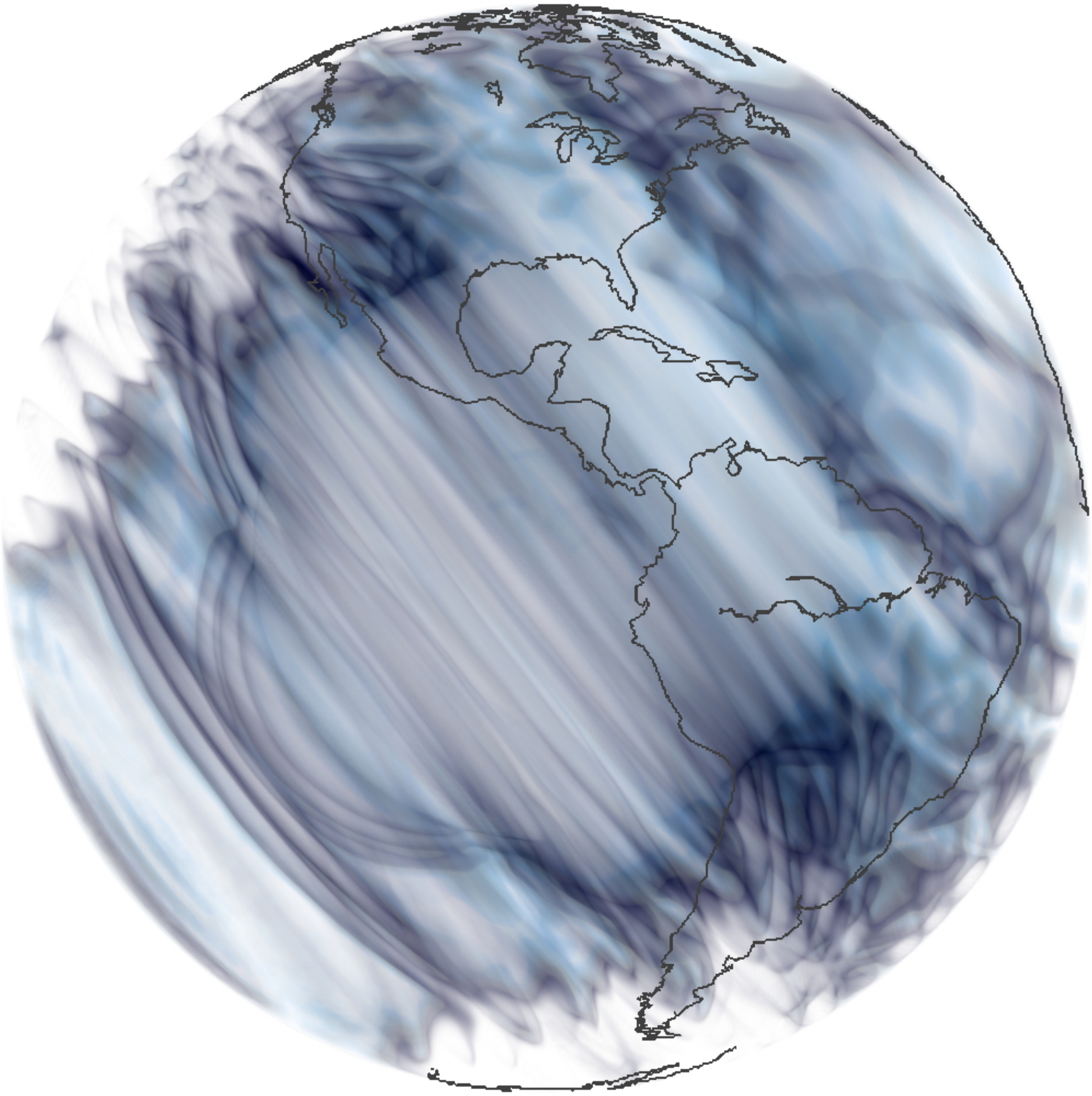
Seismic waves



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1. Basic Theorems in Dynamic Elasticity

Arial view of the San Andreas fault in Carrizo plain (north of Los Angeles)

Introduction

We assume that the concepts of stress, strain, body and surface force are understood, and give here a brief overview of dynamic elasticity in order to introduce our notation. Starting with the equation of motion, we write (ground) displacement at any location as a response to external and internal forces applied to an elastic body. Using the response of the body to an impulsive force, we then relate seismic displacement with the displacement on a geological fault.

Learn objectives

- You can describe all components involved in the wave equation.
- You know how to relate stress and strain for an elastic material.
- You are able to explain how to represent any ground displacement by the displacement on a geological fault.

1.1. The Equation of Motion

Equation of Motion, General Form

The total force acting on a continuous, incompressible (density ρ constant in time) body of volume V, bounded by a surface S, equals the sum of forces-per-unit-volume $\mathbf{f}$ applied on its interior points, and those applied on S, or

$$\int_V \mathbf{f}\,\mathrm{d}V + \int_S \hat{\mathbf{n}} \cdot \boldsymbol{\tau}\,\mathrm{d}S = \int_V \rho \frac{\partial^2}{\partial t^2}\mathbf{u}\,\mathrm{d}V, \tag{1.1}$$

where the 3×3 tensor $\boldsymbol{\tau}$ denotes stress, and $\hat{\mathbf{n}}$ is the unit vector everywhere normal to S, so that the integral over S of the dot-product $\boldsymbol{\tau} \cdot \hat{\mathbf{n}}$ equals the total *surface force* acting on the body in question. As usual $\mathbf{u}$ denotes displacement and t time. In "index" notation,

$$\int_V f_i\,\mathrm{d}V + \int_S \hat{n}_j \tau_{ji}\,\mathrm{d}S = \int_V \rho \frac{\partial^2}{\partial t^2} u_i\,\mathrm{d}V, \tag{1.2}$$

where $i = 1, 2, 3$, and summation over repeated indexes ($j = 1, 2, 3$) is intended.

Applying the *divergence theorem* to the surface integral at the left-hand side of (1.1) or (1.2), and using the symmetry of $\boldsymbol{\tau}$, which will be discussed in the next section,

$$\int_S \hat{\mathbf{n}} \cdot \boldsymbol{\tau}\,\mathrm{d}S = \int_V \nabla \cdot \boldsymbol{\tau}\,\mathrm{d}V, \tag{1.3}$$

with ∇ the Nabla operator and thus $\nabla\cdot$ the divergence operator. Equation (1.1) can be rewritten

$$\int_V \left(\rho \frac{\partial^2}{\partial t^2}\mathbf{u} - \mathbf{f} - \nabla \cdot \boldsymbol{\tau}\right) \mathrm{d}V = \mathbf{0}. \tag{1.4}$$

A volume V could always be found where (1.4) is not valid, unless

$$\rho \frac{\partial^2}{\partial t^2}\mathbf{u} - \mathbf{f} - \nabla \cdot \boldsymbol{\tau} = \mathbf{0}. \tag{1.5}$$

In index notation,

$$\rho \frac{\partial^2}{\partial t^2} u_i - f_i - \frac{\partial \tau_{ji}}{\partial x_j} = 0, \tag{1.6}$$

for $i = 1, 2, 3$.

Stress, Deformation, and the Constitutive Relation

Assuming ρ is known, an ingredient is still missing to turn (1.5) or (1.6) into a partial differential equation in $\mathbf{u}$, i.e. into a tool to predict deformation resulting from a given forcing $\mathbf{f}$: we must specify the *"rheology"* of the medium, that is to say the relationship between $\boldsymbol{\tau}$ and $\mathbf{u}$. This relationship is traditionally given in terms of stress $\boldsymbol{\tau}$ and the small-deformation 3×3 strain tensor $\boldsymbol{\epsilon}$, defined

$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right], \tag{1.7}$$

where the superscript T stands for "transpose". The tensor $\nabla\mathbf{u}$, gradient of the displacement, must not be confused with the scalar $\nabla\cdot\mathbf{u}$, divergence of the displacement. In index notation, strain is written as

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.8}$$

For example, an *elastic* and *isotropic* medium follows *Hooke's law*, where stress $\boldsymbol{\tau}$ relates to strain $\boldsymbol{\epsilon}$ as

$$\boldsymbol{\tau} = \lambda\ \mathrm{tr}(\boldsymbol{\epsilon})\ \mathbf{I} + 2\mu\boldsymbol{\epsilon}, \tag{1.9}$$

with tr denoting the *"trace"* operator (sum of the diagonal elements of a tensor), $\mathbf{I}$ the 3×3 identity matrix, and λ and μ the *Lamé's parameters.* In index form,

$$\tau_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}, \tag{1.10}$$

where δ_{ij} is the *Kronecker's symbol* ($\delta_{ij} = 1$ if $i = j$, 0 otherwise).

After substituting the expression (1.8) for ϵ_{ij} (or, in tensor notation, the expression (1.7) for $\boldsymbol{\epsilon}$) the *"constitutive relation"* (1.10) is written in the more compact form

$$\tau_{ij} = c_{ijkl}\frac{\partial u_l}{\partial x_k}, \tag{1.11}$$

where the entries of the 4-th order ($3\times3\times3\times3$) tensor $\mathbf{c}$ are simple functions of λ and μ.

Based upon both algebraic and physical considerations, it can be proved that $\mathbf{c}$ enjoys the symmetries

$$c_{ijkl} = c_{jikl}, \tag{1.12}$$

$$c_{ijkl} = c_{ijlk}, \tag{1.13}$$

and

$$c_{ijkl} = c_{klij} \tag{1.14}$$

(e.g., *Aki & Richards* 2002, section 2.2).

Equation of Motion, Elastic and Isotropic Medium

Substituting (1.7) into (1.9), and substituting into (1.5) the resulting expression for $\boldsymbol{\tau}$–or, likewise, substituting (1.8) into (1.10), and the result then into (1.6)–an explicit form for the required differential equation in $\mathbf{u}$ is found. If the compact expression (1.11) for $\boldsymbol{\tau}$ is used, the *equation of motion* reads

$$\rho\frac{\partial^2}{\partial t^2}u_i - f_i - \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_l}{\partial x_k}\right) = 0. \tag{1.15}$$

In tensor notation,

$$\rho\frac{\partial^2}{\partial t^2}\mathbf{u} - \mathbf{f} - \nabla\cdot(\mathbf{c}:\nabla\mathbf{u}) = \mathbf{0}. \tag{1.16}$$

A theoretical seismology course largely consists of presenting a variety of approaches to the solution of the differential equation (1.15) or (1.16). For our goals at this stage, the assumption that the Earth be perfectly elastic and isotropic will generally be acceptable.

1.2. Betti's Theorem

Enrico Betti (1823-1892) was an Italian "mathematician who wrote a pioneering memoir on topology, the study of surfaces and higher-dimensional spaces, and wrote one of the first rigorous expositions of the theory of equations developed by the noted French mathematician Évariste Galois" (Encyclopaedia Britannica). Incidentally, he proved that, for a displacement field $\mathbf{u}$ associated with stress $\boldsymbol{\tau}$ and force-per-unit-volume $\mathbf{f}$ satisfying equation (1.5), and a displacement field $\mathbf{v}$ likewise associated with stress $\boldsymbol{\sigma}$ and force-per-unit-volume $\mathbf{g}$ according to (1.5), both applied on a body of volume V and bounded by a surface S, the following scalar equality holds:

$$\int_V (\mathbf{f} - \rho\, \ddot{\mathbf{u}}) \cdot \mathbf{v}\, \mathrm{d}V + \int_S \mathbf{v} \cdot \boldsymbol{\tau} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \int_V (\mathbf{g} - \rho\, \ddot{\mathbf{v}}) \cdot \mathbf{u}\, \mathrm{d}V + \int_S \mathbf{u} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\, \mathrm{d}S, \tag{1.17}$$

where we have dropped the cumbersome notation $\frac{\partial}{\partial t}$ for time-derivative, in favor of the more compact dot $\dot{}$ (double dot $\ddot{}$ for second time-derivative). In index form,

$$\int_V (f_i - \rho\, \ddot{u}_i)\, v_i\, \mathrm{d}V + \int_S v_i\, \tau_{ij}\, \hat{n}_j\, \mathrm{d}S = \int_V (g_i - \rho\, \ddot{v}_i)\, u_i\, \mathrm{d}V + \int_S u_i\, \sigma_{ij}\, \hat{n}_j\, \mathrm{d}S. \tag{1.18}$$

Proof

By virtue of the divergence theorem,

$$\int_S \mathbf{v} \cdot \boldsymbol{\tau} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \int_V \nabla \cdot (\mathbf{v} \cdot \boldsymbol{\tau})\, \mathrm{d}V. \tag{1.19}$$

The S-integral at the right-hand side of (1.17) can be turned into a V-integral in exactly the same way, and (1.17) takes the form

$$\int_V [(\mathbf{f} - \rho\, \ddot{\mathbf{u}}) \cdot \mathbf{v} + \nabla \cdot (\mathbf{v} \cdot \boldsymbol{\tau})]\, \mathrm{d}V = \int_V [(\mathbf{g} - \rho\, \ddot{\mathbf{v}}) \cdot \mathbf{u} + \nabla \cdot (\mathbf{u} \cdot \boldsymbol{\sigma})]\, \mathrm{d}V. \tag{1.20}$$

Equation (1.20) can be compacted thanks to (1.5), replacing $\mathbf{f} - \rho\, \ddot{\mathbf{u}}$ with $-\nabla \cdot \boldsymbol{\tau}$ at the left-hand side and, likewise, $\mathbf{g} - \rho\, \ddot{\mathbf{v}}$ with $-\nabla \cdot \boldsymbol{\sigma}$ at the right-hand side. The result reads

$$\int_V [(\nabla \cdot \boldsymbol{\tau}) \cdot \mathbf{v} - \nabla \cdot (\mathbf{v} \cdot \boldsymbol{\tau})]\, \mathrm{d}V = \int_V [(\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{u} - \nabla \cdot (\mathbf{u} \cdot \boldsymbol{\sigma})]\, \mathrm{d}V. \tag{1.21}$$

In index notation,

$$\int_V \left[\frac{\partial \tau_{ij}}{\partial x_i}\, v_j - \frac{\partial}{\partial x_j}\, (v_i\, \tau_{ij})\right] \mathrm{d}V = \int_V \left[\frac{\partial \sigma_{ij}}{\partial x_i}\, u_j - \frac{\partial}{\partial x_j}\, (u_i\, \sigma_{ij})\right] \mathrm{d}V. \tag{1.22}$$

Notice that

$$\frac{\partial}{\partial x_j}\, (v_i\, \tau_{ij}) = \frac{\partial v_i}{\partial x_j}\, \tau_{ij} + v_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial v_i}{\partial x_j}\, \tau_{ij} + v_j \frac{\partial \tau_{ij}}{\partial x_i}, \tag{1.23}$$

where the latter equality holds because the stress tensors $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are by definition – equations (1.8) and (1.10) – symmetric. $\frac{\partial}{\partial x_j}(u_i\,\sigma_{ij})$ at the right-hand side can be rewritten in a similar fashion. Substituting into (1.21), we are left with

$$\frac{\partial v_i}{\partial x_j}\,\tau_{ij} = \frac{\partial u_i}{\partial x_j}\,\sigma_{ij}. \tag{1.24}$$

If one now substitutes $\tau_{ij} = c_{ijkl}\frac{\partial u_l}{\partial x_k}$, and $\sigma_{ij} = c_{ijkl}\frac{\partial v_l}{\partial x_k}$ into (1.24), and makes use of the symmetries (1.12) (1.14), the equality is verified and Betti's theorem is proved.

"Integrated" Form of Betti's Theorem

Betti's theorem (1.17) remains valid when $\mathbf{u}$, $\boldsymbol{\tau}$, $\mathbf{f}$ and $\mathbf{v}$, $\boldsymbol{\sigma}$, $\mathbf{g}$ are evaluated at different times. Consider $\mathbf{u}$, $\boldsymbol{\tau}$, $\mathbf{f}$ at time t, and $\mathbf{v}$, $\boldsymbol{\sigma}$, $\mathbf{g}$ at time $T-t$ with T an arbitrary constant, and integrate (1.17) over time from $t=-\infty$ to $t=+\infty$. Consider the terms containing the second time-derivative of $\mathbf{u}$ and $\mathbf{v}$:

$$\begin{aligned}\int_{-\infty}^{+\infty}\left[\rho\ddot{\mathbf{u}}(t)\cdot\mathbf{v}(T-t)-\rho\ddot{\mathbf{v}}(T-t)\cdot\mathbf{u}(t)\right]\mathrm{d}t = &\quad \rho\int_{-\infty}^{+\infty}\frac{\partial}{\partial t}\big[\dot{\mathbf{u}}(t)\cdot\mathbf{v}(T-t)+\dot{\mathbf{v}}(T-t)\cdot\mathbf{u}(t)\big]\,\mathrm{d}t\\ = &\quad \rho\big[\dot{\mathbf{u}}(t)\cdot\mathbf{v}(T-t)+\dot{\mathbf{v}}(T-t)\cdot\mathbf{u}(t)\big]_{-\infty}^{+\infty}\end{aligned}\tag{1.25}$$

After evaluating [...] at $\pm\infty$, the expression (1.25) becomes

$$\rho\big[\{\dot{\mathbf{u}}(\infty)\cdot\mathbf{v}(T-\infty)+\dot{\mathbf{v}}(T-\infty)\cdot\mathbf{u}(\infty)\}-\{\dot{\mathbf{u}}(-\infty)\cdot\mathbf{v}(T+\infty)+\dot{\mathbf{v}}(T+\infty)\cdot\mathbf{u}(-\infty)\}\big]. \tag{1.26}$$

If we assume that there exists a time T_0 before which $\mathbf{u}$ and $\mathbf{v}$, and hence $\dot{\mathbf{u}}$ and $\dot{\mathbf{v}}$, are constant and equal to $\mathbf{0}$, then $\mathbf{v}(T-\infty)=\dot{\mathbf{v}}(T-\infty)=\dot{\mathbf{u}}(-\infty)=\mathbf{u}(-\infty)=\mathbf{0}$, and subsequently expression (1.26) must also equal zero.

We can then cancel out terms with second time-derivatives of $\mathbf{u}$ and $\mathbf{v}$ and find the time-integrated form of Betti's theorem, valid only for displacement fields with a quiescent past:

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}t\int_V\mathbf{f}(t)\cdot\mathbf{v}(T-t)\,\mathrm{d}V+\int_{-\infty}^{+\infty}\mathrm{d}t\int_S\mathbf{v}(T-t)\cdot\boldsymbol{\tau}(t)\cdot\hat{\mathbf{n}}\,\mathrm{d}S=\\ \int_{-\infty}^{+\infty}\mathrm{d}t\int_V\mathbf{g}(T-t)\cdot\mathbf{u}(t)\,\mathrm{d}V+\int_{-\infty}^{+\infty}\mathrm{d}t\int_S\mathbf{u}(t)\cdot\boldsymbol{\sigma}(T-t)\cdot\hat{\mathbf{n}}\,\mathrm{d}S.\end{aligned}\tag{1.27}$$

In index form,

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}t\int_V f_i(t)\,v_i(T-t)\,\mathrm{d}V+\int_{-\infty}^{+\infty}\mathrm{d}t\int_S v_i(T-t)\,\tau_{ij}(t)\,\hat{n}_j\,\mathrm{d}S=\\ \int_{-\infty}^{+\infty}\mathrm{d}t\int_V g_i(T-t)\,u_i(t)\,\mathrm{d}V+\int_{-\infty}^{+\infty}\mathrm{d}t\int_S u_i(t)\,\sigma_{ij}(T-t)\hat{n}_j\,\mathrm{d}S.\end{aligned}\tag{1.28}$$

1.3. Green's Function and its Reciprocity Relations

Betti's theorem (1.17) and (1.27) may seem like an abstract mathematical curiosity, but it is at the basis of the quantitative study of earthquakes and their effects. The idea that an earthquake be the result of rupture on a fault was substantiated after the 1906 San Francisco earthquake, the first large earthquake to occur in a well surveyed region, so that displacement between points both close and far removed from the fault was accurately mapped. It was then realized that Betti's theorem could be employed to establish a mathematical relationship between the said rupture and observed seismic oscillations.

Green's Problem

In his book *An Essay on the Applications of Mathematical Analysis to the Theories of Electricity and Magnetism* (1828) the self-taught British mathematician George Green (1793-1841) introduced the concept of *"Green's problem"*, or the problem of finding the response of a system to an excitation impulsive in both space and time. In our case, this amounts to solving (1.15) after substituting

$$f_i = \delta_{im}\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-T) \qquad (i=1,2,3), \tag{1.29}$$

with $\delta(...)$ denoting Dirac's function, and repeating the exercise for $m = 1,2,3$. In other words, it amounts to finding the responses of the medium to three ($m = 1,2,3$) different forces, all with amplitude $\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-T)$ but each directed like a different reference axis. The solution must be a function of the location ξ and time T of the forcing.

If the three resulting expressions (1.29) for f_i are substituted into (1.15) or (1.16), we find three vectorial equations, which we can compact into one tensorial equation in the *"Green's function"* $\mathbf{G} = \mathbf{G}(\mathbf{x}, t; \xi, T)$,

$$\rho\frac{\partial^2}{\partial t^2}\mathbf{G} - \delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-T)\,\mathbf{I} - \nabla\cdot(\mathbf{c}:\nabla\mathbf{G}) = \mathbf{0}. \tag{1.30}$$

It should be clear from (1.30) that the m-th column of the 3×3 tensor $\mathbf{G}$ coincides with the response $\mathbf{u}$ of the medium to an impulse $\mathbf{f}$ directed like the m-th axis. In index notation, (1.30) reads

$$\rho\frac{\partial^2}{\partial t^2}G_{im} - \delta_{im}\,\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-T) - \frac{\partial}{\partial x_j}\left(c_{jikl}\frac{\partial G_{lm}}{\partial x_k}\right) = 0 \qquad (m=1,2,3). \tag{1.31}$$

Green's problem (1.30) or (1.31) has been solved analytically for certain simple media (Earth models); it has been solved empirically, analyzing seismic observations; it can now be solved numerically for almost arbitrarily complicated (and realistic) media, by many different techniques. In this lecture, we are not concerned with its solution, but with the powerful applications of its solution, however determined, in combination with Betti's theorem. Let us first point out a few properties of $\mathbf{G}$, following from its definition.

Green's Function: Reciprocity in Time

If the system described by (1.30) is set into motion by an impulse at time T, and the boundary conditions are independent of time, it follows that $\mathbf{G}$ must depend only on the time difference $t-T$, and not on the absolute times t and T. Then, $\mathbf{G}$ at time $t = t_2$ following an impulse at time $T = t_1$ (hence $t - T = t_2 - t_1$) must equal $\mathbf{G}$ obtained from an impulse at time $T = -t_2$, and measured at time $t = -t_1$ (hence, again, $t - T = t_2 - t_1$), or

$$\mathbf{G}(\mathbf{x}, t_2; \boldsymbol{\xi}, t_1) = \mathbf{G}(\mathbf{x}, -t_1; \boldsymbol{\xi}, -t_2), \tag{1.32}$$

for any t_1, t_2, and boundary conditions constant in time.

Green's Function: Reciprocity in Space

Consider Betti's theorem in the form (1.28), applied to a medium with "homogeneous boundary conditions" on S (for example, let S be free of stresses–the Earth's surface is in the first approximation free of stresses), so that the S-integrals cancel out. Let $\mathbf{f}$ and $\mathbf{g}$ be impulsive forces analogous to (1.29), $f_i = \delta_{im}\delta(\mathbf{x} - \boldsymbol{\xi_1})\delta(t - T_1)$ parallel to the m-th reference axis, and $g_i = \delta_{in}\delta(\mathbf{x} - \boldsymbol{\xi_2})\delta(t - T_2)$ parallel to the n-th one. Notice that, by definition of $\mathbf{G}$, it follows for the displacement fields $\mathbf{u}$ and $\mathbf{v}$ generated respectively by $\mathbf{f}$ and $\mathbf{g}$ that $u_i(\mathbf{x}, t) = G_{im}(\mathbf{x}, t; \boldsymbol{\xi_1}, T_1)$, and $v_i(\mathbf{x}, t) = G_{in}(\mathbf{x}, t; \boldsymbol{\xi_2}, T_2)$ $(i = 1, 2, 3)$. Substituting in what is left of (1.28),

$$\int_{-\infty}^{+\infty} \mathrm{d}t \int_V \delta_{im}\delta(\mathbf{x}-\boldsymbol{\xi_1})\delta(t-T_1)\, v_i(T-t)\, \mathrm{d}V = \int_{-\infty}^{+\infty} \mathrm{d}t \int_V \delta_{in}\delta(\mathbf{x}-\boldsymbol{\xi_2})\delta(T-t-T_2)\, u_i(t)\, \mathrm{d}V, \tag{1.33}$$

or

$$\int_{-\infty}^{+\infty} \mathrm{d}t \int_V \delta(\mathbf{x} - \boldsymbol{\xi_1})\delta(t - T_1)\, G_{mn}(\mathbf{x}, T - t; \boldsymbol{\xi_2}, T_2)\, \mathrm{d}V =$$
$$\int_{-\infty}^{+\infty} \mathrm{d}t \int_V \delta(\mathbf{x} - \boldsymbol{\xi_2})\delta(T - t - T_2)\, G_{nm}(\mathbf{x}, t; \boldsymbol{\xi_1}, T_1)\, \mathrm{d}V. \tag{1.34}$$

Integrating over time and V, and making use of the properties of Dirac's delta,

$$G_{mn}(\boldsymbol{\xi_1}, T - T_1; \boldsymbol{\xi_2}, T_2) = G_{nm}(\boldsymbol{\xi_2}, T - T_2; \boldsymbol{\xi_1}, T_1). \tag{1.35}$$

Choosing $T_1 = T_2 = 0$,

$$G_{mn}(\boldsymbol{\xi_1}, T; \boldsymbol{\xi_2}, 0) = G_{nm}(\boldsymbol{\xi_2}, T; \boldsymbol{\xi_1}, 0), \tag{1.36}$$

a purely spatial reciprocity. Choosing, instead, $T = 0$,

$$G_{mn}(\boldsymbol{\xi_1}, -T_1; \boldsymbol{\xi_2}, T_2) = G_{nm}(\boldsymbol{\xi_2}, -T_2; \boldsymbol{\xi_1}, T_1), \tag{1.37}$$

a space-time reciprocity.

1.4. Representation Theorem

Once the Green's function for the Earth or for a chosen region of the Earth is known, we combine it with Betti's theorem (1.27) or (1.28) as follows. In equation (1.27), replace $\mathbf{g}(\mathbf{x}, t)$ with the second-order tensor $\delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - T_0)\,\mathbf{I}$. This is equivalent to rewriting (1.27) three times, after equating the vector $\mathbf{g}(\mathbf{x}, t)$ to each column of $\delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - T_0)\,\mathbf{I}$, but we prefer a more compact notation. Notice also that $\delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - T_0)\,\mathbf{I}$ coincides with the forcing term in equation (1.30), defined by (1.29): it follows, by definition of Green's problem, that $\mathbf{v}(\mathbf{x},t) = \mathbf{G}(\mathbf{x},t;\boldsymbol{\xi},T_0)$ and $\boldsymbol{\sigma}(\mathbf{x},t) = \mathbf{c} : \nabla\mathbf{G}(\mathbf{x},t;\boldsymbol{\xi},T_0)$. Substituting into (1.27),

$$\int_{-\infty}^{+\infty} \mathrm{d}t \int_V \mathbf{f}(\mathbf{x},t)\cdot\mathbf{G}(\mathbf{x},T-t;\boldsymbol{\xi},T_0)\,\mathrm{d}V + \int_{-\infty}^{+\infty} \mathrm{d}t \int_S \{[\mathbf{c} : \nabla\mathbf{u}(\mathbf{x},t)]\cdot\hat{\mathbf{n}}\}\cdot\mathbf{G}(\mathbf{x},T-t;\boldsymbol{\xi},T_0)\,\mathrm{d}S =$$
$$\int_{-\infty}^{+\infty} \mathrm{d}t \int_V \delta(\mathbf{x}-\boldsymbol{\xi})\delta(T-t+T_0)\,\mathbf{I}\cdot\mathbf{u}(\mathbf{x},t)\,\mathrm{d}V + \int_{-\infty}^{+\infty} \mathrm{d}t \int_S \mathbf{u}(\mathbf{x},t)\cdot\{[\mathbf{c} : \nabla\mathbf{G}(\mathbf{x},T-t;\boldsymbol{\xi},T_0)]\cdot\hat{\mathbf{n}}\}\,\mathrm{d}S. \tag{1.38}$$

The integral in the first term at the right-hand side clearly collapses to $\mathbf{u}(\boldsymbol{\xi},T+T_0)$; we can then rewrite (1.38)

$$\begin{aligned}
\mathbf{u}(\boldsymbol{\xi},T+T_0) &= \int_{-\infty}^{+\infty} \mathrm{d}t \int_V \mathbf{f}(\mathbf{x},t)\cdot\mathbf{G}(\mathbf{x},T-t;\boldsymbol{\xi},T_0)\,\mathrm{d}V \\
&+ \int_{-\infty}^{+\infty} \mathrm{d}t \int_S \{[\mathbf{c} : \nabla\mathbf{u}(\mathbf{x},t)]\cdot\hat{\mathbf{n}}\}\cdot\mathbf{G}(\mathbf{x},T-t;\boldsymbol{\xi},T_0)\,\mathrm{d}S \\
&- \int_{-\infty}^{+\infty} \mathrm{d}t \int_S \mathbf{u}(\mathbf{x},t)\cdot\{[\mathbf{c} : \nabla\mathbf{G}(\mathbf{x},T-t;\boldsymbol{\xi},T_0)]\cdot\hat{\mathbf{n}}\}\,\mathrm{d}S
\end{aligned} \tag{1.39}$$

(representation theorem). Equation (1.39) states that $\mathbf{u}(\mathbf{x},t)$ can be calculated from any forcing term $\mathbf{f}$ once the Green's function of the medium in question has been found, and if $\mathbf{u}(\mathbf{x},t)$ is known on S. In practice, (1.39) is a machine to calculate the displacement of a medium given $\mathbf{f}$, however complicated, and the boundary conditions.

When applying (1.39) to calculate the displacement field following an earthquake, S must equal the combination of the outer surface of the Earth with a surface Σ formed by the combination of two adjacent surfaces coinciding with the opposite faces of the earthquake fault. $\mathbf{u}$ (not $\boldsymbol{\tau}$) will generally be discontinuous across Σ, so it is necessary to treat Σ as a portion of S, i.e. of the boundary of the medium. Equation (1.39) is then further simplified considering that

(i) the outer surface of the Earth is stress-free,

(ii) there are no force densities $\mathbf{f}$ significant to our problem acting on the interior of the Earth, as gravity and apparent rotational forces can be safely neglected in the first approximation, and

(iii) we can ask that $\mathbf{G}$ be continuous across Σ ($\mathbf{G}$ is the response to a point source, and there is therefore no reason for $\mathbf{G}$ not to be continuous across Σ).

The requirement (ii) is legitimate because the system might still be set in motion by boundary conditions (prescribed displacement on the fault). Because of (iii), and the continuity of $\boldsymbol{\tau} = \mathbf{c} : \nabla\mathbf{u}$, the first S-integral at the right-hand side of (1.39) cancels out; because of (ii), the V-integral at the right-hand side of (1.39) cancels out; because of (i), the integral over S reduces to an integral over Σ; equation (1.39) subsequently collapses to

$$\mathbf{u}(\boldsymbol{\xi},T+T_0) = \int_{-\infty}^{+\infty} \mathrm{d}t \int_\Sigma \mathbf{u}(\mathbf{x},t)\cdot\{[\mathbf{c} : \nabla\mathbf{G}(\mathbf{x},T-t,\boldsymbol{\xi},T_0)]\cdot\hat{\mathbf{n}}\}\,\mathrm{d}S, \tag{1.40}$$

or

$$u_n(\boldsymbol{\xi}, T + T_0) = \int_{-\infty}^{+\infty} \mathrm{d}t \int_\Sigma u_i(\mathbf{x}, t) \hat{n}_j c_{ijkl} \frac{\partial}{\partial x_k} G_{ln}(\mathbf{x}, T - t, \boldsymbol{\xi}, T_0)\, \mathrm{d}S. \tag{1.41}$$

Compare this result with *Aki and Richards*, equation (3.2), choosing $T_0 = 0$, using the reciprocity equation (1.36), and swapping t with T and $\mathbf{x}$ with $\boldsymbol{\xi}$ (only a change in notation).

Equation (1.40) states that knowing the displacement on the fault (values of $\mathbf{u}(\mathbf{x}, t)$ to be integrated at the right-hand side) is enough to determine displacement everywhere. It is therefore an important recipe to calculate ground displacement due to an arbitrary distribution of rupture displacement on a fault, in particular interesting for kinematic source descriptions and finite-fault source inversions.

References and Further Reading

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A. Math review

A.1. Vectors, tensors, gradients, Gauss's theorem

In this seismology course, basically all geometries can be described by Cartesian coordinates in a Euclidean space. In the following, we give a very brief review of vector calculus.

A **vector u** can be written as

$$\mathbf{u} = u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}} + u_z\hat{\mathbf{z}} \tag{A.1}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the x, y and z directions.

The **dot product** of two vectors is a scalar and defined as

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_x v_x + u_y v_y + u_z v_z \\ &= u_i v_i \end{aligned} \tag{A.2}$$

where we use the index notation i assuming it takes the values 1, 2 and 3 for the x, y and z components, respectively. Note that we also use the *summation convention*, i.e., repeated indices in a product are summed over.

A **second-order tensor A** is a linear form that produces one vector from another, such as

$$\begin{aligned} \mathbf{u} &= \mathbf{A}\mathbf{v} \\ u_i &= A_{ij} v_j \end{aligned} \tag{A.3}$$

where we sum again over indices $j = 1, 2, 3$.

The **double-dot product** of two second-order tensors is a scalar defined as

$$\mathbf{A} : \mathbf{B} = A_{ij} B_{ij} \tag{A.4}$$

where we use the summation convention again over i and j. We will often use the double-dot product between a 4th-order tensor $\mathbf{c}$ and and a second-order tensor $\mathbf{A}$ like

$$(\mathbf{c} : \mathbf{A})_{ij} = c_{ijkl} A_{kl} \tag{A.5}$$

summing over indices k and l, resulting in a second-order tensor.

The **gradient** of a scalar field $\nabla\lambda$ is a vector field defined by the partial derivatives in x, y and z directions, i.e.,

$$\begin{aligned} \nabla\lambda &= \frac{\partial\lambda}{\partial x}\hat{\mathbf{x}} + \frac{\partial\lambda}{\partial y}\hat{\mathbf{y}} + \frac{\partial\lambda}{\partial z}\hat{\mathbf{z}} \\ (\nabla\lambda)_i &= \partial_i\lambda \end{aligned} \tag{A.6}$$

where we use the notation ∂_i as shorthand for $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ for $i = 1, 2, 3$ respectively.

The **gradient** of a vector field $\nabla\mathbf{v}$ is a tensor field defined by

$$\begin{aligned} \mathbf{A} &= \nabla\mathbf{v} \\ A_{ij} &= \partial_i v_j \end{aligned} \tag{A.7}$$

The **divergence** of a vector field, written as $\nabla \cdot \mathbf{v}$ is a scalar field defined by

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\ &= \partial_i v_i \end{aligned} \tag{A.8}$$

using the summation convection.

The **divergence** of a second-order tensor field $\nabla \cdot \mathbf{A}$ is a vector field

$$(\nabla \cdot \mathbf{A})_j = \partial_i A_{ij} \tag{A.9}$$

where we again sum over indices $i = 1, 2, 3$.

The **divergence theorem**, also known as **Gauss's theorem**, equates the volume integral of a vector field to the surface integral of the orthogonal component of the vector field, i.e.,

$$\int_V \nabla \cdot \mathbf{v}\, \mathrm{d}V = \int_S \mathbf{v} \cdot \hat{\mathbf{n}}\, \mathrm{d}S \tag{A.10}$$

where $\hat{\mathbf{n}}$ is the outward normal vector to the surface. Note that the same equality can be written for the divergence of a second-order tensor field.