Assignment H1 - Solution

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1 Problem 1 (Discrete Convolution)

Solution:

$$
\begin{aligned}
(f * f)_i &= \sum_{k=-\infty}^{+\infty} f_{i-k} \cdot f_k \\
&= f_{i-0} \cdot f_0 + f_{i-1} \cdot f_1 \\
&= \frac{1}{2} f_i + \frac{1}{2} f_{i-1}
\end{aligned}
$$

$$
\begin{aligned}
(f * (f * f)_i)_i &= \sum_{k=-\infty}^{+\infty} f_{i-k} \cdot (\frac{1}{2} f_k + \frac{1}{2} f_{k-1}) \\
&= f_{i-0} \cdot \frac{1}{2} f_0 + f_{i-1} \cdot (\frac{1}{2} f_1 + \frac{1}{2} f_0) + \frac{1}{2} f_{i-2} \cdot \frac{1}{2} f_1 \\
&= \frac{1}{4} f_i + \frac{1}{2} f_{i-1} + \frac{1}{4} f_{i-2}
\end{aligned}
$$

2 Problem 2 (Properties of the Convolution)

Solution:

a) Linearity:

$$
\begin{aligned}
(\alpha \cdot f + \beta \cdot g) * \omega &= \sum_{k=-\infty}^{+\infty} (\alpha \cdot f_{i-k} + \beta \cdot g_{i-k}) \cdot \omega_k \\
&= \alpha \cdot \sum_{k=-\infty}^{+\infty} f_{i-k} \cdot \omega_k + \beta \cdot \sum_{k=-\infty}^{+\infty} g_{i-k} \cdot \omega_k \\
&= \alpha \cdot (f * \omega) + \beta \cdot (g * \omega) \qquad for\ all\ \alpha, \beta \in \mathbb{R}
\end{aligned}
$$

b) Commutativity:

$$
f * \omega = \sum_{k=-\infty}^{+\infty} f_{i-k} \cdot \omega_k = \sum_{k=-\infty}^{+\infty} \omega_k \cdot f_{i-k}
$$

define $\quad n := i - k \qquad n \in \mathbb{Z}$

$$
\begin{aligned}
\sum_{k=-\infty}^{+\infty} \omega_k \cdot f_{i-k} &= \sum_{i-n=-\infty}^{+\infty} \omega_{i-n} \cdot f_n \\
&= \sum_{n=i+\infty}^{i-\infty} \omega_{i-n} \cdot f_n \\
&= \sum_{n=-\infty}^{+\infty} \omega_{i-n} \cdot f_n \\
&= \omega * f
\end{aligned}
$$

c) Identity:

$$
f * e = \sum_{k=-\infty}^{+\infty} f_{i-k} \cdot e_k = f_i
$$

$$
e_i = \begin{cases} 1 & (i = 0) \\ 0 & (\text{else}) \end{cases}
$$