Q6

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Q1 5

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Q1. (5 points) Determine whether each statement is true or false. Circle the correct answers. No justification is needed.

(a) For all $A, B \in M_{2\times 2}(\mathbb{R})$, $\det(A+B) = \det A + \det B$.

Circle the correct answer: True False

(b) There exists $A \in M_{2\times 2}(\mathbb{R})$ such that $A^2 = A$ and $\det A = 2$.

Circle the correct answer: True False

det A det A = det A

(c) For all $\mathbb{S} \subseteq \mathbb{R}^3$ and all $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{S}$, if $\mathbb{S}$ is a 2-dimensional subspace of $\mathbb{R}^3$ and $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are all distinct, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set.

Circle the correct answer: True False

(d) For all $\mathbb{S}, \mathbb{T} \subseteq \mathbb{R}^3$, if $\mathbb{S}$ and $\mathbb{T}$ are 2-dimensional subspaces of $\mathbb{R}^3$, then $\mathbb{S} \cap \mathbb{T} \neq \{\vec{0}\}$.

Circle the correct answer: True False

(e) For all $\mathbb{S}, \mathbb{T} \subseteq \mathbb{R}^3$, if $\mathbb{S}$ is a 1-dimensional subspace of $\mathbb{R}^3$ and $\mathbb{T}$ is a 2-dimensional subspace of $\mathbb{R}^3$, then $\mathbb{S} \cup \mathbb{T} = \mathbb{R}^3$.

Circle the correct answer: True False

Q2 5

Q2. (5 points)

(a) Suppose that A and B are $n \times n$ matrices with $\det A = -2$ and $\det B = 3$. Compute $\det(A^2B^{-1}A^TB^3)$.

$= \det A \det A \cdot \frac{1}{\det B} \det A \cancel{\det B} \det B \det B$

$= (-2)(-2)(-2)(3)(3)$

$= (-8)(9)$

$= (-72)$

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Compute $(A^{-1})_{12}$.

Note A is triangular. $\det A = 1 \times 2 \times 3 \times 4 = 24$.

$C_{21} = (-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{vmatrix} = (-1)(1 \times 3 \times 4)$

$= -12$

$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$

$(A^{-1})_{12} = \frac{1}{24} \times C_{21}$

$= \frac{1}{24} \times -12$

$= \boxed{-\frac{1}{2}}$

(c) Determine all real numbers k for which the area of the parallelogram induced by the vectors $\vec{u} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ k \end{bmatrix}$ is equal to 2.

$\text{Area}(\vec{u}, \vec{v}) = \left| \det \begin{bmatrix} 5 & 3 \\ 1 & k \end{bmatrix} \right|$

$= |5k - 3|$

We want $|5k-3| = 2$.

Then,

$5k - 3 = 2$ $\qquad$ $5k - 3 = -2$

$5k = 5$ $\qquad$ $5k = 1$

$\boxed{k = 1}$ $\qquad$ $\boxed{k = \frac{1}{5}}$

Q3 5

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Q3. (5 points)

(a) Consider the 2-dimensional subspace of $\mathbb{R}^3$ defined as $\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 - 2x_2 + x_3 = 0 \right\}$.

(i) Determine a basis, $\mathcal{B}$, for $\mathbb{S}$ that includes the vector $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Let $\vec{x} \in S$. Then $\vec{x} = \begin{bmatrix} 2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. We know $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans S, and since $\vec{v} \in S$, $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans S. Consider the matrix $\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. (Scrap page) cont...

(ii) Determine a basis, $\mathcal{C}$, for $\mathbb{R}^3$ such that $\mathcal{B} \subseteq \mathcal{C}$ for your set $\mathcal{B}$ from part (i).

Consider $C = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Then $B \subseteq C$. Then,

$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus C is a linearly independent set with three vectors, which means C spans $\mathbb{R}^3$ and is a basis for $\mathbb{R}^3$.

(b) Determine all $c \in \mathbb{R}$ for which the following matrix is invertible.

$$A = \begin{bmatrix} 1 & c & c^5 \\ c & c^2 & c \\ c^5 & c & c^2 \end{bmatrix}$$

We need $\det A \neq 0$.

$\det A = \begin{vmatrix} 1 & c & c^5 \\ c & c^2 & c \\ c^5 & c & c^2 \end{vmatrix} = c^2 \begin{vmatrix} 1 & c & c^5 \\ 1 & c & 1 \\ c^4 & 1 & c^2 \end{vmatrix}$.

$\begin{vmatrix} 1 & c & c^5 \\ 1 & c & 1 \\ c^4 & 1 & c \end{vmatrix} = 1 \begin{vmatrix} c & 1 \\ 1 & c \end{vmatrix} (-1) \begin{vmatrix} c & c^5 \\ 1 & c \end{vmatrix} + c^4 \begin{vmatrix} c & c^5 \\ c & 1 \end{vmatrix}$ (cofactor expansion along 1st col)

$= (c^2 - 1) + (-1)(c^2) \begin{vmatrix} 1 & c^3 \\ 1 & 1 \end{vmatrix} + c^5 \begin{vmatrix} 1 & c^4 \\ c & 1 \end{vmatrix}$

$= (c^2 - 1) - c^2(1 - c^3) + c^5(1 - c^5)$

Hence

$\det A = c^2((c^2 - 1) - c^2(1 - c^3) + c^5(1 - c^5))$

We observe that $\det A = 0$ if $c = 0$ or $c = 1$. Thus, A is invertible if $c \neq 0, 1$, $c \in \mathbb{R}$.

Q4 5

Q4. (5 points)

(a) Let a be a non-zero real number and consider the linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(x_1, x_2, x_3) = (ax_1 + x_3, ax_2, -\frac{1}{a}x_1 + x_3)$.
Prove that $[L]$, the standard matrix of L, is an invertible matrix.

$[L] = \begin{bmatrix} a & 0 & 1 \\ 0 & a & 0 \\ -\frac{1}{a} & 0 & 1 \end{bmatrix}$ $\det[L] = a \begin{vmatrix} a & 1 \\ -\frac{1}{a} & 1 \end{vmatrix}$ (CE along second row)

$= a(a - -\frac{1}{a})$

$= a^2 + 1$

Hence, for all $a \neq 0$, $\det[L] \neq 0$, meaning by IMT, $[L]$ is invertible.

(b) Let $n \geq 2$ be an integer and $A \in M_{n\times n}(\mathbb{R})$. Suppose that the entries in each row of A sum to 0, that is, $\sum_{j=1}^{n} a_{ij} = 0$ for $i = 1, 2, \dots, n$. Prove that $\det A = 0$.

We prove by induction on n where $P(n)$ is $\forall A \in M_{n\times n}(\mathbb{R})$, $\det A = 0$.

Base case: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = ad - bc$. By the hypothesis, $a = -b$ and $c = -d$, hence $\det A = -bd - -bd = 0$.

Inductive Hyp: Assume $P(n-1)$

Inductive step: let $A \in M_{n\times n}(\mathbb{R})$. Performing cofactor expansion across the i-th row,

$\det A = a_{i1}C_{i1} + \dots + a_{in}C_{in}$.

Note each of C_{ij} is a product of (1) or (-1) with a matrix of size $n-1 \times n-1$. Thus, $\det A = 0$.

By POMI, $\det A = 0$ $\forall n \geq 2$.

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Q3 a1) Hence,

$B = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} \right\}$ also spans S. Since it is linearly dependent, it is a basis for S.

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