MATH5305 Computational Mathematics
Assignment 2

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Question 1

The instal-value problem is a separable ODE which becomes

$$\frac{du}{dt} = \epsilon^{-2}u(1-u^2)$$

$$\int \frac{1}{u}\frac{1}{1-u}\frac{1}{1+u}du = \int \epsilon^{-2}dt \tag{1}$$

Using partial fractions we see that

$$\frac{1}{u}\frac{1}{1-u}\frac{1}{1+u} = \frac{1}{u} + \frac{1}{2}\times\frac{1}{1-u} - \frac{1}{2}\times\frac{1}{1+u} \tag{2}$$

So 1 becomes

$$\begin{aligned}
\int \frac{1}{u} + \frac{1}{2}\frac{1}{1-u} - \frac{1}{2}\frac{1}{1+u}du &= \epsilon^{-2}t + C, \quad \text{where } C \text{ is a constant} \\
\ln|u| - \frac{1}{2}\ln|1-u| - \frac{1}{2}\ln|1+u| &= \epsilon^{-2}t + C \\
\ln|u| - \ln\sqrt{1-u} - \ln\sqrt{1+u} &= \epsilon^{-2}t + C \\
\ln\left(\frac{u}{\sqrt{1-u^2}}\right) &= \epsilon^{-2}t + C \\
\frac{u}{\sqrt{1-u^2}} &= Ae^{\epsilon^{-2}t}, \quad \text{where } A = e^C
\end{aligned}$$

Subbing in the initial condition gives $A = \frac{u_0}{\sqrt{1-u_0^2}}$

$$\begin{aligned}
u &= A\sqrt{1-u^2}e^{\epsilon^{-2}t} \\
u^2 + A^2u^2e^{2\epsilon^{-2}t} &= A^2e^{2\epsilon^{-2}t} \\
u^2 &= \frac{A^2e^{2\epsilon^{-2}t}}{1+A^2e^{2\epsilon^{-2}t}} \\
u(t) &= \frac{Ae^{\epsilon^{-2}t}}{\sqrt{1+A^2e^{2\epsilon^{-2}t}}}, \quad \text{where } A = \frac{u_0}{\sqrt{1-u_0^2}}
\end{aligned}$$

To find the limit as t goes to infinity, we divide the top and bottom of the fraction by $e^{\epsilon^{-2}t}$ which gives

$$u(t) = \frac{A}{\sqrt{e^{-2\epsilon^{-2}t} + A^2}}$$

$$\lim_{t\to\infty} u(t) = \frac{A}{\sqrt{A^2}} = sgn(A), \quad \text{where } sgn(.) \text{ is the sign function}$$

$$\lim_{t\to\infty} u(t) = \begin{cases} 1 & \text{if} \quad u_0 > 0 \\ 0 & \text{if} \quad u_0 = 0 \\ -1 & \text{if} \quad u_0 < 0 \end{cases}$$