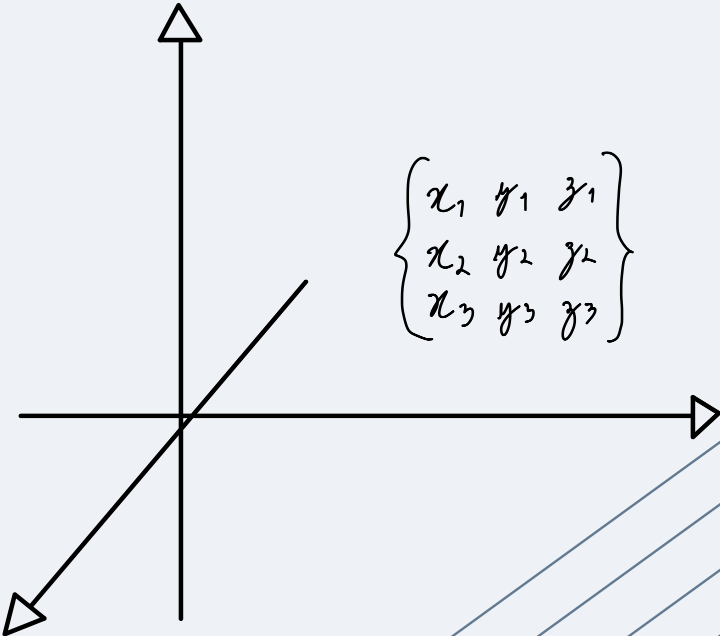
Linear Algebra



LINEAR ALGEBRA

- INTRODUCTIONS
- VECTORS
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    - PROPERTIES
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- MATRICES
    - NOTATION
    - PROPERTIES
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- EIGENVALUES & EIGENVECTORS

INTRODUCTIONS

- INSTRUCTOR
- CLASS STRUCTURE
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- CONTENT
- WHY LINEAR ALGEBRA?

VECTORS

VECTORS

- NOTATIONS
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- OPERATIONS: SCALAR MULTIPLICAT
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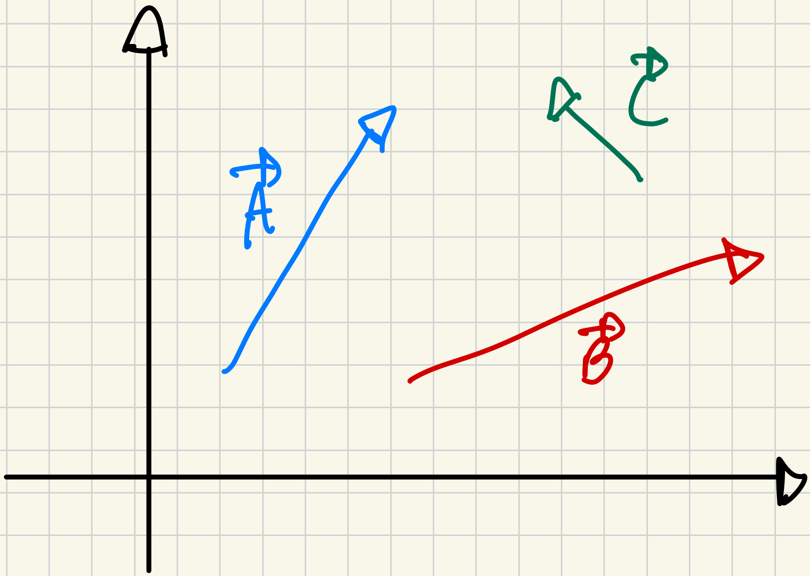
- OPERATIONS: DOT PRODUCT
- ANGLE
- PROJECTION
- OPERATIONS: CROSS PRODUCT
- LINEAR INDEPENDENCE & ORTHOGONALITY
- PYTHON APPS

VECTORS NOTATION



VECTORS HAVE:

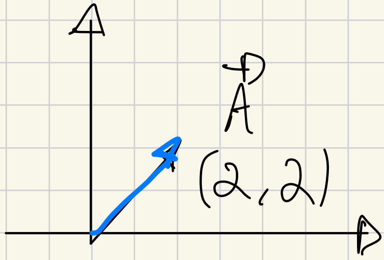
- MAGNITUDE
- DIRECTION

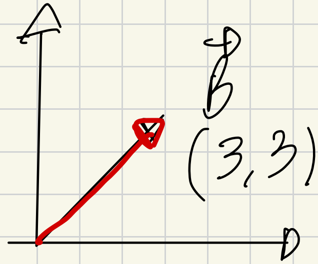
EXAMPLES:

- VELOCITY & FORCE

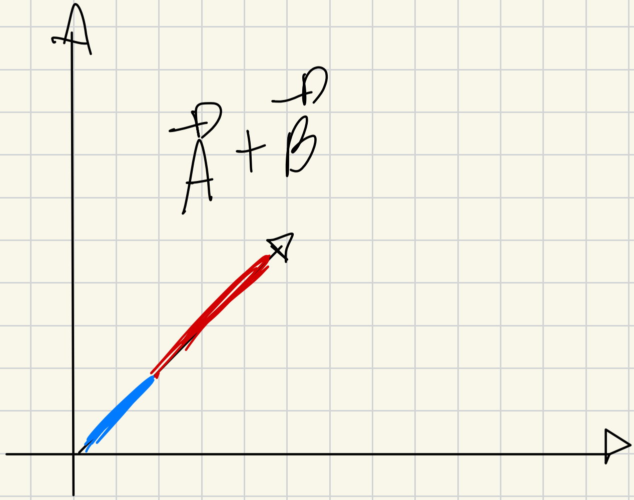
SIMPLE OPERATIOS

ADDITION & SUBSTRACTION



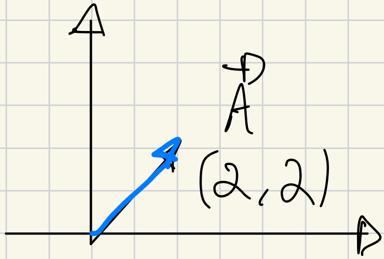


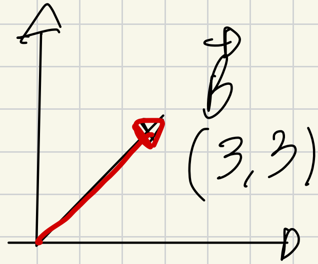
BECAUSE $\vec{A}$ & $\vec{B}$ HAVE THE SAME DIRECTION, THE MAGNITUDE OF $\vec{A}+\vec{B}$ IS THE SUM OF BOTH MAGNITUDES:



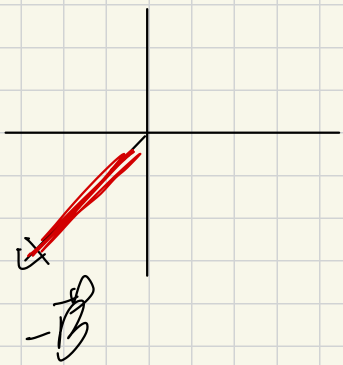
SIMPLE OPERATIOS

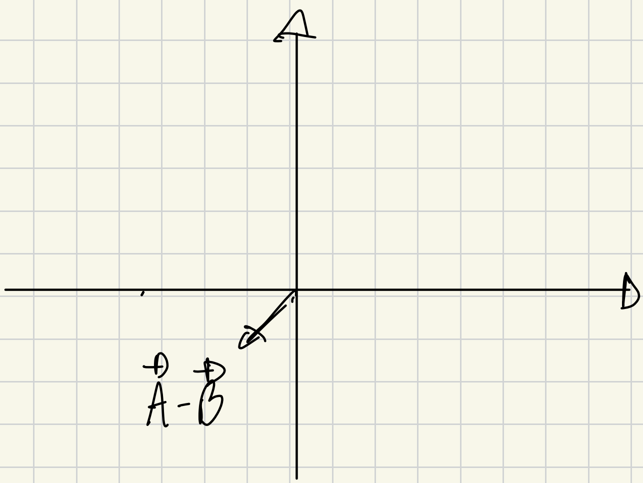
ADDITION & SUBSTRACTION



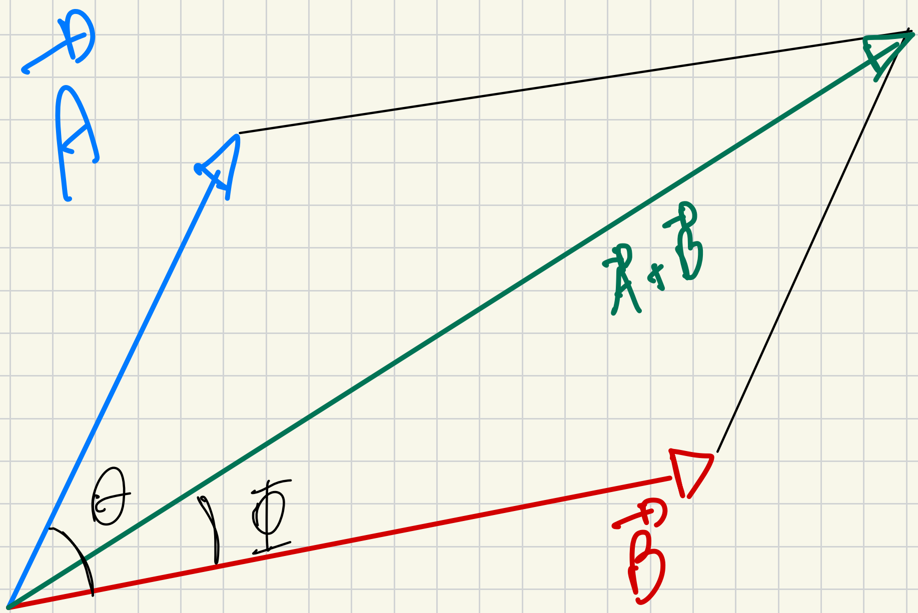


BECAUSE $\vec{A}$ & $\vec{B}$ HAVE THE SAME DIRECTION, THE MAGNITUDE OF $\vec{A}-\vec{B}$ IS THE SUBS OF BOTH MAGNITUDES:





PARALLELOGRAM LAW

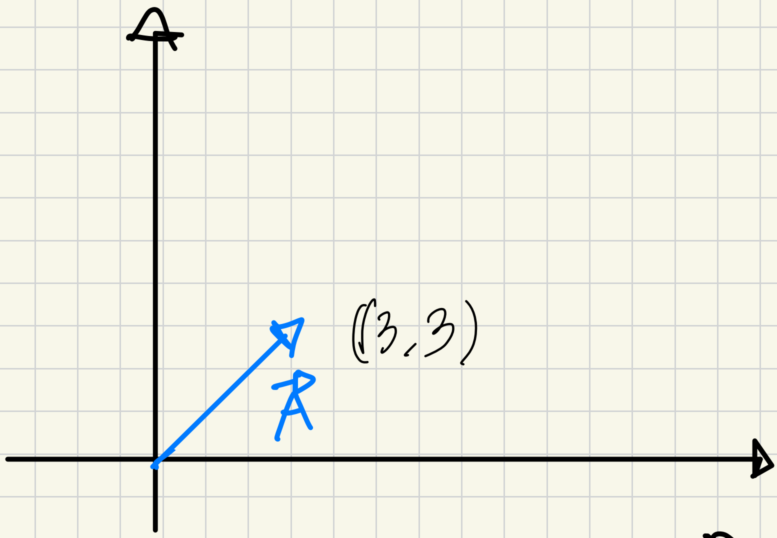


$$|\vec{A}+\vec{B}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 + 2|A|\cdot|B|\cdot\cos(\theta)}$$

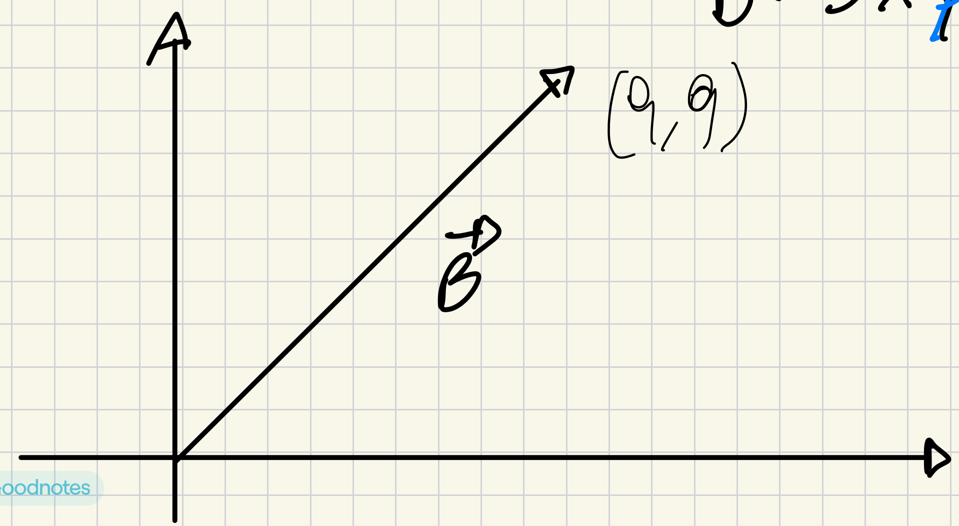
$$\Phi = \tan^{-1}\left(\frac{|\vec{A}|\sin(\theta)}{|\vec{B}|+|\vec{A}|\cos(\theta)}\right)$$

SCALAR MULTIPLICATION

ONLY THE MAGNITUDE CHANGES:

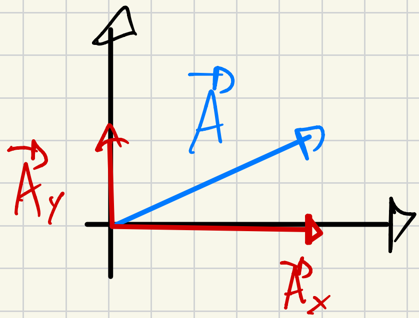


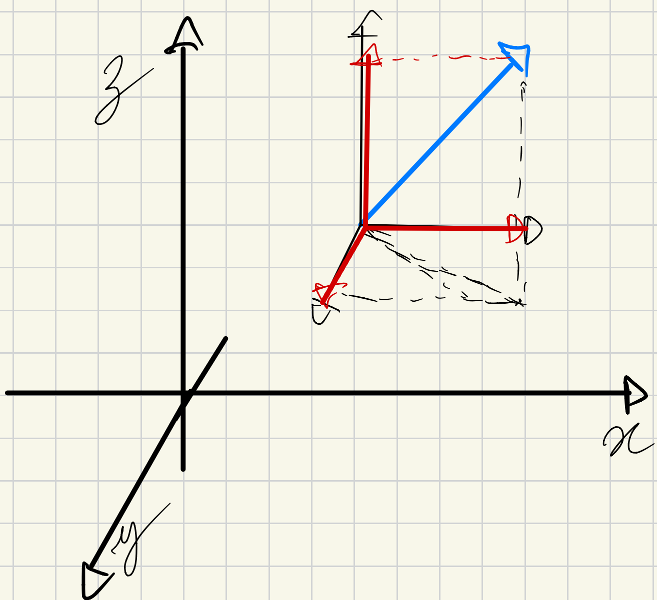
$$\vec{B} = 3 \times \vec{A}$$



DECOMPOSITION OF VECTORS

BREAKING ONE VECTOR INTO N SCALARS CAN HELP US CALCULATE MORE OPERATIONS:

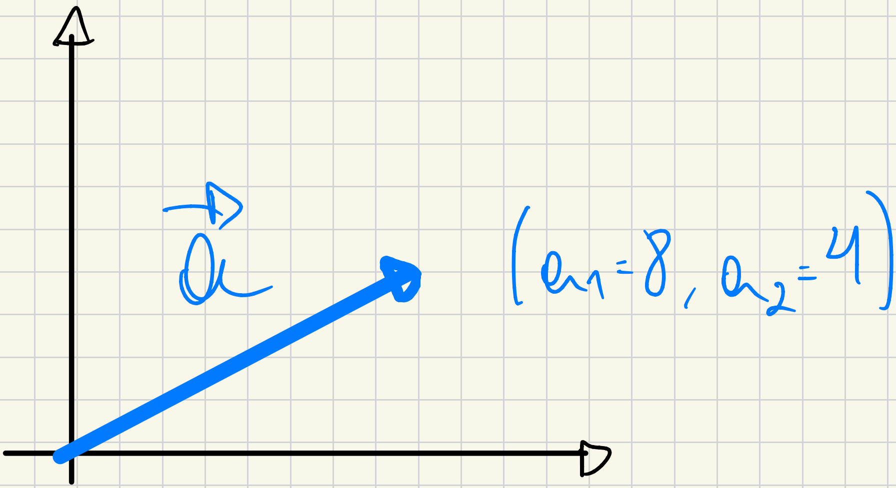




NORM

NORM OF A VECTOR REPRESENTS THE LENGTH OF A VECTOR:

$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots a_n^2}$$

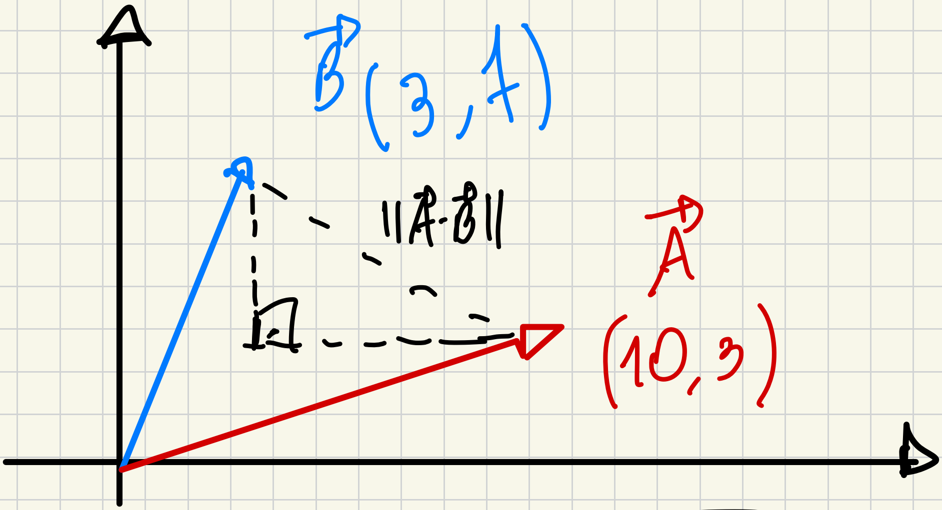


$$\|a\| = \sqrt{8^2 + 4^2} \simeq 8.94$$

np.linalg.norm()

DISTANCE

DISTANCE IS A VERY USED CALCULATION TO UNDERSTAND HOW CLOSE TO VECTORS ARE:

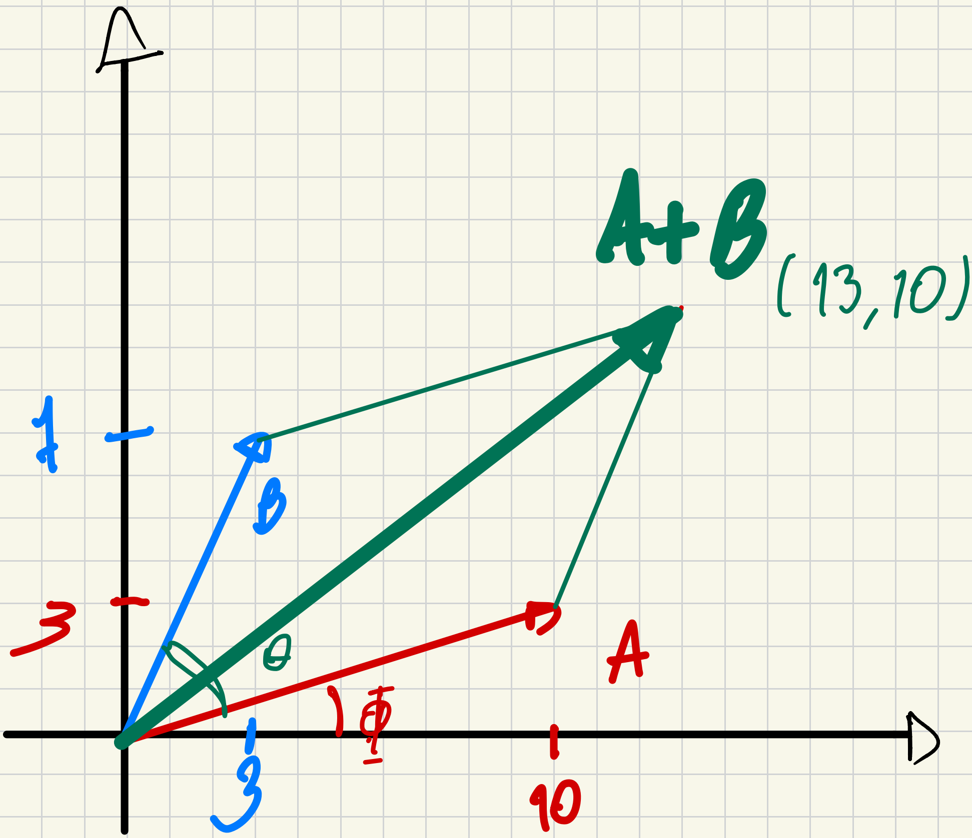


$$D(\vec{A},\vec{B}) = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2}$$

$$= \sqrt{(10-3)^2+(3-7)^2} \simeq 8.06$$

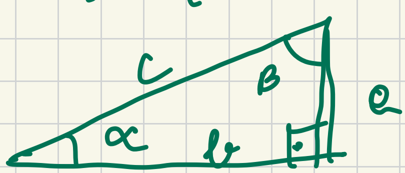
$$d(u,v) = ||u-v|| = \sqrt{(u_1-v_1)^2+\dots(u_n-v_n)^2}$$

SCRATCH (IGNORE it)



$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = c$$

$$\alpha = \operatorname{asin}\left(\frac{c}{a}\right)$$



DOT PRODUCT (SCALAR PRODUCT)

IF DECOMPOSITIONS ARE AVAILABLE:

$$u = (u_1, u_2, \dots, u_n)$$
$$v = (v_1, v_2, \dots, v_n)$$
$$u \cdot v = u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$$

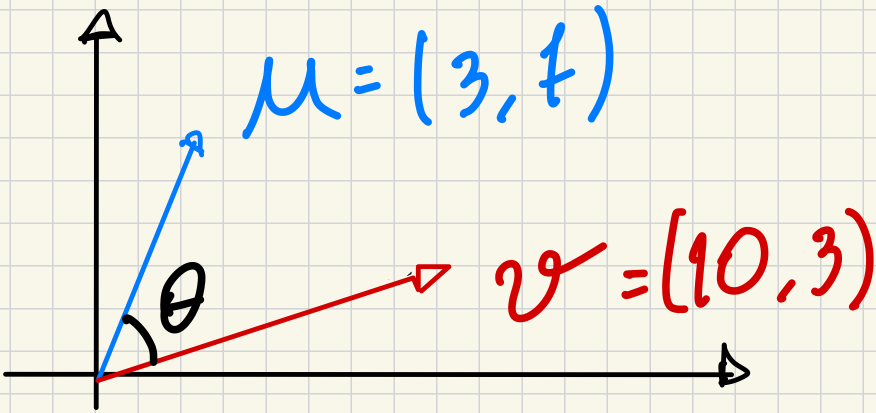
$$= u^T \cdot v$$

IF MAGNITUDES AND ANGLE:

$$u \cdot v = |u| \cdot |v| \cdot \cos(\theta)$$

np.dot(u,v)

DOT PRODUCT (SCALAR PRODUCT)



$$\mu \cdot v = 3 \cdot 10 + 7 \cdot 3 = 51$$

OR:

$$|\mu| = \sqrt{3^2 + 7^2} \simeq 7.62$$

$$|v| = \sqrt{10^2 + 3^2} \simeq 10.44$$

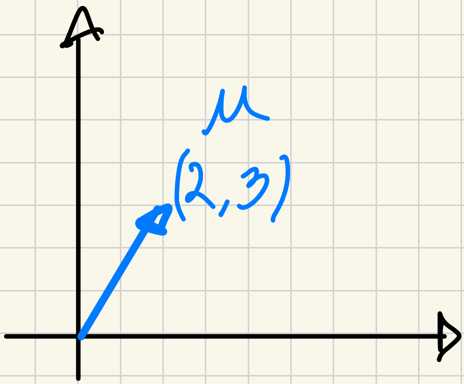
USING $\sin^{-1}\left(\frac{c}{a}\right)$ FROM

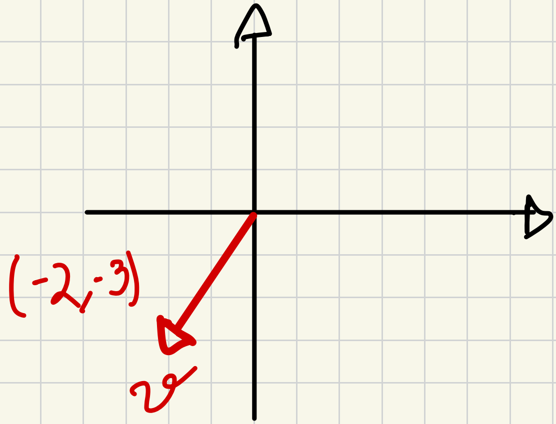
$\theta = 0.87$ RAD

$\cos(\theta) = 0.64$

$$\mu \cdot v = |\mu| \cdot |v| \cdot \cos(\theta) = 51$$

Dot Product (Scalar Product)





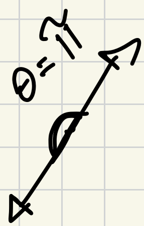
$$u \cdot v = 2\cdot(-2) + 3\cdot(-3) = -13$$

$$u \cdot v = |u| \cdot |v| \cdot \cos(\theta)$$

$$|u| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

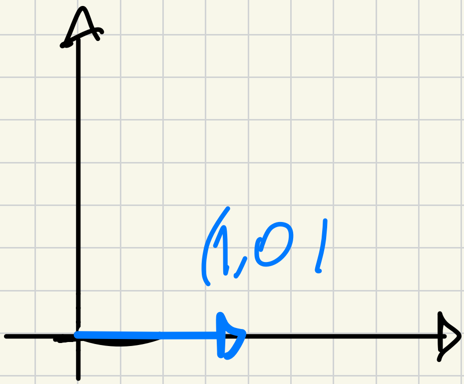
$$|v| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$$

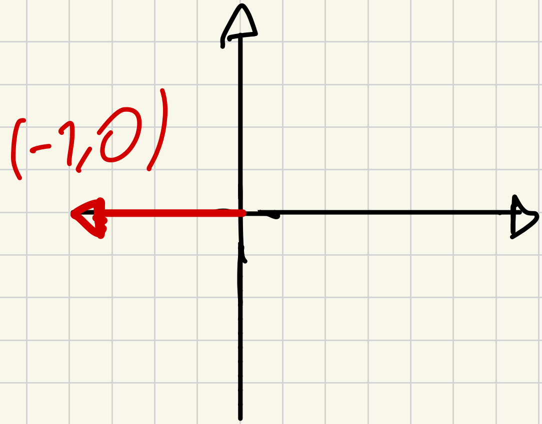
$$\cos(\theta) = -1$$



$$u \cdot v = \sqrt{13} \cdot \sqrt{13} \cdot (-1) = -13$$

Dot Product (Scalar Product)

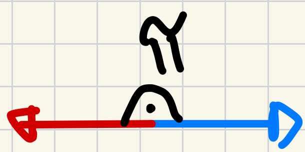




$$u \cdot v = 1 \cdot (-1) + 0 \cdot 0 = -1$$

$$u \cdot v = |u| \cdot |v| \cdot \cos(\theta)$$

$|u| = 1$

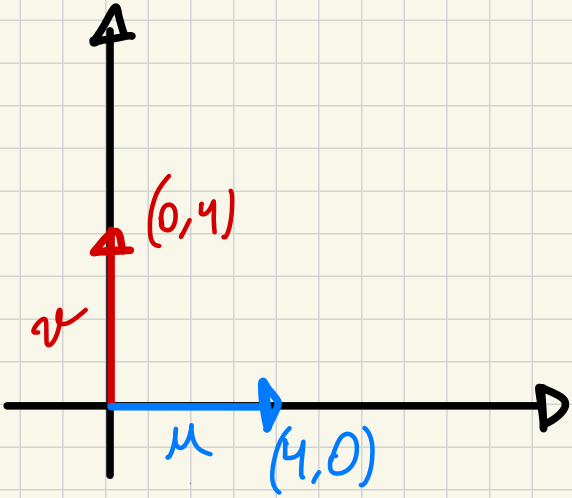


$|v| = 1$

$\cos(\theta) = -1$

$$u \cdot v = 1 \cdot 1 \cdot (-1) = -1$$

DOT PRODUCT (SCALAR PRODUCT)



$$u \cdot v = 0 \cdot 4 + 4 \cdot 0 = 0$$

$$u \cdot v = |u| \cdot |v| \cdot \cos(\theta)$$

$|u| = 4$

$|v| = 4$

$\theta = \frac{\pi}{2} \Rightarrow \cos(\theta) = 0$

$$u \cdot v = 4 \cdot 4 \cdot 0 = 0$$

ANGLE

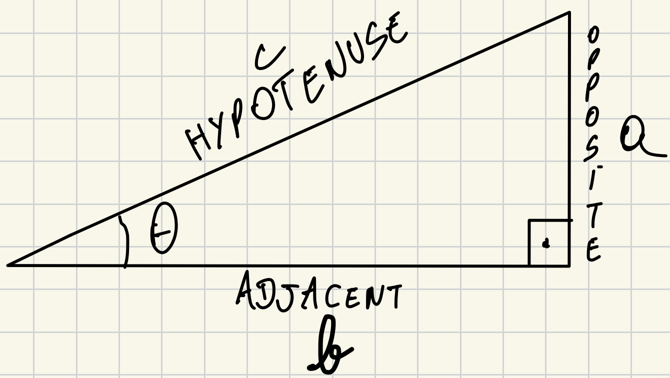
$$\measuredangle(a,b) = \text{ARCCOS}\left(\frac{a^T \cdot b}{|a| \cdot |b|}\right)$$

$$= \text{ACOS}\left(\frac{\text{DOT}(a,b)}{\text{NORM}(a) \cdot \text{NORM}(b)}\right)$$

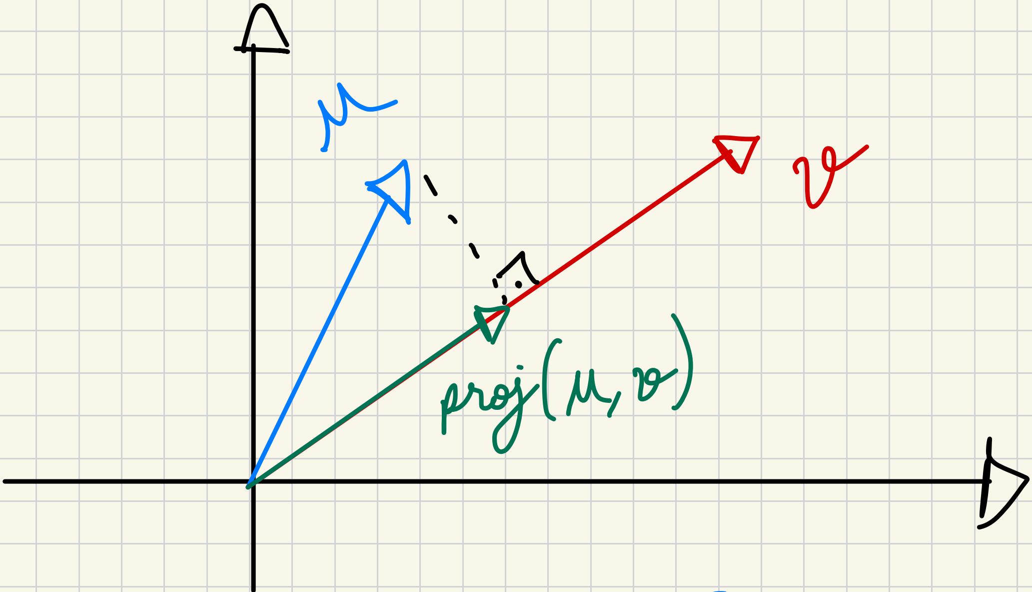
$$\text{SIN}(\theta) = \frac{a}{c}$$

$$\text{COS}(\theta) = \frac{b}{c}$$

$$\text{TAN}(\theta) = \frac{a}{b}$$



PROJECTION



$$|proj(u,v)| = \frac{u^T \cdot v}{|v|}$$

$$\overrightarrow{proj(u,v)} = \frac{|proj(u,v)|}{|v|} \cdot \vec{v}$$

PROJECTION

$u = (1, -2, 3), \quad v = (2, 4, 5)$

$proj(u, v) = ?$

$|u| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$

$|v| = \sqrt{2^2 + 4^2 + 5^2} = \sqrt{45}$

$u \cdot v = 2 - 8 + 15 = 9$

$|proj(u, v)| = \frac{9}{\sqrt{45}}$

$\overrightarrow{proj(u, v)} = \frac{|proj(u, v)|}{\sqrt{45}} \cdot (2, 4, 5) =$

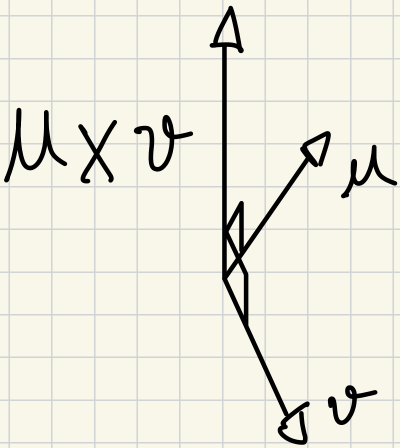
$= \frac{9}{\sqrt{45}} \cdot \frac{1}{\sqrt{45}} (2, 4, 5) = \frac{9}{45} (2, 4, 5) = \frac{1}{5} (2, 4, 5) =$

$= (0.4, 0.8, 1)$

CROSS PRODUCT

A CROSS PRODUCT OF u AND v IS A VECTOR (DIFFERENTLY THEN THE DOT PRODUCT)

THE DIRECTION WOULD BE PERPENDICULAR TO THE PLAN OF u AND v.



THE MAGNITUDE OF $u \times v$ IS:

$$|u \times v| = |u| \cdot |v| \cdot \sin(\theta)$$

θ = ANGLE BETWEEN u & v

CROSS PRODUCT

WE CAN ALSO REPRESENT THE CROSS PRODUCT AS A MATRIX:

IF i, j AND k ARE UNIT VECTORS $(1,0,0)$, $(0,1,0)$ AND $(0,0,1)$, WE CAN DECOMPOSE ANY 3D VECTOR IN:

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$
$$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

$$\vec{c} = \vec{a} \times \vec{b}$$

$$\vec{c} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\vec{c} = i\,|a_2b_3 - a_3b_2| - j\,|a_1b_3 - a_3b_1| + k\,|a_1b_2 - a_2b_1|$$

CROSS PRODUCT

$$\vec{c} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\vec{c} = i\,|a_2 b_3 - a_3 b_2| - j\,|a_1 b_3 - a_3 b_1| + k\,|a_1 b_2 - a_2 b_1|$$

$u = (1, 2, 5)$ $\quad v = (2, -4, 5)$ $\quad \boxed{u \times v = ?}$

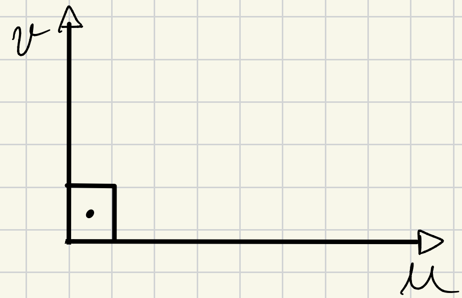
$$u \times v = \big(2\cdot 5 + 5\cdot 4,\ -(1\cdot 5 - 5\cdot 2),\ 1\cdot(-4) - 2\cdot 2\big)$$

$$= (30,\ 5,\ -8)$$

$$|u \times v| = \sqrt{30^2 + 5^2 + (-8)^2} \cong 31.45$$

$$\theta_{u\times v,\,u} = \operatorname{acos}\left(\frac{30\cdot 1 + 5\cdot 2 + (-8)\cdot 5}{31.45 \cdot \sqrt{1^2+2^2+5^2}}\right) = 90^\circ$$

LINEAR INDEPENDENCE



WE CALL 2 VECTORS ORTHOGONAL IF THE ANGLE BETWEEN THEN IS $90°$.

$u = (2, 0) \qquad v = (0, 5)$

v CAN'T BE WRITTREN AS LINEAR COMBINATION OF ANY VECTOR IN THE SAME DIMENSION OF u.

VECTORS APPLICATIONS

- CROSS PRODUCT: MAGNETIC FIELD
- DISTANCE:
  - K-MEANS CLUSTER
  - IMAGE RECOGNITION
- WORD EMBEDDINGS

MATRICES

MATRICES

- CLASSIC NOTATION
- MATRIX AS LINEAR EQUATIONS
- MATRIX VS VECTOR
- PLOTTING A MATRIX AS VECTORS
- OPERATIONS
    - ADDITION
    - SCALAR MULTIPLICATION
    - MATRIX MULTIPLICATION
- LINEAR TRANSFORMATIONS
- DETERMINANT
- RANK OF A MATRIX
- ROW ECHOLON FORM
- SINGULAR
- TRANSPOSE
- INVERSE
- EIGEN VECTORS & EIGEN VALUES.

MATRICES

MATRICES ARE A RECTANGULAR REPRENSENTATIONS OF LINEAR TRANSFORMATIONS OF A N-DIMENSION VECTOR TO A M-DIMENSION ONE:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

MATRICES

CONSIDER 3 3D LINEAR EQUATIONS

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$$

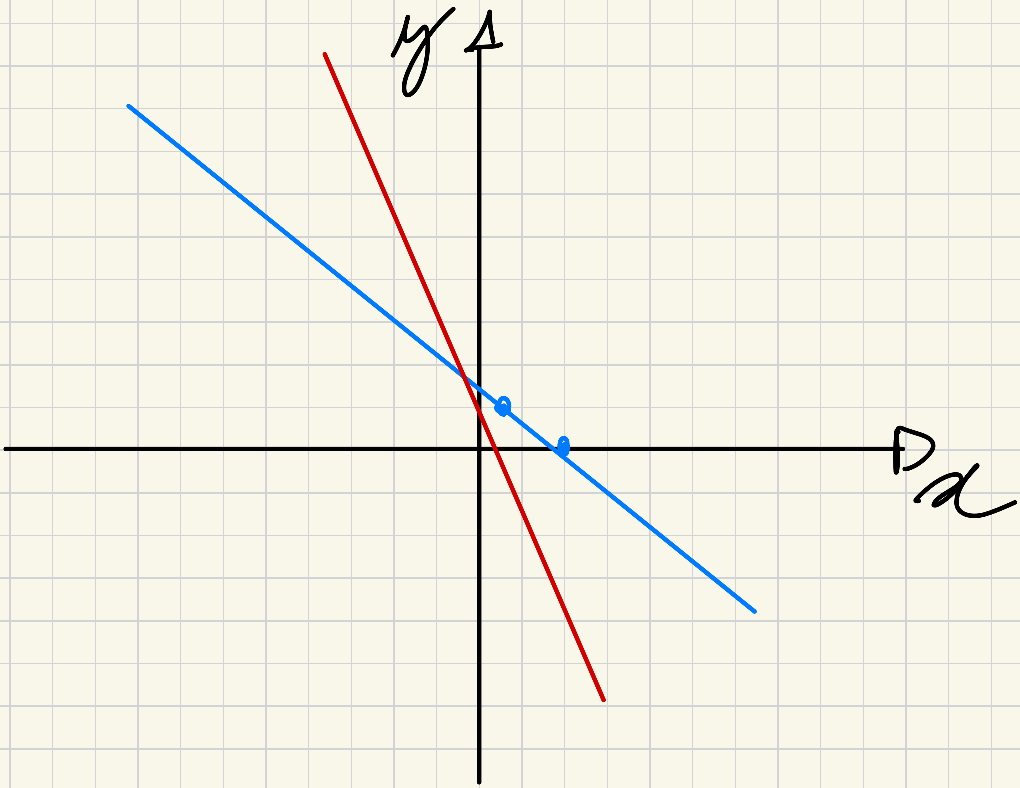
WE COULD REPRESENT IT AS:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

MATRICES

$$2x + 3y = 4$$
$$6x + 2y = 1$$

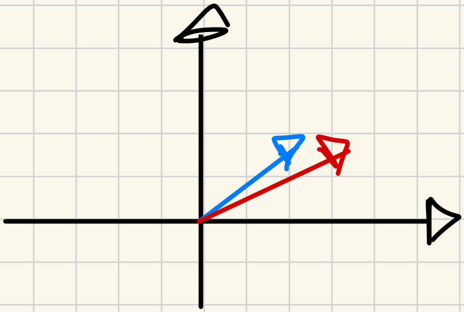
$$\begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



MATRICES

WE CAN ALSO PLOT A MATRIX AS VECTOR:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$



MATRIX PROBLEM 1

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad A(BA) = ?$$

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 4+1 \\ 1+4 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$$

$$A(BA) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 4+10 & 5+8 \\ 8+5 & 10+4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$$

MATRIX PROBLEM 2

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}_{3\times 2} \qquad AB = ?$$

$$AB = \begin{bmatrix} 2-6+1 & 1-4+1 \\ 4+3+3 & 2+2+3 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 10 & 7 \end{bmatrix}$$

Matrix Operations

Addition:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

Scalar Multiplication:

$$3 \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 \cdot a_{11} & 3 \cdot a_{12} & 3 \cdot a_{13} \\ 3 \cdot a_{21} & 3 \cdot a_{22} & 3 \cdot a_{23} \\ 3 \cdot a_{31} & 3 \cdot a_{32} & 3 \cdot a_{33} \end{bmatrix}$$

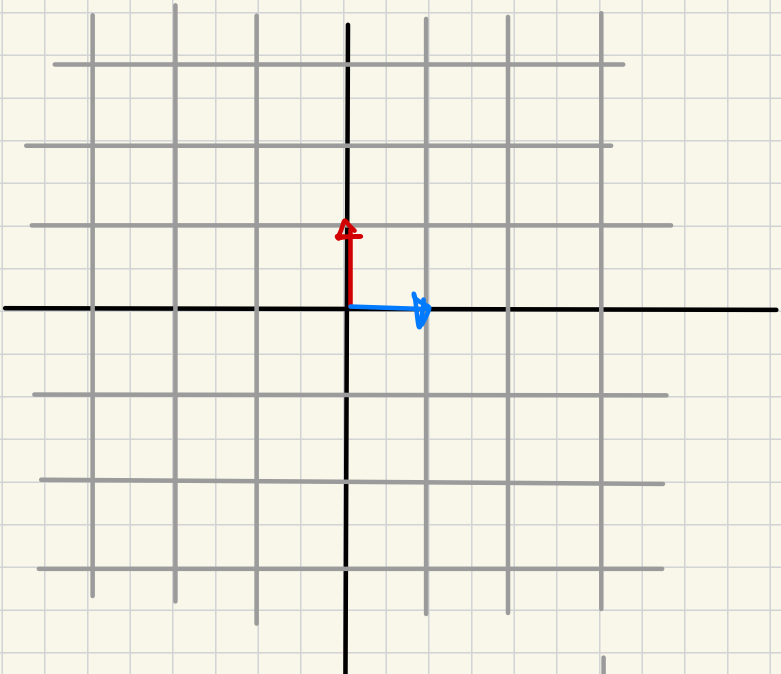
Matrix Multiplication

$$C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$$

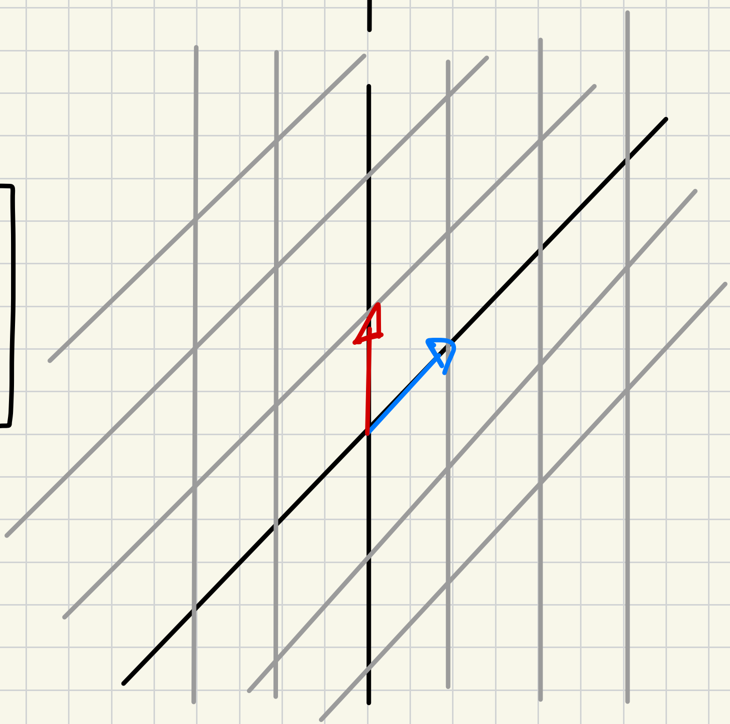
$$= \begin{bmatrix} a_{11}\cdot b_{11} + a_{12}\cdot b_{21} + a_{13}\cdot b_{31} & a_{11}\cdot b_{12} + a_{12}\cdot b_{22} + a_{13}\cdot b_{32} \\ a_{21}\cdot b_{11} + a_{22}\cdot b_{21} + a_{23}\cdot b_{31} & a_{21}\cdot b_{12} + a_{22}\cdot b_{22} + a_{23}\cdot b_{32} \end{bmatrix}$$

Linear Transformations

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



DETERMINANT

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$$

$$\boxed{DET(A) = ?}$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} \quad 2 \cdot \left(0 \cdot 5 - \left((-1) \cdot 4\right)\right) = 2 \cdot (0 + 4) = 8$$

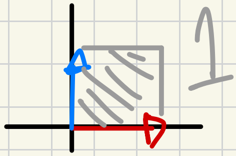
$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} \quad -3 \left(2 \cdot 5 - \left((-1) \cdot 1\right)\right) = -3 \cdot 11 = -33$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} \quad 1 \cdot (2 \cdot 4 - 0 \cdot 1) = 8$$

$$DET(A) = 8 - (-33) + 8 = 49$$

DETERMINANT

FOR: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

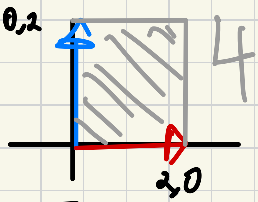


DET(A) = 1

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

(x, y) UNCHANGED

FOR: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

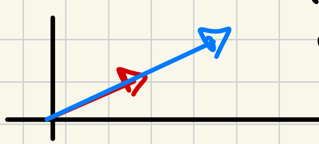


DET(A) = 4

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

(x, y) CHANGED SCALE

FOR: $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$



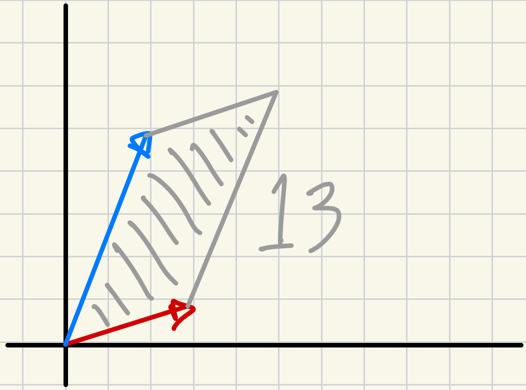
DET(A) = 0

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 2 \cdot (2x + y) \end{bmatrix}$$

x & y ARE NOW 1D

DETERMINANT

FOR: $A = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$



$$\det(A) = 15 - 2 = 13$$

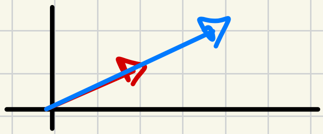
$$\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + y \\ 2x + 5y \end{bmatrix}$$

(x, y) CHANGED SCALE

RANK OF A MATRIX

FOR:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$



EVEN THOUGH A IS A 2D MATRIX, THE 2 VECTORS ARE ALLIGNED IN 1 DIMENSION.

$$\text{RANK}(A) = 1$$

ROW ECHELON FORM

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow[R_4 = R_3 + R_2]{R_3 \to R_4} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

RANK = 2

DET = ?

Row Echolon Form

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \xrightarrow{R_2 \to 2R_1 - R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 4 & 1 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 4R_1 - R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_2 + R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Rank}(A) = 3$$

SINGULAR MATRIX

A SQUARE MATRIX IS SINGULAR IF $DET(A) = 0$, HENCE IT IS NOT INVERTIBLE.

$$RANK(A) < ORDER(A)$$

SINGULAR MATRIX HAVE LINEAR DEPENDENT ROWS/COLUMNS.

TRANSPOSE

$$A = \begin{bmatrix} 2 & -9 & 3 \\ 13 & 11 & -17 \\ 3 & 6 & 15 \\ 4 & 13 & 1 \end{bmatrix}_{4\times 3}$$

$$A^T = \begin{bmatrix} 2 & 13 & 3 & 4 \\ -9 & 11 & 6 & 13 \\ 3 & -17 & 15 & 1 \end{bmatrix}_{3\times 4}$$

TRANSPOSE PROPERTIES

1) $\boxed{(A^T)^T = A}$

2) $\boxed{(A+B)^T = A^T + B^T}$

3) $\boxed{(kA)^T = k(A^T)}$

4) $\boxed{(AB)^T = B^T \cdot A^T}$

INVERSE

$$\boxed{A^{-1} = \text{INVERSE OF } A}$$

$$A \cdot A^{-1} = I$$

$$\boxed{I = \text{IDENTITY}}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}_{m \times n}$$

INVERSE

$$A^{-1} = \mathrm{adj}(A) / \det(A)$$

$$\mathrm{adj}(A) = C^T$$

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

M_{ij} = MATRIX WITHOUT i ROW AND j COLUMN

INVERSE

$$A^{-1} = adj(A) / det(A)$$

$$adj(A) = C^T$$

$$C_{ij} = (-1)^{i+j} \, det(M_{ij})$$

M_{ij} = MATRIX WITHOUT i ROW AND j COLUMN

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$C = \begin{bmatrix} +d & -c \\ -b & +a \end{bmatrix} \qquad C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$det(A) = a \cdot d - bc \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \qquad A^{-1} = \frac{1}{24 - 14} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

INVERSE

$$A^{-1} = \operatorname{adj}(A) / \det(A)$$

$$\operatorname{adj}(A) = C^T$$

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

M_{ij} = MATRIX WITHOUT i ROW AND j COLUMN

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1-4 & -(2-(-2)) & 4-(-1) \\ -(2-(-2)) & 1-1 & -(2-(-2)) \\ 4-(-1) & -(2-(-2)) & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$$

CONTINUE... $\longrightarrow$

INVERSE

$$A^{-1} = \mathrm{adj}(A) / \det(A)$$

$$\mathrm{adj}(A) = C^T$$

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

M_{ij} = MATRIX WITHOUT i ROW AND j COLUMN

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$$

$$\det(A) = 1 \cdot (1-4) - 2 \cdot (2-(-2)) + (-1)(4-(-1))$$

$$= -3 - 8 - 5 = -16$$

$$A^{-1} = \begin{bmatrix} 3/16 & 1/4 & -5/16 \\ 1/4 & 0 & 1/4 \\ -5/16 & 1/4 & 3/16 \end{bmatrix}$$

INVERSE (PROPERTIES)

1) IF A IS NONSINGULAR $(A^{-1})^{-1} = A$

2) IF A & B ARE NONSINGULAR.

$$AB^{-1} = B^{-1}A^{-1}$$

3) $(A^{T})^{-1} = (A^{-1})^{T}$

4) $A \cdot A^{-1} = A^{-1} \cdot A = I$

EIGENVECTORS & EIGENVALUES

EIGENVECTOR OF A MATRIX IS A VECTOR THAT DOESN'T CHANGE IT'S DIRECTION WHEN MULTIPLIED BY THE MATRIX:

EIGENVALUE

$$A\vec{x} = \lambda \vec{x}$$

EIGENVECTOR

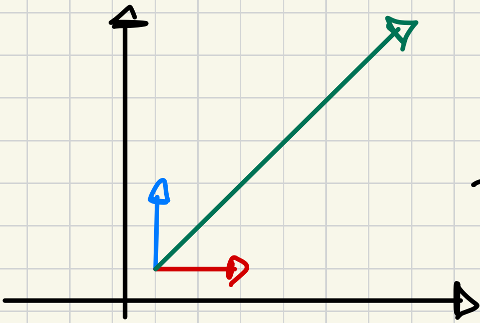
2x2 EXAMPLE:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

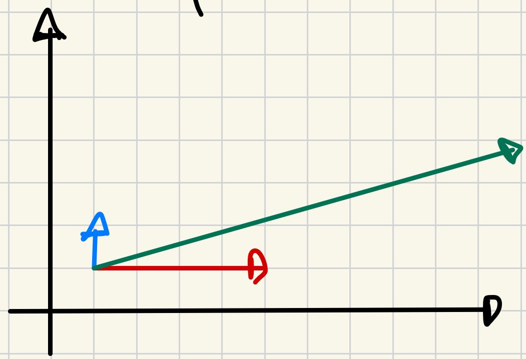
Eigenvectors & Eigenvalues

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\mu = (1, 0)$$
$$v = (0, 1)$$
$$w = (3, 3)$$



$$\xrightarrow{A\text{-vector}}$$



$$A\mu = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{x}_1 \,;\, \lambda_1 = 2}$$

$$Av = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

$$\boxed{\vec{x}_2 \,;\, \lambda_2 = 0.5}$$

$$Aw = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1.5 \end{bmatrix}$$

Not Eigen!!!

PROPERTIES OF EIGENVALUES

- $A^{+} = -A$ MATRICES HAVE IMAGINARY OR ZERO λ
- SINGULAR MATRICES HAVE $\emptyset$ λ's
- Sum OF $\lambda_1, \lambda_2, \dots \lambda_n$ is THE TRACE.
- EIGENVECTORS ARE LINEARLY INDEPENDENT
- IF A SQUARE, $\lambda = 0$ is NOT A EIGENVALUE.
- $a \cdot \lambda$ is A EIGENVALUE OF $a \cdot A$
- IF λ EIGENVALUE OF A, λ is ALSO EIGENVALUE OF A^T

CALCULATION OF $\vec{v}$ & λ

1) FIND λ's SOLVING:

$$\det(A - \lambda I) = \emptyset$$

2) FIND $\vec{v}_\lambda$ SOLVING:

$$A \cdot \vec{v}_i = \lambda_i \cdot \vec{v}_i$$

CALCULATION OF $\vec{v}$ & λ

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

$$(5-\lambda)(2-\lambda) - 4 = 0 \Rightarrow 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 6 \quad \text{OR} \quad \lambda = \underline{1}$$

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \Rightarrow -v_{11} = v_{12}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 6 v_{21} \\ 6 v_{22} \end{bmatrix} \Rightarrow v_{21} = 4 v_{22}$$

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Fim