Numerical approximations of harmonic 1-forms on real loci of Calabi-Yau manifolds

Daniel Platt (Imperial College London)
Loughborough University, 20 March 2024

Abstract: For applications in differential geometry and string theory one would like to construct Calabi-Yau manifolds of complex dimension three with the following property: it should contain a real submanifold of real dimension three that admits a harmonic nowhere vanishing 1-form. Many examples are expected to exist, but none have been proven to exist. The problem is that there is no explicit formula for the Calabi-Yau metric which makes it hard to write down the "harmonic" equation, let alone solve it. In the talk I will present numerical approximations of the Calabi-Yau metric, and numerical approximations of harmonic 1-forms, obtained by neural networks. This suggests some conjectural examples of harmonic, nowhere vanishing 1-forms. I will also show some proven non-examples, and explain the main long-term motivation for this numerical work, which is to numerically verifiably prove that there exists a genuine solution to the harmonic equation near the approximate solutions. This is work in progress, joint with Michael Douglas and Yidi Qi.

Background I: Calabi-Yau manifolds

- Calabi conjecture (Yau's theorem):
  If (Y, g, J, ω) Kähler, complex dim n with:

  $$\Omega \in \Omega^{n,0}(Y) \text{ parallel and nowhere 0 s.t.}$$

  then ex. $\phi \in C^\infty(Y)$ s.t. $\omega_{CY} = \omega + i\partial\overline{\partial}\phi$ has $\omega_{CY}^n = \Omega \wedge \overline{\Omega}$
  ($\Rightarrow$ induced metric g_{CY} is Ricci-flat)
- Example: Fermat quintic

  $$Y := \{z = [z_0 : \cdots : z_4] \in \mathbb{CP}^4 : z_0^5 + \cdots + z_4^5 = 0\}$$

  has $\Omega \in \Omega^{n,0}(Y) \Rightarrow g_{CY}$ exists

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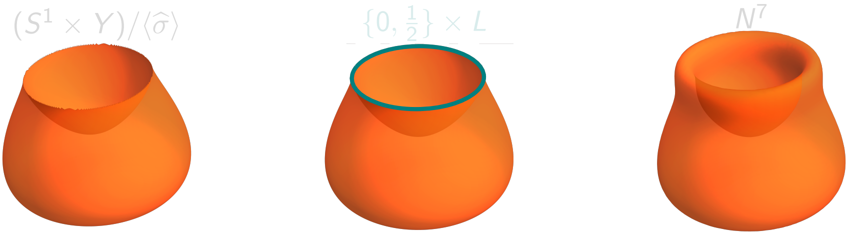
Background II: Harmonic 1-forms on Calabi-Yaus

- Let Y be Calabi-Yau 3-fold with Calabi-Yau metric g_{CY}
- $\sigma : Y \to Y$ anti-holomorphic involution, $L := fix(\sigma)$
  example: quintic with real coefficients in $\mathbb{CP}^4$ and $\sigma([z_0 : \cdots : z_4]) = [\overline{z_0} : \cdots : \overline{z_4}]$
- $S^1 \times Y$ has dimension 7 and holonomy $SU(3)$. Problem: want holonomy G_2
- Define $\widehat{\sigma} : S^1 \times Y \to S^1 \times Y$ as $(x, y) \mapsto (-x, \sigma(y))$



Theorem ([Joyce and Karigiannis, 2017])

If there exists $\lambda \in \Omega^1(L)$ harmonic w.r.t. $g_{CY}|_L$ that is nowhere 0, then there exists a resolution $N^7 \to (S^1 \times Y)/\langle\widehat{\sigma}\rangle$ with holonomy equal to G_2.

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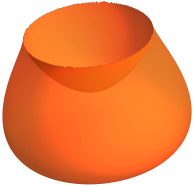

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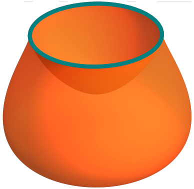

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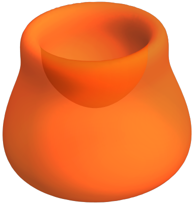

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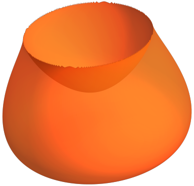

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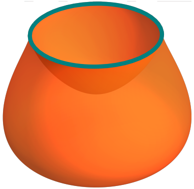

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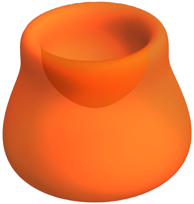

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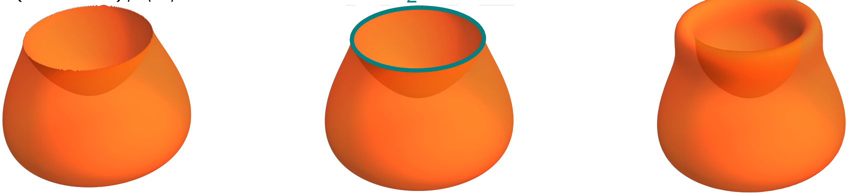

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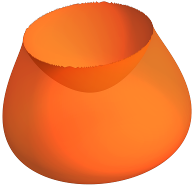

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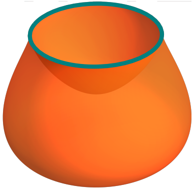

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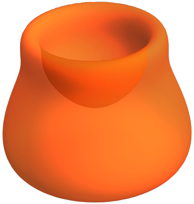

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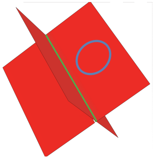

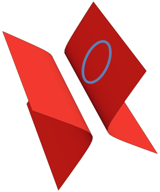

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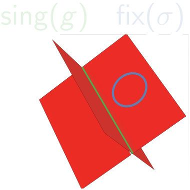

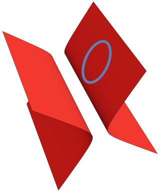

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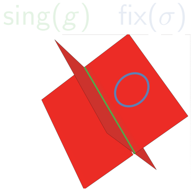

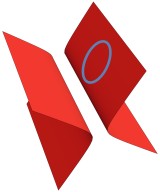

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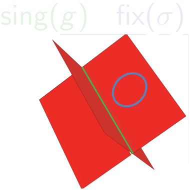

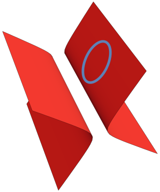

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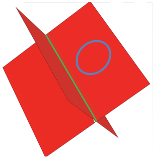

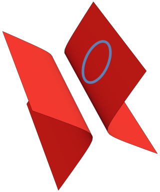

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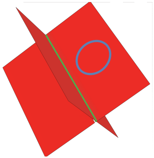
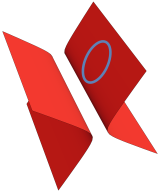
sing(g) fix(σ)





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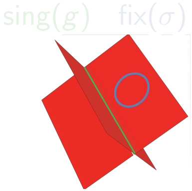

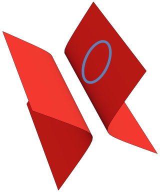

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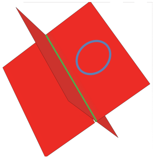

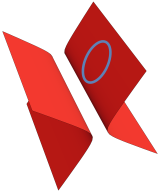

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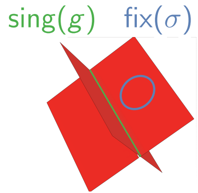

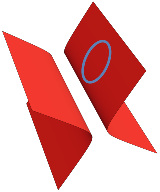

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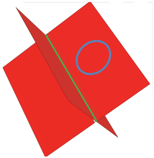
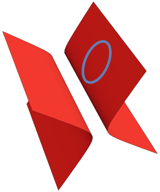
$\mathrm{sing}(g)$ $\mathrm{fix}(\sigma)$





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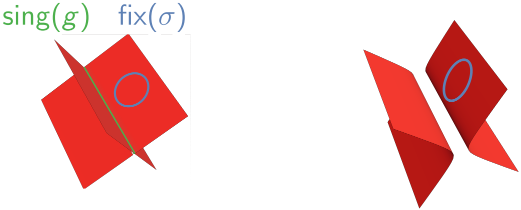

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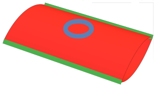

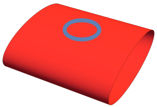

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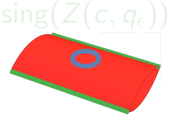

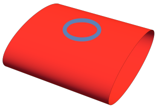

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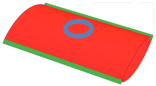

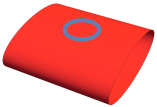

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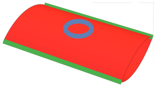

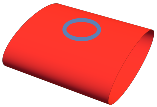

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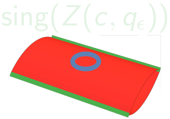

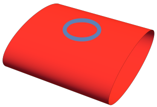

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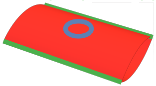

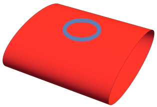

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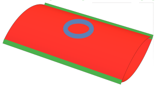

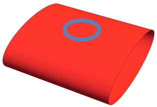

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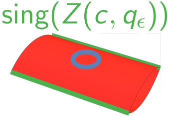

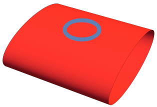

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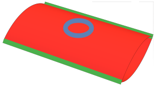

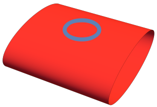

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Numerical Calabi-Yau metrics

- Holomorphic volume form locally $\Omega = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \rightsquigarrow \mathrm{vol}_\Omega := \Omega \wedge \overline{\Omega} \in \Omega^6(Y)$
- Ample line bundle $L \to Y$ and $k \in \mathbb{N}$ such that $L^{\otimes k}$ very ample
  Example: $Y \subset \mathbb{CP}^4$ quintic, $(O(1)|_Y)^{\otimes k}$
- $s_1, \ldots, s_N \in H^0(L^{\otimes k})$ basis of holomorphic sections
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- h positive definite Hermitian form on $H^0(L^{\otimes k}) \rightsquigarrow$ some Fubini-Study metric
  Kähler potential: $K = \log \sum_{i,j} h_{i,j} s^i \bar{s}^j$. Volume form: $\omega_h^3 = \mathrm{vol}_h \in \Omega^6(Y)$.

  If $\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} = 1$, then Ricci-flat
- [Donaldson, 2009]: choose h cleverly to minimise $\int_Y \left(\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1\right)^2$

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Numerical harmonic 1-forms

Example: quintic $X := Z(f) \subset \mathbb{CP}^4$, $L := \mathrm{fix}(\sigma) \subset X$

- $\xi_1, \ldots, \xi_{10} \in \Omega^1(\mathbb{RP}^4)$ closed 1-forms s.t. $T_x^*\mathbb{RP}^4 = (\xi_1(x), \ldots, \xi_{10}(x))\ \forall x$
- $p_1, \ldots, p_N$ polys, p_i degree d_i; for $\alpha = (\alpha_{1,1}, \ldots, \alpha_{10,N}) \in \mathbb{R}^{10N}$ let

$$\lambda_\alpha(x) = \sum_{\substack{i=1,\ldots,N \\ j=1,\ldots,10}} \alpha_{i,j} \frac{p_i(x)}{|x|^{d_i}} \xi_j|_L(x) \in \Omega^1(L)$$

For $x_1, \ldots, x_{100000} \in X$ find $\min_{\alpha \text{ s.t. } ||\lambda_\alpha||_{L^2}=1} \int_{x_1,\ldots,x_{100000}} |d\lambda_\alpha| + |d^*\lambda_\alpha|$

- Stone-Weierstrass $\Rightarrow$ best approximations converge to harmonic form as $N \to \infty$
- Ansatz for p_i: $A_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_{i+1}}$ linear, $sq : \mathbb{R}^k \to \mathbb{R}^k$ square each coordinate

$$p(x_0, \ldots, x_4) = A_k \circ \cdots \circ \mathrm{sq} \circ A_2 \circ \mathrm{sq} \circ A_1(x_0, \ldots, x_4)$$

- Equivalent: neural network with activation function $x \mapsto x^2$
- Approximate metric $\frac{i}{2}\partial\bar{\partial}K$ smooth+explicit $\Rightarrow$ explicitly compute $(|d\lambda_\alpha|(x_i) + |d^*\lambda_\alpha|(x_i))/\sqrt{|\lambda(x_1)|^2 + \cdots + |\lambda(x_{100000})|^2}$
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Numerical harmonic 1-forms

Example: quintic $X := Z(f) \subset \mathbb{CP}^4$, $L := \mathrm{fix}(\sigma) \subset X$

- $\xi_1, \ldots, \xi_{10} \in \Omega^1(\mathbb{RP}^4)$ closed 1-forms s.t. $T_x^*\mathbb{RP}^4 = (\xi_1(x), \ldots, \xi_{10}(x))\ \forall x$
- $p_1, \ldots, p_N$ polys, p_i degree d_i; for $\alpha = (\alpha_{1,1}, \ldots, \alpha_{10,N}) \in \mathbb{R}^{10N}$ let

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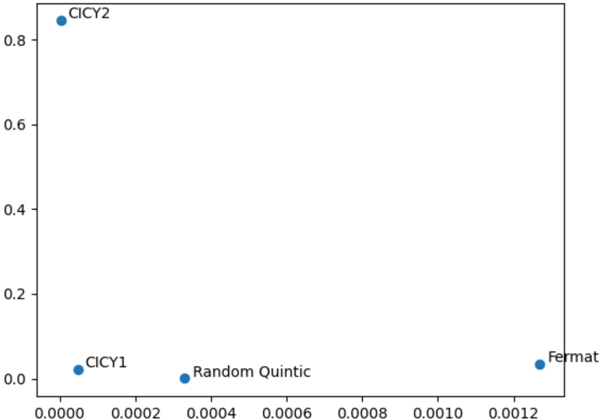
Experimental results: 1-forms and their zeros

1. **Fermat:** non-example 1; no harmonic 1-form.
2. **Random Quintic:** non-example 2; harmonic 1-form must have zeros
3. **CICY1:** conjectural example 3; large perturbation $\epsilon = \frac{1}{4}$, harmonic 1-form may have zeros
4. **CICY2:** conjectural example 3; small perturbation $\epsilon = \frac{1}{100}$, conjecture no zeros

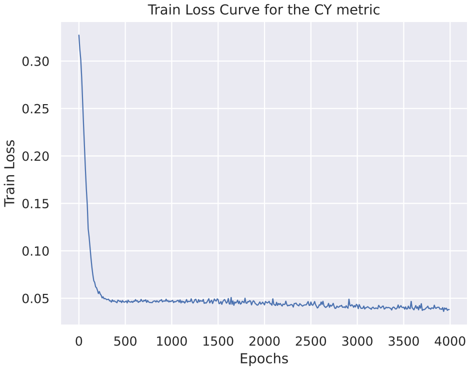
y-axis: $\min |\lambda|$

x-axis: harmonic loss $\frac{||d\lambda||_{L^1} + ||d^*\lambda||_{L^1}}{||\lambda||_{L^2}}$

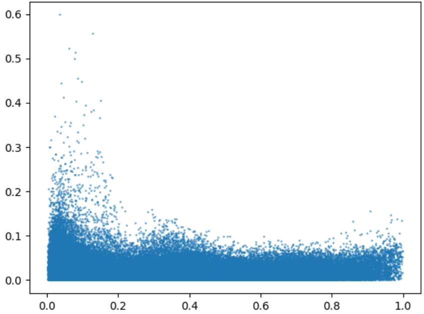


Experimental results on quintic

- $g = v \cdot f_-$ singular quintic from before, $\xi = 0.84x_0^5 + \dots$ random quintic
- Find $\epsilon > 0$ such that $g_\epsilon := g + \epsilon\xi$ has $Z_{\mathbb{R}}(g_\epsilon)$ diffeo to $Z_{\mathbb{R}}(g)$
  - $U \subset \mathbb{RP}^4$ nbhd of $Z_{\mathbb{R}}(g)$
  - $k := \min_U |Dg| > 0$, $M := \min_{\mathbb{RP}^4 \setminus U} |g| > 0$
  - if $||\epsilon_0\xi||_{C^0} < M$ and $||D\epsilon_0\xi||_{C^0} < k$, then $Z_{\mathbb{R}}(g_\epsilon)$ smooth for all $0 < \epsilon < \epsilon_0$
  - $\Rightarrow Z_{\mathbb{R}}(g_\epsilon)$ diffeo for all $0 < \epsilon < \epsilon_0$ (for us $\epsilon_0 = 0.00195503$)



Average of $|\mathrm{vol}_h / \mathrm{vol}_\Omega - 1|$ while iteratively improving vol_h



$|\mathrm{vol}_h / \mathrm{vol}_\Omega(x) - 1|$ over $\max\{v(x)/||x||^2, f_-(x)/||x||^3\}$

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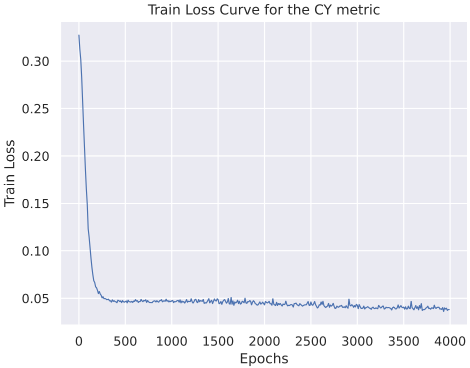

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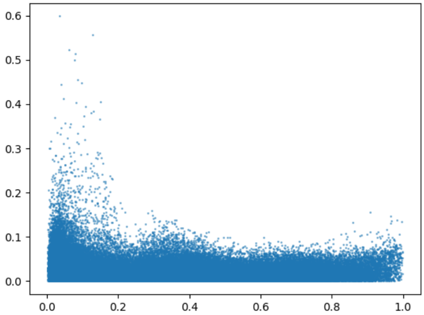

$|\mathrm{vol}_h / \mathrm{vol}_\Omega(x) - 1|$ over $\max\{v(x)/||x||^2, f_-(x)/||x||^3\}$

Experimental results on quintic

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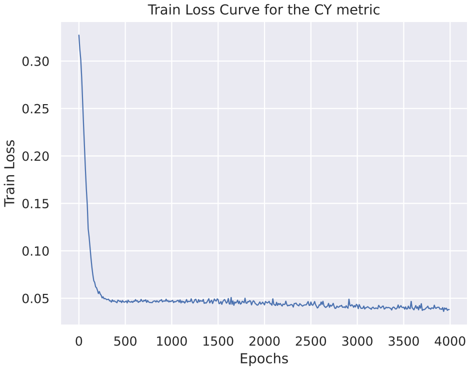

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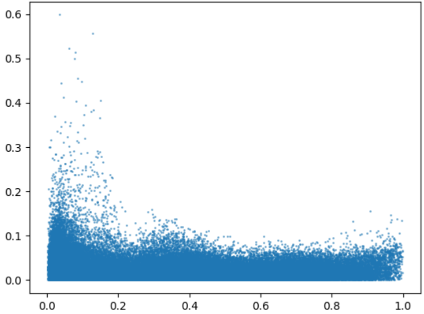
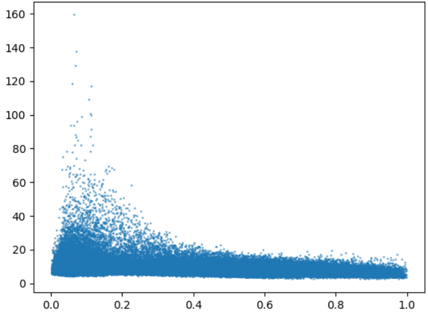


$|\mathrm{vol}_h / \mathrm{vol}_\Omega(x) - 1|$ over $\max\{v(x)/||x||^2, f_{-}(x)/||x||^3\}$

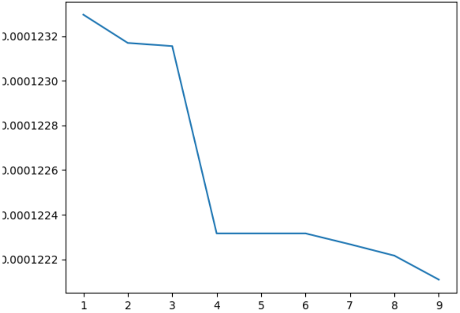
Experimental results on quintic

Neck formation



$\max_{v \in T_{||v||_{FS}=1}} ||v||_h$ over $\max\{v(x)/\,||x||^2, f_-(x)/\,||x||^3\}$

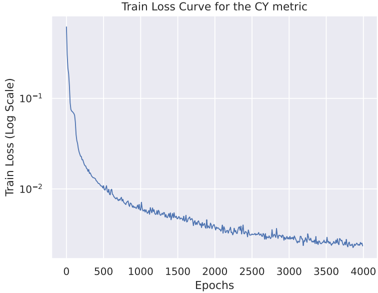
1-form has even number of zeros



k-medoid clustering loss of 500 points with smallest $|\omega|(x)$ over number of clusters (heuristic: "elbow" $k = 4$ is optimal number of clusters

Experimental results on quadric $\cap$ quartic

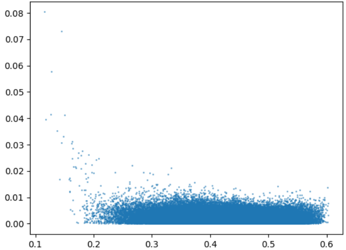
$$\epsilon = \frac{1}{4}$$

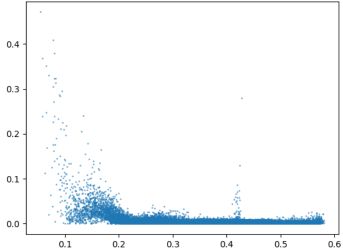


$$\epsilon = \frac{1}{100}$$

$\uparrow$ Training loss

$\downarrow$ Loss over distance from singularity





Experimental results on quadric ∩ quartic

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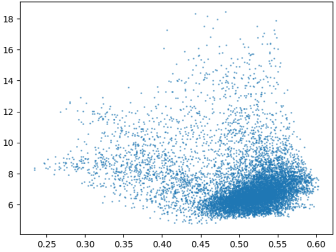
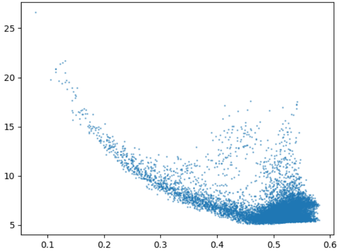


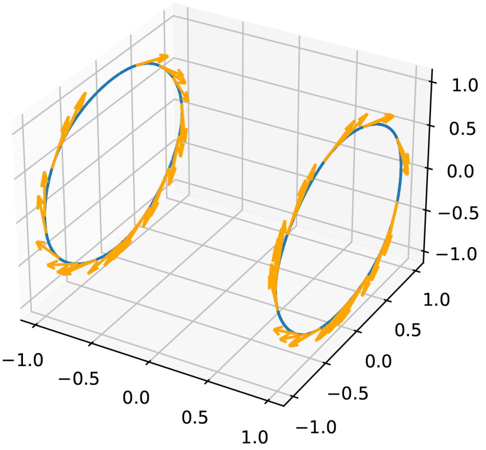
↑ Metric stretching over distance from singularity

$$\epsilon = \frac{1}{100}$$



Experimental results on quadric $\cap$ quartic

$c = -x_0^2 + x_1^2 + x_2^2$ and $q = -x_0^4 + x_3^4 + x_4^4 + x_5^4$
Set $x_0 = 1$ and $x_3 = x_4 = 0 \curvearrowright \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} \times \{\pm 1\}$
1-form restricted to this



Bonus motivation

Proposition

For all $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left|\left|\frac{\text{vol}_h}{\text{vol}_\Omega} - 1\right|\right|_{L_1^p} < \delta \Rightarrow ||g_{approx} - g_{CY}||_{L_1^p} < \epsilon.$$

Proposition

For all $\mu > 0$ *there exists* $\epsilon > 0$ *such that the following is true: for* $\lambda \in \Omega^1(L^3)$ *such that* $\Delta_{approx}\lambda = 0$ *and* $||\lambda||_{L^2, g_{approx}} = 1$ *and* $\min_L |\lambda| > \mu$ *let* $\widetilde{\lambda} \in [\lambda]$ *be the unique* Δ_{CY}*-harmonic* 1*-form. Then:*

$$||g_{approx} - g_{CY}||_{L_1^p} < \epsilon \Rightarrow |\widetilde{\lambda} - \lambda|(x) < \frac{\mu}{2} \Rightarrow |\widetilde{\lambda}|(x) > \frac{\mu}{2} \text{ for all } x \in L.$$

- Find: g_{approx} with $\left|\left|\frac{\text{vol}_h}{\text{vol}_\Omega} - 1\right|\right|_{L_1^p} < \delta$, λ with $\Delta_{approx}\lambda = 0$ and $\min_L |\lambda| > \mu$
- $\Rightarrow$ there exists nowhere vanishing g_{CY}-harmonic 1-form on L

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Thank you for the attention!

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