Conditions for eigenvalue configurations of two real symmetric matrices

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Problem

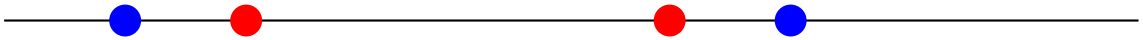
Definition (Eigenvalue Configuration)

Let $F \in \mathbb{R}^{m \times m}$, $G \in \mathbb{R}^{n \times n}$ be symmetric with distinct eigenvalues. Then their **eigenvalue configuration**, which we write as $EC(F, G)$, is the "relative locations" of the eigenvalues.

Example 1: Let $F = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ and $G = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$.

$EC(F, G) =$  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example 2: Let $F = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ and $G = \begin{bmatrix} 0 & -1 \\ -1 & 4 \end{bmatrix}$

$EC(F, G) =$  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

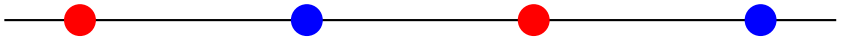
Problem

Problem

In : C, a desired eigenvalue configuration

Out: Condition on $F = [a_{ij}]$ and $G = [b_{ij}]$ such that $EC(F, G) = C$

Example:

In : $C =$  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Out: $\left(\begin{array}{cll} & D_{11} & < 0 \\ \wedge & D_{12} & > 0 \\ \wedge & D_{21} & > 0 \\ \wedge & D_{22} & < 0 \\ \wedge & D_{31} & > 0 \\ \wedge & D_{32} & < 0 \end{array}\right) \vee \left(\begin{array}{cll} & D_{11} & > 0 \\ \wedge & D_{12} & < 0 \\ \wedge & D_{21} & < 0 \\ \wedge & D_{22} & > 0 \\ \wedge & D_{31} & > 0 \\ \wedge & D_{32} & < 0 \end{array}\right) \vee \cdots$ where

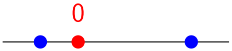
$$\begin{aligned} D_{11} &= 2(a_{11} + a_{22}) - 2(b_{11} + b_{22}) \\ D_{12} &= (a_{11} + a_{22})^2 - 2\,(a_{11} + a_{22})\,(b_{11} + b_{22}) + 4\,\left(b_{11}b_{22} - {b_{12}}^2\right) \\ &\vdots \end{aligned}$$

Problem: Background and Motivation

Generalization of Descartes' rule of signs: ([1, 2])

$$F = [0] \in \mathbb{R}^{1\times 1} \qquad \text{and} \qquad G = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

$$g(x) = \det(xI - G) = (x+1)(x-3)$$



Descartes' rule of signs:

$$\begin{aligned} g(x) &= \underbrace{x^2}_{+}\underbrace{-2x}_{-}\underbrace{-3}_{-} \\ \Longrightarrow v(+,-,-) &= 1 \\ &= \#\text{ positive roots of } g \\ &= EC(F,G). \end{aligned}$$

Applications in Science and Engineering: Many nontrivial problems in science/engineering can be reduced to this problem.

Example (network analysis [3]):

- Let F be the adjacency matrix of a graph.
- Some properties are related to the eigenvalues of F.
- How do eigenvalues change when edges are added or removed?
- Changing edges corresponds to adding a matrix U to F
- Hence, one might be interested in the eigenvalue configuration of

$$F \qquad \text{and} \qquad G = F + U.$$

Problem

Non-triviality

Why is this problem nontrivial?

Nontrivial specialized quantifier elimination problem.

Problem (Rephrased)

In : $\exists_{\alpha_1,\ldots,\alpha_m,\ \beta_1,\ldots,\beta_n} \Phi_C(\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_n,a_{ij},b_{ij})$

Out: $\Psi_C(a_{ij},b_{ij})$ equivalent to the above.

- Tarski [4] gave first algorithm for quantifier elimination of general formulas over real closed fields.
- Improvements: Collins, Hong, McCallum, Grigorev, Roy, Renegar, Canny, Brown, Strzebonski, Safey-Eldin, ..., Chen [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]

Q: Why not just use existing algorithms for quantifier elimination?

A: General algorithms not good for specialized input.

Challenge: Develop an algorithm that exploits the problem structure.

Contents

Signature Result: Setup

- Let $F = [a_{ij}] \in \mathbb{R}^{m \times m}$ and $G = [b_{ij}] \in \mathbb{R}^{n \times n}$ be symmetric with distinct (non-shared) eigenvalues.
- Label the eigenvalues $\alpha_1 \leq \cdots \leq \alpha_m$ and $\beta_1 \leq \cdots \leq \beta_n$, respectively.
- Let f denote the characteristic polynomial of F.
- We will assume that $f^{(k)}(\beta_j) \neq 0$ for all $1 \leq k \leq m - 1$.

  This is for the sake of simple presentation, but not needed logically.

Signature Result: Theorem Statement

Theorem (Signature-based Method)

We have $C = EC(F, G) \iff \operatorname{sign} D \in \Gamma_C$, where

1. $D \in \mathbb{Z}[a_{ij}, b_{ij}]^{2^m}$ is such that

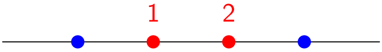
$$
\begin{aligned}
D_e &= \operatorname{coeffs} \det(xI_n - f_e(G)) && e \in \{0,1\}^m \\
f_e &= f^{(0)^{e_0}} \cdots f^{(m-1)^{e_{m-1}}} && f^{(k)} = k\text{-th derivative} \\
f &= \det(xI_m - F).
\end{aligned}
$$

2. $\Gamma_C = \{S \in (\{-,0,+\}^n)^{2^m-1} : VH^{-1}\sigma(S) = C\}$, where

$$
\begin{aligned}
\sigma(S) &= 2v(S) - n && v = \text{ sign variation count} \\
V_{t,s} &= \mathbb{1}_{v(s,+)=m-t} && t \in \{1,\ldots,m\},\ s \in \{-,+\}^m \\
H_{e,s} &= s_1^{e_1} \cdots s_m{}^{e_m} && e \in \{0,1\}^m, \qquad s \in \{-,+\}^m
\end{aligned}
$$

Signature Result: Derivation Sketch $(m = n = 2)$ (i)

Sketch of Derivation: (with full details: ≈24 pages [22])



$$C = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C_1 = 0}$$

$$0 = C_1$$

$$\Updownarrow$$

$$0 = \# \bullet \in (\bullet_1, \bullet_2)$$ by definition of EC

$$\Updownarrow$$

$$0 = \#\{\bullet : \#\{\bullet : \bullet > \bullet\} = 2 - 1\}$$

$$\Updownarrow$$

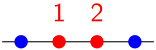
$$0 = \#\{j : \#\{i : \alpha_i > \beta_j\} = 2 - 1\}$$

$$\Updownarrow$$

$$0 = \#\{j : \#\{x : f(x) = 0 \wedge x > \beta_j\} = 2 - 1\}$$

Signature Result: Derivation Sketch $(m = n = 2)$ (ii)

$$0 = C_1$$



$$\Updownarrow$$

$$0 = \#\{j : \underbrace{\#\{x : f(x) = 0 \land x > \beta_j\}}_{\text{count using Descartes' rule of signs!}} = 2 - 1\}$$

$$\Updownarrow$$

$$0 = \#\{j : v(\underbrace{f^{(0)}(\beta_j), f^{(1)}(\beta_j), f^{(2)}(\beta_j)}_{\text{coefficients of Taylor expansion of } f(x+\beta_j)}) = 2 - 1\}$$

In general: $C_t = \#\{j : v(f^{(0)}(\beta_j), \ldots, f^{(m-1)}(\beta_j), +) = m - t\}$

Thus we have eliminated α's.

Signature Result: Derivation Sketch $(m = n = 2)$ (iii)

Q: How do we count $C_t = \#\{j : v(f^{(0)}(\beta_j), f^{(1)}(\beta_j), +) = 2 - t\}$?

A: For each sign vector $s \in \{-, +\}^2$, count the quantity

$$\#\{j : \operatorname{sign}(f^{(0)}(\beta_j), f^{(1)}(\beta_j)) = s\}$$

and group them together based on the sign variation count of s.

We get:

$$C_t = \sum_{\substack{s \in \{-,+\}^2 \\ v(s,+) = 2 - t}} \#\{j : \operatorname{sign}(f^{(0)}(\beta_j), f^{(1)}(\beta_j)) = s\}$$

In vector form:

$$C = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{V} \overbrace{\begin{bmatrix} \#\{j : \operatorname{sign}(f^{(0)}(\beta_j), f^{(1)}(\beta_j)) = (--)\} \\ \#\{j : \operatorname{sign}(f^{(0)}(\beta_j), f^{(1)}(\beta_j)) = (-+)\} \\ \#\{j : \operatorname{sign}(f^{(0)}(\beta_j), f^{(1)}(\beta_j)) = (+-)\} \\ \#\{j : \operatorname{sign}(f^{(0)}(\beta_j), f^{(1)}(\beta_j)) = (++)\} \end{bmatrix}}^{q}$$

Signature Result: Derivation Sketch $(m = n = 2)$ (iv)

Q: How do we compute the following for a given sign vector s?

$$\#\{j : \mathrm{sign}(f^{(0)}(\beta_j), f^{(1)}(\beta_j)) = s\}$$

that is, how to count roots of g subject to a sign condition on $f^{(0)}, f^{(1)}$.

A: Use Tarski [4] and Ben-Or, Kozen, and Reif ([6]):
to count these in terms of the signature of related matrices.

$$q = \overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}^{-1}}^{H^{-1}} \begin{bmatrix} \mathrm{sig}\left(f^{(0)}\right)^0 \left(f^{(0)}\right)^0 (G) \\ \mathrm{sig}\left(f^{(0)}\right)^0 \left(f^{(1)}\right)^1 (G) \\ \mathrm{sig}\left(f^{(0)}\right)^1 \left(f^{(1)}\right)^0 (G) \\ \mathrm{sig}\left(f^{(0)}\right)^1 \left(f^{(1)}\right)^1 (G) \end{bmatrix}$$

where sig is the *signature* of the matrix:
($\#$ positive eigenvalues - $\#$ negative eigenvalues).

Signature Result: Derivation Sketch $(m = n = 2)$ (v)

Q: How do we compute the signature of the following matrix?

$$f_e(G) := \left(f^{(0)}\right)^{e_0} \left(f^{(1)}\right)^{e_1} (G)$$

A: Use Descartes' rule of signs:

$$\operatorname{sig} f_e(G) = 2v(\text{coeffs}\,\text{charpoly}\, f_e(G)) - n$$

Hence:

$$C = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = V H^{-1} \begin{bmatrix} 2v(\text{coeffs}\,\text{charpoly}\, f_{00}(G)) - n \\ 2v(\text{coeffs}\,\text{charpoly}\, f_{01}(G)) - n \\ 2v(\text{coeffs}\,\text{charpoly}\, f_{10}(G)) - n \\ 2v(\text{coeffs}\,\text{charpoly}\, f_{11}(G)) - n \end{bmatrix}.$$

Notice that the coefficients of the characteristic polynomial of $f_e(G)$ involve only the entries of F and G.

So, finally we have eliminated all α's and β's.

We are DONE!

Signature Result: Theorem Statement

Theorem (Signature-based Method)

We have $C = EC(F, G) \iff \operatorname{sign} D \in \Gamma_C$, where

1. $D \in \mathbb{Z}[a_{ij}, b_{ij}]^{2^m}$ is such that

$$
\begin{aligned}
D_e &= \operatorname{coeffs}\det(xI_n - f_e(G)) & \quad e &\in \{0,1\}^m \\
f_e &= f^{(0)^{e_0}} \cdots f^{(m-1)^{e_{m-1}}} & \quad f^{(k)} &= k\text{-th derivative} \\
f &= \det(xI_m - F).
\end{aligned}
$$

2. $\Gamma_C = \{S \in (\{-,0,+\}^n)^{2^m-1} : VH^{-1}\sigma(S) = C\}$, where

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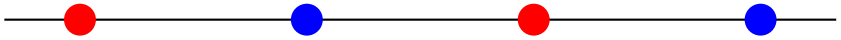
\begin{aligned}
\sigma(S) &= 2v(S) - n & \quad v &= \text{ sign variation count} \\
V_{t,s} &= \mathbb{1}_{v(s,+)=m-t} & \quad t &\in \{1,\ldots,m\},\ s \in \{-,+\}^m \\
H_{e,s} &= s_1^{e_1} \cdots s_m{}^{e_m} & \quad e &\in \{0,1\}^m, \qquad s \in \{-,+\}^m
\end{aligned}
$$

Signature Result: Example

In : $C =$  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Out: $\operatorname{sign} D \in \Gamma_C$ where

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix} \qquad \Gamma_C = \left\{ \begin{bmatrix} - & + \\ + & - \\ + & - \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \\ + & - \end{bmatrix}, \ldots \right\}$$

$$\begin{aligned} D_{11} &= 2(a_{11} + a_{22}) - 2(b_{11} + b_{22}) \\ D_{12} &= (a_{11} + a_{22})^2 - 2\,(a_{11} + a_{22})\,(b_{11} + b_{22}) + 4\left(b_{11} b_{22} - {b_{12}}^2\right) \\ &\vdots \end{aligned}$$

Contents

Symmetry Result: Setup

- Let $F = [a_{ij}] \in \mathbb{R}^{m \times m}$ and $G = [b_{ij}] \in \mathbb{R}^{n \times n}$ be symmetric with distinct (non-shared) eigenvalues.
- Label the eigenvalues $\alpha_1 \leq \cdots \leq \alpha_m$ and $\beta_1 \leq \cdots \leq \beta_n$, respectively.
- Let f denote the characteristic polynomial of F.
- We will assume that $f^{(k)}(\beta_j) \neq 0$ for all $1 \leq k \leq m - 1$.

  This is for the sake of simple presentation, but not needed logically.

Symmetry Result: Theorem Statement

Theorem (Symmetry-based Method)

We have $C = EC(F, G) \iff \operatorname{sign} D \in \Gamma_C$, where

1. $D \in \mathbb{R}[a_{ij}, b_{ij}]^m$ such that

$$D_r = \text{coeffs} \prod_{(i_1,\ldots,i_r,j)\in Y} \left(x + \prod_{p=1}^{r} (\alpha_{i_p} - \beta_j) \right)$$

2. $\Gamma_C = \{ S \in \left(\{-, 0, +\}^{\binom{m}{r} n} \right)^m : T^{-1} v(S) = C \}$ where

$$v(S) = \text{ sign variation count of } S$$

$$T_{rs} = \sum_{1 \le t \le m} \binom{m-t}{r-t} (-2)^{r-1} \binom{s}{t}.$$

Symmetry Result: Derivation Sketch $(m = n = 2)$ (i)

Sketch of Derivation: (with full details: ≈9 pages [23])

1. Consider the set

$$Y = \Big\{\text{nonempty subsets of } [m]\Big\} \times [n].$$

$$Y = \underbrace{\Big\{\text{nonempty subsets of } [m]\Big\}}_{\alpha' s} \times \underbrace{[n]}_{\beta}.$$

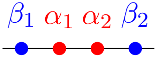
$$Y = \left\{ \begin{array}{l} \{1\} \\ \{2\} \\ \{1,2\} \end{array} \right\} \times \{1,2\} = \left\{ \begin{array}{ll} (1,1) & (12,1) \\ (1,2) & (12,2) \\ (2,1) & \\ (2,2) & \end{array} \right\}$$

2. Consider products formed from the elements of Y as follows:

$$(i_1, \ldots, i_r, j) \mapsto \prod_{p=1}^{r} \left(\alpha_{i_p} - \beta_j\right).$$

Symmetry Result: Derivation Sketch $(m = n = 2)$ (ii)

3. Observe the signs of those products.

Let $C =$ 

.

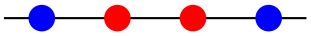
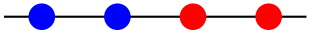
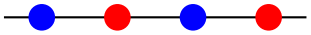
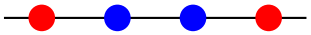
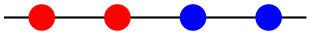
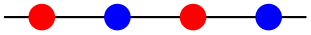
Element of Y	r	$\prod_{p=1}^{r} \left(\alpha_{i_p} - \beta_j\right)$	$\operatorname{sign} \prod_{p=1}^{r} \left(\alpha_{i_p} - \beta_j\right)$
$(1, 1)$	1	$(\alpha_1 - \beta_1)$	$+$
$(1, 2)$	1	$(\alpha_1 - \beta_2)$	$-$
$(2, 1)$	1	$(\alpha_2 - \beta_1)$	$+$
$(2, 2)$	1	$(\alpha_2 - \beta_2)$	$-$
$(12, 1)$	2	$(\alpha_1 - \beta_1)(\alpha_2 - \beta_1)$	$+$
$(12, 2)$	2	$(\alpha_1 - \beta_2)(\alpha_2 - \beta_2)$	$+$

Now, for each r, count how many $-$'s appear in the last column.

$$\begin{array}{c|c} r & \#- \\ \hline 1 & 2 \\ 2 & 0 \end{array} \implies y := \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Symmetry Result: Derivation Sketch $(m = n = 2)$ (iii)

Claim: Eigenvalue configurations and y vectors are in 1-1 correspondence.

C		y
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$		$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$		$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$		$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$		$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$		$\begin{bmatrix} 4 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$		$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Symmetry Result: Derivation Sketch $(m = n = 2)$ (iv)

In fact: $y = TC$ where $T \in \mathbb{N}^{m \times m}$ such that

$$T_{rs} = \sum_{1 \le t \le m} \binom{m-t}{r-t} (-2)^{r-1} \binom{s}{t}.$$

But who cares? Recall our goal is to write an equivalent condition to $EC(F, G) = C$ without referencing any eigenvalues.

Recall that

$$y_r = \# \left\{ (i_1, \ldots, i_r, j) \in Y : \prod_{p=1}^{r} \left(\alpha_{i_p} - \beta_j \right) < 0 \right\}.$$

This is exactly the number of negative roots of the polynomial

$$h_r := \prod_{(i_1, \ldots, i_r, j) \in Y} \left(x - \prod_{p=1}^{r} \left(\alpha_{i_p} - \beta_j \right) \right).$$

This polynomial is symmetric in α and β.

Symmetry Result: Derivation Sketch $(m = n = 2)$ (v)

1. Recall that y_r equals the number of negative roots of $h_r := \prod_{(i_1,\ldots,i_r,j)\in Y} \left(x - \prod_{p=1}^{r} \left(\alpha_{i_p} - \beta_j\right)\right)$.
2. Note that the above polynomial is symmetric in α and β.
3. The Fundamental Theorem of Symmetric Polynomials applies.
4. Thus, we express h_r in terms of the entries a_{ij} and b_{ij} of F and G.
5. Finally, use Descartes' rule of signs to count the negative roots of h_r.

We are DONE!

Symmetry Result: Theorem Statement

Theorem (Symmetry-based Method)

We have $C = EC(F, G) \iff \operatorname{sign} D \in \Gamma_C$, where

1. $D \in \mathbb{R}[a_{ij}, b_{ij}]^m$ such that

$$D_r = \text{coeffs} \prod_{(i_1,\ldots,i_r,j)\in Y} \left(x + \prod_{p=1}^{r} (\alpha_{i_p} - \beta_j) \right)$$

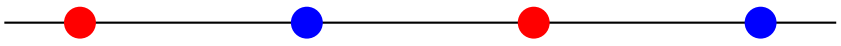
2. $\Gamma_C = \{ S \in \left(\{-, 0, +\}^{\binom{m}{r} n} \right)^m : T^{-1} v(S) = C \}$ where

$$v(S) = \text{ sign variation count of } S$$

$$T_{rs} = \sum_{1 \le t \le m} \begin{pmatrix} m - t \\ r - t \end{pmatrix} (-2)^{r-1} \begin{pmatrix} s \\ t \end{pmatrix}.$$

Remark: D are symmetric in α and β. From FTSP, they can be expressed in terms of the entries of the matrices F and G.

Symmetry Result: Example

In : $C =$  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Out: $\operatorname{sign} D \in \Gamma_C$ where

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & & \end{bmatrix}$$

$$\begin{aligned} D_{11} &= 2(a_{11} + a_{22}) - 2(b_{11} + b_{22}) \\ D_{12} &= (a_{11} + a_{22})^2 + (b_{11} + b_{22})^2 + 2(a_{11}a_{22} - a_{12}^2) \\ &\quad + 2(b_{11}b_{22} - b_{12}^2) - 3(a_{11} + a_{22})(b_{11}b_{22}) \\ &\vdots \end{aligned}$$

$$\Gamma_C = \left\{ \begin{bmatrix} + & - & + & - \\ - & + & & \end{bmatrix}, \begin{bmatrix} - & + & - & - \\ + & - & & \end{bmatrix}, \ldots \right\}$$

Contents

Comparison

Size of D: total number of coefficients of D

m	Signature	Symmetry	Descartes
arbitrary	$(2^m - 1) \cdot n$	$(2^m - 1) \cdot n$	N/A
1	n	n	n

1. Size of D for each approach is the same: $(2^m - 1) \cdot n$
2. The above two methods are asymmetric with regard to the sizes of F and G.
   - If size of F is fixed, size is linear!
   - Choose F to be smaller matrix

Contents

Future Work

Short-term:

- Allow F and G to share eigenvalues
- Find applications
- Prune redundancies from output

Long-term:

- Generalize to > 2 matrices
- Generalize to tensors

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Thank you!