Titel der Arbeit

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vorgelegt von
Name
Matrikelnummer: 123456

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Prof. Dr. Dozent

1. Introduction

Contents of the thesis.

The Hodge Decomposition theorem for compact Kähler manifolds is a fundamental theorem of the Hodge Theory. It provides a decomposition of the de Rahm cohomology groups into suitable Dolbeault cohomology groups, thus yielding a connection between the topology and the complex structure of a compact Kähler manifold.

Theorem 1.0.1 (Hodge Decomposition)**.** *For a compact Kähler manifold X, there is a direct sum decomposition*

$$H^k_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(X, \mathbb{C}).$$

The primary objective of this thesis will be the elaboration of the proof of this fundamental theorem. In order to achieve this, we will have to introduce the needed theory first. We are going to start by presenting the consequences of the existence of an almost complex structure and a compatible euclidean inner product on a real vector space. For this purpose, we will mainly use the tools of Linear Algebra.

With this, we will be able to define the local versions of the *Hodge star operator* $*$ and the *Lefschetz* and dual *Lefschetz operators* L and Λ.

Afterwards, our focus is going to shift to complex manifolds and their different tangent bundles. Although it is assumed that the reader is already familiar with the definition and basic properties of complex manifolds, we will begin with the definition and also the elaboration of the properties of hermitian manifolds, which are the complex counterparts of Riemannian manifolds.

After we have used our local findings for the operators mentioned above to define similarly named global operators for hermitian manifolds, we will also introduce an L^2*-metric* that will be used to generalize the idea of adjoint operators to *formal adjoint opertors.* We will be particularly interested in the formal adjoint operators of the exterior derivative d and the Dolbeault operators ∂ and $\overline{\partial}$, which will be noted as d^*, ∂^* and $\overline{\partial}^*$.

Those formal adjoint operators will particularly interest us because they appear in the *Kähler identities.* These identities relate the Dolbeault operators and their formal adjoints to each other using the dual Lefschetz operator.

Theorem 1.0.2. *On a compact Kähler manifold, we have the identities*

$$[\Lambda, \overline{\partial}] = -i\partial^*, \quad [\Lambda, \partial] = i\overline{\partial}^*,$$

with the Lie bracket being defined as the commutator.

Next, we are going to introduce the theory of *harmonic differential forms.* In order to do so, we will define the *Laplacians* Δ_d, Δ_∂ and $\Delta_{\overline{\partial}}$ and work out their properties. We will then use the *Kähler identities* to prove the next important theorem.

Theorem 1.0.3. *For the Laplacians $\Delta_d, \Delta_\partial$ and $\Delta_{\bar{\partial}}$ on a compact Kähler manifold, we have the following relation*

$$\frac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\bar{\partial}}.$$

Since harmonic and $\Delta_{\bar{\partial}}$-harmonic forms will be defined as forms annihilated by Δ_d and $\Delta_{\bar{\partial}}$, respectively, this theorem shows that those two notions are equivalent for Kähler manifolds. Furthermore, we will use this theorem to establish the following corollary, which will be crucial for proving the *Hodge Decomposition* theorem.

Corollary 1.0.4. *For the compact Kähler manifold X, the complex harmonic differential k-forms $\mathcal{H}^k(X)$ decompose as*

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X),$$

with $\mathcal{H}^{p,q}(X)$ being the harmonic differential forms of type (p, q).

The final statements needed for our proof of the *Hodge Decomposition* theorem will be the *Hodge Isomorphism theorems*, which enable us to apply the findings of the harmonic forms theory to the de Rahm and Dolbeault cohomologies by providing two isomorphisms.

Theorem 1.0.5 (Hodge Isomorphism theorem I)**.** *The natural mapping*

$$\mathcal{H}^k(X) \to H^k_{dR}(X, \mathbb{C}), \quad \alpha \mapsto [\alpha]$$

is an isomorphism. In particular, any class of closed forms in $H^k_{dR}(X, \mathbb{C})$ has a unique harmonic representative.

Theorem 1.0.6 (Hodge Isomorphism theorem II)**.** *The natural mapping*

$$\mathcal{H}^{p,q}(X) \to H^{p,q}_{\bar{\partial}}(X, \mathbb{C}), \quad \alpha \mapsto [\alpha]$$

is an isomorphism. In particular, any class of $\bar{\partial}$-closed forms in $H^{p,q}_{\bar{\partial}}(X, \mathbb{C})$ has a unique harmonic representative.

After these two isomorphism theorems are proven, we already have the *Hodge Decomposition* given as an isomorphism and in order to get the above *Hodge Decomposition* theorem, we only need to prove the independence of the Kähler metric.

To conclude this thesis, we will then provide an application of the *Hodge Decomposition*. We are going to show that the *Hopf surfaces*, which are compact 2-dimensional complex manifolds, can not be equipped with a Kähler metric.

Remarks on the implementation.

For the *Hodge Decomposition*, several different proofs are already known. Therefore, the idea of this thesis is the collection and the coherent presentation of one of these possible proofs from the perspective of an undergraduate student who is already familiar with some of the basic concepts of complex and algebraic geometry.

In this context, this thesis broadly follows the proof presented in *Claire Voisin's* book *Hodge Theory and Complex Algebraic Geometry I*. However, other popular sources have also influenced this thesis. Therefore, we will reference similar or equal statements in this literature whenever possible. This is done to allow for the possibility of verification and to encourage the reader to engage more deeply with the content.

Also, since multiple different notation conventions exist in complex geometry, we will try to stick to the notation suggested by *Voisin* in her book. However, we will also provide the reader with explanations for the used notation throughout the thesis so that even the unfamiliar reader will be able to follow.

Conventions.

Throughout the thesis, we are always going to limit our discussion to differentiable manifolds without border. In order to have a Riemannian metric on every differentiable manifold, we are also only going to allow paracompact manifolds.

Additionally, we are going to adhere to the following meaning of the used symbols.

$\mathbb{N}$	Natural numbers, including 0
$\subset$	Not necessarily proper subset
$\subsetneq$	Proper subset
$\hookrightarrow$	Injection or monomorphism
$\twoheadrightarrow$	Surjection or epimorphism

Also, the uppercase letters J and K will usually denote multi-indices except for one instance, when J denotes an almost complex structure. Additionally, if there are uppercase letters in the index, they also denote multi-indices. For a basis vector v_j, we will write v^j for the dual basis vector.

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I want to thank all the nice people for helping me (...)

First name last name
Freiburg im Breisgau
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Contents

2. Local Theory

This chapter aims to provide the reader with some of the essential notions and tools for the local theory needed to form a comprehensive understanding of the concept of hermitian manifolds and in particular, Kähler manifolds. The main focus will be on complex vector spaces and hermitian forms on those spaces and also the tools linear algebra provides.

Later, we are going to focus on the different tangent bundles of complex manifolds, which are collections of vector spaces that vary in a geometric way on the manifold. Therefore, most of the notions and tools we are going to introduce in this chapter will be translated into the global context later.

The main goal of this chapter will be the definition of the local Hodge star operator $*$ and the definition of the local Lefschetz and dual Lefschetz operator L and Λ. For this purpose, we will start by providing a brief overview of the topic of complexification of vector spaces. Afterwards, we will focus on euclidean and hermitian vector spaces. Finally, we will conclude with the definition of the operators mentioned above.

The reader should be aware that this chapter is based on the similarly named chapter in Daniel Huybrechts' *Complex Geometry* [Huy04], although the level of detail found there is not to be expected.

2.1. **Complexification of vector spaces.**

To simplify the notation, we are going to assume the following setting for the remainder of this section.

Setting. Let V denote a real n-dimensional vector space. Also, assume that V is an almost complex vector space, i.e. V is equipped with an endomorphism $I : V \longrightarrow V$ such that $I^2 = -\operatorname{Id}$. This endomorphism is called the almost complex structure of V.

Using this almost complex structure, we can also think of V as a complex vector space with the $\mathbb{C}$-module structure defined as $(a+ib)\cdot v = av + bI(v)$ for all $a, b \in \mathbb{R}$. For this complex vector space, we will write (V, I). With the product rule for the determinant, we can calculate

$$\det(I)^2 = \det(I^2) = \det(-\operatorname{Id}) = (-1)^n.$$

Since $\det(I)$ is real, we conclude that $n = 2m$ for some $m \in \mathbb{N}$.

Furthermore, V and (V, I) are equal as sets and if $(v_1, \dots, v_d)$ is a complex basis of (V, I), it is immediate that $(v_1, I(v_1), \dots, v_d, I(v_d))$ is a real basis of V. Therefore, their dimensions relate as

$$\dim_{\mathbb{C}}(V, I) = d = \frac{1}{2}\dim_{\mathbb{R}} V = \frac{1}{2}n = m.$$

Additionally, as an almost complex vector space, V is endowed with a natural orientation. This boils down to the fact that the real space $\mathbb{C}^m$ has a natural orientation given by the basis $(e_1, ie_1, \dots e_m, ie_m)$, with the $e_1, \dots, e_m$ being the standard basis vectors (cf. [Huy04, Corollary 1.2.3]).

At the same time, it is possible to construct a different complex vector space using V.

Definition 2.1.1. The *complexification* $V_{\mathbb{C}}$ of V is defined as $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$.

Let $(v_1, \dots, v_n)$ be a real basis of V. With the properties of the tensor product, it is $(v_1 \otimes 1, \dots, v_n \otimes 1)$ a complex basis of $V_{\mathbb{C}}$. This shows that there exists an inclusion $V \hookrightarrow V_{\mathbb{C}}$ and for the dimension of $V_{\mathbb{C}}$, we get

$$\dim_{\mathbb{C}} V_{\mathbb{C}} = n = \dim_{\mathbb{R}} V.$$

Now, the almost complex structure I can be linearly extended to an almost complex structure $I_{\mathbb{C}}$ on $V_{\mathbb{C}}$. This is defined as $I_{\mathbb{C}}(v \otimes 1 + w \otimes i) := I(v) \otimes 1 + I(w) \otimes i$, and it is evident that this linear extension also has the property $I_{\mathbb{C}}^2 = -\operatorname{Id}$. Thus, we also call this to be an almost complex structure.

Notation 2.1.2. Note that for a vector $v \otimes \lambda \in V_{\mathbb{C}}$, it is a common practice to sometimes omit the tensor product in the notation, just noting λv instead of $v \otimes \lambda$. If it is possible without confusion, we will also write I instead of $I_{\mathbb{C}}$ for the complex extension of the almost complex structure.

The following proposition shows how the two $\mathbb{C}$-module structures on $V_{\mathbb{C}}$, defined by the almost complex structure I and by multiplication with i, compare to each other.

Proposition 2.1.3 (Decomposition of $V_{\mathbb{C}}$ [Huy04, Lemma 1.2.5])**.** *For the complexification $V_{\mathbb{C}}$ we have the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ with*

$$V^{1,0} := \{v \in V_{\mathbb{C}} \mid I(v) = iv\} \quad \textit{and} \quad V^{0,1} := \{v \in V_{\mathbb{C}} \mid I(v) = -iv\}.$$

Proof. We extend the proof of the stated lemma in [Huy04]. Let $v \in V_{\mathbb{C}}$. It is $v = \frac{1}{2}(v - iI(v)) + \frac{1}{2}(v + iI(v))$. A simple calculation shows

$$I(v - iI(v)) = I(v) - I(iI(v)) = I(v) - iI^2(v) = I(v) + iv = i(-iI(v) + v)$$

and thus $\frac{1}{2}(v - iI(v)) \in V^{1,0}$. With a similar calculation we obtain $\frac{1}{2}(v + iI(v)) \in V^{0,1}$. At the same time, it holds to be $V^{1,0} \cap V^{0,1} = \{0\}$. Thus, the inclusion $V^{1,0} \oplus V^{0,1} \hookrightarrow V_{\mathbb{C}}$ is injective and with the above calculation, it is also surjective. Hence, it is a canonical isomorphism, so the decomposition is proven. □

Remark 2.1.4. We expand the argument in the proof of [Huy04, Lemma 1.2.5]. The proof of the last proposition shows that a vector $w \in V^{1,0}$ can be written as $w = v - iI(v)$ for some $v \in V_{\mathbb{C}}$. At the same time, we can split $v = x + iy$ with $x, y \in V$. Then it is

$$\begin{aligned}\overline{w} = \overline{v - iI_{\mathbb{C}}(v)} &= \overline{x + iy - i(I(x) + iI(y))} \\ &= x - iy + iI(x) + I(y) = \overline{v} + i(I(x) + -iI(y)) = \overline{v} + iI_{\mathbb{C}}(\overline{v}).\end{aligned}$$

Hence it is $\overline{w} \in V^{0,1}$. Similar calculations show that for $w \in V^{0,1}$, it is $\overline{w} \in V^{1,0}$ and $\overline{\overline{w}} = w$. Since complex conjugation is $\mathbb{R}$-linear, this already proves that $V^{1,0}$ and $V^{0,1}$ are isomorphic as real vector spaces.

Remark 2.1.5. Using the proof of the last proposition and the natural inclusion $V \hookrightarrow V_{\mathbb{C}}$, $v \mapsto v \otimes 1$, we can define an $\mathbb{R}$-linear isomorphism

$$\varphi_1 : (V, I) \longrightarrow V^{1,0}, \quad v \mapsto \big(v \otimes 1 - iI_{\mathbb{C}}(v \otimes 1)\big).$$

However, we are able to calculate

$$\varphi_1(I(v)) = I(v) \otimes 1 - iI_{\mathbb{C}}(I(v) \otimes 1) = I_{\mathbb{C}}(v \otimes 1) - iI_{\mathbb{C}}^2(v \otimes 1) = I_{\mathbb{C}}\big(v \otimes 1 - iI_{\mathbb{C}}(v \otimes 1)\big).$$

Hence, we obtain $\varphi_1(I(v)) = I_{\mathbb{C}}(\varphi_1(v)) = i\varphi_1(v)$ because $\varphi_1(v) \in V^{1,0}$. Since the $\mathbb{C}$-module structure on (V, I) is defined using I, we know that φ_1 is also a $\mathbb{C}$-linear isomorphism. At the same time, we are able to define a similar $\mathbb{R}$-linear isomorphism

$$\varphi_2 : (V, I) \longrightarrow V^{0,1} \quad v \mapsto (v \otimes 1 + iI_{\mathbb{C}}(v \otimes 1)).$$

The same calculation yields $\varphi_2(I(v)) = -i\varphi_2(v)$. Thus, φ_2 is a $\mathbb{C}$-antilinear isomorphism.

Next, we are going to define an induced almost complex structure on the dual space V^*. Because of its induced nature, this almost complex structure is also called I and it is defined as a mapping $I : V^* \longrightarrow V^*$, such that $I(f)(v) = f(I(v))$ for all $f \in V^*$ and $v \in V$. Now, the following lemma ensures the compatibility of the complexification with the dual space of V.

Lemma 2.1.6 ([Huy04, Lemma 1.2.6])**.** *It is* $(V_{\mathbb{C}})^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = (V^*)_{\mathbb{C}}$ *and it also holds to be*

$$(V^{1,0})^* = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = if(v)\ \forall v \in V\} = (V^*)^{1,0},$$
$$(V^{0,1})^* = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = -if(v)\ \forall v \in V\} = (V^*)^{0,1}.$$

Proof. It is $(V_{\mathbb{C}})^* = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ and $(V^*)_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$. In order to prove that these two spaces are equal, we have to prove the existence of a canonical isomorphism $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$.

Let $f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ and extend it to an $\mathbb{R}$-linear mapping $\tilde{f} : V_{\mathbb{C}} \longrightarrow \mathbb{C}$ by setting $\tilde{f}(v \otimes \lambda) := \lambda f(v)$ for all $v \in V$ and $\lambda \in \mathbb{C}$. This mapping is also $\mathbb{C}$-linear because we can show for all $\mu \in \mathbb{C}$

$$\tilde{f}(\mu \cdot (v \otimes \lambda)) = \tilde{f}(v \otimes \mu\lambda) = \mu\lambda f(v) = \mu \cdot \tilde{f}(v \otimes \lambda).$$

This shows that for every f, we can find a unique $\tilde{f} \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$.
Let now $g \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$. Using the inclusion $V \hookrightarrow V_{\mathbb{C}}$, we can restrict g to obtain a mapping $h : V \longrightarrow \mathbb{C}$ that is defined as $h(v) := g(v \otimes 1)$. Since g was $\mathbb{C}$-linear, h is already an $\mathbb{R}$-linear mapping.

This shows that $h \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, and since those two constructions are obviously inverse to each other, this completes the proof of the first statement.[1]

[1] This first part of the proof was created using a Math Stack Exchange post of the user *Mark* (https://math.stackexchange.com/users/470733/mark) that can be found on https://math.stackexchange.com/q/4718935 and was last checked on the 25th of August, 2023.

For the second statement, we use Remark 2.1.5 to get

$$\begin{aligned}(V^{1,0})^* &= \mathrm{Hom}_{\mathbb{C}}(V^{1,0}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}((V, I), \mathbb{C}) \\ &= \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = if(v)\ \forall v \in V\}.\end{aligned}$$

Additionally, for the other subspace, we can use the same remark to obtain

$$\begin{aligned}(V^{0,1})^* &= \mathrm{Hom}_{\mathbb{C}}(V^{0,1}, \mathbb{C}) = \mathrm{Hom}_{\overline{\mathbb{C}}}((V, I), \mathbb{C}) \\ &= \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = -if(v)\ \forall v \in V\}.\end{aligned}$$

□

Notation 2.1.7. Because of the last lemma, we will only write $V_{\mathbb{C}}^*$, omitting the brackets from now on.

2.2. **Euclidian and hermitian vector spaces.**

Later, we are going to define the notion of a hermitian manifold, i.e. a complex manifold whose holomorphic tangent space in every point is equipped with a hermitian form. In order to do so, this section will cover some fundamental statements about those forms on complex vector spaces.

For the remainder of this section, we are going to assume the following setting.

Setting. Let (V, g) be a real n-dimensional euclidean vector space, i.e. g is a positive definite symmetric bilinear form on the real space V. Also, assume that V is equipped with an almost complex structure I.

Definition 2.2.1. The inner product g is said to be *compatible with the almost complex structure* I if it holds to be $g(I(v), I(w)) = g(v, w)$ for all $v, w \in V$.

Notation 2.2.2. If the inner product g on V is compatible with the almost complex structure I, we usually only write (V, g, I).

The just-established notion of a compatible inner product gives rise to an additional notion.

Definition 2.2.3. The *fundamental form* associated to (V, g, I) is defined as the form $\omega \in \bigwedge^2 V^* \cap \bigwedge^{1,1} V^*$, such that for all $v, w \in V$ it is

$$\omega(v, w) := g(I(v), w) = -g(v, I(w)).$$

Note that the second equality is equivalent to g being compatible with the almost complex structure I. Also, note that this immediately yields $\omega(I(v), I(w)) = \omega(v, w)$.

Remark 2.2.4. The expression $\bigwedge^2 V^* \cap \bigwedge^{1,1} V^*$ has to be explained. With Lemma 2.1.6, we know that $\bigwedge^2 V^* \subset \bigwedge^2 V_{\mathbb{C}}^*$. At the same time, it is

$$\bigwedge^2 V_{\mathbb{C}}^* = \bigwedge^{2,0} V^* \oplus \bigwedge^{1,1} V^* \oplus \bigwedge^{0,2} V^* \tag{2.1}$$

(cf. [Huy04, Proposition 1.2.8 (ii), Example 1.2.34]). and for these reasons, the intersection is meaningful as it happens in $\bigwedge^2 V_{\mathbb{C}}^*$. This expression describes all the alternating real 2-forms of type $(1,1)$, i.e. alternating real 2-forms on V, that are also $\mathbb{C}$-linear in it's first argument and $\mathbb{C}$-antilinear in its second argument if viewed as forms on $V_{\mathbb{C}}$.

Now, for the fundamental form ω from the last definition to be well-defined, we have to check whether ω is indeed an alternating real 2-form on V and is also of type $(1,1)$. For the first statement, real bilinearity follows directly with the bilinearity of g. Also using the symmetry of g, we calculate for all $v, w \in V$

$$\omega(v,w) = g(I(v),w) = g(I^2(v),I(w)) = -g(v,I(w)) = -g(I(w),v) = -\omega(w,v).$$

Hence, ω is alternating and therefore a real 2-form. With (2.1), it suffices to show that the $\mathbb{C}$-bilinear extension of ω vanishes on all pairs of vectors v, w in $V^{1,0}$ or $V^{0,1}$ to prove that it is of type $(1,1)$. In the first case, i.e. $v, w \in V^{1,0}$, we calculate

$$\omega(v,w) = \omega(I(v),I(w)) = \omega(iv,iw) = i^2\omega(v,w) = -\omega(v,w).$$

Hence $\omega(v,w) = 0$. The first equation holds because the complex bilinear extension inherits this property from the real form ω. The calculation for the other case can be carried out analogously. Hence, ω is indeed of type $(1,1)$ and this establishes the well-definedness of ω.

For (V,g,I), we can also define a positive definite hermitian form on the complex space (V,I). This form is defined as

$$h : (V,I) \times (V,I) \to \mathbb{C}, \quad (v,w) \mapsto g(v,w) - i\omega(v,w).$$

Additionally, the inner product g on V can be extended sesquilinearly to a positive definite hermitian form on $V_{\mathbb{C}}$. This extension is defined as

$$h_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}, \quad (v \otimes \lambda, w \otimes \mu) \mapsto (\lambda\overline{\mu}) \cdot g(v,w).$$

See also [Huy04, p. 30] for the similar definitions. However, it has to be checked whether these two positive definite hermitian forms are well-defined.

Proposition 2.2.5 ([Huy04, Lemma 1.2.15]). *For (V,g,I), the form $h := g - i\omega$ is indeed a positive definite hermitian form on (V,I). Also, the extension $h_{\mathbb{C}}$ of the inner product g defines a positive definite hermitian form on $V_{\mathbb{C}}$.*

Proof. We expand the calculation in the proof of the stated lemma in [Huy04] and add the argument for our second statement. Let $v, w \in (V,I)$. It is $h(v,v) = g(v,v) - i\omega(v,v) = g(v,v)$ because ω is alternating and therefore $\omega(v,v) = 0$. Since g is positive definite, this proves that h is positive definite as well. Furthermore, since g is symmetric, it is also

$$h(v,w) = g(v,w) - i\omega(v,w) = g(w,v) + i\omega(w,v) = \overline{h(w,v)},$$

and it also holds to be

$$\begin{aligned} h(I(v), w) &= g(I(v), w) - i\omega(I(v), w) \\ &= g(I^2(v), I(w)) - i(g(I^2(v), w)) \\ &= -g(v, I(w)) + ig(v, w) \\ &= i\big(ig(v, I(w)) + g(v, w)\big) \\ &= i\big(- i\omega(v, w) + g(v, w)\big) = ih(v, w). \end{aligned}$$

On (V, I), the image under I corresponds to multiplication with i because the $\mathbb{C}$-module structure is defined using the almost complex structure. This proves the $\mathbb{C}$-linearity in the first argument, as the $\mathbb{R}$-linearity is already inherited from g and ω.

For the $\mathbb{C}$-antilinearity in the second argument, we combine the last two statements to get

$$h(v, I(w)) = \overline{h(I(w), v)} = \overline{ih(w, v)} = -ih(v, w).$$

This completes the proof of the first statement.

To prove the second statement, it is already clear by definition that $h_{\mathbb{C}}$ is $\mathbb{C}$-linear in its first argument and $\mathbb{C}$-antilinear in its second argument. Let $(v_1, \ldots, v_n)$ be an orthonormal basis of V with respect to the inner product g. With the properties of the tensor product, it is again $(v_1 \otimes 1, \ldots, v_n \otimes 1)$ a basis of $V_{\mathbb{C}}$. Therefore, we can write every element $u \in V_{\mathbb{C}}$ as $u = \sum_{j=1}^n \lambda_j (v_j \otimes 1) = \sum_{j=1}^n v_j \otimes \lambda_j$. We are then able to calculate

$$h_{\mathbb{C}}(u, u) = h_{\mathbb{C}}\Big(\sum_{j=1}^n v_j \otimes \lambda_j, \sum_{k=1}^n v_k \otimes \lambda_k \Big) = \sum_{j,k=1}^n \lambda_j \overline{\lambda}_k g(v_j, v_k) = \sum_{j=1}^n |\lambda_j|^2 \geq 0,$$

and $h_{\mathbb{C}}(u, u) = 0$ if and only if $u = 0$. Hence, $h_{\mathbb{C}}$ is positive definite. Furthermore, it holds to be

$$h_{\mathbb{C}}(v \otimes \lambda, w \otimes \mu) = \lambda\overline{\mu} \cdot g(v, w) = \overline{\overline{\lambda}\mu \cdot g(v, w)} = \overline{\overline{\lambda}\mu \cdot g(w, v)} = \overline{h_{\mathbb{C}}(w \otimes \mu, v \otimes \lambda)}.$$

Thus, h and $h_{\mathbb{C}}$ are both positive definite hermitian forms. □

Notation 2.2.6. In [Huy04, Lemma 1.2.17], it is shown that these two hermitian forms only differ by a factor of $\frac{1}{2}$ under the natural inclusion $(V, I) \hookrightarrow V^{1,0}$. This may be a reason for the common practice not to differentiate between h and $h_{\mathbb{C}}$ in the notation, which we will also adhere to.

Remark 2.2.7. In the last proposition, it has been proven that the compatible inner product g already defines a positive definite hermitian form on (V, I). Let now $\tilde{h}$ be an arbitrary positive definite hermitian form on (V, I). We can define the *real part* of h as follows.

$$\Re(\tilde{h})(v, w) := \frac{1}{2}\big(\tilde{h}(v, w) + \overline{\tilde{h}(v, w)}\big) = \frac{1}{2}\big(\tilde{h}(v, w) + \tilde{h}(w, v)\big)$$

With the second equality, it is obvious that this defines a real positive definite and symmetric bilinear form on V. It is also

$$\begin{aligned}\Re(\tilde{h})(I(v), I(w)) &= \frac{1}{2}\big(\tilde{h}(I(v), I(w)) + \tilde{h}(I(w), I(v))\big)\\ &= \frac{1}{2}\big(\tilde{h}(v, w) + \tilde{h}(w, v)\big)\\ &= \Re(\tilde{h})(v, w)\end{aligned}$$

and thus $\Re(\tilde{h})$ defines a compatible inner product on V. Because these constructions are inverse to each other, there is a one-to-one relation between positive definite hermitian forms on (V, I) and inner products on V that are compatible with I. See also [Voi02, Section. 3.1.1] for further information about this relation.

Later, we will also need inner products and hermitian forms on the exterior algebra spaces $\bigwedge^k V^*$ and $\bigwedge^k V_{\mathbb{C}}^*$. In the following two lemmas, we are going to construct inner products on V^* and $\bigwedge^k V$ using the existing inner product g on V, and we will eventually use those constructions to define an induced inner product on $\bigwedge^k V^*$. This construction is inspired by [Sch12, Section 11], but no proofs are provided there.

Before we begin, we need to take a look at the natural linear mapping

$$g^\flat : V \longrightarrow V^*,\ v \mapsto g(v, -).$$

With respect to the inner product g, we can choose an orthonormal basis $(v_1, \ldots, v_n)$ of V. For all $r, s \in \mathbb{N}$ with $1 \leq r, s \leq n$, we get

$$g^\flat(v_r)(v_s) = g(v_r, v_s) = \delta_{rs}.$$

Hence, $g^\flat(v_r) = v^r$ with v^r being the dual basis vector of v_r. This already shows that $g^\flat$ is an isomorphism. Let now $g^\sharp : V^* \longrightarrow V$ denote the inverse mapping of $g^\flat$. Using this mapping, we get the following lemma.

Lemma 2.2.8. *The inner product g on V induces an inner product on V^*. It is defined as*

$$\tilde{g} : V^* \times V^* \longrightarrow \mathbb{R}, \quad (v, w) \mapsto g\big(g^\sharp(v), g^\sharp(w)\big).$$

Proof. It is obvious that $\tilde{g}$ defines a bilinear mapping. Let now $(v_1, \ldots, v_n)$ be an orthonormal basis of V again. Also let $(v^1, \ldots, v^n)$ denote the corresponding dual basis of V^*. If only evaluated on those dual basis vectors, $\tilde{g}$ simplifies as

$$\tilde{g}(v^r, v^s) = g(g^\sharp(v^r), g^\sharp(v^s)) = g(v_r, v_s).$$

Thus, $\tilde{g}$ directly inherits the inner product properties from g. □

Lemma 2.2.9. *The inner product g on V induces an inner product on $\bigwedge^k V$, which is defined as*

$$\begin{gathered} g_k : \bigwedge^k V \times \bigwedge^k V \longrightarrow \mathbb{R} \\ \big(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\big) \mapsto \det\Big(\big(g(v_r, w_s)\big)_{rs}\Big). \end{gathered}$$

Proof. With the multilinearity of the determinant, it is again obvious that g_k is a bilinear mapping. Since the determinant is invariant under transposition, g_k is also symmetric. Let now $(v_1, \ldots, v_n)$ be an orthonormal basis of V again. We know that this induces a basis $(v_J)_J$ of $\bigwedge^k V$. (cf. [Lee12, Proposition 14.8].) If we only evaluate g_k on these basis vectors again, we get

$$g_k(v_J, v_K) = \det\left(\left(g(v_{j_r}, v_{k_s})\right)_{rs}\right) = \delta_{JK}.$$

This is because if $J \neq K$, there is at least one $j_l \in J$ such that $j_l \notin K$. Thus we obtain $g(v_{j_l}, v_{\tilde{k}}) = 0$ for all $\tilde{k} \in K$. Hence, the l-th column of the matrix $G := \left(g(v_{j_r}, v_{k_s})\right)_{rs}$ is everywhere zero and therefore $\det(G) = 0$. If J and K are equal however, then it is $G = \mathrm{Id}_k$ and therefore $\det(G) = 1$. With this, we have for all $\alpha := \sum_J \alpha_J v_J \in \bigwedge^k V$

$$g_k(\alpha, \alpha) = \sum_J \alpha_J^2 \, g_k(v_J, v_J) = \sum_J \alpha_J^2 \geq 0.$$

This calculation also implies that $g_k(\alpha, \alpha) = 0$ if and only if $\alpha = 0$. Thus, the statement is proven. □

The combination of the last two lemmas finally gives us an inner product on $\bigwedge^k V^*$.

Corollary 2.2.10. *The induced inner product on $\bigwedge^k V^*$ is given as*

$$\tilde{g}_k : \bigwedge^k V^* \times \bigwedge^k V^* \to \mathbb{R}$$
$$\left(v^1 \wedge \cdots \wedge v^k, w^1 \wedge \cdots \wedge w^k\right) \mapsto \det\left(\left(g(g^\sharp(v^j), g^\sharp(w^k))\right)_{jk}\right).$$

Remark 2.2.11. The two proofs of Lemmas 2.2.8 and 2.2.9 also show that the induced inner products preserve orthonormality, i.e. for an orthonormal basis $(v_1, \ldots, v_n)$ of V, the induced bases $(v^1, \ldots, v^n)$ of V^* and $(v_J)_J$ of $\bigwedge^k V$ are also orthonormal with respect to the induced inner products. Since Corollary 2.2.10 just combines these inner products, this property also holds for the induced inner product $\tilde{g}_k$ on $\bigwedge^k V^*$.

Remark 2.2.12. Similar to the construction of the positive definite hermitian form on $V_\mathbb{C}$, we are able to obtain positive definite hermitian forms $\tilde{h}_k$ on the exterior algebra spaces $\bigwedge^k V_\mathbb{C}^*$, by extending the inner products $\tilde{g}_k$ sesquilinearly (cf. [Huy04, p. 33]).

Notation 2.2.13. It is common practice to only write g and h for all these inner products and hermitian forms, respectively. This is because they have similar properties and since they are all defined on different spaces, it should always be clear from the context which form is meant.

2.3. **Local operators.**

The next section will focus on the definition of some essential operators. These are initially defined as local operators on vector spaces but are going to be used on vector bundles later. However, most of their properties can already be shown locally.

Therefore, we will assume the following setting for the remainder of this section.

Setting. Let (V, g, I) be an euclidean vector space of dimension n with a compatible almost complex structure. Also, let ω denote the fundamental form associated with g and let h denote the induced hermitian forms on (V, I) and $V_{\mathbb{C}}$. The existence of the almost complex structure I ensures $n = 2m$ fore some $m \in \mathbb{N}$.

Definition 2.3.1 (Lefschetz operator)**.** With ω the associated fundamental form to (V, g, I), the real *Lefschetz operator* is defined as the linear mapping

$$L : \bigwedge^k V^* \to \bigwedge^{k+2} V^*, \quad \alpha \mapsto \omega \wedge \alpha.$$

The complex *Lefschetz operator* on $\bigwedge^k V_{\mathbb{C}}^*$ is then defined as the $\mathbb{C}$-linear extension $L_{\mathbb{C}} : \bigwedge^k V_{\mathbb{C}}^* \to \bigwedge^{k+2} V_{\mathbb{C}}^*$, i.e. $L_{\mathbb{C}}(\beta) = \omega \wedge \beta$ for all $\beta \in \bigwedge^{k+2} V_{\mathbb{C}}^*$.

Remark 2.3.2. Note that $\omega \in \bigwedge^2 V^* \cap \bigwedge^{1,1} V^* \subset \bigwedge^2 V_{\mathbb{C}}^*$ and this is the reason for these two wedge products to be both meaningful. Also, we have to keep in mind that the definition of this operator depends on the fundamental form ω, which itself depends on the choice of the inner product g.

Furthermore, due to the fundamental form ω being of type $(1, 1)$, it is apparent that the restriction of $L_{\mathbb{C}}$ to forms of type (p, q) behaves like $L : \bigwedge^{p,q} V^* \to \bigwedge^{p+1,q+1} V^*$ and thus the complex Lefschetz operator preserves the revised structure of $\bigwedge^k V_{\mathbb{C}}^* = \bigoplus_{p+q=k} \bigwedge^{p,q} V^*$ (cf. [Huy04, Proposition 1.2.8 (ii)]).

In linear algebra, we have the notion of an adjoint operator with respect to an inner product on a vector space. Therefore, we can use our induced inner product on the exterior algebra spaces $\bigwedge^k V^*$ to define the dual of the Lefschetz operator.

Definition 2.3.3. The *dual Lefschetz operator* Λ is defined as the adjoint operator of L with respect to the inner product g, i.e. the uniquely defined mapping

$$\Lambda : \bigwedge^{k+2} V^* \to \bigwedge^{k} V^*,$$

such that for all $\alpha \in \bigwedge^{k+2} V^*$ and $\beta \in \bigwedge^k V^*$, it is $g(\Lambda(\alpha), \beta) = g(\alpha, L(\beta))$.

Remark 2.3.4. Note that this operator is indeed uniquely defined because of the non-degeneracy of the inner product g. This is because if we let $\alpha \in \bigwedge^{k+2} V^*$ and assume there would be a second operator $\widetilde{\Lambda}$ admitting the same adjunction property, such that $\Lambda(\alpha) \neq \widetilde{\Lambda}(\alpha)$, it would be for all $\beta \in \bigwedge^k V^*$

$$g(\Lambda(\alpha) - \widetilde{\Lambda}(\alpha), \beta) = g\,(\Lambda(\alpha), \beta) - g(\widetilde{\Lambda}(\alpha), \beta) = g(\alpha, L(\beta)) - g(\alpha, L(\beta)) = 0.$$

Thus, the non-degeneracy of g can be used to obtain $\Lambda(\alpha) - \widetilde{\Lambda}(\alpha) = 0$, which is a contradiction to our assumption.

To properly define the next operator now, it is necessary to discuss the existence of a volume element in $\bigwedge^n V^*$ first.

Remark 2.3.5. We already know that V has a natural orientation, which we are going to call σ for now. As mentioned in [Spi65, p. 83 below Theorem 4-6], there exists a unique volume form $\mathrm{vol} \in \bigwedge^n V^*$, such that $\mathrm{vol}(v_1, \dots, v_n) = 1$ whenever $(v_1, \dots, v_n)$ is an orthonormal basis of V that is positively orientated with respect to σ. This volume form can be given as $\mathrm{vol} = v^1 \wedge \cdots \wedge v^n$.

Now, we want to show that $\mathrm{vol} = \frac{1}{m!}\omega^m$. Therefore, let $(w_1, \dots, w_m)$ be a complex orthonormal basis of (V, I) with respect to the hermitian form h. A simple calculation proves that the induced real basis $(w_1, I(w_1), \dots, w_m, I(w_m))$ is orthonormal with respect to the inner product g on V. We can calculate for all w_j and w_k

$$\omega(w_j, I(w_k)) = g(I(w_j), I(w_k)) = g(w_j, w_k) = \delta_{jk}$$

and also

$$\omega(w_j, w_k) = g(I(w_j), w_k) = 0.$$

Let $I(w_j)^*$ denote the dual basis vector of $I(w_j)$. With the above calculation, we conclude that $\omega = \sum_{j=1}^m w^j \wedge I(w_j)^*$ and therefore it is

$$\omega^m = \left(\sum_{j=1}^m w^j \wedge I(w_j)^*\right)^m = m! \cdot \left(w^1 \wedge I(w_1)^*\right) \wedge \cdots \wedge \left(w^m \wedge I(w_m)^*\right) = m! \cdot \mathrm{vol}\,.$$

This concludes our collection of the necessary elements to define the Hodge star operator.

Definition 2.3.6 (Hodge star operator)**.** The *Hodge star operator* on $\bigwedge^k V^*$ is defined as a linear mapping $* : \bigwedge^k V^* \to \bigwedge^{n-k} V^*$, such that for all $\alpha, \beta \in \bigwedge^k V^*$, it is

$$\alpha \wedge * \beta = g(\alpha, \beta) \cdot \mathrm{vol}\,. \tag{2.2}$$

The $(n-k)$ form $*\beta$ is called the *Hodge dual* of β.

Note that (2.2) uniquely defines the Hodge dual because for $r, s \in \mathbb{N}$ with $r + s = n$, the exterior product defines a non-degenerate pairing

$$\bigwedge^r V^* \times \bigwedge^s V^* \to \bigwedge^n V^*.$$

To show that such an operator exists, we choose an orthonormal basis $(v_1, \dots, v_n)$ of V that is positively oriented with respect to the natural orientation of V. With Remark 2.2.11, we already know that the induced basis $(v^J)_J$ of $\bigwedge^k V^*$ is orthonormal as well. For an arbitrary permutation $\tau \in S_n$, we set the Hodge star operator to map as follows:

$$* \left(v^{\tau(1)} \wedge \cdots \wedge v^{\tau(k)}\right) = \mathrm{sign}(\tau) \cdot\ v^{\tau(k+1)} \wedge \cdots \wedge v^{\tau(n)}. \tag{2.3}$$

With this definition, it is

$$\begin{aligned} v^{\tau(1)} \wedge \cdots \wedge v^{\tau(k)} \wedge * \left(v^{\tau(1)} \wedge \cdots \wedge v^{\tau(k)}\right) &= \mathrm{sign}(\tau) \cdot v^{\tau(1)} \wedge \cdots \wedge v^{\tau(n)} \\ &= \mathrm{sign}(\tau)^2 \cdot v^1 \wedge \cdots \wedge v^n \\ &= 1 \cdot \mathrm{vol}\,. \end{aligned}$$

At the same time, it is for all $v^{j_1} \wedge \cdots \wedge v^{j_k} \neq \pm v^{\tau(1)} \wedge \cdots \wedge v^{\tau(k)}$

$$v^{j_1} \wedge \cdots \wedge v^{j_k} \wedge * \left(v^{\tau(1)} \wedge \cdots \wedge v^{\tau(k)}\right) = 0.$$

This is because if we assume without loss of generality that $j_1 \neq \cdots \neq j_k$, then there exists at least one $s \in \mathbb{N}$ with $(k+1) \leq s \leq n$, such that $\tau(s) \in \{j_1, \ldots, j_k\}$. This shows that the above mapping indeed explicitly defines the Hodge star. See also [Sch12, p. 56f], where we have later discovered a very similar calculation.

Proposition 2.3.7 (Properties of the Hodge star [Huy04, Proposition 1.2.20])**.** *Among others, the Hodge star operator has the following properties:*

(1) The Hodge star operator on $\bigwedge^k V^*$ is an isometric isomorphism, i.e. it is bijective and for all $\alpha, \beta \in \bigwedge^k V^*$, it is $g(\alpha, \beta) = g(*\alpha, *\beta)$.

(2) For all $\alpha \in \bigwedge^k V^*$, it is $*^2 \alpha = (-1)^k \alpha$. In particular, it is $*^{-1} = (-1)^k *$.

Proof. With the explicit definition in (2.3), it is quite easy to determine that the Hodge star operator maps any orthonormal basis to an orthonormal basis, and this proves property (1). For the proof of property (2), we use a local version of the calculation in [Voi02, Lemma 5.5]. We can apply property (1) and calculate for all $\alpha, \beta \in \bigwedge^k V^*$

$$\alpha \wedge * \beta = g(\alpha, \beta) \cdot \mathrm{vol} = g(*\alpha, *\beta) \cdot \mathrm{vol} = g(*\beta, *\alpha) \cdot \mathrm{vol} = *\beta \wedge *^2 \alpha.$$

With $n - (n-k) = k$, we know $*^2 \alpha$ is a k-form again. As $*\beta$ is a $(n-k)$ form, we obtain

$$\begin{aligned} *\beta \wedge *^2 \alpha &= (-1)^{k(n-k)} *^2 \alpha \wedge * \beta \\ &= (-1)^{k(2m-k)} *^2 \alpha \wedge * \beta \\ &= (-1)^k *^2 \alpha \wedge * \beta. \end{aligned}$$

Given that this holds for all $\beta \in \bigwedge^k V^*$, the second property is already proven. □

Additionally, there is an interesting relation between the Hodge star operator and the Lefschetz and dual Lefschetz operators, which is going to be useful later.

Lemma 2.3.8 ([Huy04, Lemma 1.2.23])**.** *For $\alpha \in \bigwedge^{k+2} V^*$ the image under the dual Lefschetz operator Λ can be explicitly calculated as*

$$\Lambda(\alpha) = \left(*^{-1} \circ L \circ *\right)(\alpha) = \left((-1)^k * \circ L \circ *\right)(\alpha).$$

Proof. We expand the proof of the given lemma in [Huy04]. Let $\beta \in \bigwedge^k V^*$. Using the definition of the Hodge star and the definition of the Lefschetz operator, we can calculate

$$g(\alpha, L\beta) \cdot \mathrm{vol} = g(L\beta, \alpha) \cdot \mathrm{vol} = L\beta \wedge * \alpha = \omega \wedge \beta \wedge * \alpha.$$

As ω is a 2-form and the wedge product is associative, we have

$$g(\alpha, L\beta) \cdot \mathrm{vol} = (-1)^{2k} \beta \wedge \omega \wedge * \alpha = \beta \wedge (\omega \wedge * \alpha).$$

Applying the definition of the Lefschetz operator again and using the definition of the Hodge star operator yields

$$\begin{aligned} g(\alpha, L\beta) \cdot \mathrm{vol} &= \beta \wedge L(*\alpha) \\ &= \beta \wedge (* *^{-1}) L(*\alpha) \\ &= \beta \wedge *(*^{-1}(L(*\alpha))) \\ &= g\big(\beta, (*^{-1}(L(*\alpha)))\big) \cdot \mathrm{vol} \\ &= g(*^{-1}(L(*\alpha)), \beta) \cdot \mathrm{vol}\,. \end{aligned}$$

Thus, we have shown the equality $g(\Lambda\alpha, \beta) = g(\alpha, L\beta) = g(*^{-1}(L(*\alpha)), \beta)$. Since this holds for all k-forms β, the non-degeneracy of g proves the first equality of the statement. The second equality then follows directly with Proposition 2.3.7 (2). □

Similar to the $\mathbb{C}$-linear extension of the Lefschetz operator, we will also need the $\mathbb{C}$-linear extension of the Hodge star operator.

Definition 2.3.9. The $\mathbb{C}$-linear extension of the Hodge star $*_{\mathbb{C}} : \bigwedge^k V_{\mathbb{C}}^* \to \bigwedge^{n-k} V_{\mathbb{C}}^*$ is defined such that for all $\alpha, \beta \in \bigwedge^k V_{\mathbb{C}}^*$, we have

$$\alpha \wedge *_{\mathbb{C}} \overline{\beta} = h(\alpha, \beta) \cdot \mathrm{vol}\,.$$

Note that this expression is meaningful because $\mathrm{vol} = \frac{1}{m!}\omega^m \in \bigwedge^n V^* \cap \bigwedge^{m,m} V^*$. Since the hermitian form h on $\bigwedge^k V_{\mathbb{C}}^*$ was previously defined as the sesquilinear extension of the inner product g on $\bigwedge^k V^*$, it is immediate that this is indeed the $\mathbb{C}$-linear extension of the real Hodge star operator.

There is also the $\mathbb{C}$-linear extension $\Lambda_{\mathbb{C}} : \bigwedge^{k+2} V_{\mathbb{C}}^* \to \bigwedge^k V_{\mathbb{C}}^*$ of the dual Lefschetz operator. Using the explicit formula in Lemma 2.3.8, this extension is given as

$$\Lambda_{\mathbb{C}} = (-1)^k *_{\mathbb{C}} \circ L_{\mathbb{C}} \circ *_{\mathbb{C}}\,.$$

We can use the same calculation as in the proof of Lemma 2.3.8 to show that this is indeed the adjoint operator to L with respect to the hermitian form h.

Notation 2.3.10. It is common practice to abuse the notation, denoting the complex extensions $L_{\mathbb{C}}$, $*_{\mathbb{C}}$ and $\Lambda_{\mathbb{C}}$ as L, $*$ and Λ, respectively.

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3. Zusammenfassung

Diese Arbeit beschäftigt sich mit der Hodge-Zerlegung für kompakte Kähler-Mannigfaltigkeiten, welche eine der zentralen Aussagen der Hodge-Theorie ist. Sie liefert eine Zerlegung der de-Rahm-Kohomologie-Gruppen in passende Dolbeault-Kohomologie-Gruppen und stellt somit eine Verbindung zwischen den topologischen Eigenschaften und der komplexen Struktur einer kompakten Kähler-Mannigfaltigkeit her.

Das Ziel dieser Arbeit ist die Ausarbeitung des Beweises dieser Zerlegung. Dafür muss die erforderlich Theorie eingeführt und erklärt werden. Dabei ist es zunächst sinnvoll, die lokale Theorie auszuarbeiten. Diese befasst sich hauptsächlich mit den Eigenschaften von euklidischen und unitären Vektorräumen im Zusammenhang mit der Existenz einer kompatiblen fastkomplexen Struktur. Dabei werden vor allem die Werkzeuge aus der Linearen Algebra gebraucht.

Mit den gesammelten Eigenschaften werden dann jeweils der Lefschetz-Operator, der duale Lefschetz-Operator und der Hodge-Stern-Operator lokal definiert.

Danach wird der Fokus zunehmend auf Mannigfaltigkeiten gelegt. Nachdem hermitesche Mannigfaltigkeiten definiert wurden, werden einige der zuvor erarbeiteten lokalen Aussagen in globale Aussagen übersetzt. Außerdem werden die entsprechenden globalen Operatoren definiert. Dabei wird jedoch angenommen, dass der Leser bereits mit den grundlegenden Begriffen und Eigenschaften von komplexen und fastkomplexen Mannigfaltigkeiten vertraut ist.

Nachdem formal adjungierte Operatoren mithilfe einer vorher definierten L^2-Metrik eingeführt wurden, werden dann die Kähler-Identitäten behandelt. Diese stellen die zuvor eingeführten globalen Operatoren in Relation zueinander und sind äußerst essenzielle Eigenschaften von Kähler-Mannigfaltigkeiten. Diese Kähler-Identitäten werden in dieser Arbeit jedoch nicht bewiesen.

Das nächste Ziel ist der Beweis der Hodge-Isomorphie-Sätze. Dafür wird die Theorie der harmonischen Differentialformen eingeführt und einige wichtige Eigenschaften werden bewiesen. Dafür werden die zuvor behandelten Kähler-Identitäten benötigt.

Danach wird mithilfe dieser Isomorphie-Sätze die Hodge-Zerlegung bewiesen. Außerdem wird gezeigt, dass diese Zerlegung unabhängig von der Wahl der Kähler-Metrik ist. Am Ende wird dann eine nützliche topologische Anwendung der Hodge-Zerlegung präsentiert.

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