Diversifiable and non-diversifiable risk in (life) insurance applications

Daniel Matheson

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- Consider a portfolio of N life insurance policies.
- Let $(X_i)_{i=1}^N$ represent some quantity of interest, where X_i is associated with the i^{th} policyholder in the portfolio. For example:
  - X_i could be 1 if the policyholder is still alive in 10 years after the policy was issued, and 0 otherwise,
  - or X_i could be the present value of the loss on the i^{th} policy.
- Regardless of what $(X_i)_{i=1}^N$ represents, we would like that the X_i are **independent and identically distributed** with mean μ and standard deviation σ.
  - This means that the N life insurance policies in the portfolio should be very similar in structure as well as policyholder profiles.

- Since $(X_i)_{i=1}^N$ are iid,

$$\mathbb{E}\left[\sum_{i=1}^N X_i\right] = N\mu \qquad \text{and} \qquad \text{Var}\left[\sum_{i=1}^N X_i\right] = N\sigma^2.$$

- By the Central Limit Theorem, we get that for reasonably large N,

$$\sum_{i=1}^N X_i \sim \mathcal{N}\left(N\mu, N\sigma^2\right) \implies \frac{\left(\sum_{i=1}^N X_i\right) - N\mu}{\sqrt{N}\sigma} \sim \mathcal{N}\left(0,1\right).$$

- In this case where $(X_i)_{i=1}^N$ are iid, the probability that $\frac{\sum_{i=1}^N X_i}{N}$ deviates from its expected value decreases to 0 as N increases. Precisely, for any $k > 0$,

$$\begin{aligned} \mathbb{P}\left[\left|\frac{\sum_{i=1}^N X_i}{N} - \mu\right| \geq k\right] &= \mathbb{P}\left[\left|\sum_{i=1}^N X_i - N\mu\right| \geq kN\right] \\ &= \mathbb{P}\left[\left|\frac{\sum_{i=1}^N X_i - N\mu}{\sqrt{N}\sigma}\right| \geq \frac{k\sqrt{N}}{\sigma}\right] \end{aligned}$$

Letting $N \uparrow \infty$, we can assume $\frac{\sum_{i=1}^N X_i - N\mu}{\sqrt{N}\sigma} \sim \mathcal{N}(0,1)$ due to CLT.

- Therefore, in the limit as $N \uparrow \infty$, the probability from the previous slide can be written as

$$\lim_{N\uparrow\infty} \mathbb{P}\left[|Z| \geq \frac{k\sqrt{N}}{\sigma}\right] = \lim_{N\uparrow\infty} 2\Phi\left(-\frac{k\sqrt{N}}{\sigma}\right) = 0,$$

where $Z \sim \mathcal{N}(0,1)$ and Φ is the standard normal CDF.

- What this means is that as N increases, the variation of the sample mean of the X_i $\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right)$ from their expected value (μ) will converge to 0, **assuming independence**.

- If the $(X_i)_{i=1}^N$ are correlated with $\rho > 0$ between each X_i, X_j with $i \neq j$, then it is not generally true that adding more policies will decrease the variation described before.
  - Because $\mathrm{Var}\left[\sum_{i=1}^N X_i\right]$ is of order N^2.
- Therefore we define diversifiable risks as follows:

Definition

The risk within our portfolio, as measured by the random variables $(X_i)_{i=1}^N$ *is said to be* **diversifiable** *if the following condition holds:*

$$\lim_{N\uparrow\infty} \frac{\sqrt{\mathrm{Var}\left[\sum_{i=1}^N X_i\right]}}{N} = 0.$$

If this condition does not hold, the risk is called **non-diversifiable**.

- In practice, most insurers sell so many contracts over all their life insurance portfolios that mortality risk can be treated in many situations as fully diversified away.
- Insurers normally employ the no-arbitrage principle to argue that the value of a deterministic payment stream (e.g premiums) should be the same as the price of the zero-coupon bonds that replicate that stream.
  - In order to do this, we need to assume that the mortality risk associated with the portfolio is diversifiable.
  - What are the conditions for this to be a reasonable assumption?

- Assumptions needed to assure mortality risk is diversifiable:
  - (i) The N lives are independent with respect to their future mortality. $\implies$ Policyholder j dying does not affect whether policyholder i will die and vice-versa, for all $i \neq j \in \{1, \dots, N\}$.
  - (ii) The survival model for each of the N lives is known. $\implies$ We know the distribution of how long each life will live.
- The cash flow for this portfolio at some future time t will depend on how many policyholders are still alive at time t and on the times of death of those still alive at time t – which are random variables, so what do we do?

- We made some assumptions that assure mortality risk is diversifiable, and so provided that N is large, the variability of a risk such as the number of survivors at any time relative to the expected number of survivors will be very small.
  - That is, for large enough N, we can very precisely measure the number of policyholders who will survive to a future time t.
  - In practice, this would mean that we can treat the cash flows of the portfolio as being essentially deterministic.

Example

For $0 \leq t \leq t+s$, let $N_{t,s}$ denote the number of deaths between ages $x+t$ and $x+t+s$ for N lives currently aged x. Show that

$$\lim_{N \uparrow \infty} \frac{\sqrt{\mathrm{Var}\left[N_{t,s}\right]}}{N} = 0.$$

Example (cont'd)

Solution: *The random variable* $N_{t,s} \sim$ *Binomial* $(N, {}_tp_x(1 - {}_sp_{x+t}))$, *where* ${}_tp_x = \mathbb{P}($ *life aged* x *survives to at least age* $x + t)$. *Hence*

$$\mathrm{Var}\left[N_{t,s}\right] = N \cdot {}_tp_x(1 - {}_sp_{x+t})\left(1 - {}_tp_x(1 - {}_sp_{x+t})\right)$$

$$\implies \frac{\sqrt{\mathrm{Var}\left[N_{t,s}\right]}}{N} = \sqrt{\frac{{}_tp_x(1 - {}_sp_{x+t})\left(1 - {}_tp_x(1 - {}_sp_{x+t})\right)}{N}} \xrightarrow[N\uparrow\infty]{} 0. \qquad \square$$

$\implies$ If we set $t = 0, s > t$, then this example shows that the mortality risk in the next s years is diversifiable by increasing the number of policies (N) in the portfolio.

- Non-diversifiable risk in life insurance is on the asset management side, mostly via interest rate risk.
  - Insurers can lock into forward rates or interest rate swaps, which removes much of their investment risk at the start of a contract – but doing so also eliminates the possibility of profits.
  - It may not be possible to find 'risk-free' investments such as interest rate swaps that have expiry dates far enough in the future – some policies such as whole life insurance could last for another 50 or 60 years.
  - If the insurer locks into the forward rates at inception, there is a risk that interest rates will move – resulting in premiums that either too high, or too low.
    - Especially in recent times, as central banks change their rates often.
- Due to these conflicting priorities, it is generally not possible to eliminate interest rate risk via diversification.
  - Most life insurance company failures occur because of problems with non-diversifiable risk related to assets.

Sources

- Dickson, David et al. Actuarial Mathematics for Life Contingent Risks (2nd Edition). Cambridge University Press, 2013.