An Introduction to Game Theory

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Work in Progress

ABSTRACT

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This paper serves as an introduction to the field of game theory. Topics are organized into three parts: two-person zero-sum games, as well as methods to solve small forms of these kinds of games; linear programming, which can be used to solve any two-person zero-sum game; and two-person nonzero-sum games, which include both non-cooperative and cooperative types. The paper also includes proofs for two important theorems.

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1. INTRODUCTION

First, consider the name of the topic: “game theory” or “theory of games.” The word “game” may bring to mind games of chess, poker, or baseball. What these have in common is competition: people compete against one another. We can group these under the broader term “competitive situations,” which includes other scenarios such as price wars, labor disputes, and superpower conflicts. Game theory is essentially the mathematics of conflict, as well as of cooperation. Its aim is to analyze situations involving competition by using scientific and mathematical methods.

Though there are earlier instances of using mathematics to illustrate games and conflict situations, game theory was nonexistent until the 20th century. The mathematicians Ernst Zermelo and Emile Borel provided contributions, but it is John von Neumann that is often considered the founder of game theory. His 1928 paper contains the proof that every zero-sum game has a solution (the famous minimax theorem) [10], while the 1944 paper *Theory of Games and Economic Behavior*, co-authored with economist Oskar Morganstern, marks the emergence of modern game theory [11].

Game theory was developed extensively in the 1950’s. The term Prisoner’s Dilemma, now a well-known problem, was introduced. Meanwhile, mathematician John Nash extended von Neumann’s 1928 result by proving the existence of equilibrium pairs in nonzero-sum games [7]. Game theory spread to a broad range of applications (political science, social psychology, evolutionary biology, and philosophy, to name a few) and is still being developed today; in fact, the 2020 Nobel Prize in Economics was awarded to two game theorists.

2. TWO-PERSON ZERO-SUM GAMES

2.1. The matrix game

In the context of game theory, a *game* is a competitive situation with the following properties:

- A finite set of *players*. The players can be people, groups of people, computer programs, nature, and so on.
- A finite set of *strategies*, or a range of choices on what to do, for each player.
- A *payoff* given to each player at each outcome, or end of the game. A payoff is a numerical value that can represent something quantitative (such as money or score) or non-quantitative (such as level of preference or happiness). A positive payoff is interpreted as a gain, while a negative payoff is considered a loss.

The game of Rock, Paper, Scissors qualifies as a game. There are two players (we will call them Roy and Colette), each with three strategies (Rock, Paper, or Scissors). The combination of strategies results in nine possible outcomes. Payoffs are determined by the rule that Rock beats Scissors, Scissors beats Paper, and Paper beats Rock. Let the payoffs be 1 for the winning player, -1 for the losing player, and 0 for a draw. We can represent these values with a payoff matrix:

$$\begin{array}{cc} & \text{Colette (Column Player)} \\ & \begin{array}{ccc} \text{Rock} & \text{Paper} & \text{Scissors} \end{array} \\ \begin{array}{rr} \text{Roy} & \text{Rock} \\ \text{(Row Player)} & \text{Paper} \\ & \text{Scissors} \end{array} & \begin{bmatrix} (0,0) & (-1,1) & (1,-1) \\ (1,-1) & (0,0) & (-1,1) \\ (-1,1) & (1,-1) & (0,0) \end{bmatrix} \end{array}$$

Figure 2.1. A payoff matrix for Rock, Paper, Scissors.

The rows represent Roy's strategy set, the columns represent Colette's strategy set, and the entries are ordered pairs of payoffs. In general, a game with *m* strategies for the row player and *n* strategies for the column player is represented by an $m \times n$ matrix. Suppose Roy chooses Paper (Row 2) while Colette picks Rock (Column 1), which results in the payoff pair (1, -1). The first value is Roy's payoff of 1 (a win), and the second is Colette's payoff of -1 (a loss).

Note that for every payoff pair in Figure 2.1, each value is a negative of the other. A game is *zero-sum* if the sum of the values for every payoff pair equals zero. That is, a gain for one player is a loss of the same amount for the other player. A game that is not zero-sum is called *nonzero-sum*. We will cover nonzero-sum games in a later section.

A zero-sum game can be written as a more compact matrix, where each entry is a single value. We illustrate this for the Rock, Paper, Scissors example in Figure 2.2. Only the row player's payoffs are indicated in the matrix entries, since the column player's payoffs are simply the negative of each entry. Zero-sum games are also known as *matrix games* because of this representation.

$$\begin{array}{cc c}
 & & \text{Colette} \\
 & & \begin{array}{ccc}\text{Rock} & \text{Paper} & \text{Scissors}\end{array} \\
\text{Roy} & \begin{array}{r}\text{Rock}\\ \text{Paper}\\ \text{Scissors}\end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}
\end{array}$$

Figure 2.2. A payoff matrix for Rock, Paper, Scissors with Roy's payoffs as the entries.

One of the goals of analyzing zero-sum games is to find an acceptable solution: assuming each player wishes to play optimally, what strategy should each choose, and what will be the outcome? By "optimally," we mean that the row player aims to maximize his payoff, and the column player aims to minimize the row player's payoff (thereby maximizing her own payoff). We also assume that the row player expects the column player to play this way, and vice versa.

Some matrix games have straightforward solutions.

DEFINITION: Let A be an $m \times n$ matrix. An entry a_{kl} is a *saddle point* if it is both a minimum in its row and a maximum in its column. That is,

$$a_{kl} \le a_{kj},\ 1 \le j \le n \quad \text{and} \quad a_{kl} \ge a_{il},\ 1 \le i \le m.$$

In Figure 2.3 below, the left matrix has a saddle point at Row 2, Column 2. Suppose Roy and Colette happened to decide on the strategies Row 2 and Column 2, respectively. If either player changes strategy (for example, if Roy switches to Row 3), the changing player will obtain a lower payoff. Thus, each player is discouraged from switching away from a strategy with a saddle point. Row 2, Column 2 is a viable solution to the game, giving a payoff of 1 (to the row player).

$$\begin{bmatrix} 6 & -3 & 2 \\ 2 & \mathbf{1} & 5 \\ -2 & -1 & 3 \end{bmatrix} \qquad \begin{array}{ccc} & & \text{Colette guesses:} \\ & & \begin{array}{cc} \text{Penny} & \text{Nickel} \end{array} \\ \text{Roy hides:} & \begin{array}{c} \text{Penny} \\ \text{Nickel} \end{array} & \begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix} \end{array}$$

Figure 2.3. Two matrix games: one with a saddle point (boldfaced), and a "Coin in the Hand" game.

Now consider the game on the right.

Coin in the Hand: The first player, Roy, hides either a penny or a nickel in his hand. The second, Colette, tries to guess which coin is in his hand. If Colette guesses correctly, Roy must give her the coin, while if she guesses incorrectly, she must pay Roy a coin of the same kind in Roy's hand. Payoffs are determined by amount of money gained (lost).

This game does not have a saddle point. Roy could attempt to guess Colette's strategy and "counterpick" his own strategy: "She might guess Nickel, so I am better off hiding Penny." However, Colette could reason the same way: "Roy will switch to hiding Penny, so I should instead guess Penny." Their reasoning might continue in an endless cycle. An alternative approach is to choose a strategy at random. Roy could flip a fair coin and choose Penny if heads, and Nickel if tails. In other words, Roy could choose Row 1 with a probability of 0.5 and Row 2 with the same probability 0.5. Colette would not be able to predict the exact strategy that Roy will use.

DEFINITION: Let A be an $m \times n$ matrix game. A *mixed strategy* for the row player is an m-tuple of probabilities $\vec{p} = (p_1, p_2, \ldots p_m)$ where each $p_i \geq 0$ and $\sum_{i=1}^{m} p_i = 1$. Similarly, A *mixed strategy* for the column player is an n-tuple of probabilities $\vec{q} = (q_1, q_2, \ldots q_n)$ where each $q_j \geq 0$ and $\sum_{j=1}^{n} q_j = 1$.

A *pure strategy* is a special case of a mixed strategy where a single p_i (q_j) equals 1. It is the same as choosing a single strategy Row *i* (Column *j*).

When the players use mixed strategies, the *expected payoff* is given by

$$E(\vec{p}, \vec{q}) = \sum_{j=1}^{n} \sum_{i=1}^{m} p_i q_j a_{ij} .$$

The expected payoff can be thought as the average payoff for each game in an extremely large number of playthroughs. In our 2×2 example, suppose Roy uses the mixed strategy $\vec{p} = (1/2\,, 1/2)$ and Colette uses $\vec{q} = (1/3\,, 2/3)$. The expected payoff would be

$$E(\vec{p}, \vec{q}) = (1/2)(1/3)(3) + (1/2)(1/3)(-3) + (1/2)(2/3)(-1) + (1/2)(2/3)(2)$$

$$= 1/2 - 1/2 - 1/3 + 2/3 = 1/3.$$

The actual payoff could be different for a single game (Roy cannot actually win 1/3 of a penny), but if the game were played repeatedly, the average payoff per game would converge to this value.

Now, Roy's goal is to maximize the expected payoff *E*. Similarly, Colette's goal is to minimize *E*. An *optimal mixed strategy* $\vec{r}$ for the row player maximizes the minimum possible *E* imposed by the column player's strategy. This "highest possible lower bound" to *E* is the *row value* or *maximin value*, defined as

$$v_r = \min_{\vec{q}} E(\vec{r}, \vec{q})$$

where $\vec{q}$ is any mixed strategy from the column player.

Meanwhile, an optimal mixed strategy $\vec{s}$ for the column player minimizes the maximum possible *E*. This "lowest possible upper bound" is the *column value* or *minimax value*, defined as

$$v_c = \max_{\vec{p}} E(\vec{p}, \vec{s})$$

where $\vec{p}$ is any mixed strategy from the row player.

Since v_r is a lower bound and v_c is an upper bound, $v_r \leq E(\vec{r}, \vec{s}) \leq v_c$. In fact, the payoff *E* is kept at a single, definite amount because $v_r = v_c$. The following important theorem is due to John von Neumann.

MINIMAX THEOREM: Every zero-sum game has a solution, namely,

- An optimal mixed strategy for each player ($\vec{r}$ and $\vec{s}$)
- And the value of the game, $v = v_r = v_c$.

2.2. Solving small games

The original proof of the minimax theorem is a non-constructive existence proof; it does not show how to find a solution. Fortunately, some games are easily solved, such as those with saddle points. If a saddle point is the entry a_{ij} in a matrix, the solution is the pure strategies Row *i* and Column *j*, giving the value $v = a_{ij}$. Other games can be reduced to a smaller form.

DEFINITION: Let *A* be an $m \times n$ matrix. Row *i dominates* Row *k* if

$$a_{ij} \geq a_{kj}, \;\; 1 \leq j \leq n.$$

Column *j dominates* Column *l* if

$$a_{ij} \leq a_{il}, \;\; 1 \leq i \leq m.$$

As an example, consider Rows 1 and 2 in the left matrix of Figure 2.4. When the payoffs are compared by column, $3 \geq 0$ and $-1 \geq -2$. Since every entry in Row 1 is greater than the corresponding entry in Row 2, Row 1 dominates Row 2. Meanwhile, Row 1 does not dominate Row 3 because $-1 \ngeq 2$ in the second column.

Dominated rows and columns are never played in an optimal mixed strategy. Leaving out the dominated rows (columns) and solving the resulting smaller matrix gives the solution for the original matrix.

$$\begin{bmatrix} 3 & -1 \\ \cancel{0} & \cancel{-2} \\ -3 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & -1 \\ -3 & 2 \end{bmatrix}$$

Figure 2.4. In the 3×2 *matrix on the left, Row 1 dominates Row 2. The matrix can be simplified to the one on the right.*

A graphical method can be used to solve any 2×2 game; we will use Coin in the Hand as out example. Assign the row probabilities p and $1 - p$ to the two rows as shown below. Additionally, assign the column probabilities q and $1 - q$ to the columns.

$$\begin{array}{ccc} & & \text{Colette} \\ & & \begin{array}{cc} q & 1-q \end{array} \\ \text{Roy} & \begin{array}{c} p \\ 1-p \end{array} & \begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix} \end{array}$$

Figure 2.5. The Coin in the Hand game. Probabilities p and q represent the mixed strategies of each player.

We begin with finding p, the optimal mixed strategy for Roy. Suppose Colette plays the pure strategy Column 1. Then Roy's payoff is 5 when $p = 0$ and -1 when $p = 1$. Plot the points (0, 5) and (1, -1) on a graph and connect the two with a line. The line gives us the expected payoff with respect to p: $E_1 = -6p + 5$. Now suppose Colette plays the pure strategy Column 2. In this case, Roy's payoff is -5 when $p = 0$ and 1 when $p = 1$. Plot the points as before. The resulting line is $E_2 = 6p - 5$.

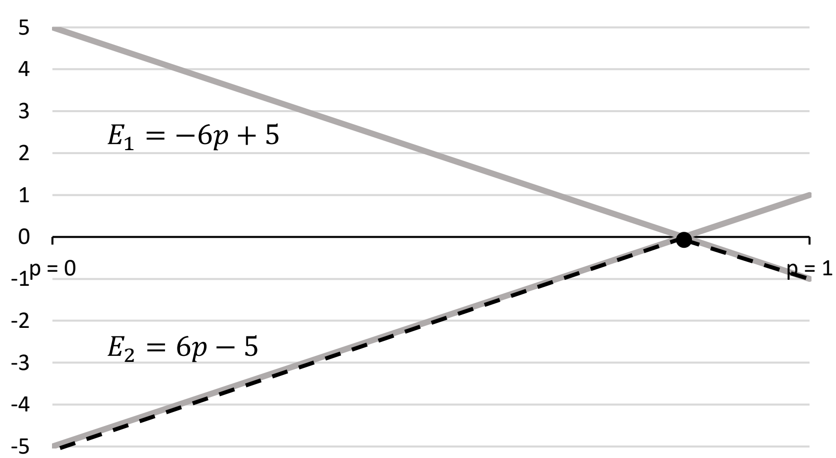

Figure 2.6. The lines representing Roy's expected payoff with respect to p.

Colette plays to minimize E, so look at only the minimum points in the graph (represented as a dashed line). The maximin value, v_r, is the maximum of this minimum (shown with a dot), and it occurs where the lines intersect. Solve for p by setting the lines E_1 and E_2 equal to each other.

$$-6p + 5 = 6p - 5 \quad \Rightarrow \quad 12p = 10 \quad \Rightarrow \quad p = 5/6\,, 1 - p = 1/6.$$

To find the value of the game, v, substitute 5/6 for p in either line (we use E_2):

$$v = 6(5/6) - 5 = 0.$$

The method for finding q is similar. If Roy plays the pure strategy Row 1, Colette's payoff is 1 when $q = 0$ and -1 when $q = 1$. Graph the line connecting the points (0, 1) and (1, -1). Graph a second line that corresponds to Roy's pure strategy Row 2.

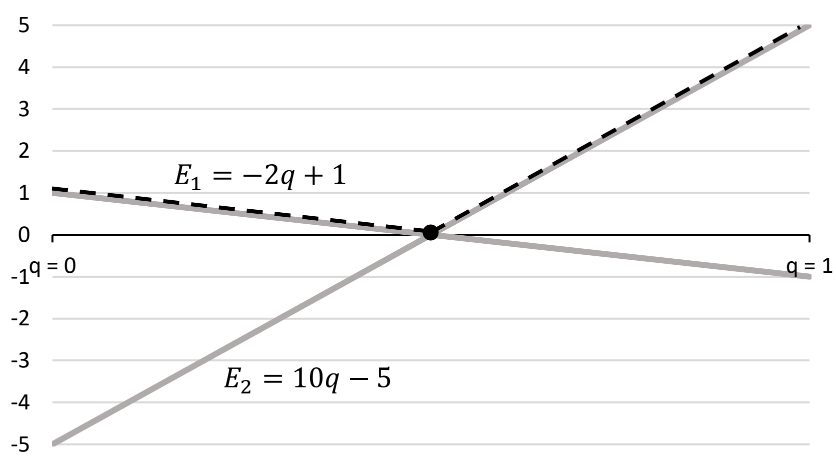

Figure 2.7. The lines representing Colette's expected payoff with respect to q.

Since Roy plays to maximize E, look at only the maximum points in the graph. The minimax value occurs where the lines intersect, so solve for q by setting the lines equal to each other.

$$-2q + 1 = 10q - 5 \;\Rightarrow\; 12q = 6 \;\Rightarrow\; q = 1/2\,, 1 - q = 1/2.$$

Thus, the solution to the game is $\vec{p} = (5/6, 1/6), \vec{q} = (1/2, 1/2)$, and $v = 0$. If Roy and Colette play these mixed optimal strategies, they will expect neither a gain nor a loss of payoff.

The graphical method can be extended to both $2 \times n$ and $m \times 2$ games. We will use the 2×3 matrix in Figure 2.8 as our example. As before, assign the probabilities *p* and $1 - p$ to the two rows, plot a line for each column, and look at the maximin value. In our example, the maximin is at the intersect of E_1 and E_3. Thus, $p = 5/11$, $1 - p = 6/11$, and $v = -4/11$. (For an $m \times 2$ game, use *q* and $1 - q$ for the two columns and plot a line for each of the *m* rows.)

$$\begin{array}{cc} & \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{array}{c} p \\ 1-p \end{array} & \begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 1 \end{bmatrix} \end{array}$$

Figure 2.8. A 2×3 *game with row probability p. The column probabilities are placeholder labels.*

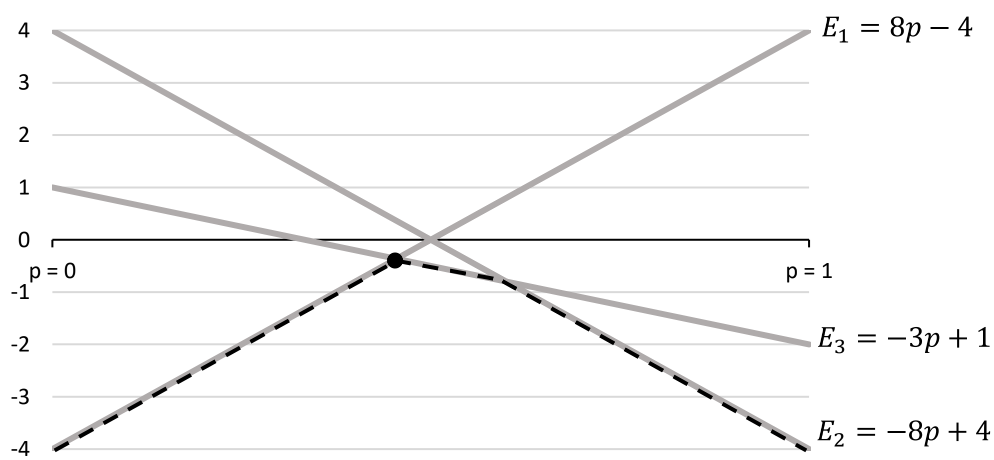

Figure 2.9. The lines representing Roy's expected payoff with respect to p for the matrix in Figure 2.8.

To find Colette's optimal mixed strategy, notice that line E_2 does not intersect at the maximin in Figure 2.9. Any line that does not intersect at that point corresponds to an *inactive* strategy, or one that has zero probability of being played. Remove the inactive columns and solve for the two remaining strategies, or *active* strategies, by assigning the probabilities q and $1-q$.

$$\begin{array}{c} \begin{array}{ccc} q & 0 & 1-q \end{array} \\ \begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 1 \end{bmatrix} \end{array}$$

Figure 2.10. Column 2 was found to be inactive, so a probability of 0 is assigned to it.

By solving for q as before, we find $q = 3/11$ and $1 - q = 8/11$ for Columns 1 and 3 respectively. The solution to the game is $\vec{p} = (5/11, 6/11)$, $\vec{q} = (3/11, 0, 8/11)$, and $v = -4/11$.

3. LINEAR PROGRAMMING

Linear programming is a mathematical method of decision-making that was developed around the same time as game theory. In fact, it was found that the theory of zero-sum games is equivalent to linear programming; solving a problem for one results in a solution for the other. Thus, any zero-sum game can be solved via an equivalent linear programming problem.

A linear program is a problem of maximizing or minimizing a linear function (the *objective function*) subject to a set of linear *constraints* or restrictions. A linear program with *n* variables typically has the following form:

$$\text{Maximize (or minimize) } f(\vec{x}) = c_1x_1 + c_2x_2 + \cdots + c_nx_n + d$$

$$\text{Subject to } a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b_1 \text{ (or)}$$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b_2.$$

3.1. Converting to a linear programming problem

Zero-sum games can be solved by converting the matrices to linear functions; the optimal values for the latter become the mixed optimal strategies for the former. In this and the following section, we will solve an example 3×3 matrix game by using a version of the simplex method developed by George Dantzig. However, the method outlined below will work for any $m \times n$ matrix game.

Our starting 3×3 matrix is the same Rock, Paper, Scissors game in Figure 2.2 but with a small twist: playing Rock against Scissors counts as two wins, represented as a payoff of 2. The value of the game, v, is currently unknown[1]. Modify A by adding a constant c to each entry so that every entry is positive. In this case, let $c = 3$ to obtain the entries in A'.

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}, \quad A' = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 3 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

Figure 3.1. Two matrices. The left represents the game "Rock, Paper, Scissors with Bigger Rock." The right is obtained by adding 3 to every entry in A.

Since the only change to the entries is the addition of a constant, the optimal mixed strategies remain the same for both A and A'. Meanwhile, the new value v' is equal to $v + c$. Although v' is still unknown, we know that since every entry in A' in positive, $v' > 0$. In other words, the row player will receive a positive payoff no matter what strategy either player uses.

The row player's problem is to find the mixed strategy $\vec{p} = (p_1, p_2, p_3)$ that maximizes v'. Each column of A' gives us a constraint for the p-variables. For example, the entries for Column

[1] One can guess with intuition that $v = 0$, since there is no difference in payoffs if the players switch roles (such a game is called *symmetric*). For games in general, however, it is difficult, if not impossible, to guess the exact value of v.

1, read from top to bottom, are 3, 4, and 1. These entries become the coefficients in the first constraint. The linear program becomes

$$\text{Maximize } v'$$

$$\text{Subject to } 3p_1 + 4p_2 + p_3 \geq v'$$

$$2p_1 + 3p_2 + 4p_3 \geq v'$$

$$5p_1 + 2p_2 + 3p_3 \geq v'$$

$$p_1 + p_2 + p_3 = 1. \tag{1}$$

(The non-negativity constraint $p_1, p_2, p_3 \geq 0$ is often implicit and left out of the set.)

We now want to modify our linear program in order to make it easier to manipulate. Let $x_i = p_i/v'$ for $i = 1, 2, 3$. Convert the *p*-variables to *x*-variables by dividing each constraint in (1) by *v*'. The division is allowed because we know $v' > 0$ in the modified matrix A'. The new constraint set is

$$3x_1 + 4x_2 + x_3 \geq 1$$

$$2x_1 + 3x_2 + 4x_3 \geq 1$$

$$5x_1 + 2x_2 + 3x_3 \geq 1$$

$$x_1 + x_2 + x_3 = 1/v'. \tag{2}$$

Maximizing *v*' is equivalent to minimizing its reciprocal 1/*v*'. Thus, the last constraint in (2) becomes the objective function: *minimize* $x_1 + x_2 + x_3 = 1/v'$.

The column player's problem, meanwhile, is to find the mixed strategy $\vec{q} = (q_1, q_2, q_3)$ that minimizes *v*'. This time, each constraint for the q-variables come from each row of A'. The linear program is therefore

$$\text{Minimize } v'$$

$$\text{Subject to } \; 3q_1 + 2q_2 + 5q_3 \leq v'$$

$$4q_1 + 3q_2 + 2q_3 \leq v'$$

$$q_1 + 4q_2 + 3q_3 \leq v'$$

$$q_1 + q_2 + q_3 = 1. \tag{3}$$

Once again, we make a change of variables by letting $y_i = q_i/v'$ for $i = 1, 2, 3$. Divide each constraint by *v*' to obtain

$$3y_1 + 2y_2 + 5y_3 \leq 1$$

$$4y_1 + 3y_2 + 2y_3 \leq 1$$

$$y_1 + 4y_2 + 3y_3 \leq 1$$

$$y_1 + y_2 + y_3 = 1/v'. \tag{4}$$

The last constraint in (4) becomes the objective function, and the column player's new goal is to *maximize* $y_1 + y_2 + y_3 = 1/v'$.

The linear program in (4) is in a special type of form called a *primal*. In addition, (2) is the *dual* of the primal. A primal and its dual form a closely related pair; in fact, the solution to the dual can be found by solving the primal, and vice versa. We continue with (4) as it is a maximizing problem.

3.2. The simplex method

The constraints are currently inequalities, but we can rewrite them as equalities by introducing *slack variables*. A slack variable is a non-negative value that, when added to the left terms of a constraint, raises the sum to the value of the right term. Add one per constraint in (4) (excluding the bottom) to obtain

$$3y_1 + 2y_2 + 5y_3 + y_4 = 1$$

$$4y_1 + 3y_2 + 2y_3 + y_5 = 1$$

$$y_1 + 4y_2 + 3y_3 + y_6 = 1. \tag{5}$$

In addition, rearrange the objective function so that the right side is 0.

$$-y_1 - y_2 - y_3 + 1/v' = 0. \tag{6}$$

A major feature of the simplex method is the tableau, which is basically the linear program in (5) and (6) written in a matrix-like form where the coefficients are the entries. The particular tableau we will use is mainly borrowed from [2]. The bottom row is the objective function and the other rows are the constraint equalities.

y_1	y_2	y_3	y_4	y_5	y_6	1/v'		
3	2	5	1	0	0	0	1	y_4
4	3	2	0	1	0	0	1	y_5
1	4	3	0	0	1	0	1	y_6
-1	-1	-1	0	0	0	1	0	1/v'

Figure 3.2. The simplex tableau for (5) and (6).

The tableau can be written in a condensed form by hiding the columns for the slack variables and 1/v'. Notice that the upper left region of the tableau in Figure 3.3 corresponds to our matrix A'.

We have mentioned that the solution to the dual problem (the row player's) can be found by solving the primal. To do so, label the left side of the tableau with the *x*-variables and the bottom with *surplus variables* for the dual's constraints. A surplus variable functions similarly to a slack variable, but it is subtracted from the left terms of the inequalities in (2).

	$\boldsymbol{y_1}$	y_2	y_3		
x_1	3	2	5	1	y_4

x_2	**4**	3	2	1	**y_5**
x_3	1	4	3	1	y_6
	-1	-1	-1	0	1/v'
	x_4	x_5	x_6		

Figure 3.3. The condensed tableau. The x-variables and surplus variables in the dual problem are also included.

We want to find the optimal set of variables, which will maximize the value corresponding to 1/v' (in the bottom right corner). The simplex method is an iterative procedure where each step converts the tableau, producing a value closer and closer to the optimal value. The algorithm to obtain an optimal tableau is as follows:

1) Choose a column j with a negative entry in the bottom row. The variable y_j becomes the *leaving variable*.
2) For each row in Column j (excluding the bottom row), divide the rightmost entry by j's entry. Choose the row i whose calculated quotient is closest to 0. The variable y_i becomes the *entering variable*, and Row i, Column j becomes the *pivot*.
3) Swap the entering variable with the leaving variable in the new tableau.
4) The entries in the new tableau are calculated by the following pivoting algorithm:

   a) If a is the pivot, then the new entry $a' = \frac{1}{a}$.

   b) If b is an entry in the same row as a, then $b' = \frac{b}{a}$.

   c) If c is an entry in the same column as a, then $c' = -\frac{c}{a}$.

   d) If d is an entry in c's row and b's column, then $d' = \frac{ad-bc}{a}$.

Old

a	$\cdots$	b
$\vdots$	$\ddots$	$\vdots$

New

$\frac{1}{a}$	$\cdots$	$\frac{b}{a}$
$\vdots$	$\ddots$	$\vdots$

Tableau

c	$\dots$	d

Tableau

$-\frac{c}{a}$	$\dots$	$\frac{ad-bc}{a}$

Figure 3.4. The pivoting algorithm where a is the pivot.

5) If every entry in the bottom row is non-negative, then the tableau is optimal. If not, repeat from Step 1.

We illustrate the simplex algorithm for the tableau in Figure 3.3. For Step 1, any of the first three columns are suitable, so we choose Column 1. For Step 2, we select Row 2 for the pivot (boldfaced) since the quotient 1/4 is closest to 0, and we swap the leaving variable y_1 with the entering variable y_5. We swap the x-variables as well. The new tableau is given in Figure 3.5.

	y_5	$\boldsymbol{y_2}$	y_3		
x_1	-3/4	-1/4	7/2	1/4	y_4
x_4	1/4	3/4	1/2	1/4	y_1
$\boldsymbol{x_3}$	-1/4	**13/4**	5/2	3/4	$\boldsymbol{y_6}$
	1/4	-1/4	-1/2	1/4	1/v'
	x_2	$\boldsymbol{x_5}$	x_6		

Figure 3.5. The tableau after pivoting Row 2, Column 1.

We then move on to Column 2 and choose Row 3 as the pivot, since 3/13 is closest to 0. After a third pivot in Column 3, Row 1 of the resulting tableau (not shown), we obtain an optimal tableau.

	y_5	y_6	y_4		
x_6	…	…	…	1/12	y_3
x_4	…	…	…	1/12	y_1
x_5	…	…	…	1/6	y_2
	1/6	1/12	1/12	1/3	1/v'
	x_2	x_3	x_1		

Figure 3.6. The optimal tableau. The entries in the upper left region are not required to be computed at this stage.

The solution to the primal is in the rightmost column.

$$y_1 = 1/12,\ y_2 = 1/6,\ y_3 = 1/12,\ \text{and}\ 1/v' = 1/3.$$

Meanwhile, the solution to the dual problem is in the bottommost row.

$$x_1 = 1/12,\ x_2 = 1/6,\ x_3 = 1/12,\ x_4 = 1/3.$$

To obtain the solution to our original matrix, we use the (rearranged) formulas $p_i = x_i v',\ q_j = y_j v'$, and $v = v' - c$, where $c = 3$. Our final solution is

$$\vec{p} = (1/4, 1/2, 1/4),\ \ \vec{q} = (1/4, 1/2, 1/4),\quad v = 0.$$

3.3. An alternative proof of the Minimax Theorem

In the previous two sections, we have illustrated the linear programming method on an example game. Here, we will use the same procedure on a general $m \times n$ game matrix, which will also serve as a proof of the minimax theorem.

THEOREM: Every two-player[1] zero-sum game has a an optimal mixed strategy for each player. Furthermore, the row value and the column value are the same; that is, $v_r = v_c = v$.

Proof. Let A be an $m \times n$ matrix game with value *v* and mixed strategies $\vec{p}$ and $\vec{q}$ for the row player and column player respectively. We organize the proof in three parts.

In Part 1, we convert the matrix game into a primal/dual pair of linear programs (henceforth called LPs). Add a constant *c* to each entry in A so that every entry is positive; call the resulting

[1] The minimax theorem applies to *n*-player zero-sum games, but we will prove only the two-player case.

matrix A' and the resulting value v', where $v' = v + c > 0$. Now let a_{ij} be an entry in A' where $1 \leq i \leq m,\ 1 \leq j \leq n$. The LP for the row player becomes

$$\text{Maximize } v'$$

$$\text{Subject to } a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{mj}p_m \geq v'$$

$$p_1 + p_2 + \cdots + p_m = 1$$

where there are *n* inequality constraints, one for each column in A'. The LP for the column player becomes

$$\text{Minimize } v'$$

$$\text{Subject to } a_{i1}q_1 + a_{i2}q_2 + \cdots + a_{in}q_n \leq v'$$

$$q_1 + q_2 + \cdots + q_n = 1$$

where there are *m* inequality constraints.

Now let $x_i = p_i/v'$ and $y_j = q_j/v'$. Rewrite each constraint in (5) and (6) in terms of x_i and y_i by dividing each term by v'. The row LP becomes the dual with *n* constraints of the form

$$a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m \geq 1$$

and with the new objective function

$$\text{Minimize } x_1 + x_2 + \cdots + x_m = 1/v'. \tag{7}$$

The column LP becomes the primal with *m* constraints of the form

$$a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n \leq 1$$

and with the new objective function

$$\text{Maximize } y_1 + y_2 + \cdots + y_n = 1/v'. \tag{8}$$

Subtract *n* surplus variables to the dual LP, one for each of the *n* constraints, to obtain equalities of the form

$$a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m - x_{m+n} = 1.$$

Similarly, add *m* slack variables for the primal LP to obtain equalities of the form

$$a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n + y_{n+m} = 1.$$

In Part 2, we solve the pair of primal/dual LPs using the simplex method. To set up a tableau for each, rearrange the objective functions in (7) and (8) so that the right side of the equality is 0. For the primal, (8) becomes $-y_1 - y_2 - \cdots - y_n + 1/v' = 0$. For the dual, we first change it to an equivalent problem of *maximizing the negative* of $-1/v'$. Thus, (7) becomes

$$\text{Maximize } x_1 + x_2 + \cdots + x_m = -1/v'$$

$$\Rightarrow x_1 + x_2 + \cdots + x_m + 1/v' = 0.$$

x_1	x_2	$\ldots$	x_m		
a_{11}	a_{21}	$\ldots$	a_{m1}	-1	x_{m+1}
a_{12}	a_{22}	$\ldots$	a_{m2}	-1	x_{m+2}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	-1	$\vdots$
a_{1n}	a_{2n}	$\ldots$	a_{mn}	-1	x_{m+n}
1	1	1	1	0	1/v'

y_1	y_2	$\ldots$	y_n		
a_{11}	a_{12}	$\ldots$	a_{1n}	1	y_{n+1}
a_{21}	a_{22}	$\ldots$	a_{2n}	1	y_{n+2}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	1	$\vdots$
a_{m1}	a_{m2}	$\ldots$	a_{mn}	1	y_{n+m}
-1	-1	-1	-1	0	1/v'

Figure 3.7. A comparison of the dual tableau (left) and the primal tableau (right).

The pair of tableaus is very similar; the dual's layout is essentially the primal's layout with the rows and columns switched. By merging the two into a single primal/dual tableau, we can solve for both problems simultaneously.

	y_1	$\ldots$	y_n		
x_1	a_{11}	$\ldots$	a_{1n}	1	y_{n+1}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	1	$\vdots$
x_m	a_{m1}	$\ldots$	a_{mn}	1	y_{n+m}

-1	-1	-1	0	1/v'
x_{m+1}	...	x_{m+n}		

Figure 3.8. The primal/dual tableau.

Follow the algorithm described in the previous section to obtain an optimal tableau. The rightmost column contains the optimal values for $\vec{y} = (y_1, y_2, \dots y_n)$; entries labeled with slack variables are ignored. If a non-slack variable is not in the column, set its probability equal to 0. The bottommost row contains the optimal values for $\vec{x} = (x_1, x_2, \dots x_m)$, while the bottom-right entry is the optimal value for both the primal and the dual.

In Part 3, we simply convert the variables back to those for the original matrix game. Obtain $\vec{p}$, $\vec{q}$, and v with the rearranged formulas $p_i = x_i v',\ q_j = y_j v'$, and $v = v' - c$; these become the solution for the game. ■

4. TWO-PERSON NONZERO-SUM GAMES

In a nonzero-sum game, such as in Figure 4.1, a payoff gain for one player does not always result in a loss for the other player. Thus, the entries in a nonzero-sum game matrix are ordered pairs of payoffs. These kinds of games are also known as *bimatrix games*.

$$\begin{array}{cc} & \text{Column Player} \\ \text{Row Player} & \begin{bmatrix} (1,0) & (-1,-3) \\ (-1,1) & (1,5) \end{bmatrix} \end{array}$$

Figure 4.1. An example of a nonzero-sum game.

Nonzero-sum games can be played either cooperatively or non-cooperatively. In a *cooperative* game, players are allowed to coordinate strategies (make binding agreements) or to share payoffs (make side payments)[1]. In a *non-cooperative* game, both actions are forbidden. Zero-sum games are non-cooperative by nature because the sum of payoffs cannot be increased through cooperation.

4.1. The non-cooperative perspective

Unlike zero-sum games, not all non-cooperative, nonzero-sum games are solvable. The game in Figure 4.1 is one example of a game with a "solution" in the sense that it provides an acceptable conclusion. Consider the payoff pair (1,5). If each player chooses their respective strategies Row 2 and Column 2, neither player can increase their payoff by switching to another strategy. The payoff pair acts like a saddle point of a zero-sum game.

It is also possible to have a "stable outcome" when the players use mixed strategies. The expected payoff for the row (column) player is calculated from the first (second) amount in each payoff pair. Call these payoffs E_r and E_c.

DEFINITION: Let A be an $m \times n$ bimatrix game. A pair of mixed strategies $\vec{r}$ and $\vec{s}$ is an *equilibrium pair* if

$$E_r(\vec{r},\vec{s}) \geq E_r(\vec{p},\vec{s}) \quad \text{and} \quad E_c(\vec{r},\vec{s}) \geq E_c(\vec{r},\vec{q})$$

where $\vec{p}$ is any mixed strategy from the row player and where $\vec{q}$ is any mixed strategy from the column player.

Equilibrium pairs are also named *Nash equilibria* after the game theorist John Nash, who also proved the following:

[1] We will cover only binding agreements in this paper.

THEOREM: Every game (both zero-sum and nonzero-sum) has at least 1 equilibrium pair of mixed strategies.

The proof is a non-constructive existence proof and does not show how to find equilibrium pairs. Though there are methods to find such pairs, they are more difficult than computing values for zero-sum games. Even when the equilibrium pairs are known, they may not provide a satisfactory "solution." We give two examples.

$$\begin{array}{cc} & \text{Colette goes to:} \\ & \begin{array}{cc} \text{Opera} & \text{Boxing} \end{array} \\ \text{Roy goes to:} \begin{array}{c} \text{Opera} \\ \text{Boxing} \end{array} & \begin{bmatrix} (4,1) & (0,0) \\ (0,0) & (1,4) \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} \text{Cooperate} & \text{Defect} \end{array} \\ \begin{array}{c} \text{Cooperate} \\ \text{Defect} \end{array} & \begin{bmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{bmatrix} \end{array}$$

Figure 4.2. Two nonzero-sum games: Battle of the Buddies[1] *(left) and Prisoner's Dilemma (right).*

Battle of the Buddies: Roy and Colette get into an argument about where to go for entertainment this evening. Roy prefers the opera, while Colette prefers a boxing match. However, neither wants to go out alone. Payoffs represent the "happiness level" of each player going out.

The left matrix in Figure 4.2 has two pure strategy equilibrium pairs: (4,1) and (1,4). However, the payoffs to each player are different, so the row player Roy prefers (4,1) while the column player Colette prefers (1,4). Suppose Roy is stubborn and decides he will go to the opera no matter what. If Colette knows this, she will benefit from going to the opera for the "better than nothing" payoff of 1. On the other hand, Colette might be equally stubborn and will stick to going

[1] This type of game is more commonly known as Battle of the Sexes, but we have borrowed the name used in [5].

for boxing in order to convince Roy to switch strategy. If both players act this way, the outcome will have the worst payoff, (0,0).

The right matrix in Figure 4.2 has a single equilibrium pair at (1,1). In fact, Row 2 (Defect) dominates Row 1 (Cooperate) with respect to Roy's payoffs, and Column 2 dominates Column 1 for Colette's payoffs. However, note the payoff pair (3,3).

DEFINITION: Let (u, v) and (u', v') be two payoff pairs. Then (u, v) *dominates* (u', v') if $u \geq u'$ and $v \geq v'$. (Payoff pair dominance is not to be confused with row or column dominance). Payoff pairs that are dominated are said to be *non-Pareto optimal*, and payoff pairs that are *not* dominated are *Pareto optimal*.

The equilibrium pair (1,1) is non-Pareto optimal because it is dominated by (3,3). Is Roy Cooperate, Colette Cooperate a more viable outcome? On the contrary, either player can increase payoff to 5 by switching away from (3,3), as it is not an equilibrium pair. Neither Roy nor Colette has an incentive to Cooperate, yet switching to Defect ultimately leads to a worse payoff. This conflict of decisions is the dilemma in the name Prisoner's Dilemma.

4.2. The cooperative perspective

If not all non-cooperative nonzero-sum games have conclusive solutions, what about cooperative ones? Consider Battle of the Buddies in Figure 4.2 again, this time allowing Roy and Colette to agree on a strategy together. They decide to flip a coin and choose Opera for both if heads, and Boxing for both if tails. The two players are pursuing a *joint strategy* by assigning a probability to each outcome. In our example, both Row 1, Column 1 and Row 2, Column 2 have a probability of 0.5 of being played, while the other two outcomes have a probability of 0.

More payoff pairs are available to the players when they use joint strategies. Let S be the *cooperative payoff region*, or the set of all payoff pairs when a joint strategy is used. S is always convex, bounded, and closed. We can graph the payoff region for Battle of the Buddies by plotting the entries and connecting the outermost points with line segments.

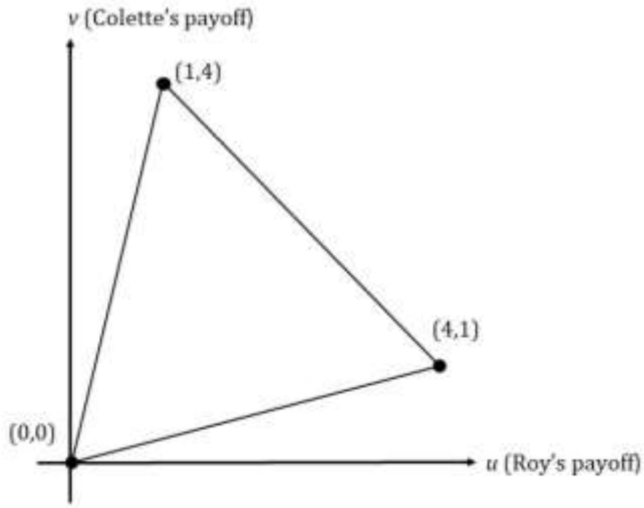

Figure 4.3. The cooperative payoff region S for Battle of the Buddies.

The problem for the players is to find the "best" payoff pair (u, v) that both can agree on. In other words, what payoff pair will be considered fair and reasonable to both players? How much payoff is each willing to lose in order to reach an agreement? John Nash developed a procedure that gives a solution to this bargaining problem. Define $\varphi(S, u_0, v_0)$ to be an *arbitration procedure* such that $\varphi(S, u_0, v_0) = (u^*, v^*)$, where

- S is the cooperative payoff region,
- (v_0, u_0) is the *status quo point*, and
- (u^*, v^*) is the arbitration pair.

The status quo point is the payoff pair that both players must accept if they reject the arbitration pair. It acts as a lower bound that the players will "default" to.

The arbitration procedure must satisfy a set of axioms, called the *Nash bargaining axioms*. The axioms can be interpreted as a set of principles ensuring fairness and consistency.

Nash Bargaining Axioms:

1) Individual Rationality: $(u^*, v^*) \geq (u_0, v_0)$.
2) Pareto Optimality: (u^*, v^*) is Pareto optimal, i.e. not dominated by any other payoff pair.
3) Feasibility: $(u^*, v^*) \in S$.
4) Independence of Irrelevant Alternatives: If $T \subset S$ and $(u_0, v_0), (u^*, v^*) \in T$, then $\varphi(T, u_0, v_0) = (u^*, v^*)$.
5) Invariance of Linear Transformations: Let T be obtained from S by the linear transformation $u' = au + b,\ v' = cv + d$ where $a, c > 0$. Then $\varphi(T, au_0 + b, cv_0 + d) = (au^* + b, cv^* + d)$.
6) Symmetry: Let S be symmetric, i.e. $(u, v) \in S$ if and only if $(v, u) \in S$. If $u_0 = v_0$, then $u^* = v^*$.

Not only does the procedure exist, it produces a unique arbitration pair for a given cooperative game.

THEOREM: There exists a unique arbitration procedure φ that satisfies the Nash bargaining axioms.

The proof actually contains the procedure for finding arbitration pairs, which we will give shortly. First, we want to define the status quo point; it is often set as the pair of maximin values.

A *maximin value* is equivalent to the row value of a zero-sum game in terms of its formula, though the values for each player are not necessarily equal. To find the maximin values, we rewrite the left bimatrix of Figure 4.2 into two matrices with single-entry payoffs. Matrix R contains Roy's payoffs, and C contains Colette's payoffs.

$$R = \begin{array}{cc} p \\ 1\text{-}p \end{array} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{array}{c} \begin{array}{cc} q & 1\text{-}q \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \end{array}$$

Figure 4.4. The two-matrix form of Battle of the Buddies with mixed strategies represented as p *and* q.

Each matrix is treated like a zero-sum game and is solved by calculating the mixed optimal strategies. Because C contains Colette's actual payoffs, not the negatives of Roy's payoffs, do not find the "column" or minimax value. The maximin value for Roy is $v_r = 4/5$, and the maximin value for Colette is $v_c = 4/5$. Thus, the status quo point is $(u_0, v_0) = (4/5, 4/5)$.

Proof. The payoff region of a game falls into one of two cases. Case 1 is the typical case.

Case 1: Suppose there exists $(u, v) \in S$ such that $u > u_0$ and $v > v_0$. Let *K* be the set of all (u, v) such that $u > u_0$, $v > v_0$. Now let (u^*, v^*) be the point that maximizes the function

$$g(u, v) = (u - u_0)(v - v_0)$$

for $(u, v) \in K$. Then define $\varphi(S, u_0, v_0) = (u^*, v^*)$.

Case 2: Suppose there is no $(u, v) \in S$ where $u > u_0$ and $v > v_0$. There are three subcases.

(a) There exists a $u > u_0$. Then define $\varphi(S, u_0, v_0) = (u^*, v_0)$ where u^* is the largest u such that $(u, v_0) \in S$.

(b) There exists a $v > v_0$. Then define $\varphi(S, u_0, v_0) = (u_0, v^*)$ where v^* is the largest v such that $(u_0, v) \in S$.

(c) Neither (a) nor (b) is true. Then define $\varphi(S, u_0, v_0) = (u_0, v_0)$. □

Since our game satisfies Case 1, we let *g* be the function we want to maximize. Our status quo point is (4/5, 4/5), and so we have

$$g(u, v) = (u - 4/5)(v - 4/5).$$

We can rewrite *g* in terms of a single variable. By Nash Axiom 2, we know that (u^*, v^*) is Pareto optimal. Graphically, this means that the pair is on the rightmost and uppermost bounding line segment. The line is easily computed to be $v = 5 - u$, and we substitute $5 - u$ for *v* to obtain

$$g(u) = (u - 4/5)(5u - 4/5) = -u^2 + 5u - 84/25.$$

Note that *g* is an upside down parabola. We can find the maximum by setting the derivative equal to 0 and solving for *u*.

$$0 = -2u + 5 \Rightarrow u = 5/2.$$

The arbitration pair is (5/2, 5/2). We see that Nash Axiom 6 is satisfied, as the payoff region is symmetric and the maximin values are equal to each other.

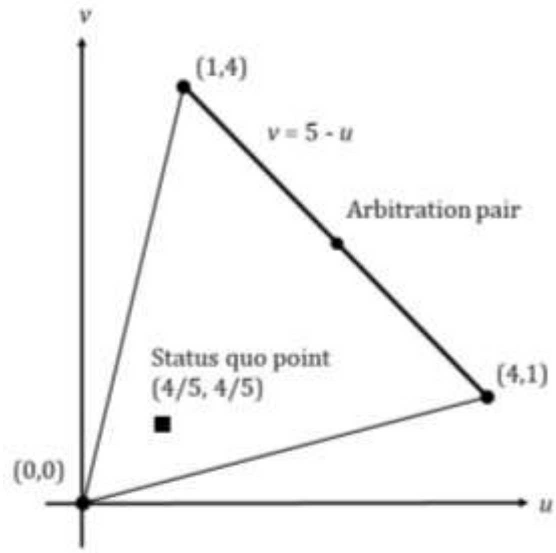

Figure 4.5. The cooperative payoff region S with status quo point and arbitration pair.

4.3. Proof of the arbitration procedure (continued)

Here, we finish the proof of the arbitration procedure started in the previous section.

Proof. We first prove that the procedure φ for Case 1 satisfies the Nash Bargaining Axioms. Axioms (1) and (3) are clearly true. For (2), suppose (u^*, v^*) is non-Pareto optimal; that is, there exists $(u, v) \in S$ such that $u \geq u^*, v \geq v^*$, and $(u, v) \neq (u^*, v^*)$. Then $(u - u_0) \geq (u^* - u_0)$ and $(v - v_0) \geq (v^* - v_0)$, which means that $g(u, v) = (u - u_0)(v - v_0) > g(u^*, v^*) = (u^* - u_0)(v^* - v_0)$, a contradiction.

For (4), the maximum value of g over $K \cap T$ is greater than or equal to the maximum value of g over K. Since $(u^*, v^*) \in T$, the two maxima are equal: $\varphi(T, u_0, v_0) = \varphi(S, u_0, v_0)$. For (5), Case 1 also holds for T with status quo point $(au_0 + b, cv_0 + d)$. Since $g(u', v') = \big(u' - (au_0 + b)\big)\big(v' - (cv_0 + d)\big) = ac(u - u_0)(v - v_0)$ and $a, c > 0$, the maximum of g is attained at $(au^* + b, cv^* + d)$. For (6), suppose S is symmetric, $u_0 = v_0$, and $u^* \neq v^*$. Then

$(v^*, u^*) \in S$ whenever $(u^*, v^*) \in S$. Then $g(u^*, v^*) = g(v^*, u^*)$. Since g has a maximum at only one point, we have reached a contradiction.

Now we prove that φ for Case 2 satisfies the same axioms. Once again, Axioms (1) and (3) are clearly true. For (2), suppose (u^*, v^*) is non-Pareto optimal. In Case 2a, $v_0 = v^* = v$, so $u > u^*$. Because u^* is defined to be the largest u, this is a contradiction. Cases 2b and 2c are similar and also lead to contradictions.

The proof for Axiom (4) is similar to that for Case 1. For (5) in Case 2a, φ is defined so that $\varphi = (au^* + b, cv_0 + d)$, and the other cases are similar. For (6), Cases 2a and 2b cannot hold if S is symmetric and $u_0 = v_0$. In Case 2c, $(u^*, v^*) = (u_0, v_0)$, so $u^* = u_0 = v_0 = v^*$.

Finally, we prove that φ is unique. Suppose there is another arbitration procedure $\bar{\varphi}$ such that $\bar{\varphi}(S, u_0, v_0) = (\bar{u}, \bar{v})$. Our goal is to show that $(\bar{u}, \bar{v}) = \bar{\varphi}(S, u_0, v_0) = \varphi(S, u_0, v_0) = (u^*, v^*)$. We begin with Case 1. Let S' be the region obtained by a linear transformation of S where

$$u' = \frac{u - u_0}{u^* - u_0}, v' = \frac{v - v_0}{v^* - v_0}.$$

The status quo point (u_0, v_0) is mapped to $(0,0)$, and the arbitration pair (u^*, v^*) is mapped to $(1,1)$. By Axiom 5, $\varphi(S', 0,0) = (1,1)$. Furthermore, every (u', v') in S' satisfies the inequality $u' + v' \leq 2$.

Now define Q to be a symmetric region where $S \subset Q$ and $a + b \leq 2$ for all $(a, b) \in Q$. Then $a \leq 1$ whenever $(a, a) \in Q$. By Axiom 6, $\bar{\varphi}(Q, 0,0) = (1,1)$, but by Axiom 4, $\bar{\varphi}(S', 0,0) = (1,1)$.

Thus, we have shown that there exists an arbitration procedure φ that satisfies the Nash bargaining axioms and is unique. ■

5. CONCLUSION

This page serves as the conclusion.

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