Numerics for harmonic 1-forms on real loci of Calabi-Yau manifolds

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Chinese University of Hong Kong, 1 Nov 2025

Abstract: Calabi-Yau manifolds are manifolds with a unique Ricci-flat metric. Even though existence is known in many cases, no explicit formulae for these metrics are known. That frequently causes problems when one wants to compute things that depend on the metric, in particular in Physics. One example in maths is the following: does there exist a harmonic 1-form on a real locus of a Calabi-Yau manifold that is nowhere vanishing? No example is known. In the talk I will explain a conjectural example and an interesting non-example. To define the manifolds, some real algebraic geometry is used. We then numerically (approximately) solve the Ricci-flat equation and the harmonic 1-form equation. It turns out that a neural network is good at that and the approximate solution is easily interpretable. This is based on arXiv:2405.19402, which is joint work with Michael Douglas, Yidi Qi, and Rodrigo Barbosa. Time permitting, I will comment on ongoing efforts to turn this into a numerically verified proof that there is a genuine solution near our approximate solution.

Numerical Calabi-Yau metrics

- $Y \subset \mathbb{CP}^4$ quintic, $\Omega \in \Omega^{3,0}(Y)$, $\mathrm{vol}_\Omega := \Omega \wedge \bar{\Omega} \in \Omega^6(Y)$
- Homogeneous deg k polys $= \mathrm{span}\{s_1, \dots, s_N\}$
- $h \in \mathbb{C}^{N \times N}$, Kähler potential $K = \log \sum_{i,j} h_{ij} s_i \overline{s_j}$, $\omega_h := i\partial\bar{\partial}K$, $\omega_h^3 = \mathrm{vol}_h \in \Omega^6(Y)$
- If $\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} = 1$, then Ricci-flat
- [Don09]: choose $h(k)$ s.t. $\frac{\mathrm{vol}_{h(k)}}{\mathrm{vol}_\Omega} \to 1$ as $k \to \infty$
- [DLQ22, HN13, DPQB25]: choose $x_1, \dots, x_{1000} \in Y$, find

$$\min_{h \in \mathbb{C}^{N \times N}} \int_{x_1, \dots, x_{1000}} \left(\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1 \right)^2$$

  Large k: can take ansatz $K =$ neural network with activation $t \mapsto t^2$
- $\rightsquigarrow \omega_h$ "approximate Calabi-Yau metric"

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Numerical harmonic 1-forms

- $Y \subset \mathbb{CP}^4$ quintic, $L = \mathrm{fix}(\sigma) \subset \mathbb{RP}^4$
- $\xi_1, \ldots, \xi_{10} \in \Omega^1(\mathbb{RP}^4)$ 1-forms s.t. $T_x^*\mathbb{RP}^4 = \mathrm{span}\{\xi_1, \ldots, \xi_{10}\}$, polys $p_1, \ldots, p_N$ of degree $d_1, \ldots, d_N$:

$$\lambda = \sum_{\substack{1 \le i \le N \\ 1 \le j \le 10}} \alpha_{ij} \frac{p_i}{|x|^{d_i}} \cdot \xi_j|_L \in \Omega^1(L) \qquad \text{for} \qquad \alpha_{ij} \in \mathbb{R}$$

- Find

$$\min_{\alpha_{ij}} \left(\int_L |\Delta\lambda| \right) / ||\lambda||_{L^2}$$

- λ "approximately harmonic 1-form"

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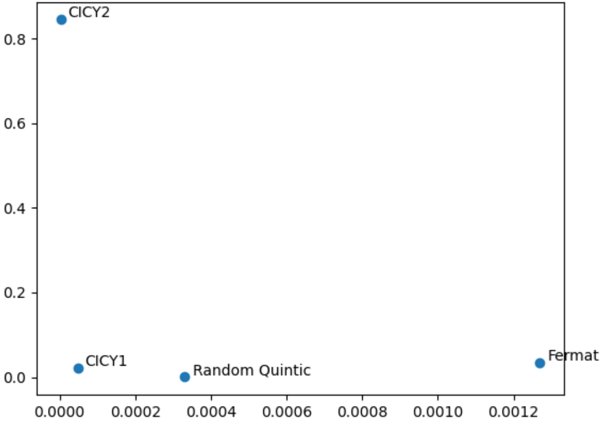
Experimental results: 1-forms and their zeros

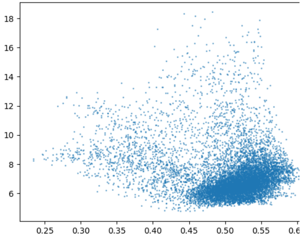
1. **Fermat:** non-example 1; no harmonic 1-form.
2. **Random Quintic:** non-example 2; harmonic 1-form must have zeros
3. **CICY1:** large perturbation $\epsilon = \frac{1}{4}$, harmonic 1-form may have zeros
4. **CICY2:** small perturbation $\epsilon = \frac{1}{100}$, conjecture no zeros

y-axis: $\min|\lambda|$

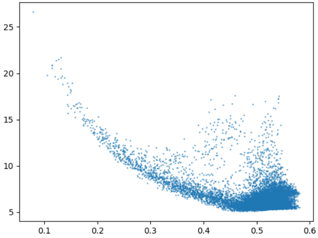
x-axis:
harmonic loss $\frac{||\Delta\lambda||_{L^1}}{||\lambda||_{L^2}}$



Formation of long necks



CICY1



CICY2

- y-axis: $\max\limits_{v \in T_x M : |v|_{FS}=1} |v|_h$

  $|\cdot|_{FS}$: length of a tangent vector in the ambient Fubini-Study metric

  $|\cdot|_h$: length of a vector in the learned approximate Calabi-Yau metric
- x-axis: approximation of the distance to the singular locus of the initial singular variety

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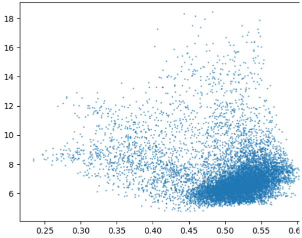

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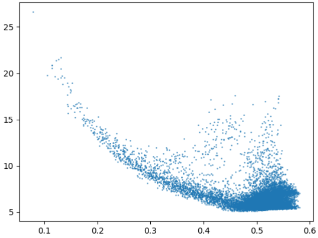

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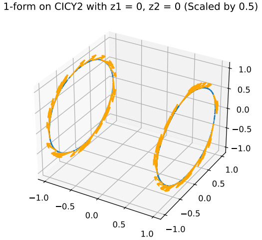
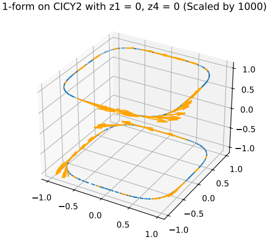
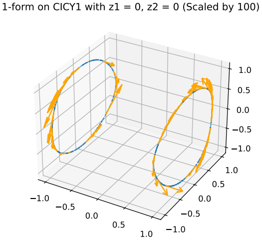
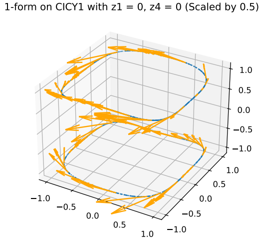
Plots of approximately harmonic 1-forms





`CICY2` (close to singular limit): λ almost zero in S^2-direction, parallel in S^1-direction





`CICY1`

Work in progress: numerically verified proof

- Have: $\Delta_{\text{approx}}\lambda$ small (even $d\lambda = 0$ and $d^*\lambda$ small)
- Want: $\Delta_{\text{CY}}\lambda = 0$
- Step 1: Get nearby $\tilde{\lambda}$ such that $\Delta_{\text{approx}}\tilde{\lambda} = 0$. Still $|\tilde{\lambda}| > 0$?
- Step 2 (harder): if $\omega_{\text{CY}} \approx \omega_{\text{approx}}$, get nearby $\tilde{\tilde{\lambda}}$ s.t. $\Delta_{\text{CY}}\tilde{\tilde{\lambda}}$. Still $|\tilde{\tilde{\lambda}}| > 0$?
- For step 1: geometry fact: $\tilde{\lambda} = \lambda - d(\Delta^{-1}(d^*\lambda))$
- If have embedding and elliptic estimate

$$||f||_{C^1} \leq c\,||f||_{W^{2,p}}\,, \qquad ||\Delta^{-1} : L^p \to W^{2,p}|| \leq c',$$

then:

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- Have: $\Delta_{\text{approx}}\lambda$ small (even $d\lambda = 0$ and $d^*\lambda$ small)
- Want: $\Delta_{\text{CY}}\lambda = 0$
- Step 1: Get nearby $\tilde{\lambda}$ such that $\Delta_{\text{approx}}\tilde{\lambda} = 0$. Still $|\tilde{\lambda}| > 0$?
- Step 2 (harder): if $\omega_{\text{CY}} \approx \omega_{\text{approx}}$, get nearby $\tilde{\tilde{\lambda}}$ s.t. $\Delta_{\text{CY}}\tilde{\tilde{\lambda}}$. Still $|\tilde{\tilde{\lambda}}| > 0$?
- For step 1: geometry fact: $\tilde{\lambda} = \lambda - d(\Delta^{-1}(d^*\lambda))$
- If have embedding and elliptic estimate

$$||f||_{C^1} \leq c\,||f||_{W^{2,p}}\,, \qquad \left|\left|\Delta^{-1} : L^p \to W^{2,p}\right|\right| \leq c',$$

then:

$$|d(\Delta^{-1}(d^*\lambda))| \leq \left|\left|\Delta^{-1}(d^*\lambda)\right|\right|_{C^1} \leq c\left|\left|\Delta^{-1}(d^*\lambda)\right|\right|_{W^{2,p}} \leq c\,c'\,||d^*\lambda||_{L^p}$$

- If $||d^*\lambda||_{L^p} \approx 10^{-5}$ smaller than $(cc')^{-1}\min|\lambda|$ then, $|\tilde{\lambda}| > 0$
- $\rightsquigarrow$ need to get c, c' explicit

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