An application of numerical techniques to rigorous proof in special holonomy

Daniel Platt
15 November 2022

Abstract: Approximations of Calabi-Yau metric are a popular tool to produce heuristics, but so far have not been leveraged to rigorously prove theorems in geometry. I present one work in progress, in which we prove that the real loci of certain Calabi-Yau manifolds admit harmonic nowhere vanishing 1-forms, which are needed for an application in G2-geometry. I will explain the proof strategy, which consists of two parts: first, I formulate an estimate for the difference between approximate metric and true Calabi-Yau metric in terms of the Ricci curvature of the approximate metric which is of independent interest. Second, I explain the connection between nowhere vanishing 1-forms with respect to the two different metrics. This is joint work with Rodrigo Barbosa, Michael Douglas, and Yidi Qi.

Pure maths and machine learning

- Use machine learning for conjecture generation
  e.g. [Davies et al., 2021]: conjecture connecting algebraic and geometric properties of knots
- Machine learning applied to pure mathematics datasets
  e.g. [He, 2017]: inputs are Calabi-Yau manifolds, outputs are their Hodge numbers (previously computed exactly)
- Numerical verification methods for PDE solutions
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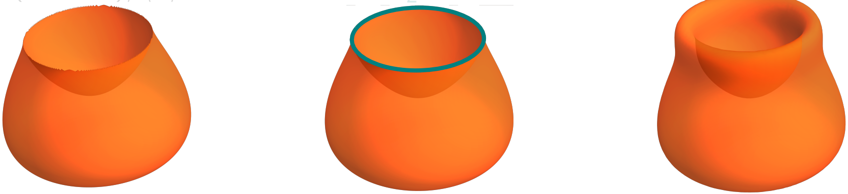
Background

- Let Y be Calabi-Yau 3-fold with Calabi-Yau metric g_{CY}
- $\sigma : Y \to Y$ anti-holomorphic involution, $L := fix(\sigma)$
  example: quintic with real coefficients in $\mathbb{CP}^4$ and $\sigma([z_0 : \cdots : z_4]) = [\overline{z_0} : \cdots : \overline{z_4}]$
- $S^1 \times Y$ has dimension 7 and holonomy $SU(3)$. Problem: want holonomy G_2
- Define $\widehat{\sigma} : S^1 \times Y \to S^1 \times Y$ as $(x, y) \mapsto (-x, \sigma(y))$

$(S^1 \times Y)/\langle\widehat{\sigma}\rangle$ $\{0, \frac{1}{2}\} \times L$ N^7



Theorem ([Joyce and Karigiannis, 2017])

If there exists $\lambda \in \Omega^1(L)$ harmonic w.r.t. $g_{CY}|_L$ that is nowhere 0, then there exists a resolution $N^7 \to (S^1 \times Y)/\langle\widehat{\sigma}\rangle$ with holonomy equal to G_2.

- Goal: check if such a 1-form exists
- First Betti number $\to$ harmonic 1-forms. Nowhere 0? Must know the metric!

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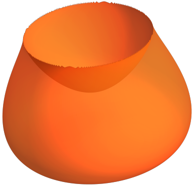

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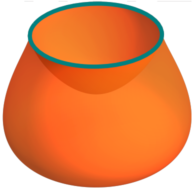

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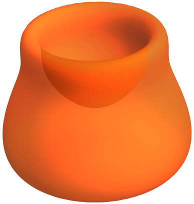

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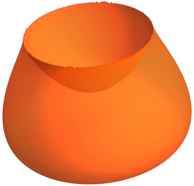

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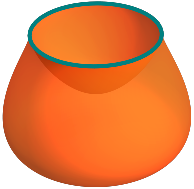

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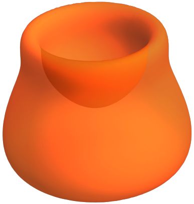

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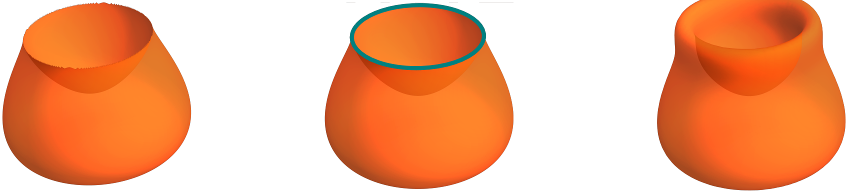

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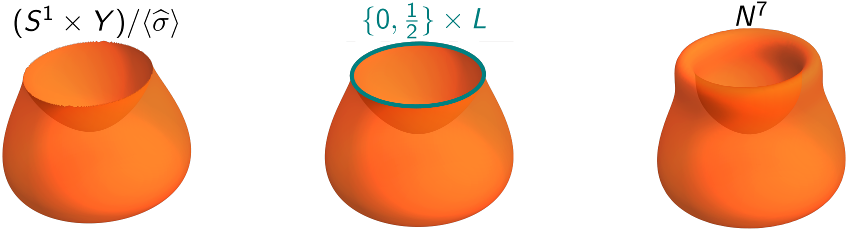

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Strategy for checking if nowhere zero harmonic $\lambda \in \Omega^1(L)$ exists

Goal: Check if L admits harmonic nowhere zero 1-form

Step 1: Approximate g_{CY} by g_{approx}

Step 2: Prove: for all $\epsilon_1 > 0$ exists $\delta_1 > 0$ such that:

$$\text{if } ||Ric(g_{approx})||_{C^0} < \delta_1, \text{ then } ||g_{CY} - g_{approx}||_{C^1} < \epsilon_1.$$

Step 3: Find $\lambda \in \Omega^1(L)$ harmonic w.r.t. g_{approx} and compute $\min_{x \in L} |\lambda(x)|$.

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$\eta \in \Omega^0(L)$ s.t. $\lambda + d\eta$ is nowhere 0 and harmonic w.r.t. g_{CY}.

Result:
- Compute g_{approx}
- Check that $||Ric(g_{approx})||_{C^0}$ is small
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- Holomorphic volume form locally $\Omega = dz^1 \wedge dz^2 \wedge dz^3 \rightsquigarrow \mathrm{vol}_\Omega := \Omega \wedge \overline{\Omega} \in \Omega^6(Y)$
- Ample line bundle $L \to Y$ and $l \in \mathbb{N}$ such that $L^{\otimes l}$ very ample
  Example: $Y \subset \mathbb{CP}^4$ quintic, $(\mathcal{O}(1)|_Y)^{\otimes l}$
- $s_1, \ldots, s_N \in H^0(L^{\otimes l})$ basis of holomorphic sections
  $\Rightarrow$ embedding $s = (s_1, \ldots, s_N) : Y \to \mathbb{CP}^{N-1}$
- h positive definite Hermitian form on $H^0(L^{\otimes l}) \rightsquigarrow$ Fubini-Study metric
  Kähler potential: $\log \sum_{i,j} h_{i,j} s^i \overline{s}^j$. Volume form: $\mathrm{vol}_h \in \Omega^6(Y)$
- [Donaldson, 2009]: choose h cleverly to approximate CY metric
  (Ignoring a constant) If $\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} = 1$, then Ricci-flat
- Choose $x_1, \ldots, x_k \in Y$ and find

$$\min_h \int_{\{x_1, \ldots, x_k\}} \left(\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1 \right)^2$$

- Convenient: there is fast machine learning software to find local minima [Douglas et al., 2022]

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  (Ignoring a constant) If $\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} = 1$, then Ricci-flat
- Choose $x_1, \dots, x_k \in Y$ and find

$$\min_h \int_{\{x_1,\dots,x_k\}} \left(\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1 \right)^2$$

- Convenient: there is fast machine learning software to find local minima [Douglas et al., 2022]

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- Holomorphic volume form locally $\Omega = dz^1 \wedge dz^2 \wedge dz^3 \rightsquigarrow \mathrm{vol}_\Omega := \Omega \wedge \overline{\Omega} \in \Omega^6(Y)$
- Ample line bundle $L \to Y$ and $l \in \mathbb{N}$ such that $L^{\otimes l}$ very ample
  Example: $Y \subset \mathbb{CP}^4$ quintic, $(O(1)|_Y)^{\otimes l}$
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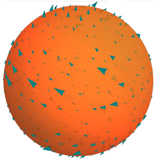

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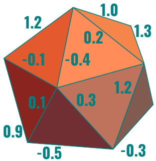

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Conjecture ([Schulz and Tsogtgerel, 2020])

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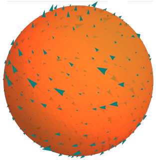

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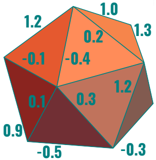

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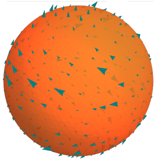
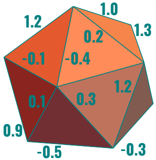
   $\mapsto$ 

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  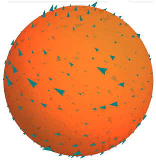

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  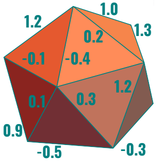

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  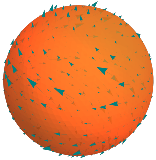

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  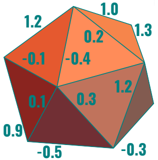

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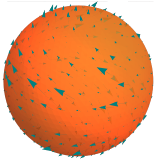

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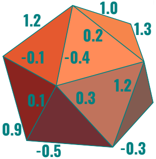

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  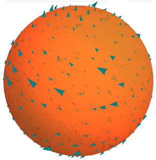

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  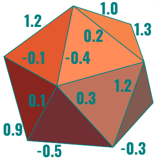

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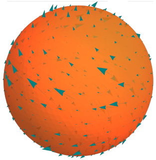

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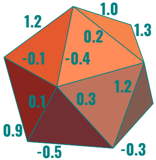

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Step 4: Perturb g_{approx}-harmonic to g_{CY}-harmonic

Theorem

For all $\epsilon_2 > 0$ exists $\delta_2 > 0$ such that:

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2. $\overset{\text{step 2}}{\Longrightarrow}$ $||g_{CY} - g_{approx}||_{C^1}$ small
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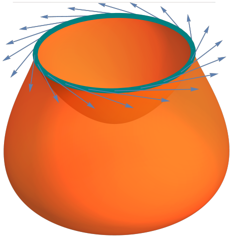
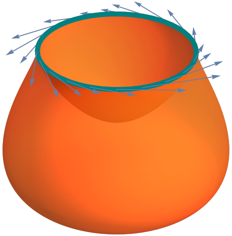


Harmonic 1-form w.r.t. g_{approx}



Perturbed 1-form, it has a worse bound from below w.r.t. g_{CY}, but still nowhere 0

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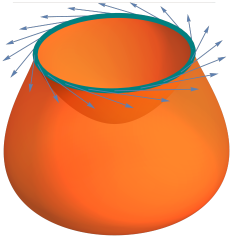

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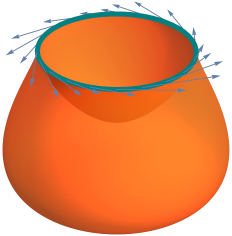

Perturbed 1-form, it has a worse bound from below w.r.t. g_{CY}, but still nowhere 0

Result

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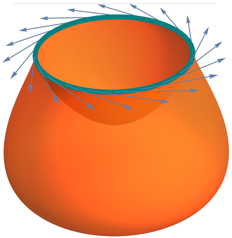

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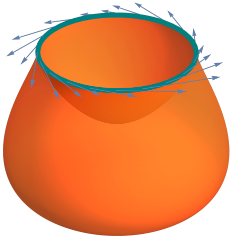

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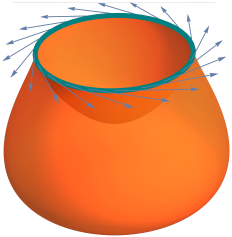

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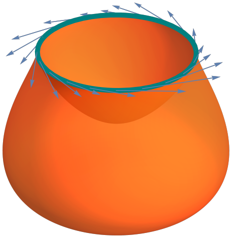

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Example 0: [Joyce and Karigiannis, 2017, Example 7.6]

- Singular Calabi-Yau
  $Y_0 := \{([w_0 : w_1], [x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 :$
  $(w_0 x_0 y_0 z_0)^2 + (w_1 x_1 y_1 z_1)^2 = 0\}$
- Real locus $Y_0(\mathbb{R}) = T^3$ smooth
  $\Rightarrow$ small perturbation Y is smooth, still has $Y(\mathbb{R}) = T^3$
- Conjecture: metric on T^3 is close to flat metric. $\Rightarrow$ exist nowhere zero 1-forms
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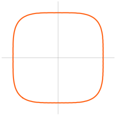
Example 1: $S^1 \times S^2$

$Q = x_1^4 + x_2^4 - 1$



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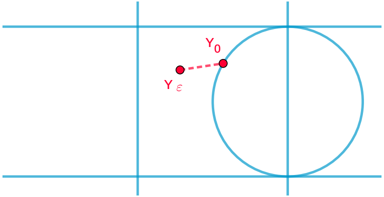

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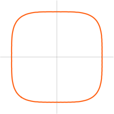

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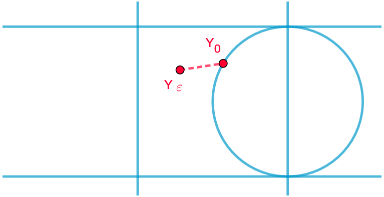

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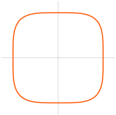

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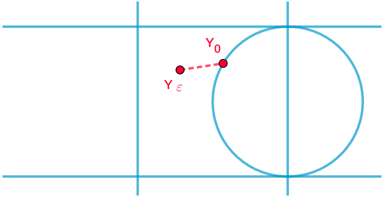

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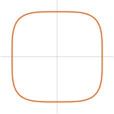

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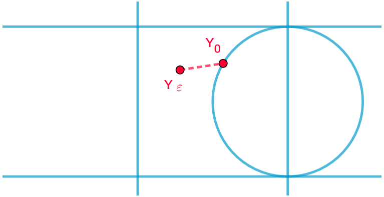

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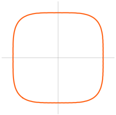

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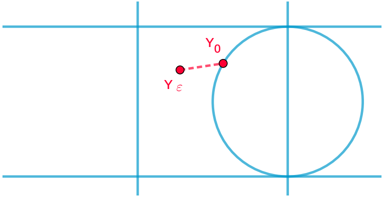

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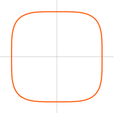

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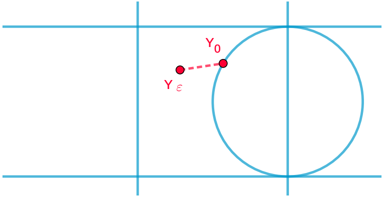

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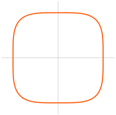

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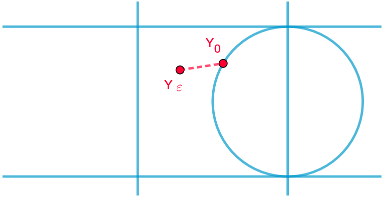

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- If C has harmonic nowhere zero 1-form $\Rightarrow$ closed nowhere zero 1-form $\Rightarrow$ Tischler's theorem: C is a fibration over S^1
- Topological condition, not satisfied for these diffeomorphism types
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- Diffeomorphism type of real cubics $C \subset \mathbb{RP}^4$ classified in [Krasnov, 2009]: Possible are $\mathbb{RP}^3 \#(S^1 \times S^2)\# \ldots \#(S^1 \times S^2)$ with $0, 1, 2, 3, 4, 5$ handles (plus $\mathbb{RP}^3 \cup S^3$ and one exotic possibility that is not understood)
- Then $Q = Z(C \cdot (x_0^2 + \cdots + x_4^2))$ quintic
- Smooth in $\mathbb{RP}^4$, perturb to be smooth in $\mathbb{CP}^4 \rightsquigarrow Y_\epsilon$
- If C has harmonic nowhere zero 1-form $\Rightarrow$ closed nowhere zero 1-form $\Rightarrow$ Tischler's theorem: C is a fibration over S^1
- Topological condition, not satisfied for these diffeomorphism types
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Thank you for the attention!

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