Solving PDEs with Newton's Method

Daniel Platt
CUHK Shenzhen, 10 Dez 2024

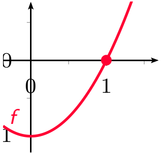
Newton's method is a way for finding zeros of functions: starting from an approximate zero, one iteratively improves it, and in favourable situations finds an exact zero. The same works for elliptic partial differential equations: one first obtains an approximate solution to the equation, and then iteratively uses the linearisation of the equation to improve the solution. In many cases one can prove that this provides an exact solution. I will introduce this technique and review two applications:(1) using pen and paper techniques to construct minimal surfaces, i.e. surfaces with minimal surface tension (joint work with S. Dwivedi and T. Walpuski); (2) using machine learning to construct the approximate solution with the goal of finding harmonic vector fields (joint work with M. Douglas, Y. Qi, R. Barbosa).

Solving PDEs with Newton's Method

1. Newton's method
2. Past research: pen and paper
3. Current research: a numerically verified proof
4. Future research: where to go from here

Newton's method

Toy example:
find zero of
$f(x) = x^2 - 1$



1. Approximate solution $x_a = 1.01 \Rightarrow$

$$f(x) = f(x_a) + f'(x_a)(x - x_a) + \frac{1}{2}f''(x_a)(x - x_a)^2$$
$$= 0.0201 + \underbrace{2.02(x - x_a)}_{=\mathscr{L}(x-x_a)} + \underbrace{(x - x_a)^2}_{=\mathscr{N}(x-x_a)}$$

2. Linear estimate: $||\mathscr{L}^{-1}|| = 1/2.02$
3. Non-linear estimate: $|\mathscr{N}(a) - \mathscr{N}(b)| = |a^2 - b^2| \leq (|a| + |b|) \cdot |a - b|$
4. $x \mapsto -\mathscr{N}\mathscr{L}^{-1}(x) - f(x_a)$ is a contraction on the set $(-1/2, 1/2)$

$$|\mathscr{N}\mathscr{L}^{-1}(a) - \mathscr{N}\mathscr{L}^{-1}(b)| < 0.49 \cdot |a - b|$$

$\Rightarrow$ Fixed point theorem: ex. fixed point y such that $y = -\mathscr{N}\mathscr{L}^{-1}(y) - f(x_a)$
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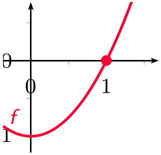

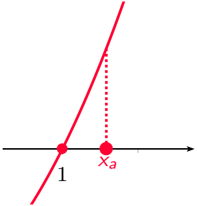

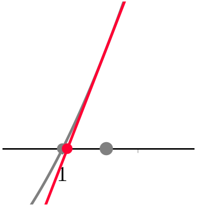

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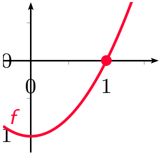

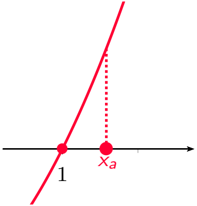

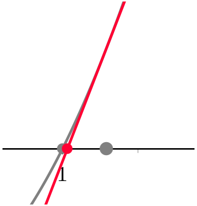

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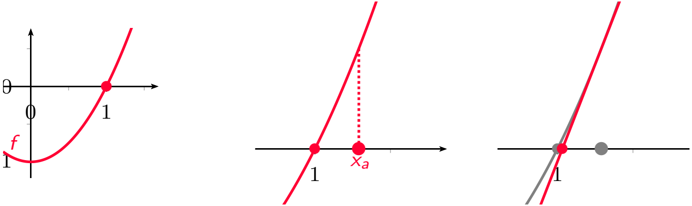

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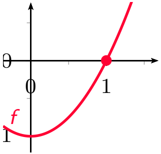

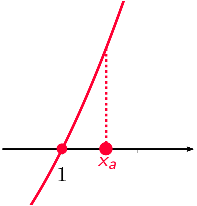

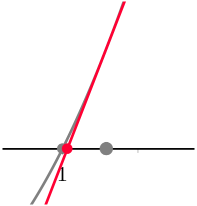

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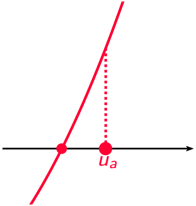
Newton's method for elliptic PDEs

Elliptic PDE $\mathscr{F}$ for function u



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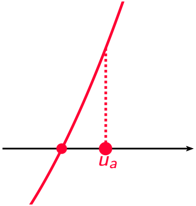

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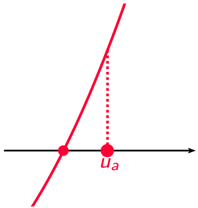

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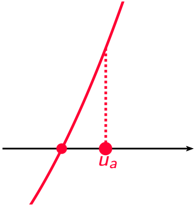

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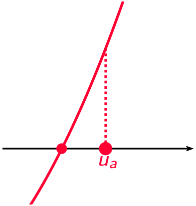

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Past research: pen and paper

- M-theory: $\mathbb{R}^4 \times M^7$ where M is Ricci-flat, used to model gravity
- particles $\leftrightarrow$ 3-dim minimal submanifolds in M
- Conjecture (2016), proved in Dwivedi, Platt and Walpuski, 2023: ex. M with arbitrarily small minimal submanifolds
- Definition: minimal submanifold $\leftrightarrow$ locally area minimising

1. For $t \in (0,1)$ have approximately minimal submanifold $u_a^t \subset M$, volume $\sim t$
   Normal vector fields $n \leftrightarrow$ submanifolds near u_a^t
   $u_a + n$ minimal surface $\Leftrightarrow \mathscr{F}(n) = \text{res} + \mathscr{L}(n) + \mathscr{N}(n) = 0$
   Platt, 2020: $|\text{res}| \leq \text{const} \cdot t^2$

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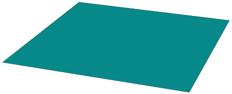 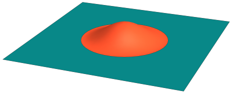

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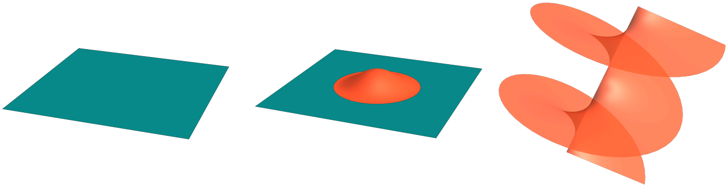

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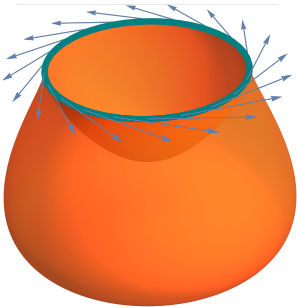

1. Approximate solution u_a, neural network $u_a : L \to \mathbb{R}^3$
   cf. Aslan, Platt and Sheard, 2023; Douglas, Platt and Qi, 2024; Ewert, Magruder, Maiboroda, Shen, Singh and Platt, 2024
   Loss function: $x_1, \dots, x_{1000} \in L$, loss $= \sum_{i=1}^{1000} |\Delta u_a(x_i)|^2$

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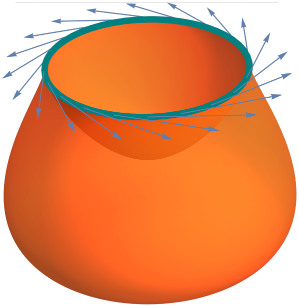

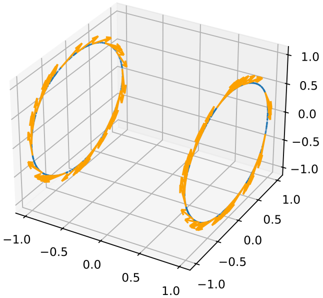

1. Approximate solution u_a, neural network $u_a : L \to \mathbb{R}^3$
   cf. Aslan, Platt and Sheard, 2023; Douglas, Platt and Qi, 2024; Ewert, Magruder, Maiboroda, Shen, Singh and Platt, 2024
   Loss function: $x_1, \ldots, x_{1000} \in L$, loss $= \sum_{i=1}^{1000} |\Delta u_a(x_i)|^2$

Current research: a numerically verified proof

1. Approximate solution u_a, neural network $u_a : L \to \mathbb{R}^3$
   Know $|\Delta u_a| \leq 10^{-3}$ is $x_1, \ldots, x_{1000}$, how about in between?
   $\rightsquigarrow$ AI safety, robustness of AI systems
   $f = \Delta u_a$, if $|\nabla f| < 1$ and $\mathrm{dist}(x_1, x_2) < 10^{-3}$,
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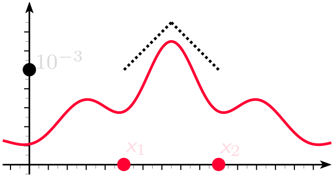

   How to apply to curved space? Estimate distances between points?
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2. Estimate for $||\Delta^{-1}||$
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3./4. as before

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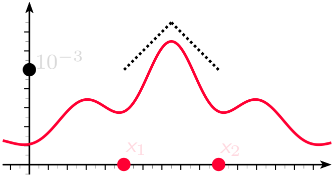

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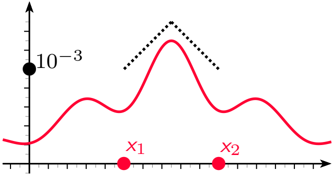

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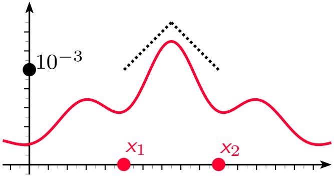

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   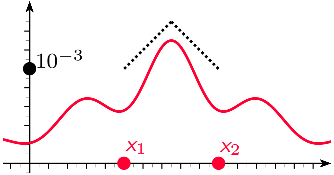

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   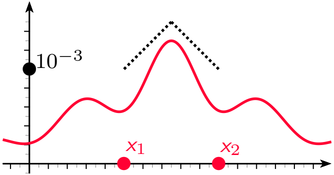

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Future research: more numerically verified proofs

- Many more problems with approximate solutions, unknown if ex. genuine solution
  Examples with $\mathscr{L} = \Delta$

1. Conjecture (1993): ex. Ricci-flat metric on $S^1 \times \mathbb{R}^4$ that is not constant in the S^1-direction (cf. Albertini, Platt and Wiseman, 2024)
2. Conjecture (2021): ex. minimal surface in 2D without symmetry
3. Theorem (Calabi-Yau 1978): ex. Ricci-flat metric on the space $\{(x, y, z, u) \in \mathbb{CP}^3 : x^4 + y^4 + z^4 + u^4 = 0\}$
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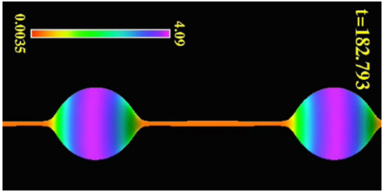

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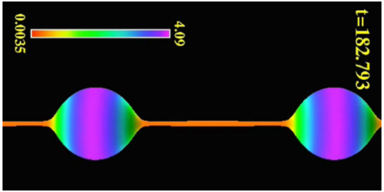

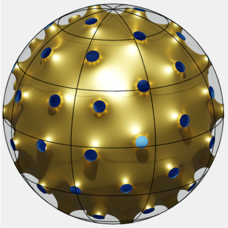

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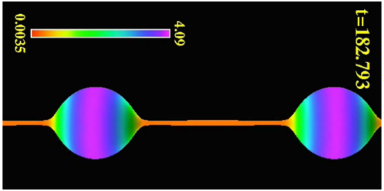
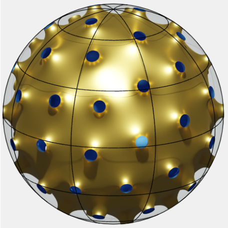
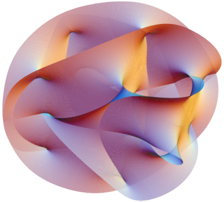
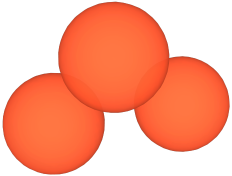






Future research: more conjecture generation using ML

1. Hilbert's 16th problem (1900): maximum number of connected components of algebraic surface in $\mathbb{R}^3$



2. Conjecture (1981): maximum number of singularities of hypersurface?

3. Question (1931): is there a metric with positive curvature on $S^2 \times S^2$?

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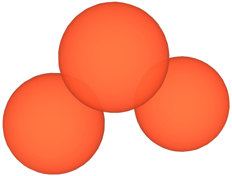

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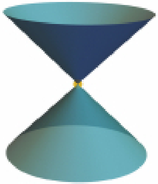

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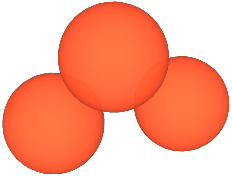

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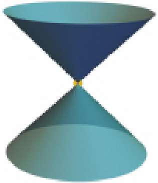

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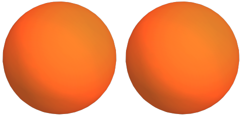

Thank you for the attention!

References I

- Albertini, E., Platt, D., & Wiseman, T. (2024). Towards a uniqueness theorem for static black holes in kaluza-klein theory with small circle size. *arXiv preprint arXiv:2410.20967*.
- Aslan, B., Platt, D., & Sheard, D. (2023). Group invariant machine learning by fundamental domain projections. *NeurIPS Workshop on Symmetry and Geometry in Neural Representations*, 181–218.
- Douglas, M. R., Platt, D., & Qi, Y. (2024). Harmonic 1-forms on real loci of calabi-yau manifolds. *arXiv preprint arXiv:2405.19402*.
- Dwivedi, S., Platt, D., & Walpuski, T. (2023). Associative submanifolds in joyce's generalised kummer constructions. *Communications in Mathematical Physics*, *401*(3), 2327–2353.
- Ewert, C., Magruder, S., Maiboroda, V., Shen, Y., Singh, P., & Platt, D. (2024). Group-invariant machine learning on the kreuzer-skarke dataset. *Physics Letters B*, *858*, 138996.
- Galdeano, M., Platt, D., Tanaka, Y., & Wang, L. (2023). Spin (7)-instantons on joyce's first examples of compact spin (7)-manifolds. *arXiv preprint arXiv:2310.03451*.

References II

Platt, D. (2020). Improved estimates for G_2-structures on the generalised kummer construction. *arXiv preprint arXiv:2011.00482*.

Platt, D. (2024). G 2-instantons on resolutions of g 2-orbifolds. *Communications in Mathematical Physics*, *405*(3), 81.