Brownian Motion and its Free Analogue

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Introduction

Brownian motion has been a mathematical object of curiosity for much of the scientific and mathematical community for over a century. It was first observed as a physical phenomenon by the Scottish botanist Robert Brown in 1827. While looking through his microscope at pollen grains suspended in water, he noticed that they moved around randomly in all directions equally, but could not explain why this was happening.

It was not until 1905 that Albert Einstein published a paper[1] which described brownian motion as the random motion of particles suspended in a fluid (liquid or gas), resulting from their collision with the atoms or molecules in the fluid. At the time, this meant that brownian motion served as further evidence to the scientific finding that fluids were indeed made of atoms, which was still up for debate by some.

Brownian motion has since become a very significant part of many areas of mathematics such as Itô calculus, the Black-Scholes model for pricing European options (which won the 1997 Nobel Prize in Economics), Physics (e.g solution to the diffusion equation), and many others ranging from Neuroscience and Biology, to Economics and Quantitative Sociology.

In this work, we propose to spend some time on this type of brownian motion in Section 1, which we will call the "classical brownian motion" - and the rest of the paper will be spent on its analogue in the Free Probability framework of Voiculescu, which we will refer to as "free brownian motion". We will briefly review the framework of free probability in Section 2.

While it remains an ongoing topic of research, we could likely find some important uses for the free analogue of such an influential stochastic process in the non-commutative setting of free probability theory. The first question we should ask ourselves, though, is whether this free brownian motion even exists, after we specify which properties it should satisfy.

There has been much work done in this area; both in proving that free brownian motion exists, and what its relationship is to the classical case (e.g [3],[4],[6]). Voiculescu and others have shown that free brownian motion can be thought of as a limit of classical brownian motions, under certain elaborate matrix constructions [3].

We will be more interested in the work done by Biane and Speicher ([2]) which outlines a method for constructing the free brownian motion using the Full Fock Space over $L^2\{[0,\infty)\}$. The work done by Kun-Hung (Rick) Hsueh in his MMATH Thesis [1] is a good source for learning Biane and Speicher's theories at a level more suitable to graduate students, and was used often as a reference for this work.

[1] *On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat*

1 Classical Brownian Motion

In order to define the classical brownian motion, we begin by briefly going over the necessary classical probability framework on which it rests.

Definition 1.1: A *probability space* $(\Omega, \mathcal{M}, P)$ is comprised of:
(1) The sample space Ω which is the set of all possible outcomes.
(2) A σ-algebra $\mathcal{M}$, which is a subset of $\mathcal{P}(\Omega)$ (power set)
(3) A probability measure P on $\mathcal{M}$.

Definition 1.2: For $(\Omega, \mathcal{M}, P)$ a probability space,
1. A *random variable* $f : \Omega \to \mathbb{F}$ is a $\mathcal{M}/\mathcal{B}(\mathbb{F})$-measurable function. Where $\mathcal{B}(\cdot)$ is the Borel σ-algebra. This can be any other σ-algebra, but for our purposes we will only need this one. Normally, $\mathbb{F}$ is taken to be $\mathbb{R}$ (for a real-valued random variable) or $\mathbb{C}$ (complex-valued).
2. A *stochastic process* on the probability space $(\Omega, \mathcal{M}, P)$ is a collection of random variables $\{Y_t\}_{t\geq 0}$.

Remark 1.3: We can think of a stochastic process as a function of two variables, $Y(t, \omega)$. Intuitively, one can think of fixing t as looking at the distribution of $\{Y_t(\omega) : \Omega \to \mathbb{F} \mid \omega \in \Omega\}$, i.e where the stochastic process will be after "time" t has passed. When we fix $\omega \in \Omega$, we can think of $Y_\omega(t)$ as a single path or realization of the stochastic process - i.e $Y_\omega(t) : [0, \infty) \to \mathbb{F}$ - which brings us into the realm of time series analysis. See the explanation and diagram on page 3 for further details.

Definition 1.4: The σ-algebra generated by a random variable $X : \Omega \to \mathbb{F}$ is $\sigma(X) = \{X^{-1}(B) \mid B \in \mathcal{B}(\mathbb{F})\}$.

Definition 1.5: Random variables $\{F_j : \Omega \to \mathbb{F} \mid j \in J\}$ are said to be *(classically) independent* if any finite subset (indexed by $I \subseteq J$ with $|I| = k$) of the σ-algebras they generate (denote $\{\mathcal{A}_i = \sigma(A_i) \mid i \in I\}$) are independent in the following sense:

$$\forall\ (A_1, \ldots, A_k) \in (\mathcal{A}_1 \times \ldots \times \mathcal{A}_k), \quad P\left(\bigcap_{i\in I} A_i\right) = \prod_{i\in I} P(A_i)$$

Theorem 1.6: There exists a stochastic process $\{B_t\}_{t\geq 0}$, called the *(Classical) Brownian Motion* such that, for the normal distribution $\mathcal{N}(\mu, \sigma^2)$,
(1) $B_0(\omega) = 0$ P-almost surely: $P(\{\omega \in \Omega \mid B_0(\omega) = 0\}) = 1$.
(2) $B_\omega(t)$ is almost everywhere continuous: for the Lebesgue measure m on $[0, \infty)$, and any fixed $\omega \in \Omega$, $m(\{t \in [0, \infty) \mid B_\omega(t) \text{ is not continuous at } t\}) = 0$.
(3) B_t has stationary increments, meaning that $(B_t - B_s) \sim \mathcal{N}(0, t - s)\ \forall\ s \leq t$ in $[0, \infty)$
(4) B_t has independent increments, meaning that $\forall\ t_1 < t_2 < \ldots < t_k$, $k \in \mathbb{N}$

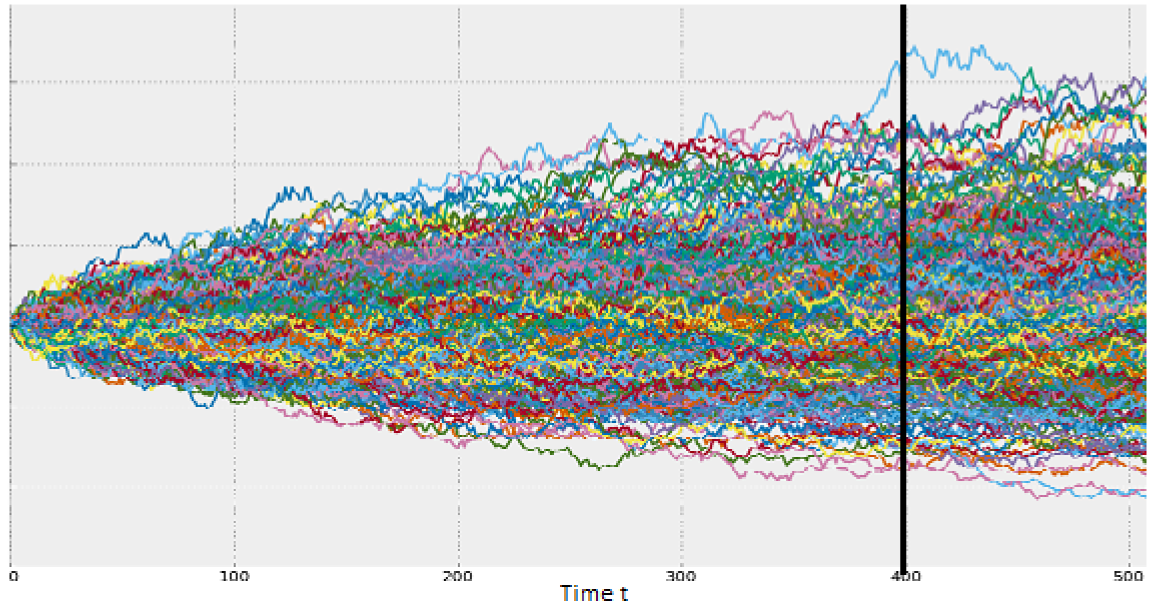
$$\left(B_{t_2} - B_{t_1}\right), \left(B_{t_3} - B_{t_2}\right), \ldots, \left(B_{t_k} - B_{t_{k-1}}\right)$$

are independent, in the classical sense. (Note that the linear combination of measurable functions is measurable, so $\left(B_{t_i} - B_{t_{i-1}}\right), i = 2, \ldots, k$ are in fact random variables)

Proof: See Donsker's Theorem [5] for a proof of existence arising from the limit of scaled discrete random walks. Another method of construction will be described later. □

Remark 1.7: Brownian Motion's paths are nowhere differentiable.

Question: In the classical case, the paths can be visualized as follows:



Where if we are restricting our attention to $B_t(\omega)$, for say $t = 400$, we are asking what the distribution is of "landing spots" of *all of the paths* from $t = 0 \to 400$ along the black line. And if we take one of the coloured paths alone, we are looking at $B_\omega(t)$ for some $\omega \in \Omega$.
We would now like to consider if such a stochastic process exists in the free probability framework. What sort of properties should we expect of this process? In the classical case, the idea of continuous paths was valid, but in the free probability framework, we have no such concept. What would a "non-commutative path" look like?
We do, however, have a free analogue of the Normal distribution via the Central Limit Theorem; the Wigner semi-circle distribution. We also have the concept of free independence, which could presumably take the place of classical independence.
With these ideas in mind, let us proceed to the free probability framework.

2 Free Probability Framework

We begin with some preliminary definitions:

Definition 2.1: A $*$-*probability space* $(\mathcal{A}, \mathbb{E})$ is comprised of:
(1) A unital $*$-algebra $\mathcal{A}$; which is a vector space over $\mathbb{C}$ that is also a ring, which has a unitary element $1_\mathcal{A}$, and it has an involution operation $* : \mathcal{A} \to \mathcal{A}$ satisfying:
  (i) $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^* \ \forall \ \alpha, \beta \in \mathbb{C}, x, y \in \mathcal{A}$.
  (ii) $(xy)^* = y^* x^* \ \forall \ x, y \in \mathcal{A}$
  (iii) $(x^*)^* = x \ \forall \ x \in \mathcal{A}$
(2) A linear functional $\mathbb{E} : \mathcal{A} \to \mathbb{C}$ such that $\mathbb{E}(1_\mathcal{A}) = 1$ and $\mathbb{E}(a^* a) \geq 0 \ \forall \ a \in \mathcal{A}$.

Definition 2.2: For $*$-probability space $(\mathcal{A}, \mathbb{E})$, an element $x \in \mathcal{A}$ is said to be *self-adjoint* if $x = x^*$.

Definition 2.3: For $*$-probability space $(\mathcal{A}, \mathbb{E})$, a self-adjoint element $x \in \mathcal{A}$ has k^{th} *moment* $\mathbb{E}(x^k)$.

Furthermore, if there exists a unique $\mu \in \text{Prob}(\mathbb{R})$ ($\mathbb{R}$ since elements are self-adjoint) such that

$$\int_{\mathbb{R}} t^k \, d\mu(t) = \mathbb{E}(x^k) \text{ and } \int_{\mathbb{R}} |t|^k \, d\mu(t) < \infty \; \forall \; k \in \mathbb{N}$$

then μ is said to be the *distribution (or law)* of x. This is often denoted $x \sim \mu$.

In this case, we say that μ has *Lebesgue density* if $\int_{\mathbb{R}} t^k d\mu(t) = \int_{\mathbb{R}} t^k f(t) \, dm(t)$ for some $f : \mathbb{R} \to \mathbb{R}$ (i.e the Radon-Nikodym derivative where m is Lebesgue meaure).

Example 2.4: The *Wigner Semi-Circle Law* with radius $R > 0$ has Lebesgue density

$$d\mu(t) = \frac{2}{\pi R^2} \sqrt{R^2 - x^2} \; \mathbb{I}_{[-R,R]}(t) dm(t)$$

Where $\mathbb{I}_{[-R,R]}$ is the indicator function of $[-R, R]$.
Furthermore, for $x \sim \mu$, the moments are

$$\mathbb{E}(x^k) = \begin{cases} 0 & \text{if } k \text{ odd} \\ \left(\frac{R}{2}\right)^k C_{k/2} & \text{if } k \text{ even} \end{cases}$$

Where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} *Catalan number*, which we will see more in depth later.
For $x \sim \mu$, we say that x is a *semi-circular element of radius* R, where if $R = 2$, we say that x is the *standard semi-circular element.*
Lastly, the *variance* of a semi-circular element is defined as $\mathbb{E}\left((x - \mathbb{E}(x))^2\right) = \mathbb{E}(x^2) = \frac{R^2}{4}$ since $E(x) = 0, (k = 1 \text{ odd})$, and $C_1 = 1$.

Definition 2.5: Let $(\mathcal{A}, \mathbb{E})$ be a $*$-probability space. The $*$-subalgebra generated by a self-adjoint element $x \in \mathcal{A}$ is denoted $\text{Alg}(f_g)$, and is defined as

$$\text{Alg}(f_g) = \text{span}\left\{ f_g^p \mid p \in \mathbb{Z} \right\} \subseteq \mathcal{A}$$

Definition 2.6: Let $(\mathcal{A}, \mathbb{E})$ be a $*$-probability space. We say that a collection $\{f_1, \dots, f_k\} \subseteq \mathcal{A}$ is *freely independent* if the $*$-subalgebras they generate are freely independent. That is: $\{A_j = \text{Alg}(f_j)\}_{j=1}^k$ are freely independent, meaning that:

$$\left\{ \begin{array}{r} \forall \; n \in \mathbb{N} \\ i_1, \dots, i_n \in \{1, \dots, k\} \; \text{ such that } i_j \neq i_{j-1} \; \forall \; j = 2, \dots n \\ a_{i_1} \in A_{i_i}, \dots, a_{i_n} \in A_{i_n} \\ \text{with } \mathbb{E}(a_{i_j}) = 0 \; \forall \; j = 1, \dots, n \end{array} \right\} \implies \left\{ \mathbb{E}(a_{i_1} a_{i_2} \dots a_{i_n}) = 0 \right\}$$

With this framework in place, we can now propose the following:

Proposition 2.7: There exists a stochastic process $\{X_t\}_{t \geq 0}$ on some $*$-probability space $(\mathcal{A}, \mathbb{E})$, called the *Free Brownian Motion* such that:
(1) X_t are self-adjoint $\forall \; t \geq 0$
(2) $X_0 = 0$
(3) X_t has stationary increments, meaning that $(X_t - X_s)$ is a semi-circular element with *variance* $(t - s) \; \forall \; 0 \leq s \leq t$.
(4) X_t has freely independent increments, meaning that $\forall \; t_1 < t_2 < \dots < t_k \in [0, \infty)$,

$$(X_{t_2} - X_{t_1}), (X_{t_3} - X_{t_2}), \dots, (X_{t_k} - X_{t_{k-1}})$$

are freely independent. (Note that these linear combinations $(X_{t_i} - X_{t_{i-1}}) \in \mathcal{A}$)

In order to prove the existence of this stochastic process, we need to introduce some heavier machinery.

3 The Full Fock Space

Before introducing the concept of the Full Fock Space, it is worth going over some basic definitions involved:

Definition 3.1:
(1) A *Hilbert space* is an inner product space which is complete with respect to the metric induced by said inner product.
(2) The *Direct Product* of vector spaces $V_1, V_2, \ldots, V_n$ is denoted as $V_1 \oplus V_2 \oplus \ldots \oplus V_n$ and is defined as

$$V \oplus V_2 \oplus \ldots \oplus V_n = \bigoplus_{i=1}^{n} V_i = \{(v_1, v_2, \ldots, v_n) \mid v_1 \in V_1, v_2 \in V_2, \ldots, v_n \in V_n\}$$

(3) The *Tensor Product* of vector spaces $V_1, V_2, \ldots, V_n$ over $\mathbb{C}$ is denoted as $V_1 \otimes V_2 \otimes \ldots \otimes V_n = \bigotimes_{i=1}^{n} V_i$ and is itself a vector space over $\mathbb{C}$ which is spanned by elements of the form $v_1 \otimes \ldots \otimes v_n$ with $v_i \in V_i \;\forall\; i$, called the *simple tensors of length n*.
For vector space A, denote $A^{\otimes n} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{n \text{ times}}$, and $A^{\otimes 0} = \mathbb{C}\Omega$ where Ω is the *vacuum vector*.
Then, for any $a \in A, a^{\otimes 0} = \Omega$; i.e the vacuum vector is a simple tensor of length zero.

Definition 3.2: For a Hilbert space $\mathcal{H}$, the *Full Fock Space*, denoted $\mathcal{F}(\mathcal{H})$ is defined as:

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n} = \Omega\mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \ldots$$

The inner product on $\mathcal{F}(\mathcal{H})$ is defined as:

$$\begin{aligned}\langle v, w \rangle_{\mathcal{F}(\mathcal{H})} &= \langle (v_1 \otimes v_2 \otimes \ldots), (w_1 \otimes w_2 \otimes \ldots) \rangle_{\mathcal{F}(\mathcal{H})} \\ &= \prod_{n=1}^{\infty} \langle v_n, w_n \rangle_{\mathcal{H}}\end{aligned}$$

Remark 3.3: If the Hilbert space $\mathcal{H}$ has an orthonormal basis $\{e_i \in \mathcal{H} \mid i \in I\}$, with I an arbitrary index set. Then the set $\{\Omega\} \cup \{e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n} \in \mathcal{F}(\mathcal{H}) \mid i_1, i_2, \ldots i_n \in I, n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{F}(\mathcal{H})$.

The Fock Space will serve us well in our explicit construction of the Free Brownian Motion, but we also need to define some operators on this space before we can get there.

Definition 3.4: Fix an arbitrary Hilbert space $\mathcal{H}$, and $h \in \mathcal{H}$.
(1) Define the n^{th} *left creation operator* as

$$\begin{aligned}&L_n^+(h) : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes(n+1)} \\ &L_n^+(h)(v_1 \otimes \ldots \otimes v_n) = h \otimes v_1 \otimes \ldots \otimes v_n \qquad \forall\; v_1 \otimes \ldots \otimes v_n \in \mathcal{H}^{\otimes n}\end{aligned}$$

(2) Define the *left creation operator* as

$$\begin{aligned}&L^+(h) : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}) \\ &L^+(h) = \bigoplus_{n=1}^{\infty} L_n^+(h)\end{aligned}$$

(3) Define the n^{th} *left annihilation operator* as

$$L_n^-(h) : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes (n-1)}$$
$$L_n^+(h)(v_1 \otimes \ldots \otimes v_n) = \langle v_1, h\rangle_{\mathcal{H}} \cdot v_2 \otimes \ldots \otimes v_n \qquad \forall\ v_1 \otimes \ldots \otimes v_n \in \mathcal{H}^{\otimes n}$$

(4) Define the *left annihilation operator* as

$$L^-(h) : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$$
$$L^-(h) = \bigoplus_{n=1}^{\infty} L_n^-(h)$$

Note that these operators are linear, and also bounded, since for fixed $h \in \mathcal{H}$,

$$\left\|L^+(h)\right\| = \sup_{\|v\|=1} \left\|L^+(h)v\right\| = \sup_{\|v\|=1} \sqrt{\langle (h \otimes v_1 \otimes \ldots), (h \otimes v_1 \otimes \ldots)\rangle_{\mathcal{F}(\mathcal{H})}}$$
$$= \sqrt{\langle h, h\rangle_{\mathcal{H}}} \prod_{n=1}^{\infty} \sqrt{\langle v_n, v_n\rangle_{\mathcal{H}}} = \|h\| \, \|v\| = \|h\| < \infty$$

And a similar proof can be done for $L^-(h)$.

Remark 3.5: We should take note of the way in which these operators behave with respect to one another. It is clear what the creation operator does: it simply "shifts" everything to the right and adds $h \in \mathcal{H}$ in the front as a tensor product. But the annihilation operator does not simply remove $v_1 \in \mathcal{H}$ as one would expect; a piece of v_1 remains via the inner product $\langle v_1, h\rangle_{\mathcal{H}}$. What this means is that we get the following relationship:

$$L^-(h)L^+(h) = \langle h, h\rangle_{\mathcal{H}} \mathcal{I}$$

where $\mathcal{I}$ is the identity operator on $\mathcal{F}(\mathcal{H})$.
Note also that $L^-(h)(c\Omega) = 0\ \forall\ c \in \mathbb{C}$, and $L^+(h)(0) = 0$.

Definition 3.6: We will define a linear functional $\mathbb{E}(\circ)$ on $\mathcal{F}(\mathcal{H})$ as:

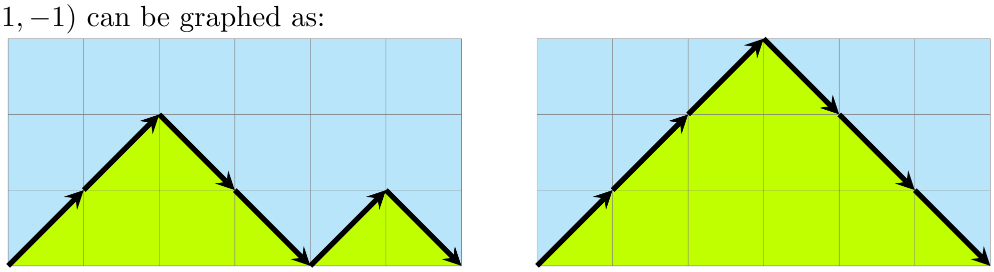
$$\mathbb{E}(\circ) = \langle \circ\Omega, \Omega\rangle_{\mathcal{F}(\mathcal{H})} = \begin{cases} 1 & \text{if } \circ = \mathcal{I} \\ 0 & \text{otherwise} \end{cases}$$

We need one last concept in order to enter into the main results of the paper, Dyck Paths.

4 Dyck Paths

Definition 4.1: A *Dyck Path* of length $k \in \mathbb{N}$ is a k-tuple $(\lambda_1, \ldots, \lambda_k) \in \{1, -1\}^k$ such that
(1) $\lambda_1 + \ldots + \lambda_j \geq 0 \quad \forall\ 1 \leq j \leq k$
(2) $\lambda_1 + \ldots + \lambda_k = 0$

Example 4.2: Dyck Paths can be visualized, for example, the Dyck Paths $(1, 1, -1, -1, 1, -1,)$ and $(1, 1, 1, -1, -1, -1)$ can be graphed as:



Note that the key feature of a Dyck Path is that it must return to the "x-axis" by the end of its trajectory, and it cannot ever go below it. *From now on, let us call this "x-axis" the base of the Dyck Path.* This is the graphical interpretation of conditions (1) and (2) in Definition 4.1. Additionally, we note that there are no Dyck Paths of length $(2k+1), k \in \mathbb{N}$ since it could never return to its base in $(2k+1)$ moves.

Lemma 4.3: The number of Dyck Paths of length $2k, k \in \mathbb{N}$, denoted $|D_{2k}|$, is the k^{th} Catalan number, $C_k = \frac{1}{k+1}\binom{2k}{k}$. For the proof see [1] pp 15-25.

In the following Lemmas,, let $\mathcal{H}$ be a Hilbert space with $h \in H$, $\mathcal{F}(\mathcal{H})$ be the Full Fock Space, and $L^+(h), L^-(h)$ be the creation and annihilation operators respectively. Finally, let $\mathbb{E}(\cdot)$ be the linear functional we defined in Definition 3.6.

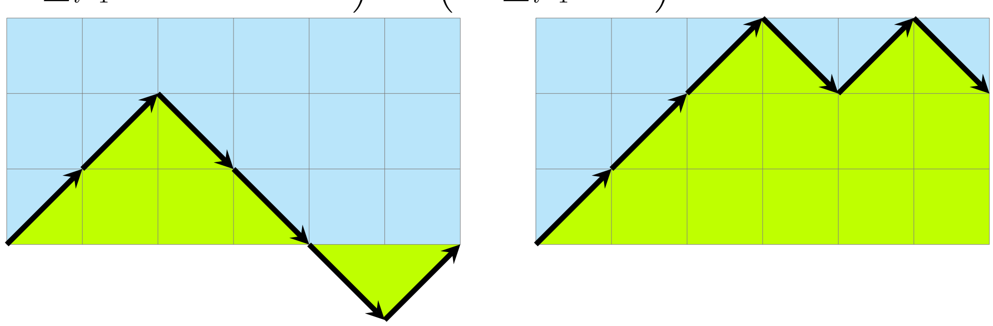
Lemma 4.4: Let $k \in \mathbb{N}$. For $(\epsilon_1, \dots, \epsilon_k) \in \{+, -\}$, set

$$\lambda_j = \begin{cases} 1 & \text{if } \epsilon_j = + \\ -1 & \text{if } \epsilon_j = - \end{cases}$$

Then we have the following:

$$\begin{aligned} \mathbb{E}\left(L^{\epsilon_k}(h) \dots L^{\epsilon_1}(h)\right) &= \langle L^{\epsilon_k}(h) \dots L^{\epsilon_1}(h)\Omega, \Omega\rangle_{\mathcal{F}(\mathcal{H})} \\ &= \begin{cases} \langle h, h\rangle_{\mathcal{H}}^{k/2} & \text{if } (\lambda_1, \dots, \lambda_k) \text{ is a Dyck Path} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Proof: In order to relate $(L^{\epsilon_k}(h) \dots L^{\epsilon_1}(h))$ to a Dyck path, we think of these operators as a *bracket system* where each "opening bracket" $(L^+(h))$ must have a "closing bracket" that follows it $(L^-(h))$.
And from Remark 3.5 we know that $L^-(h)L^+(h) = \langle h, h\rangle_{\mathcal{H}}\mathcal{I}$.
Let us first consider what happens if $(\epsilon_1, \dots, \epsilon_k)$ form a set $(\lambda_1, \dots, \lambda_k)$ which violate the requirements of Dyck Paths as found in Definition 4.1. There are two possibilities, which violate$\left(1 : \sum_{i=1}^{j} \lambda_i \geq 0 \ \forall\ 1 \leq j \leq k\right)$ and $\left(2 : \sum_{i=1}^{k} \lambda_i = 0\right)$ in Definition 4.1 respectively:



What happened in the first case? We had two L^+ creation operators, followed by two L^- annihilation operators, giving us $\langle h, h\rangle_{\mathcal{H}}^2$, then we applied another annihilation operator; which gives us 0 as explained in Remark 3.5. In other words; we closed a bracket (L^-) before we even opened one (L^+) which doesn't make any sense.
And so in this case, or any case where $\sum_{i=1}^{j} \lambda_i < 0$ for some $1 \leq j \leq k$,

$$\mathbb{E}\left(L^{\epsilon_k}(h) \dots L^{\epsilon_1}(h)\right) = \langle 0\Omega, \Omega\rangle_{\mathcal{F}(\mathcal{H})} = 0$$

In the second case, we did not return to the base of our Dyck Path diagram, i.e $\sum_{i=1}^{k} \lambda_i > 0$ (the < 0 case is covered by the first case). In terms of brackets, this means we have brackets that were never closed. So there are more creation operators than annihilation operators.

In this specific case, we had the system $\{\{\{\}\{\}$. So the last two sets of brackets cancel eachother out to give $\langle h,h\rangle_{\mathcal{H}}^2$, but then we are left with two L^+ operators. Hence

$$\mathbb{E}\left(L^{\epsilon_k}(h)\dots L^{\epsilon_1}(h)\right) = \langle\langle h,h\rangle_{\mathcal{H}}^2 h\otimes h,\Omega\rangle = 0$$

And hence in general, it is easy to see that in this case we get, for $n\in\mathbb{Z}_{\geq 1}$:

$$\mathbb{E}\left(L^{\epsilon_k}(h)\dots L^{\epsilon_1}(h)\right) = \langle\langle h,h\rangle_{\mathcal{H}}^{(k-n)/2} h^{\otimes n},\Omega\rangle = 0$$

Now the pattern should be clear: we know that we only get a non-zero value for $\mathbb{E}(\cdot)$ when $(L^{\epsilon_k}(h)\dots L^{\epsilon_1}(h))$ is some multiple of the identity operator $\mathcal{I}$. This corresponds to when all brackets systems are proper; i.e for every opened bracket, there is a closed one. Or in other words, for every creation operator L^+, there is an annihilation operator L^- which comes *after* it. So in $(L^{\epsilon_k}(h)\dots L^{\epsilon_1}(h))$, there must be $k/2$ creation operators and $k/2$ annihilation operators in the aforementioned order. Clearly such a bracket system is exactly corresponding to a Dyck path, and in this case we get

$$\begin{aligned}\mathbb{E}\left(L^{\epsilon_k}(h)\dots L^{\epsilon_1}(h)\right) &= \langle\langle h,h\rangle_{\mathcal{H}}^{k/2}\mathcal{I}\Omega,\Omega\rangle_{\mathcal{F}(\mathcal{H})}\\ &= \langle h,h\rangle_{\mathcal{H}}^{k/2}\langle\Omega,\Omega\rangle_{\mathcal{F}(\mathcal{H})} = \langle h,h\rangle_{\mathcal{H}}^{k/2}\end{aligned}$$

As we wanted to show. □

Lemma 4.5: Let $X(h) = L^+(h) + L^-(h)$. Then

$$\begin{aligned}\mathbb{E}\left((X(h))^k\right) &= \langle X(h)^k\Omega,\Omega\rangle_{\mathcal{F}(\mathcal{H})}\\ &= \begin{cases}(\langle h,h\rangle)_{\mathcal{H}}^{k/2}C_{k/2} & \text{if } k \text{ even}\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Proof:

$$\begin{aligned}\langle X(h)^k\Omega,\Omega\rangle_{\mathcal{F}(\mathcal{H})} &= \langle\left(L^+(h)+L^-(h)\right)^k\Omega,\Omega\rangle_{\mathcal{F}(\mathcal{H})}\\ &= \langle\left(\sum_{\epsilon_1,\dots\epsilon_k\in\{+,-\}} L^{\epsilon_k}(h)\dots L^{\epsilon_1}(h)\right)\Omega,\Omega\rangle_{\mathcal{F}(\mathcal{H})}\\ &= \sum_{\epsilon_1,\dots\epsilon_k\in\{+,-\}}\langle L^{\epsilon_k}(h)\dots L^{\epsilon_1}(h)\Omega,\Omega\rangle_{\mathcal{F}(\mathcal{H})}\end{aligned}$$

by Lemma 4.4, sum only contributes if $(\epsilon_1,\dots,\epsilon_k)$ form a Dyck path

$$\text{i.e } \langle L^{\epsilon_k}(h)\dots L^{\epsilon_1}(h)\Omega,\Omega\rangle_{\mathcal{F}(\mathcal{H})} = \begin{cases}\langle h,h\rangle_{\mathcal{H}}^{k/2} & \text{if } (\lambda_1,\dots,\lambda_k) \text{ is a Dyck Path}\\ 0 & \text{otherwise}\end{cases}$$

$$\text{where } \lambda_j = \begin{cases}1 & \text{if } \epsilon_j = +\\ -1 & \text{if } \epsilon_j = -\end{cases}$$

$$\begin{aligned}\Rightarrow \langle X(h)^k\Omega,\Omega\rangle_{\mathcal{F}(\mathcal{H})} &= \langle h,h\rangle_{\mathcal{H}}^{k/2}\cdot|D_k|\\ &= \begin{cases}\langle h,h\rangle_{\mathcal{H}}^{k/2}C_{k/2} & \text{if } k \text{ even} \quad \text{by Lemma 4.3}\\ 0 & \text{if } k \text{ odd since no Dyck Paths of odd length}\end{cases}\end{aligned}$$

□

Lemma 4.6: $X(h) = L^+(h) + L^-(h)$ as previously defined is self-adjoint.

Proof: Let $v, w \in \mathcal{F}(\mathcal{H})$ be simple tensors, and the result can easily be extended by linearity.

$$\begin{aligned}
\langle X(h)v, w\rangle_{\mathcal{F}(\mathcal{H})} &= \langle \left((L^+(h) + L^-(h) \right) (v_1 \otimes v_2 \otimes \ldots), (w_1 \otimes w_2 \otimes \ldots) \rangle_{\mathcal{F}(\mathcal{H})} \\
&= \langle h \otimes v_1 \otimes \ldots, w_1 \otimes w_2 \otimes \ldots \rangle_{\mathcal{F}(\mathcal{H})} \\
&\quad + \langle \langle v_1, h\rangle_{\mathcal{H}} v_2 \otimes v_3 \otimes \ldots, w_1 \otimes w_2 \otimes \ldots \rangle \\
&= \langle w_1, h\rangle_{\mathcal{H}} \prod_{n=1}^{\infty} \langle v_n, w_{n+1}\rangle_{\mathcal{H}} + \langle v_1, h\rangle_{\mathcal{H}} \prod_{n=1}^{\infty} \langle v_{n+1}, w_n\rangle_{\mathcal{H}} \\
&= \langle (v_1 \otimes \ldots), (h \otimes w_1 \otimes \ldots)\rangle_{\mathcal{F}(\mathcal{H})} + \langle (v_1 \otimes \ldots), (\langle w_1, h\rangle_{\mathcal{H}} w_2 \otimes \ldots)\rangle \\
&= \langle v, L^+(h)w\rangle_{\mathcal{F}(\mathcal{H})} + \langle v, L^-(h)w\rangle_{\mathcal{F}(\mathcal{H})} = \langle v, X(h)w\rangle
\end{aligned}$$

□

5 Construction of the Free Brownian Motion

Theorem 5.1: Consider the Hilbert space $\mathcal{L}^2([0,\infty)) = \left\{ f : [0,\infty) \to \mathbb{R} \mid \int_{[0,\infty)} |f|^2 \, dm < \infty \right\}$. And let $\mathcal{F}(\mathcal{H})$ be the Full Fock Space over $\mathcal{H}$.

Define the stochastic process $X_t = X(h_t)$, with $h_t = \mathbb{I}_{[0,t)} \in \mathcal{L}^2(\mathbb{R}^+)$. Then X_t is the Free Brownian Motion satisfying all of the properties required in Proposition 2.7 which were:

(1) X_t are self-adjoint $\forall\, t \geq 0$

(2) $X_0 = 0$

(3) $(X_t - X_s)$ is a semi-circular element with *variance* $(t - s)$ $\forall\, 0 \leq s \leq t$.

(4) X_t has freely independent increments.

Proof: We first note that X_t are indeed self-adjoint by Lemma 4.6.

And then by Lemma 4.5,

$$\begin{aligned}
\mathbb{E}(X_t^k) &= \mathbb{E}(X(h_t)^k) \\
&= \begin{cases} \langle h_t, h_t\rangle^{k/2} C_{k/2} & \text{if } k \text{ even} \\ 0 & \text{otherwise} \end{cases} \\
\text{such that } \|h_t\|_{\mathcal{L}^2} &= \int_{\mathbb{R}^+} \mathbb{I}^2_{[0,t)}(x)dx = \sqrt{t} = \sqrt{\langle h_t, h_t\rangle} \\
&\Rightarrow \langle h_t, h_t\rangle = t \;\forall\, t \in [0,\infty) \\
&\Rightarrow \mathbb{E}(X_t^k) = \begin{cases} t^{k/2} C_{k/2} & \text{if } k \text{ even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

These moments of X_t are exactly the moments of the Wigner semi-circle law with radius $R = 2\sqrt{t}$ as in Example 2.4. And $\mathrm{Var}(X_t) = E(X_t^2) = \frac{R^2}{4} = t$ as required of $(X_t - X_0) = X_t$.

We note that $X_0 = 0$ since we showed L^+, L^- are bounded operators in Definition 3.4 (and hence continuous), therefore;

$$\begin{aligned}
X_0 &= \lim_{t\to 0} X_t = \lim_{t\to 0} X\left(\mathbb{I}_{[0,t)}\right) \\
&= X\left(\lim_{t\to 0} \mathbb{I}_{[0,t)}\right) = X(0) = 0
\end{aligned}$$

For $s < t$ in $[0, \infty)$,

$$\begin{aligned}(X_t - X_s) &= \left(L^+\left(\mathbb{I}_{[0,t)}\right) + L^-\left(\mathbb{I}_{[0,t)}\right)\right) - \left(L^+\left(\mathbb{I}_{[0,s)}\right) + L^-\left(\mathbb{I}_{[0,s)}\right)\right) \\ &= \left(L^+\left(\mathbb{I}_{[0,t)}\right) - L^+\left(\mathbb{I}_{[0,s)}\right)\right) + \left(L^-\left(\mathbb{I}_{[0,t)}\right) - L^-\left(\mathbb{I}_{[0,s)}\right)\right) \\ &= L^+\left(\mathbb{I}_{[0,t)} - \mathbb{I}_{[0,s)}\right) + L^-\left(\mathbb{I}_{[0,t)} - \mathbb{I}_{[0,s)}\right) \quad \text{by linearity} \\ &= L^+\left(\mathbb{I}_{[s,t)}\right) + L^-\left(\mathbb{I}_{[s,t)}\right) \\ &= \begin{cases} (t-s)^{k/2} C_{k/2} & \text{if } k \text{ even} \\ 0 & \text{otherwise} \end{cases} \quad \text{as before}\end{aligned}$$

Hence $(X_t - X_s)$ follows Wigner's semicircle law with $R = 2\sqrt{t-s}$ and has variance $t - s$ as desired.
Note that we can assert that these random variables follow the Wigner Semi-circle law because this law is completely determined by its moments; due to its compact support $[-R, R]$.
One fact remains to be proved; that $\{X_t\}_{t\geq 0}$ has freely independent increments. □

Lemma 5.2: Let $r \in \mathbb{N}$ and $\{h_1, \ldots, h_r\} \subseteq \mathcal{H}$ be a collection of mutually orthogonal vectors. Then for the random variable $X(h) = L^+(h) + L^-(h)$ as defined previously on the Full Fock Space, the collection $\{X(h_1), \ldots, X(h_r)\}$ is freely independent.

Proof: See [1] Theorem 3.12 pp 32. □

With Lemma 5.2, we can finish the proof of Theorem 5.1:

Proof of 5.1 cont'd: Let $t_1 < t_2 < \ldots < t_k$ in $[0, \infty)$ for some $k \in \mathbb{N}$
Then the functions $(h_{t_2} - h_{t_1}), \ldots, \left(h_{t_k} - h_{t_{k-1}}\right)$ are orthogonal, because for any $j \neq \ell$ in $\{1, \ldots, k\}$,

$$\begin{aligned}\langle \left(h_{t_j} - h_{t_{j-1}}\right), \left(h_{t_\ell} - h_{t_{\ell-1}}\right)\rangle_{\mathcal{L}^2} &= \langle \left(\mathbb{I}_{[0,t_j)} - \mathbb{I}_{[0,t_{j-1})}\right), \left(\mathbb{I}_{[0,t_\ell)} - \mathbb{I}_{[0,t_{\ell-1})}\right)\rangle_{\mathcal{L}^2} \\ &= \langle \left(\mathbb{I}_{[t_{j-1},t_j)}\right), \left(\mathbb{I}_{[t_{\ell-1},t_\ell)}\right)\rangle_{\mathcal{L}^2} \\ &= \int_{\mathbb{R}^+} \mathbb{I}_{[t_{j-1},t_j)} \mathbb{I}_{[t_{\ell-1},t_\ell)} \, dm \\ &= m\left([t_{j-1}, t_j) \cap [t_{\ell-1}, t_\ell)\right) = m(\emptyset) = 0\end{aligned}$$

Hence, by Lemma 5.2, $\left\{X\left(h_{t_2} - h_{t_1}\right), \ldots, X\left(h_{t_k} - h_{t_{k-1}}\right)\right\}$ form a family of free random variables.
That is, $(X_{t_2} - X_{t_1}), \ldots, \left(X_{t_k} - X_{t_{k-1}}\right)$ are freely independent.
So we have proven every aspect of Proposition 2.7 with regards to X_t as defined. So it is the Free Brownian Motion we were looking for. □

Remark 5.3: Using a similar construction, with the Bosonic Fock Space, one can also show that the Classical Brownian Motion exists, giving an alternative to Donker's Theorem. There is no space to include such a derivation in this paper, but it is laid out in detail in [1] pp 33-42.

6 Next Steps

What would come next in this line of thinking is extending the aforementioned stochastic integration (Itô Calculus) to the concept of stochastic integration over the Free Brownian Motion. This was done by Rolland & Speicher in [2], and Hsueh [1] also covers some of this material.

What could also be of interest is exploring the relationship between Classical and Free Brownian Motion; what do they have in common, and is there an “in between” Brownian Motion, and could we also perform stochastic integration over that stochastic process? So-called q-Brownian Motion attemps to answer this question (see [6]).

References

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