# STAT 946: Stochastic Differential Equations, Final Exam

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**Note:** In all questions, $\{B_t\}_{t\in\mathbb{R}^+}$ is the 1-dimensional standard Brownian motion. The filtration $\{F_t\}_{t\in\mathbb{R}^+}$ used for the SDEs is always the filtration generated by the driving process.

**Question 1:** Give an example for each of the following, and give an explanation:
(a) A continuous-time martingale $\{X_t\}_{t\in\mathbb{R}^+}$ which is not a Markov process.
(b) A continuous-time martingale $\{X_t\}_{t\in\mathbb{R}^+}$ which converges almost-surely to a limit $X_\infty$ as $t\to\infty$, but does not converge in $\mathcal{L}^2$.

*Solution:*

(a) Consider the stochastic process $\{X_t\}_{t\in\mathbb{R}^+}$ defined as:

$$X_t = \int_0^t \operatorname{sgn}(B_s)dB_s$$

This is a martingale as described in Theorem 3.2.1 in class,
because $f(t,\omega) = \operatorname{sgn}(B_t(\omega)) \in V_{[0,T]}\ \forall\ T \in \mathbb{R}^+$, i.e
(1) $\operatorname{sgn}(B_t)$ is $\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}$-measurable,
(2) $\operatorname{sgn}(B_t)$ is $\mathcal{F}_t$-adapted (since $\mathcal{F}_s \subseteq \mathcal{F}_t\ \forall\ s \le t$), and
(3) $\mathbb{E}\left[\int_0^T \operatorname{sgn}^2(B_t)dt\right] = T < \infty\ \forall\ T\in\mathbb{R}^+$.
However, $\{X_t\}_{t\in\mathbb{R}^+}$ is *not* Markov, because for $A\in\mathcal{B}(\mathbb{R})$, and $\{F_t\}_{t\in\mathbb{R}^+}$ the filtration generated by the Brownian motion,

$$\begin{aligned}\mathbb{P}\left(X_t \in A \mid \mathcal{F}_s\right) &= \mathbb{P}\left(\int_0^t \operatorname{sgn}(B_r)dB_r \in A \mid \mathcal{F}_s\right)\\ &\neq \mathbb{P}\left(\int_0^t \operatorname{sgn}(B_r)dB_r \in A \mid \sigma(X_s)\right)\end{aligned}$$

because simply conditioning on $\sigma(X_s)$ does not mean that we know the current value of $\operatorname{sgn}(B_s)$ or $B_s$, as we would with conditioning on $\mathcal{F}_s$, because $\mathcal{F}_s \supseteq \mathcal{F}_r\ \forall\ r \le s$.
That is, the value of $\operatorname{sgn}(B_s)$ depends on all of the values $\{\operatorname{sgn}(B_r) \mid r\in[0,s]\}$ which we do not get any information about, other than their total integral with respect to $B_t$, from $\sigma(X_s)$. So the two probabilities will not necessarily be equal. Hence the process $\{X_t\}_{t\in\mathbb{R}^+}$ is a martingale which does not have the Markov property.

(b) Consider the random variables $\{X_n\}_{n\in\mathbb{N}}$ where $X_n \overset{\text{iid}}{\sim} \text{Unif}(-1,1)$. Hence $\mathbb{E}(X_n) = 0\ \forall\ n$, $\operatorname{Var}(X_n) = \frac{1}{12}(1-(-1))^2 = \frac{1}{3}$, and $\mathbb{E}(X_n^2) = \operatorname{Var}(X_n) + \mathbb{E}(X_n)^2 = \frac{1}{3}$.
Now define the process $\{M_t\}_{t\in\mathbb{R}^+}$ as:

$$M_t := \sum_{n\in\mathbb{N}\cap[0,t]} X_n \qquad \forall\ t\in\mathbb{R}^+$$

$$\Rightarrow \mathbb{E}\left[|M_t|\right] = \mathbb{E}\left[\left|\sum_{n\in\mathbb{N}\cap[0,t]} X_n\right|\right] \le \mathbb{E}\left[\sum_{n\in\mathbb{N}\cap[0,t]} |X_n|\right] \le \mathbb{E}\left[\lfloor t\rfloor\right] = \lfloor t\rfloor < \infty\ \forall\ t\in\mathbb{R}^+$$

So $M_t \in \mathcal{L}^1\ \forall\ t \in \mathbb{R}^+$. Note also that for $\mathcal{F}_s = \sigma\left(X_1, \ldots, X_{\lfloor s \rfloor}\right)$ and $s \leq t$,

$$\begin{aligned}
\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[\sum_{n \in \mathbb{N} \cap [0,t]} X_n \mid \mathcal{F}_s\right] \\
&= \sum_{n \in \mathbb{N} \cap [0,s]} X_n + \mathbb{E}\left[\sum_{n \in \mathbb{N} \cap (s,t]} X_n\right] \\
&= \sum_{n \in \mathbb{N} \cap [0,s]} X_n + \sum_{n \in \mathbb{N} \cap (s,t]} \mathbb{E}\left[X_n\right] = \sum_{n \in \mathbb{N} \cap [0,s]} X_n = M_s
\end{aligned}$$

So $\{M_t\}_{t \in \mathbb{R}^+}$ is a martingale which converges almost surely to $M_\infty = \sum_{n \in \mathbb{N}} X_n$.
However, $M_\infty \notin \mathcal{L}^2$, so $\{M_t\}_{t \in \mathbb{R}^+}$ cannot converge in $\mathcal{L}^2$:

$$\begin{aligned}
\mathbb{E}\left[M_\infty^2\right] &= \mathbb{E}\left[\left(\sum_{n \in \mathbb{N}} X_n\right)^2\right] \\
&= \mathbb{E}\left[\sum_{n \in \mathbb{N}} X_n^2 + 2 \sum_{n<m} X_n X_m\right] \\
&= \mathbb{E}\left[\sum_{n \in \mathbb{N}} X_n^2\right] + 2\mathbb{E}\left[\sum_{n<m} X_n X_m\right]
\end{aligned}$$

Where $\mathbb{E}\left[\sum_{n \in \mathbb{N}} X_n^2\right] = \lim_{k \to \infty} \sum_{n=1}^k \mathbb{E}(X_n^2) = \lim_{k \to \infty} \frac{k}{3} = +\infty$ by Monotone Convergence Theorem. So $\mathbb{E}\left[M_\infty^2\right] = +\infty \implies M_\infty \notin \mathcal{L}^2$.
(or if $\mathbb{E}\left[\sum_{n<m} X_n X_m\right] = -\infty$, then $\mathbb{E}\left[M_\infty^2\right]$ is not defined, so $M_\infty \notin \mathcal{L}^2$)
Hence $\{M_t\}_{t \in \mathbb{R}^+}$ is a martingale which converges almost surely, but not in $\mathcal{L}^2$.

**Question 2:** Consider the SDE

$$dX(t) = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} X_2(t) \\ \ln\left(1 + X_1^2(t) + X_2^2(t)\right) \end{bmatrix} dt + \begin{bmatrix} 0 & 1 \\ 1 & X_2(t) \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}$$

with the initial condition $X(0) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1, x_2 \in \mathbb{R}$. $B_1, B_2$ are two independent standard one-dimensional Brownian motions.
(a) Show that this SDE has a unique strong solution.
(b) Find the generator of this solution.

*Solution:*

(a) We showed in class that a linear SDE of the form

$$\begin{aligned}
dX_t &= b(t,x)dt + \sigma(t,x)dB_t, \\
X_0 &= x
\end{aligned}$$

has a unique strong solution under the following conditions for any $T > 0$:

$$|b(t,x)| + |\sigma(t,x)| \leq c\,(1 + |x|)\ \ \forall\ x \in \mathbb{R}^n, t \in [0,T] \text{ for some } c \in \mathbb{R}^+ \tag{1}$$

$$|(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D\,|x-y|\ \ \forall\ x,y \in \mathbb{R}^n, t \in [0,T] \text{ for some } D \in \mathbb{R}^+ \tag{2}$$

But we are dealing with vector valued functions now, so instead of $|\cdot|$ we will use $\|\cdot\|_2$, i.e

$$\|x\|_2 = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}} \ \ \forall\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$
$$\|A\|_2 = \left(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2\right)^{\frac{1}{2}} \ \ \forall\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{M}^{2\times 2}(\mathbb{R})$$

So we want to show the following $\forall\ x, y \in \mathbb{R}^2, t \in [0,T]$ and some constants $c, D \in \mathbb{R}^+$:

$$\|b(t,x)\|_2 + \|\sigma(t,x)\|_2 \le c\,(1 + \|x\|_2) \tag{3}$$
$$\|b(t,x) - b(t,y)\|_2 + \|\sigma(t,x) - \sigma(t,y)\|_2 \le D\,\|x-y\|_2 \tag{4}$$
$$(3) = \left(x_2^2 + \left(\ln\left(1 + x_1^2 + x_2^2\right)\right)^2\right)^{\frac{1}{2}} + \|\sigma(t,x)\|_2 \tag{5}$$

$$\text{note that } \ln\left(1 + x_1^2 + x_2^2\right) \overset{\dagger}{=} \ln(1+y^2) = \int_0^y \frac{2z}{1+z}dz \quad \text{by Fund. Thm. of Calc.}$$
$$\dagger\text{: change of variables } y^2 = x_1^2 + x_2^2.$$
$$\le \int_0^y 2dz = 2y = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}$$
$$\text{and } \|\sigma(t,x)\|_2 = \left\| \begin{bmatrix} 0 & 1 \\ 1 & x_2 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & x_2 \end{bmatrix} \right\|_2$$
$$\le \left\| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} 0 & 0 \\ 0 & x_2 \end{bmatrix} \right\|_2 = \sqrt{2} + \left(x_2^2\right)^{\frac{1}{2}} \ \ (\Delta \text{ ineq})$$
$$\le \sqrt{2}\left(1 + \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}\right) = \sqrt{2}\,(1 + \|x\|_2)$$
$$\Rightarrow (5) \le \left(x_2^2 + \left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} + \sqrt{2}\,(1 + \|x\|_2)$$
$$= \left(x_1^2 + 2x_2^2\right)^{\frac{1}{2}} + \sqrt{2}\,(1 + \|x\|_2) \le 2\sqrt{2}\,(1 + \|x\|_2)$$
$$\Rightarrow c = 2\sqrt{2} \text{ satisfies } (3)$$

$$(4) = \|b(t,x) - b(t,y)\|_2 + \|\sigma(t,x) - \sigma(t,y)\|_2$$
$$= \left((x_2 - y_2)^2 + \left(\ln\left(1 + x_1^2 + x_2^2\right) - \ln\left(1 + y_1^2 + y_2^2\right)\right)^2\right)^{\frac{1}{2}} + \left\| \begin{bmatrix} 0 & 0 \\ 0 & (x_2 - y_2) \end{bmatrix} \right\|_2$$

$$\text{where } \left| \ln\left(1 + \underbrace{x_1^2 + x_2^2}_{z^2}\right) - \ln\left(1 + \underbrace{y_1^2 + y_2^2}_{w^2}\right) \right| = \left| \int_w^z \frac{\partial}{\partial t} \ln(1+t^2)\ dt \right| \quad \text{by Fund. Thm. of Calc.}$$
$$= \left| \int_w^z \frac{2t}{1+t^2}\ dt \right| \le 2\,|z - w|$$
$$= 2\left| \left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - \left(y_1^2 + y_2^2\right)^{\frac{1}{2}} \right| = 2\ \Big|\, \|x\|_2 - \|y\|_2 \,\Big|$$
$$\le 2\,\|x - y\|_2 \text{ by reverse triangle inequality}$$

$$\Rightarrow (4) \leq \left((x_2 - y_2)^2 + (2\|x-y\|_2)^2\right)^{\frac{1}{2}} + \left((x_2-y_2)^2\right)^{\frac{1}{2}}$$
$$\leq \sqrt{5}\|x-y\|_2 + \|x-y\|_2 = \left(1+\sqrt{5}\right)\|x-y\|_2$$
$$\Rightarrow D = \left(1+\sqrt{5}\right) \text{ satisfies (4).}$$

Therefore, there exists a unique strong solution to the SDE.

(b) In order to find the generator, we refer to a theorem from class and use the fact that $b(t,x), \sigma(t,x)$ are continuous on $(\mathbb{R}^+ \times \mathbb{R}^2)$, so we have that the solution $\{X_t\}_{t\in\mathbb{R}^+}$ of the SDE is a diffusion process with drift $b(t,x)$ and diffusion matrix $a(t,x) = \sigma\sigma^T$.
We also know that the generator of such a process is given by:

$$\mathcal{L}_t = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(t,x)\frac{\partial}{\partial x_i} \tag{6}$$

In this case, $b(t,x) = \begin{bmatrix} x_2(t) \\ \ln\left(1+x_1^2(t)+x_2^2(t)\right)\end{bmatrix}, \sigma(t,x) = \begin{bmatrix} 0 & 1 \\ 1 & x_2(t)\end{bmatrix}$ so the diffusion matrix is

$$a(t,x) = \sigma\sigma^T = \begin{bmatrix} 0 & 1 \\ 1 & x_2(t)\end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & x_2(t)\end{bmatrix}$$
$$= \begin{bmatrix} 1 & x_2(t) \\ x_2(t) & 1+x_2^2(t)\end{bmatrix}$$

Therefore the generator of $\{X_t\}_{t\in\mathbb{R}^+}$, by (6), is

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + x_2(t)\frac{\partial^2}{\partial x_1\partial x_2} + x_2(t)\frac{\partial^2}{\partial x_2 \partial x_1} + \left(1+x_2^2(t)\right)\frac{\partial^2}{\partial x_2^2}\right)$$
$$+ x_2(t)\frac{\partial}{\partial x_1} + \ln\left(1+x_1^2(t)+x_2^2(t)\right)\frac{\partial}{\partial x_2}$$
$$= \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + 2x_2(t)\frac{\partial^2}{\partial x_1\partial x_2} + \left(1+x_2^2(t)\right)\frac{\partial^2}{\partial x_2^2}\right) + x_2(t)\frac{\partial}{\partial x_1} + \ln\left(1+x_1^2(t)+x_2^2(t)\right)\frac{\partial}{\partial x_2}$$

(the second line if dealing with functions in $\mathcal{C}^2$, giving equal mixed partials)

**Question 3:** Suppose a stochastic process $\{Y_t\}_{t\in\mathbb{R}^+}$ vanishes at $t=0$ (i.e $Y_0 = 0$) and satisfies

$$dY_t = B_t dB_t$$

Find $\mathbb{E}\left([Y]_t\right)$ where $[Y]_t$ is the quadratic variation of $\{Y_t\}_{t\in\mathbb{R}^+}$.

*Solution:* For a semi-martingale $X = X_0 + M + A$, with $X_0 \in \mathcal{F}_0$, $M$ being a continuous local martingale vanishing at 0, and $A$ being a finite variation process vanishing at 0, we have that $[X]_t = [M]_t$, because:

$$[X]_t = [X_0 + M + A]_t$$
$$= [X_0 + M + A, X_0 + M + A]_t$$
$$= [X_0, X_0]_t + 2[X_0, M]_t + 2[X_0, A]_t + 2[M, A]_t + [M, M]_t + [A, A]_t$$
$$= [M, M]_t = [M]_t$$

Since all other co-quadratic variations are zero, because any co-quadratic variation with one of the processes being of finite variation is zero. We quickly prove this result before moving on:
WLOG consider the partition $\tau_n^k = k \cdot 2^{-n}; n, k \in \mathbb{N}$. Then

$$A_n(t,w) := \sum_{k\in\mathbb{N}} \left[X\left(t\wedge\tau_{k+1}^n,\omega\right) - X\left(t\wedge\tau_k^n,\omega\right)\right]\left[Y\left(t\wedge\tau_{k+1}^n,\omega\right) - Y\left(t\wedge\tau_k^n,\omega\right)\right] \xrightarrow{\mathcal{L}^2} [X,Y]_t$$

$$V_n(t,w) := \sum_{k\in\mathbb{N}} \left|Y\left(t\wedge\tau_{k+1}^n,\omega\right) - Y\left(t\wedge\tau_k^n,\omega\right)\right| \xrightarrow{\mathcal{L}^2} V\left[Y(\cdot,\omega);[0,t]\right], \text{ the total variation of } Y \text{ on } [0,t]$$

The proof of these two statements is far too long to include here, but we will use these to show that if $\{Y_t\}_{t\in\mathbb{R}^+}$ is a finite variation process, and $X$ is a continuous process then $[X,Y]_t = 0$:

$$\begin{aligned} A_n(t,w) &:= \sum_{k\in\mathbb{N}} \left[X\left(t\wedge\tau_{k+1}^n,\omega\right) - X\left(t\wedge\tau_k^n,\omega\right)\right]\left[Y\left(t\wedge\tau_{k+1}^n,\omega\right) - Y\left(t\wedge\tau_k^n,\omega\right)\right] \\ &\le \max_{k\in\mathbb{N}} \left|X\left(t\wedge\tau_{k+1}^n,\omega\right) - X\left(t\wedge\tau_k^n,\omega\right)\right| \sum_{k\in\mathbb{N}} \left|Y\left(t\wedge\tau_{k+1}^n,\omega\right) - Y\left(t\wedge\tau_k^n,\omega\right)\right| \end{aligned}$$

Taking limit as $n\to\infty$ on both sides gives that

$$\max_{k\in\mathbb{N}} \left|X\left(t\wedge\tau_{k+1}^n,\omega\right) - X\left(t\wedge\tau_k^n,\omega\right)\right| \xrightarrow{\mathcal{L}^2} 0$$

since $X$ is cts on compact [0,t], so unif. cts $\Rightarrow$ unif. (a.s) converge $\Rightarrow$ $\mathcal{L}^2$ converge

$$\text{and } \sum_{k\in\mathbb{N}} \left|Y\left(t\wedge\tau_{k+1}^n,\omega\right) - Y\left(t\wedge\tau_k^n,\omega\right)\right| \xrightarrow{\mathcal{L}^2} [Y(\cdot,\omega);[0,t]] < \infty \text{ by assumption}$$

$$\begin{aligned} &\Rightarrow A_n(t,w) \xrightarrow{\mathcal{L}^2} 0 \\ &\Rightarrow [X,Y]_t = 0 \quad \text{since } \mathcal{L}^2 \text{ limits unique} \end{aligned}$$

Write $Y_t$ in the following form:

$$Y_t = Y_0 + \int_0^t B_s dB_s = \int_0^t B_s dB_s \tag{7}$$

Where $\int_0^t B_s dB_s$ is a continuous local martingale vanishing at 0, so $Y_t$ is a semi-martingale with $Y_0, A = 0$.
Therefore we have that

$$[Y]_t = \left[\int_0^t B_s dB_s\right]_t = [B\cdot B]_t \tag{8}$$

To find $[B\cdot B]_t$ we use Itô's integration by parts formula, using the fact that $(B\cdot B)_t$ is a continuous local martingale vanishing at 0, hence it is also a semi-martingale with initial value 0 and finite variation process $A=0$:

$$\begin{aligned} [B\cdot B]_t &= [B\cdot B, B\cdot B]_t \\ &= (B\cdot B)_t^2 - (B\cdot B)_0^2 - 2\int_0^t (B\cdot B)_s d(B\cdot B)_s \\ &= \left(\int_0^t B_s dB_s\right)^2 - 0 - 2\int_0^t \left(\int_0^s B_r dB_r\right) B_s dB_s \end{aligned}$$

$$= \left(\int_0^t B_s dB_s\right)^2 - 2\int_0^t \underbrace{\left(\frac{1}{2}\left(B_s^2 - s\right)\right)}_{\text{from Assignment 1}} B_s dB_s$$

$$\Rightarrow [Y]_t = [B \cdot B]_t = \left(\int_0^t B_s dB_s\right)^2 - \int_0^t \left(B_s^3 - sB_s\right) dB_s$$

Hence we get that

$$\begin{aligned}
\mathbb{E}\left([Y]_t\right) = \mathbb{E}\left([B \cdot B]_t\right) &= \mathbb{E}\left[\left(\int_0^t B_s dB_s\right)^2 - \int_0^t \left(B_s^3 - sB_s\right) dB_s\right] \\
&= \mathbb{E}\left[\left(\int_0^t B_s dB_s\right)^2\right] - \mathbb{E}\left(\int_0^t \left(B_s^3 - sB_s\right) dB_s\right) \\
&= \underbrace{\mathbb{E}\left(\int_0^t B_s^2 ds\right)}_{\text{by Itô's Isometry}} - \mathbb{E}\left(\int_0^t \left(B_s^3 - sB_s\right) dB_s\right)
\end{aligned}$$

We can apply Itô's Isometry because $B_t \in V$, i.e it is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable, $\mathcal{F}_t$-adapted, and $\mathbb{E}\left(\int_0^T B_t^2 dt\right) = \frac{1}{2}(T^2) < \infty \ \forall \ T \in \mathbb{R}^+$.

$$\begin{aligned}
&= \int_0^t \mathbb{E}\left(B_s^2\right) ds - \underbrace{\mathbb{E}\left(\int_0^t \left(B_s^3 - sB_s\right) dB_s\right)}_{\text{martingale}^\dagger} \quad \text{(Fubini thm)} \\
&= \int_0^t s ds \quad \text{since } \mathbb{E}(B_s^2) = s \ \forall \ s \in \mathbb{R}^+ \\
\Rightarrow \mathbb{E}\left([Y]_t\right) &= \int_0^t s ds = \frac{1}{2}t^2
\end{aligned}$$

†: $\int_0^t (B_s^3 - sB_s) dB_s$ is a martingale by Thm 3.2.1 since $B_t^3 - sB_t \in V_{[0,T]} \ \forall \ T \in \mathbb{R}^+$, i.e it is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable, $\mathcal{F}_t$-adapted, and $\mathbb{E}\left[\int_0^T (B_s^3 - sB_s)^2 ds\right] < \infty \ \forall \ T \in \mathbb{R}+$ (see below). So it has expectation zero.

$$\begin{aligned}
\mathbb{E}\left[\int_0^T (B_s^3 - sB_s)^2 ds\right] &= \int_0^T \mathbb{E}\left[B_s^6 - 2sB_s^4 + s^2 B_s^2\right] ds \quad \text{(Fubini thm)} \\
&= \int_0^T 15s^3 - 2s(3s^2) + s^2(s) ds < \infty
\end{aligned}$$

using the $6^{\text{th}}, 4^{\text{th}}$, and $2^{\text{nd}}$ moments of the Normal distribution since $B_s \sim \mathcal{N}(0, s)$.

$\implies \mathbb{E}\left([Y]_t\right) = \frac{1}{2}t^2$

**Question 4:**

(a) Solve the SDE

$$dX_t = bX_t dt + dB_t$$

with the initial condition $X_0 = x$ such that $b, x \in \mathbb{R}$.

(b) Find the variance of $X_t$

*Solution:*

(a) This is a linear SDE of the form

$$dX_t = (\alpha(t) + \beta(t)X_t)dt + (\gamma(t) + \delta(t)X_t)dB_t$$
$$\text{such that } \begin{cases} \alpha(t) = 0, \ \beta(t) = b \\ \gamma(t) = 1, \ \delta(t) = 0 \end{cases}$$

which are clearly all adapted processes. So we know the solution is of the form

$$X_t = \underbrace{\exp\left\{\int_0^t \left(\beta(s) - \frac{1}{2}\delta^2(s)\right) ds + \int_0^t \delta(s)dB_s\right\}}_{u(t)} \cdot \left(X_0 + \int_0^t \frac{\alpha(s) - \delta(s)\gamma(s)}{u(s)} ds + \int_0^t \frac{\gamma(s)}{u(s)} dB_s\right)$$

$$\Rightarrow X_t = \exp\left(\int_0^t b ds\right) \cdot \left(x + 0 + \int_0^t \frac{1}{u(s)} dB_s\right)$$

$$\Rightarrow X_t = e^{bt}\left(x + \int_0^t e^{-bs} dB_s\right)$$

(b)

$$\begin{aligned}
\mathrm{Var}(X_t) &= \mathbb{E}\left(X_t^2\right) - \left(\mathbb{E}\left(X_t\right)\right)^2 \\
&= \underbrace{\mathbb{E}\left[\left(e^{bt}\left(x + \int_0^t e^{-bs} dB_s\right)\right)^2\right]}_{(1)} - \underbrace{\left(\mathbb{E}\left[e^{bt}\left(x + \int_0^t e^{-bs} dB_s\right)\right]\right)^2}_{(2)} \\
(1) &= \mathbb{E}\left[e^{2bt}x^2 + e^{2bt}\left(\int_0^t e^{-bs} dB_s\right)^2 + 2xe^{2bt}\int_0^t e^{-bs} dB_s\right] \\
&= e^{2bt}x^2 + e^{2bt}\mathbb{E}\left[\left(\int_0^t e^{-bs} dB_s\right)^2\right] + 2xe^{2bt}\mathbb{E}\left[\int_0^t e^{-bs} dB_s\right] \\
(2) &= \left(\mathbb{E}\left[e^{bt}x + e^{bt}\int_0^t e^{-bs} dB_s\right]\right)^2 \\
&= e^{2bt}x^2 + e^{2bt}\left(\mathbb{E}\left[\int_0^t e^{-bs} dB_s\right]\right)^2 + 2xe^{2bt}\mathbb{E}\left[\int_0^t e^{-bs} dB_s\right] \\
\Rightarrow (1) - (2) &= e^{2bt}\left\{\mathbb{E}\left[\left(\int_0^t e^{-bs} dB_s\right)^2\right] - \left(\mathbb{E}\left[\int_0^t e^{-bs} dB_s\right]\right)^2\right\} \\
&= e^{2bt}\left\{\mathbb{E}\left[\int_0^t e^{-2bs} ds\right] - 0\right\} \quad \text{by Itô's Isometry and } e^{-bs} \in V,
\end{aligned}$$

so $\int_0^t e^{-bs} dB_s$ is a martingale, therefore has expectation 0.

$$\begin{aligned}
&= e^{2bt}\int_0^t e^{-2bs} ds = e^{2bt}\frac{e^{-2bs}}{-2b}\Big|_0^t \\
\Rightarrow \mathrm{Var}(X_t) &= \frac{e^{2bt}}{2b}\left(1 - e^{-2bt}\right)
\end{aligned}$$

**Question 5:** For any one-dimensional continuous semimartingale vanishing at 0, $\boldsymbol{X} = \{X_t\}_{t\in\mathbb{R}^+}$, define $\mathcal{E}(\boldsymbol{X})$ to be the unique strong solution of the SDE

$$dZ_t = Z_t dX_t$$

with the initial condition $Z_0 = 1$.
(a) Show that

$$\mathcal{E}(\boldsymbol{X})_t = \exp\left(X_t - \frac{1}{2}\left[X\right]_t\right)$$

(b) Let $\boldsymbol{X}, \boldsymbol{Y}$ be two one-dimensional continuous semimartingales vanishing at 0. Show that

$$\mathcal{E}(\boldsymbol{X})\mathcal{E}(\boldsymbol{Y}) = \mathcal{E}\left(\boldsymbol{X} + \boldsymbol{Y} + [\boldsymbol{X}, \boldsymbol{Y}]\right)$$

*Solution:*

(a) We saw a similar example in class, the SDE:

$$\begin{aligned} dX_t &= rX_t dt + \sigma X_t dB_t \\ \text{had solution } X_t &= X_0 e^{\left(1-\frac{1}{2}\sigma^2\right)t+\sigma B_t} \end{aligned}$$

To solve this SDE, we used Itô's formula on the function $\ln(X_t)$. We will proceed with a similar method in this question, but using the version of Itô's formula for semi-martingales. i.e for $f(x) = \ln(x)$ applied to $Z_t$:

$$\begin{aligned} \ln(Z_t) - \underbrace{\ln(Z_0)}_{=\ln(1)=0} &= \int_0^t \frac{\partial}{\partial x} f(Z_s) dZ_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(Z_s) d\left[Z\right]_s \\ \Rightarrow \ln(Z_t) &= \int_0^t \frac{1}{Z_s} dZ_s + \frac{1}{2}\int_0^t -\frac{1}{Z_s^2} d\left[Z\right]_s \end{aligned}$$

We will show that $d\left[Z\right]_s = Z_s^2 d\left[X\right]_s$ to finish the proof. For the semi-martingale $X = M + A$ ($M$ a continuous local martingale vanishing at 0, and $A$ a continuous finite variation process vanishing at 0), we know that

$$Z_t = \int_0^t Z_s dX_s = \int_0^t Z_s dM_s + \int_0^t Z_s dA_s, \tag{9}$$

$$\text{and } \left[Z\right]_t = \left[M\right]_t \quad \text{as proven in question 3} \tag{10}$$

The definition of the stochastic integral of $Z$ over $M$ is the unique (within indistinguishability) continuous local martingale vanishing at 0 which satisfies the following condition for any continuous local martingale $\{\psi_t\}_{t\in\mathbb{R}^+}$:

$$\left[\int_0^\cdot Z_s dM_s, \psi\right]_t = \int_0^t Z_s d\left[M, \psi\right]_s$$

However, $Z_t = \int_0^t Z_s dM_s$ is itself a continuous local martingale, so we can consider the following:

$$\begin{aligned}
[Z]_t &= \left[\int_0^\cdot Z_s dM_s, \int_0^\cdot Z_s dM_s\right]_t \\
&= \int_0^t Z_s d\left[M, \int_0^\cdot Z_r dM_r\right]_s \\
&= \int_0^t Z_s d\left[\int_0^\cdot Z_r dM_r, M\right]_s \quad \text{(co-quadratic variation is symmetric)}
\end{aligned}$$

$$\begin{aligned}
\text{but } \left[\int_0^\cdot Z_r dM_r, M\right]_s &= \int_0^s Z_r d[M]_r \quad \text{by def'n again and } [M,M]_t = [M]_t \\
\Rightarrow d\left[\int_0^\cdot Z_r dM_r, M\right]_s &= Z_s d[M]_s \\
\Rightarrow [Z]_t &= \int_0^t Z_s (Z_s d[M]_s) = \int_0^t Z_s^2 d[M]_s \\
&= \int_0^t Z_s^2 d[X]_s \quad \text{since } [X]_s = [M]_s
\end{aligned}$$

Therefore, returning to our Itô Formula derivation:

$$\begin{aligned}
\ln(Z_t) &= \int_0^t \frac{1}{Z_s} dZ_s - \frac{1}{2}\int_0^t \frac{1}{Z_s^2} d[Z]_s \\
&= \int_0^t \frac{1}{\cancel{Z_s}} \underbrace{\cancel{Z_s} dX_s}_{\text{by SDE}} - \frac{1}{2}\int_0^t \frac{1}{\cancel{Z_s^2}} \cancel{Z_s^2} d[X]_s = X_t - \frac{1}{2}[X]_t \\
\Rightarrow Z_t &= \exp\left(X_t - \frac{1}{2}[X]_t\right)
\end{aligned}$$

(b)

$$\mathcal{E}(X)\mathcal{E}(Y) = \exp\left(X_t - \frac{1}{2}[X]_t\right)\exp\left(Y_t - \frac{1}{2}[Y]_t\right) \text{ by part (a)}$$

Note that $(X + Y + [X,Y])$ is also a continuous semi-martingale vanishing at 0, because $[X,Y]$ is a finite variation process vanishing at 0, so we can apply part (a):

$$\begin{aligned}
\mathcal{E}(X+Y+[X,Y]) &= \exp\left(X_t + Y_t + [X,Y]_t - \frac{1}{2}[X+Y+[X,Y]]_t\right) \\
\text{but } [X+Y+[X,Y]]_t &= [X+Y+[X,Y], X+Y+[X,Y]]_t \\
&= [X]_t + [Y]_t + 2\underbrace{[X,[X,Y]]_t}_{=0} + 2\underbrace{[Y,[X,Y]]_t}_{=0} + 2\underbrace{[[X,Y],[X,Y]]_t}_{=0} + 2[X,Y]_t
\end{aligned}$$

Where the terms above are zero because $[X,Y]_t$ is a finite variation process (see proof in Question 3). Hence,

$$\begin{aligned}
\mathcal{E}(X+Y+[X,Y]) &= \exp\left(X_t + Y_t + [X,Y]_t - \frac{1}{2}\left([X]_t + [Y]_t + 2[X,Y]_t\right)\right) \\
&= \exp\left(X_t + Y_t - \frac{1}{2}\left([X]_t + [Y]_t\right)\right) = \mathcal{E}(X)\mathcal{E}(Y)
\end{aligned}$$

**Question 6:** Consider the PDE

$$\frac{\partial g(t,x)}{\partial t} = 2\frac{\partial^2 g(t,x)}{\partial x^2} \qquad \forall\ t>0, x\in\mathbb{R} \tag{11}$$

with the initial condition $g(0,x)=f(x)$ where $f\in C_0^2$ is given. From general theory, it is known that there exists a unique bounded solution. Write down this solution in terms of a Brownian Motion.

*Solution:* In order to find this solution, we use the Feynman-Kac formula, which states that:
For $f\in C_0^2, q\in C_b$, i.e $q$ bounded, $g(t,x)=\mathbb{E}^x\left[\exp\left(-\int_0^t q(X_s)ds\right)f(X_t)\right]$ solves the PDE

$$\begin{aligned}\frac{\partial g}{\partial t} &= \mathcal{L}g - qg,\ \forall\ t>0, x\in\mathbb{R}^n \qquad (12)\\ g(0,x) &= f(x)\end{aligned}$$

when $\mathcal{L}$ is the generator of the Itô diffusion resulting as the solution of $dX_t = b(X_t)dt+\sigma(X_t)dB_t$, which as we saw previously is written as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(t,x)\frac{\partial}{\partial x_i}\\ \text{for } a(t,x) &= \sigma\sigma^T\end{aligned}$$

To get (11) in the form as in (12), we observe that $q=0$ and we must find the Itô Diffusion which has generator $\mathcal{L}=2\frac{\partial^2}{\partial x^2}$, which simply amounts to $a=a_{11}=4\Rightarrow\sigma(X_t)=2$, i.e

$$\begin{aligned} dX_t &= 2dB_t\\ \Rightarrow X_t &= 2\int_0^t dB_t = 2B_t \quad \text{since } B_0=0\\ \implies g(t,x) &= \mathbb{E}^x\left[f\left(2B_t\right)\right]\end{aligned}$$