In Wang (1995), titled **"Insurance pricing and increased limits ratemaking by proportional hazards transform"**, the author proposes a new premium principle for insurance pricing which accounts for risk loadings imposed by a proportional decrease in the hazard rates. We summarize this work here, following the structure of the paper closely.

As explained in Wang (1995), "insurance is a practice of exchanging a contingent claim for a fixed payment called [the] premium. The principle of assigning premiums according to the underlying risk is an essential element of actuarial science." That is, by assigning a premium to a particular contingent claim, the insurer is quantifying their perception of the underlying risk of that claim. There are methods to calculate the premium, called *premium principles* which are normally based on the expected loss, and the standard deviation or variance of that loss.

A new approach: Proportional Hazard Transforms

In casualty insurance, a risk X is a non-negative random loss with distribution function $F_X(t) = \mathbb{P}(X \leq t)$, density $f_X(t) = \frac{d}{dt}F_X(t)$, and survival function $S_X(t) = 1 - F_X(t)$. We borrow the concept of a hazard rate from life insurance, replacing the 'lifetime' of the insured life by the claim size of the contingent claim. We define this hazard rate for the risk X as:

$$\mu_X(t) = \lim_{\Delta t \downarrow 0} \frac{F_X(t+\Delta t) - F_X(t)}{\Delta t\,(1 - F_X(t))} = \frac{f_X(t)}{1 - F_X(t)} = -\frac{d}{dt} \ln S_X(t).$$

Note that this means $S_X(t) = \exp\left(-\int_0^t \mu_X(s)\, ds\right)$, hence the hazard rate uniquely characterizes X. Intuitively, the hazard rate of X at t represents the likelihood that the loss X is (infinitesimally) larger than t, given that it is at least t. Therefore, if the insurer were to use a lower hazard rate, they would be more risk-averse in the sense that they believe the loss X will be larger than '$t + \Delta t$' given that it is at least t. Hence we can impose a 'safety margin' by reducing the hazard rate by a multiple:

$$\mu_Y(t) = \frac{1}{\rho}\mu_X(t), \quad \rho \geq 1, \quad t \geq 0, \tag{1}$$

which gives us a random variable Y such that $S_Y(t) = S_X(t)^{\frac{1}{\rho}}$. Since the hazard rates μ_X, μ_Y uniquely characterize the underlying risks X and Y respectively, equation (1) also defines the mapping $\Pi_\rho : X \mapsto Y$, which we call the *proportional hazards transform.*

The risk-adjusted premium

For a risk X with survival function $S_X(t)$, we define the *risk-adjusted premium* as

$$\pi_\rho(X) = \mathbb{E}\left[\Pi_\rho(X)\right] = \int_0^\infty S_X(t)^{\frac{1}{\rho}}\, dt, \quad \rho \geq 1. \tag{2}$$

We call this ρ the *risk-averse index.* Note that when $\rho = 1$, $\pi_1(X) = \int_0^\infty S_X(t)\, dt = \mathbb{E}[X]$, which is the net expected loss that is often used to calculate premiums. See Section 3 of Wang (1995) for some examples using common loss distributions.

This risk-adjusted premium has many desirable properties such as linearity and preservation of stochastic order. The list of properties is listed in Section 4 of Wang (1995).

Premium allocation among layers and reinsurance

The most significant aspect of this new premium principle is that it can be used to explain the mechanism of (re)insurance without the traditional method of utility functions. To do this, we note that the risk-adjusted premium is an increasing function of ρ:

$$\rho_1 > \rho_2 \implies \pi_{\rho_1}(X) > \pi_{\rho_2}(X),$$

and that the risk-adjusted premium $\pi_\rho(X)$ represents both the inherent risk of X itself, as well as the degree of risk-aversion present for the 'decision-maker' whose risk-averse index ρ we are considering. Because $\pi_\rho(X)$ is increasing in ρ, then we must have

$$\begin{aligned}\rho_0[\text{index for policy-holder}] &> \rho_1[\text{index for insurer}] \\ &> \rho_2[\text{index for reinsurer}] \\ \iff \pi_{\rho_0}(X) &> \pi_{\rho_1}(X) > \pi_{\rho_2}(X),\end{aligned}$$

which is of course a necessary condition for the policy-holder, insurer, and/or reinsurer to participate in the insurance contract according to their own risk appetite. This explains the mechanism of the insurance market without the use of utility theory.

What is also significant with this risk-adjusted premium is that the underlying risk X can easily be broken down into layers and priced separately. We define such a layer $(a, b]$ of a given risk X by a stop-loss cover:

$$\mathbb{I}_{(a,b]} = \begin{cases} 0, & 0 \leq X < a; \\ (X-a), & a \leq X < b; \\ (b-a), & X > b. \end{cases} \quad \text{where } S_{I_{(a,b]}}(t) = \begin{cases} S_X(a+t), & 0 \leq t < b-a; \\ 0, & t \geq b-a. \end{cases}$$

Then we can take X and divide it into layers $\{(x_i, x_{i+1}], i = 0, 1, \ldots\}$ with $0 = x_0 < x_1 < \ldots$ In this case, we get that

$$\pi_\rho(X) = \sum_{i=0}^{\infty} \pi_\rho\left(\mathbb{I}_{(x_i,x_{i+1}]}\right).$$

This is particularly helpful in reinsurance, where the insurer wants to offload the risk of very large losses to a reinsurer. To this end, suppose the insurer has risk-averse index ρ_1 and the reinsurer ρ_2, with $\rho_1 > \rho_2 \geq 1$. This means that $\pi_{\rho_1}(X) - \pi_{\rho_2}(X) > 0$, so the reinsurer sees a potential for profit due to their higher risk appetite.

Increased limits ratemaking

In general practice, insurers will set a *basic limit* $\alpha > 0$ (typically \$25,000 according to Wang) which can be used as a guideline for pricing larger limits, called *increased limits*. This is dome by calculating a multiplying factor called the *increased limits factors* (ILFs). For increased limit $\omega > \alpha$, this is defined as

$$\mathrm{ILF}(\omega) = \frac{\mathbb{E}[X;\omega] + \mathrm{RL}_{(0,\omega]}}{\mathbb{E}[X;\alpha] + \mathrm{RL}_{(0,\alpha]}},$$

where $\mathbb{E}[X;\beta]$ is the expectation of X over all $X \leq \beta$ and (Risk Loading) $\mathrm{RL}_{(d,\omega]} = \pi_{\rho_1}\left(\mathbb{I}_{(d,\omega]}\right) - \mathbb{E}\left[\mathbb{I}_{(d,\omega]}\right]$. Hence the ILFs with risk loading with risk-adjusted premiums becomes:

$$\mathrm{ILF}(\omega) = \begin{cases} \frac{\pi_{\rho_1}\left(\mathbb{I}_{(0,\omega]}\right)}{\pi_{\rho_1}\left(\mathbb{I}_{(0,\alpha]}\right)} & \text{without reinsurance,} \\ \frac{\pi_{cm}\left(\mathbb{I}_{(0,\omega]}\right)}{\pi_{cm}\left(\mathbb{I}_{(0,\alpha]}\right)} & \text{with reinsurance,} \end{cases}$$

where $\pi_{cm}(X) = \pi_{\rho_1}\left(\mathbb{I}_{(0,d]}\right) + C\pi_{\rho_2}\left(\mathbb{I}_{d,\omega]}\right)$ is the *competitive market premium* (see Section 7 for details on finding retention level d and pricing factor C) and for an infinitesimal layer $(x, x+\epsilon]$,

$$\pi_{cm}\left(\mathbb{I}_{(x,x+\epsilon]}\right) = \begin{cases} \pi_{\rho_1}\left(\mathbb{I}_{(x,x+\epsilon]}\right), & x < d; \\ C\pi_{\rho_2}\left(\mathbb{I}_{(x,x+\epsilon]}\right), & x \geq d. \end{cases}$$

Hence by using the optimal reinsurance allocations given in Section 7 of Wang (1995), the risk-adjusted premium can be used to allocate premiums to separate insurance layers for the risk X, and these can easily be priced using the ILFs.

Sources:

Wang, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics 17* (1), 43 – 54.