There are three key models/assumptions/components in the burst calculator:

- A simple model, that works at sea-level (only), gives us a relation between (mass of helium) and (sea level ascent rate).
- The ascent rate is approximately constant wrt. height
- A basic pressure model for the Earth, coupled with the 'fact' that the pressure & temperature inside & outside the balloon are equal, giving a relationship between (mass of helium) and (burst altitude).

All of these assumptions are wrong, to varying degrees.

0 Letters

We know these things & constants

$$\begin{aligned} m_B &= \text{mass of balloon} & m_P &= \text{mass of payload} \\ \rho_{G,0} &= \text{density of balloon gas at sea level} & \rho_{A,0} &= \text{density of air at sea level} \\ C_D &= \text{drag coefficient} & g &= 9.81 \\ a_0 &= \text{initial altitude} & r_B &= \text{burst radius of balloon} \\ P_0 &= \text{pressure at sea level} \end{aligned}$$

and are interested in these (related) things, some of which will be specified, and some will be calculated

$$\begin{aligned} V_0 &= \text{volume of balloon at launch} & a_B &= \text{burst altitude} \\ v_0 &= \text{instantaneous ascent rate just after launch} & T &= \text{time until burst} \end{aligned}$$

we have these trivial relationships

$$\begin{aligned} V_0 &= \frac{4}{3}\pi r_0^3 & V_B &= \frac{4}{3}\pi r_B^3 \\ m_G &= V_0 \rho_{G,0} \quad \text{mass of gas} & m &= m_B + m_P + m_G \\ A_0 &= \pi r_0^2 \quad \text{cross-sectional area at launch} \end{aligned}$$

1 Sea level ascent rate

The balloon reaches equilibrium/terminal velocity almost instantly, so we can calculate this by equating forces vertically.

$$\begin{aligned} \text{Gravity} \qquad & -mg \\ \text{Buoyancy} \qquad & \rho_{A,0} g V_0 \\ \text{Drag} \qquad & -\frac{1}{2}\rho_{A,0} C_D A_0 v_0^2 \end{aligned}$$

We can solve for v_0 in terms of r_0:

$$v_0 = \sqrt{\frac{\rho_{A,0} g V_0 - mg}{\frac{1}{2}\rho_{A,0} C_D A_0}} = \sqrt{\frac{2g((\rho_{A,0} - \rho_{G,0})(\frac{4}{3}\pi r_0^3) - m_B - m_P)}{\rho_{A,0} C_D \pi r_0^2}}$$

We can also solve for r_0 in terms of v_0; we need to solve this cubic in r_0:

$$8\pi g(\rho_{A,0} - \rho_{G,0})r_0^3 - 3\pi\rho_{A,0}C_D v_0^2 r_0^2 - 6(m_B + m_P)g = 0$$

Note that if you differentiate wrt. r_0 and look for turning points, you should find two: one at 0, and one at some $r_0 > 0$. Because that cubic is negative at $r = 0$, we know that it has exactly one real root, which moreover is positive.

2 Ascent rate is constant

This is approximately correct. It's quite close to a straight line (try plotting it; then try numerically differentiating and plotting that; see example data for NOVA 26 at http://goo.gl/DSeVZV).

It's worth noting that the model described in the first section does not even remotely hold above sea level; in fact it predicts that the ascent rate will be directly proportional to $\sqrt{r}$ (where r is the radius of the balloon). Since our balloons pop at over 3 times the launch radius, and the ascent rate is fairly constant, this is not consistent with reality.

This paper `http://www.atmos-meas-tech.net/4/2235/2011/amt-4-2235-2011.pdf` presents a substantially more advanced model (after introducing the above model and explaining where it is wrong), however using their model in practice is incredibly computationally intensive.

Anyway, assuming ascent rate is constant, we have

$$T = (a_B - a_0)/v_0$$

3 Bursting

We have a model for the Earth's atmosphere; a function $\rho(a)$ that gives air pressure for a certain altitude. TODO: details; is it just the US Standard Model?

We assume that the pressure and temperature inside the balloon is equal to the pressure and temperature outside of the balloon.

The balloon gas, and the amount of air displaced by the balloon, both obey the ideal gas law $PV = nRT$. By various assumptions, $\frac{PV}{RT}$ is equal for both gases (R is a constant). Therefore, the number of moles of balloon gas n_G is equal to the number of moles of air displaced n_A. But n_G is constant wrt. time and altitude, so n_A is constant wrt. altitude. In particular, the mass of air displaced remains constant.

This relates volume V at altitude a to air density:

$$\rho_{A,0}V_0 = \rho(a)V$$

The balloon bursts when the radius is r_B, i.e., when its volume is V_B, so

$$\rho_{A,0}r_0^3 = \rho(a_B)r_B^3$$

There's a discussion of how the temperature outside and inside differs in the paper linked above. It is however close enough (we think).