The Additive and Multiplicative Effects Network Model

Seminar on Statistical Modeling of Social Networks

Daniel A. Seussler Becerra

14. / 15. Juli 2021

Introduction

- Networks are ubiquitous in Social Science applications.
- Possible Research Questions:
  - Network statistics as reciprocity, number of communities, etc..
  - Estimate structural parameters, including associated uncertainty.
  - Identify drivers of longitudinal changes in networks.

Nomenclature

- Vertices or Nodes, named actors in a network.
- Edges, Dyads or ties, the relationship between two actors.
  - Can be valued, ordered, or binary. As well as directed or undirected.

Consider the set of dyadic observations for the actors $1, \dots, n$:

$$\{y_{ij} : 1 \leq i, j \leq n,\ i \neq j\}$$

where y_{ij} is the observed (valued) relationship from actor i to j. It can be written as a *sociomatrix* $Y \in \mathbb{R}^{n \times n}$, i.e.,

$$Y := \begin{bmatrix} NA & y_{12} & \dots & y_{1n} \\ y_{21} & NA & \dots & \dots \\ \dots & \dots & \dots & y_{n-1\,n} \\ y_{n1} & \dots & y_{n\,n-1} & NA \end{bmatrix}.$$

Note: There is *a priori* no reason to assume (conditional) independence of the observations y_{ij}!

First-order network effects

- Actor-specific heterogeneity to send or receive ties in the network.

Second-order network effects

- Dyad-level network effects. Reciprocity, for example.

Third-order network effects

- Triad-level network effects. Balance, Transitivity and Clustering, for example.
- Commonly described by proverbs:
    - *my friends' friend is also my friend*, and
    - *the enemy of my enemy is my friend.*

Implications for Statistical Modeling

Issue: We cannot assume (conditional) independence of observations in the presence of statistical network effects. Ordinary Regression approaches will yield biased estimates.

Solution: Capture statistical network effects in an extended error component of a generalized linear model. In Bayesian fashion done in the AME framework.

AME Applications in Social Science

The AME framework has been applied to a variety of problems in the social sciences:

- Dorff, Gallop, and Minhas (2020) study the implications of Boko Haram entering the Nigerian civil conflict.
- Gade, Hafez, and Gabbay (2019) apply the AME network model on the Syrian civil conflict.
- Minhas, Hoff, and Ward (2016) use a VAR extension to model international relations.
- Kim et al. (2018) present a longitudinal extension to the AME framework to study UN voting patterns over time.

Theory

First-order effects

Let $\{\mu_i : 1, \ldots, n\}$ be the the set of rowmeans, $\{\eta_i : 1, \ldots, n\}$ of columnmeans[1]. Then,

$$sd.rowmean(Y) = \frac{1}{n}\sum_{i=1}^{n}\left(\mu_i - \bar{\mu}_i\right)^2, \text{ and } sd.colmean(Y) = \frac{1}{n}\sum_{i=1}^{n}\left(\eta_i - \bar{\eta}_i\right)^2.$$

Note: Presence of actor-specific heterogeneity can also be tested using an ordinary ANOVA F-test for comparison of group means.

[1]Undefined (NA) observations are dropped.

Second-order effects

Reciprocity in the network can be described by the correlation of the relationship between actor i and j, to the relationship of actor j to i, if $i \neq j$. Then,[2]

$$dyadic.dependency(Y) = corr\left(vec(Y), vec\left(Y^T\right)\right).$$

Naturally, for an undirected network dyadic dependency is 1.

[2] Here, $vec(A)$ is a $mn \times 1$ column vector, which is obtained by stacking the columns of $A \in \mathbb{R}^{n \times m}$ on top of another.

Third-order effects

are patterns among three nodes. Let $E = Y - \bar{Y}$. Then,

$$cycle.dependency(Y) = \frac{1}{n(n-1)(n-2)} \sum_{i \neq j \neq k \neq i} E_{ij} E_{jk} E_{ki},$$

$$trans.dependency(Y) = \frac{1}{n(n-1)(n-2)} \sum_{i \neq j \neq k \neq i} E_{ij} E_{jk} E_{ik}.$$

Naturally, for an undirected network cycle dependency equals transitive dependency. Optionally the statistics can be standardized with $sd(vec(Y))^3$.

Alternatively, E can be defined as the residual of an ordinary regression.

Let,

$$\mu_{ij} = \mathbb{E}(y_{ij}) = h(\theta_{ij})$$
$$\theta_{ij} = \beta^T \mathbf{x}_{ij}$$

where h is an invertible response function, $g = h^{-1}$ the link function. Assuming a normal density for the conditional distribution, we yield the identity as link, i. e., $\mu_{ij} = \theta_{ij}$ and:

$$y_{ij} \mid x_{ij} \sim N(\mu_{ij}, \sigma^2)$$

- Idea: Model actor-specific heterogeneity with additive random effects.
- For this, define an extended error component e_{ij}.
- Also known as the Social Relations Regression Model (SRRM).

Let μ be a mean or linear predictor $\beta^T \mathbf{x}_{ij}$, then:

$$\begin{aligned} y_{ij} &= \mu + e_{ij} \\ e_{ij} &= a_i + b_j + \epsilon_{ij} \\ \{(a_1, b_1), \dots, (a_n, b_n)\} &\sim N_2(\mathbf{0}, \Sigma_{ab}), \\ \{(\epsilon_{ij}, \epsilon_{ji}) : \; i \neq j\} &\sim N_2(\mathbf{0}, \Sigma_\epsilon), \end{aligned}$$

$$\Sigma_{ab} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix}, \quad \Sigma_\epsilon = \sigma_\epsilon^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

for $\{1 \leq i, j \leq n, i \neq j\}$.

Straightforward interpretation of the covariance structure:

recprocity	ρ
within-row variance	$Cov[Y_{ik}, Y_{il}] = \sigma_a^2$
within-column variance	$Cov[Y_{ik}, Y_{lk}] = \sigma_b^2$
row-column variance w/o reciprocity	$Cov[Y_{ik}, Y_{kl}] = \sigma_{ab}$
row-column variance w/ reciprocity	$Cov[Y_{ij}, Y_{ji}] = 2\sigma_{ab} + \rho\sigma^2$

Multiplicative Effects: Stochastic Equivalence and Homophily

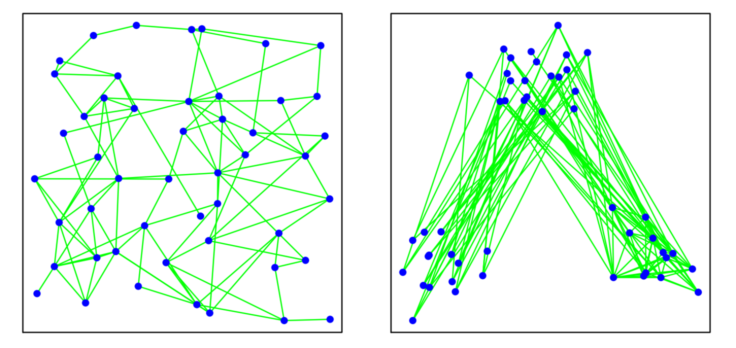

Figure 1: Homophily (left) and Stochastic Equivalence (right).

- Real networks exhibit different amounts of *both* stochastic equivalence and homophily. Image taken from Hoff (2007).

Remark: The Latent Space Model

- Every actor i has a latent space position z_i.
- Then, the likelihood of tie formation between *two* actors is taken to be associated to their relative distance in a latent space.
- E. g. the distance in L^1 or L^2.

The model is

$$\begin{aligned}\eta_{ij} &= \log \text{odds}(y_{ij} = 1 \mid z_i, z_j, \mathbf{x}_{ij}, \gamma, \beta) \\ &= \gamma + \beta^T \mathbf{x}_{ij} - |z_i - z_j|\end{aligned}$$

with the component $|z_i - z_j|$.

Issue: The latent space component does confound stochastic equivalence and homophily.

Solution: Hoff (2007) defines the multiplicative component as

$$\alpha(\mathbf{u}_i, \mathbf{v}_j) = \mathbf{u}_i^T \mathbf{v}_j$$

where $\mathbf{u}_i, \mathbf{v}_j$ are sender and receiver-specific latent factors of dimension k. For a symmetric sociomatrix, L is introduced as an $r \times r$ diagonal matrix, i.e.,

$$\alpha(\mathbf{u}_i, \mathbf{u}_j) = \mathbf{u}_i^T L \mathbf{u}_j$$

to properly generalize symmetric matrices. Furthermore, let

$$\{(\mathbf{u}_1, \mathbf{v}_1), \dots, (\mathbf{u}_n, \mathbf{v}_n)\} \sim N_{2r}(\mathbf{0}, \Psi).$$

See Hoff (2007) for why this component generalizes both the latent space and latent class model.

Taking additive and multiplicative effects into account, let

$$\begin{aligned}
y_{ij} &= \beta^T \mathbf{x}_{ij} + e_{ij} \\
e_{ij} &= a_i + b_j + \alpha(\mathbf{u}_i, \mathbf{v}_j) + \epsilon_{ij} \\
\{(a_1, b_1), \dots, (a_n, b_n)\} &\sim N_2(\mathbf{0}, \Sigma_{ab}), \\
\{(\epsilon_{ij}, \epsilon_{ji}) : \ i \neq j\} &\sim N_2(\mathbf{0}, \Sigma_\epsilon), \\
\{(\mathbf{u}_1, \mathbf{v}_1), \dots, (\mathbf{u}_n, \mathbf{v}_n)\} &\sim N_{2r}(\mathbf{0}, \Psi), \\
\Sigma_{ab} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix}, &\quad \Sigma_\epsilon = \sigma_\epsilon^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.
\end{aligned}$$

for $\{1 \leq i, j \leq n, i \neq j\}$.

For an appropiate M we can write these equations in matrices:

$$Y = M(X, \beta) + \mathbf{a}\mathbf{1}^T + \mathbf{1}\mathbf{b}^T + UV^T$$

and

$$Y = M(X, \beta) + \mathbf{a}\mathbf{1}^T + \mathbf{1}\mathbf{a}^T + ULU^T$$

in the symmetric case.

Next: Parameter Estimation with MCMC sampling

Objective: Approximate posterior distribution of $\beta, \Sigma, \sigma^2, \rho, \mathbf{a}, \mathbf{b}$ and $\mathbf{u}, \mathbf{v}$ with Gibbs sampling.

For appropriate $\beta_0, Q_0, v_0, \Sigma_0, \eta_0$, let

$$\beta \sim N_p(\beta_0, Q_0^{-1}),$$

$$\frac{1}{\sigma^2} \sim \text{Gamma}\left(\frac{v_0}{2}, \frac{v_0 \sigma^2}{2}\right),$$

$$\Sigma^{-1} \sim \text{Wishart}\left(\frac{\Sigma_0^{-1}}{\eta_0}, \eta_0\right).$$

Algorithm 1: Gibbs Sampling for the SRRM, P. D. Hoff (2021).

Initalize unkown variables;

1. Simulate $\{\beta, \mathbf{a}, \mathbf{b}\}$ given $Y, \Sigma, \sigma^2, \rho$;
2. Simulate σ^2 given $Y, \beta, \mathbf{a}, \mathbf{b}, \rho$;
3. Simulate ρ given $Y, \beta, \mathbf{a}, \mathbf{b}, \sigma^2$;
4. Simulate Σ given $\mathbf{a}, \mathbf{b}$;
5. Simulate missing values of Y given $\beta, \mathbf{a}, \mathbf{b}, \rho, \sigma^2$ and observed values of Y.

Algorithm 2: Gibbs Sampling for the AME, P. D. Hoff (2021).

Initalize unkown variables;

1. Update $\{\beta, \mathbf{a}, \mathbf{b}, \sigma^2, \rho, \Sigma\}$ and the missing values of Y using Algorithm 1, but with Y replaced by Y - UV^T;
2. Simulate $\Psi^{-1} \sim$ Wishart $((\Psi_0 \kappa_0 + [UV]^T [UV])^{-1}, \kappa_0 + n)$, where $[UV]$ is the $n \times 2r$ matrix equal to the column-wise concatenation of U and V;
3. For each $k = 1, \dots, r$, simulate the r-th columns of U and V from their full conditional distributions.

- Extensions available to include binary, count, ordinal, and censored data as outcome variables.
- Modifications to the sampling algorithm necessary if other outcome variables are selected. (E.g. in the application case below.) See Hoff (2021) for specifics.
- Replicated or Longitudinal Network Data require extended approaches. See Hoff (2015) or Kim et al. (2018) for two different approaches.

Application Case: Interstate Defence Alliances in 2000

Interstate Defence Alliances in 2000

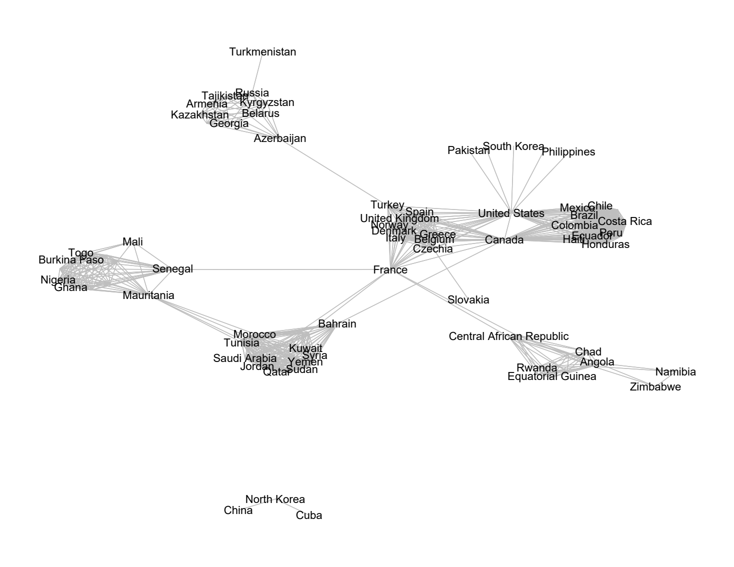

Figure 2: The interstate defence alliance network in the year 2000.

Network Statistics i

- 158 countries included.[3]
- 767 observed interstate alliances.
- 55 countries have no alliance recorded.
- Median number of alliances is 9.
- Network density is 0.061.

[3] I exclude the Yemen Arab Republic, Yemen People's Republic, German Democratic Republic, German Federal Republic, Yugoslavia, and Czechoslovakia as former countries.

$$sd.rowmean(Y) = 0.0596$$
$$dyadic.dependency(Y) = 1$$
$$cycle.dependency(Y) = 0.3881$$

Remark: An ANOVA F-test for equality of row- or columnmeans yields an F-statistic of 10.967. The Null Hypothesis of equality of row- or columnmeans is thus rejected.

Conclusion: Significant network effects present!

1. *Nodal Covariates:*[4]
    - Log Gross Domestic Product (GDP) in per capita terms.
2. *Dyadic Covariates:*
    - Geographic factors
    - Political and Military similarity
    - Past Conflicts
    - Economic Dependence
    - Cultural Similarity

[4] Adding the *Composite Index of National Capability* (CINC) as a measure for military capability, appended in the alliances data set, did not improve model fit during preliminary modeling. It is therefore omitted from the analysis.

1. $\mathit{GeoDistance}_{ij}$ is the logarithmic geographic distance between two countries measured by the distance of the respective capitals. If two countries share a border, the log distance is set 0.
2. $\mathit{CulturalSim}_{ij}$ is a dichotomous variable which takes the value 1, if the most spoken language of two countries is the same, 0 else.[5]

[5] I use data from the replication files of Warren (2010) and choose the year 1985, since the year 2000 did not seem to be accurate.

3. $EconomicDep_{ij}$ is a measure[6] for the economic dependence of two states, calculated by the share of Imports and Exports to the respective GDP:

$$EconomicDep_{ij} = min\left(\frac{Trade_{ij}}{GDP_i}, \frac{Trade_{ij}}{GDP_j}\right) \cdot 100\%$$

4. $SharedAllies_{ij}$ is the number of shared allies between country i and j.

[6] To construct this index, I take data from the Correlates of War Project on trade flows (Barbieri and Keshk (2016)) and The World Bank (2021). Missing data points for GDP are imputed on individual basis, negative values for trade are set to 0.

5. $ConflictInd_{ij}$ is an indicator variable that takes the value 1 if a militarized interstate dispute was recorded in the preceding 10 years between country i and j, 0 else.
6. $PoliticalSim_{ij}$ is a measure for the political dissimilarity of country i and j and is constructed using data from the Polity IV Index:[7]

$$PoliticalSim_{ij} = |POLITY_i - POLITY_j|$$

Note that a high value in *PoliticalSim* corresponds to a high *dissimilarity*.

[7] The Polity IV index classifies countries on a 21 point scale from -10 to 10, where -10 to -6 corresponds to autocracies, -5 to 5 to anocracies, and 6 to 10 to democracies.

7. $\boldsymbol{CapabilityRat_{ij}}$ is a capability ratio and defined as the log of the relative *Composite Index of National Capability* of both countries[8], with the stronger state (indicated by the subscript s) as the numerator:

$$CapabilityRat_{ij} = log\left(\frac{CINC_s}{CINC_w}\right).$$

Remark: All dyadic covariates are symmetric.

[8]At the provided precision, three countries are indistinguishable from 0. I impute these data points with 0.5 times of the second to lowest value.

First, to validate the Network Approach, the following specifications are compared:

- Probit Regression vs. Additive Effects model
- Additive Effects model vs. Multiplicative Effects model
- Multiplicative Effects model vs. AME model

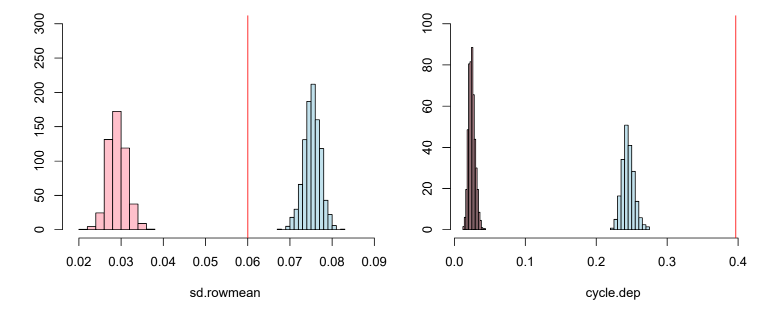

Figure 3: Posterior GOF statistics for the Probit Regression (pink) vs. the Additive Effects model (blue).

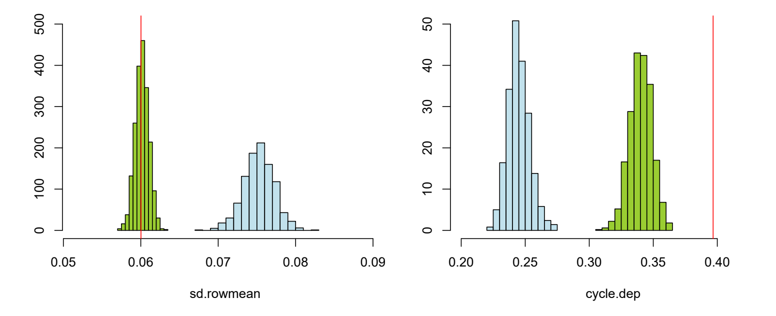

Figure 4: Posterior GOF statistics for the the Additive Effects model (blue) vs. Multiplicative Effects (green).

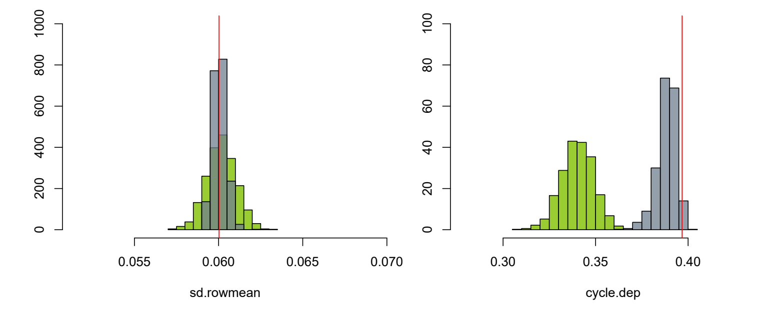

Figure 5: Posterior GOF statistics for the Multiplicative Effects model (green) vs. full AME model (grey).

	AME Network Model (R=2)			Probit Regression		
	pmean	psd	p - value	pmean	psd	p - value
Intercept	-33.186	6.369	0	-0.199	0.144	0.166
GDP (log p.c.).node	1.232	0.303	0	0.044	0.008	0
GeoDistance.dyad	-0.511	0.11	0	-0.191	0.008	0
CulturalSim.dyad	0.484	0.898	0.59	0.69	0.082	0
EconomicDep.dyad	0.454	0.236	0.055	-0.024	0.031	0.437
SharedAllies.dyad	0.261	0.067	0	0.105	0.037	0.005
ConflictInd.dyad	0.201	0.443	0.65	-0.05	0.05	0.32
PoliticalSim.dyad	-0.283	0.059	0	-0.083	0.005	0
CapabilityRat.dyad	0.003	0.143	0.981	-0.074	0.014	0
AME Components						
va	26.493	9.52				
ve	1	0				

[1] Effective Sample Size of the AME Network Model (R=2): 14, 46, 41, 124, 52, 5, 10, 30, 74.

[2] Effective Sample Size of the Probit Regression: 1000, 1000, 1095, 1000, 1000, 7, 624, 1000, 929.

	Additive Effects			Multiplicative Effects (R=2)		
	pmean	psd	p - value	pmean	psd	p - value
Intercept	-0.331	1.738	0.849	-7.136	1.683	0
GDP (log p.c.).node	0.014	0.115	0.905	0.319	0.096	0.001
GeoDistance.dyad	-0.386	0.019	0	-0.396	0.048	0
CulturalSim.dyad	0.524	0.154	0.001	0.239	0.351	0.497
EconomicDep.dyad	0.38	0.103	0	0.009	0.066	0.893
SharedAllies.dyad	1.177	0.106	0	0.05	0.027	0.062
ConflictInd.dyad	-0.263	0.167	0.116	-0.194	0.276	0.482
PoliticalSim.dyad	-0.141	0.009	0	-0.197	0.025	0
CapabilityRat.dyad	-0.162	0.029	0	0.001	0.073	0.985
AME Components						
va	4.76	1.051				
ve	1	0				

[1] Effective Sample Size of the Additive Effects Model: 755, 672, 133, 1000, 41, 2, 53, 123, 264.

[2] Effective Sample Size of the Multiplicative Effects Model: 51, 52, 25, 270, 745, 19, 26, 75, 109.

Caveat: The p - value is calculated under the assumption of an approximately normal posterior distribution, i.e., $z = \frac{pmean}{psd} \sim N(0,1)$. Yet, the p - value should be interpreted with caution, as the density plots do not always suggest approximate normality.

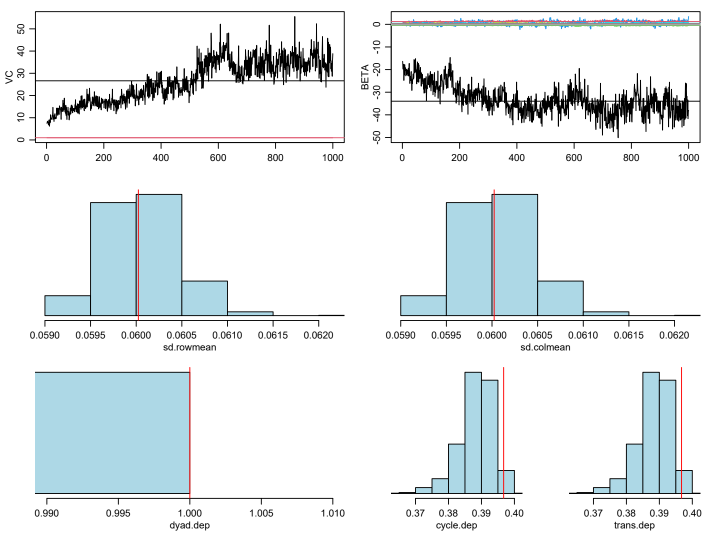

Figure 6: R-package 'amen' output for the AME Rank 2 network model, full specification.

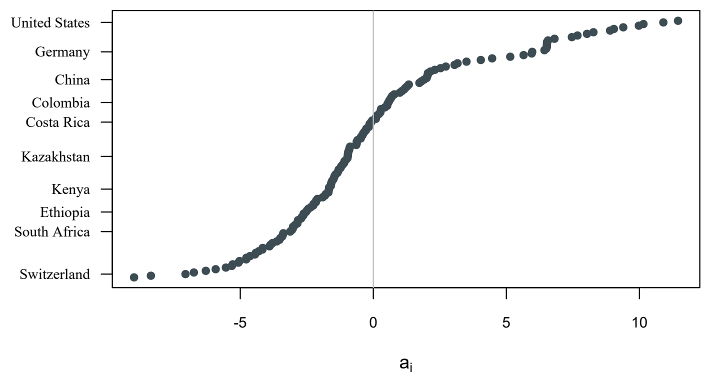

Figure 7: Additive Random Effects in the AME network model with rank 2. Values are posterior mean estimates for the additive random effects. AE effects capture first-order network effects, i.e., actor-specific heterogeneity in the propensity to form a network tie.

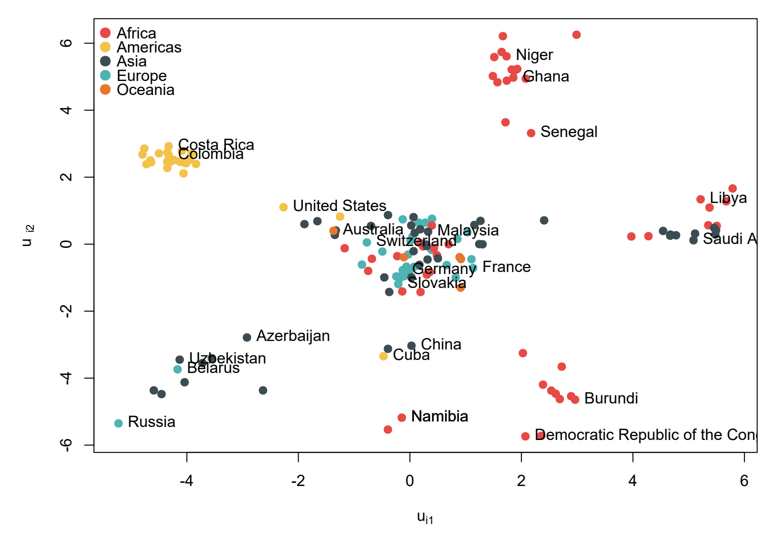

Figure 8: Visualization of the latent factor space in the AME network model with rank 2. Values are posterior mean estimates for the multiplicative random effects.

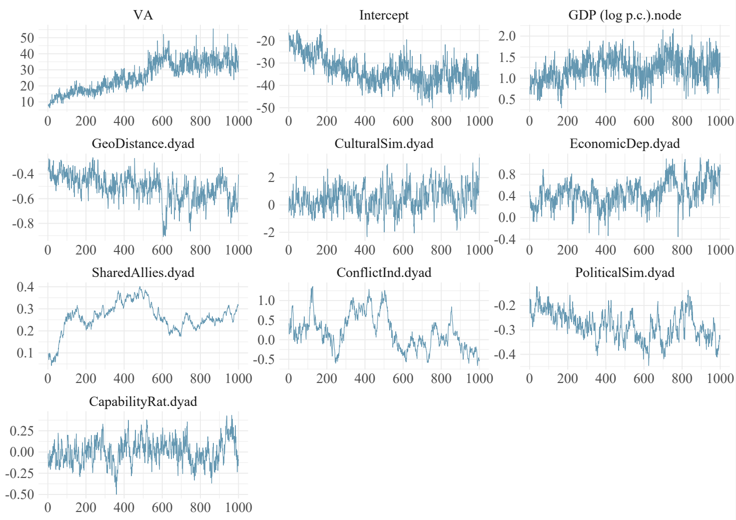

Figure 9: Trace plots for the AME rank 2 model. Trending and multimodality is visible for several parameter trace plots of the MCMC estimation.

Result: Select AME model based on the posterior goodness of fit comparison to the other approaches.

But, trace plots exhibit problematic aspects and effective sample size is very low.

Next:

- Increasing the dimension of the latent factor, for example R = 5.
- Dropping the Intercept.
- Dropping covariates with lowest effective Sample Size, i. e., *SharedAllies* and *ConflictInd*.
- Leaving only the 'significant' covariates *Intercept, GeoDistance, EconomicDep, SharedAllies*, and *PoliticalSim*.
- Longer Estimation Times (I. e., 4x Iteration and Burn-in Time)

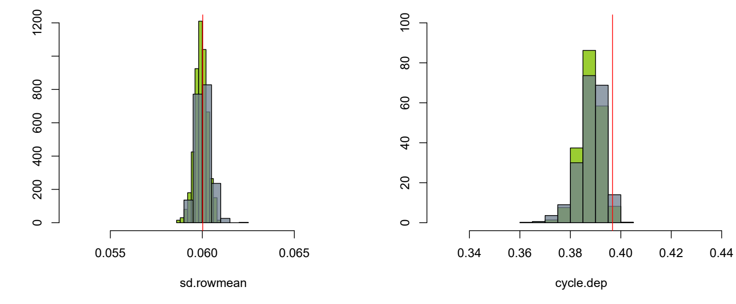

Figure 10: Comparison of posterior goodness of fit statistics for the AME rank 2 (grey) and the AME rank 5 (green) model.

	AME (R=2, selected)			AME (R=2, 4x)		
	pmean	psd	p - value	pmean	psd	p - value
Intercept	-18.382	10.263	0.073	-37.519	6.168	0
GDP (log p.c.).node	0.478	0.482	0.321	1.196	0.345	0.001
GeoDistance.dyad	-0.752	0.115	0	-0.681	0.129	0
CulturalSim.dyad				0.439	1.205	0.715
EconomicDep.dyad	0.416	0.23	0.07	0.714	0.366	0.051
SharedAllies.dyad	0.252	0.056	0	0.294	0.128	0.022
ConflictInd.dyad				-0.383	0.585	0.513
PoliticalSim.dyad	-0.3	0.07	0	-0.375	0.071	0
CapabilityRat.dyad				0.025	0.211	0.907
AME Components						
va	25.65	11.908		54.731	18.034	
ve	1	0		1	0	

[1] Effective Sample Size: 2, 3, 39, 41, 6, 16.

[2] Effective Sample Size: 45, 73, 103, 193, 75, 2, 11, 56, 122.

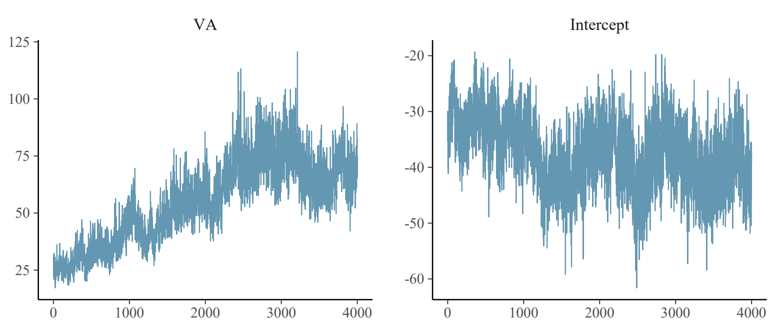

Figure 11: Trace plots for estimated posterior variance of the additive effect and the intercept. Results based on 400,000 iterations and a burn-in period of 100,000. Chains were thinned by keeping every 100^{th} sample.

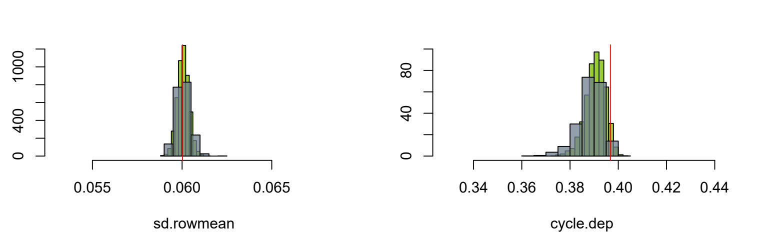

Figure 12: Posterior Goodness of Fit statistics for model estimates with 100,000 iterations (grey) and 400,000 iterations (green). A narrower distribution can be identified for the green model.

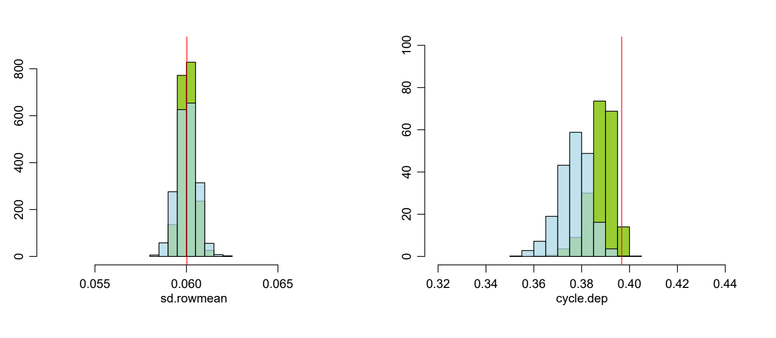

Figure 13: Posterior Goodness of Fit Statistics for the AME rank 2 with all covariates (green) and no covariates (blue) specified.

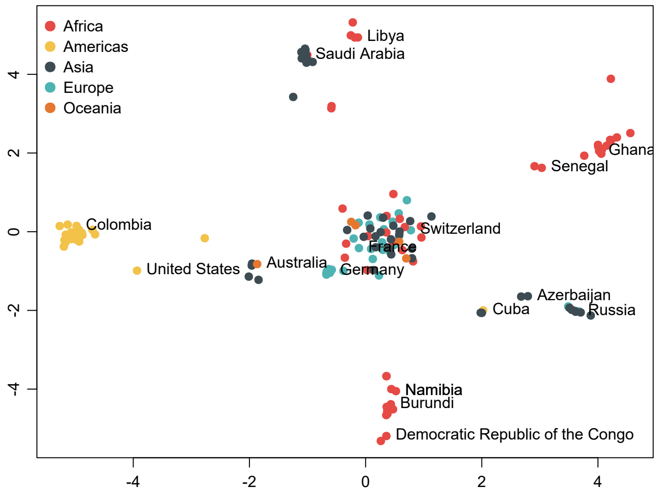

Figure 14: Visualization of the latent factor space in the AME network model with rank 2, no fixed effects specified. Values are posterior mean estimates for the multiplicative random effects.

Ideas on improving MCMC Sampling

1. Only changes in dyadic covariates above, but, persistent trend in the variance of the additive effect and the intercept could indicate:

    - an over-identified model, or
    - actor-specific heterogeneity is *not sufficiently* taken into account.
    - However, no evident political or economic nodal covariate available.

Idea: Construct an index of vote patterns in the United Nations, along the lines of the outcome variable in Kim et al. (2018). This is left for further exploration.

2. Wide posterior intervals indicate a high uncertainty about the parameter estimates of the intercept and the variance of the additive effects.

- May create issues with MCMC sampling.
- Possible Solution: specify different starting values and/or stronger priors.

Idea: Network density ranges from 0.046 to 0.057 in the time-frame 1981 - 2000. Stronger priors could be constructed with this information.

Discussion & End

Summary

The AME network model provides an easy-to-use and interpretable approach to the statistical modeling of networks.

Application Case: Effects of covariates on interstate alliance formation are directionally consistent with international relations literature.

However,

- Issues with the MCMC estimation persist.
- Selection of covariates is difficult, posterior goodness of fitness statistics serve only up to a point in model selection.
- Difficult to pin down issues in the sampling mechanism and establish valid inference.
- Could be due to the low density, symmetric, binary network with large heterogeneity.

Still, a versatile approach to network analysis!

Thank you very much for your attention!

Barbieri, Katherine, and Omar M. G. Omar Keshk. 2016. “Correlates of War Project Trade Data Set Codebook, Version 4.0.” 2016. http://correlatesofwar.org.

Dorff, Cassy, Max Gallop, and Shahryar Minhas. 2020. “Networks of Violence: Predicting Conflict in Nigeria.” *The Journal of Politics* 82 (2): 476–93. https://doi.org/10.1086/706459.

Gade, Emily Kalah, Mohammed M Hafez, and Michael Gabbay. 2019. “Fratricide in Rebel Movements: A Network Analysis of Syrian Militant Infighting.” *Journal of Peace Research* 56 (3): 321–35. https://doi.org/10.1177/0022343318806940.

Hoff, Peter D. 2007. “Modeling Homophily and Stochastic Equivalence in Symmetric Relational Data.” In *Proceedings of the 20th International Conference on Neural Information Processing Systems*, 657–64. NIPS’07. Red Hook, NY, USA: Curran Associates Inc.

———. 2015. “Multilinear tensor regression for longitudinal relational data.” *The Annals of Applied Statistics* 9 (3): 1169–93. https://doi.org/10.1214/15-AOAS839.

———. 2021. "Additive and Multiplicative Effects Network Models." *Statistical Science* 36 (1): 34–50. https://doi.org/10.1214/19-STS757.

Kim, Bomin, X. Niu, D. Hunter, and X. Cao. 2018. "A Dynamic Additive and Multiplicative Effects Model with Application to the United Nations Voting Behaviors." *arXiv: Applications*.

Minhas, Shahryar, Peter D. Hoff, and Michael D. Ward. 2016. "A New Approach to Analyzing Coevolving Longitudinal Networks in International Relations." *Journal of Peace Research* 53 (3): 491–505. https://doi.org/10.1177/0022343316630783.

The World Bank. 2021. "GDP (Current USD)." data retrieved from World Development Indicators, http://data.worldbank.org/indicator/NY.GDP.MKTP.CD, (Accessed on 12/06/2021).

Warren, T. Camber. 2010. "The Geometry of Security: Modeling Interstate Alliances as Evolving Networks." *Journal of Peace Research* 47 (6): 697–709. https://doi.org/10.1177/0022343310386270.