Ray theory   High-frequency approximation

Let's consider the wave equation for an elastic isotropic medium

$$\rho \ddot{\underline{u}} = (\lambda+\mu)\underline{\nabla}(\underline{\nabla}\cdot\underline{u}) + \mu\underline{\nabla}^2\underline{u} + (\underline{\nabla}\cdot\underline{u})\underline{\nabla}\lambda + (\underline{\nabla}\mu)(\underline{\nabla}\underline{u}) + (\underline{\nabla}\underline{u})\cdot(\underline{\nabla}\mu)$$

where the Lamé parameter $\lambda = \lambda(\underline{r})$, $\mu = \mu(\underline{r})$ can vary in space.

We can substitute a monochromatic traveling wave

$$\underline{u}(\underline{r},t) = \underbrace{\underline{U}(r)}_{\text{local amplitude}} e^{i\omega[\underbrace{\sigma(r)}_{\text{phase factor}} - t]}$$

ω: angular frequency

with spatial derivation $\underline{\nabla}\underline{u}(r,t) = \underline{\nabla}\,\underline{U}(r) e^{i\omega[\sigma(r)-t]} - i\omega\,\underline{U}(r)\nabla\sigma(r)e^{i\omega[\sigma(r)-t]}$

time derivation $\partial_t^2 \underline{u}(\underline{r},t) = -\underline{U}(r)\omega^2 e^{i\omega[\sigma(r)-t]}$

and find terms proportional to ω_0, ω and ω^2

independent of ω

The high-frequency approximation neglects ω_0 terms and sets the others to zero, as it looks for solutions independent of ω $\underline{U}(r)$, $\sigma(r)$

ω^2 terms: $\left(\frac{\lambda+\mu}{\rho}\underline{\nabla}\sigma\underline{\nabla}\sigma + \frac{\mu}{\rho}\nabla\sigma\cdot\nabla\sigma\,\underline{\underline{I}} - \underline{\underline{I}}\right)\cdot\underline{U} \overset{!}{=} 0$

has non-trivial solutions for

eikonal equations
$$\begin{cases} \underline{\nabla}\sigma(\underline{r})\cdot\underline{\nabla}\sigma(\underline{r}) = \dfrac{\rho(\underline{r})}{\lambda(\underline{r})+2\mu(\underline{r})} & \longrightarrow \text{P-wave rays} \\[2ex] \underline{\nabla}\sigma(\underline{r})\cdot\underline{\nabla}\sigma(\underline{r}) = \dfrac{\rho(r)}{\mu(r)} & \longrightarrow \text{S-wave rays} \end{cases}$$

ω terms: set to zero $\longrightarrow$ "transport" equations for $\underline{U}(r)$