the graph G^* defined in case (c). We observe that G^* is T_2^0-free, T_2^1-free and T_5-free, because every vertex of degree at least four in G^* is obtained by identifying pole vertices of diamonds. Recall that G^* is 3-colorable if and ony if G is 3-colorable. Hence, the subcases $p = 0$ and $p = 1$ of (e) and the subcase $H = T_5$ of (f) hold.

remaining eight subcases of (e) and subcase $\mathbf{H = T_6}$ of (f). As before, we reduce from 3-Coloring for graphs of maximum degree at most four. Let $G = (V, E)$ be a graph of maximum degree at most four. To complete the proof of (e), let H be a graph that contains T_2^p as a subgraph for some $2 \leq p \leq 9$. Recall that the graph G_ℓ defined in case (d) is is 3-colorable if and only if G is 3-colorable. We choose $\ell = \lceil \frac{p-1}{2} \rceil$. Then G_ℓ is H-free, and the remaining subcases of (e) hold. As an aside, note that for $p \geq 10$, there exists no ℓ such that G_ℓ is T_2^p-free, because for all $\ell \geq 1$ we can "map" the degree-3 vertex t of T_2^p on a degree-4 vertex in G_ℓ that corresponds to an original degree-4 vertex of G. Then we will either find in G_ℓ a suitable vertex u that is in a diamond or that is a degree-4 vertex that corresponds to an original degree-4 vertex of G, such that we can "map" the degree-4 vertex s of T_2^p to u in order to obtain a subgraph in G_ℓ that is isomorphic to T_2^p. Hence, the case $p \geq 10$ is still open.

Now let H be a graph that contains T_6 as a subgraph. We choose $\ell = 1$. Then G_2 is H-free, and the corresponding subcase of (f) holds.

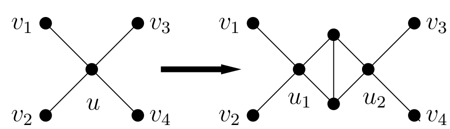

Fig. 4. The balanced-diamond operation.

remaining two subcases of (f). As before, we reduce from 3-Coloring for graphs of maximum degree at most four. Let $G = (V, E)$ be a graph of maximum degree at most four. The last graph operation that we use is displayed in Figure 4. For a vertex $u \in V$ with four neighbors $v_1, \ldots, v_4$, we do as follows. We remove u and add two new vertices u_1 and u_2. We make u_1 adjacent to v_1 and v_2, whereas we make u_2 adjacent to v_3 and v_4. Finally, we add two more vertices that together with u_1 and u_2 form a diamond, in which u_1 and u_2 are the poles. We call this operation the *balanced-diamond* operation. Note that we only define this operation on vertices of degree four (we refer to the paper of Kamiński and Lozin [9] for a more general variant called diamond implementation). Because any 3-coloring gives the poles of a diamond the same color, the resulting graph is 3-colorable if and only if G is 3-colorable.

To complete the proof of (f), let H be a graph that contains T_3 or T_4 as a subgraph. We apply the balanced-diamond operation on each vertex of degree four in G. Then the resulting graph G' is H-free. Moreover, by our observation,

G' is 3-colorable if and only if G is 3-colorable. This concludes the proof of Theorem 2. □

3 The proof of Theorem 3

Let G be a graph. A graph H is a *minor* of G if H can be obtained from a subgraph of G by a sequence of edge contractions. We first prove Theorem 3(a).

Theorem 3(a). *Let H be a fixed forest with $\Delta(H) \leq 3$, such that each connected component of H has at most one vertex of degree three. Then* Coloring *can be solved in polynomial time for H-free graphs.*

Proof. Let G be a graph. If $|V_H| = 1$, then the statement of the theorem holds. Suppose that $|V_H| \geq 2$. Let $H_1, \ldots, H_p$ be the connected components of H. Consider a connected component H_i for some $1 \leq i \leq p$. Because $\delta(H) \leq 3$, we find that $\delta(H_i) \leq 3$. Moreover, by our assumption, H_i contains at most one vertex of degree 3. Then H_i is either a path or a subdivided star, in which the centre vertex has degree 3. As such, H_i is a subgraph of G if and only if H_i is a minor of G. Consequently, H is a subgraph of a graph G if and only if H is a minor of G. By a result of Bienstock et al. [2], every graph that does not contain H as a minor has path-width at most $|V_H| - 2$. Hence G has path-width at most $|V_H| - 2$. Because H is fixed, this implies that G has bounded path-width, and consequently, bounded treewidth. Because Coloring can be solved in linear time for graphs of bounded treewidth as shown by Arnborg and Proskurowski [1], the result follows. □

Theorem 3(a) limits the remaining cases of Theorem 3(b) to those graphs H that are a forest on at most 7 vertices and that contain a vertex of degree 4 or two vertices of degree at least 3. Moreover, our goal is to show polynomial-time solvability for such cases, and a graph is H-free if it is H'-free for any subgraph H' of H. This narrows down our case analysis to the trees $H_1, \ldots, H_5$ shown in Fig. 5. We consider each such tree, but we first give some auxiliary results.

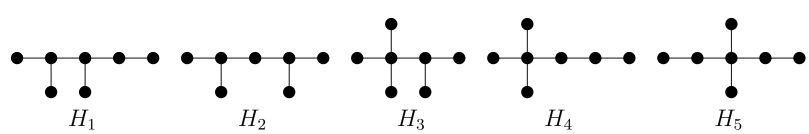

Fig. 5. The trees $H_1, \ldots, H_5$.

Observation 1 *Let G be a graph with $|V_G| \geq 2$. Let $u \in V_G$ with $d_G(u) < k$ for some integer $k \geq 1$. Then G is k-colorable if and only if $G - u$ is k-colorable.*

We say that a vertex u of a graph G is *universal* if $G = G[N_G[u]]$, i.e., if u is adjacent to all other vertices of G.

Observation 2 *Let u be a universal vertex of a graph G with $|V_G| \geq 2$. Let $k \geq 2$ be an integer. Then G is k-colorable if and only if $G - u$ is $(k-1)$-colorable.*

A vertex u of a connected graph G with at least two vertices is a *cut-vertex* if $G-u$ is disconnected. A maximal connected subgraph of G with no cut-vertices is called a *block* of G.

Observation 3 *Let G be a connected graph, and let k be a positive integer. Then G is k-colorable if and only if each block of G is k-colorable.*

Let (G, k) be an instance of Coloring. We apply the following preprocessing rules recursively, and as long as possible. If after the application of a rule we can apply some other rule with a smaller index, then we will do this.

Rule 1. Find all connected components of G and consider each of them.

Rule 2. Check if G is 1-colorable or 2-colorable. If so, then stop considering G.

Rule 3. If $|V_G| \geq 2$, $k \geq 3$, and G has a vertex u with $d_G(u) \leq 2$, take $(G-u, k)$.

Rule 4. If $|V_G| \geq 2$, $k \geq 3$, and G has a universal vertex u, take $(G-u, k-1)$.

Rule 5. If G is connected, then find all blocks of G and consider each of them.

We obtain the following lemma.

Lemma 1. *Let G be an n-vertex graph that together with an integer $k \geq 3$ forms an instance of* Coloring. *Applying rules 1–5 recursively and exhaustively takes polynomial time and yields a set I of at most n instances, such that (G, k) is a yes-instance if and only if every instance of I is a yes-instance. Moreover, each $(G', k') \in I$ has the following properties:*

(i) $|V_{G'}| \leq |V_G|$;
(ii) $\delta(G') \geq 3$;
(iii) G' *has no universal vertices;*
(iv) G' *is 2-connected;*
(v) $3 \leq k' \leq k$;
(vi) if G is H-free for some graph H, then G' is H-free as well.

Proof. Let G be an n-vertex graph and let $k \geq 3$ be an integer. We first show that applying rules 1–5 recursively and exhaustively takes polynomial time. Rule 1 takes linear time, because we only have to find the connected components of G. Rule 2 takes linear time, because G is 1-colorable if and only if G has no edges, and G is 2-colorable if and only if G is bipartite. Rules 3 and 4 take linear time, because we only need to check the degree of each vertex. Rule 5 takes linear time, because we only need to find the set of blocks of G. Because the size of G decreases after applying Rule 3 or Rule 4, our procedure terminates.

The number of instances created only increases after applying Rule 1 or Rule 5. Because the total number of blocks of all connected components is at most n, the set I has size at most n.

We now show that rules 1-5 are correct. Rule 1 is correct, because G is k-colorable if and only if each connected component of G is k-colorable. Clearly, Rule 2 is correct as well. Rule 3 is correct due to Observation 1. Rule 4 is correct due to Observation 2. Rule 5 is correct due to Observation 3. Hence, our procedure creates a set I of at most n instances, such that (G, k) is a yes-instance if and only if each instance of I is a yes-instance. Note that (G, k) is a yes-instance if $I = \emptyset$, as in that case G is 2-colorable, and consequently, k-colorable, due to one or more applications of Rule 2.

Let (G', k') be an instance of I. Then $|V_{G'}| \leq |V_G|$ because we only decreased the size of G. This proves (i). By Rule 3, G' has minimum degree at least 3. This proves (ii). By Rule 4, G' has no universal vertices. This proves (iii). By Rule 5, G' is 2-connected. This proves (iv). By our assumption, $k \geq 3$. By Rule 2, $k' \geq 3$. We have $k' \leq k$, because we only decreased k. This proves (v). Because we only removed vertices from G, we find that G' is a subgraph of G. Hence, if G is H-free for some graph H, then G' is H-free. This proves (vi). □

3.1 The Cases $H = H_1$ and $H = H_2$

We first give some extra terminology. Let $G = (V, E)$ be a graph. We let $\omega(G)$ denote the size of a maximum clique in G. The *complement* of G is the graph $\overline{G}$ with vertex set V, such that any two distinct vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. If $\chi(F) = \omega(F)$ for any induced subgraph F of G, then G is is called *perfect*. Let C_r denote the cycle on r vertices. We will use the Strong Perfect Graph Theorem proved by Chudnovsky et al. [4]. This theorem tells us that a graph is perfect if and only if it does not contain C_r or $\overline{C}_r$ as an induced subgraph for any odd integer $r \geq 5$.

Lemma 2. *Let G be a 2-connected graph with $\delta(G) \geq 3$ that has no universal vertices. If G is H_1-free or H_2-free, then G is perfect.*

Proof. Note that H_1 and H_2 are both subgraphs of $\overline{C}_r$ for any $r \geq 7$. Moreover, $C_5 = \overline{C_5}$. Then, by the Strong Perfect Graph Theorem [4], we are left to prove that G contains no induced cycle C_r for any odd integer $r \geq 5$. To obtain a contradiction, assume that G does contain an induced cycle $C = v_0 v_1 \cdots v_{r-1} v_{r-1} v_0$ for some odd integer $r \geq 5$.

First suppose that G is H_1-free. Let $0 \leq i \leq r-1$ and consider the path $v_i v_{i+1} \cdots v_{i+3} v_{i+4}$, where the indices are taken modulo r. Since $\delta(G) \geq 3$, v_{i+1} and v_{i+2} each have at least one neighbor in $V' = V \setminus \{v_0, \ldots, v_{r-1}\}$, say v_{i+1} is adjacent to some vertex u and v_{i+2} is adjacent to some vertex v. Because G is H_1-free, $u = v$, and moreover, $|N(v_{i+1}) \cap V'| = |N(v_{i+2}) \cap V'| = 1$. Because $0 \leq i \leq r-1$ was taken arbitrarily, we deduce that the vertices $v_0, \ldots, v_{r-1}$ are all adjacent to the same vertex $u \in V'$ and to no other vertices in V'. Because G is 2-connected, u is not a cut-vertex. Hence, $V' = \{u\}$. However, then u is a universal vertex. This is a contradiction.

Now suppose that G is H_2-free. By the same arguments and the fact that r is odd, we conclude again that there exists a universal vertex $u \in V'$. This is a contradiction. □

We are now ready to prove that Coloring is polynomial-time solvable for H_1-free and for H_2-free graphs. Let G be a graph, and let $k \geq 1$ be an integer. If $k \leq 2$, then Coloring is even polynomial-time solvable for general graphs. Suppose that $k \geq 3$. Then, by Lemma 1, we may assume without loss of generality that G is 2-connected, has $\delta(G) \geq 3$ and does not contain any universal vertices. Lemma 2 then tells us that G is perfect. Because Grötschel et al. [8] showed that Coloring is polynomial-time solvable for perfect graphs, our result follows.

3.2 The Case $H = H_3$

We first give some additional terminology. We say that we *identify* two distinct vertices $u, v \in V_G$ if we first remove u, v and then add a new vertex w by making it (only) adjacent to the vertices of $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$.

Consider the graphs $F_1, \ldots, F_4$ shown in Fig. 6. We call the vertices x_1, x_2 of F_1, x_1, x_2, x_3 of F_2 and x_1, x_2, y_1, y_2 of F_3 and F_4 the *pole vertices* of the corresponding graph F_i, whereas the other vertices of F_i are called *centre vertices*. We say that a graph G *properly contains* F_i for some $1 \leq i \leq 4$ if G contains F_i as an induced subgraph, in such a way that centre vertices of F_i are only adjacent to vertices of F_i, i.e., the subgraph F_i is connected to other vertices of G only via its poles.

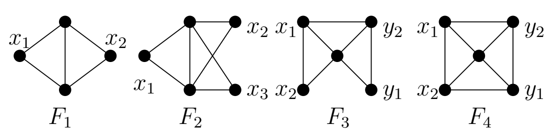

Fig. 6. The graphs F_1, F_2, F_3, F_4.

For our result we need one additional rule that we apply on a graph G.

Rule 6. If G properly contains F_i for some $1 \leq i \leq 4$, then remove the centre vertices of F_i from G and identify the pole vertices of F_i as follows:

- if $i = 1$, then identify x_1 and x_2;
- if $i = 2$, then identify x_1, x_2, and x_3;
- if $i = 3$ or $i = 4$, then identify x_1 and y_1, and also identify x_2 and y_2.

We prove that Coloring can be solved in polynomial time for H_3-free graphs as follows. Let G be an H_3-free graph, and let $k \geq 1$ be an integer.

Case 1. $k \leq 2$.
Then Coloring can be solved in polynomial time even for general graphs.

Case 2. $k \geq 3$.
By Lemma 1, we may assume without loss of generality that $\delta(G) \geq 3$ and that G contains no universal vertices. In Lemma 3 (stated after this case analysis) we

show that $\Delta(G) \leq 4$. Then Brooks' Theorem (cf. [5]) tells us that G is 4-colorable unless $G = K_5$. In the latter case, G is 5-colorable.

Case 2a. $k \geq 5$**.**
Then (G, k) is a yes-answer.

Case 2b. $k = 4$**.**
Then (G, k) is a yes-answer if and only if $G \neq K_5$.

Case 2c. $k = 3$**.**
We show in Lemma 4 that an application of Rule 6 on G yields an H_3-free graph that is 3-colorable if and only if G is 3-colorable. We apply Rule 6 exhaustively. This takes polynomial time, because each application of Rule 6 takes linear time and reduces the size of G. In order to maintain the properties of having minimum degree at least 3 and containing no universal vertices, we apply Lemma 1 after each application of Rule 6. Hence, afterward, we have found in polynomial time a (possibly empty) set $\mathcal{G}$ of at most n graphs, such that G is 3-colorable if and only if each graph in $\mathcal{G}$ is 3-colorable. Moreover, each $G' \in \mathcal{G}$ is H_3-free, has minimum degree at least 3, contains no universal vertices, and in addition, does not properly contain any of the graphs $F_1, \ldots, F_4$. Then, by Lemma 3, each $G' \in \mathcal{G}$ has $\Delta(G') \leq 4$. As a consequence, we may apply Lemma 5. This lemma tells us that a graph $G' \in \mathcal{G}$ is 3-colorable if and only if it does not contain K_4 as a subgraph. As we can check the latter condition in polynomial time and $|\mathcal{G}| \leq n$, i.e., we have at most n graphs to check, our result follows.

What is left to do is to state and prove Lemmas 3–5.

Lemma 3. *Let G be an H_3-free graph with no universal vertices. If $\delta(G) \geq 3$, then $\Delta(G) \leq 4$.*

Proof. Let $G = (V, E)$ be an H_3-free graph with no universal vertices. Suppose that $\delta(G) \geq 3$. To obtain a contradiction assume that $d_G(u) \geq 5$ for some vertex $u \in V$. Because G has no universal vertices, there is a vertex $v \in N_G(u)$ such that v has a neighbor $x \in V \setminus N_G[u]$. Because $d_G(v) \geq \delta(G) \geq 3$, we deduce that v has another neighbor $y \notin \{u, x\}$. Because $d_G(u) \geq 5$, we also deduce that u has three neighbors z_1, z_2, z_3 neither equal to v nor to y. However, the subgraph of G with vertices $u, v, x, y, z_1, z_2, z_3$ and edges $uz_1, uz_2, uz_3, uv, vx, vy$ is isomorphic to H_3. This is a contradiction, because G is H_3-free. $\square$

Lemma 4. *Let G be an H_3-free graph with $\delta(G) \geq 3$ and $\Delta(G) \leq 4$. Let G' be the graph obtained from G after one application of Rule 6. Then G' is 3-colorable if and only if G is 3-colorable. Moreover, G' is H_3-free.*

Proof. Let G be an H_3-free graph with $\delta(G) \geq 3$ and $\Delta(G) \leq 4$ that properly contains a graph F_i for some $1 \leq i \leq 4$. Let G' be the graph obtained from G after applying Rule 6 with respect to F_i.

We first prove that G' is 3-colorable if and only if G is 3-colorable. First suppose that G' is 3-colorable. Consider a 3-coloring of G'. We color all vertices in $V \setminus V_{F_i}$ by the same colors as in G', the pole vertices of F_i are colored by the same color as the vertex obtained from them by the identification. It remains to observe that if $i = 1$ or $i = 2$, then the neighbors of the two centre vertices are colored by one color, and if $i = 3$ or $i = 4$, then the neighborhood of the unique centre vertex is colored by two colors. Hence, we can safely color the centre vertices of F_i. Now suppose that G is 3-colorable. Because in any 3-coloring of F_i the identified vertices are necessarily colored with the same color, G' is 3-colorable as well.

Now we show that G' is H_3-free. To obtain a contradiction, assume that G' has a subgraph H isomorphic to H_3. Let u be the vertex of degree four in H, and let v be the vertex of degree three. Because G is H_3-free, at least one of u, v must be obtained by identifying pole vertices of F_i.

First suppose that u is not obtained by identifying pole vertices of F_i. Then v must be obtained by identifying pole vertices of F_i. Then, in G, we find that u is adjacent to a vertex v' that is a pole vertex of F_i and that corresponds to v in G' by the identification of pole vertices. Moreover, because u has degree 4 in G', we find that u has three other neighbors z_1, z_2, z_3 not equal to v' in G that are not identified with each other or with v' after applying Rule 6; one of them may still be a pole vertex in the case that $i = 3$ or $i = 4$, but then such z_i is identified with some vertex of G not in $\{v', z_1, z_2, z_3\} \setminus \{z_i\}$. Also, z_1, z_2, z_3 cannot be centre vertices of F_i, as centre vertices are removed by Rule 6.

Because u is in G' and Rule 6 removes centre vertices of F_i, we find that u is not a centre vertex of F_i. Because u is not a pole vertex of F_i either, this means that $u \in V \setminus V_{F_i}$. If $i = 1$ or $i = 2$, then let w_1 and w_2 be the two centre vertices of F_i. Then the subgraph of G with vertices $u, v', w_1, w_2, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, v'w_1, v'w_2$ is isomorphic to H_3. This is a contradiction. Hence, $i = 3$ or $i = 4$.

Let w be the unique centre vertex of F_i and assume that $v' \in \{x_1, x_2\}$. Let v'' denote the other vertex of $\{x_1, x_2\}$. If none of the vertices z_1, z_2, z_3 is in V_{F_i}, then the subgraph of G that has vertices $u, v', v'', w, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, v'w, v'v''$ is isomorphic to H_3. This is a contradiction. Therefore, one of the vertices z_1, z_2, z_3, say z_1, is a pole vertex of F_i. Note that z_2 and z_3 are not in F_i, as we already deduced. We also deduced that z_1 is not identified with v'. Suppose that $z_1 \in \{y_1, y_2\}$. Then again the subgraph of G that has vertices $u, v', v'', w, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, v'w, v'v''$ is isomorphic to H_3, which is a contradiction. Hence, $z_1 \in \{x_1, x_2\}$. If $z_1 = x_1$, then $v' = x_2$. Then the subgraph of G with vertices $u, v', w, y_2, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, z_1w, z_1y_2$ is isomorphic to H_3. If $z_1 = x_2$, then $v' = x_1$. Then the subgraph of G with vertices $u, v', w, y_2, z_1, z_2, z_3$ and edges $uv', uz_1, uz_2, uz_3, v'w, v'y_2$ is isomorphic to H_3. Both cases are not possible. We conclude that u must be obtained by identifying pole vertices, namely x_1 and x_2 if $i = 1$, x_1, x_2, x_3 if $i = 2$, and we may assume without loss of generality that u is obtained by identifying x_1 and y_1 if $i = 3$ or $i = 4$.

First suppose that $i = 1$. Because $\Delta(G) \leq 4$ and $d_{G'}(u) = 4$, each pole x_j must have two neighbors s_1^j and s_2^j in G that are not in F_1 for $j = 1, 2$. Because G' contains H_3, one of the vertices $s_1^1, s_2^1, s_1^2, s_2^2$, say s_1^1, has two neighbors

t_1 and t_2 in G that are not in $V_{F_1} \cup \{s_1^1, s_2^1, s_1^2, s_2^2\}$. Let w_1 and w_2 denote the two centre vertices of F_1. We find that the subgraph of G with vertices $s_1^1, s_2^1, t_1, t_2, w_1, w_2, x_1$ and edges $x_1s_1^1, x_1s_2^1, x_1w_1, x_1w_2, s_1^1t_1, s_1^1t_2$ is isomorphic to H_3. This is a contradiction.

Now suppose that $i = 2$. Because $\Delta(G) \leq 4$ and $d_{G'}(u) = 4$, one pole, say x_1, has two neighbors s_1 and s_2 in G that are not in F_2. Let w_1 and w_2 denote the two centre vertices of F_2. We find that the subgraph of G with vertices $s_1, s_2, w_1, w_2, x_1, x_2, x_3$ and edges $x_1s_1, x_1s_2, x_1w_1, x_1w_2, w_1x_2, w_1x_3$ is isomorphic to H_3. This is a contradiction.

Finally suppose that $i = 3$ or $i = 4$. Recall that we assume that $u \in V_H$ was obtained by identifying x_1 and y_1. Then, because $d_{G'}(u) = 4$ and $\Delta(G) \leq 4$, we find that $i = 3$ and that y_1 has two neighbors s_1 and s_2 in G that are not in F_3. Let w denote the centre vertex of F_3. We find that the subgraph of G with vertices $s_1, s_2, w, x_1, x_2, y_1, y_2$ and edges $y_1s_1, y_1s_2, y_1y_2, y_1w, wx_1, wx_2$ is isomorphic to H_3. This is a contradiction. We conclude that u cannot be obtained by identifying pole vertices. This completes the proof of Lemma 4. $\square$

Lemma 5. *Let G be an H_3-free graph with $\delta(G) \geq 3$ and $\Delta(G) \leq 4$ that does not properly contain any of the graphs $F_1, \ldots, F_4$. Then G is 3-colorable if and only if G is K_4-free.*

Proof. Let $G = (V, E)$ be an H_3-free graph with $\delta(G) \geq 3$ and $\Delta(G) \leq 4$ that does not properly contain any of the graphs $F_1, \ldots, F_4$. First suppose that G is 3-colorable. This immediately implies that G is K_4-free.

Now suppose that G is K_4-free. If $\Delta(G) \leq 3$, then Brooks' Theorem (cf. [5]) tells us that G is 3-colorable unless $G = K_4$, which is not the case. Hence, we may assume that G contains at least one vertex of degree four. To obtain a contradiction, assume that G is a minimal counter-example, i.e., $\chi(G) \geq 4$ and $G - v$ is 3-colorable for all $v \in V$.

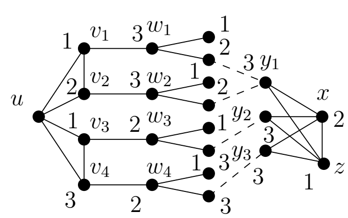

Fig. 7. The structure of the graph G. We note that neighbors of $w_1, \ldots, w_4$ not equal to $v_1, \ldots, v_4$ may not be distinct.

Let u be a vertex of degree four in G, and let $N_G(u) = \{v_1, v_2, v_3, v_4\}$. We first show the following four claims.

(a) $G[N_G(u)]$ is C_3-free;
(b) $G[N_G(u)]$ contains no vertex of degree three;

(c) $G[N_G(u)]$ is not isomorphic to P_4;
(d) $G[N_G(u)]$ is not isomorphic to C_4.

Claims (a)–(d) can be seen as follows. If $G[N_G(u)]$ contains C_3 as a subgraph, then $G[N_G[u]]$, and consequently, G contains K_4 is a subgraph of G. This proves (a). If $G[N_G(u)]$ contains a vertex of degree three, then G properly contains F_2 as G is C_3-free due to (a). This proves (b). If $G[N_G(u)]$ is isomorphic to P_4, then G properly contains F_3. This proves (c). If $G[N_G(u)]$ is isomorphic to C_4, then G properly contains F_4. This proves (d).

Because G is H_3-free, each v_j has at most one neighbor in $V \setminus N_G[u]$. Because $\delta(G) \geq 3$, this means that $G[N_G(u)]$ contains no isolated vertices. Then, by claims (a)–(d), we find that $G[N_G(u)]$ contains exactly two edges. Moreover, $d_G(v_j) = 3$ for $j \in \{1, \ldots, 4\}$ as $\delta(G) \geq 3$.

We assume without loss of generality that v_1v_2 and v_3v_4 are edges in G. Let w_j be the neighbor of v_j in $V \setminus N_G[u]$ for $j = 1, \ldots, 4$. We note that $w_1 \neq w_2$ and $w_3 \neq w_4$, as otherwise G properly contains F_1.

Recall that $G - u$ is 3-colorable. Let c be an arbitrary 3-coloring of $G - u$. We show that the following two claims are valid for c up to a permutation of the colors $1, 2, 3$.

(1) $c(v_1) = c(v_3) = 1$, $c(v_2) = 2$ and $c(v_4) = 3$;
(2) $c(w_1) = c(w_2) = 3$ and $c(w_3) = c(w_4) = 2$;

Claims (1) an (2) can be seen as follows. If c uses at most two different colors on $v_1, \ldots, v_4$, then we can extend c to a 3-coloring of G, which is not possible as $\chi(G) \geq 4$. Hence, c uses three different colors on $v_1, \ldots, v_4$. Then we may assume without loss of generality that $c(v_1) = c(v_3) = 1$, $c(v_2) = 2$ and $c(v_4) = 3$. This proves (1). We now prove (2). In order to obtain a contradiction, assume that $c(w_1) \neq c(w_2)$. Because $c(v_2) = 2$, we find that $c(w_2) = 1$ or $c(w_2) = 3$. If $c(w_2) = 1$, then we change the color of v_2 into 3, contradicting (1). Hence, $c(w_2) = 3$. Then, as $c(v_1) = 1$, we obtain $c(w_1) = 2$. However, we can now change the colors of v_1 and v_2 into 3 and 1, respectively, again contradicting (1). We conclude that $c(w_1) = c(w_2)$. Hence, $c(w_1) = c(w_2) = 3$. By the same arguments, we find that $c(w_3) = c(w_4)$. Hence, $c(w_3) = c(w_4) = 2$. This proves (2).

The facts that $w_1 \neq w_2$ and $w_3 \neq w_4$ together with Claim (2) imply that w_1, w_2, w_3, w_4 are four distinct vertices. We observe that $d_G(w_j) = 3$ for $j = 1, \ldots, 4$, as otherwise H_3 is a subgraph of G. See Fig. 7 for an illustration. In this figure we also indicate that w_1, w_2 have neighbors colored with colors 1 and 2, and that w_3, w_4 have neighbors colored with colors 1 and 3, as otherwise we could recolor $w_1, \ldots, w_4$ such that $c(w_1) \neq c(w_2)$ or $c(w_3) \neq c(w_4)$, and hence we would contradict Claim (2). We may also assume without loss of generality that c is chosen in such a way that the set of vertices with color 1 is maximal, i.e., each vertex with color 2 or 3 has a neighbor with color 1.

Consider the subgraph Q of $G-u$ induced by the vertices colored with colors 2 and 3. We claim that the vertices w_1 and v_2 are in the same connected component of Q. To show this, suppose that there is a connected component Q' of Q that contains w_1 but not v_2. Then we recolor all vertices of Q' colored 2 with color 3 and all vertices of Q' colored 3 with color 2. We obtain a 3-coloring of $G-u$ such that w_1 and w_2 are colored by distinct colors, contradicting Claim (2). Using the same arguments, we conclude that w_3 and v_4 are in the same connected component of Q. Now we show that all the vertices w_1, v_2, w_3, v_4 are in the same connected component of Q. Suppose that there is a connected component Q' of Q that contains w_1, v_2 but not w_3, v_4. Then we recolor all vertices of Q' colored 2 with color 3 and all vertices colored 3 with color 2. We obtain a 3-coloring of $G-u$ such that w_1, w_2, w_3, w_4 are colored with the same color, contradicting Claim (2).

We observe that $d_Q(w_1) = d_Q(v_2) = d_Q(w_3) = d_Q(v_4) = 1$. Then, because w_1, v_2, w_3, v_4 belong to the same connected component of Q, we find that Q contains a vertex x with $d_Q(x) \geq 3$.

Let $y_1, \ldots, y_r$ denote the neighbors of x in Q for some $r \geq 3$. Because $y_1, \ldots y_r$ are colored with the same color, they are pairwise non-adjacent. Because $\Delta(G) \leq 4$, we find that $r \leq 4$. First suppose that $r = 4$. Because $d_G(y_1) \geq 3$ as $\delta(G) \geq 3$ and $y_1, \ldots, y_4$ are pairwise non-adjacent, y_1 has at least two neighbors in $V \setminus N_G[x]$. However, then G contains H_3 as a subgraph. This is a contradiction. Now suppose that $r = 3$. Recall that the set of vertices with color 1 is maximal. Hence x is adjacent to a vertex z with color 1. Because G is H_3-free and $d_G(y_i) \geq 3$ for $i = 1, 2, 3$, we find that z is adjacent to y_1, y_2, y_3. However, since $\Delta(G) \leq 4$, this means that $G[N_G[z]]$ is isomorphic to F_2. Consequently, G properly contains F_2. This contradiction completes the proof of Lemma 5. $\square$

3.3 The Cases $H = H_4$ and $H = H_5$

For these cases we replace Rule 4 by a new rule. Let $G = (V, E)$ be a graph and k be an integer.

Rule 4*. If $k \geq 3$ and $V \setminus N_G[u]$ is an independent set for some $u \in V$, take $(G[N_G(u)], k-1)$.

We prove that Coloring can be solved in polynomial time for H_4-free graphs and for H_5-free graphs in the following way. Let $G = (V, E)$ be a graph, and let $k \geq 1$ be an integer. If $k \leq 2$, then Coloring can be solved in polynomial time even for general graphs. Now suppose that $k \geq 3$. Lemma 6 (stated afterward) shows that Rule 4* is correct. Moreover, an application of Rule 4* takes linear time and reduces the number of vertices of G by at least one. Hence, we can replace Rule 4 by Rule 4* in Lemma 1. Due to this, we may assume without loss of generality that G is 2-connected and has $\delta(G) \geq 3$, and moreover, that $V \setminus N_G[u]$ contains at least two adjacent vertices for all $u \in V$. Then Lemma 7 tells us that $\Delta(G) \leq 3$. By using Brooks' Theorem (cf. [5]) we find that G is 3-colorable, unless $G = K_4$. Hence, (G, k) is a yes-answer when $k \geq 4$, whereas (G, k) is a yes-answer when $k = 3$ if and only if $G \neq K_4$.

What is left to do is to state and prove Lemmas 6 and 7.

Lemma 6. *Let $k \geq 2$ be an integer, and let u be a vertex of a graph $G = (V, E)$ such that $V \setminus N_G[u]$ is an independent set. Then G is k-colorable if and only if $G[N_G(u)]$ is $(k-1)$-colorable.*

Proof. First suppose that G is k-colorable. Let c be a k-coloring of G. Then the vertices of $N_G(u)$ are colored with at most $k-1$ colors, which are different from $c(u)$. Hence, $G[N_G(u)]$ is $(k-1)$-colorable. Now suppose that $G[N_G(u)]$ is $(k-1)$-colorable. Then we extend this coloring to a k-coloring of G by coloring $V \setminus N_G(u)$ with a new color. $\square$

Lemma 7. *Let $G = (V, E)$ be a 2-connected graph with $\delta(G) \geq 3$ such that $V \setminus N_G[u]$ contains at least two adjacent vertices for all $u \in V$. If G is H_4-free or H_5-free, then $\Delta(G) \leq 3$.*

Proof. Let $G = (V, E)$ be a 2-connected graph with $\delta(G) \geq 3$ such that $V \setminus N_G[u]$ contains at least two adjacent vertices for all $u \in V$. Suppose that G is H_4-free or H_5-free. To obtain a contradiction, assume that G has a vertex u with $d_G(u) \geq 4$. Let $N_G(u) = \{z_1, z_2, z_3, z_4\}$. By our assumption, $V \setminus N_G[u]$ contains two adjacent vertices v and w. We choose v and w in a such way that at least one of them, say v, is adjacent to a vertex in $N_G(u)$, say to z_1.

First suppose that G is H_4-free. However, then the subgraph of G with vertices $u, v, w, z_1, z_2, z_3, z_4$ and edges $uz_1, uz_2, uz_3, uz_4, z_1v, vw$ is isomorphic to H_4. This is a contradiction.

Now suppose that G is H_5-free. Because G is 2-connected, G contains a path P from w to u that neither uses v nor z_1. Let v' be the vertex of P that is in $V \setminus N_G[u]$ and that is adjacent to a neighbor of u, say to z_2. Then the subgraph of G with vertices $u, v, v', z_1, z_2, z_3, z_4$ and edges $uz_1, uz_2, uz_3, uz_4, z_1v, z_2v'$ is isomorphic to H_5. This is a contradiction. $\square$

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