Normal modes of the Earth

For long-period waves (> 100 s period), self-gravity effects become important and we can write the linearized equations of motion as

$$\underbrace{\rho_0 \, \partial_t^2 \underline{u}(r,t) = \nabla \cdot \tau'(r,t) - \rho_0 \nabla \phi'(r,t) - \rho' \nabla \phi_0(r,t)}$$

$$\rho_0 \, \partial_t^2 \underline{u}(r,t) = \nabla \cdot \tau'(r,t) \underbrace{- \rho_0 \nabla \phi'(r,t) - \rho' \nabla \phi_0(r,t)}_{\text{gravity effects}}$$

with $\begin{cases} \nabla^2 \phi' = 4\pi G \rho' & \text{compressible} \\ \nabla^2 \phi' = 0 & \text{incompressible } (\rho' = 0) \end{cases}$

SNREI (Spherically symmetric, Non-Rotating, Elastic Isotropic) Earth

In frequency domain:

$$-\rho_0 \, \omega^2 \underline{u}(r,\omega) = \underline{\nabla} \cdot \underline{\underline{\tau}}'(r,\omega) - \rho_0 \nabla \phi'(r,\omega) - \rho' \nabla \phi_0(r,\omega)$$

We separate time & space and radial & horizontal components

$$\underline{u}(\underline{r},\omega) = \underbrace{U_{lm}(r,\omega)\, \underline{P}_{lm}(\vartheta,\phi) + V_{lm}(r,\omega)\, \underline{B}_{lm}(\vartheta,\phi)}_{\text{spheroidal}} + \underbrace{W_{lm}(r,\omega)\, \underline{C}_{lm}(\vartheta,\phi)}_{\text{toroidal}}$$

$$\phi(\underline{r},\omega) = P_{lm}(r,\omega)\, Y_{lm}(\vartheta,\phi)$$

where $Y_{lm}(\vartheta,\phi)$: spherical harmonics (angular degree l, azimuthal order m)

$\underline{P}_{lm}, \underline{B}_{lm}, \underline{C}_{lm}$: vector spherical harmonics

Assuming small perturbation, i.e., $\underline{\underline{\tau}} = \underline{\underline{\tau}}_0 + \underline{\underline{\tau}}'$ with respect to hydrostatic pressure $\underline{\underline{\tau}}_0 = -p_0 \underline{\underline{I}}$, and density $\rho = \rho_0 + \rho'$ with $\rho' = -\underline{\nabla} \cdot (\rho_0 \underline{u})$ leads to a system

$$\left.\begin{cases} \frac{1}{r^2} \frac{d}{dr} \left[r^2 (\lambda + 2\mu) \dot{U} + \dots \right. & = 0 \\ \frac{1}{r^2} \frac{d}{dr} \left[\mu r^2 (\dot{V} - \dots \right. & = 0 \end{cases}\right\} \text{spheroidal}$$

$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}[\mu r^2(\dot{W} - \ldots & = 0 \quad \} \text{ toroidal} \leftarrow \text{decoupled} \\ \dot{P} + \frac{2}{r}P - \ldots & = -4\pi G \rho U - \ldots \end{cases}$$

For each (l, m), we find a discrete set of eigenfrequencies ω_{nlm} associated with eigenfunctions U_{lm}, V_{lm} and W_{lm}.

Properties of SNREI:

- toroidal-spheroidal decoupling:
  W_{lm} (toroidal) independent of U_{lm} & V_{lm} (spheroidal) and so are eigenfrequencies
  ω^T_{nlm} (toroidal) with n = overtone number
  ω^S_{nlm} (spheroidal) $\quad n=0$ fundamental mode
  $n=1$ 1. overtone
- degeneracy: independent of m $(-l \leq m \leq l)$, i.e., all $2l+1$ modes are the same of angular degree l: ${}_nT_l = W^n_{lm}(r)\, C_{lm}(\vartheta, \varphi)$ multiplet

Rotation, ellipticity and 3D heterogeneity remove the degeneracy
$\Rightarrow$ for each (n, l) there are $2l+1$ singlets with different frequencies

Green's tensor & synthetic seismograms:

Modes $s_k(r)$, $k=(n,l,m)$ and eigenfrequency ω_k form a complete basis such that we can write the Green's tensor as

$$\underline{\underline{G}}(r, r'; t) = \sum_k \frac{1}{\omega_k} \underline{s}_k(r) \underline{s}_k(r') \sin(\omega_k t)$$

"mode summation"