

## Homework 2: Linear Regression

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In class we studied the linear regression model  $\mathbf{y} = \mathbf{X}\boldsymbol{\theta}^* + \boldsymbol{\epsilon}$ , under the *homoskedastic* assumption  $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ . In this homework you will derive the same results for the slightly more general *heteroskedastic* model where  $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}^*)$ . Each subproblem is worth 10 points.

**Problem 2.1.** Derive an expression for the coefficient vector  $\boldsymbol{\theta}$  that minimizes the mean squared error, i.e.,

$$\arg \min_{\boldsymbol{\theta}} \|\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\|_2^2.$$

**Problem 2.2.** Derive an expression for the maximum likelihood estimator (MLE) of  $\boldsymbol{\theta}^*$ , i.e.,

$$\arg \max_{\boldsymbol{\theta}} \mathbb{P}(\mathbf{y}, \mathbf{X} | \boldsymbol{\theta}, \boldsymbol{\Sigma}^*)$$

**Problem 2.3.** What is the distribution of the MLE of  $\boldsymbol{\theta}^*$ ?

**Problem 2.4.** Given a new sample with feature vector  $\mathbf{x}$ , what is the MLE of the response,  $\hat{y}$ ?

**Problem 2.5.** Given a new sample with feature vector  $\mathbf{x}$ , what is the distribution of the MLE  $\hat{y}$ ?

**Problem 2.6.** Derive an expression for the MLE of  $\boldsymbol{\Sigma}^*$ , i.e.,

$$\arg \max_{\boldsymbol{\Sigma}} \mathbb{P}(\mathbf{y}, \mathbf{X} | \boldsymbol{\theta}^*, \boldsymbol{\Sigma})$$

**Problem 2.7.** Consider the following vector  $\mathbf{y}$ , containing information about glucose level of four individuals, and the following data matrix  $\mathbf{X}$  containing information about height and weight of the corresponding individuals:

$$\mathbf{y} = \begin{bmatrix} 110 \\ 140 \\ 180 \\ 190 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 180 & 150 \\ 150 & 175 \\ 170 & 165 \\ 185 & 210 \end{bmatrix}.$$

Given these data

- What are your maximum likelihood estimates of  $\boldsymbol{\Sigma}^*$  and  $\boldsymbol{\theta}^*$ ?
- Given a new sample with feature vector  $\mathbf{x} = [175 \ 170]^T$ , what is the maximum likelihood estimate of its response  $\hat{y}$ ?
- Derive a 95% confidence interval for  $\hat{y}$ .
- Would you conclude that height is a significant feature for this model? Why?
- Would you conclude that weight is a significant feature for this model? Why?