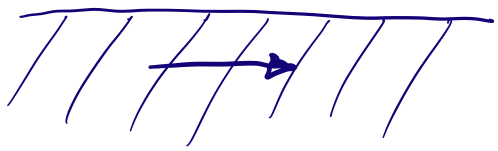
Surface waves

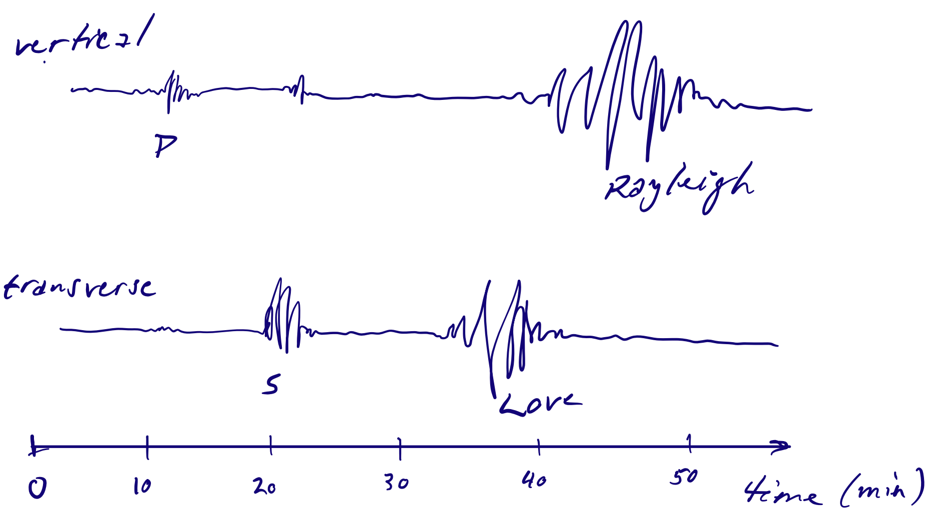


Surface waves need a free surface; they exist as the constructive interference of incoming & reflected waves.

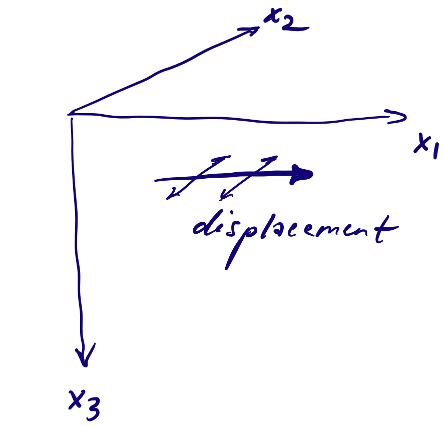
Two distinct types of surface waves are observed:

- Love waves: displacement parallel to free surface and perpendicular to direction of propagation

- Rayleigh waves: displacement in a plane perpendicular to Love-wave displacement



Love waves :



Let's rewrite the equations of motion

$$\rho \, \partial_t^2 \underline{u} - \underline{\nabla} \cdot \underline{\underline{T}} = \underline{f}$$

only looking at displacement in x_2-direction, propagating along x_1 for an isotropic elastic medium; the homogeneous equation states

$$\rho \, \partial_t^2 u_i = \frac{\partial \tau_{ij}}{\partial x_j} = \mu \left(\frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2} \right)$$

since $\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

With a Love-wave ansatz

$$u(r,t) = \begin{pmatrix} 0 \\ h(x_3) \, e^{i(kx_1 - \omega t)} \\ 0 \end{pmatrix}$$

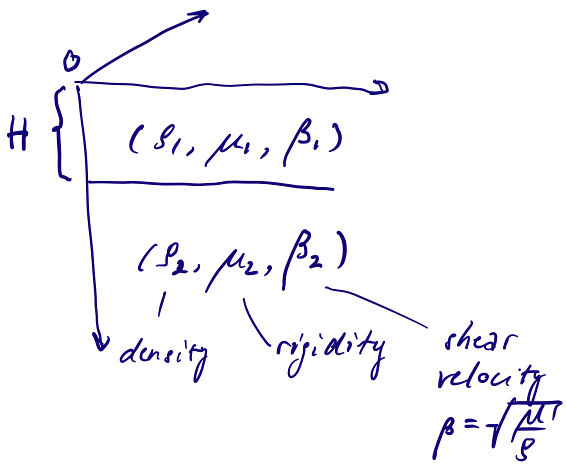
this leads to

$$\boxed{\rho \, \partial_t^2 u_2 = \mu \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_3^2} \right)}$$

and $u_1 = u_3 = 0$ and $\frac{\partial u_2}{\partial x_2} = 0$,

motion in x_2-direction only

Layer over halfspace



With the Love-wave ansatz from above, where we separate the (radial) dependency of x_3 by a radial eigenfunction $h(x_3)$,

$$u(\underline{r}, t) = \begin{pmatrix} 0 \\ h(x_3)\, e^{i(kx_1 - \omega t)} \\ 0 \end{pmatrix}$$

we find the general solution

$$u(\underline{r}, t) = \begin{pmatrix} 0 \\ \left[A_i e^{-\sqrt{k^2 - \frac{\omega^2}{\beta_i^2}}\, x_3} + B_i e^{+\sqrt{k^2 - \frac{\omega^2}{\beta_i^2}}\, x_3} \right] e^{i(kx_1 - \omega t)} \\ 0 \end{pmatrix}$$

The following boundary conditions will be imposed:

(i) $\underline{u} \longrightarrow 0$ for $x_3 \to \infty$

(ii) no stress at free surface

(iii) displacement & traction are continuous at interface $x_3 = H$

This helps constraining the variables A_i, B_i, ω, k:

for (i): $\mathrm{Re}\left(-\sqrt{k^2 - \frac{\omega^2}{\beta_2^2}}\right) < 0$ and $B_2 = 0$

or

$\mathrm{Re}\left(+\sqrt{k^2 - \frac{\omega^2}{\beta_2^2}}\right) < 0$ and $A_2 = 0$

for (ii): $\tau_{31} = \tau_{32} = \tau_{33} = 0$ where $\tau_{32} = \mu\left(\frac{\partial u_2}{\partial x_3}\right) \longrightarrow A_1 = B_1$

for (iii): $u_2(\underline{r}, t) = 2A_1 \cos\left(i\sqrt{k^2 - \frac{\omega^2}{\beta_1^2}}\, x_3\right) e^{i(kx_1 - \omega t)}$ in layer 1

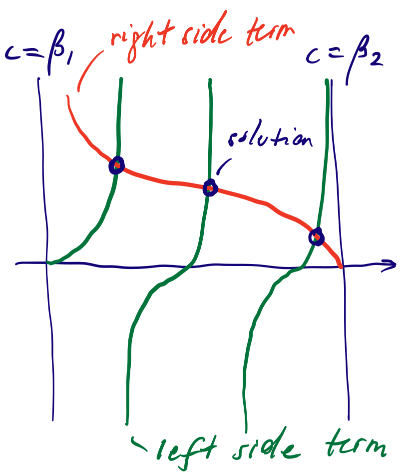
$u_2(\underline{r}, t) = A_2\, e^{-\sqrt{k^2 - \frac{\omega^2}{\beta_2^2}}\, x_3}\, e^{i(kx_1 - \omega t)}$ in layer 2

together with:

continuity at $x_3 = H$ / continuity of traction $\mu\left(\frac{\partial u_2}{\partial x_3}\right)$:

$$\begin{cases} 2A_1 \cos\left(i\sqrt{k^2 - \frac{\omega^2}{\beta_1^2}}\right) H = A_2\, e^{-\sqrt{k^2 - \frac{\omega^2}{\beta_2^2}}\, H} \\ 2\mu_1 A_1\, i\sqrt{k^2 - \frac{\omega^2}{\beta_1^2}}\, \sin\left(i\sqrt{k^2 - \frac{\omega^2}{\beta_1^2}}\, H\right) = \mu_2 \sqrt{k^2 - \frac{\omega^2}{\beta_2^2}}\, A_2\, e^{-\sqrt{k^2 - \frac{\omega^2}{\beta_2^2}}\, H} \end{cases}$$

$\longrightarrow$ this becomes an eigenvalue problem for variables (A_1, A_2, ω, k):



$$\underbrace{\tan\left(\omega H \sqrt{\frac{1}{\beta_1^2} - \frac{1}{c^2}}\right)}_{\text{left}} = \underbrace{\frac{\mu_2 \sqrt{\frac{1}{c^2} - \frac{1}{\beta_2^2}}}{\mu_1 \sqrt{\frac{1}{\beta_1^2} - \frac{1}{c^2}}}}_{\text{right}}$$

with $c = \frac{\omega}{k}$ phase velocity

Note that this system has only real solutions for $\beta_1 < c < \beta_2$

In particular, there is no solution for $\beta_1 = \beta_2$, i.e., just a homogeneous medium. It requires a layer with $\beta_1 < \beta_2$.

Cut-off frequency: $\omega_{c_n} = \dfrac{n\pi}{H\sqrt{\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}}}$

only waves with $\omega \geq \omega_{c_n}$ will have n modes (or more)

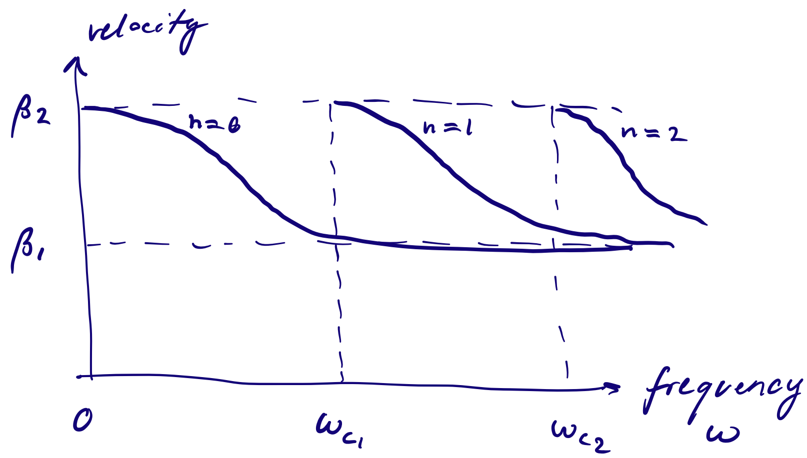
Example: typical crust in continent

$$\left.\begin{aligned} H &= 35 \text{ km} \\ \beta_1 &= 3.5 \text{ km/s} \\ \beta_2 &= 4.5 \text{ km/s} \end{aligned}\right\}$$

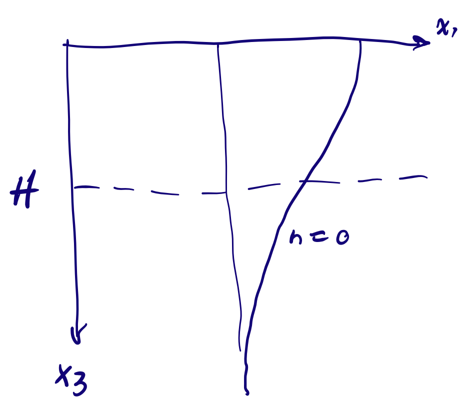
cut-off for first overtone $\omega_{c_1} \sim 0.08$ Hz

corresponds to a period of ~13 s

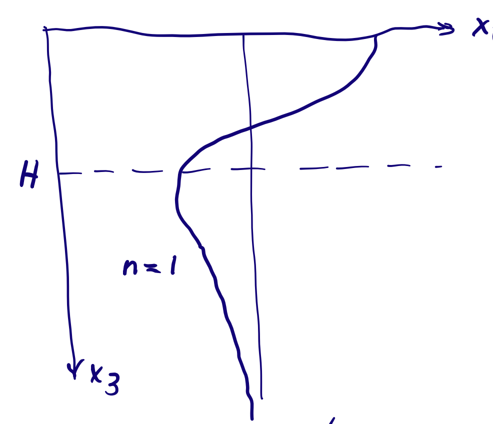
There are no 1. overtones at periods longer than 13 s phase velocities



radial eigenfunctions



fundamental mode



1. overtone

Amplitudes depend on depth & type of source excitation. For example, shallow earthquakes excite mostly fundamental modes at long-periods