Designing and Modeling Complex Systems with Category Theory

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May 2024

Introduction

Category theory studies composition. By studying and abstracting notions of composition, categories manage to encapsulate a shockingly flexible and wide-reaching language for describing and working with structure of all sorts. Much like the study of sets and the set membership of elements turns out to be capable of formalizing all of mathematics, so too does category theory, except from a bird's eye point of view: the details of a particular field go out of focus and only its high level structural patterns remain. This is useful. High levels of abstraction provide high levels of generality, and a general, common language of how things are structured is useful for both organizing thought and sharing it.

Because of this generality, category theory also has some potential for re-formulating difficult problems in other fields or even opening a broader class of thinking and phenomena to mathematical rigor. This is an active field of research. There are people using the tools of category theory to formalize and investigate everything from quantum mechanics and linguistics to philosophy and systems theory.

It's this last application to systems theory that we'll be looking more deeply at in this text. Though we're merely wandering at the edge of the work being done, we'll see how to use category theory to build the specifications for, and investigate the behaviors of, dynamical systems built from the composition of smaller, simpler systems. We'll do this by working through a case study for both discrete and differential deterministic equations. For our first case study we'll build a state machine to control the behavior of agents in a game. For our second case study we'll build a differential system for modeling a complicated food web.

With even this partial, introductory view of the subject matter it should be easy to see the promise of this approach to facilitate thinking about and designing models for complex systems.

Category Theory Basics

The first step to understanding category theory is to understand what a category is. If you're new to higher mathematics it's time to take a deep breath. This will seem like a lot.

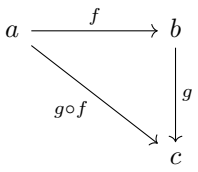
Definition 1 (Category)**.** A category $\mathcal{C}$ is defined by the following:

1. $\mathcal{C}$ contains a collection of objects $\mathsf{ob}(\mathcal{C})$. We'll denote that an object is in a category using set notation: $c \in \mathcal{C}$.

2. For any two objects $a, b \in \mathcal{C}$ there is a collection of morphisms, or arrows, between those objects $\mathcal{C}(a, b)$ called the homset. This is short for "set of homomorphisms." We'll denote an element $f \in \mathcal{C}(a, b)$ using function notation: $f : a \to b$.

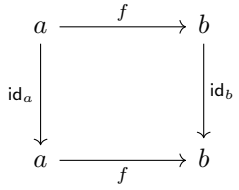
3. Every object $a \in \mathcal{C}$ has a morphism to itself $\mathsf{id}_a : a \to a$ called its identity. This morphism doesn't do anything. It's like multiplying a number by 1.

4. For every two morphisms $f : a \to b$ and $g : b \to c$ there's a third morphism $g \circ f : a \to c$ that's their composition. The circle is the symbol for function composition and $g \circ f$ is read "g after f."

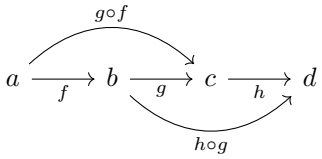


These objects and morphisms are then under the following two constraints:

1. (Unitality) Any morphism $f : a \to b$ can be composed with the identity morphisms of a and b such that $f \circ \mathsf{id}_a = \mathsf{id}_b \circ f = f$. This makes the following diagram commute. What this means is that both paths from a to b here are equivalent.



2. (Associativity) For any morphisms $f : a \to b$, $g : b \to c$, and $h : c \to d$, $h \circ (g \circ f) = (h \circ g) \circ f$. Since it doesn't matter what order we apply the morphisms, we write this $h \circ g \circ f$.

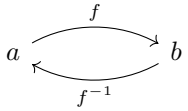


Let's break this down a little bit. We have a collection of objects with connections between them, and a couple of rules. We know that every object is connected to itself. And we know that if object a is connected to b, and b is connected to c, then a is connected to c through b.[1] That's it. Categories are honestly pretty simple. But from these humble beginnings we're going to climb very high.

A Quick Note on Isomorphism

When most of us think of two things being "the same" in mathematics we think of equalities like $1 + 1 = 2$. Equality is not the only useful notion of sameness in mathematics though. When thinking about whether two objects in a category are meaningfully "the same," a more useful notion than equality is that of isomorphism. An isomorphism between objects means that, while they may not be exactly the same, as in equality, there exists a way to move between the two objects without losing any information.

Definition 2 (Isomorphism)**.** Two objects in a category are isomorphic if there exists a morphism $f : a \to b$ for which there is another morphism $f^{-1} : b \to a$ such that $f^{-1} \circ f = \mathsf{id}_a$ and $f \circ f^{-1} = \mathsf{id}_b$. Such a morphism is called an isomorphism.



Another way to think about isomorphism is as a renaming. Two objects are isomorphic if it's possible to represent one as the other by a simple, and reversible, relabeling. For instance, an example of two sets that are isomorphic are the sets $\{a, b, c\}$ and $\{1, 2, 3\}$. By relabeling $a \mapsto 1$, $b \mapsto 2$, and $c \mapsto 3$, we have represented the first set by means of the second set. Specifically, we've labeled the first three letters of the alphabet by the numbers 1, 2, and 3.

Functors and Categories of Categories

Most mathematical structures have some way of mapping one instance of that structure to another instance. Functions map sets to sets. Linear transformations map vector spaces to other vector spaces. And group homomorphisms map

[1] For people with some background in math or logic this might be familiar as the transitive property.

groups to other groups. Since categories are a mathematical structure, it makes sense to ask if there exists a way to map between them. Such a map would have to preserve all of the things that make a category a category: identity morphisms and composite morphisms, and the rules of unitality and associativity. We call such a map between categories a functor.

Definition 3 (Functor)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is a map between categories $\mathcal{C}$ and $\mathcal{D}$ such that the following hold:

1. For any object $a \in \mathcal{C}$ there is an object $Fa \in \mathcal{D}$.
2. For any morphism $f : a \to b$ between objects a and b in $\mathcal{C}$ there is a morphism $Ff : Fa \to Fb$ between objects Fa and Fb in $\mathcal{D}$.
3. For all objects $a \in \mathcal{C}$ and $Fa \in \mathcal{D}$, $F\mathsf{id}_a = \mathsf{id}_{Fa}$.
4. For any composition of morphisms $g \circ f$ in $\mathcal{C}$, $F(g \circ f) = Fg \circ Ff$ in $\mathcal{D}$.

There are some things to note here. The first is that, like sets and functions, the mapping of objects from one category to another is unique. Every object in the source category is mapped to exactly one object in the target category. The same goes for morphisms. The second is that identity morphisms get mapped to identity morphisms and compositions get mapped to compositions.

Let's check that these rules are enough to preserve unitality and associativity. Let categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $F : \mathcal{C} \to \mathcal{D}$ between them be given.

To test that unitality is preserved by the functor, we'll start with the unitality definition in $\mathcal{C}$, that $f \circ \mathsf{id}_a = \mathsf{id}_b \circ f = f$ for objects $a, b \in \mathcal{C}$ and morphism $f : a \to b$, and show that this unitality condition remains true for the objects and morphism under the functor in $\mathcal{D}$.

Proof. Let a functor $F : \mathcal{C} \to \mathcal{D}$ be given, with objects $a, b \in \mathcal{C}$ and morphism $f : a \to b$.

$$
\begin{aligned}
F(\mathsf{id}_b \circ f) &= F\mathsf{id}_b \circ Ff && \text{by the functor composition rule} && (1)\\
&= \mathsf{id}_{Fb} \circ Ff && \text{by the functor identity rule} && (2)\\
&= Ff && \text{by the definition of the identity} && (3)\\
&= Ff \circ \mathsf{id}_{Fa} && \text{by the definition of the identity} && (4)\\
&= Ff \circ F\mathsf{id}_a && \text{by the functor identity rule} && (5)\\
&= F(f \circ \mathsf{id}_a) && \text{by the functor composition rule} && (6)
\end{aligned}
$$

Thus the functor preserves unitality, since $F(\mathsf{id}_b \circ f) = F(f \circ \mathsf{id}_a) = Ff$. □

Now let's check that associativity is preserved.

Proof. Let a functor $F : \mathcal{C} \to \mathcal{D}$ be given, with objects $a, b, c, d \in \mathcal{C}$ and morphisms $f : a \to b$, $g : b \to c$, and $h : c \to d$.

$$\begin{aligned} F(h \circ (g \circ f)) &= Fh \circ F(g \circ f) && \text{by the functor composition rule} && (7)\\ &= Fh \circ (Fg \circ Ff) && \text{by the functor composition rule} && (8)\\ &= (Fh \circ Fg) \circ Ff && \text{by associativity in the category } \mathcal{D} && (9)\\ &= F(h \circ g) \circ Ff && \text{by the functor composition rule} && (10)\\ &= F((h \circ g) \circ f) && \text{by the functor composition rule} && (11) \end{aligned}$$

Thus the functor preserves associativity, since $F(h \circ (g \circ f)) = F((h \circ g) \circ f)$. □

With functors in our toolbox, it's now possible to construct a category where the objects themselves are categories and the morphism between them are functors. This, along with our next step, will prove to be extremely powerful.

A Quick Note on Avoiding Paradox

For anyone who has taken a course in logic or set theory, that last paragraph might be ringing some alarm bells. Whenever we have recursion in our structure like this, there's the opportunity for paradox to sneak in. The category of categories is no exception.

To avoid the paradox of whether the category of categories contains itself, category theorists employ formal notions of "size" to differentiate distinct universes of categories. These formalisms can get pretty technical though, so we'll be ignoring them in this paper. For us it will be enough to assume that whenenever we talk about a category whose objects are also categories, our object categories are in some way "smaller" than the category that they're a part of, and thus of a different type.[2]

Functor Categories and Natural Transformations

For our next, and last, step up into abstraction let's talk about functor categories. Functor categories are categories whose objects are functors and whose morphisms are maps called natural transformations.

Since natural transformations map functors to other functors, we know that they must preserve the essential "functor-ness" of functors. This "functor-ness" is a bit harder to pin down than the "category-ness" that functors themselves preserve. We need to see what natural transformations do to functors, but also what they do to the objects and arrows of the categories that the functors are acting on.

Definition 4 (Natural Transformation)**.** A natural transformation $\alpha : F \Rightarrow G$ is a map between functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ such that the following holds:

[2] In general, this "smallness" will be that the category's collection of objects forms a set.

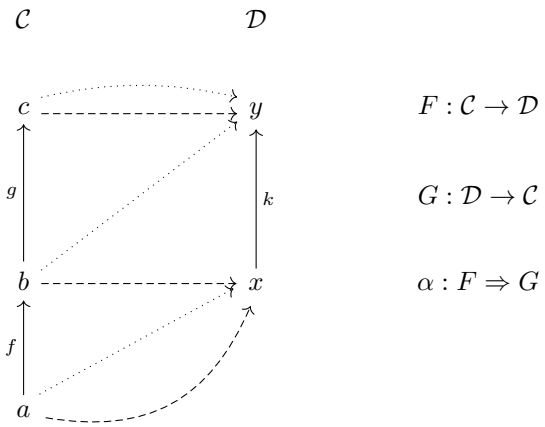
1. For each object $c \in \mathcal{C}$, there is a morphism $\alpha_c : Fc \to Gc$ in $\mathcal{D}$. This is called the c-component of α.

2. For every morphism $f : a \to b$ in $\mathcal{C}$, $\alpha_b \circ Ff = Gf \circ \alpha_a$. This is called the naturality condition, and it preserves functoriality.

The naturality condition ensures that the following diagram commutes.

$$\begin{array}{ccc} Fa & \xrightarrow{\alpha_a} & Ga \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fb & \xrightarrow[\alpha_b]{} & Gb \end{array}$$

This definition is short but there's a whole extra layer of stuff to keep track of since we added a layer of abstraction. Let's get a better idea of what this definition captures by using a picture.



Here we see two simple categories, $\mathcal{C}$ and $\mathcal{D}$, and two functors F (the dashed lines) and G (the dotted lines) between them. Let's dig into this picture a bit and pick apart the functors and the natural transformation.

First, let's note all the important mappings of the functors and the component morphisms of the natural transformation.

$$\begin{array}{lll} Fa = x & Fb = x & Fc = y \\ Ff = \mathsf{id}_x & Fg = k & F(g \circ f) = k \end{array}$$

$$\begin{array}{lll} Ga = x & Gb = y & Gc = y \\ Gf = k & Gg = \mathsf{id}_y & G(g \circ f) = k \end{array}$$

$$\begin{aligned} \alpha_a &: Fa \to Ga & \alpha_a &= \mathsf{id}_x \\ \alpha_b &: Fb \to Gb & \alpha_b &= k \\ \alpha_c &: Fc \to Gc & \alpha_c &= \mathsf{id}_y \end{aligned}$$

Next, let's check that the functors preserve composition.[3]

$$F(g \circ f) = k = k \circ \mathsf{id}_x = Fg \circ Ff \quad \checkmark$$

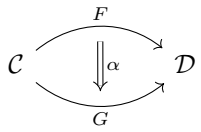
$$G(g \circ f) = k = \mathsf{id}_y \circ k = Gg \circ Gf \quad \checkmark$$

And now let's look at the natural transformation $\alpha : F \Rightarrow G$ and make sure that it satisfies the naturality condition for the morphisms $f, g \in \mathcal{C}$.

$$\alpha_b \circ Ff = k \circ \mathsf{id}_x = Gf \circ \alpha_a \quad \checkmark$$

$$\alpha_c \circ Fg = \mathsf{id}_y \circ k = Gg \circ \alpha_b \quad \checkmark$$

Everything is working as expected. Hopefully this helped to unpack some of the complexity behind the natural transformation definition. In a real example, our functors would be defined equationally and we'd prove these properties for arbitrary morphisms instead of proving them for each morphism individually like we did here. We'd also represent α diagramatically in the following condensed form.

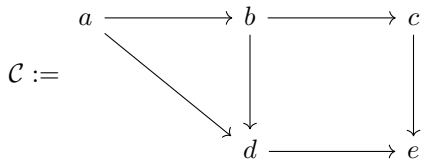


This specifies all of the information we need to work with the natural transformation in a succint form.

Duality

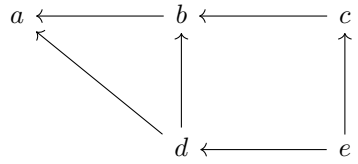
The last thing we'll cover in this section is the very simple, and very powerful, property of duality. Every category has a dual category. The objects in this dual category are exactly the same as in the original category, but every arrow is turned around: its source becomes its target and its target becomes its source. Let's look at a simple example.

Let $\mathcal{C}$ be the category below.



[3] You should be able to convince yourself that they preserve identities.

It's dual category then we write as $\mathcal{C}^{op}$ and define as follows.

$\mathcal{C}^{op} :=$ 

And that's it. One reason duality turns out to be so powerful is that it doubles our effectiveness. For every structure we find in the next section, we get the dual structure essentially for free. Duality also plays an important role in representability and the Yoneda lemma which are major features of category theory that we'll get to in the next section.

Structure between Categories

A category in isolation, like a set in isolation, doesn't give us much. It's only when we begin adding relations and mappings between categories that the theory comes into its own. These will include categorical versions of some familiar structures, like embeddings, equivalence relations, and isomorphism. But it will also include some structures that might seem strange and perplexing at first: representability, adjunctions, and monads.

Subcategories

Let's start with the simplest structure first, that of a subcategory relation. Like any of the other "sub-" relations from other fields of mathematics, the subcategory relation encapsulates a notion of one category being in some way contained inside of another. This means that the pattern of objects and morphisms that make up the subcategory are faithfully present in the other category. Our first step will be to define this "faithfulness."

Definition 5 (Faithful Functor)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is faithful if for each $x, y \in \mathcal{C}$, the map $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is injective.

In other words, when a functor between categories is faithful, it means that every morphism between objects x, y in the source category $\mathcal{C}$ is mapped to exactly one morphism between Fx, Fy in the target category $\mathcal{D}$. No morphisms in the source get collapsed into the same morphism in the target. All of them are "there."

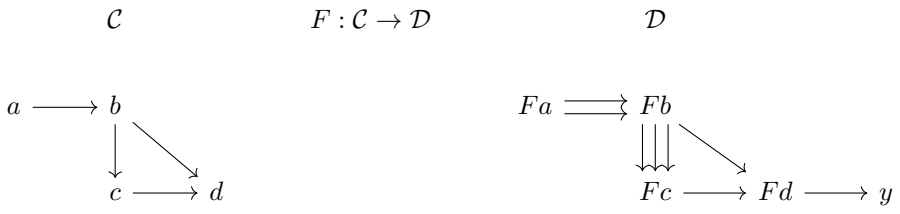
However, this doesn't mean that objects can't be collapsed. For that we need another condition on our functor.

Definition 6 (Injective on Objects)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is injective on objects if for each $x \in \mathcal{C}$, there is a unique $y \in \mathcal{D}$ such that $y = Fx$.

With these two conditions we can define our subcategory relation.

Definition 7 (Subcategory)**.** A category $\mathcal{C}$ with a functor $F : \mathcal{C} \to \mathcal{D}$ that is both faithful and injective on objects is a subcategory of the category $\mathcal{D}$. A functor F of this type is called an **embedding** of $\mathcal{C}$ into $\mathcal{D}$.

Let's take a look at this using the two small categories below with a functor F between them.



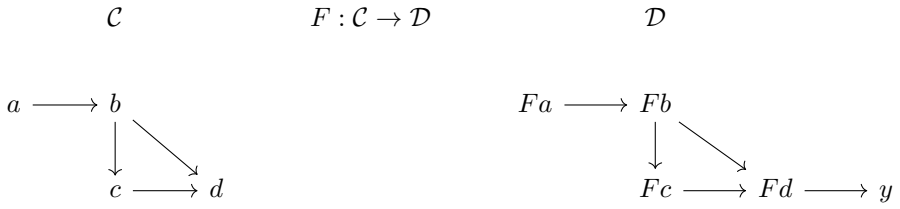
You should be able to convince yourself that $\mathcal{C}$ is a subcategory of $\mathcal{D}$ using the definitions of subcategory and functor above. As expected we get injectivity on morphisms and objects. But what if we also have surjectivity on morphisms? To look into this lets define another property that a functor can have.

Definition 8 (Full Functor)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is full if for each $x, y \in \mathcal{C}$, the map $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is surjective.

This gives us a stronger version of the subcategory relation called a full subcategory that rejects any extraneous morphisms in the target category.

Definition 9 (Full Subcategory)**.** A category $\mathcal{C}$ with a functor $F : \mathcal{C} \to \mathcal{D}$ that is full, faithful, and injective on objects is a full subcategory of the domain $\mathcal{D}$. A functor F of this type is called a **full embedding** of $\mathcal{C}$ into $\mathcal{D}$.

Let's take a look at an example by paring down our category D from the previous example.



Again, convince yourself that this is indeed a full embedding.

Equivalence Relations between Categories

Even more familiar than the subcategory relation, but less simple in the categorical context, are equivalence relations. When comparing the "sameness" of categories there are several relations we can define. In order of strictest to loosest these are equality, isomorphism, and equivalence. Equality and isomorphism are actually quite simple. Unfortunately they're both mostly useless in practice.

Definition 10 (Equality of Categories)**.** Two categories are equal if they contain exactly the same objects and morphism.

Definition 11 (Isomorphism of Categories – Subcategories)**.** Two categories are isomorphic is they're both subcategories of each other.

This is similar to the situation with sets where mutual subsets define an equality of sets. By ensuring that every object and morphism in the source has a unique object and morphism in the target that it gets mapped to under the embedding $F : \mathcal{C} \to \mathcal{D}$ and vice versa under the embedding $G : \mathcal{D} \to \mathcal{C}$, we ensure that our mappings between sets and objects are bijective and that our categories are isomorphic.

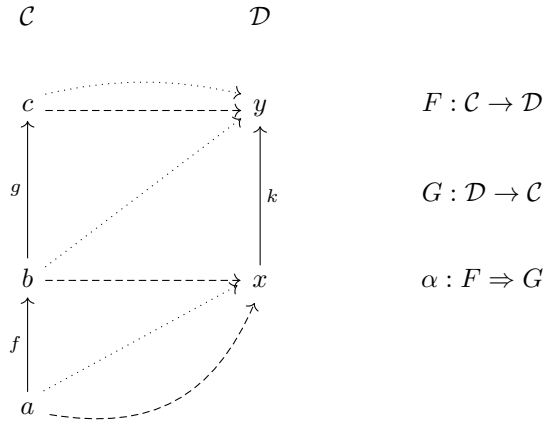
Another way to define an isomorphism between categories is as an isomorphism in the category of categories and functors.

Definition 12 (Isomorphism of Categories – Category of Categories)**.** Given two categories $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}$ and a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if $G \circ F = \mathsf{id}_{\mathcal{C}}$ and $F \circ G = \mathsf{id}_{\mathcal{D}}$.

More useful than these is the notion of categorical equivalence and the notion of natural isomorphism that it relies on.

Definition 13 (Natural Isomorphism)**.** A natural isomorphism is a natural transformation $\alpha : F \Rightarrow G$ where all of the c-components of the transformation $\alpha_c : Fc \to Gc$ are isomorphisms.

Let's try to get a handle on what this means by looking at the image we used to explore natural transformations.



As before, the components of α are the following:

$$\alpha_a = \mathsf{id}_x$$

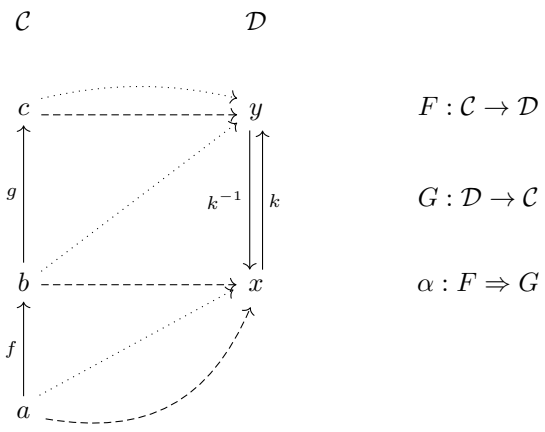
$$\alpha_b = k$$

$$\alpha_c = \mathsf{id}_y$$

Our task now is to figure out which, if any, of these component morphisms are isomorphism. For both identity morphisms the answer is yes: $\mathsf{id}_x \circ \mathsf{id}_x = \mathsf{id}_x$. In general, if you do nothing and then do nothing again you've still not done anything.[4] The case is different for k. There simply is no morphism from y to x in $\mathcal{D}$ that could be the inverse of k, so k cannot be an isomorphism. This means our natural transformation α is not a natural isomorphism.

[4] This is the kind of high-octane thinking that math opens up to us. Breathe it in.

Let's make a small addition and try again.



With this new diagram we've explicitly added an inverse of k, making it an isomorphism. Since every component morphism of α is now an isomorphism, α itself is now a natural isomorphism. With natural isomorphisms in hand we can define an equivalence between categories.

Definition 14 (Categorical Equivalence)**.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent, written $\mathcal{C} \simeq \mathcal{D}$, if there exist two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms η and ϵ such that $\eta : \mathsf{id}_{\mathcal{C}} \cong G \circ F$ and $\epsilon : \mathsf{id}_{\mathcal{D}} \cong F \circ G$.

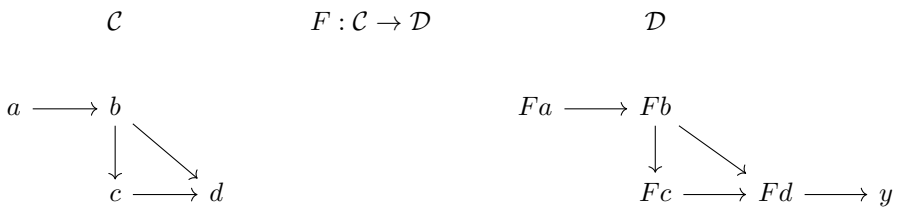
It might seem difficult to understand what's going on beneath this definition. Let's try again using another property on functors.

Definition 15 (Essentially Surjective Functor)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective on objects if for every object $y \in \mathcal{D}$ there exists an object $x \in \mathcal{C}$ such that $Fx \cong y$.

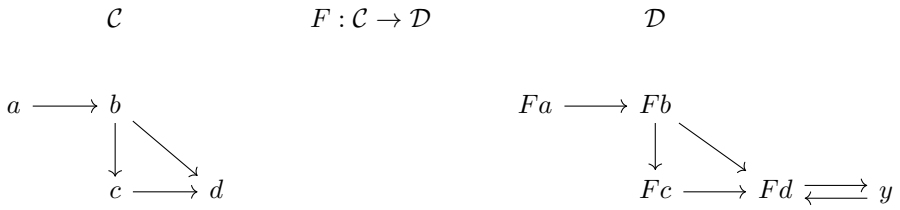
Definition 16 (Categorical Equivalence, revisited)**.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists a functor F between them that is faithful, full, and essentially surjective on objects.

Let's look more deeply at this functor and what each of its properties guarantees. Being essentially surjective on objects means that every object in our target category has some object from the source category that maps to it (up to isomorphism). The combination of a full and faithful functor guarantees that for every set of morphisms between objects in our source category will be isomorphic to the set of morphisms between the image of those objects in the target category. In other words, we have an isomorphism between morphisms and a surjection between objects (up to isomorphism). If we look at our full

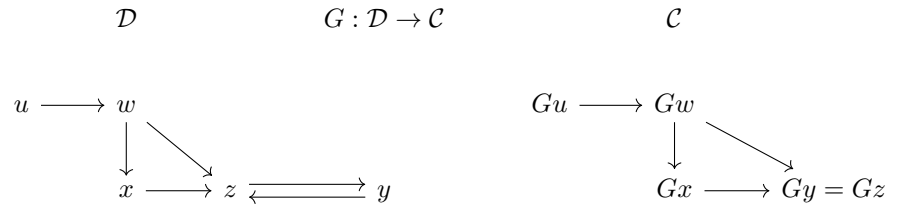
subcategory example from above, are $\mathcal{C}$ and $\mathcal{D}$ equivalent?



We know that F is full and faithful by definition of a full subcategory. But it's not surjective on objects since there's no object in $\mathcal{C}$ that maps to y in $\mathcal{D}$. Let's make a quick modification so that our two categories can be equivalent.



Our collections of objects are now surjective up to isomorphism, so our functor is full, faithful, and essentially surjective and our two categories are equivalent. Since categorical equivalence is an equivalence relation we know that the relation must be reflexive, symmetric, and transitive. It's trivial to see that the relation is reflexive since the identity functor on a category meets all of our requirements, so we know a category is always equivalent to itself. But how about symmetry? Let's look at this using our example categories above and introduce a new functor G.



We can map y and z onto the same object G and keep the functor essentially surjective. But is it still full and faithful? Naively we might think they aren't, since id_y, id_z, and both parts of the isomorphism between y and z all get mapped to id_{Gy}, which certainly doesn't seem injective. But G is full and faithful. Double checking that this is the case is a great opportunity to see why full and faithful functors are defined in terms of homsets.

Proof. Given a functor $G : \mathcal{D} \to \mathcal{C}$ as defined above, for G to be full and faithful there must be a bijection between the homset of any two objects in $\mathcal{D}$ and the homset of their images in $\mathcal{C}$. Since most of the objects and morphisms we're

concerned with above obviously meet this requirement by inspection, let's focus on $y, z \in \mathcal{D}$ and the isomorphism between them, which we'll call $f : y \to z$ and $f^{-1} : z \to y$, and their images. Listing them out, we have the following homsets.

$$\mathcal{D}(y, y) = \{\mathsf{id}_y\} \cong \{\mathsf{id}_{Gy}\} = \mathcal{C}(Gy, Gy)$$

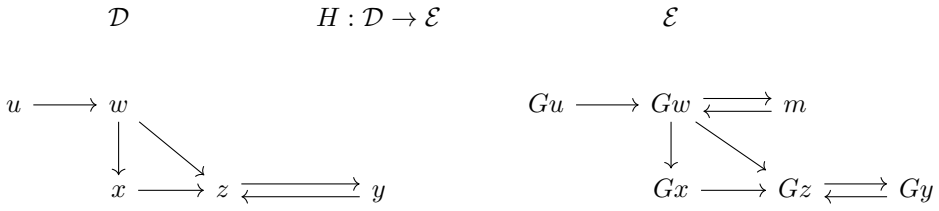
$$\mathcal{D}(z, z) = \{\mathsf{id}_z\} \cong \{\mathsf{id}_{Gz}\} = \mathcal{C}(Gz, Gz)$$

$$\mathcal{D}(y, z) = \{f\} \cong \{\mathsf{id}_{Gy}\} = \mathcal{C}(Gy, Gz)$$

$$\mathcal{D}(z, y) = \{f^{-1}\} \cong \{\mathsf{id}_{Gz}\} = \mathcal{C}(Gz, Gy)$$

So even though we have four morphisms that are all getting mapped to a single identity, each of the homsets individually is a singleton set in both the source and the target categories, meaning that they're all isomorphic, so our functor G is indeed full and faithful. □

So we now know that categorical equivalence is reflexive and symmetric, but is it transitive? To test this we'll need another small category and another functor.



You should be able to convince yourself that $\mathcal{D} \simeq \mathcal{E}$ and that $\mathcal{C} \simeq \mathcal{E}$ using the same reasoning we used for our initial equivalence $\mathcal{C} \simeq \mathcal{D}$.

Since categorical equivalence is reflexive, symmetric, and transitive we know that it does define an equivalence relation.

Representability

NOTE: Play a huge role in category theory: i.e in the Yoneda lemma
NOTE: Representable functors require that the source category $\mathcal{C}$ be locally small so that the collections of morphisms between objects in the category are sets.

Definition 17 (Covariant Representable)**.** A functor $F : \mathcal{C} \to \mathbf{Set}$ is representable if there is an object $c \in \mathcal{C}$ such that the functor $\mathcal{C}(c, -) : \mathcal{C} \to \mathbf{Set}$ is naturally isomorphic to F.

Definition 18 (Contravariant Representable)**.** A functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ is representable if there is an object $c \in \mathcal{C}$ such that the functor $\mathcal{C}(-, c) : \mathcal{C}^{op} \to \mathbf{Set}$ is naturally isomorphic to F.

Definition 19 (The Yoneda Lemma)**.** The Yoneda lemma states that for any functor $F : \mathcal{C} \to \mathbf{Set}$ where $\mathcal{C}$ is locally small and any object $c \in \mathcal{C}$, there is a bijection between the collection of natural transformations going from the functor $\mathcal{C}(c, -)$ to F and the object $Fc \in \mathbf{Set}$.

Stated another way, this means that $Hom(\mathcal{C}(c, -), F) \cong Fc$ for any functor $F : \mathcal{C} \to \mathbf{Set}$ and all objects $c \in \mathcal{C}$.

Adjunctions

Definition 20 (Forgetful)**.** A forgetful functor $F : \mathcal{C} \to \mathbf{Set}$ is a functor that sends objects in a category of structures-on-sets, like groups or preorders, to the underlying sets themselves. They "forget" the additional structure in the source category.

Definition 21 (Free)**.** A free functor $F : \mathbf{Set} \to \mathcal{C}$ is a functor that sends sets to the free construction on that set, like the free group on the set. They construct the desired structure on the set in a way that requires no choices.

Definition 22 (Adjunctions)**.** An adjunction is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that for any object $c \in \mathcal{C}$ and any object $d \in \mathcal{D}$ there is an isomorphism between the sets of functions between them and their images under the functors that is natural. In other words, that for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$ there is a natural isomorphism $\mathcal{C}(c, Gd) \cong \mathcal{D}(Fc, d)$.

In this case, the functor F is the left adjoint functor and the functor G is the right adjoint functor.

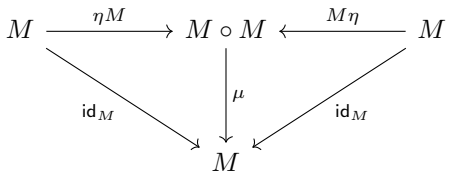
Monads

Definition 23 (Monad)**.** Given a category $\mathcal{C}$, a monad on $\mathcal{C}$ is a functor $M : \mathcal{C} \to \mathcal{C}$ with the following two natural transformations:

1. $\eta : \mathsf{id}_{\mathcal{C}} \Rightarrow M$ called the unit.
2. $\mu : M \circ M \Rightarrow M$ called the multiplication.

These natural transformation are required to make the following diagrams commute.

$$\begin{array}{ccc} M \circ M \circ M & \xrightarrow{M\mu} & M \circ M \\ \downarrow{\scriptstyle \mu M} & & \downarrow{\scriptstyle \mu} \\ M \circ M & \xrightarrow[\mu]{} & M \end{array}$$



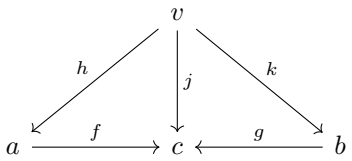
These commutative diagrams with the natural transformations η and μ create a kind of monoidal structure on the collection of endofunctors $M, M \circ M, M \circ M \circ M, etc.$.

Structure within Categories

Now that we know the basics of category theory let's look at the kinds of structures that we can find within a category and the process we use to do so. A quick word of warning: this section is going to get very abstract, but as before, the underlying intuition is fairly simple. We can think of the structure that we're looking for inside a category as some subcategory[5] of objects and morphisms, for example the one below.

$$a \xrightarrow{f} c \xleftarrow{g} b$$

Given such a subcategory, we want to find the examples of it in our category such that there's an object that relates to all the objects in our diagram such that everything commutes.



We call this new subcategory a cone since it looks a little bit like a cone with the vertex at v. We then find all examples in our category that match this cone and try to find the "best" ones: those that contain only the necessary information and nothing else. Any cones that can be factorized via some unique morphism to one of these "best" ones gets filtered out, and those that remain are the limit objects of our diagram. Regardless of the diagram/cone, all of these limit objects will be the same up to a unique isomorphism. We then say that all of these objects have the universal property specified by the cone.

Let's review this process with a picture before moving on to a more rigorous account.

$$\text{subcategory} \xrightarrow{\text{add vertex}} \text{cone} \xrightarrow{\text{find examples}} \text{cones and factorizations} \xrightarrow{\text{filter}} \text{limit objects}$$

Cones

In this last part of the section we're going to discuss a generalization of all of the previous universal properties called cones and cocones. A cone consists of any finite collection of objects and morphisms between them, combined with an additional object v called the vertex of the cone and morphisms from v to all other objects in the collection. Such a cone is universal if for all other cones of that type in the category there is a unique morphisms from their vertex to v: i.e, $k : v' \to v$ exists and is unique.

[5] This is really just a category. It's only a subcategory if it actually does exist inside our main category. I went ahead and used subcategory here to avoid having to use nebulous terms like "main category."

A cocone consists of any finite collection of objects and morphisms between them, combined with an additional object v called the vertex of the cocone and morphisms from all other objects in the collection to v. Such a cocone is universal if for all other cocones of that type in the category there is a unique morphisms from v to their vertex: i.e, $k :\to v'$ exists and is unique.

Given any cone, if there is an instance of that cone for which all other instances have a unique morphism to it, that cone is the limit. Dually for cocones and colimits. In fact, it's possible to construct a category whose objects are the cones and whose morphisms are these unique factorizations. In this category, the limit is a terminal object. Dually for cocones and colimits again, except the colimit is an initial object in the constructed category of cocones instead of a terminal object.

Categories of Cones and Comma Categories

Limits

Duality Revisited

Common Examples of Universal Properties

Definition 24 (Terminal Object)**.** Let $t \in \mathcal{C}$ be given. Then t is a terminal object in $\mathcal{C}$ if the following hold:

1. For all $x \in \mathcal{C}$, there is exactly one morphism $f : x \to t$.
2. For any other object t' in $\mathcal{C}$ that meets condition (1), there exists a unique morphism $g : t \to t'$.

Dual to terminal objects are initial objects.

Definition 25 (Initial Object)**.** Let $i \in \mathcal{C}$ be given. Then i is a initial object in $\mathcal{C}$ if the following hold:

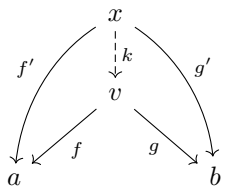
1. For all $x \in \mathcal{C}$, there is exactly one morphism $f : i \to x$.
2. For any other object i' in $\mathcal{C}$ that meets condition (1), there exists a unique morphism $g : i \to i'$.

In category theory there's a universal property that generalizes notions of multiplication called the product.

Definition 26 (Product)**.** Let $a, b, v \in \mathcal{C}$ and morphisms $f : v \to a$ and $g : v \to b$ be given. The object v is a product of a and b if the following holds:

1. For any other object x with morphisms $f' : x \to a$ and $g' : x \to b$, there is a unique morphism $k : x \to v$ such that $f \circ k = f'$ and $g \circ k = g'$.

This means the following diagram commutes:



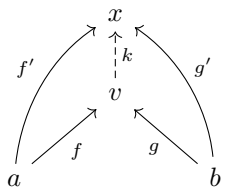
This conditions ensures that the object v is the universal or canonical object satisfying the pattern, with the morphisms of any other such object being able to be written as a sort of embellishment of the morphisms from v.

Dual to products are coproducts, which generalize the notion of addition.

Definition 27 (Coproduct)**.** Let $a, b, v \in \mathcal{C}$ and morphisms $f : a \to v$ and $g : b \to v$ be given. The object v is a coproduct of a and b if the following holds:

1. For any other object x with morphisms $f' : a \to x$ and $g' : b \to x$, there is a unique morphism $k : v \to x$ such that $k \circ f = f'$ and $k \circ g = g'$.

This means the following diagram commutes:

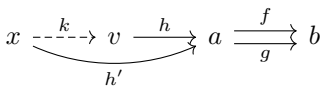


NOTE: In **Set**, equalizers pick out the subset of a set a that is the intersection of the preimages $f^{-1} : b \to a$ and $g^{-1} : b \to a$.

Definition 28 (Equalizer)**.** Let $a, b, v \in \mathcal{C}$ and let morphisms $f : a \to b$, $g : a \to b$, and $h : v \to a$ be given. The object v is an equalizer if the following hold:

1. $f \circ h = g \circ h$.
2. For any other object $x \in \mathcal{C}$ with morphism $h' : x \to a$ such that $f \circ h' = g \circ h'$, there exists a unique morphism $k : x \to v$ such that $h' = h \circ k$.

This means the following diagram commutes:

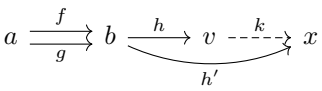


NOTE: In **Set**, coequalizers pick out the subset of a set b that is the intersection of the images $f : a \to b$ and $g : a \to b$.

Definition 29 (Coequalizer)**.** Let $a, b, v \in \mathcal{C}$ and let morphisms $f : a \to b$, $g : a \to b$, and $h : b \to v$ be given. The object v is a coequalizer if the following hold:

1. $h \circ f = h \circ g$.
2. For any other object $x \in \mathcal{C}$ with morphism $h' : b \to x$ such that $h' \circ f = h' \circ g$, there exists a unique morphism $k : v \to x$ such that $h' = k \circ h$.

This means that the following diagram commutes:

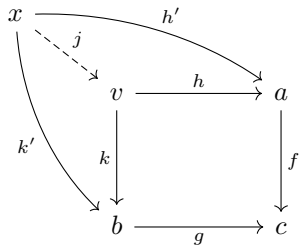


NOTE: In **Set**, pullbacks are the fiber product of two sets.

Definition 30 (Pullback)**.** Let $a, b, c, v \in \mathcal{C}$ and let morphisms $f : a \to c$, $g : b \to c$, $h : v \to a$, and $k : v \to b$ be given. The object v is a pullback if the following hold:

1. $f \circ h = g \circ k$.
2. For any other object $x \in \mathcal{C}$ with morphisms $h' : x \to a$, and $k' : x \to b$ such that $f' \circ h = g' \circ k$, there exists a unique morphism $j : x \to v$ such that $f' = f \circ j$ and $g' = g \circ j$.

This means that the following diagram commutes:

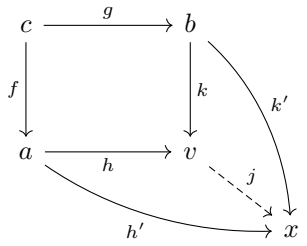


NOTE: In **Set**, pushouts are...

Definition 31 (Pushout)**.** Let $a, b, c, v \in \mathcal{C}$ and let morphisms $f : c \to a$, $g : c \to b$, $h : a \to v$, and $k : b \to v$ be given. The object v is a pushout if the following hold:

1. $h \circ f = k \circ g$.
2. For any other object $x \in \mathcal{C}$ with morphisms $h' : a \to x$, and $k' : b \to x$ such that $h' \circ f = k' \circ g$, there exists a unique morphism $j : v \to x$ such that $h' = j \circ h$ and $k' = j \circ k$.

This means that the following diagram commutes:

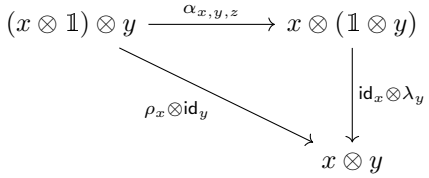


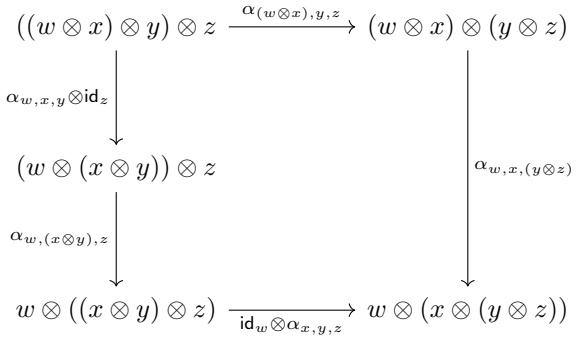
Adding Structure to Categories

Definition 32 (Monoidal Categories)**.** A category $\mathcal{C}$ is monoidal if the following exist.

1. A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the monoidal product.
2. An object $\mathbb{1} \in \mathcal{C}$ called the unit.
3. A natural isomorphism $\alpha : (a \otimes b) \otimes c \Rightarrow a \otimes (b \otimes c)$ called the associator, with components $\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$.
4. A natural isomorphism $\lambda : \mathbb{1} \otimes a \Rightarrow a$ called the left unitor with components $\lambda_x : \mathbb{1} \otimes x \to x$.
5. A natural isomorphism $\rho : a \otimes \mathbb{1} \Rightarrow a$ called the right unitor with components $\lambda_x : x \otimes \mathbb{1} \to x$.

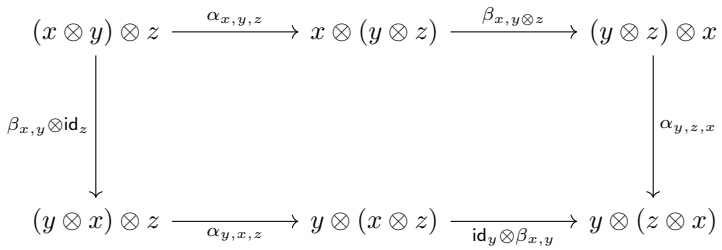
All of the above must exist such that the following two diagrams, called the triangle identity and the pentagon identity, commute.





Definition 33 (Symmetric Monoidal Category)**.** A monoidal category equipped with a natural isomorphism $\beta : a \otimes b \Rightarrow b \otimes a$, called its braiding, is symmetric if $\beta_{y,x} \circ \beta_{x,y} = \mathsf{id}_{x \otimes y}$ for all $x, y \in \mathcal{C}$ and if the following diagram, called the

hexagon identity, commutes.



Definition 34 (Cartesian Monoidal Category)**.** A monoidal category $\mathcal{C}$ is cartesian when its monoidal product is its categorical product and its monoidal unit is its terminal object. This obviously means that $\mathcal{C}$ must contain all products and have a terminal object.

An example of this would be **Set**, where the Cartesian product is the monoidal product and the singleton set is the monoidal unit.

Categorical System Theory

Systems are everywhere in science and engineering. Whether discrete or continuous, deterministic or stochastic, all such systems have two things in common: states that they can be in, and rules for how the system's current state changes. These two features alone describe what are called closed systems. Closed systems are systems that don't interact with others or with an outside environment that they're a part of. This is a hobbling limitation. In order to open these systems up we need to give them an interface. We need them to be able to accept inputs that shape the way they evolve, and we need them to be able to expose some part of their current state to their surrounding environment. This is the gift that category theory will give us. By opening our systems up, we open them to the power of composition. We can connect them.

The Category $\mathbf{Lens}_{\mathcal{C}}$

Given a cartesian category, it's possible to construct a new category of systems whose states are drawn from the objects of your base category and whose rules for updating their state are drawn from the morphism of your base category. We call this category $\mathbf{Lens}_{\mathcal{C}}$, where $\mathcal{C}$ is the base category. The objects in this category are called arenas and the morphisms between them are called lenses.

Definition 35 (Lenses)**.** Given a cartesian category $\mathcal{C}$ and objects $A^-, A^+, B^-, B^+ \in \mathcal{C}$, a lens consists of a passforward map $f : A^+ \to B^+$ and a passback map $f^\# : A^+ \times B^- \to A^-$ between two arenas as follows:

$$\begin{pmatrix} f^\# \\ f \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} B^- \\ B^+ \end{pmatrix}$$

For our purposes we'll be sticking to two such categories: the category of lenses over the category of sets and functions, $\mathbf{Lens}_{\mathbf{Set}}$, for discrete systems and the category of Euclidean spaces and smooth functions, $\mathbf{Lens}_{\mathbf{Euc}}$, for differential systems.

Definition 36 (The Category $\mathbf{Lens}_{\mathcal{C}}$)**.** Given the cartesian category $\mathcal{C}$, the category $\mathbf{Lens}_{\mathcal{C}}$ has the following properties.

1. A collection of objects called arenas. An arena $\begin{pmatrix} A^- \\ A^+ \end{pmatrix}$ is a pair of objects in $\mathcal{C}$.
2. For each pair of arenas a collection of morphisms $\begin{pmatrix} f^\# \\ f \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} B^- \\ B^+ \end{pmatrix}$ called lenses.
3. For each arena an identity lens $\begin{pmatrix} \pi_2 \\ \mathsf{id}_{A^+} \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} A^- \\ A^+ \end{pmatrix}$ where the passback map π_2 is the projection $\pi_2 : A^+ \times A^- \to A^-$.

4. For any two compatible lenses a composite lens as follows:

$$\begin{pmatrix} f^{\#} \\ f \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} B^- \\ B^+ \end{pmatrix}$$

$$\begin{pmatrix} g^{\#} \\ g \end{pmatrix} : \begin{pmatrix} B^- \\ B^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} C^- \\ C^+ \end{pmatrix}$$

$$\begin{pmatrix} g^{\#} \\ g \end{pmatrix} \circ \begin{pmatrix} f^{\#} \\ f \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} C^- \\ C^+ \end{pmatrix}$$

such that the passforward map is defined as $a^+ \mapsto g(f(a^+))$, and the passback map is defined as $(a^+, c^-) \mapsto f^{\#}(a^+, g^{\#}(f(a^+), c^-))$.

$\mathbf{Lens}_{\mathcal{C}}$ is also a monoidal category with the monoidal product being defined as follows.

1. The monoidal unit is the arena $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
2. Given lenses $\begin{pmatrix} f^{\#} \\ f \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} B^- \\ B^+ \end{pmatrix}$ and $\begin{pmatrix} g^{\#} \\ g \end{pmatrix} : \begin{pmatrix} C^- \\ C^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} D^- \\ D^+ \end{pmatrix}$, the monoidal product

$$\begin{pmatrix} f^{\#} \\ f \end{pmatrix} \otimes \begin{pmatrix} g^{\#} \\ g \end{pmatrix} : \begin{pmatrix} A^- \times C^- \\ A^+ \times C^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} B^- \times D^- \\ B^+ \times D^+ \end{pmatrix}$$

such that the passforward map is defined as $(a^+, c^+) \mapsto (f(a^+), g(c^+))$ and the passback map is defined as $((a^+, c^+), (b^-, d^-)) \mapsto ((f^{\#}(a^+, b^-)), g^{\#}(c^+, d^-))$.

References

[Che22] Eugenia Cheng. *The joy of abstraction: an exploration of math, category theory, and life*. Cambridge University Press, 2022.

[FS19] Brendan Fong and David I Spivak. *An invitation to applied category theory: seven sketches in compositionality*. Cambridge University Press, 2019.

[Gol14] Robert Goldblatt. *Topoi: the categorial analysis of logic*. Elsevier, 2014.

[Lei14] Tom Leinster. *Basic category theory*, volume 143. Cambridge University Press, 2014.

[LR03] F William Lawvere and Robert Rosebrugh. *Sets for mathematics*. Cambridge University Press, 2003.

[Mil18] Bartosz Milewski. *Category theory for programmers*. Blurb, 2018.

[ML13] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[Mye22] David Jaz Myers. Categorical systems theory. *Manuscript in preparation*, 2022.

[nLa24] nLab authors. symmetric monoidal category. `https://ncatlab.org/nlab/show/symmetric+monoidal+category`, February 2024. Revision 54.

[Rie17] Emily Riehl. *Category theory in context*. Courier Dover Publications, 2017.

[Ros22] Daniel Rosiak. *Sheaf theory through examples*. MIT Press, 2022.