COMPLEX MANIFOLDS

Contents

1. Introduction

A complex manifold is about the same thing as a differentiable manifold, but everywhere you see the word "diffeomorphism" replace it with "holomorphic isomorphism" or "biholomorphism".

Just as the theory of differentiable functions in one (or several) variables differs greatly from the theory of holomorphic functions in one (or several) variables, so to do complex manifolds differ from differentiable manifolds.

Example 1.1. *$\mathbb{C}^n$ is a complex manifold. In fact, any open subset in $\mathbb{C}^n$ is a complex manifold.*

Example 1.2. *The sphere S^2 is homeomorphic to the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Thus, it is a compact complex manifold. More generally $\mathbb{CP}^n$ is a complex manifold.*

Example 1.3. *The torus $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{C}/\mathbb{Z}^2 \cong S^1 \times S^1$ is a complex manifold. More generally, any $2n$-dimensional lattice $\Lambda \subset \mathbb{C}^n$ defines a $2n$-torus which is also a complex manifold. On the other hand, S^{2n} is not a complex manifold for $n \neq 1, 3$. It is unknown if S^6 is a complex manifold.*

Example 1.4. *Any genus g surface is a complex manifold.*

Example 1.5. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then the graph of f,*

$$\Gamma_f = \{(z, f(z))\} \subset \mathbb{C} \times \mathbb{C}$$

is a complex manifold. Given Γ_f we can recover f as follows:

$$f(z) = q\left(p^{-1}(z) \cap \Gamma_f\right)$$

where p, q are the projection onto the first and second coordinate respectively.

More in general, given any complex submanifold $\Gamma \subset \mathbb{C} \times \mathbb{C}$, we can define a "multivalued holomorphic function" by

$$f_\Gamma(z) = q\left(p^{-1}(z) \cap \Gamma\right).$$

In particular, this allows to construct the inverse of a function: let

$$\tau : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$

be defined by $\tau(z, w) = (w, z)$ and given $f\colon \mathbb{C} \to \mathbb{C}$, let

$$\Gamma_{f^{-1}} = \tau(\Gamma_f).$$

Then $f^{-1} = f_{\Gamma_{f^{-1}}}$ is the inverse of f. For example, $\log(z)$ is the multivalued holomorphic function defined as the inverse of $f(z) = \exp(z)$.

Example 1.6. *Generalising the previous example, we can consider holomorphic maps between complex manifolds:*

$$f\colon M \to N.$$

For example, given M, what are the automorphism of M:

$$\mathrm{Aut}(M) = \{f\colon M \to M \mid f \text{ is biholomorphic}\}.$$

There are many maps $f\colon \mathbb{C} \to \mathbb{C}$ but the only automorphism of $\mathbb{C}$ are affine linear maps.

The study of automorphisms of a torus $f\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ has many applications in cryptography.

Example 1.7. *Algebraic geometry is the study of the zeroes of polynomials. Given a collection of polynomials, $f_1, \dots, f_k$ in variables $x_1, \dots, x_m$, then the set*

$$\{(x_1, \dots, x_m) \in \mathbb{C}^m \mid f_1 = \cdots = f_k = 0\}$$

is called algebraic variety. If it is smooth, then it is also a complex manifold.

If $F_1, \dots, F_k$ are homogeneous polynomials in variables $x_0, \dots, x_m$ then the set

$$\{(x_0, \dots, x_m) \in \mathbb{P}^m \mid F_1 = \cdots = F_k = 0\}$$

is called projective variety. Also in this case, if it is smooth, then it is a complex manifold.

Note that a differentiable manifold X contains many compact submanifolds. For example, through every point $x \in X$, there eists a positive dimensional submanifolds. On the other hand, there are complex manifolds which do not admit any proper complex submanifolds.

Question 1.8. *What can we say about the topology of a complex manifold? Is it orientable? What can the fundamental group be? Can we list all the simply connected complex manifolds? What can the (co)homology groups look like?*

To be more concrete: The sphere is a complex manifold; is the Klein bottle a complex manifold? What about the 4-sphere? The 6-sphere? Then $2n$-sphere?

Question 1.9. *If two complex manifolds are diffeomorphic are they biholomorphic? If not, how different can they be? How many complex structures can one put on a manifold? If I am given a differentiable manifold, can we put a complex structure on it?*

This is an interesting and big question and we will only be able to answer a small part in this course.

Question 1.10. *More generally, what is the relationship between the holomorphic structure on a complex manifold and the underlying differentiable structure?*

We will cover the Hodge decomposition theorem for Kähler manifolds and the Kodaira embedding theorem. The first one gives a direct sum decomposition of complex singular cohomology of X. The second one gives conditions under which a compact complex manifold is projective.

Note that complex submanifolds of the projective spaces are algebraic.

2. Local Theory

2.1. **Holomorphic functions in several variables.** We will now recall some facts about holomorphic functions in several variables.

Notation 2.1. *We denote by $D(z_0, r)$, the complex disc of radius r centred at z_0:*

$$D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

The boundary of $D(z, r)$ will be denoted by $\partial D(z_0, r)$.

Definition 2.2. *Let $U \subset \mathbb{C}$ be an open subset. Let $f\colon U \to \mathbb{C}$ be a continuous function. Then f is* **holomorphic** *on U if for all $z_0 \in U$, the limit*

$$\lim_{z \to z_0} \frac{f(z_0) - f(z)}{z_0 - z}$$

exists.

Theorem 2.3 (Cauchy). *Let $U \subset \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be holomorphic. Let $z_0 \in U$ and assume that $D := D(z_0, r)$ is such that $\overline{D} \subset U$.*

Then

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz.$$

Notation 2.4. *Let $z_0 = (z_{0,1}, \ldots, z_{0,n}) \in \mathbb{C}^n$ and let $R = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. We will denote by $D(z_0, R)$ the* **polydisc** *centred at z with polyradius R, i..e*

$$D(z_0, R) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i - z_{0,i}| < r_i \text{ for all } i\}.$$

Definition 2.5. *Let $U \subset \mathbb{C}^n$ be an open set. Let $f\colon U \to \mathbb{C}$ be a continuous function. We say that f is* **holomorphic** *if for each $z = (z_1, \ldots, z_n) \in U$ such that $D(z, \epsilon) \subset U$ for some* *polyradius* $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$*, we have that the function in one variable*

$$f(z_1, \ldots, z_{i-1}, \cdot, z_{i+1}, \ldots, z_n) : D(z_i, \epsilon_i) \to \mathbb{C}$$

is holomorphic.

Example 2.6. *Any convergent power series in* n *variables is holomorphic.*

We will now see that also the converse is true.

Theorem 2.7 (Cauchy)**.** *Let* $U \subset \mathbb{C}^n$ *be an open set and let* $f\colon U \to \mathbb{C}$ *be holomorphic. Let* $z = (z_1, \ldots, z_n) \in U$ *such that if* $D := D(z, \epsilon)$ *for some polyradius* $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$*, then* $\overline{D} \subset U$*.*

Then, for $z' = (z'_1, \ldots, z'_n) \in D$*, we have*

$$f(z') = \frac{1}{(2\pi i)^n} \int_{\partial D(z_1, \epsilon_1)} \cdots \int_{\partial D(z_n, \epsilon_n)} \frac{f(z)}{(z_1 - z'_1) \ldots (z_n - z'_n)} dz_1 \ldots dz_n$$

The Theorem follows by induction on n, by applying Theorem 2.3 at each step. It follows:

Corollary 2.8. *Let* $U \subset \mathbb{C}^n$ *be an open set and let* $f\colon U \to \mathbb{C}$ *be holomorphic. Let* $z = (z_1, \ldots, z_n) \in U$*.*

Then there exists $D := D(z, \epsilon) \subset U$ *for some polyradius* $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ *and a power series*

$$p(w) = \sum_{m_1, \ldots, m_n \geq 0} a_{\bar{m}} (w_1 - z_1)^{m_1} \ldots (w_n - z_n)^{m_n}$$

such that p *is convergent on* D *and* $p(w) = f(w)$ *on* D*.*

Thus, holomorphic functions are complex analytic. As in the one dimensional case, the idea of the proof is to use the identity:

$$\frac{1}{1-w} = \sum w^k.$$

Definition 2.9. *Let $U \subset \mathbb{C}^n$ be an open set. A function $f\colon U \to \mathbb{C}^m$ is* **holomorphic**, *if for each projection $p_i\colon \mathbb{C}^m \to \mathbb{C}$, the function*

$$f_i = p_i \circ f\colon U \to \mathbb{C}$$

is holomorphic.

Note that if $f\colon U \to V$ and $g\colon V \to W$ are holomorphic, then the composition $g \circ f\colon U \to W$ is also holomorphic.

Definition 2.10. *Let $U \subset \mathbb{C}^n$ be an open set. A holomorphic function $f\colon U \to \mathbb{C}^m$ is* **biholomorphic** *at a point $z \in U$ if there exists a neighborhood $z \in V \subset U$ such that $f\colon V \to f(V)$ is bijective and $f^{-1}\colon f(V) \to V$ is holomorphic.*

We say that f if biholomorphic if it is a bijection and biholomorphic at all points $z \in U$.

Note that, in the assumptions above, $f(V)$ is automatically an open set of $\mathbb{C}^m$.

Example 2.11. *Let A be an invertible $n \times n$ complex matrix. Then A defines a biholomorphism $f\colon \mathbb{C}^n \to \mathbb{C}^n$.*

Example 2.12. *Let $f(z)\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ defined by $f(z) = z^2$. Then f is a biholomorphism at each point, but it is not a biholomorphism.*

Identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, it follows that any holomorphic function is real analytic and, hence, C^∞. Thus, if $f\colon U \to V$ is a biholomorpihsm, it is also a diffemorphism and a homeomorphism.

Theorem 2.13 (Hartog's theorem)**.** *Let $n \geq 2$ and let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $\delta = (\delta_1, \ldots, \delta_n)$ such that $\epsilon_i > \delta_i > 0$ for $i = 1, \ldots, n$. Let*

$$U = D(0, \epsilon) \setminus D(0, \delta) \subset \mathbb{C}^n.$$

Then any holomorphic function on U extend to a holomorphic function on $D(0, \epsilon)$.

Note that this is false if $n = 1$. Indeed, it is enough to consider the function $f(z) = \frac{1}{z}$.

2.2. **Cauchy formula in one variable.** Let $w = x + iy \in \mathbb{C}$. Let $U \subset \mathbb{C}$ and let $f \colon U \to \mathbb{C}$ be a C^∞-function. Recall that

$$\frac{\partial f}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial f}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The function f is holomorphic if and only if $\frac{\partial f}{\partial \bar{w}} = 0$.

Let $U \subset \mathbb{C}^n$ be an open set. Consider coordinates $z_i = x_i + iy_i$ for $i = 1, \ldots, n$ and let $f \colon U \to \mathbb{C}$ be a C^∞-function. Then f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}_i} = 0$ for all $i = 1, \ldots, n$.

We also have

$$\frac{i}{2} dw \wedge d\overline{w} = \frac{i}{2}(dx + idy) \wedge (dx - idy) = dx \wedge dy$$

is the usual Lebesgue measure on $\mathbb{R}^2$. Thus, we will denote

$$dA := \frac{i}{2} dw \wedge d\overline{w}.$$

Proposition 2.14. *Let $r > 0$ and let $D := D(0, r)$. Let $f \colon U \to \mathbb{C}$ be a C^∞-function, where $U \subset \mathbb{C}^n$ is an open set containing $\overline{D}$.*

Then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{w-z}dw - \frac{1}{\pi}\int_D \frac{1}{w-z}\cdot\frac{\partial f}{\partial \bar{w}}dA.$$

Proof. We will assume, for simplicity, that $z = 0$. Since $f(w) = \frac{1}{w}$ is locally integrable at $w = 0$, we have

$$\int_D \frac{1}{w}\frac{\partial f}{\partial \bar{w}}dA = \lim_{\epsilon\to 0}\int_{D-D(0,\epsilon)}\frac{1}{w}\frac{\partial f}{\partial \bar{w}}dA.$$

Away from zero, we have

$$\frac{1}{w}\frac{\partial f}{\partial \bar{w}}dA = d\left(\frac{f(w)}{2i}\frac{dw}{w}\right).$$

Thus, by Stokes theorem, we have

$$\lim_{\epsilon\to 0}\int_{D-D(0,\epsilon)}\frac{1}{w}\frac{\partial f}{\partial \bar{w}}dA = \frac{1}{2i}\int_{\partial D} f(w)\frac{dw}{w} - \lim_{\epsilon\to 0}\frac{1}{2i}\int_{\partial D(0,\epsilon)} f(w)\frac{dw}{w}.$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{2i}\int_{\partial D(0,\epsilon)} f(w)\frac{dw}{w} = \pi f(0),$$

the result follows. □

Note that if f is holomorphic, then $\frac{\partial f}{\partial \bar{w}} = 0$ and thus, we recover Theorem 2.3.

2.3. **Rank Theorem.** Let $U \subset \mathbb{C}^n$ be an open set. Given a holomorphic function $f\colon U \to \mathbb{C}^m$, we define the **holomorphic Jacobian** of f at $z \in U$, to be the matrix J_f defined by

$$J_f(z) = \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{i,j}$$

where $f_i = p_i \circ f$ and $p_i\colon \mathbb{C}^m \to \mathbb{C}$ is the i-th projection.

Theorem 2.15 (Rank Theorem)**.** *Let $U \subset \mathbb{C}^n$ be an open set and let $f\colon U \to \mathbb{C}^m$ be a holomorphic function. Let $z \in U$ be a point such that in a neighbourhood of z the Jacobian J_f has constant rank r.*

Then there exist open subsets $z \in V \subset U$ and $f(z) \in W \subset \mathbb{C}^m$, and biholomorphisms

$$\phi\colon D(0,1)^n \to V \qquad \textit{and} \qquad \psi\colon D(0,1)^m \to W$$

such that $\psi^{-1} \circ f \circ \phi\colon D(0,1)^n \to D(0,1)^m$ is given by

$$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_r, 0, \dots, 0).$$

See Theorem 8.7 in "Holomorphic functions of several variables" by Ludger Kaup and Burchard Kaup for a proof.

Corollary 2.16 (Inverse function theorem)**.** *Let $U \subset \mathbb{C}^n$ be an open set and let $f\colon U \to \mathbb{C}^n$ be a holomorphic function. Let $z \in U$ be such that $\det(J_f(z)) \neq 0$.*

Then f is a biholomorphism at z.

Proof. By the rank theorem, there are neighbourhoods U', V of $z, f(z)$ respectively and biholomorphisms $\phi\colon D(0,1)^n \to U'$ and $\psi\colon D(0,1)^n \to V'$, so that $\psi^{-1} \circ f \circ \phi\colon D(0,1)^n \to D(0,1)^n$ is a biholomorphism. It is easy to check that the composition of two biholomorphisms is a biholomorphism. Thus, the claim follows. $\square$

Remark 2.17. *Let $f = (f_1, \dots, f_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic function. Then $\det J_f(z)\colon \mathbb{C}^n \to \mathbb{C}$ is also a holomorphic function. In particular*

$$Z := \{z \in \mathbb{C}^n \mid \det(J_f)^{-1}(0)\}$$

is a closed subset and f is a biholomorphism away from Z.

More generally, the locus where f has rank $\leq k$ is a closed subset.

2.4. **Holomorphic differential forms.**

Definition 2.18. *Let $U \subset \mathbb{C}^n$ be an open subset. A* **holomorphic vector field** *on U is an expression of the form*

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}$$

where $a_i \colon U \to \mathbb{C}$ is holomorphic function.

Let $p \geq 1$ be a positive integer. A **holomorphic p-vector field** *on U is an expression of the form $v_1 \wedge \cdots \wedge v_p$ where $v_1, \ldots, v_p$ are holomorphic vector fields on U.*

Notation 2.19. *Let $U \subset \mathbb{C}^n$ be an open subset. We denote by $H^0(U, \mathcal{O}_U)$ be the complex vector space of holomorphic functions on U and by $H^0(U, T_U)$ the collection of all the complex vector fields on U.*

Note that $H^0(U, T_U)$ is a $H^0(U, \mathcal{O}_U)$-module.

Definition 2.20. *Let $p \geq 1$. Let R be a ring and let M be an R-module. The* **p-th exterior power** *$\Lambda^p M$ is the quotient of $M^{\otimes p}$ by the relations, for each $m_1, \ldots, m_p$ and $\sigma \in S_p$,*

$$m_1 \otimes \ldots \otimes m_p - \epsilon(\sigma) m_{\sigma(1)} \otimes \ldots \otimes m_{\sigma(p)}$$

where $\epsilon(\sigma)$ is the signature of σ, defined by

$$\epsilon(\sigma) = (-1)^m,$$

where m is the number of transpositions in a decomposition of σ (note that m is not uniquely defined, but $(-1)^m$ only depends on σ).

Definition 2.21. *Let $U \subset \mathbb{C}^n$ be an open subset and let $p \geq 0$. Let $R = H^0(U, \mathcal{O}_U)$ and let M be the dual of $H^0(U, T_U)$ as a R-module, i.e. $M = \mathrm{Hom}(H^0(U, T_U), R)$. A* **holomorphic p-form** *on U is an element in $H^0(U, \Omega^p_M) := \Lambda^p M$.*

By convention, a holomorphic 0-form is just a holomorphic function $f \colon U \to \mathbb{C}$.

Let $U \subset \mathbb{C}^n$ be an open subset and let $p \geq 0$. Then a holomorphic p-form can be written as

$$\sum_{|I|=p} f_I dz_{i_1} \wedge \ldots \wedge dz_{i_p},$$

where $I = (i_1, \ldots, i_p)$ with $i_1 < \cdots < i_p$, $f_I \colon U \to \mathbb{C}$ is a holomorphic function and dz_i is the element in the dual of $H^0(U, T_U)$ defined by

$$dz_i \left(\frac{\partial}{\partial z_j} \right) = \delta_{ij}.$$

Note that $\Lambda^p H^0\left(U, \Omega^1_U\right) = H^0\left(U, \Omega^p_U\right)$ and that there exists a bilinear map

$$H^0\left(U, \Omega^p_U\right) \otimes H^0\left(U, \Omega^q_U\right) \to H^0\left(U, \Omega^{p+q}_U\right),$$

which maps $\omega_1 \otimes \omega_2$ into the **exterior product** (or wedge product) $\omega_1 \wedge \omega_2$. By linearity, it is enough to define the product for

$$\omega_1 = f dz_{i_1} \wedge \ldots \wedge dz_{i_p} \qquad \text{and} \qquad \omega_2 = g dz_{j_1} \wedge \ldots \wedge dz_{j_q}$$

and we have

$$\omega_1 \wedge \omega_2 := 0 \quad \text{if } \{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} \neq \emptyset$$

and

$$\omega_1 \wedge \omega_2 := (f \cdot g) dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q} \quad \text{otherwise.}$$

The **exterior derivative**

$$d\colon H^0(U, \Omega_U^p) \to H^0(U, \Omega_U^{p+1})$$

is defined by

$$d\left(\sum_{|I|=p} f_I dz_{i_1} \wedge \ldots \wedge dz_{i_p}\right) = \sum_{|I|=p, j=1,\ldots,n} \frac{\partial f_I}{\partial z_j} dz_j \wedge dz_{i_1} \wedge \ldots \wedge dz_{i_p}$$

Note that d is a linear map of $\mathbb{C}$-vector spaces, i.e. if ω and η are holomorphic p-forms and $a, b \in \mathbb{C}$ then

$$d(a\omega + b\eta) = ad\omega + bd\eta.$$

Proposition 2.22. *Let $U \subset \mathbb{C}^n$ be an open subset. Let $\omega \in H^0(U, \Omega_U^p)$ and $\eta \in H^0(U, \Omega_U^q)$ be holomorphic forms, then*
- *(Leibnitz rule) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$, and*
- *$d(d\omega) = 0$.*

Definition 2.23. *Let $U \subset \mathbb{C}^n$ be an open set and let $f\colon U \to \mathbb{C}^m$ be a holomorpic map. Let $f_i = p_i \circ f$ where $p_i\colon \mathbb{C}^m \to \mathbb{C}$ is the i-th projection and let $V \subset \mathbb{C}^m$ be an open set containing $f(U)$. Given the holomorphic p-form $\omega = h dz_{i_1} \wedge \cdots \wedge dz_{i_p} \in H^0(V, \Omega_V^p)$, we define the* **pull-back** *of ω by*

$$f^*\omega = h \circ f \cdot df_{i_1} \wedge \cdots \wedge df_{i_p} \in H^0(U, \Omega_U^p).$$

We can extend this map by linearity to a map

$$f^*\colon H^0(V, \Omega_V^p) \to H^0(U, \Omega_U^p).$$

Proposition 2.24. *Let $U \subset \mathbb{C}^n, V \subset \mathbb{C}^m$ and $W \subset \mathbb{C}^{m'}$ be open sets, let $f\colon U \to \mathbb{C}^m$ and $g\colon V \to \mathbb{C}^{m'}$ be holomorphic function such that $f(U) \subset V$ and $g(V) \subset W$. Let $\omega \in H^0(V, \Omega^p_V)$, $\eta \in H^0(V, \Omega^q_V)$ and $\omega' \in H^0(W, \Omega^p_W)$ be holomorphic forms.*

Then the following properties hold:

- *If $p = q$ then $f^*(\omega + \eta) = f^*\omega + f^*\eta$.*
- *$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.*
- *$df^*\omega = f^*d\omega$*
- *$f^*g^*\omega = (g \circ f)^*\omega$.*

An open subset $U \subset C^n$ can be thought as a real $2n$-dimensional manifold. Let $p, q \geq 0$ be non-negative integers and let $m = p+q$. A differentiable m-form on U is said to be a (p,q)**-form** if it can be written as

$$\sum_{i_1<\ldots<i_p, j_1<\ldots<j_q} f dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

We denote the space of (p,q) forms on U by $C^\infty(U, \Omega^{p,q}_U)$.

Note that we can take the complex conjugate of a (p,q)-form to get a (q,p)-form.

3. Complex Manifolds

Definition 3.1. *A* **complex manifold** *(or holomorphic manifold) of dimension n is a connected Haussdorff topological space X with a countable open cover $\mathcal{U} = \{U_\alpha\}$ and homeomorphisms $\phi_\alpha\colon U_\alpha \to \mathbb{C}^n$ such that the transition functions*

$$\phi_\alpha \circ \phi_\beta^{-1}\colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are biholomorphisms.

The pair (U_α, ϕ_α) *is called* **complex chart** *and the set* $\{(U_\alpha, \phi_\alpha)$ *is called* **complex atlas** *or* **complex structure**.

Note that the real dimension of a complex manifold of dimension n is $2n$.

Example 3.2.

- *An open subset* $U \subset \mathbb{C}^n$ *is a complex manifold or more in general the open set of a complex manifold is also a complex manifold.*
- *If* X *and* Y *are complex manifolds then* $X \times Y$ *is a complex manifold.*

Example 3.3 (Projective space)**.** *Let* $V = \mathbb{C}^{n+1}$ *with coordinates* $z_0, \dots, z_n$ *and let* $V^* = V \setminus \{0\}$. *Consider the relation on* V^*, *given by*

$$v \sim w \qquad \text{if} \qquad \exists \lambda \in \mathbb{C} \text{ such that } v = \lambda \cdot w.$$

Let $X = V^*/\sim$ *with quotient map* $\pi \colon V^* \to X$ *and endowed with the quotient topology.* X *comes equipped with a natural set of coordinates, called homogeneous coordinates: a point* $x \in \mathbb{P}^n$ *can be written as an* $(n+1)$*-tuple* $[x_0, \dots, x_n]$ *so that* $x_i \neq 0$ *for some* i. *Two* $(n+1)$*-tuples* $[x_0, \dots, x_n]$ *and* $[y_0, \dots, y_n]$ *define the same point on* $\mathbb{P}^n$ *if and only if there is* $\lambda \in \mathbb{C}$ *such that* $x_i = \lambda y_i$ *for all* $i = 0, \dots, n$.

Let $V_i = \{(z_0, \dots, z_n) \in V^* \mid z_i \neq 0\}$ *and let* $U_i = \pi(V_i)$. *Then* $U_i \subset X$ *is open and* $\cup U_i = X$.

Let $H_i = \{(z_0, \dots, z_n) \in V^* \mid z_i = 1\}$. *Then*

$$r_i \colon H_i \to \mathbb{C}^n \qquad (z_0, \dots, z_n) \mapsto (z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$$

is an homeomorphism and

$$q_i := p_i|_{H_i} \colon H_i \to U_i$$

is also an homeomorphism. Thus, we define

$$\phi_i = r_i \circ q_i^{-1} \colon U_i \to \mathbb{C}^n.$$

Note that

$$\phi_i([x_0, \dots, x_n]) = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

For simplicity, we consider $i = 0$ and $j = 1$. Note that

$$\phi_0(U_0 \cap U_1) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 \neq 0\}$$

and $\phi_1 \circ \phi_0^{-1}(x_1, \dots, x_n) = \left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right)$. *Thus* $\phi_1 \circ \phi_0^{-1}$ *is a biholomorphism. Clearly the same result hold for* $\phi_i \circ \phi_j^{-1}$ *for any* i, j.

Thus, X is a complex manifold, called n-dimensional **projective space**. *We will denote it by $\mathbb{CP}^n$, or $\mathbb{P}^n_{\mathbb{C}}$. It is compact because it can be constructed as a quotient of the sphere S^{2n+1}.*

From a different point of view, $\mathbb{P}^n$ can be identified with the space of complex lines passing through the origin. Furthermore, $\mathbb{P}^1$ is the sphere S^2.

Example 3.4 (Complex tori)**.** *Let $\Lambda = Z^{2n} \subset \mathbb{C}^n$ be the natural inclusion. Let $X = \mathbb{C}^n/\Lambda$ with quotient map $q \colon \mathbb{C}^n \to X$ endowed with the quotient topology. Note that X Is compact. Then at each point $x \in X$, there exists an open set $x \in U \subset X$, such that if $V = q^{-1}(U)$, then the restriction map $q|_V \colon V \to U$ is a homeomorphism. Let $\phi = q|_V^{-1}$. It follows then that (U, ϕ) is a complex chart and, by compactness, we can find finitely* *many points $x_i \in X$ such that the corresponding charts (U_i, ϕ_i) define a complex structure on X. Thus X is a compact complex manifold.*

More in general, if $\Lambda \simeq Z^{2n} \subset \mathbb{C}^n$ is a lattice then $X = \mathbb{C}^n/\Lambda$ is a compact complex manifold, called **complex torus***.*

Definition 3.5. *A continuous map $f\colon X \to Y$ between complex manifolds is said to be* **holomorphic** *if for all $y \in Y$, there is a complex chart (V_y, ψ_y), with $y \in V_y$, such that for all $x \in f^{-1}(y)$, there is a chart (U_x, ϕ_x), with $x \in U_x$, such that $\psi_y \circ f \circ \phi_x^{-1}$ is holomorphic.*

It is easy to check that the definition above does not depend on the choice of the charts.

Using the notation above, we define the Jacobian of f at x by taking the Jacobian of $\psi_y \circ f \circ \phi_x^{-1}$.

A holomorphic function on X is just a holomorphic function $f\colon X \to \mathbb{C}$.

Remark 3.6. *If X is compact then any holomorphic function is constant.*

Definition 3.7. *A holomorphic map $f\colon X \to Y$ is a* **submersion** *(resp.* **immersion***) if $\dim X \geq \dim Y =: r$ (resp. $r := \dim X \leq \dim Y$) at every point $x \in X$ the Jacobian J_f of f has maximal rank r.*

An immersion is an **embedding** *if $f\colon X \to f(X)$ is a homeomorphism.*

Example 3.8. *Consider $f\colon \mathbb{C}^1 \to \mathbb{C}^n$ given by*

$$z \mapsto (z, f_2(z), \ldots, f_n(z))$$

where $f_2, \ldots, f_n$ are holomorphic functions. Then f is an embedding.

Example 3.9. *Let $\mathbb{Z}^4 \subset \mathbb{C}^2$ be the standard lattice and let $X = \mathbb{C}^2/\mathbb{Z}^4$. Denote by $q\colon \mathbb{C}^2 \to X$ the quotient map. As in Example 3.4, X is a complex manifold. Let $\lambda \in \mathbb{C}$ and consider the immersion $f\colon \mathbb{C} \to \mathbb{C}^2$ given by*

$$x \mapsto (x, \lambda x).$$

We want to show that the composition $\overline{f} = q \circ f\colon \mathbb{C} \to X$ is also an immersion. Indeed, pick a point $x \in \mathbb{C}$ and let $z = \overline{f}(x)$. Let U be an open subset of z so that $q\colon q^{-1}(U) \to U$ is a biholomorphism. It then easily follows that $\overline{f}^{-1}(U) \to U$ has a Jacobian of rank 1.

Definition 3.10. *If $f\colon X \to Y$ is an embedding such that $f(X)$ is closed in Y then we say that X is a* **closed submanifold** *of Y. The* **codimension** *of X is $\dim Y - \dim X$.*

The following result provides a way to check if a closed subset of a complex manifold is a submanifold.

Theorem 3.11. *Let $i\colon X \to Y$ be a closed submanifold of codimension k and let n be the dimension of X.*

Then for all $x \in i(X)$ there exists an open subset $x \in U \subset Y$ and a submersion $f\colon U \to D(0,1)^k \subset \mathbb{C}^k$ such that $f^{-1}(0) = X \cap U$.

Conversely, if $X \subset Y$ is a closed subset such that for all $x \in X$ there is an open subset $x \in U \subset Y$ and a submersion $f\colon U \to D(0,1)^k$ such that $X \cap U = f^{-1}(0)$ then X is a closed submanifold.

Proof. We first prove the first part of the Theorem. Since $i\colon X \to i(X)$ is a homeomorphism, there exist an open set $U \subset Y$ containing x such that if $V = i^{-1}(U)$ then V is a holomorphic chart of X. By the rank theorem (cf. Theorem 2.15), it follows that after possibly shrinking U, there exist biholomorphisms $\phi\colon D(0,1)^n \to V$ and $\psi\colon D(0,1)^{n+k} \to U$ so that $\psi^{-1} \circ i \circ \phi$ is given by

$$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_n, 0, \dots, 0).$$

Let $\pi\colon D(0,1)^{n+k} \to D(0,1)^k$ be the projection onto the last k coordinates and let $f = \pi \circ \psi^{-1}\colon U \to D(0,1)^k$. Then f is a submersion such that $f^{-1}(0) = X \cap U$, as claimed.

We now prove the second part of the theorem. Assume that $\{(V_\alpha, \phi_\alpha)\}$ is a holomorphic atlas on Y. We want to show that $\{(V_\alpha \cap X, \phi_\alpha|_{V_\alpha \cap X})\}$ defines a complex structure on X. For each α and β, we have that

$$\phi_\alpha \circ \phi_\beta^{-1}\colon \phi_\beta(V_\alpha \cap V_\beta) \to \phi_\alpha(V_\alpha \cap V_\beta)$$

is a biholomorphism. At each point $z \in \phi_\beta(V_\alpha \cap V_\beta)$, we have an open neighbourhood U' of z and a submersion $f'\colon U' \to D(0,1)^k$ such that $U' \cap \phi_\beta(X) = (f')^{-1}(0)$.

By the rank theorem (cf. Theorem 2.15), after possibly shrinking U', we may assume that $U' = D(0,1)^n$ and $f'(z_1, \dots, z_n) = (z_1, \dots, z_k, 0, \dots, 0)$. Since the restriction of $\phi_\alpha \circ \phi_\beta^{-1}$ to the subset $\phi_\beta((V_\alpha \cap V_\beta) \cap \{z_1 = \cdots = z_k = 0\})$ is also a biholomorphism, the claim follows. ☐

Example 3.12. *Let $U \subset \mathbb{C}^n$ be an open subset and let $f_i\colon U \to \mathbb{C}$ be a holomorphic function, for each $i = 1, \dots, k$. Assume that the matrix $(\frac{\partial f_i}{\partial z_j})_{i,j}$ has rank k for all $z \in V$. Let $V :=$*

$\{f_1 = \cdots = f_k = 0\}$. Then, by the previous theorem, it follows that V is a complex submanifold of codimension k.

Note that the converse is not true. There are submanifold of codimension k in $\mathbb{C}^n$ which are defined by more than k equations.

Example 3.13. *Let $f\colon X \to Y$ be a holomorphic morphism between complex manifolds and let $W \subset X$ be a submanifold. Then $f|_W\colon W \to Y$ is holomorphic.*

Example 3.14. *The only compact complex submanifolds of $\mathbb{C}^n$ are points.*

Example 3.15. *Let $f\colon X \to Y$ be a holomorphic morphism between complex manifolds and let $\Gamma_f \subset X \times Y$ be the graph of f defined as in Example 1.5. Then Γ_f is a complex submanifold.*

Example 3.16. *The space $\mathcal{M}_{n,m}(\mathbb{C})$ of $n \times m$ complex matrices is homeomorphic to $\mathbb{C}^{nm}$ and, thus, it admits a complex structure.*

Define $\mathrm{GL}_n(\mathbb{C})$ to be the set of invertible matrices inside $\mathcal{M}_{n,n}(\mathbb{C})$. Then $\mathrm{GL}_n(\mathbb{C})$ being an open subset of $\mathcal{M}_{n,n}(\mathbb{C})$, is also a complex manifold.

Example 3.17 (Projective manifolds)**.** *Let $X = \mathbb{P}^n_{\mathbb{C}}$ and let $q\colon \mathbb{C}^{n+1} \setminus \{0\} \to X$ be the quotient map as in Example 3.3. Let $R := \mathbb{C}[x_0, \ldots, x_n]$ be the ring of complex polynomials in $(n+1)$ variables. We say that $F \in R$ is homogeneous of degree d if $f(\lambda x) = (\lambda^d \cdot f)$ for all $\lambda \in \mathbb{C}$. Let $F_i \in R$ be homogeneous polynomial of degree d_i for each $i = 1, \ldots, k$ and let*

$$W = \{x \in \mathbb{C}^{n+1} \setminus \{0\} \mid F_1(x) = \cdots = F_k(x) = 0\}.$$

Then, $W \subset \mathbb{C}^{n+1} \setminus \{0\}$ is closed and $x \in W$ if and only if $\lambda x \in W$ for all $\lambda \in \mathbb{C}$. Let $V = q(W)$. Then $q^{-1}(V) = W$ and, in particular, $V \subset \mathbb{P}^n$ is closed.

Suppose that $V \subset \mathbb{C}^{n+1} \setminus \{0\}$ is a submanifold of dimension $m+1$. We want to show that $W \subset \mathbb{P}^n_{\mathbb{C}}$ is a submanifold of dimension m. Let $\{U_i\}$ be the open cover of X defined in Example 3.3. It is enough to show that $V \cap U_i$ is a submanifold of V for each i. For simplicity, we assume that $i = n$. Note that, denoting $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we have a morphism

$$\phi_n \colon U_n \times \mathbb{C}^* \to q^{-1}(U_n)$$

given by

$$\phi_n([z_0, \ldots, z_n], t) = \left(t\frac{z_0}{z_n}, \ldots, t\frac{z_{n-1}}{z_n}, t \right).$$

It is to check that ϕ_n is well-defined and it is a bijection. Moreover, the induced morphism

$$U_n \times \mathbb{C}^* \xrightarrow{\phi_n} q^{-1}(U_n) \xrightarrow{q} U_n$$

coincides with the projection onto the first coordinate. It easily follows that $V \cap q^{-1}(U_n)$ is biholomorphic to $(W \cap U_n) \times \mathbb{C}^*$. Thus, the claim follows.

Example 3.18 (Plane curves)**.** *Let $X = \mathbb{P}^2_{\mathbb{C}}$ and let $F \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial of degree d. As in Example 3.12, $W = \{x \in \mathbb{C}^3 \setminus \{0\} \mid F(x) = 0\}$ is a complex submanifold if and only if for all $x \in W$,*

$$\frac{\partial F}{\partial x_i}(x) \neq 0 \qquad \text{for some } i = 0, 1 \text{ or } 2.$$

If $d = 1$ then $F(x_0, x_1, x_2) = ax_0 + bx_1 + cx_2$ for some $a, b, c \in \mathbb{C}$, which are not all zero. It follows that W is a complex submanifold of dimension 2 and by the previous example, the image V of W in X is a smooth complex manifold of dimension 1. It is easy to check that there exists a biholomorphism $\mathbb{P}^1_{\mathbb{C}} \to V$.

We now consider the case $f = x_0x_1 - x_2^2$. Also in this case, V is smooth and, the holomorphic map

$$f\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}} \quad \text{defined by } f([y_0, y_1]) = [y_0^2, y_1^2, y_0y_1]$$

defines a biholomorphism $\mathbb{P}^1_{\mathbb{C}} \to f(\mathbb{P}^1_{\mathbb{C}}) = V$. More in general, it is possible to show that any smooth conic, i.e. any submanifold of $\mathbb{P}^2_{\mathbb{C}}$ defined by homogeneous polynomials of degree 2, is biholomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

On the other hand, we will see that if a smooth submanifold of $\mathbb{P}^2_{\mathbb{C}}$ is defined by a homogeneous polynomial of degree ≥ 3, then it is never biholomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

Definition 3.19. *Let M be a complex manifold of dimension n and let $x \in M$. Let (U, ϕ) be a complex chart around x. Let $z_1, \dots, z_n$ be complex coordinates on $\phi(U) \subset \mathbb{C}^n$. The* **holomorphic tangent space** *T_xM to M at x is the complex vector space generated by*

$$\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$$

Note that if $f\colon M \to N$ is a holomorphic morphism between complex manifolds then at each point $x \in M$, the holomorphic

Jacobian of f defines a linear map of complex vector spaces:

$$T_xM \to T_{f(x)}N.$$

3.1. **Holomorphic forms on complex manifolds.**

Definition 3.20. *Let X be a complex manifold of dimension n with a complex structure $\{(U_\alpha, \phi_\alpha)\}$. A (global)* **holomorphic p-form** *on X is a collection of p-forms ω_α on $\phi_\alpha(U_\alpha) \subset \mathbb{C}^n$ such that if*

$$h_{\alpha,\beta} := \phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_\alpha \cap U_\alpha) \to \phi_\alpha(U_\alpha \cap U_\beta),$$

then

$$h^*_{\alpha,\beta}\omega_\beta = \omega_\alpha.$$

Notation 3.21. *We denote by $\Omega^p(X)$ or $H^0(X, \Omega^p_X)$ the space of all holomorphic p-forms on a complex manifold X. In particular, if $p = 0$, then $\mathcal{O}_X(X)$ denotes the space of all holomorphic functions on X.*

As in the local case, if X is a complex manifold then $R := \mathcal{O}_X(X)$ is a ring and, for each $p \geq 0$, $M := \Omega^p(X)$ is a R-module.

Lemma 3.22. *Let $f \colon X \to Y$ be a holomorphic map between complex manifolds. Then, for each $p \geq 0$, we have a pull-back morphism $f^* \colon \Omega^p_Y(Y) \to \Omega^p(X)$.*

The proof of the Lemma is the same as in the C^∞ case. Thus we just sketch an idea. Let $\{(V_\alpha, \psi_\alpha)\}$ be a complex structure on Y. Then we can consider a complex structure $\{(U_{\alpha,\beta}, \phi_{\alpha,\beta})\}_{\alpha,\beta}$ such that f maps $U_{\alpha,\beta}$ onto V_α for each α and β. Let $\omega \in \Omega^p(Y)$. Then

ω is defined by p-forms ω_α on $\psi_\alpha(V_\alpha)$. Thus, we can consider the pull-back

$$\omega_{\alpha,\beta} = (\phi_{\alpha,\beta} \circ f \circ \psi_\alpha)^* \omega_\alpha$$

which is a p-form on $\phi_{\alpha,\beta}(U_{\alpha,\beta})$. It is then easy to check that these p-forms are compatible on X.

Furthermore, as in the local case, if $\omega \in \Omega^p(X)$ and $\eta \in \Omega^q(X)$, then it is easy to check that the differential $d\omega \in \Omega^{p+1}(X)$ is a $(p+1)$-form and the exterior product $\omega \wedge \eta \in \Omega^{p+q}(X)$ is a $(p+q)$-form.

4. Holomorphic vector bundles

4.1. Introduction.

Definition 4.1. *Let X be a complex manifold. A* **holomorphic vector bundle** *of rank r on X is a complex manifold E together with a holomorphic morphism $\pi\colon E \to X$, an open covering $\mathcal{U} = \{U_i\}$ of X and biholomorphic morphisms*

$$\psi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$$

such that if $p_i\colon U_i \times \mathbb{C}^r \to U_i$ is the projection, then $\pi|_{\pi^{-1}(U_i)} = p_i \circ \psi_i$ for each i.

A **holomorphic line bundle** *is a vector bundle of rank 1.*

In particular, using the notation above, it follows that for each $x \in X$, the fibre $E(x) := \pi^{-1}(x)$ is a complex vector space of dimension r. Moreover, for any i, j and for any $x \in U_i \cap U_j$, the map

$$g_{i,j}(x) := (\psi_i \circ \psi_j^{-1})|_{E(x)}\colon E(x) \to E(x)$$

is an automorphism and, in particular, it is linear. Thus, we have a map

$$g_{i,j}\colon U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$$

called **transition function**.

We have

$$g_{i,j} \circ g_{j,i} = \mathrm{Id} \qquad \text{for all } x \in U_i \cap U_j$$

and

$$g_{i,j} \circ g_{j,k} = g_{i,k} \qquad \text{for all } x \in U_i \cap U_j \cap U_k.$$

Definition 4.2. *Let X be a complex manifold and let $\pi\colon E \to X$ and $\pi'\colon F \to X$ be holomorphic vector bundles on X. A* **morphism of vector bundles** *$\phi\colon E \to F$ over X is a holomorphic morphism such that $\pi = \pi' \circ \phi$ and such that, for each $x \in X$, the induced map $\phi(x)\colon E(x) \to F(x)$ is linear and the rank of $\phi(x)$ is independent of $x \in X$.*

Example 4.3 (Trivial bundle)**.** *Let X be a complex manifold and let $E = X \times \mathbb{C}^r$. Then E is a holomorphic vector bundle of rank r. E is called* **trivial bundle** *of rank r.*

Example 4.4 (Algebra of vector bundles)**.** *Let E and F be vector bundles over a complex manifold X of rank r and s respectively. Then*

- *The* **direct sum** *$E \oplus F$ is the holomorphic vector bundle over X of rank $r + s$ whose fibre over $x \in X$ is $E(x) \oplus F(x)$.*
- *the* **tensor product** *$E \otimes F$ is the holomorphic vector bundle of rank rs whose fibre over $x \in X$ is $E(x) \otimes F(x)$.*
- *the p-th* **exterior power** *$\Lambda^i E$ of E is the holomorphic vector bundle whose fibre over $x \in X$ is the exterior power*

$\Lambda^i E(x)$. *In particular, the* **determinant bundle** *of* E *is* $\det E := \Lambda^r E$. *Note that* $\det E$ *is a line bundle.*

- *The dual bundle* E^* *of* E *is the holomorphic vector bundle whose fibre over* $x \in X$ *is the dual* $E(x)^*$.
- *If* $\phi\colon E \to F$ *is a morphism of vector bundles, then the* **kernel** $\ker \phi$ *of* ϕ *is the holomorphic vector bundle whose fibre over* $x \in X$ *is the kernel of* $\phi(x)\colon E(x) \to F(x)$. *The* **cokernel** $\mathrm{coker}\phi$ *of* ϕ *is the holomorphic vector bundle whose fibre over* $x \in X$ *is the kernel of* $\phi(x)\colon E(x) \to F(x)$.

Example 4.5. *Let* $X = \mathbb{P}^n_{\mathbb{C}}$ *and define*

$\mathcal{O}(-1) := \{([x_0, \ldots, x_n], v) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \mid v = \mu\cdot(x_0, \ldots, x_n)$ *for some* $\mu \in \mathbb{C}$

Then $\mathcal{O}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$. *Let* $\pi\colon \mathcal{O}(-1) \to \mathbb{P}^n$ *be the projection to the first factor. We want to show that* $\mathcal{O}(-1)$ *is a holomorphic line bundle. Let* $\{U_i\}$ *be the open cover of* X *defined in Example 3.3. We can define*

$$\psi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}$$

by

$$\psi_i([x_0, \ldots, x_n], (v_0, \ldots, v_n)) = ([x_0, \ldots, x_n], v_i).$$

Then $\{(\pi^{-1}(U_i), \psi_i)\}$ *defines a complex structure on* $\mathcal{O}(-1)$ *and a trivialisation of* $\mathcal{O}(-1)$ *as in Definition 4.1.*

The line bundle $\mathcal{O}(1)$ *is the dual* $\mathcal{O}(-1)^*$ *of* $\mathcal{O}(-1)$ *and is called the* **tautological line bundle** *on* $\mathbb{P}^n_{\mathbb{C}}$.

Furthermore, for any $k > 0$, *we define*

$$\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k} = \mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1),$$

and for any $k < 0$, we define $\mathcal{O}(k) := \mathcal{O}(-1)^{\otimes(-k)}$. Alternatively, it is easy to show that $\mathcal{O}(k)$ is the dual of $\mathcal{O}(-k)$.

Definition 4.6. *Let $f\colon Y \to X$ be a holomorphic morphism between complex manifolds and let $\pi\colon E \to X$ be a holomorphic vector bundle of rank r, defined by the open cover $\{U_i\}$ and biholomorphic morphisms*

$$\psi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r.$$

Then the **pull-back** *f^*E of E is the holomorphic vector bundle of rank r over Y defined by:*

$$f^*E := \{(y, v) \in Y \times E \mid f(y) = \pi(v)\} \subset Y \times E.$$

Let $\pi'\colon f^*E \to Y$ be the projection. In order to show that f^*E is a holomorphic bundle, it is enough to consider the open cover $\{f^{-1}(U_i)\}$ and the holomorphic morphisms

$$\psi_i'\colon \pi'^{-1}(f^{-1}(U_i)) = \{(y, v) \in f^{-1}(U_i) \times E \mid f(y) = \pi(v)\} \to f^{-1}(U_i) \times \mathbb{C}^r$$

defined by

$$(y, v) \mapsto (y, p_2(\psi_i(v))),$$

where $p_2\colon U_i \times \mathbb{C}^r \to \mathbb{C}^r$ is the projection.

Note that, we have the isomorphism of vector spaces

$$f^*E(y) \simeq E(f(y)) \qquad \text{for all } y \in Y.$$

Notation 4.7. *Let Y be a complex manifold and let $X \subset Y$ be a complex submanifold, with inclusion map $i\colon Y \to X$. Then we denote the restriction of E on X by $E|_X = i^*E$.*

Definition 4.8. *Let $\pi\colon E \to X$ be a holomorphic vector bundle over a complex manifold X. A* **section** *of E is a holomorphic morphism $s\colon X \to E$ such that $\pi \circ s = \mathrm{Id}_X$.*

Example 4.9. *Let X be a complex manifold and let $E = X \times \mathbb{C}^r$ be the trivial bundle of rank r over X. Then any $v \in \mathbb{C}^r$ defines a section of E by*

$$s_v \colon X \to E \qquad x \mapsto (x, v).$$

Thus, if $v_1, \ldots, v_n$ is a basis of $\mathbb{C}^r$ then at each point $x \in X$, we have that $s_{v_1}(x), \ldots, s_{v_n}(x)$ is a basis of $E(x)$.

Viceversa, if E is a holomorphic vector bundle on a complex manifold X of rank r and $s_1, \ldots, s_r$ are sections such that $s_1(x), \ldots, s_r(x)$ is a basis of $E(x)$ for each $x \in X$ then E is biholomorphic to the trivial vector bundle. The biholomorphism is given by

$$X \times \mathbb{C}^r \to E \qquad (x, (v_1, \ldots, v_n)) \mapsto \sum v_i s_i(x).$$

The set $s_1(x), \ldots, s_r(x)$ is called a **frame** *of E. Note that at each point $x \in X$, there exists an open neighbourhood $U \subset X$ of x such that $E|_U$ is trivial and, in particular, it admits a frame over U. This is called a* **local frame** *of E around x.*

Example 4.10 (Tangent bundle)**.** *Let X be a complex manifold of dimension n and for each $x \in X$, let T_xX be the holomorphic tangent space of M at x. Let $T_X := \cup_{x \in X} T_xX$ with the morphism $\pi \colon T_X \to X$ such that $\pi^{-1}(x) = T_xX$ for each $x \in X$. We want to show that T_X is a vector bundle of dimension n. Let $\{(U_\alpha, \phi_\alpha)\}$ be a holomorphic atlas. Then, $\phi_\alpha(U_\alpha) \subset \mathbb{C}^n$ and for each $x \in U_\alpha$, the Jacobian of ϕ_α at x defines a linear map*

$$T_xX \to T_{\phi_\alpha(x)}\phi_\alpha(U_\alpha) \simeq < \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} >$$

which induces a holomorphic morphism

$$\psi_\alpha \colon \pi^{-1}(U_\alpha) = \bigcup_{x\in U_\alpha} T_xX \to U_\alpha \times \mathbb{C}^n.$$

Thus, T_X is a vector bundle, called the **tangent bundle** *of X. Note that $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ define a local frame.*

The **cotangent bundle** *Ω^1_X of X is the dual of T_X. For each $p \geq 1$, we denote $\Omega^p_X := \Lambda^p\Omega^1_X$. A section of T_X is called a* **holomorphic vector field** *on X. A section of Ω^p_X is a holomorphic p-form on X (cf. Definition 3.20).*

4.2. **The complexified tangent bundle.** We now investigate the relation between the differentiable and complex structure of a complex manifold. We begin by considering first the case of vector spaces.

Let V be a vector space over $\mathbb{R}$ of dimension m and let $J\colon V \to V$ be a $\mathbb{R}$-linear isomorphism. If $J^2 = -\mathrm{Id}$ then J is called an **almost complex structure** on V. Indeed J induces a structure of complex vector space where, if $\lambda = a + ib$, with $a, b \in \mathbb{R}$, then we define

$$\lambda \cdot v = av + bJ(v).$$

Note that if $v_1, \dots, v_n$ is a basis of V over $\mathbb{C}$ then

$$v_1, J(v_1), \dots, v_n, J(v_n)$$

is a basis of V over $\mathbb{R}$. In particular, $m = 2n$ is even. Viceversa, if V is a complex vector space of dimension n then it can be considered as a real vector space of dimension $2n$ and the multiplication by i defines a $\mathbb{R}$-linear map $V \to V$ that is an almost complex structure.

Let V be a real vector space of dimension $2n$ and let J be an almost complex structure on V. Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V. Then V is a vector space over $\mathbb{C}$ of dimension $2n$. We can extend J to a $\mathbb{C}$-linear map $V_{\mathbb{C}} \to V_{\mathbb{C}}$ by

$$J(v \otimes z) := J(v) \otimes z$$

for all $v \in V$ and $z \in \mathbb{C}$. Note that we still have $J^2 = -\mathrm{Id}$. In particular, it follows that J has two eigenvalues i and $-i$. Let $V^{1,0}$ be the eigenspace associated to i and let $V^{(0,1)}$ be the eigenspace associated to $-i$. Then,

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

We can also define a **conjugation** on $V \otimes_{\mathbb{R}} \mathbb{C}$ by

$$\overline{v \otimes z} := v \otimes \overline{z} \qquad \text{for } v \in V, z \in \mathbb{C}.$$

It follows that

$$V^{0,1} = \overline{V^{1,0}}.$$

Example 4.11. *Let $V = \mathbb{C}^n$ and consider coordinates $(z_1, \ldots, z_n)$ where $z_j = x_j + iy_j$ for each $j = 1, \ldots, n$. Thus, V can be seen as the real vector space $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$. The scalar multiplication by i induces a linear map*

$$J \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n} \qquad (x_1, y_1, \ldots, x_n, y_n) \mapsto (-y_1, x_1, \ldots, -y_n, x_n).$$

J is called the standard complex structure on $\mathbb{R}^{2n}$.

We can also consider $x_1, y_1, \ldots, x_n, y_n$ as coordinates of the complexified vector space $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. The extended morphism J is then given by

$$x_j \mapsto y_j \qquad y_j \mapsto -x_j.$$

Therefore $V^{1,0}$ is spanned by

$$x_j - iy_j \qquad j = 1, \ldots, n$$

and $V^{0,1}$ is spanned by

$$x_j + iy_j \qquad j = 1, \ldots, n.$$

Note that

$$\overline{x_j - iy_j} = x_j + iy_j$$

for all $j = 1, \ldots, n$.

We now want to study the relationship between the complex structure and the differentiable structure of a complex manifold. To this end, we first recall the definition of a differentiable vector bundle:

Definition 4.12. *Let X be a differentiable manifold. A* **real** *(resp.* **complex***)* **vector bundle** *of rank r on X is a differentiable manifold E together with a smooth morphism $\pi\colon E \to X$, an open covering $\mathcal{U} = \{U_i\}$ of X such that, if $K = \mathbb{R}$ (resp. $K = \mathbb{C}$), then*

- *for every $x \in X$ the fibre $E(x) = \pi^{-1}(x)$ is a r-dimensional vector space over K,*
- *for every $x \in E$ there exists an open neighborhood U of x and a diffeomorphism*

$$h\colon \pi^{-1}(U) \to U \times K^r$$

such that, if $p_1\colon U \times K^r \to U$ and $p_2\colon U \times K^r \to K^r$ denote the projections then

$$\pi|_{\pi^{-1}(U)} = p_1 \circ h$$

and for all $x \in U$,

$$p_2 \circ h \colon E(x) \to K^r$$

is an isomorphism of vector spaces over K.

Many of the definitions (e.g. pull-back, section, frames, direct sum, tangent bundle, etc...) and properties that we saw for holomorphic vector bundles extend to the case of real and complex vector bundles, by replacing holomorphic morphisms with smooth morphisms.

Let X be a complex manifold of dimension n and consider the tangent bundle T_X. We denote by $T_{X,\mathbb{R}}$ be the real tangent bundle. Let

$$T_{X,\mathbb{C}} := T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

This is a (not holomorphic) complex vector bundle of rank $2n$.

Let X be a differentiable manifold of dimension $2n$. An **almost complex struture** on X is a differentiable vector bundle isomorphsm $J \colon T_X \to T_X$ such that $J^2 = -\mathrm{Id}$.

Proposition 4.13. *Let X be a complex manifold of dimension n. Then the underlying differentiable manifold admits an almost complex structure, i.e. there exists a differentiable vector bundle isomorphism*

$$J \colon T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$$

such that $J^2 = -\mathrm{Id}$.

Proof. Let $x \in X$ and let (U, ϕ) be an holomorphic chart around x. Let $V = \phi(U) \subset \mathbb{C}^n$. We may assume that $\phi(x) = 0$. Let $z_1, \ldots, z_n$ be local holomorphic coordinates and let

$$x_i := \mathrm{Re}(z_i) \quad y_i := \mathrm{Im}(z_i) \quad i = 1, \ldots, n.$$

Then $x_1, y_1, \ldots, x_n, y_n$ defines local differentiable coordinates.

The holomorphic vector fields $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ define a local frame of T_X which gives an biholomorphism

$$T_X|_U \simeq U \times \mathbb{C}^n.$$

Similarly, the differentiable vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$ define a diffeomorphism

$$T_{X,\mathbb{R}}|_U \simeq U \times \mathbb{R}^{2n}.$$

Thus, the induced diffeomorphism

$$T_{X,\mathbb{R}}|_U \simeq U \times \mathbb{C}^n$$

defines an almost complex structure J_U on $T_{X,\mathbb{R}}|_U$.

We now want to show that the almost complex structure extends to X, i.e. J_U does not depend on the choice of the holomorphic coordinates.

Let $f\colon V \to V$ be a biholomorphism such that $f(0) = 0$. Let $z_1', \ldots, z_n'$ be local holomorphic coordinates such that

$$z_i' = f_i(z_1, \ldots, z_n).$$

Let

$$x_i' := \mathrm{Re}(z_i') \quad y_i' := \mathrm{Im}(z_i') \quad i = 1, \ldots, n.$$

Then the morphism f considered as a diffeomorphism, can be expressed in local coordinates as

$$x_i' = u_i(x_1, \ldots, x_n, y_1, \ldots, y_n) \qquad y_i' = v_i(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

where u_i and v_i are the real and imaginary parts of f_i.

The real Jacobian of this map is a transition function between the corresponding trivialisation of the tangent bundle and is given by the $(2n) \times (2n)$-matrix defined by the blocks

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}.$$

In order to show that J extends to X, it is enough check that the operator J commutes with the transition function. Since f_i is holomorphic, the Cauchy-Riemann equations (cf Section 2.2)

$$\frac{\partial f_j}{\partial \overline{z_j}} = 0$$

become

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_j}{\partial y_k} & \frac{\partial u_j}{\partial y_k} \\ -\frac{\partial u_j}{\partial y_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}.$$

On the other hand, since J is defined by the standard complex structure on $\mathbb{C}^n$, its matrix is given by the $(2n) \times (2n)$-matrix defined by the blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

along the diagonal and zero otherwise. Thus, it is easy to check that the two matrices commute. $\square$

Remark 4.14. *If M is a differentiable manifold of dimension $2n$ which admits an almost complex structure J, then it is possible to show that M is orientable. The idea is that, at any point $x \in X$, there exist vectors $v_1, \ldots, v_n \in T_xX$ such that $v_1, \ldots, v_n, J(v_1), \ldots, J(v_n)$ is a basis of the tangent space T_xX. Thus, we can define the orientation on T_xX by the ordered basis $v_1, \ldots, v_n, J(v_1), \ldots, J(v_n)$. In particular, Proposition 4.13* *implies that the underlying differentiable manifold of a complex manifold is always orientable.*

Let X be a complex manifold of dimension n and let

$$T_{X,\mathbb{C}} := T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

The almost complex structure $J\colon T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ constructed in Proposition 4.13 extends to an isomorphism

$$J \otimes_{\mathbb{R}} Id_{\mathbb{C}} : T_{X,\mathbb{C}} \to T_{X,C}$$

which we still denote by J and satisfies $J^2 = -\mathrm{Id}$.

The complex vector bundle $T_{X,\mathbb{C}}$ has rank $2n$ can be decomposed as the direct sum of two complex vector bundles of dimension n

$$T_{X,\mathrm{C}} = T_X^{1,0} \oplus T_X^{0,1} \tag{1}$$

given by the two eigenspaces of J. Moreover, the conjugation on $\mathbb{C}$ also extends to a conjugation on $T_{X,\mathbb{C}}$ and, in particular, we have

$$T_X^{0,1} = \overline{T_X^{1,0}}.$$

Let T_X be the holomorphic tangent bundle (cf. Definition 4.10). Then we have a natural inclusion

$$T_X \hookrightarrow T_{X,\mathbb{C}}.$$

We now describe the inclusion locally at a point $x \in X$. Let $z_1, \ldots, z_n$ be local coordinates of X around x and let

$$x_i := \mathrm{Re}(z_i) \quad y_i := \mathrm{Im}(z_i) \quad i = 1, \ldots, n.$$

Then, as in Section 2.2, we have

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad j = 1, \ldots, n,$$

which gives the inclusion above. Note that in these coordinates J maps $\frac{\partial}{\partial x_j}$ to $\frac{\partial}{\partial y_j}$ and $\frac{\partial}{\partial y_j}$ to $-\frac{\partial}{\partial x_j}$. In particular, $\frac{\partial}{\partial z_j}$ is an eigenvector for J. Thus, we can think of the sub-bundle $T_X^{1,0}$ to coincide with T_X.

By taking the dual, the decomposition (1), induces a decomposition

$$\Omega^1_{X,\mathbb{C}} := T^*_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

and by taking the k-th exterior power of $\Omega_{X,\mathbb{C}}$, we have

$$\Omega^k_{X,\mathbb{C}} := \bigwedge^k \Omega^1_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega_X^{p,q} \tag{2}$$

where

$$\Omega_X^{p,q} := \bigwedge_X^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}$$

is the vector bundle of (p,q)-forms.

Definition 4.15. *Let X be a complex manifold. A (p,q)-***form** *(or a* **form of type** *(p,q)) on X is a section of the complex vector bundle $\Omega_X^{p,q}$.*

Note that in local coordinates a (p,q)-form on X can be written as

$$\omega = \sum_{|I|=p,|J|=q} \alpha_{I,J} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q},$$

where $\alpha_{I,J}$ are smooth functions. To simplify the notation, we will denote

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p} \quad \text{and} \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q},$$

so that

$$\omega = \sum_{|I|=p,|J|=q} \alpha_{I,J} dz_I \wedge d\overline{z}_J.$$

Let X be a differentiable manifold. Given a vector bundle E on X, we denote by $C^\infty(X, E)$ the space of all the smooth sections of E. As in the case of holomorphic forms (cf. Section 2.4), we can consider, for every $k \geq 0$, the exterior differential

$$d \colon C^\infty\left(X, \Omega_X^k\right) \to C^\infty\left(X, \Omega_X^{k+1}\right).$$

Similarly to Proposition 2.22, the differential satisfies the Leibnitz rule and the property $d \circ d = 0$, i.e. $d(d(\omega)) = 0$ for every $\omega \in C^\infty\left(X, \Omega_X^k\right)$.

Assume now that X is a complex manifold. Then $d \otimes \mathrm{Id}_{\mathbb{C}}$ defines an exterior differential on the complexified cotangent bundle $\Omega^1_{X,\mathbb{C}}$, which we still denote by d:

$$d : C^\infty\left(X, \Omega_{X,\mathbb{C}}^k\right) \to C^\infty\left(X, \Omega_{X,\mathbb{C}}^{k+1}\right).$$

Clearly, d still satisfies the Leibnitz rule and the property $d \circ d = 0$. Let $\omega \in C^\infty(X, \Omega_X^{p,q})$. Then $d\omega \in C^\infty(X, \Omega_{X,\mathbb{C}}^{p+q+1})$. More precisely, if we write

$$\omega = \sum_{|I|=p,|J|=q} \alpha_{I,J} dz_I \wedge d\overline{z}_J,$$

where $\alpha_{I,J}$ are smooth functions, then

$$d\omega = \sum_{|I|=p,|J|=q} \partial(\alpha_{I,J}) \wedge dz_I \wedge d\overline{z}_J + \sum_{|I|=p,|J|=q} \overline{\partial}(\alpha_{I,J}) \wedge dz_I \wedge d\overline{z}_J$$

where, for every smooth function α, we write

$$\partial\alpha := \sum_{j=1}^{n} \frac{\partial\alpha}{\partial z_j} dz_j \quad \text{and} \quad \bar\partial\alpha := \sum_{j=1}^{n} \frac{\partial\alpha}{\partial \overline{z}_j} d\overline{z}_j.$$

Note that

$$d = \partial + \overline{\partial},$$

i.e. for each for each smooth function α, we may write

$$d\alpha = \partial\alpha + \overline{\partial}\alpha,$$

where $\partial\alpha \in C^\infty(\Omega_X^{1,0})$ and $\overline{\partial}\alpha \in C^\infty(\Omega_X^{0,1})$. More in general, if $\omega \in C^\infty(\Omega_X^{p,q})$, then we can define

$$\partial\omega := \sum_{|I|=p,|J|=q} \partial(\alpha_{I,J}) \wedge dz_I \wedge d\overline{z}_J \qquad \overline{\partial}\omega := \sum_{|I|=p,|J|=q} \overline{\partial}(\alpha_{I,J}) \wedge dz_I \wedge d\overline{z}_J$$

and, we have

$$d\omega = \partial\omega + \overline{\partial}\omega,$$

i.e. also in this case we may write $d = \partial + \overline{\partial}$. Note that $\partial\omega \in C^\infty(\Omega_X^{p+1,q})$ and $\overline{\partial}\omega \in C^\infty(\Omega_X^{p,q+1})$. By (2) and by linearity, we have that $\partial, \overline{\partial}$ can be extended as linear maps

$$\partial, \overline{\partial} \colon C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}}).$$

Lemma 4.16 (Leibnitz rule)**.** *Let X be a complex manifold. Let $\omega \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$ and $\eta \in C^\infty(X, \Omega^\ell_{X,\mathbb{C}})$.*

Then

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^k \omega \wedge \partial\eta$$

and

$$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^k \omega \wedge \overline{\partial}\eta.$$

Proof. By linearity, we may assume that ω has type (p,q) and η has type (p',q'), where $k = p+q$ and $\ell = p'+q'$. By the Leibnitz rule for d, we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

Thus,

$$\begin{aligned}(\partial + \overline{\partial})(\omega \wedge \eta) &= d(\omega \wedge \eta) \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \\ &= \partial\omega \wedge \eta + \overline{\partial}\omega \wedge \eta + (-1)^k \omega \wedge \partial\eta + (-1)^k \omega \wedge \overline{\partial}\eta \\ &= \big(\partial\omega \wedge \eta + (-1)^k \omega \wedge \partial\eta\big) + \big(\overline{\partial}\omega \wedge \eta + (-1)^k \omega \wedge \overline{\partial}\eta\big)\end{aligned}$$

gives a decomposition in forms of type $(p+p'+1, q+q')$ and $(p+p', q+q'+1)$ respectively. Thus, the Lemma follows. $\square$

4.3. **Dolbeaut cohomology.**

Lemma 4.17. *Let X be a complex manifold. Then the following hold:*

$$\partial^2 = 0, \quad \bar{\partial}\partial + \partial\bar{\partial} = 0, \quad \overline{\partial}^2 = 0.$$

Proof. By linearity, it is enough to check the equalities on a form ω of type (p,q). Since $d \circ d = 0$, we have

$$\begin{aligned}0 = d^2\omega &= (\partial + \overline{\partial})^2\omega \\ &= \partial^2\omega + (\bar{\partial}\partial + \partial\bar{\partial})\omega + \overline{\partial}^2\omega.\end{aligned}$$

Since $\partial^2\omega$, $(\bar{\partial}\partial + \partial\bar{\partial})\omega$ and $\overline{\partial}^2\omega$ have different type, i.e. type $(p+2,q)$, $(p+1,q+1)$ and $(p,q+2)$ respectively, they must all vanish. $\square$

For each pair (p,q), we define

$$\begin{aligned}\mathcal{Z}^{p,q}(X) &=: \operatorname{Ker}\left(\overline{\partial}\colon C^\infty(X,\Omega_X^{p,q}) \to C^\infty(X,\Omega_X^{p,q+1})\right)\\ &= \{\omega \in C^\infty(X,\Omega_X^{p,q}) \mid \overline{\partial}\omega = 0\}\end{aligned}$$

and, for each $q \geq 1$, we define

$$\begin{aligned}\mathcal{B}^{p,q}(X) &=: \operatorname{Im}\left(\overline{\partial}\colon C^\infty(X,\Omega_X^{p,q-1}) \to C^\infty(X,\Omega_X^{p,q})\right)\\ &= \{\omega \in C^\infty(X,\Omega_X^{p,q}) \mid \omega = \overline{\partial}\eta \text{ for some } \eta \in C^\infty(X,\Omega_X^{p,q-1})\}\end{aligned}$$

For convenience, we define $\mathcal{B}^{p,0} = 0$ for each p.

By the previous Lemma, it follows that

$$\mathcal{B}^{p,q}(X) \subset \mathcal{Z}^{p,q}(X)$$

for each p,q.

Thus, we may define

$$H^{p,q}(X) := \mathcal{Z}^{p,q}(X)/\mathcal{B}^{p,q}(X).$$

The group $H^{p,q}(X)$ is called **Dolbeaut cohomology group** of X. If it is finite dimensional, then their dimension

$$h^{p,q}(X) := \dim H^{p,q}(X)$$

is called **Hodge number** of X.

We first study the groups $H^{p,0}(X)$.

Proposition 4.18. *Let X be a complex manifold and let $p \geq 0$ then we have an isomorphism*

$$H^{p,0}(X) \simeq H^0(X,\Omega_X^p)$$

(cf. Notation 3.21).

In particular, if X is compact, then $H^{0,0}(X) = \mathbb{C}$.

Proof. We have

$$H^{p,0}(X) = \mathcal{Z}^{p,0}(X) = \{\omega \in C^\infty(X, \Omega_X^{p,0}) \mid \overline{\partial}\omega = 0\}.$$

Let $\omega \in C^\infty(\Omega_X^{p,0})$ such that $\overline{\partial}\omega = 0$. Locally, we may write

$$\omega = \sum_{|I|=p} \alpha_I dz_I,$$

where α_I is a smooth function for each I. Then

$$0 = \overline{\partial}\alpha = \sum_{|I|=p} \frac{\partial}{\partial \overline{z}_j} \alpha_I d\overline{z}_j \wedge dz_I.$$

Since, locally the forms $d\overline{z}_j \wedge dz_I$ are linearly independent, it follows that

$$\frac{\partial}{\partial \overline{z}_j} \alpha_I = 0 \qquad \text{for every } I, j.$$

Thus, α_I is holomorphic for every I. In particular, ω is a holomorphic section of Ω_X^p. Similarly, if ω is a holomrphic section of Ω_X^p then $[\omega] \in H^{p,0}(X)$.

The last part of the Proposition, follows from Remark 3.6. $\square$

5. Connections, curvature and metrics

5.1. **Connection.**

Definition 5.1. *Let X be a differentiable manifold and let $\pi\colon E \to X$ be a complex vector bundle over X. A* **connection** *on E is a $\mathbb{C}$-linear map*

$$\nabla\colon C^\infty(X, E) \to C^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes E\right)$$

that satisfies the Leibnitz rule:

$$\nabla(f\omega) = f \cdot \nabla\sigma + df \otimes \sigma \qquad \textit{for any } f \in C^\infty(X), \sigma \in C^\infty(X, E).$$

Note that, in general, ∇ is not a C^∞-linear map.

We now want to study a connection ∇ on a vector bundle $\pi\colon E \to X$ of rank r, locally around a point. Let $x \in X$ and let $U \subset X$ be an open set containing x and which trivialise E, i.e. there exists a frame $s_1, \dots, s_r$ of E on U. Any section σ of $E|_U$ can be written as $\sum f_j s_j$ where $f_1, \dots, f_r$ are smooth functions on U. Thus, by the Leibnitz rule, we have

$$\nabla(\sigma) = \sum_{j=1}^{r} \nabla(f_j s_j) = \sum_{j=1}^{r} \left(f_j \cdot \nabla(s_j) + df_j \otimes s_j\right). \tag{3}$$

We may write

$$\nabla(s_j) = \sum_{i=1}^{r} a_{i,j} \otimes s_i$$

where $a_{i,j} \in C^\infty(\Omega^1_{X,\mathbb{C}})$. Thus, the connection is completely determined by the $(r \times r)$-matrix of 1-forms

$$A = (a_{i,j}).$$

We can write (3) in the form

$$\nabla(\sigma) = A \cdot f + df, \tag{4}$$

where $f = (f_1, \dots, f_r)$ and $df = (df_1, \dots, df_r)$.

Note that A depends on the choice of the frame. Suppose that $s'_1, \dots, s'_r$ is another local frame of E on the open set $U' \subset X$. Thus, the restriction of E to $U \cap U'$ admits two trivialisation and if σ is a section of $E|_{U \cap U'}$, we may write $\sigma = \sum f'_j s'_j$ and, with respect to this trivialisation, we have

$$\nabla(\sigma) = A' \cdot f' + df' \tag{5}$$

where $f' = (f'_1, \dots, f'_r)$ and A' is the matrix associated to the frame $s'_1, \dots, s'_r$. Assume that

$$g\colon (U \cap U') \times \mathbb{C}^r \to (U \cap U') \times \mathbb{C}^r$$

is the transition function from the trivialisation on U' to the trivialisation on U. Then

$$f = g \cdot f'.$$

Thus, if Dg denotes the differential of g, we have

$$df = d(g \cdot f') = Dg \cdot f' + g \cdot df' = g \cdot (g^{-1} \cdot Dg \cdot f' + df'),$$

and, with respect to the frame $s_1, \dots, s_r$, we may write

$$\begin{aligned}\nabla(\sigma) = A \cdot f + df &= A \cdot g \cdot f' + g \cdot (g^{-1} \cdot Dg \cdot f' + df') \\ &= g \cdot ((g^{-1} \cdot Dg + g^{-1} \cdot A \cdot g) \cdot f' + df').\end{aligned}$$

It follows that, with respect to the frame $s'_1, \dots, s'_r$, we may write

$$\nabla(\sigma) = (g^{-1} \cdot Dg + g^{-1} \cdot A \cdot g) \cdot f' + df'.$$

Thus, by (5), we have

$$A' = g^{-1} \cdot Dg + g^{-1} \cdot A \cdot g. \tag{6}$$

For each $p \geq 1$, the connection ∇ induces a $\mathbb{C}$-linear map

$$C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{p+1}_{X,\mathbb{C}} \otimes E)$$

by requiring the Leibnitz rule to hold, i.e.

$$\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge \nabla\sigma$$

for any $\omega \in C^\infty(X, \Omega^p_{X,\mathbb{C}})$ and $\sigma \in C^\infty(X, E)$. We will continue to denote such a map by ∇.

Let us assume, as before, that around a point $x \in X$, the sections $s_1, \dots, s_r$ define a local frame of E. Let A be the matrix associated to ∇ with respect to this frame. Recall that A is a $(r \times r)$-matrix of 1-forms $(a_{i,j})$. Then, with respect to this frame, a section $\eta \in C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E)$ can be written as $\eta = \sum_{i=1}^r f_i \otimes s_i$ where $f_1, \dots, f_r$ are p-forms.

Then, if $f = (f_1, \dots, f_r)$, the representation (4) extends to

$$\nabla(\eta) = A \wedge f + df \tag{7}$$

where

$$A \wedge f := \left(\sum_j a_{i,j} \wedge f_j \right)_{j=1,\dots,r}$$

and $df := (df_1, \dots, df_r)$.

We now want to study the operator $\nabla^2 = \nabla \circ \nabla$. We denote by DA the differential of A, which is the $(r \times r)$ matrix of 2-forms $(da_{i,j})$. Moreover, we denote by $A \wedge A$ the $(r \times r)$-matrix of 2-forms

$$A \wedge A = \left(\sum_{k=1}^r a_{i,k} \wedge a_{k,j} \right).$$

Thus, given a local section $\sigma = \sum f_j s_j$, where $f_1, \dots, f_r$ are smooth functions and denoting $f = (f_1, \dots, f_r)$, we have

$$\begin{aligned}
\nabla^2(\sigma) &= \nabla(A \cdot f + df) \\
&= A \wedge (A \cdot f + df) + d(A \cdot f + df) \\
&= A \wedge A \cdot f + A \wedge df + DA \cdot f - A \wedge df + d^2 f \\
&= (A \wedge A + DA) \cdot f.
\end{aligned}$$

Thus, if we define $\Theta_\nabla := A \wedge A + DA$, we have that

$$\nabla^2(\sigma) = \Theta_\nabla \cdot \sigma. \tag{8}$$

The matrix Θ_∇ is called the **curvature** of ∇ with respect to the frame $s_1, \ldots, s_r$. Note that Θ_∇ defines a C^∞-linear map

$$C^\infty(X, E) \to C^\infty(X, E \otimes \Omega^2_{X,\mathbb{C}}).$$

5.2. **Hermitian metric.** Recall the following definition:

Definition 5.2. *Let V be a complex vector space. A* **Hermitian inner product** *on V is a function*

$$\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$$

such that

- $\langle u, v \rangle = \overline{\langle v, u \rangle}$ *for any $u, v \in V$,*
- *linear in the first factor,*
- $\langle v, v \rangle \geq 0$ *for all v and equality holds if and only if $v = 0$.*

Definition 5.3. *Let X be a differentiable manifold and let $\pi \colon E \to X$ be a complex vector bundle on X. A* **Hermitian metric** *h on E is defined by a Hermitian inner product*

$$\langle \cdot, \cdot \rangle_x \colon E(x) \times E(x) \to \mathbb{C}$$

at each fibre $E(x)$ such that for any open set $U \subset X$ and for any pair of sections on U, $s, t \in C^\infty(U, E)$, the function

$$\langle s(\cdot), t(\cdot) \rangle \colon U \to \mathbb{C} \qquad x \mapsto \langle s(x), t(x) \rangle_x$$

is smooth. A complex vector bundle E equipped with a Hermitian metric h is called a **Hermitian vector bundle** *(E, h).*

We now want to describe a Hermitian metric locally around a point. Let $\pi \colon E \to X$ and h be as in the definition above and let $x \in X$. Let $s_1, \ldots, s_r$ be a local frame for E in a neighbourhood U of x. The Hermitian metric with respect to such a frame is represented by the $(r \times r)$-matrix of smooth functions $H = (h_{i,j})$, given by

$$h_{i,j}(x) := \langle s_i(x), s_j(x) \rangle_x.$$

Thus, if $\sigma, \sigma' \in C^\infty(U, E)$ are sections which, with respect to the frame $s_1, \ldots, s_r$, are represented by $f = (f_1, \ldots, f_r)$ and $f' = (f'_1, \ldots, f'_r)$ respectively, then

$$\langle \sigma(x), \sigma'(x) \rangle_x = f^t \cdot H \cdot \overline{f}'.$$

Note that, since h is Hermitian, we have $H^t = \overline{H}$. As above, suppose that $s'_1, \ldots, s'_r$ is another local frame of E on the open set $U' \subset X$ and assume that

$$g\colon (U \cap U') \times \mathbb{C}^r \to (U \cap U') \times \mathbb{C}^r$$

is the transition function from the trivialisation on U' to the trivialisation on U. Then, it is easy to check that

$$H' = g^t \cdot H \cdot \overline{g}.$$

Proposition 5.4. *Every complex vector bundle $\pi\colon E \to X$ admits a Hermitian metric h.*

Before proving the proposition, we recall the definition of a partition of the unity:

Definition 5.5. *Let M be a manifold and let $\mathcal{U} = \{U_\alpha\}$ be an open covering. A* **partition of unity** *with respect to $\mathcal{U}$ is a collection of smooth functions $f_\alpha\colon M \to [0,1]$ such that*

(1) $\mathrm{Supp}(f_\alpha) \subset U_\alpha$ *for all α (in particular, $f_\alpha = 0$ outside U_α),*
(2) $\sum_\alpha f_\alpha(x) = 1$ *for all $x \in M$, and*
(3) *for all $x \in M$, there exists an open nieghbourhood V of x such that* $\mathrm{Supp(f_\alpha) \cap V \neq 0}$ *for only finitely many α.*

It can be shown that if M is a manifold and $\mathcal{U} = \{U_\alpha\}$ is an open cover of M, then there exists a partition of the unity $\{f_\alpha\}$ with respect to such a cover.

Proof of Proposition 5.4. Let $\{U_i\}$ be a trivialising cover for E, and let h_i be an Hermitian metric on the restriction of E to U_i. Let f_i be a partition of unity with respect to the open cover $\{U_i\}$. Then we may define

$$h = \sum f_i h_i.$$

It is clear that, for every $x \in X$, this defines a Hermitian inner product on $E(x)$. Moreover, if $\sigma, \sigma' \in C^\infty(U, E)$, then the function

$$x \mapsto \langle \sigma(x), \sigma'(x) \rangle_x = \sum f_i \langle \sigma(x), \sigma'(x) \rangle_{i,x}$$

is smooth. Thus, h is a Hermitian metric on E. $\square$

Let $\pi \colon E \to X$ be a Hermitian vector bundle of rank r. Then, for each $p, q \geq 0$, the Hermitian metric induces a bilinear map

$$\begin{aligned} C^\infty(X, E \otimes \Omega^p_{X,\mathbb{C}}) \times C^\infty(X, E \otimes \Omega^q_{X,\mathbb{C}}) &\to C^\infty(X, \Omega^{p+q}_{X,\mathbb{C}}) \\ (\sigma, \tau) &\mapsto \{\sigma, \tau\} \end{aligned}$$

which is locally defined as follows. Let $x \in X$ and let $s_1, \dots, s_r$ be a local frame of E in an open set U of X containing x. If $\sigma \in C^\infty(X, E \otimes \Omega^p_{X,\mathbb{C}})$ and $\tau \in C^\infty(X, E \otimes \Omega^p_{X,\mathbb{C}})$, then locally we can write

$$\sigma = \sum_{i=1}^r \sigma_i \otimes s_i \qquad \text{and} \qquad \tau = \sum_{i=1}^r \tau_i \otimes s_i$$

where σ_i and τ_i are smooth p-forms and q-forms on U respectively. Let H be the matrix associated to the Hermitian metric h with respect to the frame $s_1, \dots, s_r$. Then we define:

$$\{\sigma, \tau\} := \sigma^t \cdot H \cdot \overline{\tau} := \sum_{i,j=1}^{r} h_{i,j} \cdot \sigma_i \wedge \overline{\tau_j}.$$

Note that $\{\sigma, \tau\}$ is a smooth $(p+q)$-form on U.

Definition 5.6. *Let E be a Hermitian vector bundle on a manifold X and let ∇ be a connection on E. We say that ∇ is* **Hermitian** *(or* **compatible** *with the Hermitian metric on E) if the following Leibnitz rule holds:*

$$d\{\sigma, \tau\} = \{\nabla\sigma, \tau\} + (-1)^p\{\sigma, \nabla\tau\}$$

for all $\sigma \in C^\infty(X, E \otimes \Omega^p_{X,\mathbb{C}})$ and $\tau \in C^\infty(X, E \otimes \Omega^q_{X,\mathbb{C}})$.

As before, let $x \in X$ and let $s_1, \dots, s_r$ be a local frame for an Hermitian vector bundle E on X, around x. By the Gram-Schmidt procedure, after possibly replacing the local frame, we may assume that $s_1, \dots, s_r$ is **orthonormal**, i.e.

$$\langle s_i, s_j \rangle = \delta_{i,j} \qquad \text{for any } i, j = 1, \dots, r.$$

The corresponding trivialisation of E will be called **isometric**. Note that, with respect to such a frame, the matrix H representing the Hermitian metric is the identity matrix.

Let ∇ be a connection and let A be the $(r \times r)$-matrix associated to ∇ with respect to the orthonormal frame $s_1, \dots, s_r$ (cf. Section 5.1). Then

Lemma 5.7. *With the same assumptions as above, ∇ is Hermitian if and only*

$$\bar{A}^t = -A$$

i.e. the matrix A is anti-autodual.

Proof. Since $s_1, \ldots, s_r$ is an orthormal frame, the matrix H associated to the Hermitian metric with respect to the metric is the identity matrix. Thus,

$$\{\sigma, \tau\} = \sigma^t \wedge \bar{\tau}.$$

It follows,

$$d\{\sigma, \tau\} = (d\sigma)^t \wedge \bar{\tau} + (-1)^p \sigma^t \wedge d\bar{\tau}.$$

By (7), we have

$$\begin{aligned} \{\nabla\sigma, \tau\} &= \{A \wedge \sigma + d\sigma, \tau\} \\ &= (-1)^p \sigma^t \wedge A^t \wedge \overline{\tau} + d\sigma^t \wedge \overline{\tau} \end{aligned}$$

and

$$\begin{aligned} \{\sigma, \nabla\tau\} &= \{\sigma, A \wedge \tau + d\tau\} \\ &= \sigma^t \wedge \overline{A} \wedge \overline{\tau} + \sigma^t \wedge d\overline{\tau}. \end{aligned}$$

Thus, the Leibnitz rule implies

$$\sigma^t \wedge (A^t + \overline{A}) \wedge \overline{\tau} = 0$$

for all σ, τ. Thus, the claim follows. $\square$

More in general, if locally around a point $x \in X$, we have that H and A are the matrices associated to a metric and a connection respectively, with respect to some local frame $s_1, \ldots, s_r$, then the connection is compatible to the metric if and only if

$$DH = A^t \cdot H + H \cdot \overline{A}, \tag{9}$$

where $DH = (dh_{i,j})$ denotes the differential of $H = (h_{i,j})$.

Theorem 5.8. *Let X be a manifold and let E be a Hermitian vector bundle on X. Then X admits a compatible connection ∇.*

5.3. **Holomorphic vector bundles.** We now apply the theory of connections and Hermitian metrics to the case of holomorphic vector bundles.

Proposition 5.9. *Let X be a complex manifold and let $\pi\colon E \to X$ be a holomorphic vector bundle of rank r.*

Then, for each $q \geq 0$, there is a $\mathbb{C}$-linear map

$$\overline{\partial}_E\colon C^\infty(X, \Omega_X^{0,q} \otimes E) \to C^\infty(X, \Omega_X^{0,q+1} \otimes E)$$

which satisfies the Leibnitz rule and $\overline{\partial}_E^2 = 0$.

Moreover, if σ is a holomorphic section of $\Omega_X^{0,q} \otimes E$ then $\overline{\partial}_E(\sigma) = 0$.

Proof. Let $x \in X$ be a point and let $s_1, \ldots, s_r$ be a local holomorphic frame defined over an open set U of X containing x. Let $\sigma \in C^\infty(X, \Omega_X^{0,q} \otimes E)$. Then locally we may write

$$\sigma = \sum f_i \otimes s_i$$

where $f_1, \ldots, f_r \in C^\infty(U, \Omega_X^{0,q})$. Then we define

$$\overline{\partial}_E(\sigma) = \sum_{i=1}^r \overline{\partial}(f_i) \otimes s_i \in C^\infty(U, \Omega_X^{q+1,0} \otimes E).$$

Let us assume now that $s'_1, \ldots, s'_r$ defines another holomorphic local frame on an open set $U' \subset X$ and let

$$g\colon (U \cap U') \times \mathbb{C}^r \to (U \cap U') \times \mathbb{C}^r$$

be the transition function from the trivialisation on U' to the trivialisation on U. Since g is defined by holomorphic functions, we have that $\overline{\partial} g = 0$. Proceeding similarly as in the steps to obtain (6), we see that if $\sigma = \sum f_i' \otimes \sigma_i'$, then

$$\sum_{i=1}^{r} \overline{\partial}(f_i) \otimes s_i = \sum_{i=1}^{r} \overline{\partial}(f_i') \otimes s_i'.$$

Thus ∂_E is independent of any local choice and it extends to X.

If σ is a holomorphic section of E then, locally around x, the functions $f_1, \ldots, f_r$ are holomorphic. In particular $\overline{\partial}(f_i) = 0$. Thus $\overline{\partial}_E(\sigma) = 0$. □

Viceversa, if $\nabla\colon C^\infty(X, E) \to C^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes E\right)$ is a connection on E, then, by composing with the projections

$$\Omega^1_{X,\mathbb{C}} \to \Omega^{1,0}_X \qquad \text{and} \qquad \Omega^1_{X,\mathbb{C}} \to \Omega^{0,1}_X$$

we can decompose

$$\nabla = \nabla^{1,0} + \nabla^{0,1}$$

where

$$\nabla^{1,0}\colon C^\infty(X, E) \to C^\infty\left(X, \Omega^{1,0}_X \otimes E\right)$$

and

$$\nabla^{0,1}\colon C^\infty(X, E) \to C^\infty\left(X, \Omega^{0,1}_X \otimes E\right).$$

Theorem 5.10. *Let X be a complex manifold and let (E, h) be a Hermitian holomorphic vector bundle of rank r.*

Then there is a unique connection

$$\nabla_E\colon C^\infty(X, E) \to C^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes E\right)$$

such that

$$\nabla_E^{0,1} = \overline{\partial}_E$$

and such that ∇_E is compatible with the metric.

We call ∇_E the **Chern connection** of (E, h). Moreover the curvature $\Theta_E := \Theta_{\nabla_E}$, defined as in (8), is called **Chern curvature** of (E, h).

Proof. As above, we will define the collection locally and then show that the connection is independent of any choice made and so it extends to X.

Let $x \in X$ be a point and let $s_1, \ldots, s_r$ be a local holomorphic frame defined over an open set U of X containing x. Let $H = (h_{i,j})$ be the matrix defining the metric on U with respect to $s_1, \ldots, s_r$ and denote by ∂H the $(r \times r)$-matrix of $(1, 0)$-forms on U defined by

$$\partial H = (\partial h_{i,j})$$

We then define

$$A := \overline{H}^{-1} \cdot \partial \overline{H}, \tag{10}$$

and we consider the connection ∇_E on $E|_U$ defined by A, so that if $\sigma = \sum f_i \sigma_i$ is a section of E on U, where $f_1, \ldots, f_r$ are smooth functions, and $f = (f_1, \ldots, f_r)$ then, as in (4), we have

$$\nabla_E(f) = A \cdot f + df.$$

Note that $A = (a_{i,j})$ is defined by $(1, 0)$-forms $a_{i,j}$. In particular, it follows that on U we have

$$\nabla_E^{0,1} = \overline{\partial}_E.$$

In order to check that the connection is compatible with the metric h on U, we need to check that A satisfies (9), i.e.

$$DH = A^t \cdot H + H \cdot \overline{A}.$$

We have

$$\begin{aligned} A^t \cdot H &= (\overline{H}^{-1} \cdot \partial \overline{H})^t \cdot H \\ &= \partial \overline{H}^t \cdot (\overline{H}^{-1})^t \cdot H \\ &= \partial H \cdot H^{-1} \cdot H = \partial H \end{aligned}$$

where the third equality follows from the fact that $\overline{H} = H^t$, since h is Hermitian. Similarly,

$$H \cdot \overline{A} = H \cdot (H^{-1} \cdot \overline{\partial} H) = \overline{\partial} H.$$

Since $dH = \partial H + \overline{\partial} H$, it follows that (9) holds and the connection is compatible with the metric.

We now show that such connection is unique on U. Let ∇ be a connection on E_U which is compatible with the metric and such that $\nabla^{0,1} = \overline{\partial}_E$. Let $B = (b_{i,j})$ be the matrix associated to ∇ respect to the frame $s_1, \dots, s_r$. We may write $B = B^{(1,0)} + B^{(0,1)}$ where $B^{(1,)}$ (resp. $B^{(0,1)}$) is the $(r \times r)$-matrix obtained by taking the $(1,0)$ (resp. $(0,1)$) components of $b_{i,j}$. If σ is a section of E, then on U, we have $\sigma = \sum_{i=1}^r f_i s_i$ where $f_1, \dots, f_r$ are smooth functions on U and if $f = (f_1, \dots, f_r)$, we have

$$\overline{\partial} f + B^{0,1} \cdot f = \nabla^{0,1}(f) = \overline{\partial} f.$$

It follows that $B^{(0,1)} = 0$, i.e. $B = B^{(1,0)}$. Since ∇ is compatible with the metric, (9) implies that

$$DH = B^t \cdot H + H \cdot \overline{B}.$$

It follows that

$$B = \overline{H}^{-1} \cdot \left(\overline{(DH)} - \overline{(B^t H)} \right).$$

Thus,

$$B = B^{(1,0)} = \overline{H}^{-1} \cdot \partial \overline{H} = A.$$

It follows that $\nabla = \nabla_E$ on U, i.e. the connection is unique on U. This also implies that if ∇'_E is a connection on a different open set U' of X, which is compatible with the Hermitian metric and such that $\nabla_E'^{(0,1)} = \overline{\partial}_E$, then, on the intersection $U \cap U'$, the connection ∇_E must coincide with the connection ∇'_E. It follows that ∇_E extends uniquely on X. Thus, the Theorem follows. $\square$

Corollary 5.11. *Let (E, h) be a Hermitian holomorphic vector bundle of rank r on a complex manifold X. Let ∇_E be the Chern connection on (E, h) and Θ_E its curvature. Let A be the matrix representing ∇_E with respect to some local holomorphic frame $s_1, \dots, s_r$.*

Then,

(1) *A is of type $(1,0)$ and $\partial A = -A \wedge A$.*
(2) *Locally $\Theta_E = \overline{\partial} A$ and, in particular, Θ_E is of type $(1,1)$.*
(3) *$\overline{\partial}\Theta_E = 0$.*

Proof. Let H be the matrix representing the metric h with respect to the local holomorphic frame $s_1, \dots, s_r$.

We first prove (1). As in the proof of Theorem 5.10, it follows that A is of type $(1,0)$ and

$$A = \overline{H}^{-1} \cdot \partial \overline{H}.$$

We have

$$0 = \partial(\overline{H} \cdot \overline{H}^{-1}) = \overline{H} \cdot \partial(\overline{H}^{-1}) + \partial \overline{H} \cdot \overline{H}^{-1}.$$

Thus, since $\partial^2\overline{H} = 0$, we have

$$\begin{aligned}\partial A &= \partial(\overline{H}^{-1} \cdot \partial\overline{H}) \\ &= \partial\overline{H}^{-1} \wedge \partial\overline{H} \\ &= -(\overline{H}^{-1} \cdot \partial\overline{H} \cdot \overline{H}^{-1}) \wedge \partial\overline{H} \\ &= -(\overline{H}^{-1} \cdot \partial\overline{H}) \wedge (\overline{H}^{-1} \cdot \partial\overline{H}) = -A \wedge A.\end{aligned}$$

We now prove (2). Recall that, by definition, $\Theta_E = A \wedge A + DA$. Thus, by (1) we have

$$\begin{aligned}\Theta_E &= A \wedge A + \partial A + \overline{\partial} A \\ &= A \wedge A + \overline{\partial} A - A \wedge A = \overline{\partial} A.\end{aligned}$$

Finally, we have

$$\overline{\partial}\Theta_E = \overline{\partial}^2 A = 0$$

and also (3) follows. □

Recall that in the C^∞ case, we could apply the Gram-Schmidt orthonormalization process to produce a local frame of a Hermitian vector bundle where the matrix defining the metric was simply the identity matrix. On the other hand, the operations in the Gram-Schmidt process aren't holomorphic in general and so we can't in general hope to produce a holomorphic frame where the metric is given by the identity matrix. Nevertheless, we have:

Proposition 5.12. *Let X be a complex manifold of dimension n and let (E, h) be a Hermitian holomorphic vector bundle of rank r on X. Let $x \in X$.*

Then there exists an open neighbourhood U of x with local holomorphic coordinates $z_1, \ldots, z_n$ so that $x = (0, \ldots, 0)$ and a local holomorphic frame $s_1, \ldots, s_r$ on U such that, if H is the matrix representing the metric h with respect to the local holomorphic frame $s_1, \ldots, s_r$, then

(1) $H(z) = \mathrm{Id} + \mathcal{O}(|z^2|)$ *and*

(2) $\Theta_E(0) = -\partial\overline{\partial}\ \overline{H}(0)$

where Θ_E is the Chern curvature of E.

Proof. We first prove (1). Let U be an open neighbourhood of x such that there exists a local holomorphic frame $t_1, \ldots, t_r$ for E and let H_1 be the matrix representing the metric h with respect to this frame. The matrix $H_1(0)$ is a positive definite $(r \times r)$-Hermitian matrix. Thus, by choosing a orthonormal basis for $E(x)$ with respect to h_x, it follows that there exists an invertible matrix $(r \times r)$-matrix B such that

$$B^t \cdot H_1(0) \cdot \overline{B} = \mathrm{Id}.$$

Thus, if $t'_i = B \cdot t_i$ for each $i = 1, \ldots, r$ it follows that $t'_1, \ldots, t'_r$ is also a local holomorphic frame and if H_2 is the matrix associated to the matrix h with respect to this frame, then $H_2(0) = \mathrm{Id}$. In particular,

$$H_2(z) = \mathrm{Id} + \mathcal{O}(|z|).$$

We need to perform a second change of frame. To this end, we consider a change of basis matrix of the form

$$C(z) = \mathrm{Id} + C_0(z)$$

where $C_0(z)$ is defined by linear forms in $z_1, \ldots, z_r$. The matrix H representing h with respect to this frame will be of the form

$$H = (\mathrm{Id} + C_0^t) \cdot H_2 \cdot (\mathrm{Id} + \overline{C_0}).$$

Thus, writing the coefficients of $H(z)$ in Taylor series, it follows that

$$H(z) = \mathrm{Id} + \mathcal{O}(|z^2|) \quad \text{if and only if} \quad DH(0) = 0.$$

We have $H_2(0) = \mathrm{Id}$ and

$$DH = DH_2 + D(\mathrm{Id} + C_0^t) \cdot H_2 + H_2 \cdot D(\mathrm{Id} + \overline{C_0}) + \mathcal{O}(|z|).$$

Thus,

$$\begin{aligned} DH(0) &= DH_2(0) + DC_0^t(0) + D\overline{C}_0(0) \\ &= \partial H_2(0) + DC_0^t(0) + \overline{\partial} H_2(0) + D\overline{C}_0(0). \end{aligned}$$

Thus, if we set

$$C_0 = (c_{i,j}) \qquad \text{where } c_{i,j} := -\sum_{l=1}^{n} \frac{\partial}{\partial z_l}(H_2)_{j,i}(0) \cdot z_l,$$

then

$$dc_{j,i} = -\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial}{\partial z_l}(H_2)_{i,j}(0) \left(\frac{\partial}{\partial z_k} z_l\right) dz_k$$

which implies

$$DC_0^t(0) = \partial C_0^t(0) = -\partial H_2(0)$$

and, similarly,

$$D\overline{C_0}(0) = \overline{\partial}\ \overline{C_0}(0) = -\overline{\partial} H_2(0).$$

It follows that $dH(0) = 0$. Thus, (1) follows.

We now prove (2). As in the proof of Theorem 5.10, we have

$$A = \overline{H}^{-1} \cdot \partial \overline{H}.$$

Thus, by Corollary 5.11, and since

$$H(0) = \mathrm{Id} \qquad \text{and} \qquad \partial H(0) = \overline{\partial} H(0) = 0,$$

we have

$$\begin{aligned}\Theta_E(0) &= \overline{\partial} A(0) \\ &= \overline{\partial}\left(\overline{H}^{-1} \cdot \partial \overline{H}\right)(0) = \\ &= \overline{\partial}\partial\overline{H}(0) \\ &= -\partial\overline{\partial}\ \overline{H}(0).\end{aligned}$$

Thus, (2) follows. □

5.4. **de Rham Cohomology.** Similarly as in Section 4.3, given a complex manifold X, we define, for each $k \geq 0$,

$$\begin{aligned}\mathcal{Z}^k(X) &:= \operatorname{Ker}\left(d\colon C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})\right) \\ &= \{\omega \in C^\infty(X, \Omega^k_{X,\mathbb{C}}) \mid d\omega = 0\}\end{aligned}$$

and, for each $k \geq 1$, we define

$$\begin{aligned}\mathcal{B}^k(X) &:= \operatorname{Im}\left(d\colon C^\infty(X, \Omega^{k-1}_{X,\mathbb{C}}) \to C^\infty(X, \Omega^k_{X,\mathbb{C}})\right) \\ &= \{\omega \in C^\infty(X, \Omega^k_{X,\mathbb{C}}) \mid \omega = d\eta \text{ for some } \eta \in C^\infty(X, \Omega^{k-1}_{X,\mathbb{C}})\}\end{aligned}$$

For convenience, we define $\mathcal{B}^0 = 0$.

Since $d \circ d = 0$, it follows that

$$\mathcal{B}^k(X) \subset \mathcal{Z}^k(X)$$

for each $k \geq 0$.

Thus, we may define

$$H^k(X, \mathbb{C}) := \mathcal{Z}^k(X)/\mathcal{B}^k(X).$$

The group $H^k(X)$ is called the **de Rham cohomology group** of X. If it is finite dimensional, then their dimension

$$b^k(X) := \dim H^k(X, \mathbb{C})$$

is called **Betti number** of X.

Similarly, by considering only real forms, i.e.

$$\begin{aligned}\mathcal{Z}^k_{\mathbb{R}}(X) &:= \operatorname{Ker}\big(d\colon C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}})\big)\\ &= \{\omega \in C^\infty(X, \Omega^k_{X,\mathbb{R}}) \mid d\omega = 0\}\end{aligned}$$

and, for each $k \geq 1$, we define

$$\begin{aligned}\mathcal{B}^k_{\mathbb{R}}(X) &:= \operatorname{Im}\Big(d\colon C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}}) \to C^\infty(X, \Omega^k_{X,\mathbb{R}})\Big)\\ &= \{\omega \in C^\infty(X, \Omega^k_{X,\mathbb{R}}) \mid \omega = d\eta \text{ for some } \eta \in C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}})\}\end{aligned}$$

then we can define

$$H^k(X, \mathbb{R}) := \mathcal{Z}^k_{\mathbb{R}}(X)/\mathcal{B}^k_{\mathbb{R}}(X).$$

Remark 5.13. *If X and X' are diffeomorphic complex manifolds then $H^k(X, \mathbb{C}) \simeq H^k(X', \mathbb{C})$ for any $k \geq 0$. The same result is not true for the Dalbaout cohomology groups.*

Moreover, if X is a complex manifold, then for any $k \geq 0$, we have

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

5.5. **Holomorphic line bundles.** We now consider the case of line bundles.

Let X be a complex manifold and let L be a complex line bundle with a connection ∇. Recall that the curvature Θ_L of L defines a $C^\infty(X)$-linear map.

Proposition 5.14. *Let L be a complex line bundle on a complex manifold X with a connection ∇.*

Then the curvature of ∇ defines a global 2-form $\Theta_\nabla \in C^\infty(X, \Omega^2_{X,\mathbb{C}})$ such that $d\Theta_\nabla = 0$.

Moreover, if ∇' is also a connection on L, then there exists a 1-form η on X such that

$$\Theta_{\nabla'} - \Theta_\nabla = d\eta.$$

Proof. We first study the curvature locally around a point. Let $x \in X$ and let s be a non-vanishing zero section of L defined over an open neighbourhood U of x. As in (4), there exists a 1-form a on U, such that given a section σ of L on U, we may write $\sigma = f \cdot s$ for some $f \in C^\infty(U)$ and

$$\nabla(\sigma) = f \cdot a + df.$$

Thus, as in (8), on U we have

$$\nabla^2(\sigma) = f \cdot \Theta_\nabla$$

where $\Theta_\nabla = a \wedge a + da = da$. In particular Θ_∇ is a closed 1-form on U, i.e. $d\Theta_\nabla = 0$.

Let U' be another open set on which L is trivial and assume that

$$g\colon (U \cap U') \times \mathbb{C} \to (U \cap U') \times \mathbb{C}$$

is the transition function from the trivialisation on U' to the trivialisation on U. By (6), it follows that if a' is the 1-form on U' defining ∇ then

$$a' = g^{-1}dg + a.$$

Since $dg^{-1} = -g^{-2}dg$, it follows easily that $da = da'$. Thus, Θ_∇ is a 2-form defined everywhere on X.

Assume now that ∇' is a connection on L such that on the open set U, it is defined by the 1-form b, i.e. if $\sigma = f \cdot s$ then

$$\nabla'(\sigma) = f \cdot b + df.$$

Let $\eta = b - a$. Then η is a 1-form on U. Let U' and g as above, and let b' be the 1-form defining ∇' on U'. Then we have

$$b' - a' = g^{-1}dg + b - g^{-1}dg - a = b - a.$$

It follows that if on U we define $\eta' = b' - a'$ then η coincides with η' on $U \cap U'$. Thus, η extends to a global 1-form on X. $\square$

Remark 5.15. *The previous result implies that if L is a complex line bundle on a complex manifold X then we can define*

$$c_1(L) := \left[\frac{i}{2\pi}\Theta_\nabla\right] \in H^2(X, \mathbb{C})$$

where ∇ is any connection on L. $c_1(L)$ is called the **first Chern class** *of L.*

Let us assume now that X is a complex manifold and (L, h) is a holomorphic Hermitian line bundle. Let ∇_L be the Chern connection and Θ_L be its curvature form. Let $x \in X$ and let s be a non-vanishing section of L on a neighbourhood U of x. Locally the metric is defined by a smooth function $h(z) = \langle s, s\rangle_z$ for $z \in U$. Since h is Hermitian, we have that $\overline{h} = h$, i.e. $h\colon U \to \mathbb{R}$ is a real valued function on U.

Since h is non-vanishing, we can write $h = e^{-\phi}$ for some smooth function $\phi\colon U \to \mathbb{R}$. The function ϕ is called the **weight of the metric** with respect to the frame defined by s. By (10), if a is the 1-form associated to ∇_L on U with respect to s, we have

$$a = h^{-1}\partial h = e^{\phi} \cdot \partial e^{-\phi} = -\partial\phi.$$

Thus, the curvature is locally given by

$$\Theta_L = da = -\overline{\partial}\partial\phi = \partial\overline{\partial}\phi.$$

In particular, Θ_L is a $(1,1)$-form. Note that, by Lemma 4.17, we have

$$\overline{i\Theta_L} = -i\overline{\Theta}_L = i\partial\overline{\partial}\phi = -i\overline{\partial}\partial\phi = i\Theta_L.$$

Thus, $i\Theta_L$ is invariant under conjugation, i.e. $i\Theta$ is a real $(1,1)$-form, and

$$\frac{i}{2\pi}\Theta_L = \frac{i}{2\pi}\partial\overline{\partial}\phi.$$

Remark 5.16. *If ϕ is the weight of a metric on a line bundle L on X then $-\phi$ is the weight of a metric on the dual L^{-1} of L.*

Definition 5.17. *Let X be a complex manifold and let L be a holomorphic line bundle on X. We say that L is* **positive** *if there exists a Hermtian metric h on L such that if ϕ is a local weight of h, then the matrix*

$$\left(\frac{\partial^2}{\partial z_i \partial \overline{z}_j}\phi\right)$$

is positive definite. If L^{-1} is positive then we say that L is **negative**.

Example 5.18 (Trivial line bundle)**.** *Assume that X is a complex manifold and $L = X \times \mathbb{C}$ is the trivial line bundle. Then we may choose $\phi = 1$ to be the constant function. In particular $\Theta_L = 0$.*

Example 5.19 (Fubini-Study metric)**.** *Let $X = \mathbb{P}^n_{\mathbb{C}}$ and let $\mathcal{O}(-1)$ be the line bundle defined as in Example 4.5. Consider* *the open set* $U_i \subset \mathbb{P}^n_{\mathbb{C}}$ *defined by* $U_i = \{[x_0, \ldots, x_n] \mid x_i \neq 0\}$ *and on* U_i*, let*

$$\phi_i([x_0, \ldots, x_n]) = -\log(\sum_{k=1}^{n} \frac{|x_k|^2}{|x_i|^2}).$$

The transition function

$$g_{i,j} \colon (U_i \cap U_j) \times \mathbb{C} \to (U_i \cap U_j) \times \mathbb{C}$$

is given by

$$g_{i,j} = x_i/x_j.$$

Let $h_i = e^{-\phi_i}$*, then*

$$h_j = g_{i,j} \cdot h_i \cdot \overline{g_{i,j}}.$$

Thus $e^{-\phi_i}$ *defines a metric on* $\mathcal{O}(-1)$*. Let* $\psi_i = -\phi_i$*. Then* $h_i = e^{-\psi_i}$ *defines a metric on* $\mathcal{O}(1)$*, called* **Fubini-Study metric**. *We want to show that such a metric admits positive curvature. Indeed, assume for simplicity that* $i = 0$ *and let* $z_i = x_i/x_0$ *be coordinates on* U_0*. Thus, we may write* $\psi_i = \log(1+\sum |z_i|^2)$*. We have*

$$\frac{\partial^2}{\partial z_k \partial \overline{z}_l}\psi_0 = \frac{\delta_{k,l}(1 + \sum |z_i|^2) - z_l \cdot \overline{z_k}}{(1 + \sum |z_i|^2)^2}.$$

Fix $z \in \mathbb{C}^n$*, and let*

$$T = \left(\frac{\partial^2}{\partial z_k \partial \overline{z}_l}\psi_0(z)\right).$$

We want to show that T *is positive definite. Let* $\langle \cdot, \cdot \rangle$ *be the standard inner product on* $\mathbb{C}^n$ *and let* $\| \cdot \|$ *be the norm induced* *by it. For each $w \in \mathbb{C}^n$, we have*

$$\langle Tw, w\rangle = \frac{\left(1+\|z\|^2\right)\|w\|^2 - |\langle z, w\rangle|^2}{\left(1+\|z\|^2\right)^2}.$$

The Cauchy-Schwarz inequality implies

$$|\langle z, w\rangle|^2 \leq \|z\|^2\|w\|^2.$$

Thus,

$$\langle Tw, w\rangle \geq \frac{\|w\|^2}{\left(1+\|z\|^2\right)^2} \geq 0$$

and the equality holds if and only $w = 0$. Thus T is positive definite and $\mathcal{O}(1)$ is a positive line bundle.

6. Kähler manifolds

6.1. **Introduction.** Let V be a complex vector space of dimension n and let $V_{\mathbb{R}}$ be the underlying real vector space of dimension $2n$. Let $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ and let W be the dual of V. Then a a real $(1,1)$-form on V is an element of $W^{1,1}$ which is invariant under conjugation. If h is a Hermitian metric on V then h defines a real $(1,1)$-form on $V_{\mathbb{C}}$.

Definition 6.1. *Let $V \subset \mathbb{C}^n$ be an open set. A real $(1,1)$-form*

$$\omega = \frac{i}{2}\sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\overline{z}_k$$

on V is said to be **positive** *if the matrix (h_{jk}) is positive definite.*

Note that if ω is a positive definite $(1,1)$ form on a complex manifold X, then it defines a Hermitian metric on the tangent bundle $T_{\mathbb{C}^n,\mathbb{C}}$.

Moreover, as in Section 2.2, if we denote $z_j = x_j + iy_j$, where x_j and y_j are real coordinates, then

$$\frac{i}{2}dz_j \wedge d\overline{z}_j = dx_j \wedge dy_j.$$

Thus, if ω is a positive $(1,1)$ form, then

$$\omega^n = \det(h_{i,j})dx_1 \wedge dy_1 \wedge \dots dx_n \wedge dy_n$$

is a non-vanishing (n,n)-form. Thus, ω^n is a volume form on V.

More in general, we define:

Definition 6.2. *Let X be a complex manifold of dimension n and let $\omega \in C^\infty(X, \Omega_X^{1,1})$ be a real $(1,1)$ form. Then ω is said to be* **positive** *if for any $x \in X$ there exists a holomorphic chart (U, ϕ) with $\phi\colon U \to V \subset \mathbb{C}^n$ such that $(\phi^{-1})^*\omega$ is a positive $(1,1)$-form on V.*

In particular, a positive $(1,1)$ form ω on a complex manifold X of dimension n defines an Hermitian metric on $T_{X,\mathbb{C}}$ and is such that ω^n is a non-vanishing volume form on X.

Definition 6.3. *Let X be a complex manifold. Let ω be a positive $(1,1)$-form which is compatible with an Hermitian metric on T_X. Then ω is* **Kähler** *if $d\omega = 0$.*

We say that X is **Kähler** *if it admits a a Kähler $(1,1)$-form ω.*

Example 6.4. *Let $X = \mathbb{C}^n$ and let $\omega = \frac{i}{2}\sum dz_j \wedge d\overline{z}_j$. Then ω is a positive real $(1,1)$-form such that $d\omega = 0$. Thus, $\mathbb{C}^n$ is a Kähler manifold.*

Note that a Kälher manifold admits more than one Kälher form. Thus, we usually consider pairs (X, ω) where ω is a Kähler form on the complex manifold X.

Example 6.5. *Let $\Lambda \subset \mathbb{C}^n$ be a lattice and let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Let $q\colon \mathbb{C}^n \to X$ be the quotient map. Then the $(1,1)$-form ω in the previous example is invariant with respect to any translation, i.e. for any $\lambda \in \Lambda$ if we define*

$$\phi\colon \mathbb{C}^n \to \mathbb{C}^n \qquad x \mapsto x + \lambda,$$

then $\phi^*\omega = \omega$. Thus, it descends to a positive real $(1,1)$ form ω' which is closed, i.e. $q^*\omega' = \omega$. Thus, complex tori are Kähler manifolds.

Example 6.6. *Let $X = \mathbb{P}^n_{\mathbb{C}}$. Then the Fubini-Study metric on $\mathcal{O}(1)$ (cf. Example 5.19) defines a Kähler metric on X. Thus X is Kähler.*

Example 6.7. *Let X be a complex manifold of dimension 1 then X is Kähler.*

Example 6.8. *Let X be a Kähler manifold and let $Y \subset X$ be a complex submanifold. Then Y is also Kähler. In particular, complex projective manifolds are Kähler.*

Let (X, ω) be a compact Kälher manifold of dimension n. Then, by the Leibnitz rule, it follows that for each $k \geq 0$, also the (k, k) form ω^k is closed, i.e. $d\omega^k = 0$. Thus, $[\omega^k] \in H^{2k}(X, \mathbb{C})$. We will show that

$$[\omega^k] \neq 0 \qquad \text{for } k = 1, \dots, n.$$

Indeed, assume that $\omega^k = d\eta$ for some $\eta \in H^{2k-1}(X, \mathbb{C})$. Then by Stoke's theorem we have

$$0 < \int_X \omega^n = \int_X d\left(\eta \wedge \omega^{n-k}\right) = \int_{\partial X} \eta \wedge \omega^{n-k} = 0$$

which gives a contradiction. Thus, the claim follows.

Since $d\omega^k = 0$, we also have that $\overline{\partial}\omega^k = 0$ and therefore $[\omega^k] \in H^{k,k}(X)$.

Example 6.9 (Hopf Manifolds)**.** *Pick* $\lambda \in \mathbb{C}$ *such that* $0 < |\lambda| < 1$*. Consider the group action*

$$\mathbb{Z} \times (\mathbb{C}^n \backslash \{0\}) \to \mathbb{C}^n \backslash \{0\} \qquad (m, z) \mapsto \lambda^m z.$$

Consider the quotient

$$q \colon \mathbb{C}^n \backslash \{0\} \to (\mathbb{C}^n \backslash \{0\}) / \mathbb{Z} =: X.$$

Then X *is a complex manifold, called* **Hopf manifold***. Moreover, since* $\mathbb{C}^n \backslash \{0\}$ *is diffeomorphic to* $S^{2n-1} \times \mathbb{R}_{>0}$*, it follows that* X *is diffeomorphic to* $S^{2n-1} \times S^1$*.*

In particular, if $n \geq 2$*, then* $H^2(X, \mathbb{C}) = 0$*. Thus,* X *is not Kähler.*

6.2. **Hodge $\star$ operator.** Let W be a real vector space of dimension m with an inner product

$$\langle \cdot, \cdot \rangle \colon W \times W \to \mathbb{R}.$$

Then $\langle \cdot, \cdot \rangle$ induces an inner product on $\wedge^k W$ for each $k \geq 0$, given by

$$\langle v_1 \wedge \cdots \wedge v_k, v_1' \wedge \cdots \wedge v_k' \rangle = \det\left(\langle v_i, v_j' \rangle\right).$$

In particular, up to orientation, there is a unique $\omega \in \wedge^m W$ such that $\langle \omega, \omega \rangle = 1$. Then, for each $k \geq 0$, there exists a linear map

$$\star\colon \wedge^k W \to \wedge^{m-k} W$$

such that, if $\alpha, \beta \in \wedge^k W$ then

$$\alpha \wedge (\star \beta) = \langle \alpha, \beta \rangle \omega.$$

If $e_1, \dots, e_m$ is a orthonormal basis of W then the following easy properties hold:

(1) $\star 1 = \omega$,
(2) $\star e_1 = e_2 \wedge \cdots \wedge e_m$,
(3) $\star \omega = 1$,
(4) $\star e_i = (-1)^{i-1} e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_m$.

More in general if $I \subset \{1, \dots, m\}$ and I^c is the complement of I then

$$\star e_I = \epsilon(\sigma) e_{I^c}$$

where $\epsilon(\sigma)$ is the signature of the permutation

$$\sigma\colon (1, \dots, m) \mapsto (I, I^c)$$

(cf. Definition 2.20). In particular, it follows that if $\omega \in \Lambda^k W$ then

$$\star \star \omega = (-1)^{k(m-k)} \omega. \tag{11}$$

Assume now that V is a complex vector space of dimension n with an Hermitian metric $\langle \cdot, \cdot \rangle$. Then, for each $k \geq 0$, we can extend the Hodge $\star$ operator to $V_{\mathbb{C}}$ to a $\mathbb{C}$-linear map

$$\star\colon \wedge^k V_{\mathbb{C}} \to \wedge^{2n-k} V_{\mathbb{C}}$$

so that if $\alpha, \beta \in \wedge^k V_{\mathbb{C}}$ then

$$\alpha \wedge \overline{\star\beta} = \langle \alpha, \beta \rangle \omega. \tag{12}$$

Note that, in particular, $\overline{\star\beta} = \star\overline{\beta}$.

Let X be a complex manifold of dimension n and let (E, h) be a Hermitian vector bundle on X. Recall that in Section 5.2, for each $p, q \geq 0$, we defined the bilinear map

$$\{\ ,\ \}\colon C^\infty(X, E \otimes \Omega^p_{X,\mathbb{C}}) \times C^\infty(X, E \otimes \Omega^q_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{p+q}_{X,\mathbb{C}}).$$

We now consider the case $p = q = 0$. Let $E = \Omega^k_{X,\mathbb{C}}$ for some $k \geq 0$. Then the Kähler metric induces a metric on $\Omega^k_{X,\mathbb{C}}$. Indeed, ω induces a metric on T_X. Locally around $x \in X$, let $v_1, \dots, v_n$ be a orthonormal frame and let $v_1^*, \dots, v_n^*$ be the corresponding dual frame. Then, we may define a metric on $\Omega^1_{X,\mathbb{C}}$ around x such that $v_1^*, \dots, v_n^*$ is an orthonormal frame. It follows easily that such a choice is canonic and therefore it extends globally to X.

Thus, the cup product becomes

$$\{\ ,\ \}\colon C^\infty(X, \Omega^k_{X,\mathbb{C}}) \times C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X).$$

Lemma 6.10. *If (X, ω) is a Kähler manifold of dimension n then, for each $k \geq 0$, there exists a linear map*

$$\star\colon C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{2n-k}_{X,\mathbb{C}})$$

such that for all $\alpha, \beta \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$, we have

$$\alpha \wedge \overline{\star\beta} = \{\alpha, \beta\}\omega^n.$$

Let E be a Hermitian vector bundle on a complex Kähler manifold X. A section s of E is said to have **compact support** if there exists a compact subset K of X such that s is zero outside K. Thus, we may define

$$C_c^\infty(X,E) := \{s \in C^\infty(X,E) \mid s \text{ has compact support } \}.$$

Clearly, if X is compact, then $C_c^\infty(X,E) = C^\infty(X,E)$.

We may define a **L^2-norm** on $C_c^\infty(X,E)$ by considering the product

$$(\alpha,\beta) := \int_X \{\alpha,\beta\}\omega^n. \tag{13}$$

Let E and F be Hermitian vector bundles on a compact complex Kähler manifold (X,ω).If $P\colon C_c^\infty(X,E) \to C_c^\infty(X,F)$ is a $\mathbb{C}$-linear map, then the **adjoint map** $P^*\colon C_c^\infty(X,F) \to C_c^\infty(X,E)$ is a linear map such that

$$(P\alpha,\beta) = (\alpha, P^*\beta)$$

for any $\alpha \in C_c^\infty(X,E)$ and $\beta \in C_c^\infty(X,F)$.

In particular, if $d\colon C^\infty(X,\Omega^k_{X,\mathbb{C}}) \to C^\infty(X,\Omega^{k+1}_{X,\mathbb{C}})$ is the differential, then we define

$$d^*\colon C_c^\infty(X,\Omega^{k+1}_{X,\mathbb{C}}) \to C_c^\infty(X,\Omega^k_{X,\mathbb{C}})$$

to be its adjoint.

Lemma 6.11. *Let (X,ω) be a Kähler manifold of dimension n and let $\beta \in C_c^\infty(X,\Omega^{k+1}_{X,\mathbb{C}})$ for some $k \geq 0$.*

Then

$$d^*\beta = -\star d \star \beta.$$

Proof. Let $\alpha \in C_c^\infty(X,\Omega^k_{X,\mathbb{C}})$. Then, by Lemma 6.10 and by the Leibnitz rule, we have

$$\{d\alpha,\beta\}\omega^n = da \wedge \overline{\star\beta} = d(\alpha\wedge\overline{\star\beta}) - (-1)^k\alpha\wedge d\overline{\star\beta}.$$

Thus, by Stokes' theorem and by (11), we get

$$\begin{aligned}(d\alpha,\beta) &= \int_X \{d\alpha,\beta\}\omega^n \\ &= (-1)^{k+1}\int_X \alpha\wedge d\overline{\star\beta} \\ &= (-1)^{k(2n-k)+k+1}\int_X \alpha\wedge\star\star\, d\overline{\star\beta} \\ &= -\int_X \alpha\wedge\star\star\, d\overline{\star\beta} \\ &= -\int_X \{\alpha,\star\, d\overline{\star\beta}\}\omega^n = \\ &= -\int_X \{\alpha,\overline{\star\, d\,\star\beta}\}\omega^n = (\alpha,-(\star\, d\star\beta)).\end{aligned}$$

Thus, the claim follows. $\square$

6.3. **Harmonic forms.**

Definition 6.12. *Let X be a complex manifold. Then we define the* **Hodge-de Rham operator** *of X as*

$$\Delta = dd^* + d^*d.$$

Notice that, for all $k\geq 0$, we have a $\mathbb{C}$-linear map

$$\Delta\colon C_c^\infty(X,\Omega^k_{X,\mathbb{C}})\to C_c^\infty(X,\Omega^k_{X,\mathbb{C}}).$$

We say that $\alpha\in C^\infty(X,\Omega^k_{X,\mathbb{C}})$ is **harmonic** *if $\Delta\alpha=0$.*

We denote by $\mathcal{H}^k(X)$ the space of harmonic k-forms, i.e.

$$\mathcal{H}^k(X) = \{\alpha\in C_c^\infty(X,\Omega^k_{X,\mathbb{C}}) \mid \Delta\alpha = 0\}.$$

Lemma 6.13. *Let (X,ω) be a Kähler manifold and let $\alpha \in C_c^\infty(X,\Omega^k_{X,\mathbb{C}})$ for some $k \geq 0$.*

Then α is harmonic if and only if $d\alpha = d^*\alpha = 0$.

Proof. Clearly if $d\alpha = d^*\alpha = 0$, then α is harmonic.

Assume now that $\Delta\alpha = 0$. Then

$$\begin{aligned} 0 = (\Delta\alpha,\alpha) &= (d^*d\alpha,\alpha) + (dd^*\alpha,\alpha) \\ &= (d\alpha,d\alpha) + (d^*\alpha,d^*\alpha). \end{aligned}$$

Thus, $d\alpha = d^*\alpha = 0$. □

Example 6.14. *Let $X = \mathbb{C}^n$ and let $\omega = \frac{i}{2}\sum dz_j \wedge d\overline{z}_j$. As in Example 6.4, (X,ω) is Kähler. Let $f \in C_c^\infty(X)$. Then, we have $d^*f = 0$. Thus,*

$$\Delta f = d^*df.$$

Write $z_j = x_j + iy_j$ for $j = 1,\ldots,n$. Then it is easy to check that

$$\Delta f = \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) f,$$

i.e. Δ is the usual Laplacian.

Lemma 6.15. *Let (X,ω) be a Kähler manifold. Then Δ and $\star$ commute, i.e.*

$$\Delta \star \alpha = \star\Delta\alpha$$

for all $\alpha \in C_c^\infty(X,\Omega^k_{X,\mathbb{C}})$.

Proof. It follows directly from Lemma 6.11 and (11). □

Lemma 6.16. *Let (X,ω) be a Kähler manifold. Then Δ is auto-adjoint, i.e.*

$$(\Delta\alpha,\beta)=(\alpha,\Delta\beta)$$

for all $\alpha,\beta\in C^\infty_c(X,\Omega^k_{X,\mathbb{C}})$.

Proof. The proof follows immediately from the definition of d^*. □

The following result follows from the theory of elliptic operators. We will omit the proof.

Theorem 6.17. *Let (X,ω) be a compact Kähler manifold.* [1] *Then, for each $k\geq 0$,*

(1) $\mathcal{H}^k(X)$ *is finite dimensional.*
(2) *We have the following orthogonal decompositions:*

$$\begin{aligned}C^\infty(X,\Omega^k_{X,\mathbb{C}})&=\mathcal{H}^k(X)\oplus\Delta(C^\infty(X,\Omega^k_{X,\mathbb{C}}))\\&=\mathcal{H}^k(X)\oplus d(C^\infty(X,\Omega^{k-1}_{X,\mathbb{C}}))\oplus d^*C^\infty(X,\Omega^{k+1}_{X,\mathbb{C}});\\\operatorname{Ker}d&=\mathcal{H}^k(X)\oplus d(C^\infty(X,\Omega^{k-1}_{X,\mathbb{C}})),\ \textit{and}\\\operatorname{Ker}d^*&=\mathcal{H}^k(X)\oplus d^*C^\infty(X,\Omega^{k+1}_{X,\mathbb{C}}).\end{aligned}$$

Note that by Lemma 6.16, it follows that $\mathcal{H}^k(X)$ is orthogonal to $\Delta(C^\infty(X,\Omega^k_{X,\mathbb{C}}))$. Moreover, if $\alpha\in C^\infty(X,\Omega^{k-1}_{X,\mathbb{C}})$ and $\beta\in d^*C^\infty(X,\Omega^{k+1}_{X,\mathbb{C}})$ then

$$(d\alpha,d^*\beta)=(d^2\alpha,\beta)=0.$$

Finally, by Lemma 6.13 harmonic forms are closed and, in particular, $\mathcal{H}^k(X)\subset\operatorname{Ker}d^*$.

The previous theorem immediately implies the following:

[1] Note that the result hold more in general for a compact closed oriented Riemannian manifold.

Corollary 6.18. *Let (X, ω) be a compact Kähler manifold. Then, for each $k \geq 0$, we have an isomorphism*

$$\mathcal{H}^k(X) \simeq H^k(X, \mathbb{C}).$$

As a consequence of the Corollary and Lemma 6.15, we obtain

Theorem 6.19 (Poincaré duality)**.** *Let (X, ω) be a compact Kähler manifold of dimension n.*

Then, for each $k \geq 0$, the Hodge $\star$ operator induces an isomorphism

$$H^k(X, \mathbb{C}) \to H^{2n-k}(X, \mathbb{C}).$$

6.4. **Harmonic (p, q)-forms.** We now want to study the Hodge $\star$ operator on (p, q)-forms.

Let X be a complex manifold of dimension n and let $x_0 \in M$. Proceeding as in the proof of Proposition 5.12, i.e. performing a suitable linear change of coordinates, we may choose local coordinates $z_1, \ldots, z_n$ around x_0 such that $dz_1, \ldots, dz_n$ is a local frame of $\Omega^{1,0}_{X,\mathbb{C}}$ such that $dz_1, \ldots, dz_n$ is orthonormal at the point x_0. Note that we cannot guarantee the existence of a local holomorphic frame such that $dz_1, \ldots, dz_n$ is orthonormal at each point (see Theorem 6.31 below in the case of Kähler manifolds).

Thus, for any $p, q \geq 0$, if

$$\eta^j = \sum_{|I|=p, |J|=q} \eta^j_{I,J} dz_I \wedge d\bar{z}_J \qquad \text{j=1,2}$$

are two (p, q)-forms on X around x_0, then at the point x_0, we have

$$\{\eta^1, \eta^2\}_{x_0} = \sum_{|I|=p, |J|=q} \eta^1_{I,J}(x_0) \cdot \overline{\eta^2_{I,J}(x_0)}.$$

Since, by (12), we have

$$\eta^1 \wedge \overline{\star\eta^2} = \{\eta^1, \eta^2\}$$

it follows that the Hodge $\star$ operator induces a $\mathbb{C}$-linear isometry

$$\star\colon \Omega_X^{p,q} \to \Omega_X^{n-q,n-p}.$$

In particular, if η is a (p,q)-form and η' is a (p',q')-form so that $p+q=p'+q'=n$, then $\eta \wedge \overline{\star\eta'}$ is of type $(n-p'+p, n-q'+q)$. Thus, it is zero unless $p=p'$ and $q=q'$.

It follows that, for each $k \geq 0$, the decomposition

$$C^\infty(X, \Omega^k_{X,\mathbb{C}}) = \bigoplus_{p+q=k} C^\infty(X, \Omega_X^{p,q})$$

is orthogonal with respect to the L^2-norm (13).

Let

$$\partial^* : C_c^\infty(X, \Omega_X^{p+1,q}) \to C_c^\infty(X, \Omega_X^{p,q})$$

be the adjoint operator of ∂ and, similarly, let

$$\bar\partial^* : C_c^\infty(X, \Omega_X^{p,q}) \to C_c^\infty(X, \Omega_X^{p,q-1})$$

be the adjoint operator of $\bar\partial$. Then, similarly to Lemma 6.11, we have

$$\partial^* = -\star\bar\partial\star \qquad \text{and} \qquad \bar\partial^* = -\star\partial\star. \tag{14}$$

Thus, we can define the corresponding **Laplacian operators** by

$$\Delta_\partial := \partial\partial^* + \partial^*\partial \qquad \text{and} \qquad \Delta_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

In particular, for any form α, we say that α is Δ_∂-**harmonic** (resp. $\Delta_{\bar\partial}$-**harmonic**) if $\Delta_\partial\,\alpha = 0$ (resp $\Delta_{\bar\partial}\,\alpha = 0$).

Similarly to Lemma 6.13, we have

Lemma 6.20. *Let (X, ω) be a Kähler manifold and let $\alpha \in C_c^\infty(X, \Omega_X^{p,q})$ for some $p, q \geq 0$.*
Then
(1) *α is Δ_∂-harmonic if and only if $\partial\alpha = \partial^*\alpha = 0$.*
(2) *α is $\Delta_{\bar\partial}$-harmonic if and only if $\bar\partial\alpha = \bar\partial^*\alpha = 0$.*

We denote by $\mathcal{H}^{p,q}(X)$ the space of $\Delta_{\bar\partial}$-harmonic (p,q)-forms, i.e.

$$\mathcal{H}^{p,q}(X) = \{\alpha \in C_c^\infty(X, \Omega_X^{p,q}) \mid \Delta_{\bar\partial}\,\alpha = 0\}.$$

Similarly to Theorem 6.17, we have

Theorem 6.21. *Let (X, ω) be a compact Kähler manifold. Then, for each $p, q \geq 0$,*
(1) *$\mathcal{H}^{p,q}(X)$ is finite dimensional.*
(2) *We have the following orthogonal decompositions:*

$$\begin{aligned} C^\infty(X, \Omega_X^{p,q}) &= \mathcal{H}^{p,q}(X) \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,q})) \\ &= \mathcal{H}^{p,q}(X) \oplus \overline{\partial} C^\infty(X, \Omega_X^{p,q-1}) \oplus \overline{\partial}^* C^\infty(X, \Omega_X^{p,q+1}). \\ \operatorname{Ker} \overline{\partial} &= \mathcal{H}^{p,q}(X) \oplus \overline{\partial}(C^\infty(X, \Omega_X^{p,q-1})), \ \textit{and} \\ \operatorname{Ker} \overline{\partial}^* &= \mathcal{H}^{p,q}(X) \oplus \overline{\partial}^* C^\infty(X, \Omega_X^{p,q+1}). \end{aligned}$$

Thus, similarly to Corollary 6.18, we get

Corollary 6.22. *Let (X, ω) be a compact Kähler manifold. Then, for each $p, q \geq 0$, we have an isomorphism*

$$\mathcal{H}^{p,q}(X) \simeq H^{p,q}(X).$$

Recall that $H^{p,q}(X)$ denotes the Dolbeaut cohomology group of X (cf. Section 4.2).

6.5. **Lefschetz Operator.**

Definition 6.23. *Let (X, ω) be a Kähler manifold. We define the* **Lefschetz operator** *as the $\mathbb{C}$-linear map*

$$L\colon C_c^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C_c^\infty(X, \Omega^{k+2}_{X,\mathbb{C}})$$

given by

$$\eta \mapsto \eta \wedge \omega.$$

We denote by

$$\Lambda\colon C_c^\infty\left(X, \Omega^{k+2}_{X,\mathbb{C}}\right) \to C_c^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right)$$

its adjoint operator.

Lemma 6.24. *Let (X, ω) be a Kähler manifold.*

Then, for any $k \geq 0$ and for any $\beta \in C_c^\infty\left(X, \Omega^{k+2}_{X,\mathbb{C}}\right)$, we have

$$\Lambda\ \beta = (-1)^k \star L \star \beta.$$

Proof. Since ω is a real $(1,1)$-form, it is enough to prove the equality for a real form $\beta \in C_c^\infty(X, \Omega^{k+2}_{X,\mathbb{R}})$.

Let $\alpha \in C_c^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right)$. Then

$$\int_X \{L\alpha, \beta\}\omega^n = \int_X L\alpha \wedge \star\beta = \int_X \omega \wedge \alpha \wedge \star\beta.$$

Since $\star\star = (-1)^k$, it follows that

$$\int_X \omega\wedge\alpha\wedge\star\beta = \int_X \alpha\wedge\star(-1)^k\star L\star\beta = \int_X \left\{\alpha, (-1)^k \star L \star \beta\right\}\omega^n.$$

Thus,

$$(L\alpha, \beta) = (\alpha, (-1)^k \star L \star \beta)$$

and the claim follows. $\square$

Definition 6.25. *Let $p\colon E \to X$ and $q\colon F \to X$ be complex vector bundles of rank r and r' respectively on a complex manifold X and let*

$$P\colon C_c^\infty(X,E) \to C_c^\infty(X,F)$$

be a $\mathbb{C}$-linear map.

Then P is a **differential operator of order** *d if, for any $x \in M$, there exist an open neighbourhood U of x with coordinates $x_1,\dots,x_n$ and with frames*

$$s_1,\dots,s_r \qquad \text{and} \qquad t_1,\dots,t_{r'}$$

for E and F on U respectively such that for any smooth functions $f_1,\dots,f_r \in C^\infty(U)$, we have

$$P\left(\sum_{i=1}^r f_i s_i\right) = \sum_{I,i,j} P_{I,i,j}\cdot \frac{\partial f_j}{\partial x_I}\cdot t_i$$

where $P_{I,i,j} \in C^\infty(U)$ and $P_{I,i,j} = 0$ if $|I| > d$ and at least one $P_{I,i,j}$ is not zero for some $|I| = d$.

It is easy to check that the condition does not depend on the choice of the coordinates and the trivialisations.

Notation 6.26. *Let A, B be differential operators of degree a and b on a complex manifold M. Then the* **Lie bracket** *is a differential operator of degree $a+b$ defined by*

$$[A,B] := AB - (-1)^{ab}BA.$$

Definition 6.27. *Let X be a complex manifold and let v be a vector field, i.e. a section of $C^\infty(X, T_{X,\mathbb{C}})$. For any k-form ω*

on X, the **contraction** *of ω with v is the $(k-1)$-form on X defined by*

$$v \lrcorner\, \omega(v_1, \ldots, v_{k-1}) := \omega(v, v_1, \ldots, v_{k-1})$$

for each vector field $v_1, \ldots, v_{k-1}$.

Example 6.28. *Let $U \subset \mathbb{C}^n$ be an open set. Then*

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$$

is a frame for T_U and $dz_1, \ldots, dz_n$ is the dual frame for Ω^1_U. For each index $I = (i_1, \ldots, i_k)$ and for each $m \in \{1, \ldots, n\}$, we have

$$\frac{\partial}{\partial z_m} \lrcorner\, dz_I = \begin{cases} 0 & \text{if } m \notin \{i_1, \ldots, i_k\} \\ (-1)^{l-1} dz_{i_1} \wedge \ldots \widehat{dz_{i_l}} \ldots \wedge dz_{i_k} & \text{if } m = i_l. \end{cases}$$

It easily follows that if $v \in C^\infty(U, T_U)$, $\alpha \in C^\infty(U, \Omega^p_{U,\mathbb{C}})$ and $\beta \in C^\infty(U, \Omega^q_{U,\mathbb{C}})$, then

$$v \lrcorner\, (\alpha \wedge \beta) = (v \lrcorner\, \alpha) \wedge \beta + (-1)^p \alpha \wedge (v \lrcorner\, \beta).$$

6.6. **Kähler identities.**

Example 6.29. *Let $U \subset \mathbb{C}^n$ be an open set and let*

$$\omega = \frac{i}{2} \sum dz_j \wedge d\overline{z}_j$$

(cf. Example 6.4). Then

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$$

is a holomorphic orthonormal basis of T_U. Let $\alpha \in C_c^\infty(U, \Omega_X^{p,q})$. We may write

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J} dz_I \wedge d\overline{z}_J.$$

Then

$$\begin{aligned}\overline{\partial}^* \alpha &= -\sum_{k=1}^n \sum_{|I|=p,|J|=q} \frac{\partial \alpha_{I,J}}{\partial z_k} \frac{\partial}{\partial \overline{z}_k} \lrcorner\, dz_I \wedge d\overline{z}_J \\ &= -\sum_{k=1}^n \frac{\partial}{\partial \overline{z}_k} \lrcorner \frac{\partial}{\partial z_k} \alpha,\end{aligned}$$

where we denote

$$\frac{\partial}{\partial z_k}\alpha = \sum_{|I|=p,|J|=q} \frac{\partial}{\partial z_k}\alpha_{I,J} dz_I \wedge d\overline{z}_J.$$

Lemma 6.30. *Let $U \subset \mathbb{C}^n$ be an open set and let*

$$\omega = i \sum dz_j \wedge d\overline{z}_j.$$

Then

$$[\bar{\partial}^*, L] = i\partial.$$

Proof. Let $\alpha \in C^\infty(U, \Omega^k_{X,\mathbb{C}})$ be a k-form on U. By linearity, we may assume that

$$\alpha = \alpha_{I,J} dz_I \wedge d\overline{z}_J$$

for some indices I, J such that $|I| = p$, $|J| = q$ and $\alpha_{I,J} \in C^\infty(U)$. As in Example 6.29, we have

$$\overline{\partial}^* \alpha = -\sum_{k=1}^n \frac{\partial}{\partial \overline{z}_k} \lrcorner \frac{\partial}{\partial z_k} \alpha.$$

Thus, we have

$$[\bar{\partial}^*, L]\alpha = -\sum_{k=1}^{n} \frac{\partial}{\partial \overline{z}_k} \lrcorner \frac{\partial}{\partial z_k}(\omega \wedge \alpha) - \omega \wedge \left(-\sum_{k=1}^{n} \frac{\partial}{\partial \overline{z}_k} \lrcorner \frac{\partial}{\partial z_k} \alpha \right)$$

By the definition of ω it follows that

$$\frac{\partial}{\partial z_k}(\omega \wedge \alpha) = \omega \wedge \frac{\partial}{\partial z_k} \alpha$$

By the definition of ω, we also have

$$\frac{\partial}{\partial \overline{z}_k} \lrcorner \omega = -i dz_k.$$

Thus, as in Example 6.28, since ω is a 2-form, we have

$$\begin{aligned} \frac{\partial}{\partial \overline{z}_k} \lrcorner \left(\omega \wedge \frac{\partial}{\partial z_k} \alpha \right) &= \left(\frac{\partial}{\partial \overline{z}_k} \lrcorner \omega \right) \wedge \frac{\partial}{\partial z_k} \alpha + \omega \wedge \left(\frac{\partial}{\partial \overline{z}_k} \lrcorner \frac{\partial}{\partial z_k} \alpha \right) \\ &= -i dz_k \wedge \frac{\partial}{\partial z_k} \alpha + \omega \wedge \left(\frac{\partial}{\partial \overline{z}_k} \lrcorner \frac{\partial}{\partial z_k} \alpha \right). \end{aligned}$$

Thus,

$$[\bar{\partial}^*, L]\alpha = \sum_{k=1}^{n} i dz_k \wedge \frac{\partial}{\partial z_k} \alpha = i \partial \alpha.$$

Thus, the claim follows. □

Our goal is to extend the result of the previous Lemma to any Kähler manifold. To this end, we need to show that any Kähler form behaves locally like the standard Kähler metric on $\mathbb{C}^n$, at least up to order one:

Theorem 6.31. *Let (X, ω) be a Kähler manifold and let $x \in X$.*

Then, locally around x we can find holomorphic coordinates such that if

$$\omega = \frac{i}{2}\sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\overline{z_k}$$

then

$$h_{j,k} = \delta_{j,k} + \mathcal{O}(|z|^2). \tag{15}$$

The coordinates $z_1, \ldots, z_n$ are called **normal coordinates** for the Kähler form ω. Note that if ω is a positive real $(1,1)$-form satisfying (15), then $d\omega = 0$ and therefore ω is Kähler.

Proof. As in Section 6.4, we can find holomorphic local coordinates $z_1, \ldots, z_n$ around X such that $dz_1, \ldots, dz_n$ is a frame of Ω^1_X which is orthonormal at x. Thus, we may write

$$\omega = \frac{i}{2}\sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\overline{z_k}$$

where $h_{j,k} = \delta_{j,k} + \mathcal{O}(|z|)$. Thus, there exist $a_{jkl}, a'_{jkl} \in \mathbb{C}$ such that

$$h_{j,k} = \delta_{j,k} + \sum_{l=1}^{n} \left(a_{jkl} z_l + a'_{jkl}\overline{z_l}\right) + \mathcal{O}\left(|z|^2\right). \tag{16}$$

Since $(h_{j,k})$ is Hermitian, we have

$$\overline{a_{kjl}} = a'_{jkl} \qquad \text{for any } j,k,l. \tag{17}$$

Since ω is closed, we have

$$a_{jkl} = a_{lkj} \qquad \text{for any } j,k,l. \tag{18}$$

Define

$$\xi_k = z_k + \frac{1}{2}\sum_{j,l=1}^{n} a_{jkl} z_j z_l \qquad k = 1, \dots, n.$$

Then, by the Inverse Function Theorem (cf. Corollary 2.16), $\xi_1, \dots, \xi_n$ are local holomorphic coordinates and, by (18), we have

$$\begin{aligned} d\xi_k &= dz_k + \frac{1}{2}\sum_{j,l=1}^{n} a_{jkl}\left(z_j dz_l + z_l dz_j\right) \\ &= dz_k + \frac{1}{2}\sum_{j,l=1}^{n} \left(a_{jkl} + a_{lkj}\right) z_l dz_j \\ &= dz_k + \sum_{j,l=1}^{n} a_{jkl} z_l dz_j. \end{aligned}$$

Thus,

$$i\sum_{k=1}^{n} d\xi_k \wedge d\overline{\xi_k} = i\sum_{k=1}^{n} dz_k \wedge d\overline{z_k} + i\sum_{j,k,l=1}^{n} \left(\overline{a_{jkl}\overline{z_l}} dz_k \wedge d\overline{z_j} + a_{jkl} z_l dz_j \wedge d\overline{z_k}\right) + \mathcal{O}\left(|z|^2\right)$$

By (17), we have

$$\sum_{j,k,l=1}^{n} \overline{a_{jkl}\overline{z_l}} dz_k \wedge d\overline{z_j} = \sum_{j,k,l=1}^{n} \overline{a_{kjl}\overline{z_l}} dz_j \wedge d\overline{z_k} = \sum_{j,k,l=1}^{n} a'_{jkl}\overline{z_l} dz_j \wedge d\overline{z_k}.$$

Thus,

$$i\sum_{k=1}^{n} d\xi_k \wedge \overline{d\xi_k} = i\sum_{j,k=1}^{n} \left(\delta_{j,k} + \sum_{l=1}^{n} a_{jkl} z_l + a'_{jkl}\overline{z_l}\right) dz_j \wedge d\overline{z_k} + \mathcal{O}\left(|z|^2\right).$$

By (16), it follows that (15) holds. Thus, the Theorem follows. □

Theorem 6.32 (Kälher identities)**.** *Let (X, ω) be a Kähler manifold.*

Then

(1) $[\overline{\partial}^*, L] = i\partial$

(2) $[\partial^*, L] = -i\overline{\partial}$

(3) $[\Lambda, \overline{\partial}] = -i\partial^*$

(4) $[\Lambda, \partial] = i\overline{\partial}^*$.

Proof of Theorem 6.32. We first prove (1). By (14), we have

$$\bar{\partial}^* = - \star \partial \star .$$

It follows that, locally at each point $x \in X$, the operator $[\overline{\partial}^*, L]$ depends on ω only up to order one. But, locally around $x \in X$, we can consider normal coordinates $z_1, \dots, z_n$, as in Theorem 6.31. Thus, (1) follows from Lemma 6.30.

We now prove (3). Let α and β be (p, q)-forms, then by (1) we have

$$([\Lambda, \overline{\partial}]\alpha, \beta) = (\alpha, [\overline{\partial}^*, L]\beta) = (\alpha, i\partial\beta) = (-i\alpha, \partial\beta) = (-i\partial^*\alpha, \beta).$$

Thus, (3) follows.

Since $\overline{L} = L$, we have that (1) implies (2) and, since $\overline{\Lambda} = \Lambda$, we have that (3) implies (4). □

Theorem 6.33. *Let (X, ω) be a Kähler manifold.*

Then

$$\Delta = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.$$

Proof. Since $d = \partial + \overline{\partial}$, we have

$$\Delta = (\partial + \bar{\partial})\left(\partial^* + \bar{\partial}^*\right) + \left(\partial^* + \bar{\partial}^*\right)(\partial + \bar{\partial}).$$

By (4) of Theorem 6.32, we have

$$\overline{\partial}^* = -i[\Lambda, \partial] = -i\Lambda\partial + i\partial\Lambda.$$

Thus, since $\partial^2 = 0$, we have

$$(\partial + \bar{\partial})\left(\partial^* + \bar{\partial}^*\right) = \partial\partial^* + \bar{\partial}\partial^* - i\partial\Lambda\partial - i\bar{\partial}\Lambda\partial + i\bar{\partial}\partial\Lambda.$$

Similarly, we have

$$\left(\partial^* + \bar{\partial}^*\right)(\partial + \bar{\partial}) = \partial^*\partial + \partial^*\bar{\partial} + i\partial\Lambda\partial + i\partial\Lambda\bar{\partial} - i\Lambda\partial\bar{\partial}.$$

By (3) of Theorem 6.32, we have

$$\partial^*\overline{\partial} = i[\Lambda, \overline{\partial}]\,\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial} = -i\overline{\partial}[\Lambda, \overline{\partial}] = -\overline{\partial}\partial^*.$$

Thus, using (3) of Theorem 6.32 again, we obtain

$$\begin{aligned}
\Delta &= \partial\partial^* - i\bar{\partial}\Lambda\partial + i\bar{\partial}\partial\Lambda + \partial^*\partial + i\partial\Lambda\bar{\partial} - i\Lambda\partial\bar{\partial} \\
&= \Delta_\partial - i\Lambda\partial\bar{\partial} - i\bar{\partial}\Lambda\partial + i\partial\Lambda\bar{\partial} + i\bar{\partial}\partial\Lambda \\
&= \Delta_\partial + i\Lambda\partial\bar{\partial} - i\bar{\partial}\Lambda\partial + i\partial\Lambda\bar{\partial} - i\partial\bar{\partial}\Lambda \\
&= \Delta_\partial + i([\Lambda, \bar{\partial}]\partial + \partial[\Lambda, \bar{\partial}]) \\
&= \Delta_\partial + i(-i\partial^*\partial - i\partial\partial^*) = 2\Delta_\partial.
\end{aligned}$$

Therefore, the first equality follows.

The second equality follows by similar calculations. □

6.7. **Hodge Decomposition.**

Lemma 6.34. *Let (X, ω) be a Kähler manifold and let α be a (p, q)-form on X.*

Then, also $\Delta\alpha$ is a (p, q)-form.

Proof. It follows from the definition of Δ_∂, that $\Delta_\partial \alpha$ is a (p,q)-form. Thus, the Lemma follows from Theorem 6.33. $\square$

Theorem 6.35. *Let (X,ω) be a Kähler manifold.*

Then, for any $k \geq 0$, we have a decomposition

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X),$$

and, for any $p,q \geq 0$, we have

$$\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}.$$

Proof. Let $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$ be a k-form. Then we may write

$$\alpha = \sum_{p+q=k} \alpha_{p,q}$$

where $\alpha \in C^\infty(X, \Omega^{p,q}_X)$. Thus, by Lemma 6.34, it follows that α is harmonic if and only if $\alpha_{p,q}$ is harmonic. This implies the first claim.

Now, let $\beta \in C^\infty(X, \Omega^{p,q}_X)$. Then, by Theorem 6.33, we have that

$$\Delta\beta = 0 \Leftrightarrow \Delta_\partial \beta = 0 \Leftrightarrow \overline{\Delta_\partial \beta} = 0 \Leftrightarrow \Delta\overline{\beta} = 0.$$

Thus the second claim holds. $\square$

We can finally prove:

Theorem 6.36 (Hodge Decomposition Theorem)**.** *Let X be a compact Kähler manifold.*

Then we have the Hodge decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{k=p+q} H^{p,q}(X)$$

and

$$H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

Proof. The Theorem follows immediately by combining together Corollary 6.18, Corollary 6.22 and Theorem 6.35. □

Example 6.37. *Assume that $X = \mathbb{P}^n_{\mathbb{C}}$. It follows from topology (using the Mayer-Vietoris sequence) that*

$$H^k(X, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } k = 2, 4, \ldots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

Recall that X is Kähler (cf. Example 6.6). Let ω be a Kähler form on X. We know that $0 \neq [\omega^p] \in H^{p,p}(X)$ for all $p = 1, \ldots, n$ (cf. Section 6.1). Thus, Theorem 6.36 implies

$$H^{p,q}(X) = \begin{cases} \mathbb{C} & \text{if } p = q = 1, 2, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

6.8. **Bott-Chern cohomology.** Let (X, ω) be a Kähler manifold. Recall that the operators $\Delta, \Delta_\partial, \Delta_{\bar\partial}$ depends on the Hodge $\star$-operator and, thus, on the Kähler metric ω.

On the other hand, we show below that the decomposition provided by the Hodge Decomposition Theorem (cf. Theorem 6.36) is independent of the choice of the metric.

Let X be a complex manifold and let $\alpha \in C^\infty(X, \Omega_X^{p-1,q-1})$ for some $p, q \geq 1$. Then

$$d(\partial\bar\partial\alpha) = \partial\partial\bar\partial\alpha - \bar\partial\bar\partial\partial\alpha = 0.$$

Thus, we may define the **Bott-Chern cohomology groups** of X as:

$$H^{p,q}_{BC}(X) = \frac{\{\alpha \in C^\infty(X, \Omega^{p,q}_X) \,|\, d\alpha = 0\}}{\partial\bar\partial C^\infty(X, \Omega^{p-1,q-1}_X)}.$$

Note that, for all $p, q \geq 1$ we have an induced map

$$\Phi\colon \{\alpha \in C^\infty(X, \Omega^{p,q}_X) \,|\, d\alpha = 0\} \to H^{p+q}(X, \mathbb{C}).$$

Let $\beta \in C^\infty(X, \Omega^{p-1,q-1}_X)$. Then,

$$\partial\bar\partial\beta = d\partial\beta.$$

Thus, Φ induces a map, which we will still denote by Φ,

$$\Phi\colon H^{p,q}_{BC}(X) \to H^{p+q}(X, \mathbb{C}). \tag{19}$$

Moreover, if $\alpha \in C^\infty(X, \Omega^{p-1,q-1}_X)$ then, similarly as above, we have

$$\overline{\partial}(\partial\bar\partial\alpha) = 0.$$

Thus, we have an induced map

$$H^{p,q}_{BC}(X) \to H^{p,q}(X).$$

Lemma 6.38 ($\partial\bar\partial$-Lemma)**.** *Let X be a compact Kähler manifold. Let α be a d-closed (p, q)-form which is ∂-exact (resp. $\overline{\partial}$-exact).*

Then there exists a $(p-1, q-1)$-form η such that $\alpha = \partial\bar\partial\eta$.

Proof. Since α is d-closed, it follows that

$$\partial\alpha = \overline{\partial}\alpha = 0.$$

Assume that α is ∂-exact. Then there exists $\beta \in C^\infty(X, \Omega^{p-1,q}_X)$ such that $\alpha = \partial\beta$.

By Theorem 6.21, we have the decomposition

$$C^\infty(X,\Omega_X^{p-1,q}) = \mathcal{H}^{p-1,q}(X) \oplus \overline{\partial} C^\infty(X,\Omega_X^{p-1,q-1}) \oplus \overline{\partial}^* C^\infty(X,\Omega_X^{p-1,q+1}).$$

Thus, we may write

$$\beta = \beta_1 + \bar{\partial}\beta_2 + \bar{\partial}^*\beta_3$$

where $\beta_1 \in \mathcal{H}^{p-1,q}(X)$, $\beta_2 \in C^\infty(X,\Omega_X^{p-1,q-1})$ and $\beta_3 \in C^\infty(X,\Omega_X^{p-1,q+1})$. Thus,

$$\alpha = \partial\beta = \partial\beta_1 + \partial\bar{\partial}\beta_2 + \partial\bar{\partial}^*\beta_3. \tag{20}$$

Since $\beta_1 \in \mathcal{H}^{p-1,q}(X)$, it follows that β_1 is $\Delta_{\overline{\partial}}$-harmonic. By Theorem 6.33, we have that β_1 is Δ_∂-harmonic. By Lemma 6.20, it follows that $\partial\beta_1 = 0$.

By (4) of Theorem 6.32, we have $[\Lambda, \partial] = i\overline{\partial}^*$. Thus,

$$\overline{\partial}^*\partial = -i[\Lambda,\partial]\,\partial = i\partial\Lambda\partial = i\partial[\Lambda,\partial] = -\partial\overline{\partial}^*.$$

Thus, (20) becomes

$$\alpha = \partial\bar{\partial}\beta_2 - \overline{\partial}^*\partial\beta_3 = -\bar{\partial}\partial\beta_2 - \overline{\partial}^*\partial\beta_3.$$

Since $\overline{\partial}\alpha = 0$, it follows that

$$\overline{\partial}\,\overline{\partial}^*\partial\beta_3 = 0.$$

By Lemma 6.20, it follows that $\overline{\partial}^*\partial\beta_3$ is $\Delta_{\overline{\partial}}$-harmonic.

By Theorem 6.21, we have

$$\operatorname{Ker}\overline{\partial}^* = \mathcal{H}^{p,q}(X) \oplus \overline{\partial}^* C^\infty(X,\Omega_X^{p,q+1}).$$

In particular, since $\overline{\partial}^*\partial\beta_3$ is both in $\mathcal{H}^{p,q}(X)$ and in $\overline{\partial}^* C^\infty(X,\Omega_X^{p,q+1})$, it must be zero.

Thus $\alpha = \partial\bar{\partial}\beta_2$ and the claim follows.

The case α is $\bar{\partial}$-exact is similar. $\square$

Theorem 6.39. *Let X be a compact Kähler manifold.*
Then there are isomorphisms

$$H^{p,q}_{BC}(X) \to H^{p,q}(X)$$

and

$$\bigoplus_{p+q=k} H^{p,q}_{BC}(X) \to H^k(X, \mathbb{C}).$$

Proof. We first prove the first claim. Let $\Phi\colon H^{p,q}_{BC}(X) \to H^{p,q}(X)$ be the map defined in (19). By Corollary 6.22, any element $[\alpha'] \in H^{p,q}(X)$ can be represented by a unique $\Delta_{\overline{\partial}}$-harmonic form α, such that $[\alpha] = [\alpha']$ in $H^{p,q}(X)$. By Theorem 6.33, α is Δ-harmonic and, by Lemma 6.13, we have $d\alpha = 0$. Thus $[\alpha] \in H^{p,q}_{BC}(X)$ and Φ is surjective.

We now show that Φ is injective. Let $[\beta] \in H^{p,q}_{BC}(X)$ such that $\Phi([\beta]) = 0$. In particular, β is d-closed. Since $[\beta] = 0$ in $H^{p,q}(X)$, it follows that β is $\overline{\partial}$-exact. By the $\partial\bar{\partial}$-Lemma (cf. Lemma 6.38), there exists a $(p-1, q-1)$-form η such that $\beta = \partial\bar{\partial}\eta$. In particular $[\beta] = 0$ in $H^{p,q}_{BC}(X)$. Thus, Φ is injective.

The first claim and the Hodge Decomposition Theorem (cf. Theorem 6.36) immediately imply the second claim. $\square$

In particular, from the Theorem, it follows that the decomposition given by Theorem 6.36 is independent on the metric.

6.9. **Lefschetz decomposition.** The goal of this Section is to show that if X is a Kähler manifold of dimension k, then for any $k = 0, \dots, n$, the Lefschetz operator induces an isomorphism of de Rham cohomology

$$H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

as in the Poincaré duality.

To this end, we define

$$\mathcal{H}^k(X,\mathbb{R}) = \{\alpha \in C_c^\infty(X,\Omega^k_{X,\mathbb{R}}) \mid \Delta\alpha = 0\}.$$

Similarly to Corollary 6.18, we have the following:

Corollary 6.40. *Let (X,ω) be a compact Kähler manifold. Then, for each $k \geq 0$, we have an isomorphism*

$$\mathcal{H}^k(X,\mathbb{R}) \simeq H^k(X,\mathbb{R}).$$

Lemma 6.41. *Let (X,ω) be a Kähler manifold of dimension n. Then, for any $\alpha \in C^\infty(X,\Omega^k_{X,\mathbb{C}})$, we have*
(1) $[L,\Lambda]\alpha = (k-n)\alpha$, *and*
(2) $[\Delta, L]\alpha = 0$.

Proof. We now prove (1). It is enough to check the claim at a point $x \in X$. We may assume that $z_1,\ldots,z_n$ are local holomorphic coordinates around x such that $dz_1,\ldots,dz_n$ form an orthonormal basis of Ω^1_X at the point x. Thus, we just need to check the identity $[L,\Lambda]\alpha = H\alpha$ holds on a Hermitian vector space $V_{\mathbb{C}}$, which is the complexified of a vector space V of dimension n.

We proceed by induction on n. Assume that $n = 1$. Then $[L,\Lambda](1) = -\Lambda L(1) = -1$, $[\Lambda, L]\eta = 0$ for all 1-forms η on V and $[L,\Lambda]\omega = L\Lambda\omega = \omega$. Thus, the claim holds.

If $n > 1$, then we may consider an orthonormal decomposition $V = W_1 \oplus W_2$ where $\dim W_1 = n-1$ and $\dim W_2 = 1$. We have that the form ω splits as $\omega = \omega_1 + \omega_2$ where ω_i is a form on W_i. Thus, $\Lambda = \Lambda_1 + \Lambda_2$, where Λ_i is the dual of the Lefschetz operator on W_i.

Let $\eta \in \Lambda^k V^*$ for some $k \geq 0$. Since

$$\Lambda^k V^* = \Lambda^k W_1^* \oplus (\Lambda^{k-1} W_1^* \otimes W_2^*),$$

we may write $\eta = \eta_1 + \eta_2$ where

$$\eta_1 \in \Lambda^k W_1^* \qquad \text{and } \eta_2 \in \Lambda^{k-1} W_1^* \otimes W_2^*.$$

Thus, by induction, we have

$$[L, \Lambda]\eta_1 = [L_1, \Lambda_1]\eta_1 + \eta_1 \cdot [L_2, \Lambda_2]1 = (k-(n-1))\eta_1 - \eta_1 = (k-n)\eta_1.$$

Similarly, we may assume $\eta_2 = \eta_2' \otimes \eta_2''$ where $\eta_2' \in \Lambda^{k-1} W_1^*$ and $\eta_2'' \in W_2^*$. By induction, we have

$$\begin{aligned}[L, \Lambda]\eta_2 &= [L_1, \Lambda_1]\eta_2' \otimes \eta_2'' + \eta_2' \otimes [L_2, \Lambda_2]\eta_2'' \\ &= ((k-1) - (n-1)\eta_2 + 0 = (k-n)\eta_2.\end{aligned}$$

Thus, the claim follows.

We now prove (2). Since, by Theorem 6.33, we have $\Delta = 2\Delta_\partial$, it is enough to check that $[\Delta_\partial, L] = 0$. Since ω is a closed real $(1,1)$-form, we have $\partial\omega = 0$. It follows that for any form α, we have

$$\partial L\alpha = \partial(\omega \wedge \alpha) = \omega \wedge \partial\alpha = L\partial\alpha.$$

Thus, $\partial L = L\partial$. Thus, since the Kähler identity (cf. (2) of Theorem 6.32) implies

$$\partial^* L - L\partial^* = [\partial^*, L] = -i\bar{\partial},$$

we have

$$\begin{aligned}[\Delta_\partial, L] &= \partial\partial^* L - L\partial\partial^* + \partial^*\partial L - L\partial^*\partial \\ &= \partial\partial^* L - \partial L\partial^* + \partial^* L\partial - L\partial^*\partial \\ &= -i\partial\bar{\partial} - i\bar{\partial}\partial = 0.\end{aligned}$$

Thus, the claim follows. $\square$

Proposition 6.42. *Let (X, ω) be a Kähler manifold of dimension n.*

Then the Lefschetz operator induces an isomorphism of real vector bundles

$$L^{n-k}\colon \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}}$$

for any $k = 0, \ldots, n$.

Proof. Since ω is a real $(1,1)$-form, the morphism

$$L^{n-k}\colon \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}} \qquad \alpha \mapsto \omega \wedge \alpha$$

is well-defined. Note that

$$\operatorname{rk}\, \Omega^k_{X,\mathbb{R}} = \operatorname{rk}\, \Omega^{2n-k}_{X,\mathbb{R}} = \binom{2n}{k}.$$

Thus, it is sufficient to prove that for every open set $U \subset X$, the morphism

$$L^{n-k}\colon C^\infty(U, \Omega^k_{X,\mathbb{R}}) \to C^\infty(U, \Omega^{2n-k}_{X,\mathbb{R}}) \qquad \alpha \mapsto \omega \wedge \alpha$$

is injective.

For any $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$, it follows from Lemma 6.41 that

$$[L, \Lambda]\alpha = (k - n)\alpha.$$

Thus, for every positive integer r, we have

$$\begin{aligned} [L^r, \Lambda]\, \alpha &= L(L^{r-1}\Lambda - \Lambda L^{r-1})\alpha + (L\Lambda - \Lambda L)L^{r-1}\alpha \\ &= L\left[L^{r-1}, \Lambda\right]\alpha + [L, \Lambda]L^{r-1}\alpha \\ &= L\left[L^{r-1}, \Lambda\right]\alpha + (k + 2r - 2 - n)L^{r-1}\alpha. \end{aligned}$$

Thus, it follows easily by induction that

$$[L^r, \Lambda]\, \alpha = (r(k - n) + r(r - 1))L^{r-1}\alpha. \tag{21}$$

We claim that L^r in injective for all $r \in \{0, \dots, n-k\}$. To this end, we proceed by double induction over $k \in \{0, \dots, n\}$ and $r \in \{0, \dots, n-k\}$. If $k \geq 0$ and $r = 0$ then L^0 is the identity on $C^\infty(U, \Omega^k_{X,\mathbb{R}})$ and it is injective.

Assume now that $k \geq 0$ and $r \geq 1$ and $\alpha \in C^\infty(U, \Omega^k_{X,\mathbb{R}})$ is such that $L^r\alpha = 0$. By (21), we have

$$L^r\Lambda\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha.$$

Note that, by assumption $r \leq n-k$ and therefore

$$r(k-n) + r(r-1) \neq 0.$$

Thus,

$$L^{r-1}(L\Lambda\alpha - (r(k-n) + r(r-1))\alpha) = 0.$$

By induction on r, we have that

$$L\Lambda\alpha - (r(k-n) + r(r-1))\alpha = 0.$$

Note that if $k \leq 1$ then $\Lambda\alpha = 0$ and, thus, $\alpha = 0$.

Assume now that $k \geq 2$ and let $\beta = \Lambda\alpha$. Then $\beta \in C^\infty(U, \Omega^{k-2}_{X,\mathbb{R}})$ and

$$L^{r+1}\beta = L^r((r(k-n) + r(r-1))\alpha) = 0.$$

Thus, by induction on k it follows that $\beta = 0$. Thus $\alpha = 0$ and the claim follows. $\square$

Theorem 6.43 (Hard Lefschetz Theorem)**.** *Let (X, ω) be a compact Kähler manifold of dimension n.*

Then the map

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism for every $k = 0, \dots, n$.

Moreover, we have an isomorphism

$$L^{n-k} : H^k(X, \mathbb{C}) \to H^{2n-k}(X, \mathbb{C})$$

which decomposes as

$$L^{n-p-q} \colon H^{p,q}(X) \to H^{n-q,n-p}(X)$$

for all $p, q = 0, \dots, n$*.*

Proof. By Poincaré duality (cf. Theorem 6.19 and Remark 5.13), the vector spaces $H^k(X, \mathbb{R})$ and $H^{2n-k}(X, \mathbb{R})$ have the same dimension. Thus, it is enough to show that L^{n-k} is injective.

By Lemma 6.41, we have that $[\Delta, L] = 0$. Thus, it follows that, for any $k \geq 0$, if $\alpha \in C^\infty(U, \Omega^k_{X,\mathbb{R}})$ s harmonic, then $L(\alpha)$ is also harmonic. Thus, the induced morphism

$$L^{n-k} \colon \mathcal{H}^k(X, \mathbb{R}) \to \mathcal{H}^{2n-k}(X, \mathbb{R})$$

is well-defined. By Corollary 6.40, it is enough to show that such a morphism is injective. The claim follows immediately from Proposition 6.42.

The proof of the second part of the Theorem is similar. □

Definition 6.44. *Let* (X, ω) *be a compact Kähler manifold and let* $\alpha \in C^\infty(U, \Omega^k_{X,\mathbb{R}})$ *for some* $k \in \{0, \dots, n\}$*. Then* α *is* **primitive** *if*

$$L^{n-k+1}\alpha = 0.$$

Lemma 6.45. *Let* X *be a compact Kähler manifold, let* $k \in \{0, \dots, n\}$ *and let* $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$

Then α *is primitive if and only if* $\Lambda\alpha = 0$*.*

Proof. By (21), we have that, for all $r \geq 0$,

$$[L^r, \Lambda]\, \alpha = (r(k-n) + r(r-1))L^{r-1}\alpha.$$

Thus, if $r = n - k + 1$, we have

$$L^{n-k+1}\Lambda\alpha = \Lambda L^{n-k+1}\alpha.$$

Assume that $L^{n-k+1}\alpha = 0$. Then $L^{n-k+1}\Lambda\alpha = 0$. We have that $\Lambda\alpha \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$. Thus, as in the Proof of Proposition 6.42, we have that L^r is injective, for any $r \le n - k + 2$. Thus, $\Lambda\alpha = 0$.

Assume now that $\Lambda\alpha = 0$ for some $\alpha \neq 0$. Let $r > 0$ minimal such that $L^r\alpha = 0$. Then by (21), we have

$$0 = [L^r, \Lambda]\,\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha.$$

Thus, $k-n+r-1 = 0$, i.e. $r = n-k+1$. Thus, $L^{n-k+1}\alpha = 0$. □

Proposition 6.46. *Let* (X, ω) *be a compact Kähler manifold of dimension* n *and let* $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$ *for some* $k \in \{0, \dots, n\}$.

Then there exists a unique decomposition

$$\alpha = \sum_{r\ge 0} L^r\alpha_r$$

where $\alpha_r \in C^\infty(X, \Omega^{k-2r}_{X,\mathbb{R}})$ *is primitive.*

Proof. We first show the existence. To this end, we proceed by induction on k. If $k = 0$ or $k = 1$, then $\Lambda(\alpha) = 0$. Thus, Lemma 6.45 implies that α is primitive and there is nothing to prove.

Assume now that $k \ge 2$. By Proposition 6.42 there exists $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$, such that

$$L^{n-k+2}\beta = L^{n-k+1}\alpha.$$

Then, $L^{n-k+1}(\alpha - L\beta) = 0$, i.e. $\alpha_0 = \alpha - L\beta$ is primitive. By induction, we may write $\beta = \sum_{r\ge 0} L^r\beta_r$ where $\beta_r \in C^\infty(X, \Omega^{k-2-2r}_{X,\mathbb{R}})$

is primitive. Thus

$$\alpha = \alpha_0 + \sum_{r\geq 1} L^r\beta_r$$

and the claim follows.

We now prove uniqueness. We proceed again by induction on k. If $k = 0$ the claim is obvious. Let $k \geq 1$ and assume that

$$\sum_{r\geq 0} L^r\beta_r = 0$$

for some primitive forms $\beta_r \in C^\infty(X, \Omega^{k-2r}_{X,\mathbb{R}})$. We want to show that $\beta_r = 0$ for all $r \geq 0$.

Since β_0 is a primitive k-form, we have that $L^{n-k+1}\beta_0 = 0$. Thus,

$$0 = L^{n-k+1}\sum_{r\geq 0} L^r\beta_r = L^{n-k+1}\sum_{r>0} L^r\beta_r = L^{n-k+2}\sum_{r>0} L^{r-1}\beta_r.$$

Since $L^{r-1}\beta_r \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$, Proposition 6.42 implies

$$\sum_{r>0} L^{r-1}\beta_r = 0.$$

Thus, by induction, we have that $\beta_r = 0$ for all $r > 0$. Since $\sum_{r\geq 0} L^r\beta_r = 0$, we also have $\beta_0 = 0$ and the claim follows. □

Definition 6.47. *Let (X, ω) be a compact Kähler manifold of dimension n and let $k \in \{0, \ldots, n\}$. We say that $[\alpha] \in H^k(X, \mathbb{R})$ is* **primitive** *if $L^{n-k+1}[\alpha] = 0$. We denote by*

$$H^k(X, \mathbb{R})_{\text{prim}} \subset H^k(X, \mathbb{R})$$

the subspace of primitive classes.

As a consequence of Proposition 6.46, we obtain:

Theorem 6.48 (Lefschetz Decomposition Theorem). *Let (X,ω) be a compact Kähler manifold of dimension n and let $k \in \{0,\dots,n\}$.*

Then
$$H^k(X,\mathbb{R}) = \bigoplus_{r\geq 0} L^r H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}}.$$

Remark 6.49. *As a consequence of the Lefschetz Decomposition Theorem, it follows that if X is a compact Kähler manifold of dimension $n \geq 2$, then for any $k \in \{0,\dots,n-2\}$, we have that the Betti number of X (cf. Section 5.4) satisfy*
$$b_{k+2} \geq b_k.$$

Similarly, using de Rham cohomology with complex coefficients, we have that if $p,q \in \{0,\dots,n-1\}$, then the Hodge numbers of X (cf. Section 4.3) satisfy
$$h^{p,q}(X) \leq h^{p+1,q+1}(X).$$

7. Sheaves

Definition 7.1. *Let X be a topological space. A* **presheaf** *$\mathcal{F}$ of abelian groups on X is a contravariant functor from the category of open subsets of X to the category of abelian groups, i.e. if $U \subset X$ and $V \subset U$ are open subsets then we have a group morphism $\rho_{U,V}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ such that if $W \subset V$ is also an open subset, then we have $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$. The map $\rho_{U,V}$ is called* **restriction map***.*

A **sheaf** *is a presheaf $\mathcal{F}$ which satisfies the following properties:*

(1) *If* $\{U_i\}$ *is an open cover of* $U \subset X$ *and* $s \in \mathcal{F}(U)$ *is such that* $\rho_{U,U_i}(s) = 0$ *for all* i *then* $s = 0$.
(2) *If* $\{U_i\}$ *is an open cover of* $U \subset X$ *and* $s_i \in \mathcal{F}(U_i)$ *is such that*
$$\rho_{U_i,U_i\cap U_j}(s_i) = \rho_{U_j,U_i\cap U_j}(s_j)$$
for all i, j, *then there exists* $s \in \mathcal{F}(U)$ *such that* $\rho_{U,U_i}(s) = s_i$.

We will often write $s|_V$ instead of $\rho_{U,V}(s)$ in analogy with the restriction of a function defined on an open set U, restricted to an open subset $V \subset U$.

Definition 7.2. *Let* $\mathcal{F}$ *be a sheaf on a topological space* X *and let* $x \in X$ *be a point. The* **stalk** *of* $\mathcal{F}$ *at* x, *denoted* $\mathcal{F}_x$, *is the direct limit*
$$\mathcal{F}_x := \lim_{x\in U} \mathcal{F}(U).$$

In the notation above, an element of $\mathcal{F}_x$ consists of a pair (U, s) where $U \subset X$ is an open subset and $s \in \mathcal{F}(U)$. We have that $(U, s) \sim (U', s')$ if and only if there exists $W \subset U \cap U'$ such that $s|_W = s'|_W$. An element of $\mathcal{F}_x$ is called a **germ** of $\mathcal{F}$ at x.

Given a sheaf $\mathcal{F}$ on a topological space X and an open set $U \subset X$, we will sometime denote $H^0(U, \mathcal{F}) := \mathcal{F}(U)$.

Example 7.3. *Let* X *be a holomorphic manifold. Define* $\mathcal{O}_X(U)$ *to be the ring of holomorphic functions* $f: U \to \mathbb{C}$. *It is easy to check that* $\mathcal{O}_X$ *is a sheaf.*

Moreover, for each $p \geq 0$, *let* $\Omega^p_X(U)$ *to be the complex vector space of holomorphic* p*-forms on* U. *Note that* $\Omega^p_X(U)$ *is a* $\mathcal{O}_X(U)$*-module. Also in this case, it is easy to check that* Ω^p_X

is a sheaf. Similarly the presheaf T_X of holomorphic vector fields is a sheaf.

Definition 7.4. *Let X be a complex manifold and let $\mathcal{F}$ be a sheaf on X. If $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for all open set $U \subset X$ such that the restriction morphisms respects the module structure, then we say that $\mathcal{F}$ is an $\mathcal{O}_X$-**module**.*

Example 7.5. *Let X be a complex manifold. Then $\mathcal{O}_X$, T_X and Ω^p_X are $\mathcal{O}_X$-modules.*

More in general, let $\pi\colon E \to X$ be a holomorphic vector bundle. Let $\mathcal{E}(U)$ be the complex vector space of holomorphic section of E over U. Then $\mathcal{E}$ is a $\mathcal{O}_X$-module.

Definition 7.6. *Let $\mathcal{F}, \mathcal{G}$ be two (pre)-sheaves of abelian groups on a complex manifold X. A* **(pre)-sheaf morphism** *$\phi\colon F \to \mathcal{G}$ is a collection of group morphisms $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ which respect the restriction morphisms.*

A sheaf morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ is called a **morphism of $\mathcal{O}_X$-modules**, *if the morphism ϕ_U is $\mathcal{O}_X(U)$-linear for each open set $U \subset X$.*

Note that a morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ on a topological space X induces a morphism of stalks $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$. We say that ϕ is **injective** (resp. surjective) if ϕ_x is injective (resp. surjective) for all $x \in X$.

Lemma 7.7. *Suppose that $\phi\colon \mathcal{F} \to \mathcal{G}$ is an injective morphism of sheaves on a topological space X. Then $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open sets $U \subset X$.*

Proof. Let $s \in \mathcal{F}(U)$ be such that $\phi_U(s) = 0$. Then, for all $x \in U$, we have $\phi_x(U, s) = 0$, i.e. there exists $V_x \subset U$ open subset containing x such that $s|_{V_x} = 0$. In particular, $\{V_x\}$ is an open cover of U. Thus, by the definition of sheaf, it follows that $s = 0$. $\square$

Proposition 7.8 (Sheafification)**.** *Let $\mathcal{F}$ be a presheaf. Then there exists a sheaf $\mathcal{F}^+$ and a morphism of pre-sheaves $i\colon \mathcal{F} \to \mathcal{F}^+$ such that for any sheaf $\mathcal{G}$ and for any morphism of pre-sheaves $f\colon \mathcal{F} \to \mathcal{G}$ there exists a morphism of sheaves $f^+\colon \mathcal{F}^+ \to \mathcal{G}$ such that $f^+ \circ i = f$.*

Note that if $\mathcal{F}$ is a sheaf then $\mathcal{F}^+ = \mathcal{F}$. The proof of the Proposition is left as an exercise.

Definition 7.9. *A sheaf $\mathcal{F}$ on a complex manifold X is called* **locally free of rank** *k at $x \in X$ if there is a neighbourhood U of x and an isomorphism $\mathcal{F}|_U \simeq \mathcal{O}_U^{\oplus r}$. The sheaf $\mathcal{F}$ is called* **locally free** *if it is locally free at all $x \in X$.*

Example 7.10. *Clearly, if X is a complex manifold and r is a positive integer, then $\mathcal{O}_U^{\oplus r}$ is locally free.*

Proposition 7.11. *Let X be a complex manifold and let $\pi\colon E \to X$ be a holomorphic vector bundle. Let $\mathcal{E}$ be the sheaf of holomorphic sections of E (cf. Example 7.5).*

Then $\mathcal{E}$ is locally free.

Proof. Let $\mathcal{E}$ be the sheaf of sections of E. By definition of locally free sheaves, we just need to check that, given $x \in X$, the sheaf $\mathcal{E}$ is locally free around x. Let U be an open set containing x and which trivialise E and let $s_1, \dots, s_r$ be a frame of E on U. Then any section σ of E on U can be written in a unique way in the form $\sigma = \sum f_i s_i$ where f_i is a holomorphic function on U. This yields an isomorphism $\mathcal{O}_U^{\oplus r} \to \mathcal{E}|_U$. Thus, $\mathcal{E}$ is locally free. $\square$

Note that in the above proof if $g_{i,j}\colon (U_i\cap U_j)\times\mathbb{C}^r \to (U_i\cap U_j)\times\mathbb{C}^r$ is the transaction map from the trivialisation on U_j to the trivialisation on U_i, and $\varphi_i\colon \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus r}$ is the induced isomorphism, then $\varphi_i \circ \varphi_j^{-1} = g_{ij}$.

Proposition 7.12. *Let $\mathcal{E}$ be a locally free sheaf on a complex manifold X. There there is a holomorphic vector bundle $\mathbb{V}(\mathcal{E})$ on X whose sheaf of sections is $\mathcal{E}$.*

Proof. Let $\{U_i\}$ be a cover of X such that $\varphi_i\colon \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus r}$ is an isomorphism. On $U_i \cap U_j$ we get isomorphisms

$$\phi_i \circ \phi_j^{-1}\colon \mathcal{O}_{U_i\cap U_j}^{\oplus r} \to \mathcal{O}_{U_i\cap U_j}^{\oplus r}.$$

An isomorphism is given by a holomorphic map $g_{i,j}\colon U_i \cap U_j \to \mathrm{GL}_r(\mathbb{C})$. It is easy to check that $g_{i,j}$ define an holomorphic vector bundle on X and that the sheaf of sections of $\mathbb{V}(\mathcal{E})$ is exactly $\mathcal{E}$. $\square$

Proposition 7.13. *Let X be a complex manifold.*

(1) *Let $f\colon \mathcal{E}_1 \to \mathcal{E}_2$ be a morphism of locally free sheaves on X. Then there is a morphism of the induced vector bundles $\mathbb{V}(f)\colon \mathbb{V}(\mathcal{E}_1) \to \mathbb{V}(\mathcal{E}_2)$.*

   If $g\colon \mathcal{E}_2 \to \mathcal{E}_3$ is also a morphism of locally free sheaves on X then $\mathbb{V}(g\circ f) = \mathbb{V}(g)\circ\mathbb{V}(f)$.

(2) *Let $F\colon E_1 \to E_2$ be a morphism of holomorphic vector bundles on X. Then there is a morphism of the induced sheaves $\tilde{F}\colon \mathcal{E}_1 \to \mathcal{E}_2$.*

   If $G\colon E_2 \to E_3$ is also a morphism of holomorphic vector bundles on X then $(G\circ F)^{\sim} = \tilde{G}\circ\tilde{F}$.

(3) *If $f\colon \mathcal{E}_1 \to \mathcal{E}_2$ is a morphism of locally free sheaves on X then $(\mathbb{V}(f))^{\sim} = f$ and, similarly, if $F\colon E_1 \to E_2$ is*

a morphism of holomorphic vector bundles on X then $\mathbb{V}(\tilde{F}) = F$.

Proof. Let $f\colon \mathcal{E}_1 \to \mathcal{E}_2$ be a morphism of locally free sheaves on X. By assumption, for each $x \in X$, there exist an open set $U \subset X$ such that

$$\mathcal{E}_1|_U \simeq \mathcal{O}_U^{\oplus r} \qquad \text{and} \qquad \mathcal{E}_1|_U \simeq \mathcal{O}_U^{\oplus r'}$$

Thus, a morphism $f\colon \mathcal{E}_1 \to \mathcal{E}_2$ is given by a collection of morphisms

$$f_U\colon \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus r'}.$$

Since a morphism $\mathcal{O}_U \to \mathcal{O}_U$ is defined by a holomorphic function $U \to \mathbb{C}$, it follows that f_U is defined by a $r' \times r$-matrix of holomorphic functions on U. In particular, if $U, V \subset X$ are two open sets, then these matrices are compatible with the isomorphism

$$\mathcal{O}_{U\cap V}^{\oplus r} \to \mathcal{O}_{U\cap V}^{\oplus r} \qquad \mathcal{O}_{U\cap V}^{\oplus r'} \to \mathcal{O}_{U\cap V}^{\oplus r'}$$

constructed in the proof of Proposition 7.12. Thus, the functions f_U define a morphism

$$\mathbb{V}(f_U)\colon \mathbb{V}(\mathcal{E}_1)|_U \to \mathbb{V}(\mathcal{E}_2)|_U$$

which is compatible with the transition functions of $\mathbb{V}(\mathcal{E}_1)$ and $\mathbb{V}(\mathcal{E}_2)$. Thus, it induces a morphism of vector bundles $\mathbb{V}(f)\colon \mathbb{V}(\mathcal{E}_1) \to \mathbb{V}(\mathcal{E}_2)$.

The rest of the proof is similar and it is left as an exercise. $\square$

From the Proposition it follows that the category of holomorphic vector bundles on X coincides with the category of locally free sheaves on X. For these reason, it is common to use the same notation for the two objects. E.g. if X is a complex manifold, then we denote by $\mathcal{O}_X, T_X, \Omega^p_X$ both the sheaves and the vector bundles on X.

Similarly to the case of vector bundles, we can take operations on sheaves. Indeed, let X be a complex manifold. If $\mathcal{F}, \mathcal{G}$ are sheaves on X, then we can construct the **direct sum** sheaf $\mathcal{F} \oplus \mathcal{G}$ so that

$$\mathcal{F} \oplus \mathcal{G}(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$$

for every open set $U \subset X$.

If $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$-modules, then we define the **tensor product** $\mathcal{F} \otimes \mathcal{G}$ to be the sheafification of the presheaf defined by

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

Similarly, we define the **dual** of $\mathcal{F}$ as the sheafification of the presheaf defined by

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{O}_X(U)),$$

is the set of $\mathcal{O}_X(U)$-linear morphism $\mathcal{F}(U) \to \mathcal{O}_X(U)$.

It is easy to check that if $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves then their direct sum, tensor product and duals are also locally free sheaves.

Definition 7.14. *Let $f\colon X \to Y$ be a holomorphic morphism between complex manifolds and let $\mathcal{F}$ be a sheaf on X. Then we define the* **push-forward** *$f_*\mathcal{F}$ of $\mathcal{F}$ as the sheaf given by*

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

for all open sets $U \subset Y$.

Definition 7.15. *Let $f\colon X \to Y$ be a holomorphic morphism between complex manifold and let $\mathcal{G}$ be a sheaf on Y. Then we define a sheaf $f^{-1}\mathcal{G}$ by, for any open set $U \subset X$,*

$$f^{-1}\mathcal{G}(U) = \lim_{f(U)\subset V} \mathcal{G}(V)$$

where the limit is taken over the open sets V in Y containing $f(U)$.

Note that, in the definitions above, if $\mathcal{F}$ is a $\mathcal{O}_X$-module then $f_*\mathcal{F}$ is a $\mathcal{O}_Y$-module. On the other hand, $f^{-1}\mathcal{G}$ is not a $\mathcal{O}_X$-module in general. It is a $f^{-1}\mathcal{O}_Y$-module, i.e. $f^{-1}\mathcal{G}(U)$ is a $(f^{-1}\mathcal{O}_Y)(V)$-module for every open set V of U.

Thus, we may define

$$f^*\mathcal{G} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

By construction, $f^*\mathcal{G}$ is a $\mathcal{O}_X$-module, called the **pull-back** of $\mathcal{G}$.

Proposition 7.16. *Let $f\colon X \to Y$ be a holomorphic morphism between complex manifolds and let $\mathcal{G}$ be a locally free sheaf on Y.*

Then $f^*\mathcal{G}$ is locally free.

Proof. Let $\{U_i\}$ be a trivialising open cover of Y, i.e. for each i there exists an isomorphism $\mathcal{O}_{U_i}^{\oplus r} \to \mathcal{G}|_{U_i}$. Then the induced map $\mathcal{O}_{f^{-1}(U_i)}^{\oplus r} \to f^*\mathcal{G}|_{f^{-1}(U_i)}$ is also an isomorphism and since $\{f^{-1}(U_i)\}$ is an open cover of X, the claim follows. $\square$

8. References

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