Exercise: Units of the wave equation

mass $\times$ acceleration $=$ forces

$[kg]$ $\left[\frac{m}{s^2}\right]$ $[N] = \left[\frac{kg\, m}{s^2}\right]$ Newton

We mostly use the wave equation written as

$$\rho\, \partial_t^2 \underline{u} - \underline{\nabla} \cdot \underline{\underline{T}} = \underline{f}$$

What are the corresponding units?

The material properties are given by

- density ρ
- bulk & shear modulus, Young modulus, Lamé parameters

$\rightarrow$ what follows for the elastic tensor $\underline{\underline{\underline{\underline{c}}}}$?

Consider Hooke's law

$$\underline{\underline{T}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}$$ with strain $\underline{\underline{\varepsilon}}$ and stress $\underline{\underline{T}}$

What are the units of $\underline{\underline{T}}$ and $\underline{\underline{\varepsilon}}$?

Response:

$$\rho\, \partial_t^2 \underline{u} - \underline{\nabla} \cdot \underline{\underline{T}} = \underline{f}$$

$$\left[\frac{kg}{m^3}\right] \left[\frac{m}{s^2}\right] \quad \left[\frac{1}{m}\right]\left[\frac{N}{m^2}\right] \quad \left[\frac{N}{m^3}\right]$$

with f: force per unit volume

- bulk & shear modulus, Young modulus, Lamé parameter

$[GPa] = [10^9\, Pa]$ with $[Pa] = \left[\frac{N}{m^2}\right]$

Pascal

unit of pressure

typical values for rocks:

$\lambda, \mu \sim 20 - 120$ GPa

For isotropic media

$$c_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

has units $[Pa]$

Considering Hooke's law

$$\underline{\underline{T}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}$$

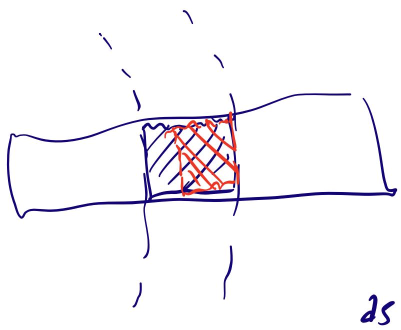
$$[Pa] = \left[\frac{N}{m^2}\right] \quad [Pa]\ [\cdot]$$

with strain $\underline{\underline{\varepsilon}} = \frac{1}{2}\left[\underline{\nabla}\underline{u} + (\underline{\nabla}\underline{u})^T\right]$

$[\cdot]\ \left[\frac{1}{m}\right][m]$

is dimensionless

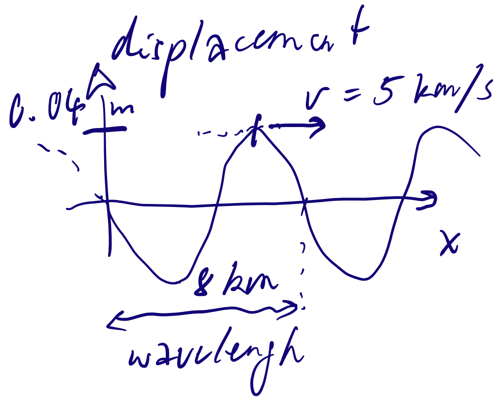
<u>Example on strain:</u>



Strain is the change in length compared to original length

$$\varepsilon \sim \frac{\text{change of length}}{\text{original length}}$$

as estimate $\varepsilon \sim \frac{4A}{\lambda}$ — max. amplitude; λ — wavelength



Let's assume a harmonic wave

$$u(x,t) = A \sin(\omega t - kx)$$

ω: angular frequency
k: wavenumber

with wavelength $\lambda = v\,T$,
period $T = \frac{2\pi}{\omega}$ and thus

$$\lambda = \frac{2\pi}{k}$$

Given (λ, v): period $T = \frac{\lambda}{v}$
$$= \frac{8\,km}{5\,km/s} = 1.6\,s$$

frequency $\omega = \frac{2\pi}{T} \sim 3.9\,(rad)$

wavenumber $k = \frac{2\pi}{\lambda}$
$\sim 0.8\,(km^{-1})$

Harmonic wave

$$u(x,t) = A \sin(\omega t - kx)$$
$$= A \sin\left(\frac{2\pi}{\lambda}(v t - x)\right)$$

Consider strain $\varepsilon = \frac{1}{2}(\nabla u + (\nabla u)^T)$

with

$$\frac{\partial}{\partial x} u = -\frac{2\pi}{\lambda} A \cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$$

leads to $\varepsilon_{max} \sim \frac{2\pi A}{\lambda} = \frac{2\pi \, 0.04 m}{8 \, km}$

$$\sim 3 \cdot 10^{-5}$$