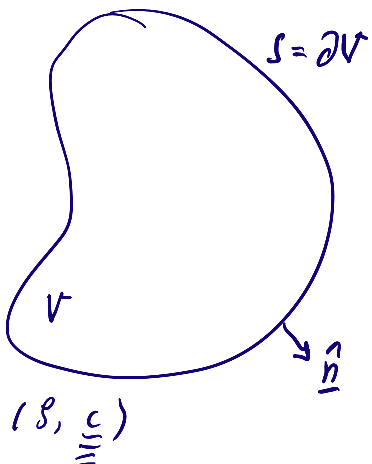
Basic theorems in dynamic elasticity



What is the displacement $\underline{u}$, given a source $\underline{f}$ and material properties $(\rho, \underline{\underline{\underline{\underline{c}}}})$?

From Newton's (second) law $\underline{F} = m\,\ddot{\underline{u}}$, we can write

$$\underbrace{\int_V \underline{f}\, dx^3}_{\text{body forces}} + \underbrace{\int_{\partial V} \hat{\underline{n}} \cdot \underline{\underline{\tau}}\, dx^2}_{\text{surface forces}} = \int_V \rho\, \ddot{\underline{u}}\, dx^3$$

which leads to the equations of motion

$$\boxed{\rho\, \ddot{\underline{u}} - \underline{\nabla} \cdot \underline{\underline{\tau}} = \underline{f}}$$

ρ : density

$\underline{\underline{\underline{\underline{c}}}}$: elastic tensor

The constitutive relationship relates stress with displacement. For elastic bodies, it is described by Hooke's law

$$\underline{\underline{\tau}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left[\underline{\nabla}\underline{u} + (\underline{\nabla}\underline{u})^T\right]$$

strain

We can thus write the equations of motion for an elastic medium as

$$\boxed{\rho\,\ddot{u} - \underline{\nabla}\cdot\left(\underline{\underline{\underline{\underline{c}}}} : \underline{\nabla}\,\underline{u}\right) = \underline{f}}$$

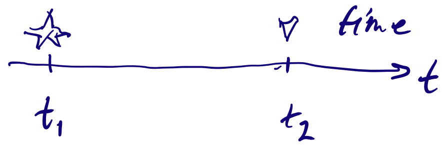
Note that the complexity of wave propagation relates to the complexity of the medium.

<u>Betti's theorem</u> : Determines a body's response to any forcing once its response to a specific forcing is found.

Betti states that displacement $\underline{u}$ is a unique solution to a given initial stress $\underline{\underline{\tau}}$ and force $\underline{f}$.

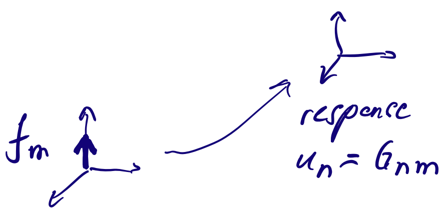
That is, if $(\underline{\underline{\tau}}, \underline{f}) = (\underline{\underline{\sigma}}, \underline{g})$ then we must have $\underline{u}(\underline{x}, t) = \underline{v}(\underline{x}, t)$ as well.

<u>Green's function:</u> Find the response to an excitation impulsive in space & time.



reciprocity in time:

$$\underline{\underline{G}}(\underline{x}, t_2 ; \underbrace{\underline{\xi}, t_1}_{\substack{\text{source at } \xi \\ \text{at time } t_1}}) = \underline{\underline{G}}(\underline{x}, -t_1 ; \underbrace{\underline{\xi}, -t_2}_{\substack{\text{source at } \xi \\ \text{at time } -t_2}})$$



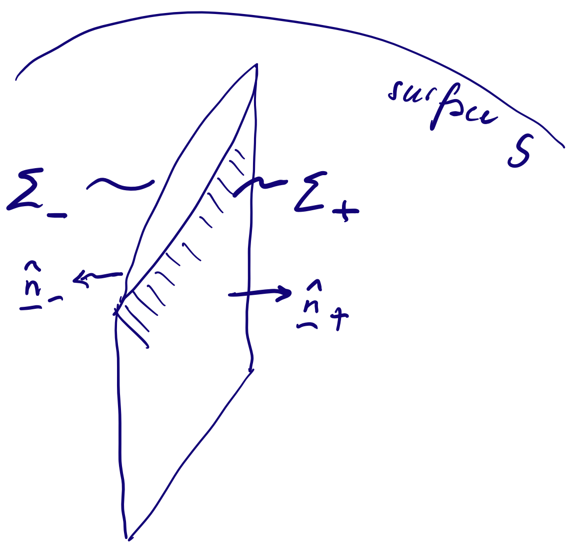
reciprocity in space:

$$\underbrace{G_{mn}}_{} (\underbrace{\underline{x}}_{}, t_2 ; \underline{\xi}, t_1) = G_{nm}(\underline{\xi}, t_2 ; \underline{x}, t_1)$$

switching location and component

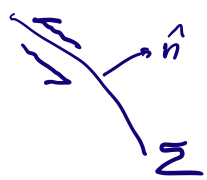
Representation theorem: Knowing displacement on the fault is enough to determine the displacement anywhere

$$\underbrace{\underline{u}(\underline{x},t)}_{\substack{\text{displacement}\\ \text{at receiver away}\\ \text{from fault}}} = \int_{-\infty}^{+\infty} d\tau \int_{\Sigma} \underbrace{[u(\xi,\tau)]}_{\substack{\text{displacement}\\ \text{on fault}}}\cdot\Big(\underbrace{[\underline{\underline{\underline{c}}} : \underline{\nabla}\,\underline{\underline{G}}(x,t-\tau;\underline{\xi},0)]}_{\text{stress } \sigma}\cdot\underline{\hat{n}}\Big)\,d\Sigma$$



fault $\Sigma = \Sigma_- \cup \Sigma_+$

welded boundary, no opening at fault:



$\underline{\underline{G}}$ and $\underline{\hat{n}}\cdot\underline{\underline{\tau}} = \underline{\hat{n}}\cdot(\underline{\underline{\underline{c}}}:\underline{\nabla}\underline{u})$

are continuous across Σ

Displacement written as convolutions,

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(T)\, g(t-T)\, dT,$$

becomes

$$\underline{u} = \underbrace{S}_{\text{source}} * \underbrace{P}_{\substack{\text{propagation} \\ \hat{=}\ \underline{\underline{G}}}} * \underbrace{I}_{\substack{\text{instrument} \\ \text{response}}}$$

Index notation: Aki & Richards (3.2)

$$u_n(\underline{x},t) = \int_{-\infty}^{\infty} dT \int_{\Sigma} \underbrace{[u_i(\underline{\xi},T)]}_{\text{slip function}} c_{ijpq}\, n_j \frac{\partial}{\partial \xi_q} G_{np}(\underline{x}, t-T; \underline{\xi}, 0)\, d\Sigma$$