Homework 2. Fourier Transform of a Gaussian and Convolution

Note that your written answers can be brief but include plots wherever meaningful/necessary.

1. In class we have looked at the Fourier transform of continuous functions and we have shown that the Fourier transform of a delta function (an impulse) is equally weighted in all frequencies. A sharp function in the time domain yields a very broad function in the frequency domain, and vice versa. This is a form of the uncertainty principle and a very important concept in signal analysis – it is not possible to have a very narrowband signal of short duration. We can demonstrate this by considering the Gaussian or normal distribution that we will encounter later on in the class

The Gaussian distribution is given by

$$g(t) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\tau} \exp\left(-\frac{t^2}{2\tau^2}\right), \quad -\infty < t < \infty \tag{1}$$

This function has a mean of zero and a root mean squared (RMS) deviation (a measure of its width) of τ, which is defined by

$$\sqrt{\int_{-\infty}^{\infty} t^2 g(t) dt \Big/ \int_{-\infty}^{\infty} g(t) dt} \,. \tag{2}$$

It turns out that the Fourier transform of a Gaussian is another Gaussian (showing so requires the use of complex variable theory).

(i) Write a program that returns the Gaussian distribution given input arguments of a vector of times `t` and a value of RMS deviation `tau`. Plot an example.

(ii) Write a second function to calculate equation (2) and verify that the input Gaussian distribution has an RMS deviation of τ (Note: `numpy.trapz` in python or `trapz` in Matlab is useful here). Because you are doing the integration numerically you will not show this exactly but you should be able to get pretty close provided your input times are fairly finely sampled and extend from -5τ to 5τ)

(iii) The Fourier transform of (1) is:

$$G(\omega) = \exp\left(-\frac{\omega^2\tau^2}{2}\right). \tag{3}$$

Write a function to calculate the amplitude spectrum and explore qualitatively with the help of plots how changes in the width of the Gaussian function you create in the time domain (i.e., variations in parameter τ) affect the width of the amplitude spectrum.

(iv) By applying the function you wrote in (2) to the amplitude spectra of the Gaussian function with different values of τ, deduce a relationship between the width of a Gaussian in the time domain and the width of its amplitude spectrum.

2. On Canvas under the week 3 module find the file “maunaloa_weekly.csv”. This contains CO2 measurements at Mauna Loa (the famous “Keeling Curve”) at weekly (dt=7 days) intervals.

(i) Plot the data. Qualitatively answer: Are there any periodicities in the data? If so at what periods? Sketch what you would expect the Fourier amplitude spectrum of the data would look like.

(ii) A common means to smooth data is to compute an N-point running mean. That is, each point is taken as an average of N points centered upon the point of interest – it is simplest to chose N odd with $N = 2M + 1$. Explain why this is equivalent to convolving the data with a time series of N points with values of $1/N$.

(iii) Smooth the temperature data with this method using the the Python function `numpy.convolve()` or the *MATLAB* `conv` command for different choices of N. What is the best choice of N to get rid of yearly fluctuations? Why is this value best? What happens at either end of the time series after convolving?

(vi) We can also smooth the temperature data by convolving it with a Gaussian. Create a Gaussian function using the command you wrote for the last exercise whose RMS deviation τ is equivalent to the half-width of the boxcar. Multiply the Gaussian by a constant so that the sum of all the points in unity. Perform convolution in the time domain. Can you see any significant differences to the result from smoothing with a boxcar?