Anna Fino: Eternal solutions to the G2-Laplacian flow

Overall plan: A closed G_2-structure on a 7-manifold M is given by a d-closed differential 3-form satisfying a non-degeneracy condition. In the lectures we will study closed G_2-structures satisfying the extremally Ricci-pinched (ERP) condition introduced by Bryant. In particular we will study the properties of compact manifolds satisfying the ERP condition and the behaviour of the Laplacian flow starting from an ERP closed G_2-structure. Moreover, we will review the known examples of compact 7-manifolds admitting ERP closed G_2-structures. The lectures are related to the following paper:
A. Fino, A. Raffero, A class of eternal solutions to the G_2-Laplacian flow. J. Geom. Anal. 31 (2021), no. 5, 4641–4660.

Talk 1.1: Closed G_2-structures on smooth manifolds

Abstract: In the lecture we will review stable 3-forms in dimension seven and closed G_2-structures on 7-manifolds, following §2.1 and §2.2 of the paper.

Literature:

- N. Hitchin, The geometry of three-forms in six and seven dimensions, `https://arxiv.org/abs/math/0010054v1`.
- R. L. Bryant, Some remarks on G_2-structures, Proceedings of Gokova Geometry-Topology Conference 2005, Gokova Geometry/Topology Conference (GGT), Gokova, 2006, pp. 75-109.
- M. Fernández, A. Gray, Riemannian manifolds with structure group G_2, Annali di Mat. Pura Appl. 32 (1982), 19-45.

Talk 1.2: Associative and Coassociative submanifolds

Abstract: In this lecture we will review the notion of a calibration as introduced by Harvey–Lawson, and review calibrated and co-calibrated submanifolds of 7-manifolds endowed with a G_2-structure. In particular we will see that calibrated submanifolds are minimal and classify them in $\mathbb{R}^7$, and we will see examples of associative/coassociative submanifolds. We will follow §4.1, §4.3, §12.1 and §12.3 in Joyce's book (without the bits on deformation theory).

Literature:

- R. Harvey, H.B. Lawson Jr., Calibrated geometries, Acta Math. 148, 47–157 (1982)
- D.D. Joyce, Riemannian Holonomy Groups and Calibrated Geometry, Oxford Graduate Texts in Mathematics, vol. 12. Oxford University Press, Oxford (2007).
- J. Lotay, Calibrated Submanifolds, Fields Institute Communications "Lectures and Surveys on G2 manifolds and related topics" `https://arxiv.org/abs/1810.08709`

Talk 1.3: The G_2-Laplacian flow and Laplacian solitons

Abstract: Following §2.3 of the paper we will review the G_2-Laplacian flow for closed G_2-structures, showing its relationship with a natural volume functional and its main properties

(short time existence and the evolution of the induced metric). Moreover, we will review soliton solutions of the Laplacian flow, proving some non-existence results in the compact case.

Literature:

- R. L. Bryant, Some remarks on G_2-structures, Proceedings of Gokova Geometry-Topology Conference 2005, Gokova Geometry/Topology Conference (GGT), Gokova, 2006, pp. 75-109.
- R. L. Bryant, F. Xu, Laplacian flow for closed G_2-structures: Short time behavior, arXiv:11d01.2004
- J. D. Lotay, Y. Wei: Laplacian flow for closed G_2 structures: Shi-type estimates, uniqueness and compactness. Geom. Funct. Anal. 27(1), 165–233 (2017)
- S. Karigiannis, Flows of G_2-structures.I. Q. J. Math. 60(4),487–522 (2009)
- C. Lin, Laplacian solitons and symmetry in G_2-geometry. J. Geom. Phys. 64, 111–119 (2013)
- L. Bedulli, L. Vezzoni, A remark on the Laplacian flow and the modified Laplacian co-flow in G_2-geometry. Ann. Global Anal. Geom. 58 (2020), no. 3, 287–290.

Talk 1.4: Extremally Ricci-Pinched Closed G_2-Structures

Abstract: Following §3 in the paper we will review the notion of extremally Ricci-pinched closed G_2-structures introduced by Bryant, and prove restrictions on such structures on compact manifolds.

Literature:

- R. L. Bryant, Some remarks on G_2-structures, Proceedings of Gokova Geometry-Topology Conference 2005, Gokova Geometry/Topology Conference (GGT), Gokova, 2006, pp. 75-109.
- R. Cleyton, S. Ivanov, Curvature decomposition of G_2-manifolds, J. Geom. Phys. 58(10) (2008), 1429–1449 .

Talk 1.5: Laplacian Flow Starting from an ERP Closed G_2-Structure

Abstract: Following §4 and §5 in the paper we will explicitly describe the solution of the G_2-Laplacian flow starting from an ERP closed G_2-structure on a compact 7-manifold and we will investigate its properties. In particular, we will show that the solution exists for all real times and that it remains ERP. We will also study the asymptotic behaviour of the ERP solution.

Talk 1.6 (by the mentor): Examples of 7-manifolds endowed with an ERP closed G_2-structures

Abstract: In the lecture we will present the known constructions of compact 7-manifolds admitting an ERP closed G_2-structure.

Anusha M. Krishnan: Ricci flow invariant curavture conditions

Overall plan: In many of the geometric applications of the Ricci flow, an important first step is identifying curvature conditions in an initial metric that continue to hold as the metric is evolved under the flow. Even better are conditions that "pinch" towards stronger curvature conditions, which allows for drawing topological conclusions. This series of talks will explore results from the literature about Ricci flow invariant curvature conditions, and some applications. To complete the picture, we will also see some curvature conditions that are not preserved under the Ricci flow. The sections below lay out the plan for each talk. Feel free to use your discretion as to how much detail to provide, and where to only explain the statements of theorems and skip proofs.

Talk 2.1: Introduction to the Ricci flow

Abstract: In this talk we will see definitions and some first examples of the Ricci flow. We will see the statements of short time existence and uniqueness of the Ricci flow on closed manifolds. We will see how curvatures evolve under the flow. We will see the statement of the scalar maximum principle.

Literature: Sections 1.1, 1.2.1, 1.3.1, Section 5 (only Theorems 5.2.1, 5.2.2), Sections 2.5, 3.1 in [PT].

- [PT] Peter Topping – Lectures on the Ricci flow. `http://homepages.warwick.ac.uk/maseq/RFnotes.html`

Talks 2.2 and 2.3: The tensor maximum principle and applications in dimension 3

Abstract: We will use the maximum principle to draw conclusions about scalar curvature evolution under the Ricci flow. We will see Hamilton's tensor maximum principle, stated in the form of an ODE-PDE theorem. As an application, we will see that the conditions of positive sectional curvature and positive Ricci curvature are preserved under the Ricci flow on closed 3-manifolds. We will see Hamilton's theorem that a compact simply-connected 3-manifold with positive Ricci curvature is diffeomorphic to the 3-sphere.

Literature: For talk 2.2, sections 3.2, 9.1, 9.2, 9.3, 9.6 (only the statement of Theorem 9.6.1) in [PT]. For talk 2.3, section 9.7, 10.1 (only the statement of Theorem 10.1.1) in [PT].

- [PT] Peter Topping – Lectures on the Ricci flow. `http://homepages.warwick.ac.uk/maseq/RFnotes.html`

Talk 2.4: An application of the maximum principle in higher dimensions

Abstract: We will see the theorem of Böhm and Wilking that a compact Riemannian manifold with finite fundamental group and nonnegative sectional curvature, also admits a metric with positive Ricci curvature. We will see some of the ideas that go into the proof of this result. We will also see the other side of the picture: an example where sec ¿ 0 is not preserved under the Ricci flow.

Literature: Introduction and section 3 of [BW].

- [BW] C. Böhm and B. Wilking - Nonnegatively curved manifolds with finite fundamental group admit metrics with positive Ricci curvature. Geom. Funct. Anal. 17 (2007), no. 3, 665–681.

Talk 2.5: A Lie algebraic approach to Ricci flow invariant curvature conditions

Abstract: In this talk we will see a Lie algebraic approach to Ricci flow invariant curvature conditions that provides a unified proof of the previously known invariant nonnegativity conditions.

Literature: Introduction (can skip the parts that relate to Theorems 3 and 4) and Section 1 of [Wil].

- [Wil] B.Wilking - A Lie algebraic approach to Ricci flow invariant curvature conditions and Harnack inequalities, J. Reine Angew. Math., 679 (2013), 223–247.

Talk 2.6 (by the mentor): Outlook

Abstract: We will survey other results relating to curvature evolution along the Ricci flow, including curvature conditions not invariant under the flow.

Ben Lambert: An Introduction to Lagrangian Mean Curvature Flow

Overall plan: In general Mean Curvature Flow (MCF) - the method of deforming a surface to decrease its area at the fastest possible rate - can be very badly behaved, with a large variety of singularities. However, Lagrangian Mean Curvature Flow (LMCF) has an array of surprisingly nice identities coming from the extra structure of a background Calabi–Yau manifold (which, for this week, will be $\mathbb{C}^n$). One motivating factor is the intriguing Thomas–Yau conjecture which states conditions under which LMCF is hoped to exist for all time and converge to Special Lagrangians, a particular kind of minimal surface. The aim of the below talks is to give an introduction to (Lagrangian) Mean Curvature Flow, describe some of the key quantities used, and work through some of the statements in "Lagrangian Mean Curvature Flow with boundary" (by C. Evans, myself and A. Wood). However I am happy to amend this schedule according to interest/background of the speakers (and expect to do this). Please do email me (`b.s.lambert@leeds.ac.uk`) to have a chat about any of the below topics!

Talk 3.1: Introduction to Mean Curvature Flow

Abstract: In this talk Mean Curvature Flow will be introduced, and we will see some first examples of the flow. Via applications of the maximum principle, we will meet some initial properties such as that compact mean curvature flow always produces singularities, that curvature controls imply higher order estimates and curvature growth. Finally we will introduce type I and type II singularities. If there is time, we will meet the monotonicity formula and talk about special solutions to the flow.

Literature: I suggest initially following selected topics from [FS] section 3, but I note that for higher codimension we will technically need evolutions from [KS].

- [FS] Felix Schulze – Introduction to Mean Curvature Flow `https://www.felixschulze.eu/images/mcf_notes.pdf`
- [KS] Knut Smoczyk – Mean Curvature Flow in Higher Codimension - An introduction and survey. `https://arxiv.org/abs/1104.3222`
- [AW] Albert Wood – Singularities of Lagrangian Mean Curvature Flow (PhD thesis) `https://discovery.ucl.ac.uk/id/eprint/10109495/`
- [CM] Carlo Mantegazza, Lecture Notes on Mean Curvature Flow (book)
- [KE] Klaus Ecker, Regularity Theory for Mean Curvature Flow (book)

Talk 3.2: Lagrangians and Lagrangian Mean Curvature Flow

Abstract: We introduce Lagrangian manifolds in $\mathbb{C}^n$ (or more generally, Calabi-Yau manifolds), describe some of their properties and give several examples. In particular we will meet the Lagrangian angle and special Lagrangians, and give some examples of these. We will also introduce Smoczyk's Theorem, namely that a mean curvature flow starting from a Lagrangian remains Lagrangian (i.e. LMCF is well defined). We will also state (and hopefully derive) the evolution equation for the Lagrangian angle.

Literature: Essentially following selected topics from [JL]. I include [JL2] for reference. [AW] section 2.4.2 is also a good reference.

- [JL] Jason Lotay – Introduction to Lagrangian Mean Curvature Flow `https://people.maths.ox.ac.uk/lotay/JDLotay_IntroLMCF.pdf`
- [JL2] Jason Lotay – Calibrated Geometry and Geometric Flows `https://people.maths.ox.ac.uk/lotay/JDLotay_cggf2017.pdf`
- [AW] Albert Wood – Singularities of Lagrangian Mean Curvature Flow (PhD thesis) `https://discovery.ucl.ac.uk/id/eprint/10109495/`

Talk 3.3: Monotonicity methods for MCF

Abstract: Many fundamental results in (L)MCF arise from the monotonicity formula, and in this talk we will discuss several of these. We will define the Gaussian ratios and Gaussian density and understand more on the structure of Type I singularities. We will prove White's regularity theorem and demonstrate several applications. As a result we will see that zero Maslov LMCF does not form Type I singularities.

Literature: I would suggest following [FS] to begin with, but with additions from Chapter 4 of [KE] and (a modification of the end of) [JL].

- [FS] Felix Schulze – Introduction to Mean Curvature Flow `https://www.felixschulze.eu/images/mcf_notes.pdf`
- [KE] Klaus Ecker, Regularity Theory for Mean Curvature Flow (book)
- [CM] Carlo Mantegazza, Lecture Notes on Mean Curvature Flow (book)
- [JL] Jason Lotay – Introduction to Lagrangian Mean Curvature Flow `https://people.maths.ox.ac.uk/lotay/JDLotay_IntroLMCF.pdf`

Talk 3.4: A boundary condition for LMCF

Abstract: In this talk we will introduce LMCF with boundary. Boundary conditions that preserve the Lagrangian condition are exceptional - most will not, and here we prove that a mixed Dirichlet–Neumann boundary condition does do this. The method is to first have a general solution to MCF with a possibly "different" boundary condition and then show that this is Lagrangian. This is section 4 of the paper, but we will prove this only in the easier case in which $\alpha = 0$.

Literature: This is section 4 of [CEBLAW] (some of section 3 is also required), and is also in section 4 in [AW]. Additional useful suggestions available from me!

- [CEBLAW] Chris Evans, Ben Lambert and Albert Wood – Lagrangian Mean Curvature Flow with Boundary" `https://arxiv.org/abs/1911.04977`
- [AW] Albert Wood – Singularities of Lagrangian Mean Curvature Flow (PhD thesis) `https://discovery.ucl.ac.uk/id/eprint/10109495/`

Talk 3.5: Equivariant LMCF with boundary

Abstract: In the previous talk, we defined a boundary condition, but is it any good? To test this, we will consider equivariant almost calibrated LMCF with a Lawlor neck boundary condition. We will see that in this case, LMCF does the nicest thing you could hope for: it exists for all time and converges to a Special Lagrangian as $t \to \infty$.

Literature: This is following section 5.1 and 5.2 of [CEBLAW] or equivalently 6.1 of [AW]

- [CEBLAW] Chris Evans, Ben Lambert and Albert Wood – Lagrangian Mean Curvature Flow with Boundary" `https://arxiv.org/abs/1911.04977`
- [AW] Albert Wood – Singularities of Lagrangian Mean Curvature Flow (PhD thesis) `https://discovery.ucl.ac.uk/id/eprint/10109495/`

Talk 3.6 (by mentor): Further perspectives on LMCF

Abstract: Essentially this talk will be propaganda for LMCF. The exact content depends on the previous talks but I will aim to bring together some of the topics of the week and talk about other results in the literature, particularly in the almost calibrated setting.
