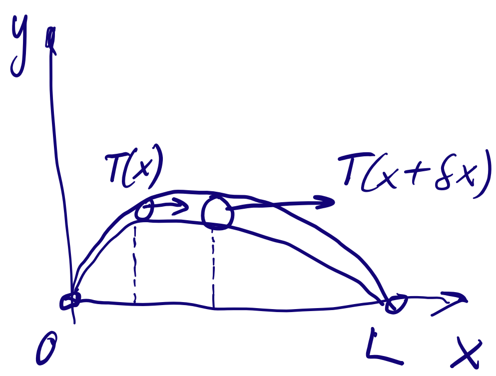
Normal modes of an ideal string



Let's look at displacement $u(x,t)$ along y-direction with a string fixed 0/L. The governing equation becomes

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)$$

with wave speed $c = \sqrt{\frac{\mu}{\rho}}$

Boundary conditions: $u(0,t) = 0$ and $u(L,t) = 0$

Separation of variables: $u(x,t) = \underbrace{\xi(x)}_{\text{spatial}} \cdot \underbrace{\sigma(t)}_{\text{time}}$

leads to $\underbrace{c^2 \frac{\ddot{\xi}(x)}{\xi(x)}}_{\text{spatial}} = \underbrace{\frac{\ddot{\sigma}(t)}{\sigma(t)}}_{\text{temporal}} = \lambda$ constant

for all x and all t

solution of spatial term: $\xi(x) = b \sin\left(\frac{\sqrt{|\lambda|}}{c} x\right)$

$= b \sin\left(\frac{k\pi}{L} x\right) \quad k = 1, 2, 3, \ldots$

eigenvalues $\lambda = \lambda_k$

$= -\left(\frac{k\pi c}{L}\right)^2$

solution of temporal term: $\sigma(t) = a' \cos\left(\frac{k\pi c}{L} t\right) + b' \sin\left(\frac{k\pi c}{L} t\right)$

Standing waves: solutions for each k

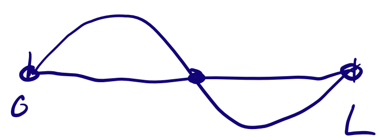
$$u_k(x,t) = \xi_k(x)\,\sigma_k(t) = \alpha_k \cos\left(\frac{k\pi c}{L}t\right)\sin\left(\frac{k\pi}{L}x\right) + \beta_k \sin\left(\frac{k\pi c}{L}t\right)\sin\left(\frac{k\pi}{L}x\right)$$

properties: - $\xi_k(x)$ are orthogonal functions

- $u(x,t)$ can be written as the sum over all eigenfunctions $u_k(x,t)$

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} u_k(x,t)}$$

- nodes: $u_n\left(\frac{L}{k\pi}, t\right) = 0$ at $x = \frac{nL}{k}$ for all times t, $n = 0, 1, 2, .., k$



$k = 2 \rightarrow$ nodes $x = \frac{0L}{2} = 0$

$x = \frac{1}{2}L$

$x = \frac{2}{2}L = L$

eigenfunctions $u_k(x,t)$ are called "normal modes" with eigenfrequency $\left(\frac{k\pi c}{L}\right)$