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LOCKEAN BELIEFS, DUTCH BOOKS, AND SCORING SYSTEMS (DRAFT)

1 INTRODUCTION

The Lockean thesis is a way of connecting all-or-nothing beliefs to graded beliefs (Foley 1992). On the Lockean thesis to have an all-or-nothing belief in a proposition is just to assign that proposition a credence at or above a certain threshold.[1] One of the interesting and controversial features of Lockeanism is that it does not require that one's beliefs either be closed under logical implication or even be logically consistent. For a failure of closure note that at any threshold less than one it is possible, on the Lockean thesis, to believe two atomic propositions, P, Q, while not believing their conjunction, $P\&Q$.[2] For a failure of consistency note that at a threshold of .6 it is possible to believe two propositions P and Q, while also believing the negation of their conjunction, $\neg(P\&Q)$.[3]

So, the Lockean thesis (at thresholds less than one) does not require one's beliefs be either logically closed or logicall consistent. We might want to ask what constraints the Lockean thesis *does* put on one's beliefs. Some constraints are well-known: for example, if the threshold for belief is greater than .5, then, on the Lockean thesis, one cannot hold pairwise inconsistent beliefs.[4] Such an observation, however, does not yield necessary and sufficient conditions for a set of beliefs to be that of an agent whose beliefs satisfy the Lockean thesis. That is, the following two statements are not equivalent.[5]

(a) A set of propositions, B, is pairwise consistent.

(b) There is a probability function that assigns each member of B a probability greater than .5.

I am grateful to Gary Chamberlain, Nilanjan Das, Kevin Dorst, Kenny Easwaran, Branden Fitelson, Wes Holliday, Matt Mandelkern, Richard Pettigrew and Linda Rothschild for comments and discussion. Special thanks to two anonymous referees at *Review of Symbolic Logic*.

[1] Sometimes the Lockean thesis is framed as one about what one ought to believe, rather than about what one does believe. The difference is not essential here.

[2] See Leitgeb (2014) for an exploration of constraints, other than having a threshold of 1, on credences and thresholds that ensure closure (and consistency).

[3] Assume you assign a credence of .6 to P and .6 to Q, then you can assign at most .8 to $\neg(P\&Q)$.

[4] See, e.g., Hawthorne & Bovens (1999) for this and related observations.

[5] The relevant notions are made more precise in the next section.

In this paper, I present two types of characterizations of sets of beliefs for Lockean agents. I draw on a pair of related traditions for characterizing graded beliefs. The first tradition is that of the *Dutch book argument* for probabilism. In this tradition, graded beliefs can be characterized by how they rationalize bets. Very roughly speaking, the Dutch book argument establishes that a set of numerically graded beliefs is probabilistically coherent if and only if there is no collection of bets it rationalizes that lead to a sure loss.[6] The second tradition is that of the *accuracy scoring argument* for probabilism. In this tradition, numerical credences are given scores depending on their accuracy.[7] The standard genre of result in this area is to establish that the numerical credences that are probabilistically coherent are equivalent to those that are not dominated by another set of credences in all worlds.

So, in the case of graded beliefs, both Dutch book arguments and accuracy scoring arguments are used to characterize probabilistically coherent beliefs (as opposed to numerical beliefs that don't satisfy the axioms of probability theory). What I show here is that these arguments can *also* be used to characterize all-or-nothing beliefs satisfying the Lockean hypothesis. In the accuracy tradition there is already a small literature partially devoted to understanding how the Lockean thesis fares vis-a-vis scoring arguments.[8] The relationship I draw between the Dutch book arguments and the Lockean hypothesis is, as far as I know, novel.

Here is the plan of this paper: In the next section, §2 , I define beliefs and credences and explain what it means for a set of beliefs to satisfy the Lockean thesis. In the next section, §3, I discuss ways of deriving betting dispositions from full beliefs and what it means for those dispositions to be subject to a Dutch book. In §4, I define a scoring system for beliefs, and some decision-theoretic notions of dominance that apply to sets of belief. In the main section, §5, I give my results linking these different ways of thinking about beliefs. One of results here about the relationship between scoring is a proof the 'main conjecture' of Easwaran (2016). I discuss the significance of there results in §6 and outline a few directions for future work in §7.

2 BELIEF, CREDENCE AND LOCKEANISM

Let us fix a finite set of worlds W, and, corresponding to it a set of propositions P, $P = 2^W$.[9] We will also fix a threshold t, $0 \leq t \leq 1$, with respect to which we will phrase Lockeanism. We can then define a credence function c, for an agent, as a probability measure over the σ-algebra $P = 2^W$.

[6] This argument goes back to Ramsey (1926). See Pettigrew (2019) for a recent book-length discussion.

[7] See, e.g., Joyce (1998) and Pettigrew (2016*a*) . This work is, in turn, dependent on the tradition of scoring graded predictions going back at least to Brier (1950); see Gneiting & Raftery (2007) for a recent discussion.

[8] For example, Easwaran & Fitelson (2015), Pettigrew (2016*b*), Easwaran (2016), Dorst (2019).

[9] *Notation*: 2^X is the powerset of X.

An agent's beliefs, B, is a subset of P. B is LOCKEAN COMPATIBLE iff there is a (probabilistically coherent) credence function c such that $c(b) \geq t$ for all $b \in B$. *Note*: as this definition makes clear B being Lockean compatible is relative to a choice of W and t, but, since we are focusing on one case, we leave that relativity implicit in this and related definitions and discussion.

To say that B is LOCKEAN COMPATIBLE is not equivalent to saying that B can be a Lockean's total set of beliefs. For example, suppose P includes the propositions 'it's raining', 'it's snowing' and 'it's raining or it's snowing'. The set of beliefs that just contains 'it's raining' is Lockean compatible. But that belief cannot be one's *only* belief, since if you believe it's raining you must also believe 'it's raining or it's snowing' on the Lockean thesis (since the probability you assign the latter is greater than or equal to what you assign the former).

We'll define a stronger notion as well. B is LOCKEAN COMPLETE iff there is a credence function c such that $c(b) \geq t$ iff $b \in B$. We will also use a weakening of this last notion to allow cases where belief is optional at the threshold t. We say B is ALMOST LOCKEAN COMPLETE iff there is a credence function c such that for all b if $c(b) > t$ then $b \in B$ and if $c(b) < t$ then $b \notin B$. This notion is useful for relating the Lockean notion to Dutch books and dominance arguments. The idea behind this last definition is that t represents the indifference point between belief and non-belief, at which either belief or disbelief is compatible with one's rationality.[10]

3 DUTCH BOOK ARGUMENT DEFINITION

Dutch book arguments traditionally assume that an agent's betting behavior ought to be determined by their credences. Dutch book arguments are used to argue for the thesis that their credences are probabilistic (something we assumed above). The idea of the argument is that credences determine one's betting preferences (or ought to). The arguments typically assume some divisible good in which an agents utility is linear (conventionally assumed to be dollars). One can then argue that if an agent has a credence x in a proposition p then she ought to be willing to buy or sell a bet that pays out \$1 if p is true and \$0 otherwise (what we'll call this bet *a* \$1 *bet for p*) for the price of \$$x$. (The agent also ought to be willing to buy the bet for any lower value and sell it for any higher value.) A Dutch book for an agent would be a set of bets the agent is willing to buy or sell that guarantee that the agent loses money no matter what the actual world is. (The Dutch book then guarantees that the bookie makes money.) The most basic Dutch book theorems show, roughly, that an agent is susceptible to a Dutch book if and only if the agent's credence function is not a probability measure.

Expected utility theory provides a standard way of linking rational betting dispositions to credences. Indeed the very idea of credences or subjective proba-

[10]This notion is used extensively by Easwaran (2016).

bility is often considered to be determined by or to determine betting behavior.[11] But how do we link betting behavior to beliefs? The Lockean thesis provides a kind of bridge between beliefs and bets via credences. On the Lockean view an agent ought to believe p if and only if she assigns p a credence greater than t. So the agent, then, for each belief b ought to be willing to buy a \$1 bet for b for \$$t$. If there is a collection of such bets that Maria is willing to buy that guarantee her a loss then she is subject to a ONE-WAY DUTCH BOOK.

A one-way Dutch book only takes advantage of what the set of beliefs, B, the agent *does* has, it does not exploit those propositions she *does not* belief. In these propositions ($P \backslash B$) the agent must have less than t credence. She, thus, ought to be willing to *sell* \$1 bets on each proposition in $P \backslash B$ for \$$t$ each. A TWO-WAY DUTCH BOOK is a collection for the agent of bets in B she is willing to buy at \$$t$ and a collection of bets in $P \backslash B$ she is willing to sell at \$$t$ that guarantee her a loss.

4 ACCURACY SCORING ARGUMENT DEFINITIONS

Another perspective associates beliefs with epistemic 'scores', depending on the accuracy of their beliefs. We'll consider a scoring system for beliefs following Easwaran & Fitelson (2015) and Easwaran (2016). The basic idea is that one's total beliefs at a world result in a score depending on whether your beliefs are true are not. We'll make the assumption that each belief incurs a cost if false and accrues a reward if true. I'll assume that the ratio of the cost to the reward is constant for all propositions. For reasons that will become clear, we'll assume that the cost for a false belief is t while the reward for a true belief is $1 - t$. Different propositions can be worth more or less. So we'll assume that there is a weighting function $f : P \to \mathbb{R}^+$, and that the score for a believing p is $-f(p)$, if p is false, and $f(p)(1-t)$, if p is true. Thus, one's total score at any world w for having beliefs B can be stated as follows:

$$S(w,B) = \sum_{b \in B : w \in b} (1-t) f(b) - \sum_{b \in B : w \notin b} t f(b)$$

We will think of the agent as choosing her beliefs in an effort to maximize her expected score. It follows, then, that she should believe a proposition p if she assigns it a credence greater than t to it and she should not believe it p if she assigns it a credence less than t, and she should be indifferent between believing it or not believing it if she assigns it t.[12] An agent with probabilistically coherent credences who maximizes expected score will thus have to choose an almost Lockean belief set. By describing an agent's beliefs in a decision-theoretic setting we can apply standard notions for decision theory to her beliefs. For instance,

[11] Beginning with Ramsey (1926).

[12] To see this note that her expected value for believing p with a credence $c(p)$ is $(1-t)c(p) - t(1-c(p))$, and this quantity is > 0 iff $c(p) > t$ and is < 0 iff $c(p) < t$.

one set of beliefs B STRONGLY DOMINATES another set of beliefs, B' if for all worlds w, $S(w,B) > S(w,B')$. A set of beliefs B WEAKLY DOMINATES another set of beliefs, B' if for all worlds w, $S(w,B) \geq S(w,B')$ and there is some world w' such that $S(w,B) > S(w,B')$. A set of beliefs B is not weakly/strongly dominated, if there is no set B' that weakly/strongly dominates it.[13] Note that the dominance relation is dependent on the weighting function (as well as t).

5 RESULTS

To facilitate our results I'll re-describe the various notions defined in the previous sections using linear algebra. Since we are only discussing a finite set of worlds, it will be possible to enumerate both all the worlds $W = \{w_1 \ldots w_m\}$ as well as all the propositions $P = \{p_1 \ldots p_n\}$.[14] We will also consider a candidate set of beliefs $B : B \subset P$ which we enumerate $\{b_1 \ldots b_o\}$.

We associate a $m \times o$ matrix with respect to W, B, and t as follows:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \ldots & a_{1o} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & \ldots \end{bmatrix}$$

Where:

$$a_{ij} = \begin{cases} 1-t \text{ if } w_i \text{ is in } b_j \\ -t \text{ otherwise} \end{cases}$$

Now consider an o-dimensional column vector $\mathbf{x}$. Now we multiply $\mathbf{x}$ by $\mathbf{A}$:

$$\begin{aligned} \mathbf{Ax} &= \begin{bmatrix} a_{11} & \ldots & a_{1o} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & \ldots \end{bmatrix} \begin{bmatrix} x_1 \\ \ldots \\ x_o \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{o} a_{1i}x_i \\ \ldots \\ \sum_{i=1}^{o} a_{mi}x_i \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i:w_1 \in b_i} x_i(1-t) - \sum_{i:w_1 \notin b_i} x_i(t) \\ \ldots \\ \sum_{i:w_m \in b_i} x_i(1-t) - \sum_{i:w_m \notin b_i} x_i(t) \end{bmatrix} \end{aligned}$$

We see that the ith coordinate of $\mathbf{Ax}$ is the payoff at world w_i of buying the bets for each of the beliefs $b_1 \ldots b_o$ with stakes $x_1 \ldots x_n$. If there is a non-negative $\mathbf{x}$ that makes $\mathbf{Ax}$ everywhere negative, then the beliefs are subject to a Dutch book.

Given any credence function c, we can correspond to it a non-negative m-dimensional vector $\mathbf{c}$ such that $c_i = c(\{w_i\})$. Each non-negative m-dimensional vector whose coordinates sum to 1 uniquely picks out a probability function by the inverse of the mapping above. Now consider multiplying such a vector in

[13]In decision theoretic terminology, if a move is not weakly dominated. it is *admissible*, Easwaran (2016) calls admissible belief sets *strongly coherent*.

[14]Of course, $n = 2^m$.

row format to $\mathbf{A}$ as follows:

$$\begin{aligned}\mathbf{c}^T\mathbf{A} &= \left[c_1,\dots,c_m\right]\begin{bmatrix} a_{11} & \dots & a_{1o} \\ \dots & \dots & \dots \\ a_{m1} & \dots & \dots \end{bmatrix} \\ &= \left[\sum_{i:w_i\in b_1} c_i(1-t) - \sum_{i:w_i\notin b_1} c_i(t), \dots, \sum_{i:w_i\in b_1} c_i(1-t) - \sum_{i:w_i\notin b_n} c_i(t)\right] \\ &= \left[c(b_1)(1-t)-(1-c(b_1))(t), \dots, c(b_n)(1-t)-(1-c(b_n))(t)\right]\end{aligned}$$

We can see that the ith coordinate of $\mathbf{c}^T\mathbf{A}$ is greater than or equal to 0 iff $c(b_i) \geq t$.

This representation allow us to concisely state both what it is for a set of beliefs to be subject to a Dutch book and to be Lockean compatible:[15]

$$\exists \mathbf{x} \geq 0 : \mathbf{A}\mathbf{x} < 0 \tag{1}$$

$$\exists \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0}, \text{ and } \mathbf{y}\mathbf{A} \geq 0 \tag{2}$$

We can see that (1) is true iff the agent is subject to a one-way Dutch book. Likewise (2) is true iff there is a probability function that c that is Lockean compatible with B.[16]

We can also relate (1) to the score-theoretic perspective: (1) is true iff there is some set of weights that guarantees a loss in every world from holding the belief set B.[17]

We can now use the following result from linear algebra (proved in the appendix) to get us our Dutch book results:

Theorem 1. *Where* $\mathbf{x}$ *is an n-dimensional column vector,* $\mathbf{y}$ *is a m-dimensional row vector, and* $\mathbf{A}$ *is a* $m \times n$ *matrix:*

$$(\nexists \mathbf{x} \geq \mathbf{0} : \mathbf{A}\mathbf{x} < \mathbf{0}) \longleftrightarrow (\exists \mathbf{y} \geq \mathbf{0} : y \neq 0 \textit{ and } \mathbf{y}\mathbf{A} \geq \mathbf{0})$$

Our first theorem immediately follows:

Theorem 2. *The following three statements are equivalent:*

(a) There is no one-way Dutch book on B

(b) On some weighting function, holding B yields a negative score at every world.

(c) B is Lockean compatible.

Proof. Theorem 1 establishes that (1) is false if and only if (2) is true. (1) is true iff there is a one-way Dutch book on B iff holding B guarantees a loss at some weighting function, while (2) holds iff B is Lockean compatible. □

[15] Note that as is standard in linear algebra $\mathbf{x} < 0$ means each coordinate of $\mathbf{x}$ is less than 0.

[16] Note that we do explictly not require $\mathbf{y}$ to sum to 1. However, it is easy to see that 2 is equivalent to $\exists \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0}, \sum_{i=1}^{m} y_i = 1,$ and $\mathbf{y}\mathbf{A} \geq 0$.

[17] The (partial) weighting function associated with $\mathbf{x}$ on B is simply $f(b_i) = \mathbf{x}_i$. On any weighting function that agrees with f, then, B will guarantee a loss in each world.

We now move on to results that relate Lockeanism to two-way Dutch books and score-theoretic dominance.

Consider the following $m \times n$ matrix $\mathbf{D}$, with respect to W, P, and t, which can be used for two-way Dutch books:

$$\mathbf{D} = \begin{bmatrix} d_{11} & \dots & d_{1n} \\ \dots & \dots & \dots \\ d_{m1} & \dots & \dots \end{bmatrix}$$

Where:

$$d_{ij} = \begin{cases} 1-t \text{ if } i \text{ is in } p_j \text{ and } p_j \text{ in } B \\ -(1-t) \text{ if } i \text{ is in } p_j \text{ and } p_j \text{ in } B \\ -t \text{ if } i \text{ is not in } p_j \text{ and } p_j \text{ in } B \\ t \text{ if } i \text{ is not in } p_j \text{ and } p_j \text{ not in } B \end{cases}$$

Consider now these two statements about $\mathbf{D}$:

$$\exists \mathbf{x} : \mathbf{x} \geq 0 \text{ and } \mathbf{Dx} < 0 \tag{3}$$

$$\exists \mathbf{y} : y \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0}, \text{ and } \mathbf{yD} \geq 0 \tag{4}$$

Simple calculations, parallel to those above, establish that while (3) is equivalent to there being a two-way Dutch book on the belief set, (4) is equivalent to B being an almost Lockean complete belief set.

Less straightforward is the following:

Lemma 1. *(3) is true iff there is a weighting function on which B is strictly dominated.*

Proof. Left to right: Note that $-\mathbf{D}$ is the matrix associated with having the beliefs $P \backslash B$. So $-\mathbf{Dx}$ is the score for having $P \backslash B$ on the weighting function $f : f(p_i) = \mathbf{x}_i$. If $\mathbf{Dx} < 0$, then $-\mathbf{Dx} > 0$. Thus, in this case, $P \backslash B$ strictly dominates B on f, as it has a positive score while B has a negative score.

Right-to-left: suppose that B is strictly dominated by some other set of beliefs B', on the weighting function f. In this case $B' \backslash B$ must strictly dominate $B \backslash B'$. So, on the weighting function f' :

$$f'(p_i) = \begin{cases} 0 \text{ if } p_i \text{ in } B \cap B' \text{ or } p_i \in P \backslash (B \cup B') \\ f(p_i) \text{ otherwise} \end{cases} .$$

Note that on f', $P \backslash B$ strictly dominates B. Let $\mathbf{x} = \{f'(p_1) \dots f'(p_n)\}$. In this case then $\mathbf{Dx} < 0$, and $-\mathbf{Dx} > 0$. □

Applying Theorem 1 to (3) and (4) then gives us our second characterization theorem for Lockeanism:

Theorem 3. *The following three statements are equivalent:*

(a) There is no two-way Dutch book on B

(b) There is no weighting function on which B is strictly dominated.

(c) B is almost Lockean complete.

The following conjecture from Easwaran (2016) follows directly from Theorem 3.

Lemma 2 (Easwaran's Conjunecture)**.** *If for all possible positive weighting functions the score of B is not weakly dominated by some other B', then B is almost Lockean complete.*

Proof. Given Theorem 3 what we need to show is that if B is not weakly dominated on any positive weighting function, then there is no weighting function on which B is strongly dominated. We'll prove the contrapositive. Suppose B is strongly dominated by B' on some weighting function, f. Let f' be a positive weighting function such that

$$f'(p) = \begin{cases} f(p) \text{ if } f(p) > 0 \\ 1 \text{ otherwise} \end{cases} .$$

Then B is strongly dominated on f' by B'':

$$B'' = \{p : p \in B'\} \cap \{p : p \in B \text{ and } f(p) = 0\}.$$

Since B is strongly dominated on f' weighting function it is also weakly dominated on f. □

6 DISCUSSION

What is the significance of the above results for our understanding of the nature of full belief? Theorems 2 and 3 can be used to motivate the Lockean thesis itself. If beliefs are taken to either guide betting behavior or should be subject to the scoring system above for their accuracy, then the results in the previous section give insight into why beliefs should conform to (a version of) the Lockean thesis. That is, of course, a big 'if' in the previous sentence, but it is at least worth exploring this kind of argument for Lockeanism. One thing to note about this way of taking the results is that while parallel to dutch book and accuracy arguments for probabilism they have a slightly different status. Pretty much everyone accepts that if we have precise numerically graded beliefs they should satisfy the laws of probability. Thus, the interests of the dutch book arguments and the accuracy scoring arguments is mainly foundational. With respect to full beliefs the Lockean view remains extremely controversial: probably the most prominent view in philosophical circles is rather that beliefs should be closed under logical implication. Thus these arguments to motivate the more permissive Lockean views might have more than merely foundational status, they might actually buttress a controversial view about what our full beliefs ought to be like.

Relatedly, these also support the Easwaran's (2016) project to take full belief as basic rather than numerical credences. Easwaran agruges that if one accepts the genre of scoring system discussed in section §4, one might have beliefs that are not logically consistent, but they will still be almost Lockean complete. Such sets of beliefs can thus be rationalized using probabilistic tools without taking numerical credence itself as a basic attitude. Easwaran's conjecture which we prove here as Lemma 2 support this claim since it shows that if your beliefs are not dominated on the scoring system in a certain respect than they are almost Lockean complete.

7 FURTHER DIRECTIONS

7.1 CONNECTIONS TO WALD'S COMPLETE CLASS THEOREM

It is worth noting that an alternative characterization of almost Lockean completeness can likely be found using Wald's Complete Class Theorem.[18] I only gesture towards the characterization here, leaving a full development for another occasion. Wald's Complete Class Theorem establishes that in a certain class of decision problem any strategy that is admissible (i.e. not weakly dominated) is a possible strategy of a Bayesian agent (i.e. one maximizing expected reward subject to her credences). Wald's Complete Class theorem does not *directly* apply to the choice problem here: for Easwaran (2016) shows that there are some belief sets that are not weakly dominated at some weighting function but that are *also* not almost Lockean complete. The reason Wald's Complete Class theorem does not apply is that it requires the class of strategies to be convex: i.e. if two strategies are available to an agent then any probabilistic mix of them also is. If we had allowed mixed strategies in our decision problem, then the Complete Class Theorem would apply and any strategy that was not weakly dominated by any mixed strategy would be almost Lockean complete. However, the main point of modeling outright belief is to have a notion of belief that contrasts with graded belief. If we allowed mixed strategies involving outright belief, it is not clear that we have no introduced an analogue of graded beliefs. Still I think it can be shown by applying Wald's Complete Class Theorem that the outright beliefs that are almost Lockean complete will be just those that are not weakly dominated by any *mixture* of other outright beliefs. Future work might investigate how we ought to think about mixed strategies in a system of outright belief and how such mixed strategies relate to graded beliefs.

[18] I am grateful to Gary Chamberlain here for pointing out the relevance of Wald's Complete Class Theorem. See, e.g., Ferguson (1967) for details.

7.2 EXTENSIONS WITH VARYING THRESHOLDS FOR EACH PROPOSITION

Note that here we assumed that the threshold t, the betting odds, and the scoring system ratios of cost to rewards, were the same for every proposition. A simple extension of these results could handle cases in which the odds are allowed to vary for each proposition. In addition to this last extension, future work might explore entirely distinct scoring systems and see whether they also give rise to Lockean patterns of beliefs.[19]

7.3 CONNECTION TO ANOTHER CHARACTERIZATION

In this paper I characterized the Lockean belief sets by way of their relationship to bets and accuracy scoring systems. Fernando (1998, Theorem 4, p. 230) gives a different characterization of Lockeanism that is based on qualitative properties of sets of beliefs.[20] It would be interesting to explore the connections between this result and those presented here.

APPENDIX: PROOF OF THEOREM 1

We start with this widely known result:[21]

$$(\text{there is no } \mathbf{x} \geq \mathbf{0} \text{ such that } \mathbf{Ax} \geq \mathbf{b}) \longleftrightarrow (\text{there is } \mathbf{y} \leq \mathbf{0} \text{ with } \mathbf{yA} \leq \mathbf{0} \text{ and } \mathbf{yb} < 0)$$

We can get the following by taking a universal instantiation of the both sides of the biconditional. (i.e. going from ‘ p iff q’ → ‘(for all x p) iff (for all x q)’).

$$(\forall \mathbf{b} > 0, \nexists \mathbf{x} \geq \mathbf{0} : \mathbf{Ax} \geq \mathbf{b}) \longleftrightarrow (\forall \mathbf{b} > \mathbf{0}, \exists \mathbf{y} \leq \mathbf{0} : \mathbf{yb} < 0 \text{ and } \mathbf{yA} \geq 0)$$

This can be simplified to:

$$(\nexists \mathbf{x} \geq \mathbf{0} : \mathbf{Ax} > \mathbf{0}) \longleftrightarrow (\exists \mathbf{y} \leq \mathbf{0} : \mathbf{y} \neq \mathbf{0} \text{ and } \mathbf{yA} \geq \mathbf{0})$$

We can then switch signs on the right-hand side to get:

$$(\nexists \mathbf{x} \geq \mathbf{0} : \mathbf{Ax} > \mathbf{0}) \longleftrightarrow (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y} \neq 0 \text{ and } \mathbf{yA} \leq \mathbf{0})$$

Theorem 1 follows immediately (by substitution of $\mathbf{A}$ with $-\mathbf{A}$):

$$(\nexists \mathbf{x} \geq \mathbf{0} : \mathbf{Ax} < \mathbf{0}) \longleftrightarrow (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y} \neq 0 \text{ and } \mathbf{yA} \geq \mathbf{0})$$

[19] See Dorst (2019) for some interesting observations in this regard.

[20] Wes Holliday pointed out to me that Fernando's result can also be found as a direct consequence of work in Adams (1965) see also (see also Fishburn 1969).

[21] This is known as Farkas's Lemma and can be proved using the separating hyperplane theorem. For statements and proof see Matoušek & Gärtner (2007, pp. 89–92), Strang (2006, p. 441–42), and Wikipedia contributors (2019).

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