Numerical approximations of harmonic 1-forms on real loci of Calabi-Yau manifolds

Daniel Platt (Imperial College London)
Loughborough University, 20 March 2024

Abstract: For applications in differential geometry and string theory one would like to construct Calabi-Yau manifolds of complex dimension three with the following property: it should contain a real submanifold of real dimension three that admits a harmonic nowhere vanishing 1-form. Many examples are expected to exist, but none have been proven to exist. The problem is that there is no explicit formula for the Calabi-Yau metric which makes it hard to write down the "harmonic" equation, let alone solve it. In the talk I will present numerical approximations of the Calabi-Yau metric, and numerical approximations of harmonic 1-forms, obtained by neural networks. This suggests some conjectural examples of harmonic, nowhere vanishing 1-forms. I will also show some proven non-examples, and explain the main long-term motivation for this numerical work, which is to numerically verifiably prove that there exists a genuine solution to the harmonic equation near the approximate solutions. This is work in progress, joint with Michael Douglas and Yidi Qi.

Background I: Calabi-Yau manifolds

- Calabi conjecture (Yau's theorem):
  If (Y, g, J, ω) Kähler, complex dim n with:

  $$\Omega \in \Omega^{n,0}(Y) \text{ parallel and nowhere 0 s.t.}$$

  then ex. $\phi \in C^\infty(Y)$ s.t. $\omega_{CY} = \omega + i\partial\overline{\partial}\phi$ has $\omega_{CY}^n = \Omega \wedge \overline{\Omega}$
  ($\Rightarrow$ induced metric g_{CY} is Ricci-flat)
- Example: Fermat quintic

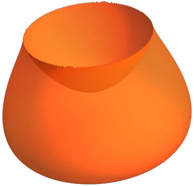
  $$Y := \{z = [z_0 : \cdots : z_4] \in \mathbb{CP}^4 : z_0^5 + \cdots + z_4^5 = 0\}$$

  has $\Omega \in \Omega^{n,0}(Y) \Rightarrow g_{CY}$ exists

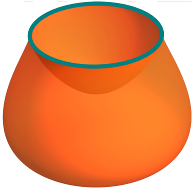
Background II: Harmonic 1-forms on Calabi-Yaus

- Let Y be Calabi-Yau 3-fold with Calabi-Yau metric g_{CY}
- $\sigma : Y \to Y$ anti-holomorphic involution, $L := fix(\sigma)$
  example: quintic with real coefficients in $\mathbb{CP}^4$ and $\sigma([z_0 : \cdots : z_4]) = [\overline{z_0} : \cdots : \overline{z_4}]$
- $S^1 \times Y$ has dimension 7 and holonomy $SU(3)$. Problem: want holonomy G_2
- Define $\widehat{\sigma} : S^1 \times Y \to S^1 \times Y$ as $(x, y) \mapsto (-x, \sigma(y))$

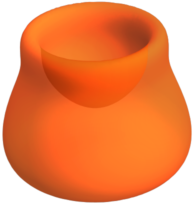
$(S^1 \times Y)/\langle\widehat{\sigma}\rangle$



$\{0, \frac{1}{2}\} \times L$



N^7



Theorem ([Joyce and Karigiannis, 2017])

If there exists $\lambda \in \Omega^1(L)$ harmonic w.r.t. $g_{CY}|_L$ that is nowhere 0, then there exists a resolution $N^7 \to (S^1 \times Y)/\langle\widehat{\sigma}\rangle$ with holonomy equal to G_2.

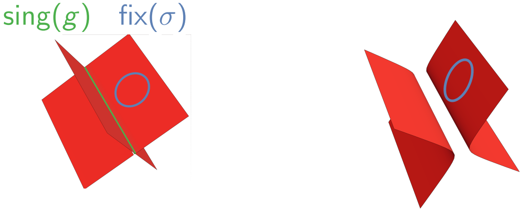
- Goal: check if such a 1-form exists
- First Betti number $\to$ harmonic 1-forms. Nowhere 0? Must know the metric!

Non-example 1: Fermat Quintic

- $Y := \{z = [z_0 : \cdots : z_4] \in \mathbb{CP}^4 : z_0^5 + \cdots + z_4^5 = 0\}$
- $\sigma([z_0 : \cdots : z_4]) = [\overline{z_0} : \cdots : \overline{z_4}]$
- $$\mathbb{RP}^3 \overset{\cong}{\to} L = \mathrm{fix}(\sigma) = \{x = [x_0 : \cdots : x_4] \in \mathbb{RP}^4 : x_0^5 + \cdots + x_4^5 = 0\}$$
  $$[x_0 : \cdots : x_4] \mapsto \left[x_0 : \cdots : x_4 : -\sqrt[5]{x_0^5 + \cdots + x_4^5}\right]$$
- $b^1(\mathbb{RP}^3) = 0 \Rightarrow$ no harmonic 1-form on L

Non-Example 2: small complex structure limit

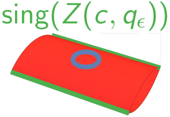
- Another quintic in $\mathbb{CP}^4$:
  1. [Krasnov, 2009] $f_\pm = (x_0(x_1^2 + x_2^2 + x_3^2 - x_4^2) - (x_1^3 + x_2^3 + x_3^3 - \frac{1}{2}x_4^3) \pm \epsilon x_0^3)$ smoothing of ordinary double point $(1:0:0:0:0)$ Has $Z(f_+) \cong \mathbb{RP}^3$, $Z(f_-) \cong \mathbb{RP}^3 \# S^1 \times S^2$
  2. $v := (x_0^2 + \cdots + x_4^2)$ and $g = v \cdot f_-$ has $Z_\mathbb{R}(g) = Z_\mathbb{R}(f_-) \subset \mathbb{RP}^4$ so $\sigma : \mathbb{CP}^4 \to \mathbb{CP}^4$, $x \mapsto \overline{x}$ has $b^1\left(\mathrm{fix}(\sigma)|_{Z(g)}\right) = 1$
  3. Take smoothing $g_\epsilon := g + \epsilon\xi$, where ξ generic poly
  4. But: ex. incompressible $S^2 \Rightarrow$ [Jaco, 1980] $Z_\mathbb{R}(g_\epsilon) \cong Z_\mathbb{R}(g)$ is no fibration over S^1 $\Rightarrow$ [Tischler, 1970] no closed nowhere zero 1-form $\Rightarrow$ any harmonic 1-form has zeros (even number of zeros by Poincaré–Hopf theorem and $\chi(Z_\mathbb{R}(g_\epsilon)) = 0$)

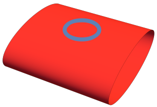


- [Tian and Yau, 1990] Calabi-Yau metrics on $Z(f_-) \setminus \mathrm{sing}(g)$ and $Z(v) \setminus \mathrm{sing}(g)$
- [Sun and Zhang, 2019] glue these to metric on smoothing $Z(g_\epsilon)$
- More non-examples from other cubics. Examples from other Fanos?

Conjectural example 3: quadric intersect quartic

- Construction of quadric intersect quartic in $\mathbb{CP}^5$, also Calabi-Yau
  1. Circle $c_{aff} = x_1^2 + x_2^2 - 1$, quartic $q_{aff} = x_3^4 + x_4^4 + x_5^4 - 1$
     Projectivise: $c = -x_0^2 + x_1^2 + x_2^2$ and $q = -x_0^4 + x_3^4 + x_4^4 + x_5^4$
  2. c and q have SO(2)-symmetry:
     $[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : \cos(t)x_1 - \sin(t)x_2 : \sin(t)x_1 + \cos(t)x_2 : x_3 : x_4 : x_5]$
     Generic smoothings c_δ and q_ϵ of c and q
  3. $\mathbb{RP}^5 \supset Z_{\mathbb{R}}(c, q_\epsilon) \cong S^1 \times S^2$ smooth, $Z(c, q_\epsilon) \subset \mathbb{CP}^5$ singular
     $\mathrm{sing}(Z(c, q_\epsilon)) = \mathrm{sing}(c) \cap Z(q_\epsilon)$





- Fantasy: $Z(c, q_\epsilon)$ has singular Calabi-Yau metric $g_0 \Rightarrow$ if q_ϵ is SO(2)-invariant, then Infinitesimal SO(2)-action gives Killing field for $g_0 \Rightarrow$ (Ricci-flat) parallel vector field $\Rightarrow$ parallel 1-form for $g_0 \Rightarrow$ nearly parallel 1-form for g_δ

Numerical Calabi-Yau metrics

- Holomorphic volume form locally $\Omega = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \rightsquigarrow \mathrm{vol}_\Omega := \Omega \wedge \overline{\Omega} \in \Omega^6(Y)$
- Ample line bundle $L \to Y$ and $k \in \mathbb{N}$ such that $L^{\otimes k}$ very ample
  Example: $Y \subset \mathbb{CP}^4$ quintic, $(O(1)|_Y)^{\otimes k}$
- $s_1, \ldots, s_N \in H^0(L^{\otimes k})$ basis of holomorphic sections
  $\Rightarrow$ embedding $s = (s_1, \ldots, s_N) : Y \to \mathbb{CP}^{N-1}$
- h positive definite Hermitian form on $H^0(L^{\otimes k}) \rightsquigarrow$ some Fubini-Study metric
  Kähler potential: $K = \log \sum_{i,j} h_{i,j} s^i \overline{s}^j$. Volume form: $\omega_h^3 = \mathrm{vol}_h \in \Omega^6(Y)$.

  If $\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} = 1$, then Ricci-flat
- [Donaldson, 2009]: choose h cleverly to minimise $\int_Y \left(\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1\right)^2$

	$\left\|\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1\right\|$	Comment
[Donaldson, 2009]	10^{-2}	$n = 2$, needs symmetries
[Headrick and Nassar, 2013]	10^{-14}	$n = 3$, needs symmetries
[Larfors et al., 2022]	10^{-2}	$n = 3$, not C^0, complete intersections+torics
[Douglas et al., 2022]+ours	10^{-4}	$n = 3$, quintics+complete intersections

Numerical harmonic 1-forms

Example: quintic $X := Z(f) \subset \mathbb{CP}^4$, $L := \mathrm{fix}(\sigma) \subset X$

- $\xi_1, \ldots, \xi_{10} \in \Omega^1(\mathbb{RP}^4)$ closed 1-forms s.t. $T^*_x\mathbb{RP}^4 = (\xi_1(x), \ldots, \xi_{10}(x))\ \forall x$
- $p_1, \ldots, p_N$ polys, p_i degree d_i; for $\alpha = (\alpha_{1,1}, \ldots, \alpha_{10,N}) \in \mathbb{R}^{10N}$ let

$$\lambda_\alpha(x) = \sum_{\substack{i=1,\ldots,N \\ j=1,\ldots,10}} \alpha_{i,j} \frac{p_i(x)}{|x|^{d_i}} \xi_j|_L(x) \in \Omega^1(L)$$

For $x_1, \ldots, x_{100000} \in X$ find $\min\limits_{\alpha \text{ s.t. } ||\lambda_\alpha||_{L^2}=1} \int_{x_1,\ldots,x_{100000}} |d\lambda_\alpha| + |d^*\lambda_\alpha|$

- Stone-Weierstrass $\Rightarrow$ best approximations converge to harmonic form as $N \to \infty$
- Ansatz for p_i: $A_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_{i+1}}$ linear, $sq : \mathbb{R}^k \to \mathbb{R}^k$ square each coordinate

$$p(x_0, \ldots, x_4) = A_k \circ \cdots \circ \mathrm{sq} \circ A_2 \circ \mathrm{sq} \circ A_1(x_0, \ldots, x_4)$$

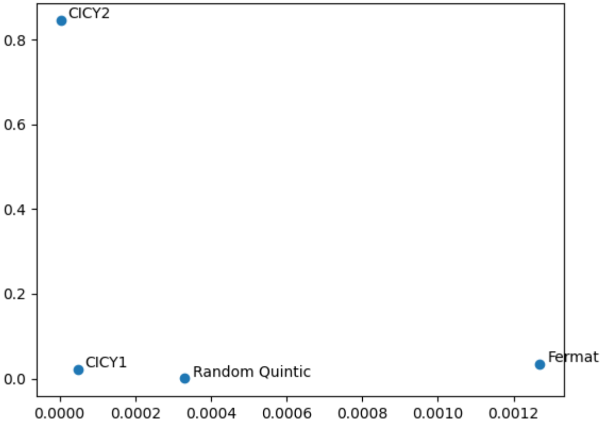
- Equivalent: neural network with activation function $x \mapsto x^2$
- Approximate metric $\frac{i}{2}\partial\overline{\partial}K$ smooth+explicit $\Rightarrow$ explicitly compute $(|d\lambda_\alpha|(x_i) + |d^*\lambda_\alpha|(x_i))/\sqrt{|\lambda(x_1)|^2 + \cdots + |\lambda(x_{100000})|^2}$
  $\Rightarrow$ minimise with `tensorflow`

Experimental results: 1-forms and their zeros

1. **Fermat:** non-example 1; no harmonic 1-form.
2. **Random Quintic:** non-example 2; harmonic 1-form must have zeros
3. **CICY1:** conjectural example 3; large perturbation $\epsilon = \frac{1}{4}$, harmonic 1-form may have zeros
4. **CICY2:** conjectural example 3; small perturbation $\epsilon = \frac{1}{100}$, conjecture no zeros

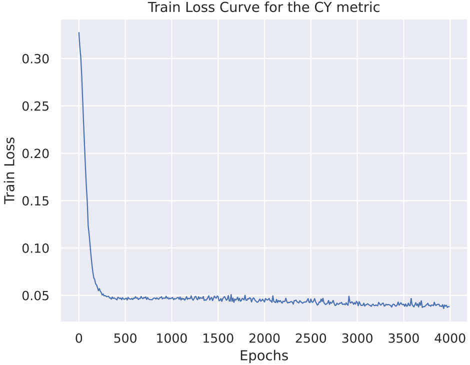
y-axis: min $|\lambda|$

x-axis: harmonic loss $\frac{||\mathrm{d}\lambda||_{L^1} + ||\mathrm{d}^*\lambda||_{L^1}}{||\lambda||_{L^2}}$

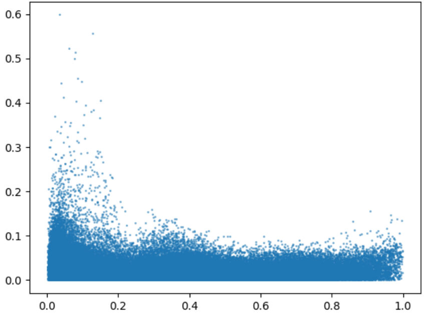


Experimental results on quintic

- $g = v \cdot f_-$ singular quintic from before, $\xi = 0.84x_0^5 + \ldots$ random quintic
- Find $\epsilon > 0$ such that $g_\epsilon := g + \epsilon\xi$ has $Z_{\mathbb{R}}(g_\epsilon)$ diffeo to $Z_{\mathbb{R}}(g)$
  - $U \subset \mathbb{RP}^4$ nbhd of $Z_{\mathbb{R}}(g)$
  - $k := \min_U |Dg| > 0$, $M := \min_{\mathbb{RP}^4 \setminus U} |g| > 0$
  - if $||\epsilon_0 \xi||_{C^0} < M$ and $||D\epsilon_0 \xi||_{C^0} < k$, then $Z_{\mathbb{R}}(g_\epsilon)$ smooth for all $0 < \epsilon < \epsilon_0$
  - $\Rightarrow Z_{\mathbb{R}}(g_\epsilon)$ diffeo for all $0 < \epsilon < \epsilon_0$ (for us $\epsilon_0 = 0.00195503$)



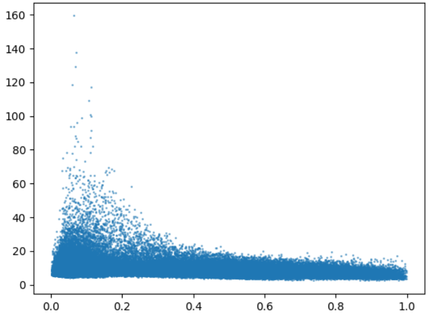
Average of $|\mathrm{vol}_h / \mathrm{vol}_\Omega - 1|$ while iteratively improving vol_h



$|\mathrm{vol}_h / \mathrm{vol}_\Omega(x) - 1|$ over $\max\{v(x)/\,||x||^2, f_-(x)/\,||x||^3\}$

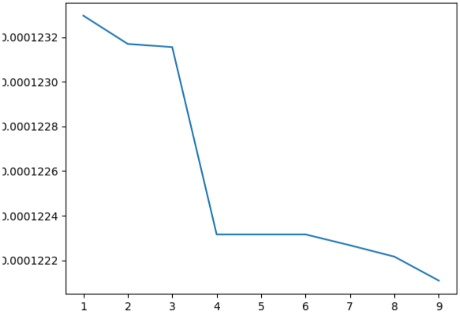
Experimental results on quintic

Neck formation



$\max_{v \in T_{||v||_{FS}=1}} ||v||_h$ over $\max\{v(x)/\,||x||^2, f_-(x)/\,||x||^3\}$

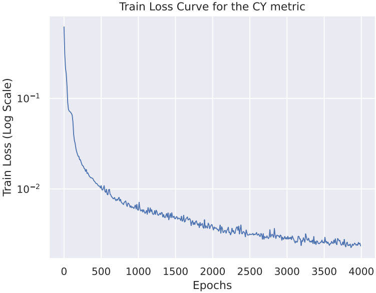
1-form has even number of zeros



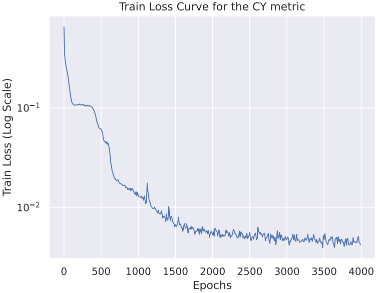
k-medoid clustering loss of 500 points with smallest $|\omega|(x)$ over number of clusters (heuristic: "elbow" $k=4$ is optimal number of clusters

Experimental results on quadric ∩ quartic

$$\epsilon = \frac{1}{4}$$

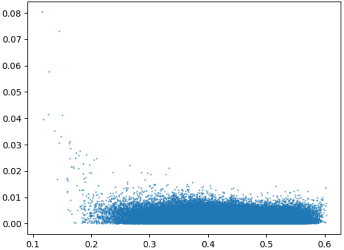


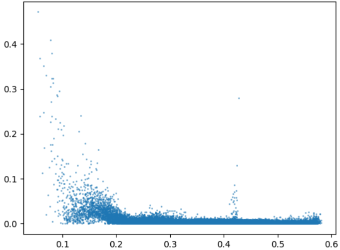
$$\epsilon = \frac{1}{100}$$



↑ Training loss

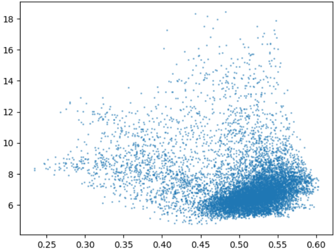
↓ Loss over distance from singularity





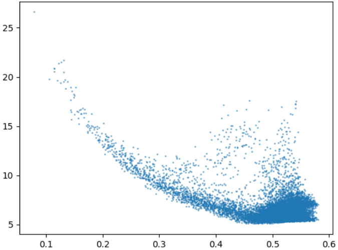
Experimental results on quadric ∩ quartic

$\epsilon = \frac{1}{4}$



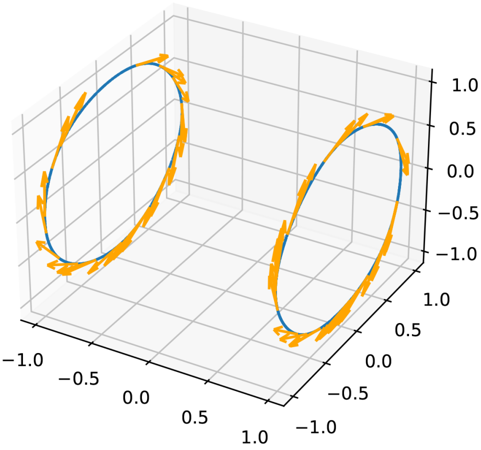
↑ Metric stretching over distance from singularity

$\epsilon = \frac{1}{100}$



Experimental results on quadric ∩ quartic

$c = -x_0^2 + x_1^2 + x_2^2$ and $q = -x_0^4 + x_3^4 + x_4^4 + x_5^4$
Set $x_0 = 1$ and $x_3 = x_4 = 0 \curvearrowright \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} \times \{\pm 1\}$
1-form restricted to this



Bonus motivation

Proposition

For all $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left|\left|\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1\right|\right|_{L^p_1} < \delta \;\Rightarrow\; ||g_{approx} - g_{CY}||_{L^p_1} < \epsilon.$$

Proposition

For all $\mu > 0$ *there exists* $\epsilon > 0$ *such that the following is true:*
for $\lambda \in \Omega^1(L^3)$ *such that* $\Delta_{approx}\lambda = 0$ *and* $||\lambda||_{L^2, g_{approx}} = 1$ *and* $\min_L |\lambda| > \mu$ *let* $\widetilde{\lambda} \in [\lambda]$ *be the unique* Δ_{CY}*-harmonic* 1*-form. Then:*

$$||g_{approx} - g_{CY}||_{L^p_1} < \epsilon \;\Rightarrow\; |\widetilde{\lambda} - \lambda|(x) < \frac{\mu}{2} \;\Rightarrow\; |\widetilde{\lambda}|(x) > \frac{\mu}{2} \text{ for all } x \in L.$$

- Find: g_{approx} with $\left|\left|\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1\right|\right|_{L^p_1} < \delta,$ $\quad \lambda$ with $\Delta_{approx}\lambda = 0$ and $\min_L |\lambda| > \mu$
- $\Rightarrow$ there exists nowhere vanishing g_{CY}-harmonic 1-form on L

Thank you for the attention!

References I

Donaldson, S. K. (2009).
Some numerical results in complex differential geometry.
Pure Appl. Math. Q., 5(2, Special Issue: In honor of Friedrich Hirzebruch. Part 1):571–618.

Douglas, M., Lakshminarasimhan, S., and Qi, Y. (2022).
Numerical calabi-yau metrics from holomorphic networks.
In *Mathematical and Scientific Machine Learning*, pages 223–252. PMLR.

Headrick, M. and Nassar, A. (2013).
Energy functionals for calabi-yau metrics.
In *Journal of Physics: Conference Series*, volume 462, page 012019. IOP Publishing.

Jaco, W. H. (1980).
Lectures on three-manifold topology.
Number 43. American Mathematical Soc.

References II

Joyce, D. and Karigiannis, S. (2017).
A new construction of compact G_2-manifolds by gluing families of Eguchi-Hanson spaces.
ArXiv e-prints.

Krasnov, V. A. (2009).
On the topological classification of real three-dimensional cubics.
Mat. Zametki, 85(6):886–893.

Larfors, M., Lukas, A., Ruehle, F., and Schneider, R. (2022).
Numerical metrics for complete intersection and kreuzer–skarke calabi–yau manifolds.
Machine Learning: Science and Technology, 3(3):035014.

Sun, S. and Zhang, R. (2019).
Complex structure degenerations and collapsing of calabi-yau metrics.
arXiv preprint arXiv:1906.03368.

References III

Tian, G. and Yau, S.-T. (1990).
Complete kähler manifolds with zero ricci curvature. i.
Journal of the American Mathematical Society, 3(3):579–609.

Tischler, D. (1970).
On fibering certain foliated manifolds overs1.
Topology, 9(2):153–154.