SCHEDULING

FLORIAN DAHMS*, ANNA-LENA FRANK†, SEBASTIAN KÜHN†, AND DANIEL PÖHLE†

Abstract. In this article, we discuss the optimization of a linearized traffic flow network model based on conservation laws. We present two solution approaches. One relies on the classical Lagrangian formalism (or adjoint calculus), whereas another one uses a discrete mixed-integer framework. We show how both approaches are related to each other theoretically. Numerical experiments are accompanied to show the quality of solutions.

Key words. traffic networks, network optimization

AMS subject classifications. 90B20, 90C11

1. Introduction.

Modeling, simulation and optimization of traffic flow networks based on partial differential equations have been investigated intensively during the last years, see for instance for an overview [4, 11, 15, 16, 17, 19, 21].

For optimization purposes, different applications such as optimal routing of traffic at intersections [11, 15, 16], traffic light control [12], evacuation planning [13, 14] and air traffic control [1] are of interest. Since in all problems the underlying optimization problem is restricted by nonlinear partial differential equations, relaxed models with simplified dynamics have been investigated. In this context, two different solution approaches emerge. On the one hand, nonlinear continuous optimization techniques have been successfully applied. To compute the optimum, the first order optimality system is derived and solved by a descent type method, see [18, 22, 25]. On the other hand, suitable discretization of the dynamics leads to linear network flow models that have been widely considered in the field of combinatorial optimization [2, 8, 9, 20].

In fact, there exist a few research results comparing both optimization tools, see [5, 7, 10, 11, 26]. One might assume that a one-to-one relation between nonlinear continuous and discrete optimization techniques holds true. This is especially the case when the governing dynamics in the network are linear. Then, appropriate numerical discretizations can be chosen, such that the original optimization problem either leads to numerically solving the finite-dimensional optimality system, i.e. the so-called *discretize-then-optimize approach*, or the interpretation as a mixed-integer programming model (MIP). However, it remains the question of detecting local or global optima. We know from the theory of linear programming, in particular the strong duality theorem [3, 23, 24], that under certain circumstances a global optimum can be reached. This is usually not the case for the adjoint calculus. The solution of the optimality system via gradient methods often gets stuck in local optima.

In this work we want to close the gap between the two solution procedures that first appear to be quite varied. To do so, we start with the traffic flow network model where the evolution of traffic density on roads is governed by the linearized Lighthill-Whitham-Richards (LWR) equations, see [6]. For the coupling conditions at the intersections, we stick to the ones presented by Coclite-Garavello-Piccoli [4] but do not use a predefined distribution matrix since these parameters will be obtained as solutions of the optimization problem. In a next step, we discretize the full network model in space and time and formally derive the adjoint equations and the

*???, ???, Frankfurt, Germany (dahms@???.de)

†I.NMF 32, DB Netz AG, Frankfurt am Main, Germany (Anna-Lena.Frank@deutschebahn.com, Sebastian.Se.Kuehn@deutschebahn.com, Daniel.Poehle@deutschebahn.com)

mixed-integer model (MIP) as well. We show the equivalence of both approaches by comparing the dual variables of the MIP and the adjoint variables. A formal, detailed proof for a simple network is given. Finally, we compare the optimization results of the MIP and the adjoint equations for different networks and objective functions numerically.

The paper is organized as follows: Section 2 contains a description of the network and the models used for describing flow on roads. Section 3 introduces the optimization problem to be considered. Herein, we also present the main theorem of this article, which we prove in the following sections. In section **??**, the adjoint equations for the former optimization problem are derived. Thereafter, the mixed-integer model is introduced in section 4. In section **??**, we conclude the proof of our main theorem. Numerical examples are given in section 5.

2. Traffic Flow Network Model.

3. Optimization Problem.

4. Mixed Integer Program (MIP).

5. Numerics.

Acknowledgements. This work was financially supported by BMI, Project DigiKap, FKZ ???.

REFERENCES

[1] A. M. Bayen, R. L. Raffard, and C. J. Tomlin, *Adjoint-based control of a new eulerian network model of air traffic flow*, IEEE Transactions on Control Systems Technology, 14 (2006), pp. 804–818.

[2] M. Carey and E. Subrahmanian, *An approach to modelling time-varying flows on congested networks*, Transportation Research Part B: Methodological, 34 (2000), pp. 157–183.

[3] V. Chvátal, *Linear Programming*, W. H. Freeman and Company, New York, 1983.

[4] G. M. Coclite, M. Garavello, and B. Piccoli, *Traffic flow on a road network*, SIAM Journal on Mathematical Analysis, 36 (2005), pp. 1862–1886.

[5] C. D'Apice, S. Göttlich, M. Herty, and B. Piccoli, *Modeling, Simulation and Optimization of Supply Chains: A Continuous Approach*, SIAM book series on Mathematical Modeling and Computation, 2010.

[6] C. Dapice, R. Manzo, and B. Piccoli, *Packet flow on telecommunication networks*, SIAM Journal on Mathematical Analysis, 38 (2006), pp. 717–740.

[7] P. Domschke, B. Geissler, O. Kolb, J. Lang, A. Martin, and A. Morsi, *Combination of nonlinear and linear optimization of transient gas networks*, INFORMS Journal on Computing, 23 (Fall 2011), pp. 605–617.

[8] L. Fleischer and E. Tardos, *Efficient continuous-time dynamic network flow algorithms*, Operations Research Letters, 23 (1998), pp. 71–80.

[9] L. R. Ford and D. R. Fulkerson, *Constructing maximal dynamic flows from static flows*, Operations Research, 6 (1958), pp. 419–433.

[10] A. Fügenschuh, B. Geissler, A. Martin, and A. Morsi, *The transport pde and mixed-integer linear programming*, in Models and Algorithms for Optimization in Logistics, Cynthia Barnhart, Uwe Clausen, Ulrich Lauther, and Rolf H. Möhring, eds., no. 09261 in Dagstuhl Seminar Proceedings, Dagstuhl, Germany, 2009, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany.

[11] A. Fügenschuh, M. Herty, A. Klar, and A. Martin, *Combinatorial and continuous models for the optimization of traffic flows on networks*, SIAM Journal on Optimization, 16 (2006), pp. 1155–1176.

[12] S. Göttlich, M. Herty, and U. Ziegler, *Modeling and optimizing traffic light settings on road networks*, preprint, (2012).

[13] S. Göttlich, S. Kühn, J.P. Ohst, S. Ruzika, and M. Thiemann, *Evacuation dynamics influenced by spreading hazardous material*, Networks and Heterogeneous Media (NHM), 6 (2011), pp. 443–464.

[14] H. W. Hamacher and S. A. Tjandra, *Mathematical modelling of evacuation problems: A state of the art*, Berichte des Fraunhofer ITWM, 24 (2001), pp. 1–38.

[15] M. Herty and A. Klar, *Modeling, simulation, and optimization of traffic flow networks*, SIAM Journal on Scientific Computing, 25 (2003), pp. 1066–1087.

[16] ———, *Simplified dynamics and optimization of large scale traffic networks*, Mathematical Models and Methods in Applied Sciences (M3AS), 14 (2004), pp. 579–601.

[17] H. Holden and N. H. Risebro, *A mathematical model of traffic flow on a network of unidirectional roads*, SIAM Journal on Mathematical Analysis, 26 (1995), pp. 999–1017.

[18] C. T. Kelley, *Iterative methods for optimization*, Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[19] A. Klar, R.D. Kühne, and R. Wegener, *Mathematical models for vehicular traffic*, Surveys on Mathematics for Industry, 6 (1996), pp. 215–239.

[20] E. Köhler and M. Skutella, *Flows over Time with Load-Dependent Transit Times*, SIAM Journal on Optimization, 15 (2005), p. 1185.

[21] M. J. Lighthill and G. B. Whitham, *On Kinematic Waves. II. A Theory of Traffic Flow on Long Crowded Roads*, Royal Society of London Proceedings Series A, 229 (1955), pp. 317–345.

[22] S. G. Nash and A. Sofer, *Linear and Nonlinear Programming*, McGraw-Hill, New York, 1996.

[23] G. L. Nemhauser and L. A. Wolsey, *Integer and combinatorial optimization*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, 1999.

[24] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, Chichester, 1986.

[25] P. Spellucci, *Numerische Verfahren der Nichtlinearen Optimierung*, Birkhäuser-Verlag, Basel, 1993.

[26] D. Sun, I. S. Strub, and A. M. Bayen, *Comparison of the performance of four eulerian network flow models for strategic air traffic management*, Networks and Heterogeneous Media, 2 (2007), p. 569.