Riemannian metric

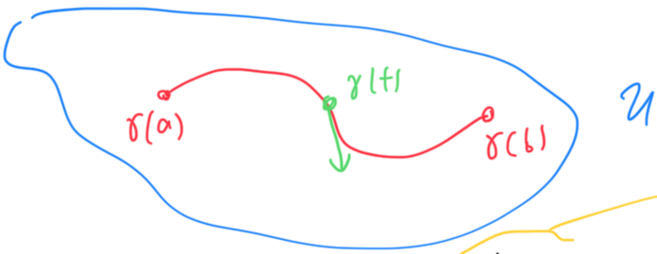
$U \subset \mathbb{R}^n$ **Def:** Riemannian metric on U:

$g : U \longrightarrow \mathbb{R}^{n \times n}$ s.t. symmetric, pos. def. in each

$v \cdot g(x) \cdot v^T \geq 0 \quad \forall x \in U$

Def: curve is a map $\gamma : [a,b] \longrightarrow U$

Tangent vector at time $t \in [a,b]$ is $\gamma'(t) \in \mathbb{R}^n$



length of vector $v \in \mathbb{R}^n$ at $x \in U$: $\quad |v|_{g(x)} := \sqrt{v \cdot g(x) \cdot v^T}$

length of γ is $\quad L(\gamma) := \int_a^b |\gamma'(t)|_{g(\gamma(t))} \, dt$

Example: $\quad g : U \longrightarrow \mathbb{R}^{2 \times 2}$ is called "standard Euclidean metric"

$\quad x \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\gamma : [0, 2\pi] \to \mathbb{R}^2$

$\quad t \longmapsto (\cos(t), \sin(t))$

$\Rightarrow \gamma'(t) = (-\sin t, \cos t)$

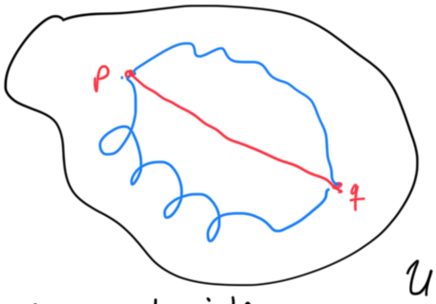
$$L(\gamma) = \int_0^{2\pi} |\gamma'(t)|_{g(\gamma(t))} \, dt = \int_0^{2\pi} |(-\sin(t), \cos(t))|_{g(\gamma(t))} \, dt$$

$$= \int_0^{2\pi} \sqrt{(-\sin t, \cos t) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}} \, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi$$

Def: $p, q \in U$: distance from p to q is

$$d^g(p,q) := \inf_{\substack{\gamma : [a,b] \to U \text{ curve} \\ \text{s.t. } \gamma(a) = p,\ \gamma(b) = q}} L(\gamma)$$



Example: $\quad S^1 := [0, 2\pi] / \sim \quad$ where $\sim$ identifies endpoints

def of $\sim$: $0 \sim 2\pi$, $2\pi \sim 0$, $t \sim t$ for all $t \in [0, 2\pi]$

$p = [0]$, $q = [1]$

$d(p,q) = 1$

$p = [6]$, $q = [1]$

$d(p,q) = 2\pi - 6 + 1 \approx 1.28$

<u>Prop</u>: For $p, q \in [0, 2\pi]$ have $d^g(p,q) = \min(|p-q|, 2\pi - |p-q|)$.

<u>Def</u>: g Riem metric $U \subset \mathbb{R}$,
h Riem metric $V \subset \mathbb{R}^k$, then
$F: U \to V$ <u>metric isometry</u> if $d^h(F(p), F(q)) = d^g(p,q)$
for all $p, q \in U$.