NOTES OF JUNE 16 2025

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1. Monad given by an operad

Let f be an operad[1], and for each object X let X^j denote its $j-$th tensor power. Then, let $\mathtt{f} : \mathcal{C} \to \mathcal{C}$ be given by

$$\mathtt{f}\ X = \bigoplus_{j \geq 0} f(j) \otimes_{k[\Sigma_j]} X^j$$

1.1. Tensor product over non-commutative rings.

The definition of the monad associated to an operad involves tensor product over the group ring $k[\Sigma_j]$, which is non-commutative. I will write down what I think the definitions should be[2].

Definition 1. If the ring k is commutative, then there is no distinction between left- and right- modules. Therefore, the tensor product $A \otimes_k B$ of two $k-$modules is defined as

$$k^{\oplus A \times B}/W,$$

where W is the sub-module generated by the elements

$$\begin{aligned}
(a + a', b) - ((a, b) + (a', b)) \\
(a, b + b') - ((a, b) + (a, b')) \\
x(a, b) - (xa, b) \\
x(a, b) - (a, xb).
\end{aligned}$$

Then, we let $a \otimes b$ denote the equivalence class of (a, b) in this quotient.

However, if we let R denote an arbitrary ring[3] and we let T denote some set, then the "direct sum" $R^{\oplus T}$ can only be guaranteed to make sense as an abelian group, since the left- and right- module structures need not coincide.

Definition 2. Let R be a non-commutative ring, A a right $R-$module, and B a left $R-$module. Define their *tensor product over R* as

$$A \otimes_R B := \mathbf{Z}^{\oplus A \times B}/W,$$

where W is the subgroup generated by the elements

$$\begin{aligned}
(a + a', b) - ((a, b) + (a', b)) \\
(a, b + b') - ((a, b) + (a, b')) \\
(ar, b) - (a, rb)
\end{aligned}$$

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[1] Over the category $\mathcal{C}$ used by May: differential $\mathbf{Z}-$graded $k-$modules.

[2] After confirming these, I will hopefully also add some *motivation* for their use.

[3] Rings for us are fixed to have 1, for the more general case we use the term *algebra*.

In the case where the non-commutative ring R contains a "nice" commutative ring k as a subring —as in differential operators and group rings—, the tensor product is defined as the quotient of the free $k-$module modulo the analogous submodule. Explicitly, the relations for $A \otimes_{k[G]} B$ are those of $k-$bilinearity and

$$(ag, b) - (a, gb),$$

where A is a $k-$module with a right $G-$action and B is a $k-$module with a left $G-$action.

2. Questions

- What is $R \otimes_R R$ when R is not commutative?
- What is the defining mapping property of the abelian group $A \otimes_R B$ when R is not commutative?