Modeling Complex Systems with Category Theory

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Objective and Scope of the Project

Scope of the Project

My objective for this project was to learn about the brand new field of categorical systems theory , which applies the pure math field of category theory to the study of arbitrary systems , and then use the tools that I learned to model a complex real-world system.

But I didn't know category theory or any complex systems...

The Actual Scope of the Project

1. Learn Category Theory
2. Learn Categorical Systems Theory
3. Write a software package for it in Python
4. Learn about a real world system (Transcription Networks)
5. Model it and simulate the results
6. Live to tell you all about it

Scope of this Presentation

Keep things high-level and descriptive and move fast.

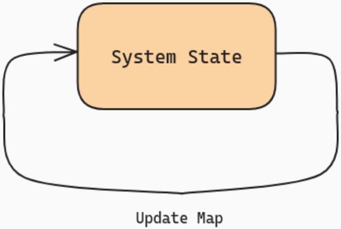
1. Tell you what a system is.
2. Tell you what a category is.
3. Tell you how they can work together.
4. Show you a complex system and how I modeled it.
5. Talk about the simulation results.

Systems

What is a System?

A system is a thing that changes. At it's simplest, a system consists of the following two things.

1. A set of states that the system can be in.
2. A map that updates the system's state based on the state that it's currently in.



Examples of Systems

The set of natural numbers with the successor function is a system.

$$\begin{aligned} S : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto n + 1 \end{aligned}$$

Examples of Systems

An ordinary differential equation is also a system.

$$\frac{dx}{dt} = \kappa x$$

Here the set of states is the set of real numbers $\mathbb{R}$ and the update map is the differential equation itself: given the current state, x, the differential equation tells us how to change it.

What's the Problem

The problem with these systems is that they're closed.

They don't interact with each other. And they don't interact with their environment.

So let's open them up with category theory.

Categories

Category Theory Basics

Category theory is a theory of structure and composition.

It uses the mathematical object of a category to encapsulate the notion of associative composition.

With this alone it creates a language capable of formalizing all of mathematics.

What is a Category?

Definition (Category)
A category $\mathcal{C}$ is defined by the following:

1. $\mathcal{C}$ contains a collection of objects $\mathrm{ob}(\mathcal{C})$.
2. For any two objects $a, b \in \mathcal{C}$ there is a collection of morphisms, $f : a \to b$, between those objects called the homset, $\mathcal{C}(a, b)$.
3. Every object $a \in \mathcal{C}$ has a morphism to itself $\mathrm{id}_a : a \to a$ called its identity.
4. For every two morphisms $f : a \to b$ and $g : b \to c$ there's a third morphism $g \circ f : a \to c$ that's their composition.

What is a Category?

These objects and morphisms are then under two constraints: unitality and associativity.

Definition (Category cont. Unitality)
Any morphism $f : a \to b$ can be composed with the identity morphisms of a and b such that $f \circ \mathrm{id}_a = \mathrm{id}_b \circ f = f$.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "\mathrm{id}_a"'] & b \arrow[d, "\mathrm{id}_b"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}
$$

These objects and morphisms are then under two constraints: unitality and associativity.

Definition (Category cont. Associativity)
For any morphisms $f : a \to b$, $g : b \to c$, and $h : c \to d$, $h \circ (g \circ f) = (h \circ g) \circ f$. Since it doesn't matter what order we apply the morphisms, we write this $h \circ g \circ f$.

$$\begin{tikzcd}
a \arrow[r, "f"'] \arrow[rr, bend left, "g\circ f"] & b \arrow[r, "g"'] \arrow[rr, bend right, "h\circ g"'] & c \arrow[r, "h"'] & d
\end{tikzcd}$$

Monoidal Categories

To build systems compositionally we'll also need a way to place them in parallel.

Similar to a binary operation on a set, we can have a binary operation on a category.

This construction is captured in the notion of a monoidal category.

Definition (Monoidal Categories)
A category $\mathcal{C}$ is monoidal if the following exist.

1. A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the monoidal product.
2. An object $\mathbb{1} \in \mathcal{C}$ called the unit.
3. A natural isomorphism $\alpha : (a \otimes b) \otimes c \Rightarrow a \otimes (b \otimes c)$ called the associator, with components $\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$.
4. A natural isomorphism $\lambda : \mathbb{1} \otimes a \Rightarrow a$ called the left unitor with components $\lambda_x : \mathbb{1} \otimes x \to x$.
5. A natural isomorphism $\rho : a \otimes \mathbb{1} \Rightarrow a$ called the right unitor with components $\lambda_x : x \otimes \mathbb{1} \to x$.

Definition (Monoidal Categories)
All of the above must exist such that the following two diagrams, called the triangle identity and the pentagon identity, commute.

$$\begin{tikzcd}
(x \otimes \mathbb{1}) \otimes y \arrow[r, "\alpha_{x,y,z}"] \arrow[dr, "\rho_x \otimes \mathrm{id}_y"'] & x \otimes (\mathbb{1} \otimes y) \arrow[d, "\mathrm{id}_x \otimes \lambda_y"] \\
 & x \otimes y
\end{tikzcd}$$

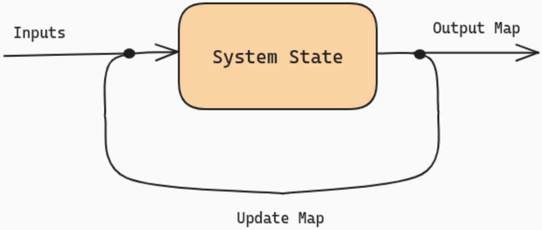
Definition (Monoidal Categories)
All of the above must exist such that the following two diagrams, called the triangle identity and the pentagon identity, commute.

$$\begin{tikzcd}
((w \otimes x) \otimes y) \otimes z \arrow[r, "\alpha_{(w\otimes x),y,z}"] \arrow[d, "\alpha_{w,x,y}\otimes \mathrm{id}_z"'] & (w \otimes x) \otimes (y \otimes z) \arrow[dd, "\alpha_{w,x,(y\otimes z)}"] \\
(w \otimes (x \otimes y)) \otimes z \arrow[d, "\alpha_{w,(x\otimes y),z}"'] & \\
w \otimes ((x \otimes y) \otimes z) \arrow[r, "\mathrm{id}_w \otimes \alpha_{x,y,z}"'] & w \otimes (x \otimes (y \otimes z))
\end{tikzcd}$$

Composing Systems

What is a Open System?

An open system is as a system with information coming in and information coming out, and with these two information streams being connected by feedback.



Case Study

Results

Conclusion