The Additive and Multiplicative Effects Model*

Seminar on Statistical Modeling of Social Networks

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Abstract

This seminar paper introduces the Additive and Multiplicative Effects Network Model as proposed in P. D. Hoff (2021). First, a general introduction to network and relational data is given along with the statistical challenges of dyadic data analysis. Second, the Additive and Multiplicative Effects model ist introduced and motivated. I then apply this to data from the Alliances Treaty and Obligations Project (ATOP) in the year 2000. *Keywords: Social Networks, Dyadic Data, Bayesian estimation, International Relations.*

*Replication files are available on Github (http://github.com/danielseussler). **Current version**: June 01, 2021; **Corresponding author**: daniel.seussler@gmail.com

1 Introduction

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2 Theory

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Network Effects

- 1st-order: Sender (Receiver) Effects
- 2nd-order: Reciprocity
- 3rd-order: Homophily & Stochastic Equivalence
- System-level - changing actor composition

The Social Relations Model

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$$\begin{aligned}
y_{ij} &= \mu + e_{ij} \\
e_{ij} &= a_i + b_j + \epsilon_{ij} \\
\{(a_1, b_1), \ldots, (a_n, b_n)\} &\sim N(0, \Sigma_{ab}) \\
\{(\epsilon_{ij}, \epsilon_{ji}) :\ i \neq j\} &\sim N(0, \Sigma_\epsilon),\ \text{where} \\
\Sigma_{ab} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} &\quad \Sigma_\epsilon = \sigma_\epsilon^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}
\end{aligned}$$

The Latent Factor Model

$$\mathbf{u}_i, \mathbf{v}_j \in \mathbb{R}^k \quad i, j \in \{1, \ldots, n\}$$

$$\begin{aligned}
\gamma(\mathbf{u}_i, \mathbf{v}_j) &= \mathbf{u}_i^T D \mathbf{v}_j \\
&= \sum_{k \in K} d_k u_{ik} v_{jk} \\
D &\text{ is a } K \times K \text{ diagonal matrix}
\end{aligned}$$

Bringing it all together: The AME Network Model

$$\begin{aligned}
y_{ij,t} &= g(\theta_{ij,t}) \\
\theta_{ij,t} &= \beta^T \mathbf{X}_{ij,t} + e_{ij,t} \\
e_{ij,t} &= a_i + b_j + \epsilon_{ij} + \alpha(\mathbf{u}_i, \mathbf{v}_j)\ \text{, where} \\
\alpha(\mathbf{u}_i, \mathbf{v}_j) &= \mathbf{u}_i^T \mathbf{D} \mathbf{v}_j = \sum_{k \in K} d_k u_{ik} v_{jk}
\end{aligned}$$

Parameter Estimation and Goodness of Fit

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Modeling Longitudinal Data

3 Application case: The Alliances Data Set

In this section the above introduced additive and multiplicative effects model is applied to the network of international defense alliance network. The data is an excerpt from the data used in S. J. Cranmer, Desmarais, and Menninga (2012) and S. Cranmer, Desmarais, and Kirkland (2012) and contains the yearly defense network as a time series from 1981 - 2000.[1] It furthermore includes information on military capabilites, political regime types, geographic borders and interstate conflict. From here on out, I refer to this data set as the `alliances` data set.

The analysis is structured as follows. First, restricted to a cross-sectional study, I examine the alliances structure with respect to the network statistics and preliminary models to motivate the application of statistical network approaches. Second, a full specification of an AME-model is presented, along with the selection of appropiate parameters. Third, I extend briefly the analysis to include a longitudinal analysis of the interstate alliances network capturing the years 1981-2000. The analysis was implemented with the R-Software (R Core Team (2021)) and the AMEN R-Package (P. Hoff, Fosdick, and Volfovsky (2020)).

Network Statistics

A visualization of the interstate alliance network is provided in 1.

Table 1: Goodness of Fit Statistics for the Year 2000

sd.rowmean	sd.colmean	dyad.dep	cycle.dep	trans.dep
0.057	0.057	1	0.389	0.389

f dlkjdhf klajdhf kljdhf

SRM & Latent Factor Analysis

We first fit the model without the SRM Terms. As illustrated in Fig, the model performs quite bad.

Modeling interstate alliances

Now that we have established the validity of the additive and multiplicative effects approach in this application case, we extend the model with dyadic and extra-dyadic effects. I follow Warren (2010) in his definition of variables to obtain a model that is both sensible and relevant to the theoretic discussion of interstate alliance networks.

Hence, as dyadic effects I include the geographic distance, a capability ratio, a measure for political similarity, a conflict dummy,

[1] The data set `alliances` can be conveniently accessed as network data in the R-Package `xergm.common`.

Table 2: Network Statistics x of the Alliances Network 2000

	x
Size	164.00
Edgecount	767.00
Dyadcount	13366.00
Density	0.06

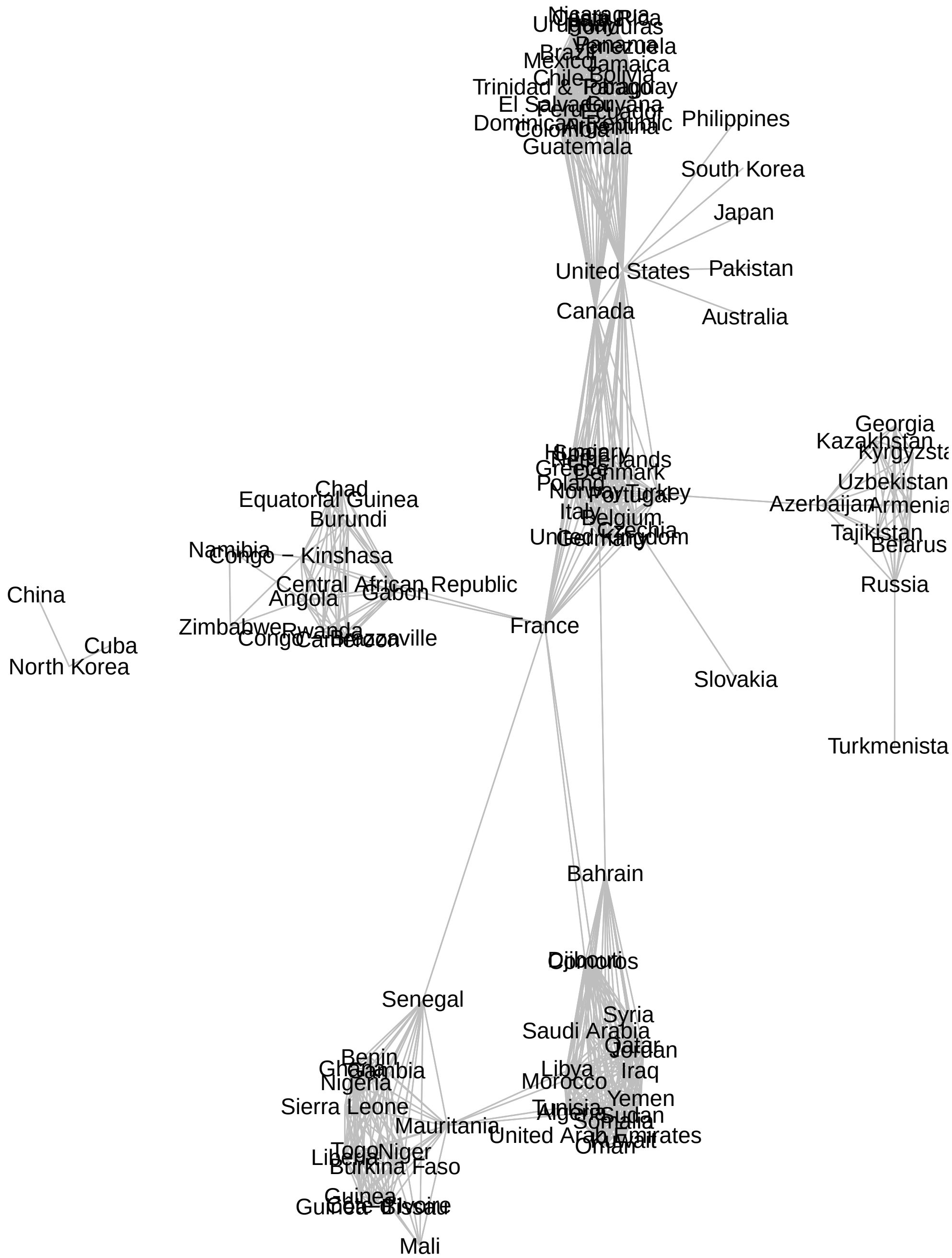

Figure 1: Interstate Alliances in the year 2000. Countries with no interstate alliance recorded are ommitted.

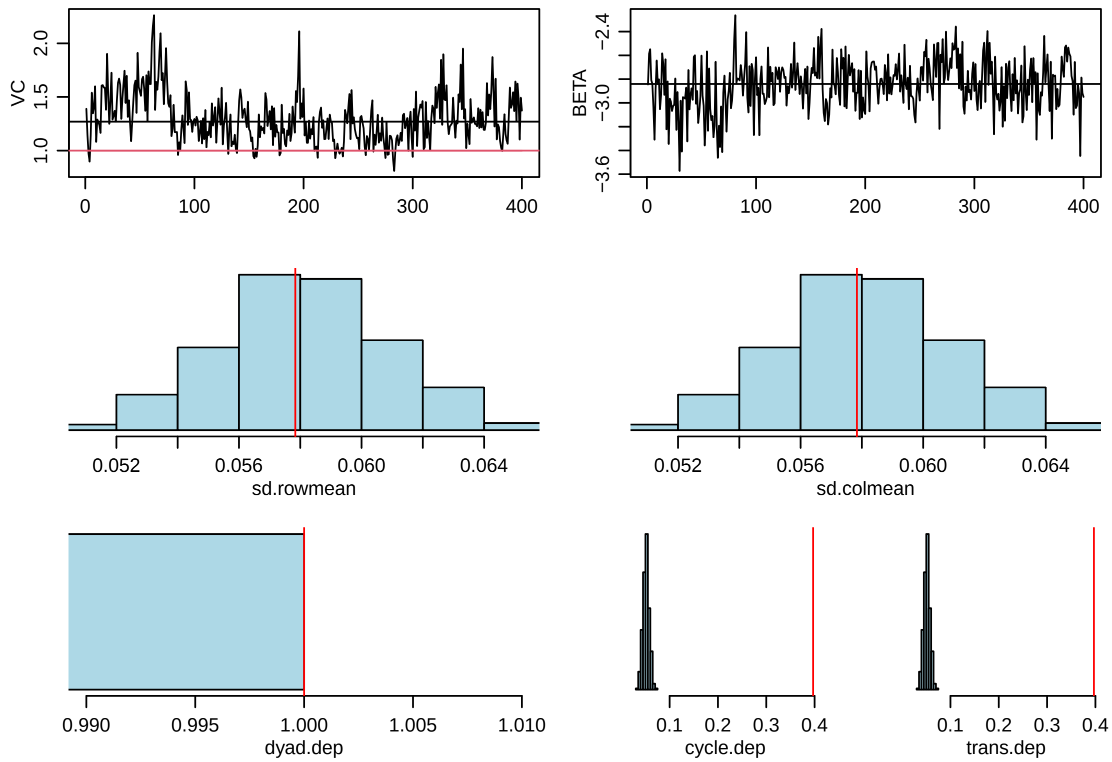

Figure 2: Goodness of Fit Statistics for the model without SRM Terms (above) and with SRM Terms (below). As indicated by the dependence structure, the second model performs better, taking sender and receiver effects into account.

$$CapRat_{ij} = \log\left(\frac{CINC_S}{CINC_W}\right)$$

$$PolSim_{ij} = \frac{\Delta - |POLITY_i - POLITY_j|)}{\Delta}, \text{ where } \Delta = \max_{ij}|POLITY_i - POLITY_j|$$

Extension: Modeling longitudinal data

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Table 3: Estimated Effects for AME Replicated Data

Statistic	N	Mean	St. Dev.
intercept	400	−6.461	0.796
cinc.node	400	105.681	10.316
polity.node	400	0.010	0.003
contigMat.dyad	400	1.086	0.052
lNet.dyad	400	0.457	0.237
LSP.dyad	400	0.062	0.016
warNet.dyad	400	−0.136	0.082
va	400	5.358	1.793
ve	400	1.000	0.000

4 Conlusions

Table 4: Estimated Effects for AME Replicated Data

Statistic	N	Mean	St. Dev.
intercept	400	−6.461	0.796
cinc.node	400	105.681	10.316
polity.node	400	0.010	0.003
contigMat.dyad	400	1.086	0.052
lNet.dyad	400	0.457	0.237
LSP.dyad	400	0.062	0.016
warNet.dyad	400	−0.136	0.082
va	400	5.358	1.793
ve	400	1.000	0.000

Table 5: Estimated Effects for AME Replicated Data

	pmean	psd	z-stat	p-val
intercept	-6.4606	0.7957	-8.1197	0
cinc.node	105.6805	10.3157	10.2446	0
polity.node	0.0099	0.003	3.2562	0.0011
contigMat.dyad	1.0858	0.0518	20.9772	0
lNet.dyad	0.4569	0.2369	1.9284	0.0538
LSP.dyad	0.0624	0.0165	3.7894	2e-04
warNet.dyad	-0.1361	0.0823	-1.6533	0.0983
va	5.3579	1.7935	-	-
ve	1	0	-	-

MCMC Estimates of 500 Burn-In and 10000 Draws.

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