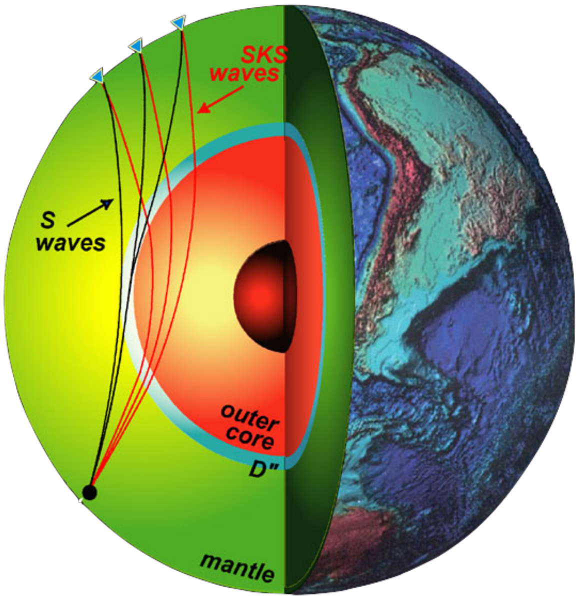
6 Brief recapitulation of ray theory and the eikonal equation



Ray paths for S and SKS waves are shown.
This particular phase pair is used for studying the very base of the mantle, the D" region.

Introduction

The representation theorem stipulates that, once the Green's problem is solved, knowing the initial displacement on a seismic fault is sufficient to determine the displacement everywhere in a given Earth model. The *Green's problem* is the relatively simple set of differential equations

$$\rho \frac{\partial^2}{\partial t^2}\mathbf{u} - \nabla \cdot (\mathbf{c} : \nabla \mathbf{u}) = \mathbf{f}, \tag{6.1}$$

with $\mathbf{f}$ a force impulsive in both space and time (remember that gravity and apparent rotational forces are neglected). Equation (6.1) is typically solved after determining the solution to its homogeneous ($\mathbf{f} = \mathbf{0}$) version

$$\rho \frac{\partial^2}{\partial t^2}\mathbf{u} - \nabla \cdot (\mathbf{c} : \nabla \mathbf{u}) = \mathbf{0}. \tag{6.2}$$

Solutions $\mathbf{u}(\mathbf{x}, t)$ to (6.2) can be found that depend on sets of arbitrary coefficients; the latter are then determined from the application of boundary conditions (stress-free outer surface) and of the impulsive forcing term.

In this chapter and the following ones, we shall focus on the problem of finding homogeneous solutions to the momentum equation, and we shall see how body- and surface-wave (in the ray-theory approximation) and normal-mode functions satisfy (6.2) and, after selection of the arbitrary parameters, (6.1).

Learn objectives

- You know the characteristics of a traveling wave.
- You are able to explain how to obtain the eikonal equations.
- You can discuss assumptions and validity regime of ray theory.

6.1 Traveling waves

Waves in one-dimensional space

In a one-dimensional space, consider any function of distance x and time t of the form $a(x, t) = a(x + ct)$, with c an arbitrary constant. The value of $a(x, t)$ at a given point x changes with time t in such a way that the overall shape of $a(x + ct)$ does not vary, but is uniformly translated over a distance $c\,\delta t$.

In fact, because of the generality of $a(x, t)$, looking for values of $(x,\, t)$ for which a takes a given value is equivalent to requiring $x + ct = b$, with b a constant; at time $t = t_0$ it follows that $x = b - ct_0$; at time $t = t_1$, $x = b - ct_1$; this is independent of $(x,\, b)$, and therefore equivalent to a translation $\delta x = c(t_1 - t_0)$ (i.e. with speed c) of the whole "waveform" $a(x, t)$.

A function with this property is called *"traveling wave"*. Should $a(x, t)$ describe a displacement, be careful not to confuse such displacement (particle motion) with the "apparent" translation associated with the propagation of the wave.

To better understand the concept of *traveling wave*, let us consider the simple example of a monochromatic sinusoidal traveling wave $\cos(kx + \omega t)$. Compare it with the function $\cos(kx)\cos(\omega t)$ (see figure 6.1).

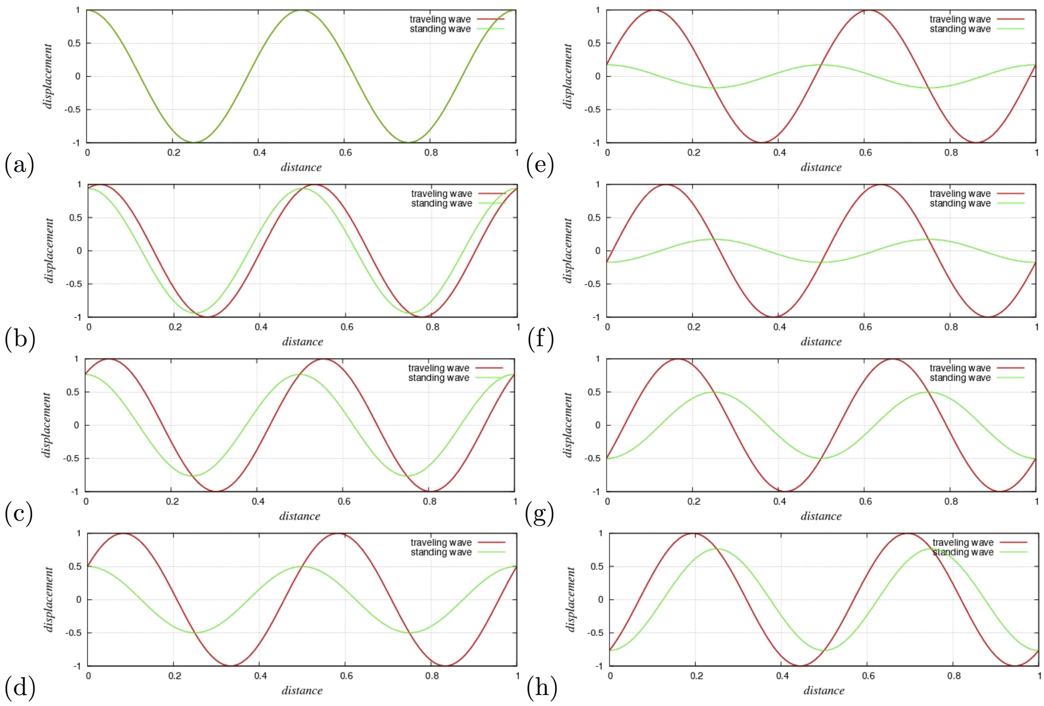


Figure 6.1: Displacement at increasing times (frames (a) through (h)) associated with a traveling wave (red) and a standing wave (green).

At time $t = 0$ the two functions are coincident, but their evolution in time is very different. In particular, as we know from trigonometry, $\cos(kx)\cos(\omega t) = 0$ independently of t for $x = \{\pi/2k, 3\pi/2k, 5\pi/2k, ...\}$. Those points are called "nodes" and $\cos(kx)\cos(\omega t)$ is an example of "standing wave". In this chapter we shall stick to traveling waves, but we will encounter standing waves again later.

Comparing $\cos(kx+\omega t)$ to the general function $a(x+ct)$, and in view of the above considerations on traveling waves, we can interpret the ratio $c=\omega/k$ as the *"phase velocity"* of the sinusoidal wave, "phase" being essentially defined by a fixed, "propagating" constant value of $a(x+ct)$. In the case of a periodic function like the cosine, the quantity ω also has a simple physical interpretation; if instead of considering the propagation of the waveform as a whole we sit at a fixed point x_0, we will notice that this point periodically returns to its initial position; the period of the cosine is 2π; after a time $t=2\pi/\omega$ the displacement at $x=x_0$ (or elsewhere) must return to its initial value; we therefore call ω the *"angular frequency"* of the wave.

Waves in three-dimensional space

In the more general, three-dimensional case, a *monochromatic* sinusoidal traveling wave can be written

$$\mathbf{u}=\mathbf{U}(\mathbf{r})\,\mathrm{e}^{\mathrm{i}\omega[\sigma(\mathbf{r})-t]}. \tag{6.3}$$

The function $\omega\,\sigma(\mathbf{r})$ is the three-dimensional counterpart of $k\,x$ as defined in the one-dimensional case. Imagine to select one possible value b of $\sigma(\mathbf{r})-t$: this defines the "phase"; at any time t, the corresponding "wavefront", or the locus of all points in space that are oscillating at that phase, coincides with the values of r that satisfy $\sigma(\mathbf{r})=b+t$. In three-dimensional space, the wavefront is a surface, perpendicular, at each location, to the direction of wave propagation. An example of the theoretical time-evolution of a surface-wave wavefront in a three-dimensional Earth model is visualized in figure 6.2. Once $\sigma(\mathbf{r})$ is known, the gradient of $\sigma(\mathbf{r})-t$ can be computed at any time t to determine the direction of propagation at $(\mathbf{r},t)$. Ray paths can then be identified.

6.2 Ray theory and the eikonal equation

Let us write the momentum equation for an elastic and isotropic medium in its explicit, cumbersome form,

$$\rho\ddot{\mathbf{u}}=(\lambda+\mu)\nabla\left(\nabla\cdot\mathbf{u}\right)+\mu\nabla^2\mathbf{u}+\left(\nabla\cdot\mathbf{u}\right)\nabla\lambda+\left(\nabla\mu\right)\cdot\left(\nabla\mathbf{u}\right)+\left(\nabla\mathbf{u}\right)\cdot\left(\nabla\mu\right), \tag{6.4}$$

Equation (6.4) holds for heterogeneous media, i.e. with λ and μ functions of $\mathbf{r}$. In index notation,

$$\rho\ddot{u}_i=(\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)+\mu\frac{\partial^2 u_i}{\partial x_k^2}+\frac{\partial\lambda}{\partial x_i}\frac{\partial u_k}{\partial x_k}+\frac{\partial\mu}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right), \tag{6.5}$$

with $i=1,2,3$, where x_1, x_2, x_3 and u_1, u_2, u_3 denote the components along the cartesian axes of $\mathbf{r}$ and $\mathbf{u}$, respectively, and summation is intended over repeated indexes (Einstein's notation).

Following, for example, *Červeny* (1985) equation (6.4) or (6.5) is solved by substituting a traveling-wave trial solution into it. Displacement $\mathbf{u}$ in (6.4) is replaced by its expression (6.3), and the resulting equation is solved for the unknown functions $\mathbf{U}(\mathbf{r})$ and $\sigma(\mathbf{r})$.

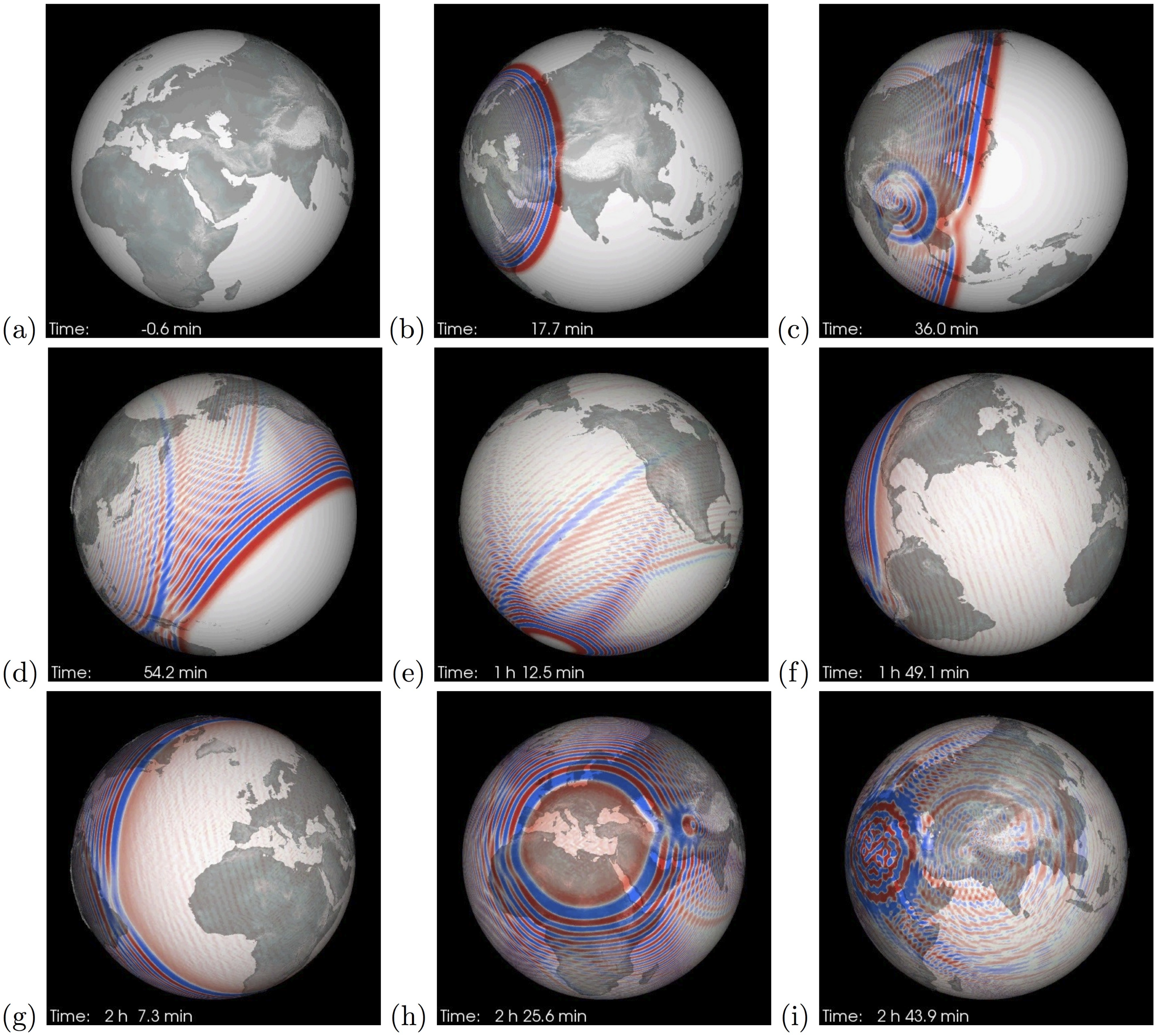


Figure 6.2: Numerically modeled, long-period Rayleigh waves propagating in a theoretical Earth model after an earthquake in the Aegean sea. Frames (a), (b),... (i) correspond to increasing time, as marked, from the time of the earthquake to about 2 hours and 45 minutes later. Focusing at the antipodes of the epicenter and at that the epicenter itself is visible in frames (e) and (i), respectively. The model is characterized by a particularly strong upper mantle heterogeneity in Tibet.

Notice that up to this point we have required no approximation, nor have we needed to introduce the concept of seismic ray.

High-frequency approximation

The most important approximation is made now: we decide to only consider waves of **high frequency**, $\omega \gg 1$. After substituting the trial solution (6.3) and Fourier-transforming, equation (6.2) can be written as the 0 = the sum of three terms: one proportional to ω^2, one to ω, and one not explicitly related to ω (ω^0 term).

The *high-frequency approximation*, dubbed WKBJ[1], consists of neglecting the ω^0 term, and equating to zero the other two terms separately: this is legitimate as we are looking for solutions $\mathbf{U}(\mathbf{r})$ and $\sigma(\mathbf{r})$ independent of ω.

Equating to zero the term multiplying ω^2,

$$\left(\frac{\lambda+\mu}{\rho}\nabla\sigma\nabla\sigma + \frac{\mu}{\rho}\nabla\sigma\cdot\nabla\sigma\,\mathbf{I} - \mathbf{I}\right)\cdot\mathbf{U} = \mathbf{0}. \tag{6.6}$$

If we call $\mathbf{\Gamma}$ the sum of the first two terms within brackets in equation (6.6),

$$(\mathbf{\Gamma} - \mathbf{I})\cdot\mathbf{U} = \mathbf{0}. \tag{6.7}$$

From linear algebra we know that there exists a nontrivial solution to (6.7) if and only if $\mathbf{\Gamma}$ has at least one eigenvalue equal to 1 (hence $\det(\mathbf{\Gamma} - \mathbf{I}) = 0$). The eigenvalues of $\mathbf{\Gamma}$ are found to be $\frac{\lambda+2\mu}{\rho}\nabla\sigma\cdot\nabla\sigma$ and $\frac{\mu}{\rho}\nabla\sigma\cdot\nabla\sigma$; if we equate to 1 each of them separately, the *"eikonal" equations* follow:

$$\nabla\sigma(\mathbf{r})\cdot\nabla\sigma(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\lambda(\mathbf{r}) + 2\mu(\mathbf{r})}, \tag{6.8}$$

and

$$\nabla\sigma(\mathbf{r})\cdot\nabla\sigma(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\mu(\mathbf{r})} \tag{6.9}$$

from which the wavefront $\sigma(\mathbf{r})$ is determined.

Likewise, from the ω^1 term, an equation dubbed *"transport" equation*, determining the amplitude $\mathbf{U}(\mathbf{r})$, is found. From this equation the decoupling is found between *compressional* (P) and *shear* (S) waves. P-wave displacement is parallel to the direction of propagation and propagates with speed $\sqrt{\frac{\lambda+2\mu}{\rho}}$ along the ray paths identified by equation (6.8). S-wave displacement is perpendicular to the direction of propagation and propagates with speed $\sqrt{\frac{\mu}{\rho}}$ along the ray paths identified by equation (6.9). We are not going to treat the transport equation in any further detail here.

[1]WKB theory is treated in chapter 10 of *Bender and Orszag* (1978). The acronym stands for Wentzel-Kramers-Brillouin, who first devised the method. WKB theory was introduced to seismology by Harold Jeffreys, hence JWKB or WKBJ.

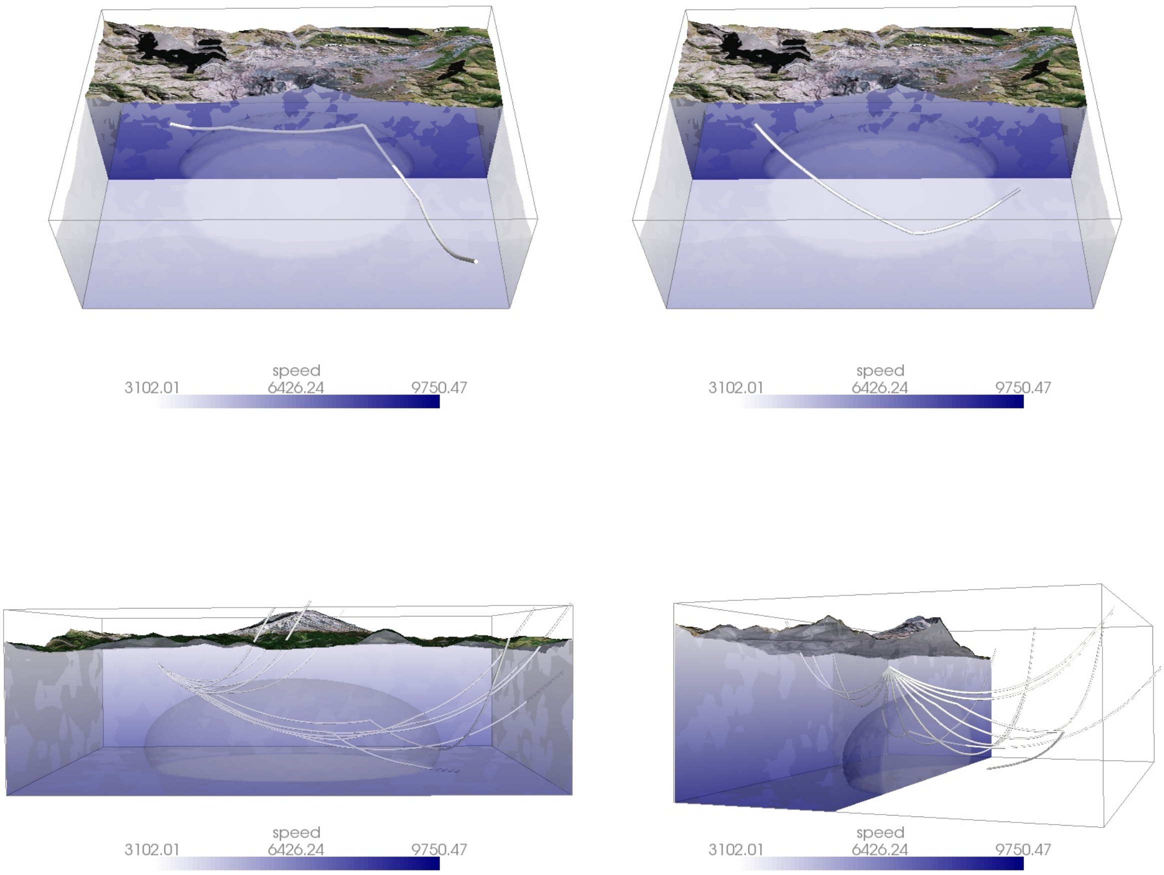


Figure 6.3: Rays traced in a three-dimensional model through a magma chamber under Mount St.Helens, showing reflection and refraction at the boundary surface.

To summarize, it is important to remember that the ray-theory ansatz (6.3) works as a solution of the momentum equation (6.4) only in the assumption that frequency be high. If this hypothesis does not hold, the ray-theory approach ceases to be valid and it becomes meaningless to speak of seismic wavefronts and seismic rays. In the real world, the ray-theory method often turns out to be a valid and useful approximation.

References and Further Reading

- Bender, C. M., and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, 1978
- Brokešova, J., *Asymptotic Ray Method In Seismology: A Tutorial*, Matfyz Press, 2006. Available online at: http://www.spice-rtn.org/library/training-material
- Červeny, The application of ray-tracing to the propagation of shear waves in complex media, in *Seismic Shear Waves. Part A: Theory*, ed. by G. Dohr, Geophysical Press, 1985.