Normal modes of the Earth

For long-period waves (> 100 s period), self-gravity effects become important and we can write the linearized equations of motion as

$$\rho_0\, \partial_t^2 \underline{u}(r,t) = \nabla \cdot \tau'(r,t) \underbrace{- \rho_0 \nabla \phi'(r,t) - \rho' \nabla \phi_0(r,t)}_{\text{gravity effects}}$$

with $\begin{cases} \nabla^2 \phi' = 4\pi G \rho' & \text{compressible} \\ \nabla^2 \phi' = 0 & \text{incompressible } (\rho' = 0) \end{cases}$

where we have introduced the gravitational potential $\phi(r)$ and the linearized forms

$$\rho(r,t) = \rho_0(r) + \rho'(r,t)$$
$$\phi(r,t) = \phi_0(r) + \phi'(r,t)$$
$$\tau(r,t) = \tau_0(r) + \tau'(r,t)$$

with small perturbations in density ρ, potential ϕ, and stress τ

In general, the gravitational potential is governed by Poisson's equation

$$\nabla^2 \phi(r,t) = 4\pi G \rho(r,t)$$

SNREI (Spherically symmetric, Non-Rotating, Elastic Isotropic) Earth

In frequency domain:

$$-\rho_0 \omega^2 \underline{u}(r,\omega) = \underline{\nabla} \cdot \underline{\underline{\tau}}'(r,\omega) - \rho_0 \nabla\phi'(r,\omega) - \rho' \nabla\phi_0(r,\omega)$$

We separate time & space and radial & horizontal components

$$\underline{u}(\underline{r},\omega) = \underbrace{U_{lm}(r,\omega)\underline{P}_{lm}(\vartheta,\phi) + V_{lm}(r,\omega)\underline{B}_{lm}(\vartheta,\phi)}_{\text{spheroidal}} + \underbrace{W_{lm}(r,\omega)\underline{C}_{lm}(\vartheta,\phi)}_{\text{toroidal}}$$

$$\phi(\underline{r},\omega) = P_{lm}(r,\omega)\, Y_{lm}(\vartheta,\phi)$$

where $Y_{lm}(\vartheta,\phi)$: spherical harmonics (angular degree l, azimuthal order m)

$\underline{P}_{lm}$, $\underline{B}_{lm}$, $\underline{C}_{lm}$: vector spherical harmonics

Assuming small perturbation, i.e., $\underline{\underline{\tau}} = \underline{\underline{\tau}}_0 + \underline{\underline{\tau}}'$ with respect to hydrostatic pressure $\underline{\underline{\tau}}_0 = -p_0 \underline{\underline{I}}$, and density $\rho = \rho_0 + \rho'$ with $\rho' = -\underline{\nabla}\cdot(\rho_0 \underline{u})$ leads to a system

$$\left\{\begin{array}{ll} \frac{1}{r^2}\frac{d}{dr}[r^2(\lambda+2\mu)\dot{U} + \ldots & = 0 \\ \frac{1}{r^2}\frac{d}{dr}[\mu r^2(\dot{V} - \ldots & = 0 \end{array}\right\} \text{spheroidal}$$

$$\left\{\begin{array}{ll} \frac{1}{r^2}\frac{d}{dr}[\mu r^2(\dot{W} - \ldots & = 0 \quad \} \text{ toroidal} \leftarrow \text{decoupled} \\ \dot{P} + \frac{2}{r}P - \ldots & = -4\pi G\rho U - \ldots \end{array}\right.$$

For each (l,m), we find a discrete set of eigenfrequencies ω_{nlm} associated with eigenfunctions U_{lm}, V_{lm} and W_{lm}.

Properties of SNREI:

- toroidal-spheroidal decoupling:

  W_{lm} (toroidal) independent of U_{lm} & V_{lm} (spheroidal) and so are eigenfrequencies

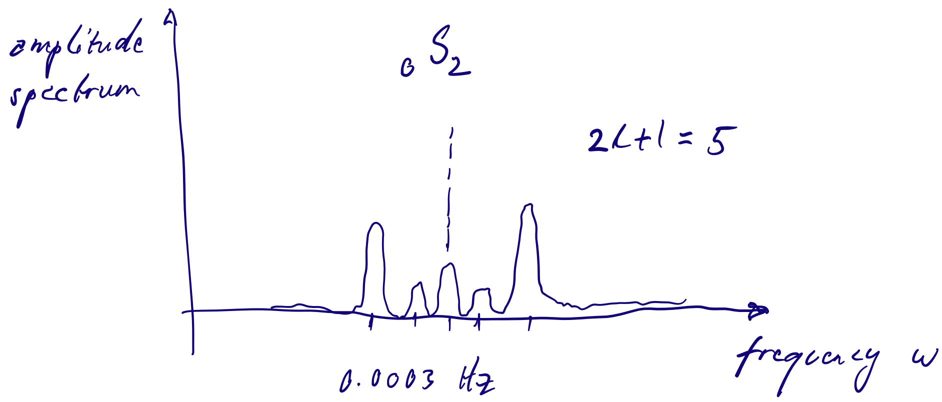
  ω^T_{nlm} (toroidal) with n = overtone number

  ω^S_{nlm} (spheroidal)   n = 0 fundamental mode; n = 1 1. overtone

- degeneracy: independent of m ($-l \leq m \leq l$), i.e., all $2l+1$ modes are the same of angular degree l: ${}_nT_l = W^n_{lm}(r)\, C_{lm}(\vartheta, \varphi)$ multiplet

Rotation, ellipticity and 3D heterogeneity remove the degeneracy

$\Longrightarrow$ for each (n, l) there are $2l+1$ singlets with different frequencies



Green's tensor & synthetic seismograms:

Modes $s_k(r)$, $k=(n,l,m)$ and eigenfrequency ω_k form a complete basis such that we can write the Green's tensor as

$$\underline{\underline{G}}(r,r';t) = \sum_k \frac{1}{\omega_k} \underline{s}_k(r)\underline{s}_k(r') \sin(\omega_k t)$$

"mode summation"

Splitting functions:

The contribution of an isolated multiplet to the displacement in the perturbed Earth is given by

$$u(t) = \mathrm{Re}\left\{ \underline{r}^T \cdot e^{i\underline{\underline{H}}t} \cdot \underline{s}\, e^{i\omega t} \right\}$$

with $\underline{r} = \{\underline{v}\cdot\underline{u}_m\}$, m entries, $\underline{s} = \{\underline{\underline{M}} : \nabla \underline{u}_m\}$, and eigenfrequency ω of degenerate multiplet.

The splitting matrix $e^{i\underline{\underline{H}}t}$ governs multiplet splitting:

$$H_{ij} = \omega \sum_{s,t} \underbrace{c_{st}}_{\text{"splitting coefficients"}} \int_\Omega Y_{li}^* Y_{st} Y_{lj}\, d\Omega$$

with corresponding splitting function $f = \sum_{s=0,2,..}^{2l} \sum_{t=-s}^{s} c_{st} Y_{st}$