COPRODUCT OF OPERADS

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1. Trees

Definition 1. An $n-$*tree* is the data (V, E, s, t) where

- V is a finite set of *vertices*,
- E is a finite set of *edges*,
- $s : E \to V \cup [1..n]$ is the *source map*[1], and
- $t : E \to V \cup \{0\}$ is the *target map*.

This data is restricted to satisfy the following conditions:

(1) It defines a graph-theoretic tree with vertices $V \cup [0..n]$ and edges E.
(2) There is exactly one $e \in E$ such that $t(e) = 0$.

We use $u \to^e v$ to denote[2] that $e \in E$ has source u and target v.

2. Free operads

Let $U : \mathrm{Op} \to \mathrm{Top}^{\mathbf{N}}$ be the forgetful functor and let F be its left adjoint. Its existence is a result of Boardman and Vogt in [1].

Definition 2. Let C be a collection. A $C-$*labelled planar* $n-$*tree* is a planar $n-$tree such that each vertex with k children is labelled by an element of C_k.

3. The construction

Given two operads O and O', the $F - U$ adjoint pair gives epimorphisms

$$\epsilon : FU\ O \to O \tag{1}$$

$$\epsilon' : FU\ O' \to O'. \tag{2}$$

If the category Op had coproducts, we would have an operad epimorphism[3]

$$\epsilon + \epsilon' : (FU\ O) + (FU\ O') \to O + O'. \tag{3}$$

Drawing inspiration[4] from the fact that left adjoints preserve colimits [2], we may define

$$(FU\ O)\ +\ (FU\ O')\ :=\ F\ ((U\ O)\ +\ (U\ O'))\,. \tag{4}$$

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[1] Here we're using the comprehension $[a..b] := \{t \in \mathbf{Z} \mid a \leq t \leq b\}$

[2] Since the vertices and edges form a tree, specifying the edge is redundant. We will keep this definition for the time being in order to fluidly converse with the authors.

[3] It being an epimorphism is not immediate, but the property is necessary for the validity of the construction.

[4] We haven't established nor cited any relevant categorical properties for operads.

and therefore compute a quotient of $F\ \ ((U\ O) + (U\ O'))$ that is a coproduct of O and O'.

4. Comments

4.1. The result by B & V is that the forgetful functor is *monadic.*

4.2. The full statement that motivates (4) is that *Right adjoints preserve limits and left adjoints preserve colimits.* It can be found in [2].

4.3. The definition (4) suggests that *"at the $FU-$level coproducts are trivially definied."*

4.4. Justify that $\epsilon + \epsilon'$ in (3) is an epimorphism.

4.5. The number of edges of a tree given by definition 1 is probably

$$(\#\ V) + n,$$

where n is the number of leaves.

References

[1] Boardman, J. M., and Vogt, R. M. *Homotopy invariant algebraic structures on topological spaces*, vol. 347 of *Lect. Notes Math.* Springer, Cham, 1973.

[2] Riehl, E. *Category theory in context.* Mineola, NY: Dover Publications, 2016.