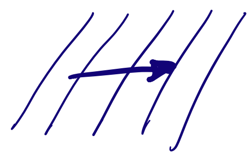
Body waves

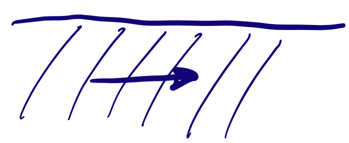
We can distinguish between 3 types of waves:

- body waves



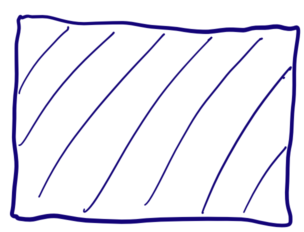
exist in any kind of media

- surface waves



need a free surface (& Love waves need layered structure)

- normal modes / free oscillations



property of finite volumes

Let's start with body waves. What can the equations of motion for an elastic medium

$$\rho \, \partial_t^2 \underline{u} - \underline{\nabla} \cdot \underline{\underline{T}} = \underline{f}$$

$$\text{with} \quad \underline{\underline{T}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}$$

tell us about the existence and characteristics of body waves?

Lamé's theorem: Consider an isotropic media where

$$\underline{\underline{T}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}} = \lambda \, \underline{\nabla} \cdot \underline{u} + \mu \, \underline{\underline{\delta}} \, \underline{\nabla} \, \underline{u}$$

with Lamé parameters λ, μ, we can write the wave equation as

$$\rho \, \partial_t^2 \underline{u} - \underbrace{(\lambda + 2\mu) \underline{\nabla}(\underline{\nabla} \cdot \underline{u})}_{\text{compressional motion}} + \underbrace{\mu \, \underline{\nabla} \times (\underline{\nabla} \times \underline{u})}_{\text{shear motion}} = \underline{f}$$

Using Helmholtz potentials, Lamé's theorem states that

- $\underline{u} = \underbrace{\underline{\nabla} \phi}_{\text{P-wave}} + \underbrace{\underline{\nabla} \times \underline{\psi}}_{\text{S-wave}}$ components

  displacement $\underline{u}$ is a superposition of a P- & S-wave

- the scalar potential ϕ and vector potential $\underline{\psi}$ follow

$$\begin{cases} \ddot{\phi} = \dfrac{\Phi}{\rho} + v_p^2 \nabla^2 \phi \\ \ddot{\underline{\psi}} = \dfrac{\underline{\Psi}}{\rho} + v_s^2 \underline{\nabla}^2 \underline{\psi} \end{cases}$$

with P-wave speed $v_p = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$

S-wave speed $v_s = \sqrt{\dfrac{\mu}{\rho}}$

[Poisson media: $\lambda = \mu \Rightarrow v_p = \sqrt{3}\, v_s$]

Note that since λ, μ are positive, it follows $v_p > v_s$

Analytical solution:

For a homogeneous fullspace, Stokes (1849) found already an analytical solution for displacement

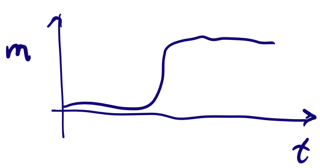
$$u_i(x,t) = \frac{3\gamma_i\gamma_j - \delta_{ij}}{4\pi\rho} \frac{1}{r^3} \int_{r/v_p}^{r/v_s} \tau\, \dot{m}(t-\tau)\, d\tau \qquad \text{near-field term}$$

$$+ \frac{\gamma_i\gamma_j}{4\pi\rho\, v_p^2} \frac{1}{r}\, \dot{m}\left(t - \frac{r}{v_p}\right) \qquad \text{far-field P-wave}$$

$$- \frac{\gamma_i\gamma_j - \delta_{ij}}{4\pi\rho\, v_s^2} \frac{1}{r}\, \dot{m}\left(t - \frac{r}{v_s}\right) \qquad \text{far-field S-wave}$$

How do near-field and far-field terms differ from each other?

- near-field term $\sim \frac{1}{r^2}$
  - static offset since it integrates over $\dot{m}$, thus $\sim m$



- far-field term $\sim \frac{1}{r}$
  - vanishes for $\dot{m} \to 0$ with a shape $\sim \dot{m}$