Linear Algebra and Regression - I

Daniel Ferreira

Introduction

Daniel Ferreira:

- 20 years studying Statistics
- Statistical Bachelor (UNICAMP / Brazil)
- 7 years at Meta (Research Scientist / Data Scientist / Marketing Science)
- 7 years teaching Data Mining and Statistical Methods at FIA (Brazil Post Graduation course)
- 7 years at SAS (Old school) - Instructor of Statistical Classes
- I love music and everything around it. I play guitar and build some electronic effects (analog mainly and some bad sounding Arduino prototypes)

Before we begin, pop into the chat:

1. What kind of work do you currently do?
2. Where is your hometown?
3. What is a fun fact about you?

Optimize Your Experience

1. Interact with your instructors via Sunday live class.
2. Don't be shy to speak up and get clarifications !
3. Solve your assignments, MCQs and other assessments and get feedback (Thursday review session)
4. Use your resources! It's your experience – what you put in is what you'll get out.

Agenda

	1st Hour (9-10 PT)	2nd Hour (10-11 PT)	3rd Hour (11-12 PT)	4th Hour (12-1 PT)
h:00 - h:05	• Why linear algebra is important for ML	• Vector demo in Python • Linear Independence • Orthogonality	• Matrix multiplication • Linear transformation • Determinant of matrix	• Inverse of matrices • Apply matrix algebra, matrix transpose, and zero and identity matrices
h:05 - h:30	• Vector notation and properties • Vector addition, scalar multiplication			
h:30 - h:55	• Vector decomposition • Norm, angle, distance, projection • Dot Product of vectors • cross Product of vectors	• Definition of matrix • Understanding matrix as system of linear equations • Understanding matrix as system of vectors • Matrix addition, scalar multiplication	• Rank of a matrix • Row echelon form • Rank of a matrix • Singular matrix • Invertible matrices • Matrix Transpose of matrices & properties	• Diagonalization • Calculation of Eigenvalues • Calculation of Eigenvectors
h:55 - h:00	Summary	Break	Break	Summary

Learning Objectives

- Learn about Vectors and Matrices and basic mathematical operations
- Learn about the various forms of visualizing matrices
- Learn about Determinant, Transpose, Inverse and Rank of a matrix
- Learn about the calculation of eigenvalues and eigenvectors

Why this Session is Important

- LA is a foundational tool for data science and engineering. Employment in math-related occupations is expected to grow 27 percent from 2019 to 2029 based on Occupational Outlook Handbook
- Vectors and Matrices are building blocks of techniques of ML & allows representing data in a structured and organized way
- LA provides a way to model complex systems and relationships between variables
- LA provides a way to optimize machine learning algorithms
- The first class of Deep Learning will start with matrix multiplication
- Recommendation Systems (like a Netflix, Ecommerce websites) use concept of Matrix Manipulations
- Remember the famous Netflix competition, any guesses what is used?
    - https://pantelis.github.io/cs301/docs/common/lectures/recommenders/netflix/

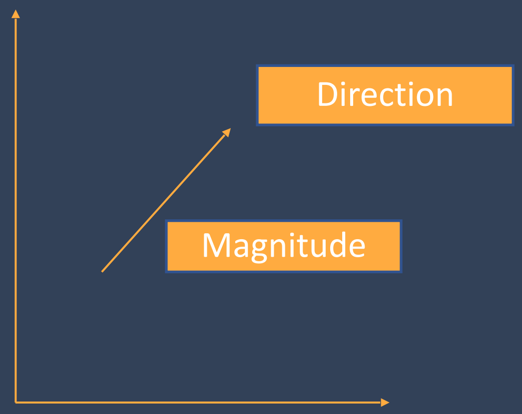
Vectors

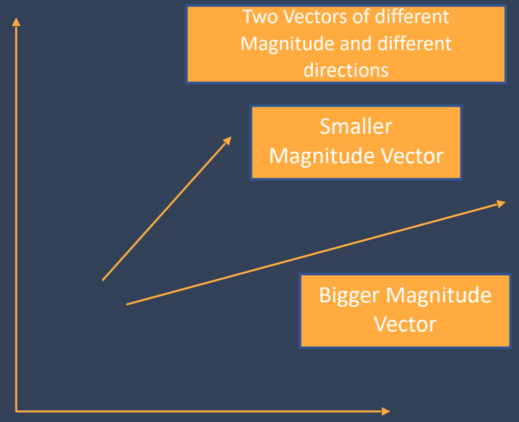
Learning Objectives

- Understand the concept of vectors and their properties.
- Perform vector operations such as addition, subtraction, and scalar multiplication.
- Calculate the norm, angle, distance and projection of vectors.
- Understand the concept of linear combinations and linear independence

What are Vectors

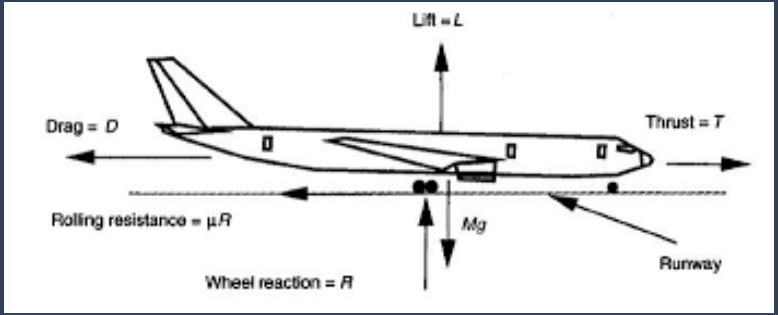
- A vector is a quantity which has both magnitude and direction. A vector quantity, unlike scalar, has a direction component along with the magnitude which helps to determine the position of one point relative to the other.

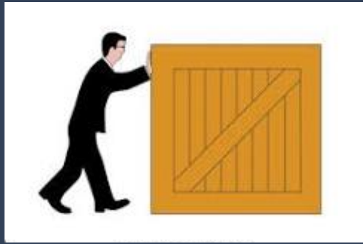




Examples of Vectors

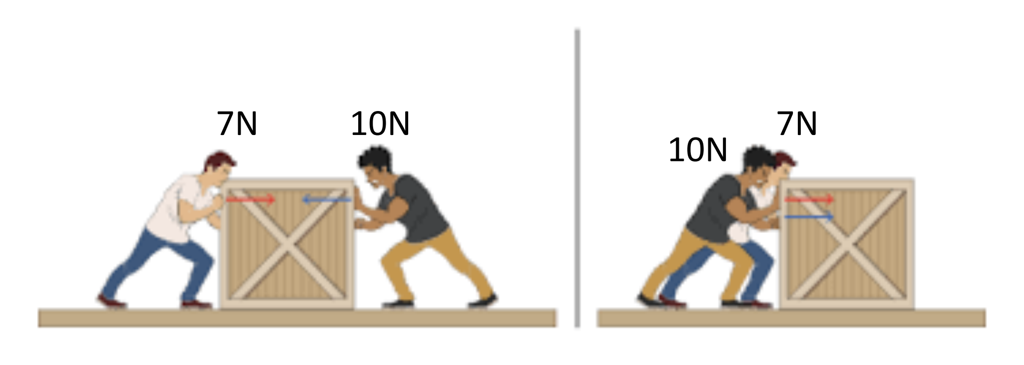
Velocity: 55 km/hr in the East Direction





Force: 70 N in the East Direction

Vector Operations- Addition & Subtraction



Two vectors acting in opposite directions having different magnitudes, the net force being applied is
In the east direction 3 N.
V1= 7N
V2=-10N
V1+V2=-3N

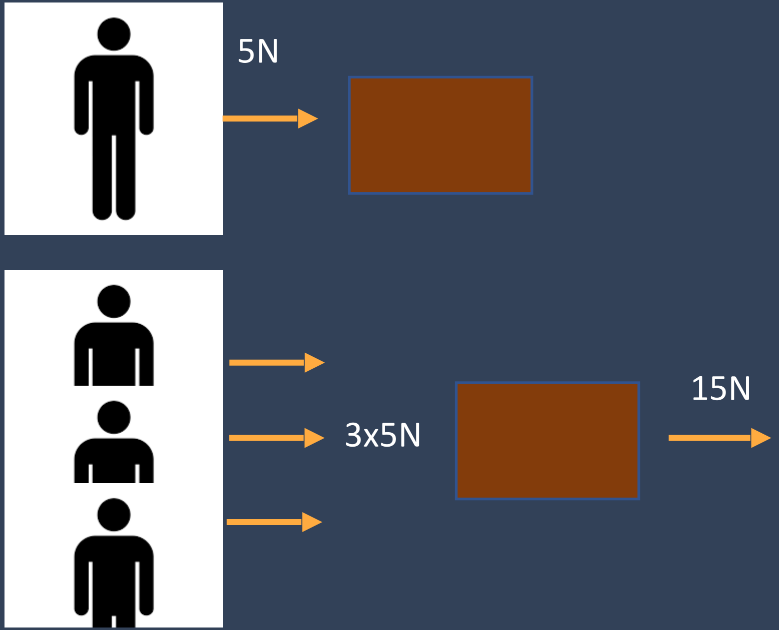
Two vectors acting in same direction having different magnitudes, the net force being applied is
In the east direction 3 N.
V1= 7N
V2=10N
V1+V2=17N

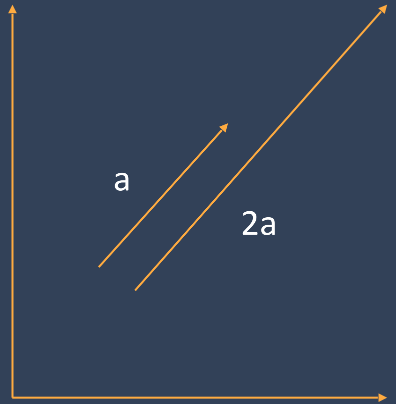
Please note vector addition is two vectors being added in the same direction, vector subtraction is adding vectors of opposite direction

Whiteboarding

Parallelogram Law
Addition of vectors u and v is the diagonal of the parallelogram formed by u and v.

Vector Operations- Scalar Multiplication





Multiple Vectors being added in the same direction is equivalent to saying that the vector is multiplied by a scalar. (Only the magnitude changes)

Let V be a vector in space R^3, where all the points are represented by ordered triples of real numbers. V= (x, y, z), k is a real number $kV=k(x,y,z)=(kx,ky,kz)$.

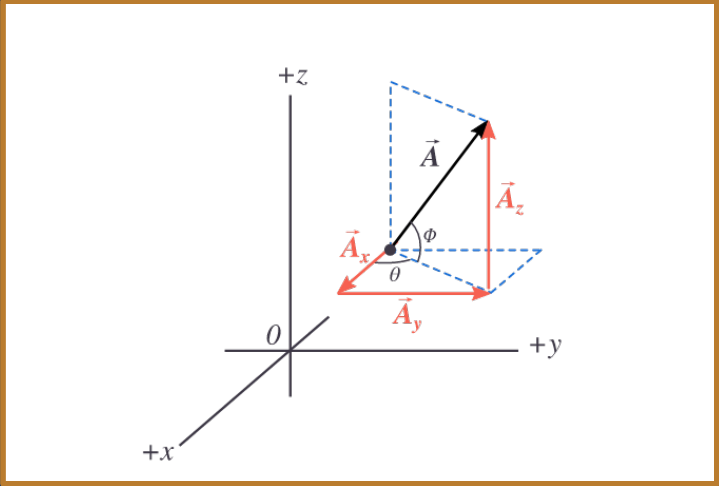
If K>0, kV retains the same direction; if k<0, kV is in the opposite direction.

Decomposition of Vectors

Vector components allow us to break a single vector quantity into two (or more) scalar quantities with which we have more mathematical experience. Vector components are used in vector algebra to add, subtract, and multiply vectors. Vectors are usually denoted on figures by an arrow.

Take for example a ball travelling in a 3-D space. It’s velocity can be decomposed into x, y, and z components and represented as follows-

$$\vec{A} = 2\hat{i} + 3\hat{j} + 5\hat{k}$$



Norm

(Euclidean) norm (length) of vector $a \in R^n$. $a=(a_1,a_2,\dots,a_n)$

$$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

```
#Compute norm in Python
import numpy as np
x = np.array([3,4,5,7])
np.linalg.norm(x)
#Output:
9.9498743710662
```

Distance

Let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$

(Euclidean) distance between n-vectors u and v:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

```
a = np.array((1, 2, 3))
b = np.array((4, 5, 6))
euclidean_distance = np.linalg.norm(a - b)
print(euclidean_distance)
#output
5.196152422706632
```

Dot Product

The dot product, also called scalar product, is a measure of how closely two vectors align, in terms of the directions they point.

Let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$

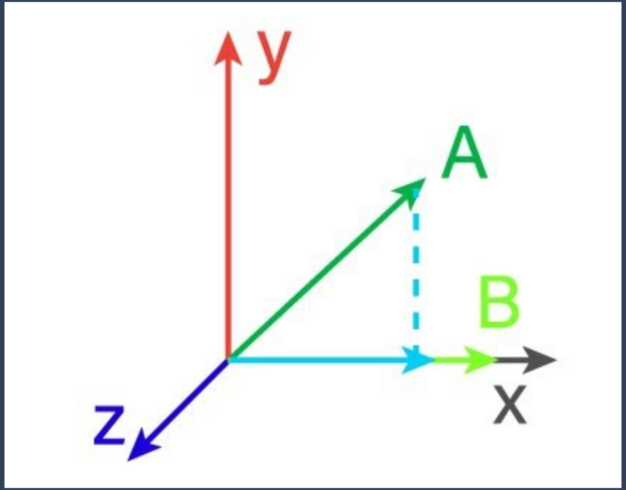
The dot product of u and v is

$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots\cdots + u_n v_n$

The scalar product of two vectors a and b of magnitude $|a|$ and $|b|$ is given as

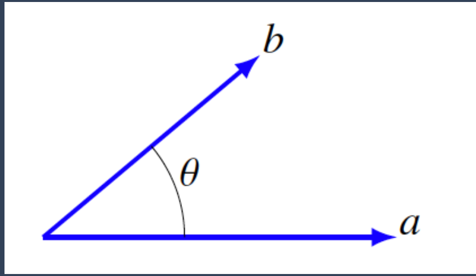
$$a \cdot b = |a||b| \cos\theta$$

where θ represents the angle between the vectors a and b taken in the direction of the vectors.



Angle

Angle between two nonzero vectors a, b defined as



$$\angle(a, b) = \arccos\left(\frac{a^T b}{\|a\| \, \|b\|}\right)$$

this is the unique value of $\theta \in [0, \pi]$ that satisfies $a^T b = \|a\| \|b\| \cos\theta$

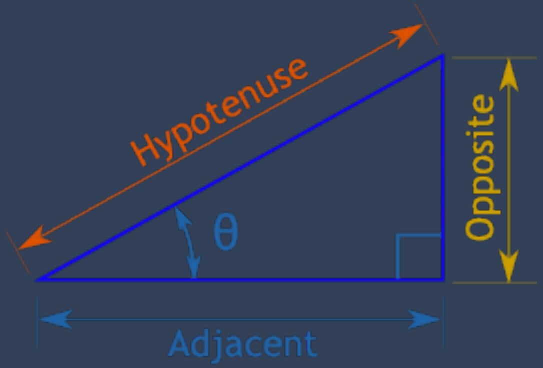
```
from numpy import dot
from numpy.linalg import norm

#define arrays
a = [0, 1]
b = [1, 0]
#calculate Cosine Similarity
cos_sim = dot(a, b)/(norm(a)*norm(b))
print(cos_sim)
#Output
0.0
```

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

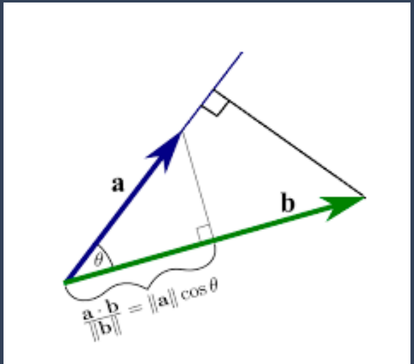
$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$



Projection

Projection vector gives the projection of one vector over another vector. The vector projection is a scalar value. You can think of the projection of a onto b as the shadow of a falling on b if the sun were to shine on b at a right angle.

$$\text{Projection of Vector a on Vector b} = \frac{a.b}{|b|}$$



The scalar projection above only gives the ***length*** of projection.

To find the ***actual vector*** that defines the projection

We will calculate the unit vector and then multiply it by the length

of the projection of a onto b.

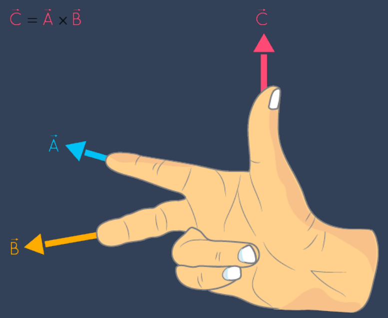
$$proj_b a = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{b}\right|^2} \vec{b}$$

Whiteboarding

Calculate projection

Cross Product

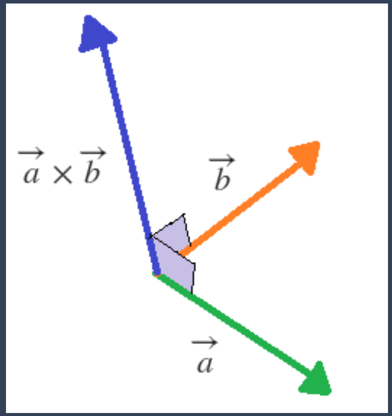
Cross product (aka vector product) of two vectors, denoted by u X v, is defined as the vector whose magnitude equals the multiplication of the magnitude of two vectors with the sine of angle between them and its direction is perpendicular to the plane which contains the two vectors which is given by the right hand thumb rule.



The magnitude of cross product of two vectors a and b of magnitude |a| and |b| is given as

$$a x b = |a||b| \sin\theta$$

where θ represents the angle between the vectors a and b taken in the direction of the vectors.

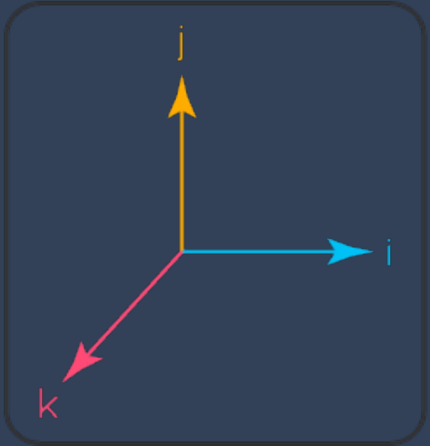


Cross Product using Matrix Notation

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = \vec{a} \times \vec{b}$

$$\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



$$\vec{c} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\boxed{\vec{c} = \hat{i}\left| a_2b_3 - a_3b_2 \right| - \hat{j}\left| a_1b_3 - a_3b_1 \right| + \hat{k}\left| a_1b_2 - a_2b_1 \right|}$$

Whiteboarding

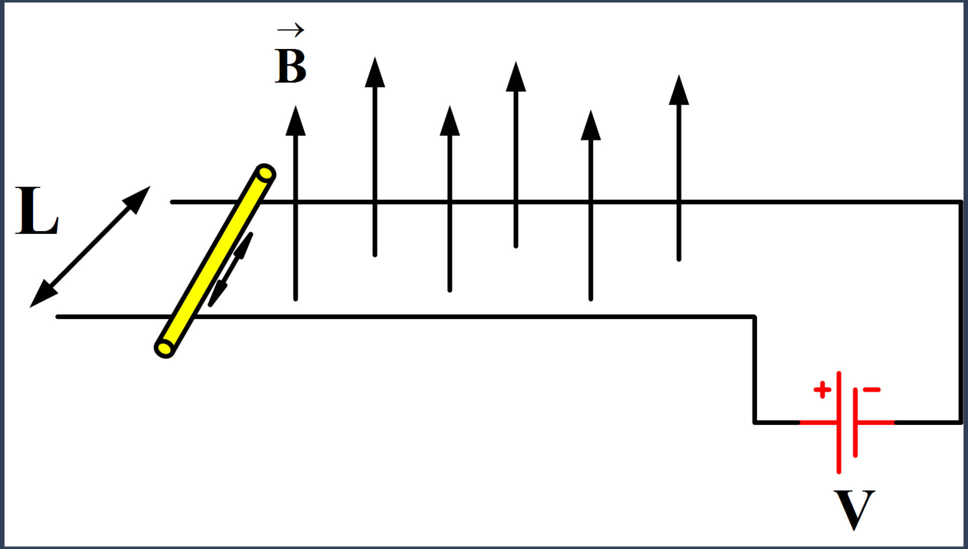
Calculate cross product

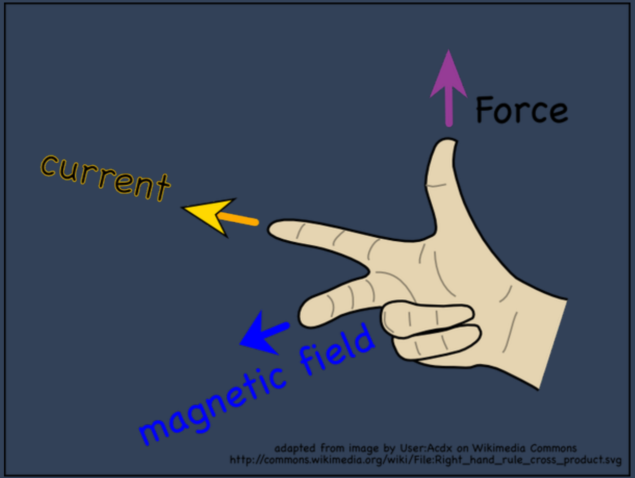
Difference between Dot and Cross Product

Dot Product	Cross Product
Dot product is product of magnitude of vectors & cosine of angle between them.	Cross product is product of magnitude of vectors & sine of angle between them.
The end result of the dot product of vectors is a scalar quantity.	The end result of the cross-product of vectors is a vector quantity.
The dot product of vectors does not have any direction because it's a scalar.	The direction of the cross product of vectors is given by the right-hand rule.
If the vectors are perpendicular to each other then their dot product is zero i.e $A . B = 0$	If the vectors are parallel to each other then their cross product is zero i.e $A \times B = 0$

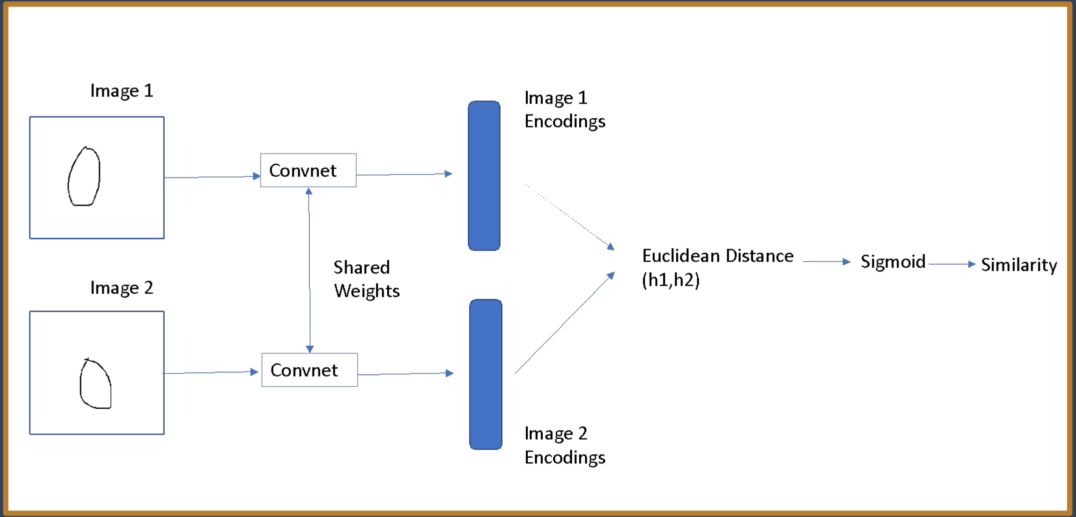
Uses of Cross Product - Magnetic Field

Consider a current carrying wire in a uniform magnetic field. Then, the direction of the force (magnetic force) on the given wire is given by the direction of cross product of the current and the magnetic field

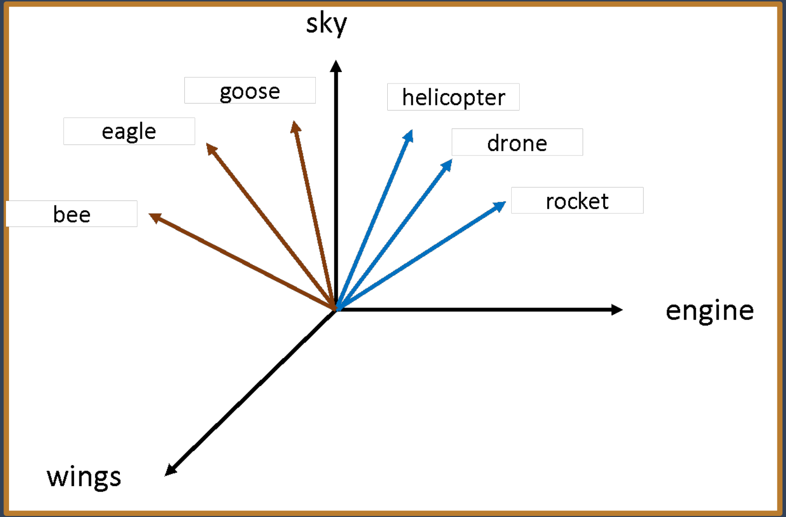




Vector Applications in Deep Learning- Embeddings



Siamese Network- Image Recognition is nothing but comparing two vectors generated from two competing images
Reference: https://towardsdatascience.com/a-friendly-introduction-to-siamese-networks-85ab17522942

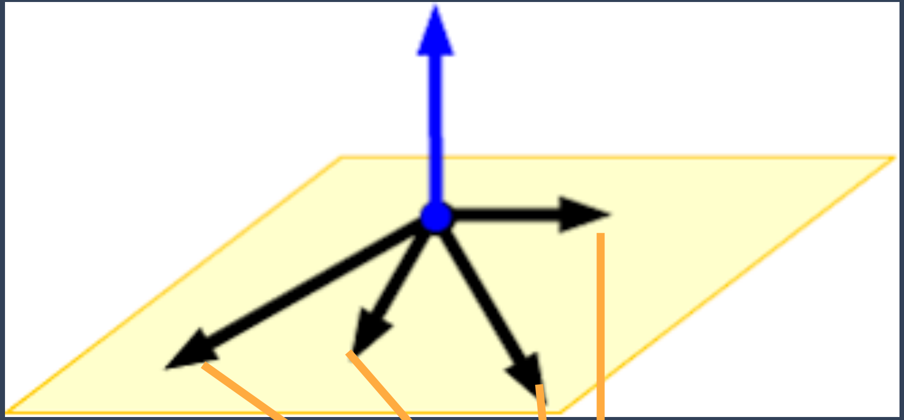


In NLP, Word Embedding- Each word is represented as a vector with the coordinates depicting concepts, this way we can find contextual understanding of words

Linear Independence and Orthogonality

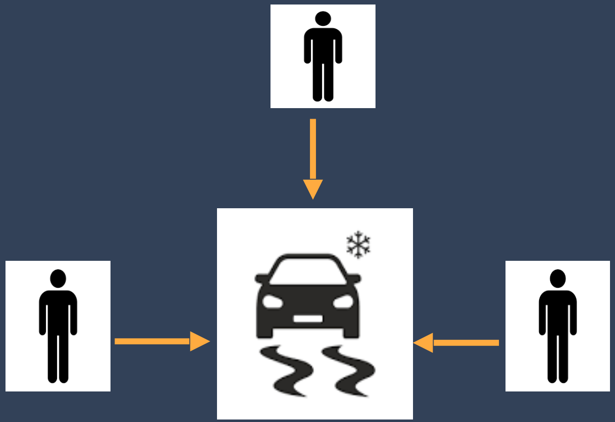
We say that 2 vectors are orthogonal if they are perpendicular to each other. Two vectors whose dot product is identically zero are called orthogonal.

A nonempty subset of nonzero vectors in R^n is called an orthogonal set if every pair of distinct vectors in the set is orthogonal. Any orthogonal set of vectors is linearly independent.



Orthogonal Vector- Independent Vector- Cannot be written as a linear combination of vectors on the yellow plane

System of vectors lying on a plane- Linearly Dependent- any vector can be written as a linear combination of the other vectors



Doesn't matter how many people push the car in either direction, the car will not get lifted unless force is applied perpendicular to the ground

Python Demo

- https://github.com/danielrferreira/classes/blob/main/linear%20algebra/Linear%20Algebra%201%20-%20Daniel%20F.ipynb

Summary

- A vector can be decomposed into its components in the i, j and k direction
- A vector projection is the vector obtained by resolving a vector in another's direction
- The dot product, also called scalar product, is a measure of how closely two vectors align
- The cross product of two vectors is perpendicular to both of them

Summary

- A vector is a quantity which has both magnitude and direction
- Vectors can be added or subtracted, or even multiplied by a scalar- all operations occur component wise
- If there exists no nontrivial linear combination of a set of vectors that equals the zero vector, the set of vectors are linearly independent

Matrices

Learning Objectives

- Understand the concept of matrices and their properties.
- Perform matrix operations such as addition, subtraction, and multiplication.
- Calculate the determinant of a matrix.
- Understand the concept of matrix rank.
- Understanding the Transpose of a Matrix

Matrix definition and notation

A matrix A is rea rectangular array of scalars presented in the form of rows and columns. It represents a linear transformation of an n dimensional vector space to an m dimensional one. It is given by an $m \times n$ array of real numbers.

Element a_{ij} is called ij-entry or ij-element in row i and column j. We denote A = [a_{ij}]

$$A= \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{11} & a_{12} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Example

$$\begin{bmatrix} 4 & -2 & 1 & -6 & 0 & 1 \\ -2 & 5 & 7 & -2 & 0 & -7 \\ 1 & 7 & 8 & 0 & 5 & -2 \end{bmatrix}$$

Matrices as a system of linear equations

Consider three linear equations in three dimensions:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Matrix representation of these linear equations

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Note that this representation can be generalized to as many variables

Matrices as a system of linear equations

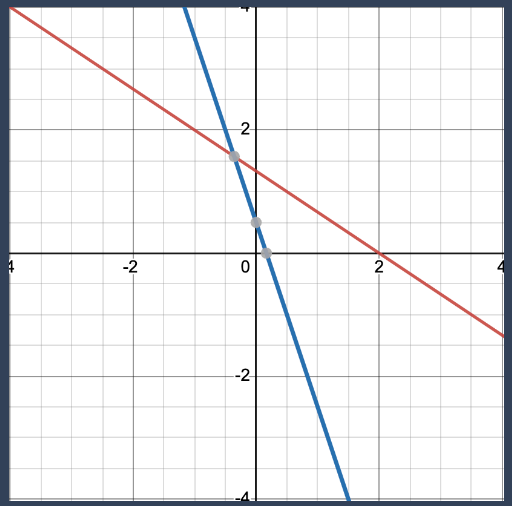
Example:

Consider the following linear equations in two Variables:

1. $2x + 3y = 4$
2. $6x + 2y = 1$

Matrix representation of these linear equations

$$\begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix}$$



Difference Between Matrix and Vector

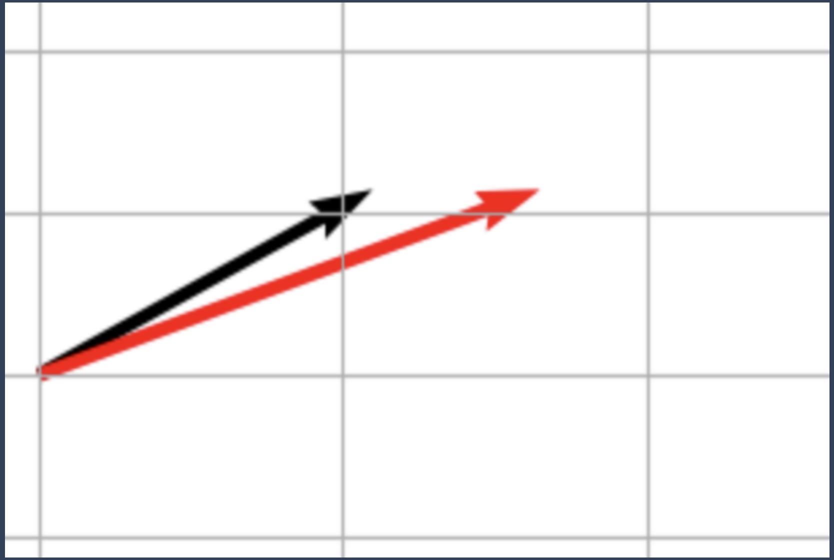
- Because a vector is just a list of numbers, we can represent it as a matrix.
- We can write vectors with matrix notation. We can always write a given vector as a row vector (1×n) or as a column vector (nx1).

$$\vec{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Matrices as System of Vectors

Example: The following matrix can be plotted as two vectors (2,1) and (3, 1)

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$



Vector 2,1 in Black

Vector 3,1 in Red

Matrix Operations

Matrix addition (of the same size). The sum of A and B is obtained by adding corresponding elements from A and B

$$A + B = \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1+3 & 3+0 \\ 2-4 & 5-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix}$$

The sum of matrices with different sizes is not defined b/c some elements from one have no corresponding elements from the other to be added to.

Matrix Operations

Scalar Multiplication: Product of Matrix A by a scalar k, kA, is the matrix obtained by multiplying each element in A by k.

$$2\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 & 2\cdot 1 \\ 2\cdot 4 & 2\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 8 & 6 \end{bmatrix}$$

Matrix Operations

-A is the negative of matrix A.

A-B is called the difference between A and B

$$A - B = A + (-1) * B$$

Matrix Operations

Matrix Multiplication:

- To perform multiplication of two matrices, we should make sure that the number of columns in the 1st matrix is equal to the rows in the 2nd matrix.

multiply $m \times p$ matrix A and $p \times n$ matrix B to get $C = AB$:

$$C_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj} = A_{i1} B_{1j} + \cdots + A_{ip} B_{pj}$$

- The product have a number of rows of the 1st matrix and a number of columns of the 2nd matrix, i.e., mxn matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 1*5 + 2*0 & 1*6 + 2*7 \\ 3*5 + 4*0 & 3*6 + 4*7 \end{bmatrix} = \begin{bmatrix} 5 & 20 \\ 15 & 46 \end{bmatrix}$$

Matrix Operations: Problem

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find $A(BA)$.

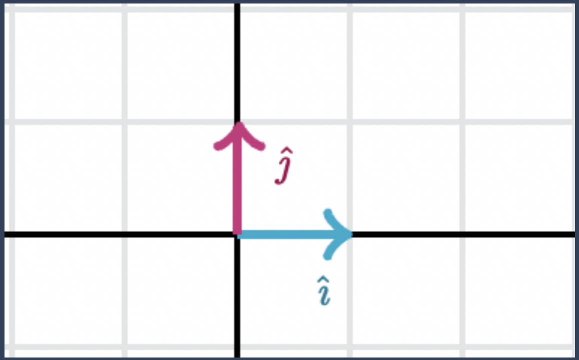
Matrix Operations: WhiteBoarding

If $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}$, write down the matrix AB.

Linear Transformation

What is the action of a matrix? What does a matrix look like?

Here is a drawing of a plane with i and j vectors (1, 0) and (0,1)



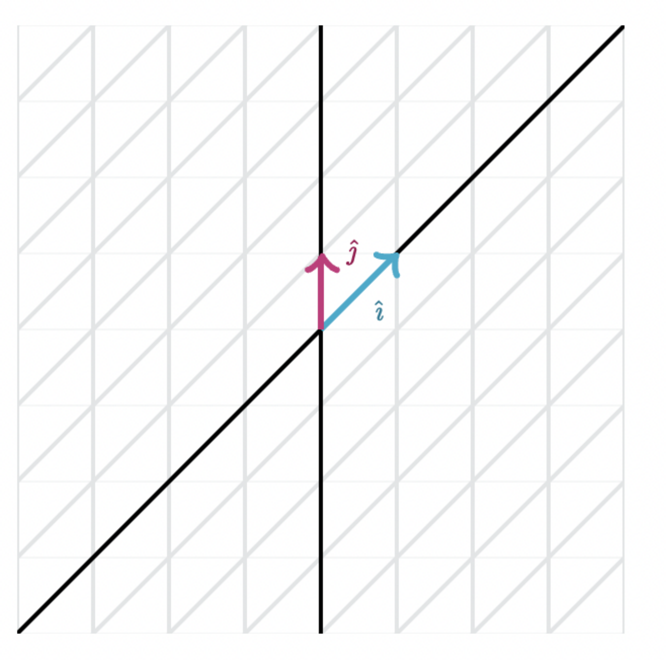
Linear Transformation

Let us consider Matrix A: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

This is how the matrix acts upon the grid:

- The columns of the matrix tell us where it moves the unit vectors i and j, which again stand for (1,0) and (0,1).
- The rest of the grid follows accordingly, always keeping grid lines parallel and evenly spaced. The origin stays frozen in place.

That means A moves i → (1,1) and j → (0,1). Here's what that looks like:



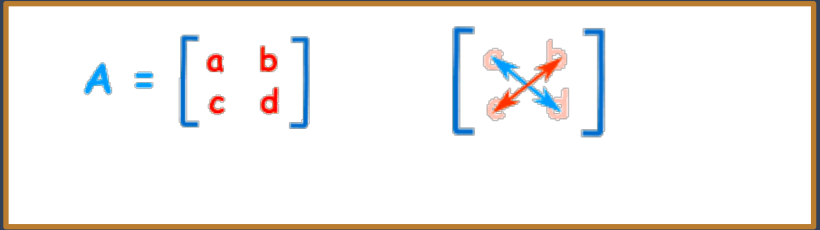
Calculation of Determinant

For a 1×1 Matrix

Let A = [a] be the matrix of order 1, then determinant of A is defined to be equal to a.

For a 2×2 Matrix

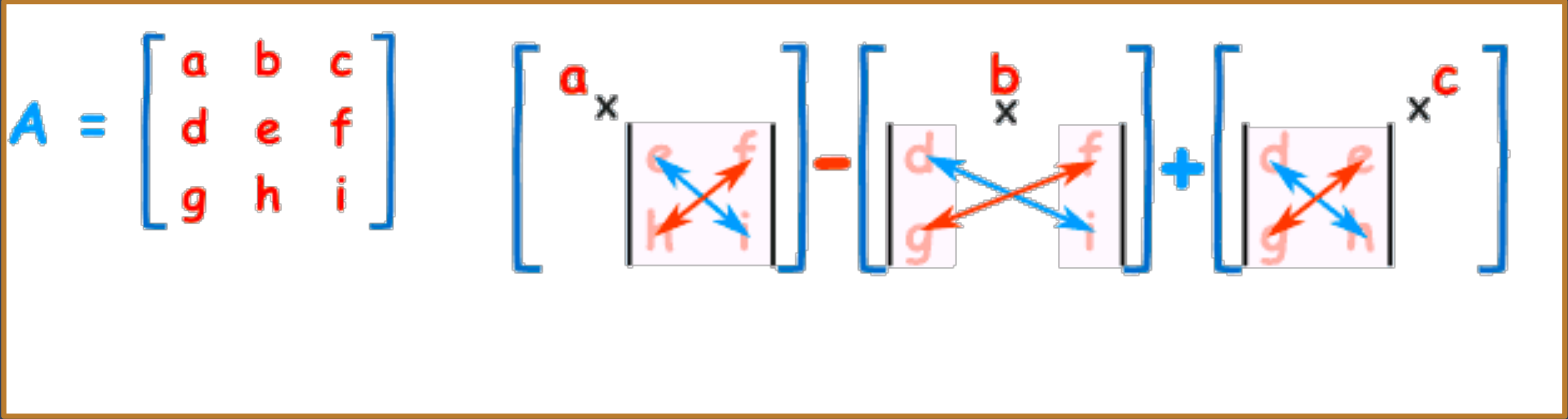
Det(A) = ad - bc



Calculation of Determinant

For a 3×3 Matrix

Det(A) = a (ei – fh) – b (di – fg) + c (dh – eg)



Calculation of Determinant

For 4×4 Matrices and Higher

The pattern continues

$$a \times \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \times \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \times \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \times \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Calculation of Determinant

Example 1

$$\det\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$$

Calculation of Determinant: Whiteboarding

Calculation of Determinant: Solution

$$\det\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} = 2 \bullet \det\begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix} - (-3) \bullet \det\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} + 1 \bullet \det\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$$

$$= 2[0-(-4)] + 3[10-(-1)] + 1[8-0]$$

$$= 2(0+4) + 3(10+1) + 1(8)$$

$$= 2(4) + 3(11) + 8$$

$$= 8 + 33 + 8$$

$$= 49 \checkmark$$

Calculation of Determinant

Example 2

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{bmatrix}$$

Calculation of Determinant: Whiteboarding

Calculation of Determinant: Solution

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} \boxed{1} & -2 & 3 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{bmatrix} - \begin{bmatrix} 1 & \boxed{-2} & 3 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -2 & \boxed{3} \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{bmatrix}$$

$$= \boxed{1} \times \begin{vmatrix} 0 & 3 \\ 5 & 4 \end{vmatrix} - \boxed{-2} \times \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + \boxed{3} \times \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix}$$

$$= 1 \times (0 - 15) + 2 \times (8 - 3) + 3 \times (10 - 0)$$

$$= 1(-15) + 2(5) + 3(10)$$

$$= -15 + 10 + 30$$

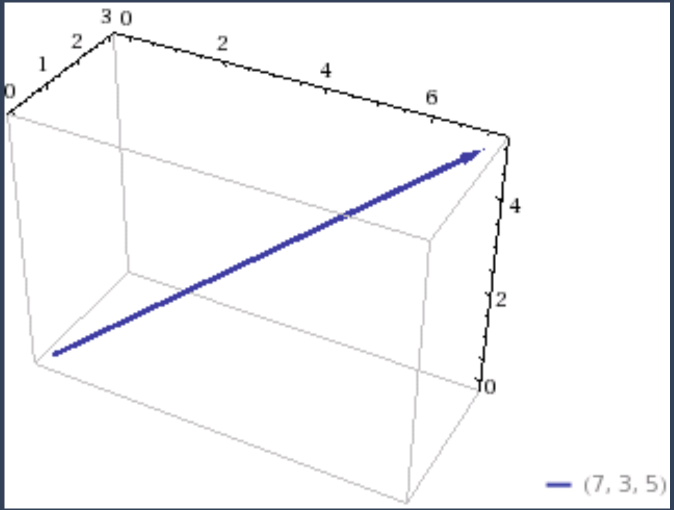
$$= 25$$

Rank of a Matrix

A matrix is a collection of vectors. You can intuitively imagine a vector as a point in space. If you want, you can draw a line from the origin to that point, and put a little arrow on the tip.

Normally, we imagine vectors in 2 or 3 dimensions:

Here, you see the vector [7, 3, 5]. Intuitively, you can imagine an arrow that goes from the origin to the point that's 7 units in the x direction, then 3 units in the y direction, and 5 units in the z direction.

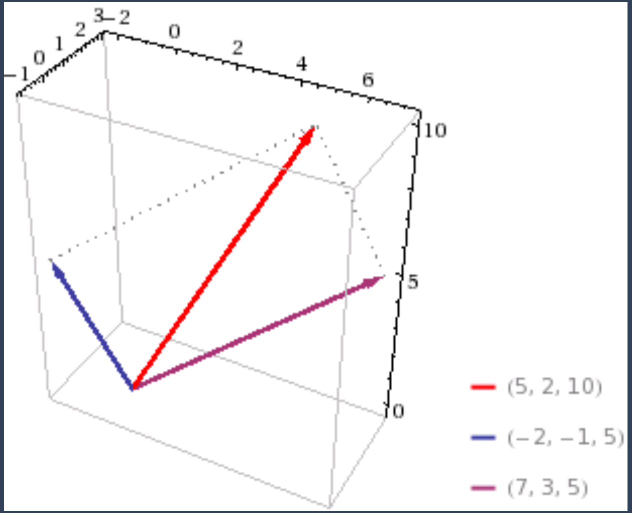


Rank of a Matrix

Now imagine just two of these 3D vectors. You can imagine adding these vectors by taking the tip of one, and using it as the origin of the other. That would give you some new vector.

Here, we add the original vector [7, 3, 5] (purple) and another vector [-2, -1, 5] (blue), and got [5, 2, 10] (red). (Recall the parallelogram law)

You may notice that the new vector lies in the plane of the original two. It's no coincidence. If we scale the original vectors by some constant, we could shorten or increase their lengths. We could even scale them by a negative amount, and point them in the opposite direction. When you add them, they'd still pick some new point in that same plane.

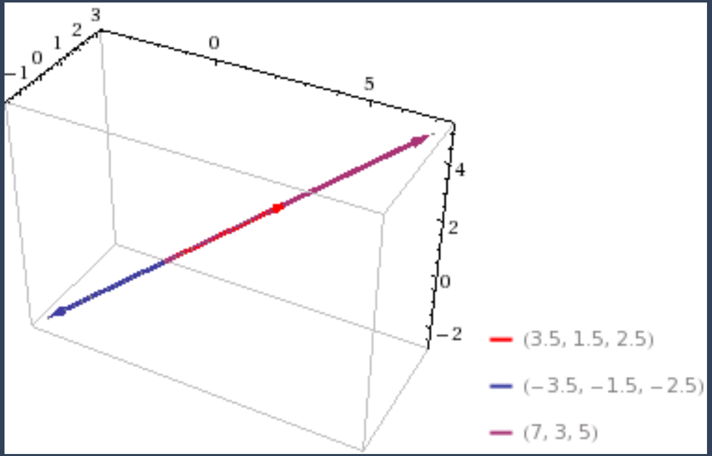


Rank of a Matrix

Now, imagine you take all possible combinations of the two vectors. The set of all possible points you could reach by scaling and adding these two vectors is the entire plane that passes through the two vectors and the origin.

When we do this, we are imagining the space that the vectors "span". In this case, it's a plane. Imagine the two vectors we started with are actually placed along the same line.

[7, 3, 5] and [-3.5, -1.5, -2.5] are two distinct vectors, but they are on the same line. No amount of combining these two would ever escape the line they both lie on. Thus, the space "spanned" by the two vectors is a single line, rather than a plane.



Row echelon form

- A matrix is in row echelon form when all its non-zero rows have a pivot, that is, a non-zero leading entry, and any pivot is to the right of the pivot in the preceding row (such that all the entries to its left and below it are equal to zero).
- An element of a matrix is a pivot if it is non-zero and all the entries to its left and below it are equal to zero.

Example The matrix

$$A = \begin{bmatrix} \underline{2} & 0 & 3 \\ 0 & 0 & \underline{1} \\ 0 & 0 & 0 \end{bmatrix}$$

is in row echelon form. It has one zero row (the third), which is below the non-zero rows. Both the first and the second row have a pivot (A_{11} and A_{23}, respectively).

Example The matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ \underline{3} & 0 & 5 & 0 \end{bmatrix}$$

is not in row echelon form because its first row is non-zero and has no pivots.

$$A = \begin{bmatrix} \underline{2} & 1 & 0 \\ 0 & \underline{1} & 1 \\ 0 & 0 & \underline{1} \\ 0 & 0 & 0 \end{bmatrix}$$

Find rank of a matrix by echelon form

- The rank is the number of linearly independent rows or columns.
- Reduce the matrix to echelon form (upper triangular matrix where all elements off diagonal are zero) through elementary row operations
- Rank is the no. of non-zero rows
- **Elementary row operations** include
- 1. Interchanging two rows.
- For example, $R_1 \leftrightarrow R_2$.
- 2. Multiplying/dividing a row by a scalar.
- For example, if the first row (all elements of the first row) is multiplied by some scalar, say 3, $R_1 \rightarrow 3R_1$.
- 3. Multiplying/dividing a row by some scalar and adding/subtracting to the corresponding elements of another row.

Row Echelon Form by Elementary Row Operations

R2 →R2+2R1

R3 →R3-R1+R2

R4 →R4-2R1

$$B = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow[\text{form}]{\text{Row echelon}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rank}(B) = 2$$

Whiteboarding: Rank

Singular Matrix

- A square matrix is said to be a singular matrix if its determinant is zero and hence it is not invertible. Det(A) = 0
- In a singular matrix, some rows and columns are linearly dependent.
- The rank of the matrix will be less than the order of the matrix.
- Order of matrix gives the dimension of the matrix and is represented as $A_{m \times n}$,

Python Demo

- https://github.com/danielrferreira/classes/blob/main/linear%20algebra/Linear%20Algebra%201%20-%20Daniel%20F.ipynb

Transpose of a Matrix

The transpose of a matrix is found by interchanging its rows into columns or columns into rows. For example, if "A" is the given matrix, then the transpose of the matrix is represented by A' or A^T.

$$M = \begin{bmatrix} 2 & -9 & 3 \\ 13 & 11 & -17 \\ 3 & 6 & 15 \\ 4 & 13 & 1 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 2 & 13 & 3 & 4 \\ -9 & 11 & 6 & 13 \\ 3 & -17 & 15 & 1 \end{bmatrix}$$

Properties of Transpose

1. Transpose of a Transpose Matrix is the matrix itself: $(A^T)^T = A$

$$N = \begin{bmatrix} 22 & -21 & -99 \\ 85 & 31 & -2\sqrt{3} \\ 7 & -12 & 57 \end{bmatrix},$$

Then

$$N' = \begin{bmatrix} 22 & 85 & 7 \\ -21 & 31 & -12 \\ -99 & -2\sqrt{3} & 57 \end{bmatrix}$$

Now,

$$(N')' = \begin{bmatrix} 22 & -21 & -99 \\ 85 & 31 & -2\sqrt{3} \\ 7 & -12 & 57 \end{bmatrix}$$

$$= N$$

Properties of Transpose

2. Addition property of Transpose: $(A + B)^T = A^T + B^T$

$$If\ P = \begin{bmatrix} 2 & -3 & 8 \\ 21 & 6 & -6 \\ 4 & -33 & 19 \end{bmatrix}\ and\ Q = \begin{bmatrix} 1 & -29 & -8 \\ 2 & 0 & 3 \\ 17 & 15 & 4 \end{bmatrix}$$

$$P + Q = \begin{bmatrix} 2+1 & -3-29 & 8-8 \\ 21+2 & 6+0 & -6+3 \\ 4+17 & -33+15 & 19+4 \end{bmatrix} = \begin{bmatrix} 3 & -32 & 0 \\ 23 & 6 & -3 \\ 21 & -18 & 23 \end{bmatrix}$$

$$(P+Q)' = \begin{bmatrix} 3 & 23 & 21 \\ -32 & 6 & -18 \\ 0 & -3 & 23 \end{bmatrix}$$

$$P' + Q' = \begin{bmatrix} 2 & 21 & 4 \\ -3 & 6 & -33 \\ 8 & -6 & 19 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 17 \\ -29 & 0 & 15 \\ -8 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 23 & 21 \\ -32 & 6 & -18 \\ 0 & -3 & 23 \end{bmatrix}$$

$$= (P+Q)'$$

Properties of Transpose

3. Multiplication by a scalar: $(kA)^T = k(A^T)$

$$If\ P = \begin{bmatrix} 2 & 8 & 9 \\ 11 & -15 & -13 \end{bmatrix}_{2\times3}\ and\ k\ is\ a\ constant,\ then\ (kP)'$$

$$= \begin{bmatrix} 2k & 11k \\ 8k & -15k \\ 9k & -13k \end{bmatrix}_{2\times3}$$

$$kP' = k\begin{bmatrix} 2 & 11 \\ 8 & -15 \\ 9 & -13 \end{bmatrix}_{2\times3} = \begin{bmatrix} 2k & 11k \\ 8k & -15k \\ 9k & -13k \end{bmatrix}_{2\times3} = (kP)'$$

We can observe that

$(kP)' = kP'$.

Properties of Transpose

4. Multiplication property of Transpose: $(AB)^T = B^T A^T$

$$A = \begin{bmatrix} 9 & 8 \\ 2 & -3 \end{bmatrix} and\ B = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$$

Let us find A×B.

$$A \times B = \begin{bmatrix} 44 & 18 \\ 5 & 4 \end{bmatrix} \Rightarrow (AB)' = \begin{bmatrix} 44 & 5 \\ 18 & 4 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 9 & 2 \\ 8 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 5 \\ 18 & 4 \end{bmatrix} = (AB)'$$

$$\therefore (AB)' = B'A'$$

Whiteboarding

Summary

- Matrices are 2-D arrays which denotes a system of linear equations or a system of vectors
- Matrices can be added, subtracted, multiplied and divided by scalars
- Matrices can be added, subtracted, multiplied to other matrices
- Matrices can also be imagined as Linear Transformation of 2-D plane and Determinant of a matrix is the scaling factor for the transformation
- Rank of a matrix is the number of linearly independent rows or columns
- Calculation and Properties of Transpose of a Matrix

Inverse of a Matrix

Learning Objectives

- Understand the concept of matrix inverse and its properties.
- Calculate the inverse of a matrix.

Inverse of a Matrix

Just like an integer *a* and inverse of *a (reciprocal)* satisfies,

$$a \times a^{-1} = 1$$

For an *n x n,* non singular matrix, A,

$$AA^{-1} = A^{-1}A = I$$

Where I is the Identity matrix.

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 1 & & & \cdot \\ \cdot & \cdot & & 1 & & \cdot \\ \cdot & \cdot & & & 1 & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}_{n\times n}$$

Calculation of Inverse

- To calculate the inverse of a matrix we use the following formula,

$$A^{-1} = adj(A) / det(A)$$

where adj(A) refers to the adjoint of a matrix A, det(A) refers to the determinant of a matrix A.

- In order to find the adjoint of a matrix A first, find the cofactor matrix of a given matrix and then take the transpose of a cofactor matrix.
- The cofactor of a matrix can be obtained as

$$C_{ij} = (-1)^{i+j} \det (M_{ij})$$

Here, Mij refers to the (i,j)th minor matrix after removing the ith row and the jth column.

Matrix Inversion Example 1: Calculate Matrix Inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1}$$

Matrix Inversion Example 1: Whiteboarding

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$.

Matrix Inversion Example 1: Solution

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$.

Then its cofactor matrix is:

$$\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

Each 2x2 determinant is obtained by multiplying diagonals and subtracting the products (from left to right).

So the cofactor matrix = $\begin{bmatrix} 1-4 & -(2+2) & 4+1 \\ -(2+2) & 1-1 & -(2+2) \\ 4+1 & -(2+2) & 1-4 \end{bmatrix}$

$$= \begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$$

By transposing the cofactor matrix, we get the adjoint matrix.

So adj A = $\begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$. Let us use the first row to find the determinant.

det A = 1 (cofactor of 1) + 2 (cofactor of 2) + (-1) cofactor of (-1)

= 1(-3) + 2(-4) + (-1)5

= -3 - 8 - 5

= -16

- **Step - 1:** Find adj A.

  We have already seen that adj A = $\begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$.

- **Step - 2:** Find det A.

  We have already seen that det A = -16

- **Step - 3:** Apply the inverse of 3x3 matrix formula A^{-1} = (adj A)/(det A).

  i.e., divide every element of adj A by det A.

  Then $A^{-1} = \begin{bmatrix} -3/-16 & -4/-16 & 5/-16 \\ -4/-16 & 0/-16 & -4/-16 \\ 5/-16 & -4/-16 & -3/-16 \end{bmatrix}$

  $= \begin{bmatrix} 3/16 & 1/4 & -5/16 \\ 1/4 & 0 & 1/4 \\ -5/16 & 1/4 & 3/16 \end{bmatrix}$.

Example 2

Find the inverse of a matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix}$

Whiteboarding

Solution

Let us find the minors of the given matrix as given below:

$$M_{1,1} = \det\begin{pmatrix} 5 & 6 \\ 2 & 9 \end{pmatrix} = 33$$

$$M_{1,2} = \det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} = -6$$

$$M_{1,3} = \det\begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix} = -27$$

$$M_{2,1} = \det\begin{pmatrix} 2 & 3 \\ 2 & 9 \end{pmatrix} = 12$$

$$M_{2,2} = \det\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} = -12$$

$$M_{2,3} = \det\begin{pmatrix} 1 & 2 \\ 7 & 2 \end{pmatrix} = -12$$

$$M_{3,1} = \det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = -3$$

$$M_{3,2} = \det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = -6$$

$$M_{3,3} = \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}: \quad -3$$

cofactors: $\begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}$

Determinant of the given matrix is

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix} = 1 \cdot 33 - 2(-6) + 3(-27)$$
$$= -36$$

Now, find the adjoint of a matrix by taking the transpose of cofactors of the given matrix.

$$\begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}^T = \begin{pmatrix} 33 & -12 & -3 \\ 6 & -12 & 6 \\ -27 & 12 & -3 \end{pmatrix}$$

Now,

$A^{-1} = (1/|A|) \text{ Adj } A$

Hence, the inverse of the given matrix is:

$$= \begin{pmatrix} -\frac{11}{12} & \frac{1}{3} & \frac{1}{12} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{3}{4} & -\frac{1}{3} & \frac{1}{12} \end{pmatrix}$$

Properties of Inverse of Matrices

1. If A is nonsingular, then $(A^{-1})^{-1} = A$
2. If A and B are nonsingular matrices, then AB is nonsingular. Thus, $(AB)^{-1} = B^{-1}A^{-1}$
3. If A is nonsingular then $(AT)^{-1} = (A^{-1})^T$
4. If A is any matrix and A^{-1} is its inverse, then $AA^{-1} = A^{-1}A = I_n$, where n is the order of matrices
5. Not all matrices have an inverse

Python Demo

- https://github.com/danielrferreira/classes/blob/main/linear%20algebra/Linear%20Algebra%201%20-%20Daniel%20F.ipynb
-

Summary

- A multiplicative matrix inverse exists and is used to divide two matrices.
- $A.A^{-1} = I$
- $A^{-1} = adj(A) / det(A)$

Eigenvectors and Eigenvalues

Agenda

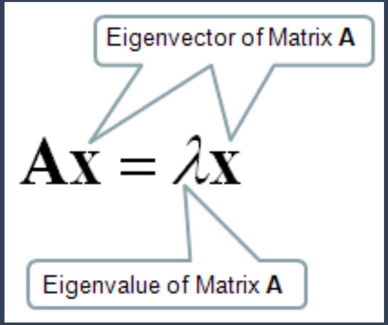
- Introduction to eigenvalues and eigenvectors
- Properties of Eigenvalues and Eigenvectors
- Calculation of Eigenvalues and Eigenvectors
- Geometric Interpretation of Eigenvectors

Learning Objectives

- Understand the concepts of eigenvalues and eigenvectors and their properties.
- Calculate eigenvalues and eigenvectors for a given matrix.
- Understand the geometric interpretation of eigenvalues and eigenvectors.

Eigenvector

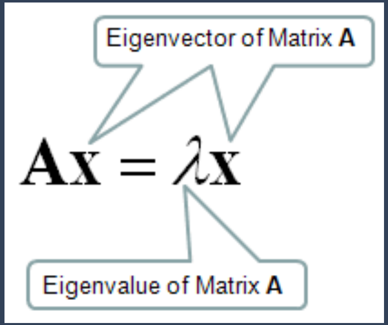
In general, the eigenvector vec{v} of a matrix A is the vector for which the following holds:



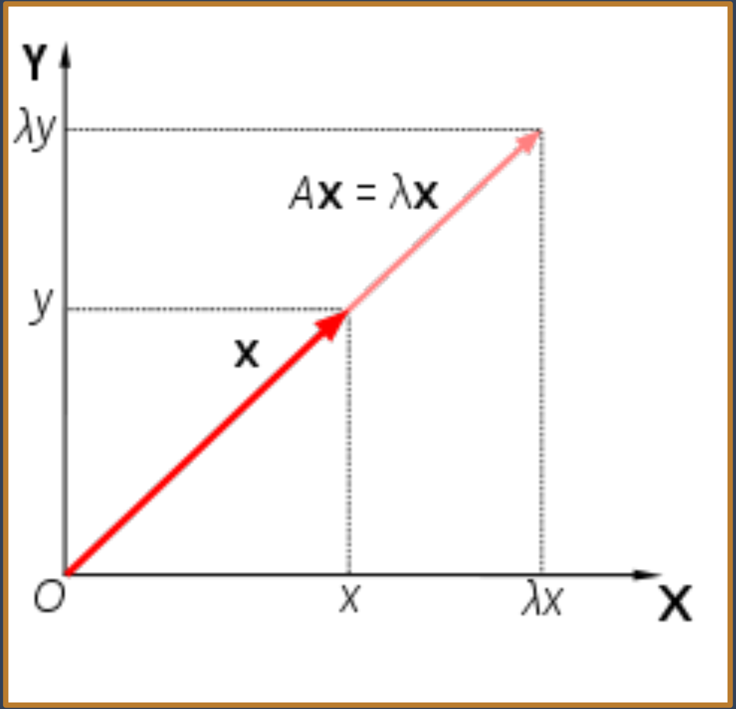
where λ is a scalar value called the ‘eigenvalue’. This means that the linear transformation A on vector vec{v} is completely defined by λ.

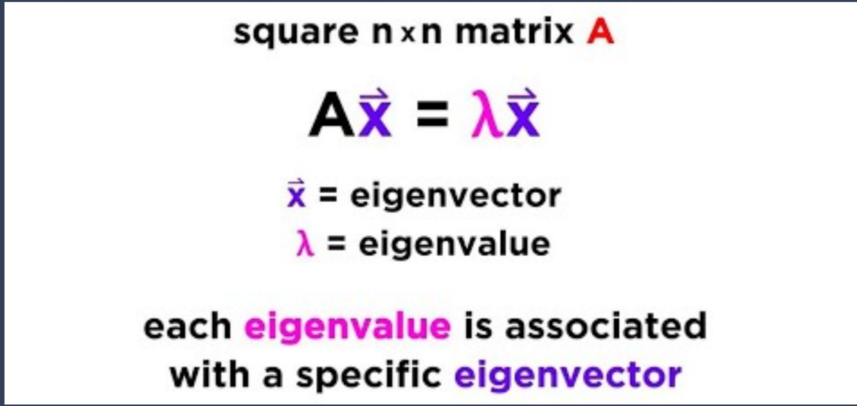
Eigenvalues

An eigenvalue of a matrix is a scalar value that represents the scaling factor of a particular vector when a transformation matrix is multiplied by that vector.



Eigenvalues and Eigenvectors

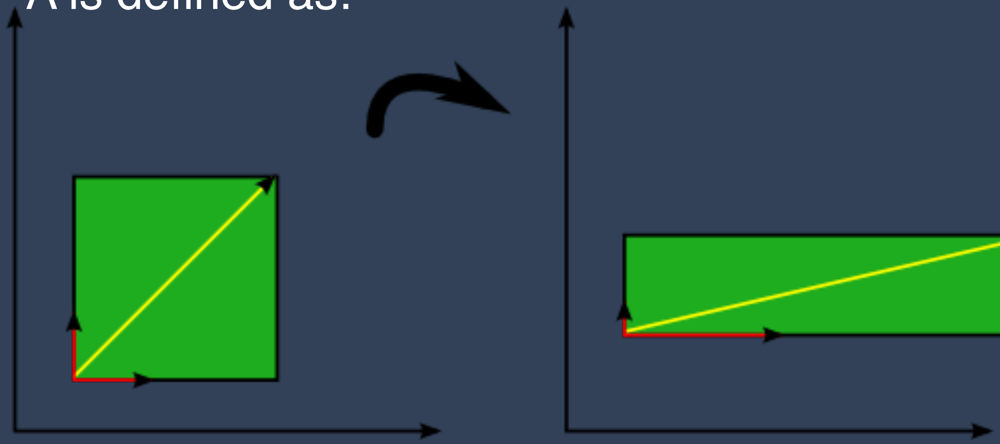




Eigenvectors

Eigenvectors (red) do not change direction when a linear transformation (e.g. scaling) is applied to them. Other vectors (yellow) do.

The transformation in this case is a simple scaling with factor 2 in the horizontal direction and factor 0.5 in the vertical direction, such that the transformation matrix A is defined as:



$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Whiteboarding

Properties of Eigenvalues

1. Eigenvalues of real symmetric matrices are always real.
2. The eigenvalues of skew-symmetric matrices are either imaginary or zero. (AT=-A)
3. Singular Matrices have Zero Eigenvalues
4. The sum of the eigenvalues of a matrix is equal to the trace (sum of diagonal elements) of the matrix
5. Eigenvectors with Distinct Eigenvalues are Linearly Independent
6. If A is a square matrix, then $\lambda = 0$ is not an eigenvalue of A
7. For a scalar multiple of a matrix: If A is a square matrix and λ is an eigenvalue of A. Then, $a\lambda$ is an eigenvalue of aA

Properties of Eigenvalues

7. **For a scalar multiple of a matrix**: If A is a square matrix and λ is an eigenvalue of A. Then, aλ is an eigenvalue of aA

$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 5$, $\lambda_2 = 2$ whereas

$\begin{pmatrix} 8 & 4 \\ 2 & 6 \end{pmatrix}$ has eigenvalues $\lambda_1 = 10$, $\lambda_2 = 4$

Properties of Eigenvalues

8. For polynomials of matrix: If A is a square matrix, λ is an eigenvalue of A and p(x) is a polynomial in variable x, then p(λ) is the eigenvalue of matrix p(A).

9. **Transpose matrix:** If A is a square matrix, λ is an eigenvalue of A, then λ is an eigenvalue of At

Calculation of Eigenvalues and Eigenvectors

From the previous slides, we know that for eigenvectors,

$$A\vec{v} = \lambda\vec{v} \qquad \text{... (1)}$$

We can rewrite this as:

$$A\vec{v} - \lambda\vec{v} = 0$$

$$\vec{v}\,(A-\lambda I) = 0 \qquad \text{... (2)}$$

Calculation of Eigenvalues and Eigenvectors

However, assuming that vec{v} is not the null-vector, equation (2) can only be defined if:

$$Det\ (A-\lambda I)= 0$$

characteristic equation

Calculation of Eigenvalues

Calculate

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}.$$

Solving using characteristic equation, we have-

$$Det \begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} = 0.$$

Calculation of Eigenvalues

Calculating the determinant gives-

$$
\begin{aligned}
&(2-\lambda)(1-\lambda)-6=0\\
\Rightarrow &2-2\lambda-\lambda-\lambda^2-6=0\\
\Rightarrow &\lambda^2-3\lambda-4=0.
\end{aligned}
$$

To solve this quadratic equation in \lambda, we find the discriminant:

$$D = b^2 - 4ac = (-3)^2 - 4 * 1 * (-4) = 9 + 16 = 25.$$

Calculation of Eigenvalues

Since the discriminant is strictly positive, this means that two different values for λ exist:

$$\lambda_1 = \frac{-b-\sqrt{D}}{2a} = \frac{3-5}{2} = -1,$$
$$\lambda_2 = \frac{-b+\sqrt{D}}{2a} = \frac{3+5}{2} = 4.$$

Calculation of Eigenvalues

We have now determined the two eigenvalues λ1 and λ2. Note that a square matrix of size N x N always has exactly N eigenvalues, each with a corresponding eigenvector. The eigenvalue specifies the size of the eigenvector.

$$\lambda_1 = -1$$

$$\lambda_2 = 4$$

Calculation of Eigenvectors

Calculating the first eigenvector

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = -1 \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}.$$

from equation 1

Writing matrix form in equivalent form:

$$\begin{cases} 2x_{11} + 3x_{12} = -x_{11} \\ 2x_{11} + x_{12} = -x_{12} \end{cases}$$

Solving which we get -

$$x_{11} = -x_{12}.$$

Calculation of Eigenvectors

Since an eigenvector simply represents an orientation (the corresponding eigenvalue represents the magnitude), all scalar multiples of the eigenvector are vectors that are parallel to this eigenvector, and are therefore equivalent (If we would normalize the vectors, they would all be equal).

Thus, instead of further solving the above system of equations, we can freely choose a real value for either x_{11} or x_{12}, and determine the other one

If we choose $x_{12} = 1$, we will get

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Following the same procedure, we can find

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Calculation of Eigenvalues - Example 2

Calculate

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

Whiteboarding

Calculation of Eigenvalues - Example 2

$|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$(5 - \lambda)(2 - \lambda) - (4)(1) = 0$

$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$(\lambda - 6)(\lambda - 1) = 0$

$\lambda = 6, \lambda = 1.$

Thus, the eigenvalues are 6 and 1

Calculation of Eigenvalues - Example 2

When $\lambda = 1$:

Substitute $\lambda = 1$ in the equation:

$(A - \lambda I)\,\mathbf{v} = O$

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x + 4y = 0$

this can be written as **-x = y**

Thus, the eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Calculation of Eigenvalues - Example 2

When $\lambda = 6$:

Substitute $\lambda = 1$ in the equation:

(A - λI) **v** = 0

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x + 4y = 0$

this can be written as **x = 4y**

Thus, the eigenvector is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Use Cases of Eigenvalues and Eigenvectors

- Component analysis is to reduce dimension space without losing valuable information. The core of PCA is built on the concept of eigenvalues and eigenvectors. The concept revolves around computing eigenvectors and eigenvalues of the covariance matrix of the features.

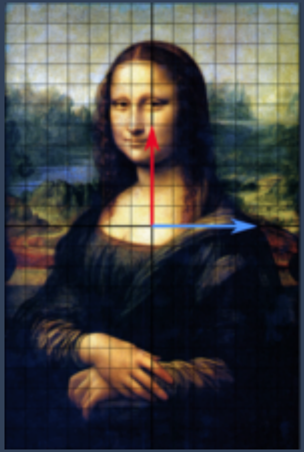
- Eigenvectors and eigenvalues are used in facial recognition techniques such as EigenFaces.

- Reduction of dimension space. The technique of Eigenvectors and Eigenvalues are used to compress the data. As mentioned above, many algorithms such as PCA rely on eigenvalues and eigenvectors to reduce the dimensions.

- Eigenvalues are used in regularisation to prevent overfitting.

Geometric Interpretation of Eigenvectors

Each point on the painting can be represented as a vector pointing from the center of the painting to that point. The linear transformation in this example is called a shear mapping. Points in the top half are moved to the right, and points in the bottom half are moved to the left, proportional to how far they are from the horizontal axis that goes through the middle of the painting. The vectors pointing to each point in the original image are therefore tilted right or left, and made longer or shorter by the transformation.

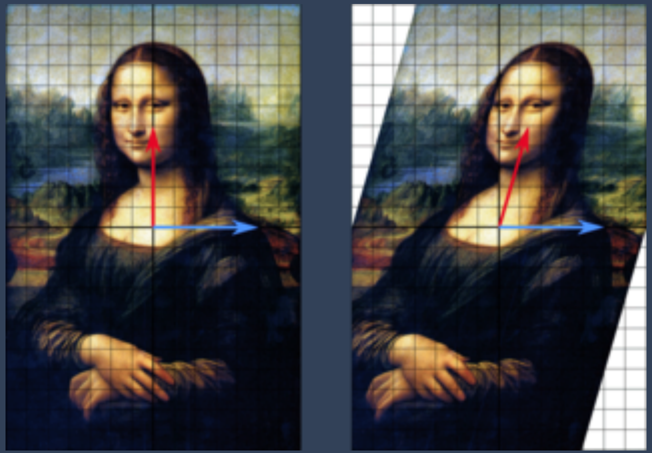


Geometric Interpretation of Eigenvectors

Points along the horizontal axis do not move at all when this transformation is applied.

Therefore, any vector that points directly to the right or left with no vertical component is an eigenvector of this transformation.

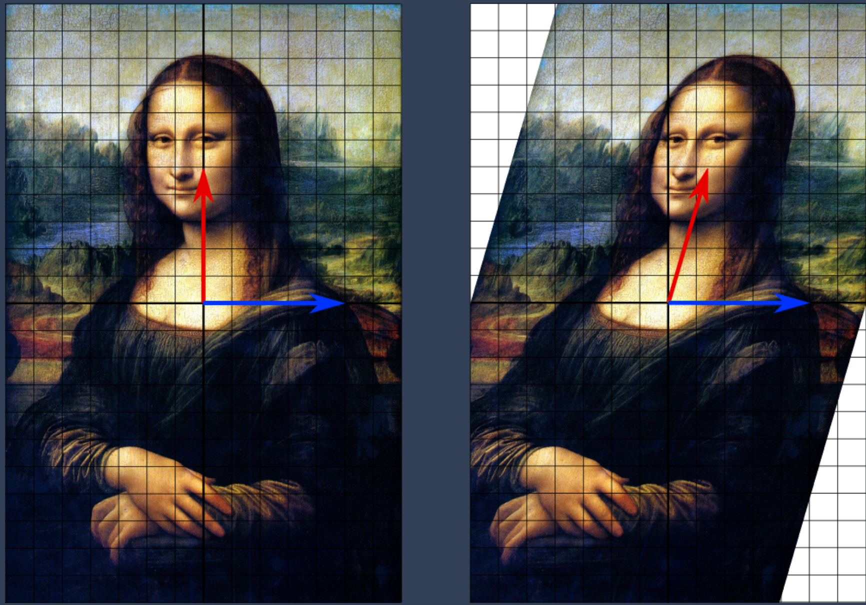
Moreover, these eigenvectors all have an eigenvalue equal to one, because the mapping does not change their length either.



Eigenvectors and Eigenvalues

In this shear mapping, the red arrow changes direction, whereas the blue arrow does not.

Here, the blue arrow is an eigenvector because it does not change direction. Also, the length of this arrow is not changed; its eigenvalue is 1.



Python Demo

- https://github.com/danielrferreira/classes/blob/main/linear%20algebra/Linear%20Algebra%201%20-%20Daniel%20F.ipynb
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Summary

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it
- The value by which the magnitude of that vector changes is called eigenvalue

References

- https://towardsdatascience.com/an-intuitive-explanation-of-vectors-23e15d35f35e
- https://intuitive-math.club/linear-algebra/matrices/
- https://byjus.com/maths/properties-of-matrices-transpose/
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- https://towardsdatascience.com/eigen-intuitions-understanding-eigenvectors-and-eigenvalues-630e9ef1f719
- https://www.youtube.com/watch?v=R13Cwgmpuxc
- https://www.youtube.com/watch?v=Ip3X9LOh2dk

Key Takeaways

- You should be able to understand and operate on Vectors and Matrices
- You should be comfortable in understanding vector dot products and cross products
- You should be able to understand Matrix Operations, Transpose Properties, Rank, Determinant Calculation, Inverse Calculation
- You should be able to understand Eigen-Vectors and Eigen-Values

Thank You