Modeling Complex Systems with Category Theory

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Objective and Scope of the Project

My objective for this project was to learn about the brand new field of categorical systems theory , which applies the pure math field of category theory to the study of arbitrary systems , and then use the tools that I learned to model a complex real-world system.

But I didn't know category theory or any complex systems...

The Actual Scope of the Project

1. Learn Category Theory
2. Learn Categorical Systems Theory
3. Learn about a real world system (Transcription Networks)
4. Learn how to use the AlgebraicDynamics library written in the Julia programming language
5. Model it and simulate the results
6. Live to tell you all about it

Scope of this Presentation

Keep things high-level and descriptive and move fast.

1. Tell you what a system is.
2. Tell you what a category is.
3. Tell you how they can work together.
4. Show you a complex system and how I modeled it.
5. Talk about the simulation results.

Systems

What is a System?

A system is a thing that changes. At it's simplest, a system consists of the following two things.

1. A set of states that the system can be in.
2. A map that updates the system's state based on the state that it's currently in.
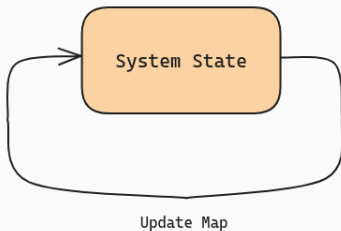
Update Map

Examples of Systems

The set of natural numbers with the successor function is a system.

$$S : \mathbb{N} \to \mathbb{N}$$

$$n \mapsto n + 1$$

An ordinary differential equation is also a system.

$$\frac{dx}{dt} = \kappa x$$

Here the set of states is the set of real numbers $\mathbb{R}$ and the update map is the differential equation itself: given the current state, x, the differential equation tells us how to change it.

What's the Problem

The problem with these systems is that they're closed.

They don't interact with each other. And they don't interact with their environment.

So let's open them up with category theory.

Categories

Category theory is a theory of structure and composition.

It uses the mathematical object of a category to encapsulate the notion of associative composition.

With this alone it creates a language capable of formalizing all of mathematics.

What is a Category?

Definition (Category)
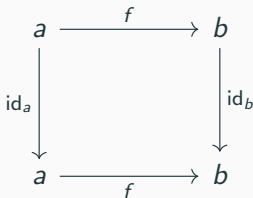A category $\mathcal{C}$ is defined by the following:

1. $\mathcal{C}$ contains a collection of objects $ob(\mathcal{C})$.

2. For any two objects a, $b \in \mathcal{C}$ there is a collection of morphisms, $f : a \to b$, between those objects called the homset, $\mathcal{C}(a, b)$.

3. Every object $a \in \mathcal{C}$ has a morphism to itself $id_a : a \to a$ called its identity.

4. For every two morphisms $f : a \to b$ and $g : b \to c$ there's a third morphism $g \circ f : a \to c$ that's their composition.

These objects and morphisms are then under two constraints: unitality and associativity.

Definition (Category cont. Unitality)
Any morphism $f : a \to b$ can be composed with the identity morphisms of a and b such that $f \circ \text{id}_a = \text{id}_b \circ f = f$.
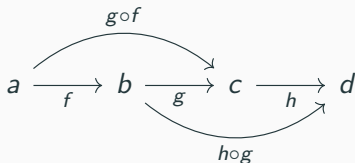
$$
\begin{array}{ccc}
a & \xrightarrow{\ f\ } & b \\
\downarrow{\scriptstyle \text{id}_a} & & \downarrow{\scriptstyle \text{id}_b} \\
a & \xrightarrow{\ f\ } & b
\end{array}
$$

These objects and morphisms are then under two constraints: unitality and associativity.
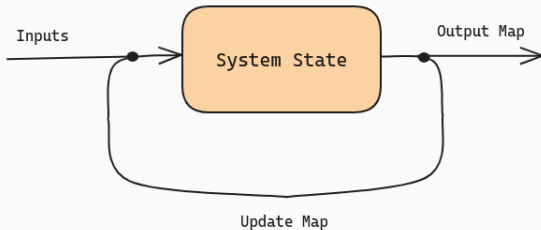
Definition (Category cont. Associativity)
For any morphisms $f : a \to b$, $g : b \to c$, and $h : c \to d$, $h \circ (g \circ f) = (h \circ g) \circ f$. Since it doesn't matter what order we apply the morphisms, we write this $h \circ g \circ f$.

$$a \xrightarrow{\ f\ } b \xrightarrow{\ g\ } c \xrightarrow{\ h\ } d$$

with arcs labeled $g \circ f$ (from a to c) and $h \circ g$ (from b to d).

Composing Systems

An open system is as a system with an interface.

Lenses and Arenas

To model this mathematically we'll use objects called arenas and morphisms called lenses.

Definition (Arenas)

An arena is a pair of objects $\begin{pmatrix} A \\ B \end{pmatrix}$ from an underlying category.

Definition

A lens is a morphism between arenas made from a pair of morphisms from the underlying category

$$\begin{pmatrix} f \\ g \end{pmatrix} : \begin{pmatrix} A \\ B \end{pmatrix} \leftrightarrows \begin{pmatrix} C \\ D \end{pmatrix}$$

where $f : B \to D$ is the output map and $g : B \times C \to A$ is the update map.

What this Gives Us

Arenas and lenses between them form a monoidal category.

This means that we can compose systems and multiply them.

By modeling our open systems as lenses, composition of lenses lets us place systems in series and multiplication of lenses lets us place systems in parallel.

Systems in Series and Parallel

Systems in Series: $\begin{pmatrix} g^\# \\ g \end{pmatrix} \circ \begin{pmatrix} f^\# \\ f \end{pmatrix}$
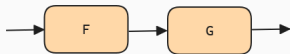
Systems in Parallel: $\begin{pmatrix} g^\# \\ g \end{pmatrix} \otimes \begin{pmatrix} f^\# \\ f \end{pmatrix}$

Case Study

Gene Transcription

Gene transcription is the process by which genes in a cell's DNA produce proteins.

The dynamics of this process for a single gene are well modeled by the following differential equation, where $f(X^*)$ is the activation function of the gene.
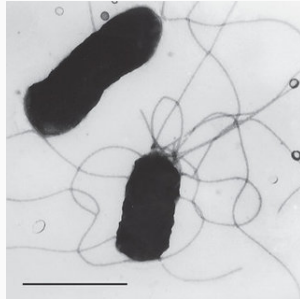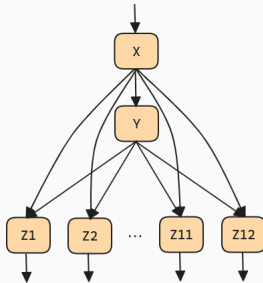
$$\frac{dY}{dt} = \beta * f(X^*) - \alpha Y$$

Activator: $f(X^*) = \dfrac{\beta X^{*n}}{K^n + X^{*n}}$ or $\beta * P(X^* > K)$

Repressor: $f(X^*) = \dfrac{\beta}{1 + \left(\frac{X^*}{K}\right)^n}$ or $\beta * P(X^* < K)$

Motor Flagella Network

Gene transcription network are connected networks of genes whose products act as transcription factors for other genes.
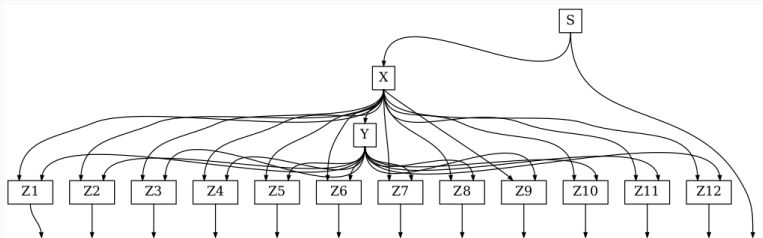
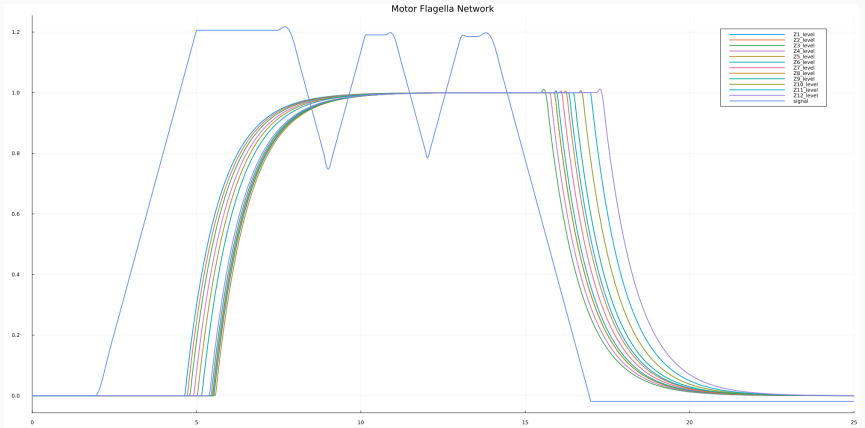Below is the motor flagella network of *e. coli* bacteria.

NEED REFERENCE for the image

Motor Flagella Network Model

Using lenses we can wire together the transcription factor dynamics as differential systems and then simulate the results.

Motor Flagella Network Simulation Results

Motor Flagella Network

Conclusion

References

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