Insurance pricing and increased limits ratemaking by proportional hazards transforms

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- Our aim is to summarize the main results of the 1995 paper **"Insurance pricing and increased limits ratemaking by proportional hazards transforms"** by Shaun Wang.

- From Wang (1995): “insurance is a practice of exchanging a contingent claim for a fixed payment called [the] premium. The principle of assigning premiums according to the underlying risk is an essential element of actuarial science”
- Premiums are calculated with a chosen *premium principle*, which is usually based on expected loss or the standard deviation/variance of the loss.

- Let X be a risk such that $X \geq 0$ with CDF F_X, density f_x and survival function $S_X = 1 - F_X$.
- Define the **hazard rate** for the risk X as

$$\mu_X(t) = \lim_{\Delta t \downarrow 0} \frac{F_X(t + \Delta t) - F_X(t)}{\Delta t \,(1 - F_X(t))} = \frac{f_X(t)}{1 - F_X(t)} = -\frac{d}{dt} \ln S_X(t).$$

- Intuitively: $\mu_X(t)$ represents the likelihood that the loss X is (infinitesimally) larger than t, given that it is at least t.

- A lower hazard rate means the insurer would be more risk averse.
- Hence a ‘safety margin’ can be imposed by reducing the hazard rate:

$$\mu_Y(t) = \frac{1}{\rho}\mu_X(t), \quad \rho \geq 1, \quad t \geq 0.$$

- This defines a random variable Y such that $S_Y(t) = S_X(t)^{\frac{1}{\rho}}$.
- Finally, this gives the mapping $\Pi_\rho : X \mapsto Y$ which we call the **proportional hazards (PH) transform**. That is, $\Pi_\rho(X) = Y$.
- See Section 3 for examples of PH transforms.

- The **Risk-Adjusted Premium** is defined as

$$\pi_\rho(X) = \mathbb{E}\left[\Pi_\rho(X)\right] = \int_0^\infty S_X(t)^{\frac{1}{\rho}}\, dt, \qquad \rho \geq 1.$$

- Note $\pi_1(X) = \int_0^\infty S_X(t)\, dt = \mathbb{E}[X]$.
- $\pi_\rho(X)$ has many desirable properties such as linearity. See Section 4 for more.

- The risk X can be broken into "layers" and priced separately. Layers are defined as:

$$\mathbb{I}_{(a,b]} = \begin{cases} 0, & 0 \leq X < a; \\ (X - a), & a \leq X < b; \\ (b - a), & X > b. \end{cases}$$

- Then $X = \sum_{i=0}^{\infty} \mathbb{I}_{(x_i, x_{i+1}]}$, for $0 = x_0 < x_1 < \ldots$.
- The risk-adjusted premium is linear with respect to this layering of X: $\pi_\rho(X) = \sum_{i=0}^{\infty} \pi_\rho \left(\mathbb{I}_{(x_i, x_{i+1}]} \right)$.

- In practice, insurers set a **basic limit** $\alpha > 0$ ($\alpha = 25000$ for Wang).
- For insured claims with higher limits $\omega > \alpha$, called **increased limits**, pricing is done by multiplying the premium of the basic limit policy by the **increased limits factors (ILFs)**:

$$\mathrm{ILF}(\omega) = \frac{\mathbb{E}\left[X;\omega\right] + \mathrm{RL}_{(0,\omega]}}{\mathbb{E}\left[X;\alpha\right] + \mathrm{RL}_{(0,\alpha]}},$$

where RL is the risk loading and $\mathbb{E}\left[X;\beta\right]$ the expectation of X over all $X \leq \beta$.

- With the risk-adjusted premium, for insurer with risk-aversion index $\rho_1 \geq 1$, we can express the risk loading as

$$\mathsf{RL}_{(0,\omega]} = \pi_{\rho_1}\left(\mathbb{I}_{(0,\omega]}\right) - \mathbb{E}\left[\mathbb{I}_{(0,\omega]}\right].$$

Hence, $\mathsf{ILF}(\omega) = \frac{\pi_{\rho_1}\left(\mathbb{I}_{(0,\omega]}\right)}{\pi_{\rho_1}\left(\mathbb{I}_{(0,\alpha]}\right)}$.

- See Section 10 for calculating the ILFs with competitive market premiums (where reinsurance is permitted and limits are high).

- Using the proportional hazard transform, with a given risk-aversion index $\rho \geq 1$, we can calculate risk-adjusted premiums.
  - These premiums take into account the risk appetite of the parties involved, which others do not.
- These premiums are linear under layering, which simplifies pricing trenches and also streamlines reinsurance (offloading trenches of losses to another party with lower risk-aversion – i.e with a lower ρ).

Thanks for listening!

Sources:

Wang, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics 17* (1), 43–54.

Example

If X is exponential with mean b, $S_X(t) = e^{-\frac{t}{b}}$ and $\mu_X(t) = \frac{1}{b}$.
Then the PH transform $Y = \Pi_\rho(X)$ also has an exponential distribution with mean ρb.
Hence $\pi_\rho(X) = \mathbb{E}\left[\Pi_\rho(X)\right] = \rho b$.