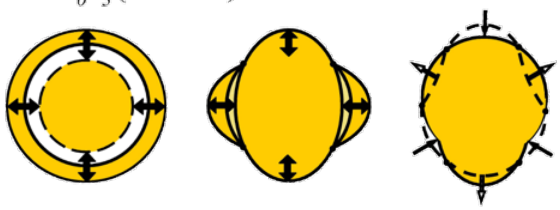
5 Normal modes

Toroidal modes ${}_0T_2$ (44.2 min), ${}_1T_2$ (12.6 min) and ${}_0T_3$ (28.4 min)



Spheroidal modes ${}_0S_0$ (20.5 min), ${}_0S_2$ (53.9 min) and ${}_0S_3$ (25.7 min)

Introduction

So far, we illustrated various approaches to the problem of predicting the displacement of the Earth (calculating theoretical seismograms) in response to a given seismic excitation (earthquake): the forward problem of global seismology.

As the Earth is bounded by an outer surface approximately free of stresses, one approach to solve the said forward problem is to treat it as a boundary-value problem like that of the guitar string; the stress-free surface of the Earth then plays the same role as the fixed ends of a string, and the requirement that the standing-wave solutions that we likewise find satisfy the no-stress condition at the outer surface results in the identification of a discrete set of suitable solutions, with associated eigenfrequencies: the *normal modes* (*free oscillations*) of the Earth.

Normal modes are one of the most popular and successful concepts in global seismology to solve the forward problem, as well as to better understand Earth's deep structure. In the following, we will first introduce normal modes by considering the motions of an elastic string, like the guitar string. We will then focus on the whole Earth and see how normal modes can be used to investigate Earth's subsurface structure.

Learn objectives

- You know how normal modes are generated.
- You can characterize toroidal and spheroidal oscillations.
- You can explain what decoupling and degeneracy mean.
- You are familiar with normal mode splitting.

5.1 Normal Modes of oscillation of an elastic string

Small oscillations of an ideal string obey the wave equation

The simplest elastic medium that I can think of is an ideal string, perfectly elastic and with zero thickness. If I pull it, all points on the string will experience the same *"tension"* (force per unit surface). I am interested in the oscillations of bounded media, like the Earth or a guitar string: I assume the string's endpoints to be fixed.

This said, let us prescribe an initial *slight* deformation of the string, limited to a single plane, so that its shape at any time t will be entirely described by a scalar function $u(x,t)$, or displacement in the y-direction. Note that, at any point x along the string, $\partial u(x,t)/\partial x \sim sin\,\theta(x)$, where θ is the angle between the string and the x-axis.

Let us consider a portion of the string of infinitesimal length δx. Dubbed T the tension, the net force acting on it in the y-direction must equal

$$F_y = T(x+\delta x) sin\,\theta(x+\delta x)\delta y\,\delta z - T(x) sin\,\theta(x)\delta y\,\delta z \tag{5.1}$$

(remember T is a force per unit surface). After dividing both sides by $\delta y\,\delta z$, the y-component of Newton's law reads

$$T(x+\delta x) sin\,\theta(x+\delta x) - T(x) sin\,\theta(x) = \rho\,\delta x\,\frac{\partial^2 u(x,t)}{\partial t^2}\,, \tag{5.2}$$

where ρ denotes density. Substituting in (5.2) the above expression for $sin\,\theta$,

$$T(x+\delta x)\frac{\partial u(x+\delta x,t)}{\partial x} - T(x)\frac{\partial u(x,t)}{\partial x} = \rho\,\delta x\,\frac{\partial^2 u(x,t)}{\partial t^2}\,. \tag{5.3}$$

After dividing both sides by δx, and remembering that T is constant with respect to x,

$$T\frac{\partial^2 u(x,t)}{\partial x^2} = \rho\,\frac{\partial^2 u(x,t)}{\partial t^2}\,. \tag{5.4}$$

The small oscillations of an ideal string obey the wave equation.

Solution of the boundary value problem in terms of modes

We look for a solution of (5.4) satisfying the boundary conditions $u(0,t)=0$, $u(L,t)=0$, where L is the length of the string.

Equation (5.4) can be solved by separation of variables. Let us introduce the unknown functions $\xi(x)$ and $\sigma(t)$ such that $u(x,t)=\xi(x)\times\sigma(t)$. Substituting into (5.4) and dividing both sides by $\xi(x)\times\sigma(t)$,

$$c^2\,\frac{\ddot{\xi}(x)}{\xi(x)} = \frac{\ddot{\sigma}(t)}{\sigma(t)}\,, \tag{5.5}$$

where I have defined the positive constant $c^2 = T/\rho$.

The left hand side depends on x only; the right hand side on t only. They have to coincide at every x and every t. It follows that each must equal the same constant, that I dub λ. The two following equations then hold,

$$\ddot{\xi}(x) = \frac{\lambda}{c^2}\,\xi(x)\,, \tag{5.6}$$

$$\ddot{\sigma}(t) = \lambda\,\sigma(t)\,. \tag{5.7}$$

Solution $\xi(x)$ of the "spatial" term

Regardless of the values taken by the unknown coefficients A and B, equation (5.6) is solved for example by

$$\xi(x) = A\exp\left(\frac{\sqrt{\lambda}}{c}\,x\right) + B\exp\left(-\frac{\sqrt{\lambda}}{c}\,x\right) \tag{5.8}$$

(please verify it by substitution of (5.8) into (5.6)). Because they are independent of t, the boundary conditions can be written $\xi(0) = \xi(L) = 0$.

For solutions other than the obvious one, $A = B = 0$ hence $\xi(x) = 0$, to be possible, $\lambda < 0$ is required in (5.8) (again, please verify it by trying to find values of A and B that satisfy the boundary conditions with $\lambda \geq 0$). We then rewrite (5.8),

$$\xi(x) = A\exp\left(i\,\frac{\sqrt{|\lambda|}}{c}\,x\right) + B\exp\left(-i\,\frac{\sqrt{|\lambda|}}{c}\,x\right)\,, \tag{5.9}$$

where i is the imaginary unit. After some algebra,

$$\xi(x) = a\cos\left(\frac{\sqrt{|\lambda|}}{c}\,x\right) + b\sin\left(\frac{\sqrt{|\lambda|}}{c}\,x\right)\,, \tag{5.10}$$

where I have introduced $a = A + B$ and $b = i(B - A)$.

Can I find values of a, b and λ that satisfy the boundary conditions $\xi(0) = \xi(L) = 0$? $\xi(0) = 0$ clearly requires $a = 0$. The other boundary condition reads

$$b\sin\left(\frac{\sqrt{|\lambda|}}{c}\,L\right) = 0\,, \tag{5.11}$$

which implies

$$\frac{\sqrt{|\lambda|}}{c}\,L = k\pi \quad (k = 1, 2, 3, ...)\,. \tag{5.12}$$

The *discrete* set of values of λ,

$$\lambda = \lambda_k = -\left(\frac{k\pi c}{L}\right)^2 \quad (k = 1, 2, 3, ...)\,, \tag{5.13}$$

that can be calculated if ρ, T and L are given, substituted into (5.10) with $a = 0$ provide the discrete set of acceptable solutions of (5.6):

$$\xi_k(x) = b\sin\left(\frac{k\pi}{L}\,x\right)\,. \tag{5.14}$$

The coefficient b remains arbitrary, and will need to be constrained by the initial conditions (the initial pull applied to the string).

Solution $\sigma(t)$ of the "temporal" term

Now that I know that the values of λ are limited to the discrete set (5.13), I might as well rewrite equation (5.7) for each k,

$$\ddot{\sigma}(t) = -\left(\frac{k\pi c}{L}\right)^2 \sigma(t)\,. \tag{5.15}$$

The parameters k, π, c and L are all real; the coefficient of $\sigma(t)$ is real and negative; (5.15) for a given value of k is then solved by

$$\sigma(t) = \sigma_k(t) = A'\mathrm{exp}\left(i\,\frac{k\pi c}{L}\,t\right) + B'\mathrm{exp}\left(-i\,\frac{k\pi c}{L}\,t\right) \tag{5.16}$$

(A' and B' are arbitrary coefficients), or, in analogy with the procedure I used above to solve for $\xi(x)$,

$$\sigma(t) = \sigma_k(t) = a'\cos\left(\frac{k\pi c}{L}\,t\right) + b'\sin\left(\frac{k\pi c}{L}\,t\right)\,, \tag{5.17}$$

with a' and b' simply related to A' and B'.

Standing waves

For each k we now have a particular solution to equation (5.4),

$$u_k(x,t) = \xi_k(x)\sigma_k(t) = \alpha\cos\left(\frac{k\pi c}{L}\,t\right)\sin\left(\frac{k\pi}{L}\,x\right) + \beta\sin\left(\frac{k\pi c}{L}\,t\right)\sin\left(\frac{k\pi}{L}\,x\right)\,, \tag{5.18}$$

where $\alpha = a'b$ and $\beta = b'b$.

I omit here the proofs of two important properties of $\xi_k(x)$ and $u_k(x,t)$, namely: (i) that the functions $\xi_k(x)$ are orthogonal, i.e.

$$\int_0^L \xi_i(x)\xi_j(x)\mathrm{d}x = \int_0^L \sin\left(\frac{i\pi}{L}\,x\right)\sin\left(\frac{j\pi}{L}\,x\right)\mathrm{d}x = \frac{L}{2}\delta_{ij}\,, \tag{5.19}$$

and (ii) that any possible solution to (5.4) can be written as a linear combination of the functions $u_k(x,t)$,

$$u(x,t) = \sum_{\mathrm{k}=1}^{\infty}\left[\alpha_k\cos\left(\frac{k\pi c}{L}\,t\right)\sin\left(\frac{k\pi}{L}\,x\right) + \beta_k\sin\left(\frac{k\pi c}{L}\,t\right)\sin\left(\frac{k\pi}{L}\,x\right)\right]\,. \tag{5.20}$$

Consider now the k-th mode $\xi_k(x) = sin\left(\frac{k\pi}{L}x\right)$. Since $sin(n\pi) = 0$ for all integer values of n, $\xi_k(x) = 0$ at $x = \frac{nL}{k}$, for all integer values of n between 0 and L. Therefore, $u_k(L/(k\pi), t) = 0$ at $x = \frac{nL}{k}$ ($n = 0, 1, 2, ..., k$) at all times t. The points $x = \frac{nL}{k}$ are called "nodes"; oscillatory functions like $u_k(x,t)$ are called *standing waves*: to distinguish them from *traveling waves*, that have no nodes as they propagate coherently in space, without changing their shape.

Starting from the *Ansatz* $u(x,t) = \xi(x)\sigma(t)$, I ended up finding a standing wave solution to equation (5.4). But remember that differential equations are not necessarily solved in only one way. I could have started from the traveling wave *Ansatz* $u = \exp[i(\gamma x - \omega t)]$; if you replace it into (5.4), you'll see that it provides a solution if $\omega^2/\gamma^2 = \rho/T$; here is another spectrum of solutions, from which we could select those that satisfy the boundary conditions.

Equation (5.4) can be solved indifferently in terms of either standing or traveling waves. In both cases we find a complete set of solutions. In other words, the actual displacement of the string for a prescribed initial pull $u(x,0)$ can be equivalently written as a linear combination of either standing or traveling waves. The same thing happens to the Earth.

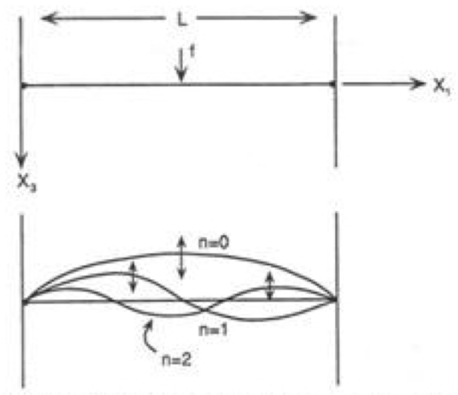


Figure 5.1: Geometry of a string under tension. Motions of the string excited by any source comprise a weighted sum of the eigenfunctions, which are solutions that satisfy the boundary conditions with discrete eigenfrequencies. The first three eigenfunctions are shown in the bottom panel. (From *Lay & Wallace*)

In analogy with the Earth, I call "normal mode" of the string each combination of a function $\xi_k(x) = sin(\frac{k\pi}{L}x)$ with the corresponding *eigenfrequency* $\frac{k\pi c}{L}$.

Response of the string to excitation

We can use the above formulation to predict the oscillations of the string, when set into motion by a given excitation. Depending on the data provided to us, the forcing term could be described by a force, to be added to F_y in eq. (5.1), or by initial conditions on displacement and velocity. Let us follow the second approach, and prescribe

$$u(x,0) = g(x)\,, \tag{5.21}$$

$$\dot{u}(x,0) = f(x)\,, \tag{5.22}$$

where the functions $g(x)$ and $f(x)$ are known. The conditions (5.21) and (5.22) determine $u(x,t)$ uniquely: after replacing u and $\dot{u}$ with (5.20) and its time-derivative, respectively, and evaluating the resulting expressions at $t = 0$, (5.21) and (5.22) become

$$\sum_{\mathrm{k}=1}^{\infty}\left[\alpha_k \sin\left(\frac{k\pi}{L}x\right)\right] = g(x)\,, \tag{5.23}$$

$$\sum_{\mathrm{k=1}}^{\infty}\left[\beta_k\,\frac{k\pi c}{L}\sin\left(\frac{k\pi}{L}\,x\right)\right]=f(x)\,. \tag{5.24}$$

The coefficients α_n and β_n, for any value of n, can now be determined multiplying both sides of both equations by $\sin(n\pi x/L)$, integrating over x from 0 to L, and making use of the orthogonality (5.19). Namely,

$$\int_0^L\sum_{\mathrm{k=1}}^{\infty}\left[\alpha_k\sin\left(\frac{k\pi}{L}\,x\right)\right]\sin\left(\frac{n\pi}{L}\,x\right)\mathrm{d}x=\int_0^L g(x)\sin\left(\frac{n\pi}{L}\,x\right)\mathrm{d}x\,, \tag{5.25}$$

$$\int_0^L\sum_{\mathrm{k=1}}^{\infty}\left[\beta_k\,\frac{k\pi c}{L}\sin\left(\frac{k\pi}{L}\,x\right)\right]\sin\left(\frac{n\pi}{L}\,x\right)\mathrm{d}x=\int_0^L f(x)\sin\left(\frac{n\pi}{L}\,x\right)\mathrm{d}x\,, \tag{5.26}$$

which, applying (5.19), can be simplified to

$$\sum_{\mathrm{k=1}}^{\infty}\left[\alpha_k\,\delta_{kn}\frac{L}{2}\right]=\int_0^L g(x)\sin\left(\frac{n\pi}{L}\,x\right)\mathrm{d}x\,, \tag{5.27}$$

$$\sum_{\mathrm{k=1}}^{\infty}\left[\beta_k\,\frac{k\pi c}{2}\delta_{kn}\right]=\int_0^L f(x)\sin\left(\frac{n\pi}{L}\,x\right)\mathrm{d}x\,, \tag{5.28}$$

or, after applying the properties of δ_{kn},

$$\alpha_n=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\,x\right)\mathrm{d}x\,, \tag{5.29}$$

$$\beta_n=\frac{2}{n\pi c}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\,x\right)\mathrm{d}x\,, \tag{5.30}$$

where in general it will be possible to calculate numerically the values of the integrals at the right hand sides. The resulting values for α_k and β_k can then be substituted into (5.20), and the displacement of the string will be determined at any time t and location x along the string.

The method followed here could be also used to solve the Green's problem associated with the string: just define $g(x)$ and $f(x)$ to represent an impulsive forcing in time and space. The resulting Green's function could then be used to determine the response of the string to any excitation, however complicated. This particular case, however, is so simple that the Green's problem approach is not particularly beneficial.

5.2 Normal Modes of the Earth

Self-gravitation and linearization of the momentum equation

Gravity

So far we have written the Earth's equation of motion (momentum conservation) in a simplified, approximate form, neglecting the effects of gravity and the perturbations to the gravity field caused by the Earth's deformation $\mathbf{u}(\mathbf{r},t)$ itself (*"self-gravitation"*). Those terms, however, turn out to be significant, and should not be neglected in a realistic simulation.

The acceleration felt at any point in space $\mathbf{r}$ due to the gravitational attraction of a distribution of mass $\rho(\mathbf{r}')$ is given by Newton's law of gravitation,

$$\ddot{\mathbf{u}}(\mathbf{r},t) = \int_V \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} G\rho(\mathbf{r}')\mathrm{d}^3\mathbf{r}', \tag{5.31}$$

with G denoting Newton's constant. Equation (5.31) is then simplified introducing a *gravitational potential* $\Phi(\mathbf{r})$ such that

$$\nabla\Phi(\mathbf{r},t) = -\ddot{\mathbf{u}}(\mathbf{r},t). \tag{5.32}$$

Then, taking the divergence of (5.31),

$$\nabla^2\Phi(\mathbf{r},t) = -\int_V \nabla\cdot\left[\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}\right] G\rho(\mathbf{r}')\mathrm{d}^3\mathbf{r}'. \tag{5.33}$$

Note that, for $\mathbf{r}' \neq \mathbf{r}$, we can write

$$\begin{aligned}
\nabla\cdot\left[\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}\right] &= \\
&= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left[\frac{(x'-x,y'-y,z'-z)}{|\mathbf{r}'-\mathbf{r}|^3}\right] \\
&= \frac{\partial}{\partial x}\left(\frac{x'-x}{|\mathbf{r}'-\mathbf{r}|^3}\right)+\frac{\partial}{\partial y}\left(\frac{y'-y}{|\mathbf{r}'-\mathbf{r}|^3}\right)+\frac{\partial}{\partial z}\left(\frac{z'-z}{|\mathbf{r}'-\mathbf{r}|^3}\right) \\
&= -\frac{3}{|\mathbf{r}'-\mathbf{r}|^3}+(x'-x)\frac{\partial}{\partial x}\left\{\frac{1}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{3/2}}\right\}+\text{etc....} \\
&= -\frac{3}{|\mathbf{r}'-\mathbf{r}|^3}+3\frac{|\mathbf{r}'-\mathbf{r}|^2}{|\mathbf{r}'-\mathbf{r}|^5},
\end{aligned} \tag{5.34}$$

which equals 0 everywhere except at $\mathbf{r}'=\mathbf{r}$. This means that the integral at the right hand side of (5.33) can be calculated over an infinitely small, spherical volume centered on $\mathbf{r}'=\mathbf{r}$; the *divergence theorem* (earlier lecture) is applied to this integral, noting that the unit-vector perpendicular to the surface of the sphere $\hat{\mathbf{n}} = \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|}$, and the surface element on the spherical surface $\mathrm{d}^2\mathbf{r}' = |\mathbf{r}'-\mathbf{r}|^2\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$, with θ and ϕ spherical coordinates. $\rho(\mathbf{r}')$ is approximately constant and equal to $\rho(\mathbf{r})$ within the sphere in question. Poisson's equation follows,

$$\nabla^2\Phi = 4\pi G\rho. \tag{5.35}$$

We account for gravity including in the usual equation of motion a body-force-density term equal to ρ times the acceleration (5.32), i.e.

$$\rho\frac{\partial^2}{\partial t^2}\mathbf{u}(\mathbf{r},t) = \nabla\cdot\boldsymbol{\tau}(\mathbf{r},t) - \rho\nabla\Phi(\mathbf{r},t). \tag{5.36}$$

Notice that in both (5.35) and (5.36) ρ changes in time as a result of the deformations $\mathbf{u}(\mathbf{r},t)$. (Compare (5.36) with (2.93) or (2.117) of *Dahlen & Tromp*, keeping in mind that Coriolis and centrifugal forces have a smaller effect and can be neglected.)

Linearization

In normal-mode literature it is then customary to simplify equations (5.36) and (5.35) by *linearization*, that is to say considering that the quantities ρ, Φ, $\boldsymbol{\tau}$ can reasonably be written

$$\rho = \rho_0 + \rho' \tag{5.37}$$

$$\Phi = \Phi_0 + \Phi' \tag{5.38}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \boldsymbol{\tau}' \tag{5.39}$$

where ρ', Φ', $\boldsymbol{\tau}$' are small perturbations with respect to the initial values ρ_0, Φ_0, $\boldsymbol{\tau}_0$. After substituting (5.37) through (5.39) into (5.36) and (5.35), neglecting second-order quantities and making use of the static equilibrium condition $\nabla \cdot \boldsymbol{\tau}(\mathbf{r}, t) = \rho\nabla\Phi(\mathbf{r}, t)$, one finds the linearized momentum equation

$$\rho_0 \frac{\partial^2}{\partial t^2}\mathbf{u}(\mathbf{r}, t) = \nabla \cdot \boldsymbol{\tau}'(\mathbf{r}, t) - \rho_0\nabla\Phi'(\mathbf{r}, t) - \rho'\nabla\Phi_0(\mathbf{r}, t) \tag{5.40}$$

(compare with eq. (3.56) or (3.60) of *Dahlen & Tromp*), and linearized Poisson's equation

$$\nabla^2\Phi' = 4\pi G\rho' \tag{5.41}$$

(sometimes further simplified with the incompressibility assumption $\rho' = 0$).

5.3 Spherically Symmetric, Non-Rotating, Elastic and Isotropic (SNREI) Earth

Models of this type provide a first approximation, relatively realistic and relatively simple, to the free oscillations of the true (slightly aspherical, slightly anisotropic, slightly anelastic) Earth. The assumptions of elasticity and isotropy are implicit in our choice of the constitutive relation between stress and deformation (see previous lectures); the neglect of *lateral* heterogeneities comes into play when horizontal gradients of ρ, λ, μ and the like are neglected, to find a simple, explicit form for the momentum and Poisson's equations (i.e., eqs. (5.50) through (5.53) below).

In analogy with most textbooks, let us Fourier-transform the Earth's linearized equation of motion (5.40) to the frequency domain,

$$-\rho_0\omega^2\tilde{\mathbf{u}}(\mathbf{r}, \omega) = \nabla \cdot \tilde{\boldsymbol{\tau}}'(\mathbf{r}, \omega) - \rho_0\nabla\Phi'(\mathbf{r}, \omega) - \rho'\nabla\Phi_0(\mathbf{r}, \omega) \tag{5.42}$$

where ω denotes frequency and the superscript $\tilde{}$ (omitted in the following) indicates the Fourier-transform of a function. Poisson's equation (5.41) holds the same form in the frequency domain.

Fourier-transforming is in a sense equivalent to separating the time-dependence from the space-dependence of the solution, like we did in the case of the guitar string; a standing-wave description of the solution follows naturally.

In practice, the *Ansatz*

$$\mathbf{u}(\mathbf{r},\omega) = U_{lm}(r,\omega)\mathbf{P}_{lm}(\theta,\phi) + V_{lm}(r,\omega)\mathbf{B}_{lm}(\theta,\phi) + W_{lm}(r,\omega)\mathbf{C}_{lm}(\theta,\phi) \tag{5.43}$$

(*no summation* over l, m) is substituted into (5.42), and

$$\Phi(\mathbf{r},\omega) = P_{lm}(r,\omega)Y_{lm}(\theta,\phi) \tag{5.44}$$

is substituted into the frequency-domain version of (5.41) (*Dahlen & Tromp*, section 8.6.1).

For practical reasons we now employ a system of spherical-polar coordinates, $\mathbf{r} = r, \theta, \phi$, with its origin at the center of the Earth. *Vector spherical harmonics* are defined

$$\mathbf{P}_{lm}(\theta,\phi) = \hat{\mathbf{r}}Y_{lm}(\theta,\phi), \tag{5.45}$$

$$\mathbf{B}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\boldsymbol{\nabla}_1 Y_{lm}(\theta,\phi), \tag{5.46}$$

$$\mathbf{C}_{lm}(\theta,\phi) = -\frac{1}{\sqrt{l(l+1)}}\left(\hat{\mathbf{r}}\times\nabla_1\right)Y_{lm}(\theta,\phi), \tag{5.47}$$

with $Y_{lm}(\theta,\phi)$ scalar spherical harmonics, and $\hat{\mathbf{r}}$ the unit vector perpendicular to the unit sphere, directed outwards. The *surface gradient* ∇_1 can be defined directly,

$$\nabla_1 = \hat{\boldsymbol{\theta}}\,\partial_\theta + \hat{\boldsymbol{\phi}}\,\frac{1}{\sin(\theta)}\partial_\phi \tag{5.48}$$

or by specifying its relation to the gradient ∇,

$$\nabla = \hat{\mathbf{r}}\partial_r + \frac{1}{r}\nabla_1. \tag{5.49}$$

The *Ansatz* (5.43), (5.44) is chosen so that the problem of finding two functions of three variables, $\mathbf{u}(r,\theta,\phi)$ and $\Phi(r,\theta,\phi)$, satisfying (5.42) and (5.41), is replaced by that of integrating, for each couple of values of l and m, a system of four second-order differential equations for $P_{lm}(r)$, $U_{lm}(r)$, $V_{lm}(r)$, and $W_{lm}(r)$ (not depending on θ or ϕ). The latter endeavour, in fact, turns out to be much easier.

Substituting (5.43) into (5.42) involves writing $\boldsymbol{\tau}'$ in terms of $\mathbf{u}$: the assumption is usually made that the initial stress $\boldsymbol{\tau}_0$ be purely hydrostatic ($\boldsymbol{\tau}_0 = -p_0\mathbf{I}$ with p_0 hydrostatic pressure), and the constitutive relation (see previous lectures) is accordingly corrected. Likewise, based on physical considerations, perturbations ρ' in density can be written in terms of unperturbed density ρ_0 and displacement $\mathbf{u}$, $\rho' = -\nabla\cdot(\rho_0\mathbf{u})$. Then, making use of the orthogonality properties of vector spherical harmonics, (5.42) and (5.41) can be written

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2(\lambda+2\mu)\dot{U} + \lambda r(2U - \sqrt{l(l+1)}V)\right] + \frac{1}{r}\left[(\lambda+2\mu)\dot{U} + \frac{\lambda}{r}\left(2U - \sqrt{l(l+1)}V\right)\right]$$

$$-3\left(\lambda+\frac{2}{3}\mu\right)\frac{1}{r}\left(\dot{U} + \frac{2}{r}U - \frac{\sqrt{l(l+1)}}{r}V\right) - \sqrt{l(l+1)}\frac{\mu}{r}\left(\dot{V} - \frac{1}{r}V + \frac{\sqrt{l(l+1)}}{r}U\right)$$

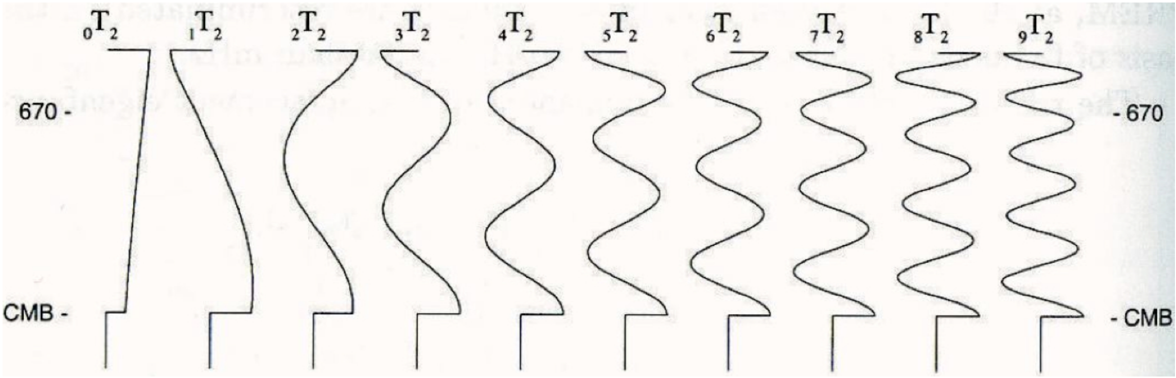


Figure 5.2: Radial eigenfunctions W^n_{lm} of the first ten "degree-two" ($l = 2$) toroidal modes in PREM. Vertical axis is depth beneath the surface of the Earth; the location of the "670" km discontinuity and core mantle boundary ("CMB") are indicated. The acronym $_nT_l$ identifies the mode, "T" standing for "toroidal". Figures included in this document are taken from chapter 8 of *Dahlen and Tromp.*

$$+\,\omega^2\rho U - \rho\left[\dot{P} + (4\pi G\rho - \frac{4g}{r})U + \frac{\sqrt{l(l+1)}g}{r}V\right] = 0, \tag{5.50}$$

$$\begin{aligned}
&\frac{1}{r^2}\frac{d}{dr}\left[\mu r^2\left(\dot{V} - \frac{1}{r}V - \frac{\sqrt{l(l+1)}}{r}U\right)\right] + \frac{\mu}{r}\left(\dot{V} - \frac{1}{r}V - \frac{\sqrt{l(l+1)}}{r}U\right)\\
&+ \frac{\sqrt{l(l+1)}\lambda}{r}\dot{U} + \frac{\sqrt{l(l+1)}\lambda + \mu}{r^2}\left(2U - \sqrt{l(l+1)}V\right)\\
&- \left[\omega^2\rho - (l(l+1) - 2)\frac{\mu}{r^2}\right]V - \frac{\sqrt{l(l+1)}\rho}{r}(P + gU) = 0,
\end{aligned} \tag{5.51}$$

$$\frac{1}{r^2}\frac{d}{dr}\left[\mu r^2\left(\dot{W} - \frac{W}{r}\right)\right] + \frac{\mu}{r}\left(\dot{W} - \frac{W}{r}\right) + \left[\omega^2\rho - \left(\sqrt{l(l+1)} - 2\right)\frac{\mu}{r^2}\right]W = 0, \tag{5.52}$$

$$\ddot{P} + \frac{2}{r}\dot{P} - \frac{l(l+1)}{r^2}P = -4\pi G\dot{\rho}U - 4\pi G\rho\left(\dot{U} + \frac{2U - \sqrt{l(l+1)}V}{r}\right), \tag{5.53}$$

where the superscript $\dot{}$ now denotes derivation with respect to r, $g = \frac{\partial}{\partial r}\Phi$, $\rho = \rho_0$, and the subscript $_{lm}$ was omitted from U_{lm} etc. for brevity. Compare with equations (8.43) through (8.45), and (8.53) from *Dahlen & Tromp.*

From these cumbersome equations two important results are apparent. They are referred to as *toroidal-spheroidal decoupling* and *degeneracy.*

Toroidal-spheroidal decoupling

While to find U_{lm}, V_{lm} and P_{lm} we must integrate simultaneously (5.50), (5.51) and (5.53), equation (5.52) can be solved independently for W_{lm}, which does not appear in the other radial equations: we say that (5.52) is *decoupled* from (5.50), (5.51) and (5.53). Inspection of equations (5.43) and (5.47) shows that displacements associated with W_{lm} are parallel to the surface of the Earth, or *toroidal*; displacements associated with U_{lm} and V_{lm} are called, instead, *spheroidal*; hence the expression *toroidal-spheroidal decoupling* [1].

In a realistic, spherically symmetric Earth model the radial equations (5.50) through (5.52) are typically solved *numerically*. As boundary conditions (regularity of the solution at the center of the Earth; no stresses acting on the outer surface) are imposed, for each couple l, m a discrete set of acceptable values of ω (eigenfrequencies) is determined. Spheroidal and toroidal equations are solved separately and independently, and give rise to two independent sets of eigenfrequencies, dubbed *spheroidal* and *toroidal* eigenfrequencies, and denoted ω^S_{nlm} and ω^T_{nlm}, respectively. Remark that we have introduced a third index n, to distinguish the infinite possible eigenfrequencies associated with a given couple l, m.

Each *spheroidal* eigenfrequency ω^S_{nlm} is associated with a couple of spheroidal eigenfunctions $U^n_{lm}(r)$, $V^n_{lm}(r)$, solutions of (5.50), (5.51) with $\omega = \omega^S_{nlm}$. Likewise, each *toroidal* eigenfrequency ω^T_{nlm} is associated with one toroidal eigenfunction $W^n_{lm}(r)$, solution of (5.52) with $\omega = \omega^T_{nlm}$.

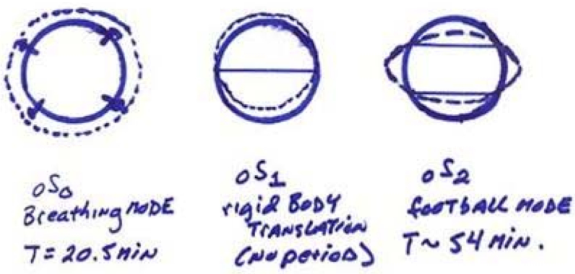


Figure 5.3: Examples of the vibrations associated with different **spheroidal** modes. (From http://web.ics.purdue.edu/∼nowack/eas557.html)

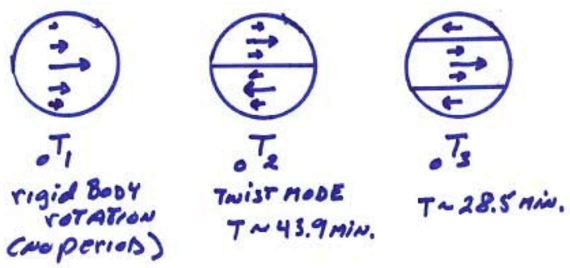


Figure 5.4: Examples of the vibrations associated with different **toroidal** modes. (From http://web.ics.purdue.edu/∼nowack/eas557.html)

The eigenfunctions $U^n_{lm}(r)$, $V^n_{lm}(r)$ with eigenfrequencies ω^S_{nlm}, or $W^n_{lm}(r)$, with eigenfrequency ω^T_{nlm}, correspond in the elastic-string example to the eigenfunctions $\xi_k(x)$, with eigenfrequencies $k\pi c/L$ (notation defined in previous lecture).

Modes associated with $n = 0$ are called *fundamental modes*. Modes associated with $n > 0$ are called *overtones*. Like in the elastic-string case, the number of "nodes" of a mode coincides

[1] Like the equations of motion, boundary conditions at the Earth's surface decouple into two equations involving U, V, and one equation involving W: see *Dahlen & Tromp*, equations (8.46) through (8.52).

with its *overtone number* n (see, for example, the toroidal eigenfunctions in figure 5.2).

Degeneracy and splitting

Unlike the harmonic degree l, the harmonic order m does *not* appear anywhere in eqs. (5.50) through (5.53). This means that the eigenfunctions U^n_{lm}, V^n_{lm}, W^n_{lm} and P^n_{lm}, and eigenvalues ω^T_{nlm} and ω^S_{nlm} are constant with respect to m. The set of $2l+1$ modes associated with a given value of l (and all integer values of m from $-l$ to $+l$) form a *multiplet*.

Degeneracy is limited to spherically symmetric (laterally homogeneous) Earth models, which are only realistic in the first approximation. When the normal modes of more realistic, slightly aspherical (laterally heterogeneous) Earth models are computed, modes and eigenfrequencies become m-dependent: an effect known as *splitting* (see for example the split eigenfrequency of mode ${}_1S_4$ in figure 5.6).

5.4 Green's tensor and synthetic seismograms

It can be shown that the eigenfunctions $U^n_{lm}(r)\mathbf{P}_{lm}(\theta,\phi)+V^n_{lm}(r)\mathbf{B}_{lm}(\theta,\phi)$ and $W^n_{lm}(r)\mathbf{C}_{lm}(\theta,\phi)$ form a *complete basis* of solutions to the boundary problem associated with (5.42) plus the no-stress boundary conditions at the outer surface, i.e. any displacement field of an Earth with stress-free outer surface can be written as their linear combination. In the following, for simplicity and in analogy with *Dahlen & Tromp*, we shall collapse the three indexes n, l, m to one index k, also used to distinguish spheroidal and toroidal modes, so that we can write all eigenfunctions indifferently as $\mathbf{s}_k(\mathbf{r})$.

It can also be shown that $\mathbf{s}_k(\mathbf{r})$ are orthonormal, or

$$\int_V \rho_0 \mathbf{s}_k(\mathbf{r}) \cdot \mathbf{s}_{k'}(\mathbf{r}) \mathrm{d}V = \delta_{kk'}. \tag{5.54}$$

We now want to use normal-mode formalism to build a mathematical tool that, given the mechanism of an earthquake (geometry of the fault and slip on the fault), will provide us with the displacement field anywhere in the Earth, at any time following the earthquake. As we have learned in a previous lecture, thanks to Betti's theorem this can be done by solving the Green's problem associated with the Earth, i.e. finding the response of the Earth to an impulsive excitation. Previously, we have represented the impulsive excitation with a force density included in the momentum equation; equivalently, it can be represented as an initial condition on the displacement. Said $\mathbf{G}(\mathbf{r},\mathbf{r}';t)$ the Green's tensor (again in the *time domain*), solving the Green's problem consists of solving eq. (5.40) after replacing $\mathbf{u}$ with $\mathbf{G}$, and with initial conditions corresponding to an impulsive source at $t=0$,

$$\mathbf{G}(\mathbf{r},\mathbf{r}';0) = \mathbf{0}, \tag{5.55}$$

$$\frac{\partial}{\partial t}\mathbf{G}(\mathbf{r},\mathbf{r}';0) = \frac{1}{\rho_0}\mathbf{I}\delta(\mathbf{r}-\mathbf{r}'). \tag{5.56}$$

Because the modes form a complete basis of eigensolutions, $\mathbf{G}$ can be written as a linear combination of modes,

$$\mathbf{G}(\mathbf{r}, \mathbf{r}'; t) = \sum_k \mathbf{s}_k(\mathbf{r}) \left[\mathbf{a}_k(\mathbf{r}') \cos(\omega_k t) + \mathbf{b}_k(\mathbf{r}') \sin(\omega_k t) \right]. \tag{5.57}$$

This is often referred to as *normal mode summation.*

Substituting (5.57) into (5.55) and (5.56) we find, respectively,

$$\sum_k \mathbf{s}_k(\mathbf{r}) \mathbf{a}_k = \mathbf{0}, \tag{5.58}$$

$$\sum_k \omega_k \mathbf{s}_k(\mathbf{r}) \mathbf{b}_k = \frac{1}{\rho_0} \mathbf{I} \delta(\mathbf{r} - \mathbf{r}'). \tag{5.59}$$

After multiplying both sides by $\mathbf{s}_k(\mathbf{r})$, integrating over the Earth's volume V, and making use of the orthonormality relation (5.54), (5.58) and (5.59) become

$$\mathbf{a}_k = \mathbf{0} \tag{5.60}$$

for all values of k, and

$$\mathbf{b}_k = \frac{1}{\omega_k} \mathbf{s}_k(\mathbf{r}'). \tag{5.61}$$

The Green's tensor of a non-rotating Earth is therefore given in terms of the normal-mode eigenfrequencies and eigenfunctions by

$$\mathbf{G}(\mathbf{r}, \mathbf{r}'; t) = \sum_k \frac{1}{\omega_k} \mathbf{s}_k(\mathbf{r}) \mathbf{s}_k(\mathbf{r}') \sin(\omega_k t). \tag{5.62}$$

After $\mathbf{G}$ has been so determined, the response of the Earth model in question to any seismic excitation can be modeled via Betti's theorem (numerically in general, analytically in some simplified cases).

5.5 Lateral heterogeneities and splitting functions.

The displacement of a slightly laterally heterogeneous ("perturbed") Earth can be written as a linear combination of the modes of a spherically symmetric ("unperturbed") Earth. For this result to be general, all unperturbed-Earth modes have to be considered.

One can assume, however, that the contribution of one multiplet to the displacement in the perturbed Earth can be written as the linear combination of unperturbed-Earth modes within that multiplet only (if this is true, the multiplet is said to be *"isolated"*).

In this assumption, the contribution of an isolated multiplet to the displacement in the perturbed Earth is written[2]

$$u(t) = \mathrm{Re} \left[\mathsf{r}^T \cdot \exp(\mathrm{i}\mathsf{H}t) \cdot \mathsf{s} \exp(\mathrm{i}\omega t) \right], \tag{5.63}$$

where Re indicates that only the real part of the expression at the right hand side is con-

[2] Eq. (1) of *He and Tromp* 1996, "Normal-mode constraints on the structure of the Earth", after neglecting the Earth's anelastic damping.

sidered; ω is the degenerate eigenfrequency of the multiplet in the unperturbed Earth; the vectors s and r, dubbed source vector and receiver vector, respectively, have $2l+1$ entries each; their m-th entries are defined $\mathbf{M} : \boldsymbol{\nabla}\mathbf{u}_m$ (source) and $\boldsymbol{\nu} \cdot \mathbf{u}_m$ (receiver). $\boldsymbol{\nu}$ is the polarization vector (notice eq. (5.63) is scalar: we consider only one component of the seismogram at a time) and the tensor $\mathbf{M}$ represents the effect of the magnitude and geometry of the seismic source.

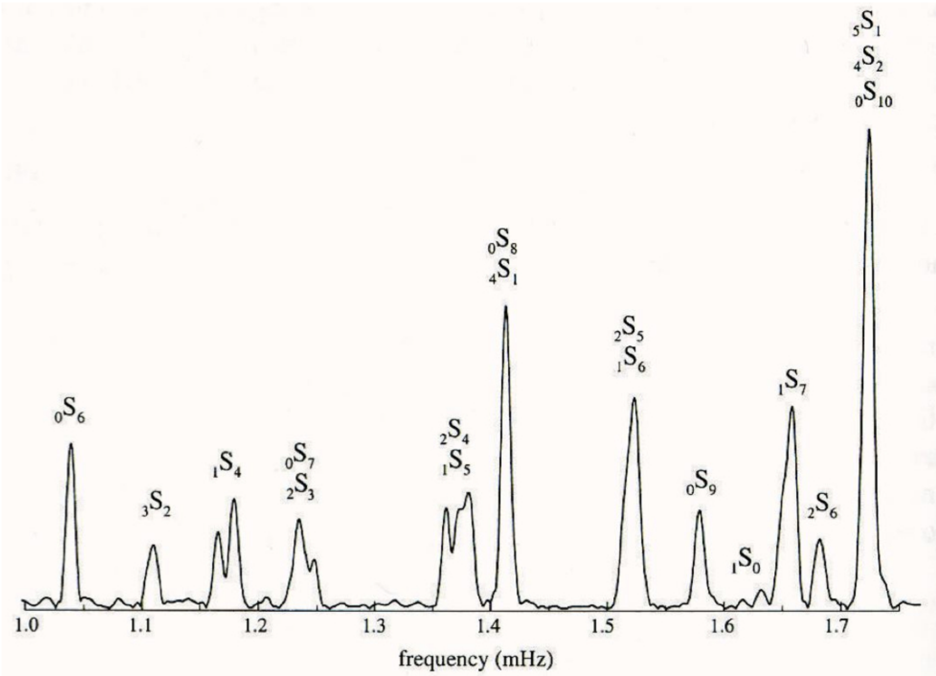


Figure 5.5: The spectrum of the Earth's long-period radial displacement. Measured at station TUC (Tucson, Arizona), after the June 9, 1994 Bolivia earthquake. Note the visible split of mode $_1S_4$. From *Dahlen & Tromp*.

Naturally, most quantities in equation (5.63) depend on the indexes n, l; I have omitted them because only one multiplet (i.e. one value of n, one value of l) is considered. For example, $\mathbf{u}_m = \mathbf{u}_{nlm}$; for a toroidal mode, $\mathbf{u}_m = W_{lm}(r)\mathbf{C}_{lm}(\theta, \phi)$.

The $(2l+1) \times (2l+1)$ matrix $\exp(\mathrm{i}\mathsf{H}t)$ is called *"splitting matrix"*, and governs multiplet splitting. In the interest of simplicity I have written it in an implicit form; I need now to specify[3]

$$\mathsf{H}_{ij} = \omega \sum_{s,t} c_{st} \int_{\Omega} Y_{li}^{*} Y_{st} Y_{lj} \mathrm{d}\Omega, \tag{5.64}$$

where Ω denotes the unit sphere, and the *splitting coefficients*[4]

$$c_{st} = \int_0^a (K_P^{st}\delta v_P^{st} + K_S^{st}\delta v_S^{st} + K_\rho^{st}\delta\rho^{st})\mathrm{d}r. \tag{5.65}$$

δv_P^{st}, δv_S^{st} and $\delta\rho^{st}$ are the st harmonic coefficients of perturbations in the Earth's compressional and shear velocity, and density, respectively; K_P^{st}, K_S^{st} and K_ρ^{st} are the corresponding

[3]Eq. (2) of *He and Tromp* 1996, after neglecting ellipticity and rotational effects.
[4]Eq. (6) of *He and Tromp* 1996, after neglecting perturbations in the Earth's internal discontinuities.

partial derivatives, or sensitivity kernels (*Seismic Tomography* course).

Replacing (5.65) into (5.64) and the resulting expression for H into (5.63), a relation is established between the measured displacement $\mathbf{u}$ and perturbations in the Earth's structure, and, in the presence of sufficient data, an inverse problem can be solved to derive the latter from the former (*Seismic Tomography* course).

The inverse problem can also be subdivided into two subsequent steps; after substituting (5.64) into (5.63), I can invert (5.63) for the splitting coefficients c_{st}. The resulting values can then be substituted into (5.65), and (5.65) in this new form can be inverted for δv_P^{st}, etc. This is feasible, of course, so long as a number of different measurements (i.e. different sources and/or different receivers) are available for the considered multiplet (the problem must be overdetermined).

In the literature, you shall often find the *splitting function*[5]

$$\sum_{s=0,2,4,\ldots}^{2l} \sum_{t=-s}^{s} c_{st} Y_{st} \tag{5.66}$$

as a convenient way to visualize normal-mode splitting. Notice that the sum in equation (5.66) is limited to even values of s. Coefficients c_{st} with s odd remain undefined here, because the integral in equation (5.64) can be proved to be 0 when s is odd (and when $t \neq i - j$). This is the main limit of the isolated-multiplet approximation: in tomographic applications, only the even harmonic order of the Earth's lateral structure can be constrained.

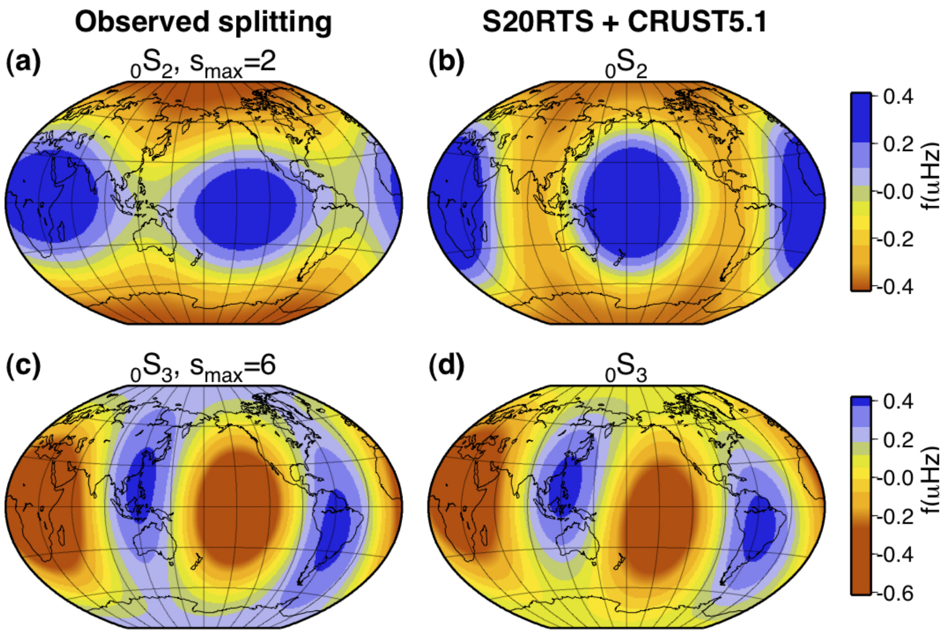


Figure 5.6: Splitting function measurements for ${}_0S_2$ and ${}_0S_3$ (From *Deuss et al 2011*).

[5]First introduced by *Woodhouse, Giardini & Li*, 1986.

Further reading

Normal modes of a string:

- Lay, T. and T. C. Wallace, *Modern Global Seismology*, Academic Press, San Diego, 1995.
- The physics of everyday stuff:
  http://www.bsharp.org/physics/guitar
- Build your own string-instrument:
  http://www.cs.helsinki.fi/u/wikla/mus/Calcs/wwwscalc.html

Normal modes of the Earth:

- Aki, K., and P. G. Richards, *Quantitative Seismology, 2nd Edition*, University Science Books, 2002: chapter 8.
- Dahlen, F. A., and J. Tromp, *Theoretical Global Seismology*, Princeton University Press, 1998: chapters 2 and 3 (conservation laws and equations of motion); section 4.1 (Green's tensor); chapter 8 (toroidal and spheroidal oscillations).
- Deuss, A., J. Ritsema, and H. van Heijst. Splitting function measurements for Earth's longest period normal modes using recent large earthquakes, *Geophys. Res. Lett.*, 38, L04303, doi:10.1029/2010GL046115, 2011.
- He, X., and J. Tromp, Normal-mode constraints on the structure of the mantle and core, *J. Geophys. Res.*, **101**, 20,053–20,082, 1996.
- Woodhouse, Giardini & Li, Evidence for inner core anisotropy from splitting in free oscillation data, *Geophys. Res. Lett.*, **13**, 1549-1552, 1986.